CERME9

Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education

Editors: Konrad Krainer and Naďa Vondrová

Organized by: Charles University in Prague, Faculty of Education

Year: 2015

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european society for research in mathematics education

Editors: Konrad Krainer, Naďa Vondrová

Publisher: Charles University in Prague, Faculty of Education and ERME

Place: Prague, Czech Republic

Year: 2015

ISBN: 978-80-7290-844-8

Recommended citation:

Author: Title of the contribution. In K. Krainer & N. Vondrová (Eds.) (2015). *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education (CERME9, 4-8 February 2015)* (pp. xx-yy). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.

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Preface

Preface

Viviane Durand-Guerrier[1] and Susanne Prediger[2]

1 Montpellier, France

2 Dortmund, Germany

Each second year, the European Society for Research in Mathematics Education (ERME) organises a conference. The 9th Congress of ERME (CERME9) took place in Prague (Czech Republic), 4–8 February 2015. A detailed report on this conference and brief remarks on further ERME activities can be found in Krainer and Vondrová (2015) who acted as the Chair of the International Programme Committee (IPC) and the Chair of the Local Organising Committee (LOC) of this conference.

It is clear per definition that CERMEs are "European", however, we are happy to welcome colleagues from all parts of the world. CERME9 was attended by 672 people from 49 countries from all over the world. With these figures, CERME9 was the largest CERME ever. This number of participants was significantly higher than what had been anticipated. Nada Vondrová, LOC chair, and all the members of the local organising committee managed to host in very good conditions all the participants. We received a great number of testimonies from participants expressing their high satisfaction for the scientific programme and the local organisation.

The programme comprised three content-related plenary activities: (a) a plenary panel on "What do we mean by cultural contexts in European Research in Mathematics Education?", held by Barbara Jaworski, Mariolina Bartolini Bussi, Edyta Nowinska, and Susanne Prediger; (b) a plenary lecture on "Research in teacher education and innovation at schools – Cooperation, competition or two separate worlds?" by Jarmila Novotná; and (c) a plenary lecture on "Understanding randomness: Challenges for research and teaching" by Carmen Batanero. The plenary lectures contributed substantially to the success of the conference.

Of central importance for each CERME are the Thematic Working Groups (TWGs). At CERME9, due to the huge number of 436 research reports and 106 posters that were accepted, 20 TWGs needed to be implemented. All TWG leaders and their co-leaders (from 23 countries) did a great job before, during, and after the conference. Among them, five acted at CERME9 their third time as a TWG leader which means that they need to be substituted according to the ERME rules for CERME10: Jeremy Hodgen, Uffe Thomas Jankvist (who also acted as the Vice-chair of IPC), Roza Leikin, Elena Nardi, and Jana Trgalova.

In particular, the presence of societies like EMS (President Pavel Exner), its Education Committee (Chair Günter Törner) and ICMI (former President Michèle Artigue), gave the conference a special flavour, as well as interesting excursions and splendid classical music during the opening and closing ceremony produced by students of the Faculty of Education, the host of CERME9.

CERME9 shows also considerable innovations regarding the organisation of the proceedings. Following a decision of the ERME Board, there will be posted on the HAL database. This will be a very valuable evolution serving the dissemination of our works for CERME9, and for the next CERME conferences. On behalf of the ERME Board and all the ERME community we address many thanks to the editors, Konrad Krainer and Nada Vondrová, for this huge work.

Although these proceedings do not contain any document related to it, let us mention another fundamental event that took place one day before the opening of the Congress: the YERME (Young European Researchers in Mathematics Education) day. This is now a constant appointment where young researchers – doctoral students or post-doctoral researchers - meet expert scholars in thematic discussion groups. At CERME9 the organization of the YERME day was coordinated by João Pedro da Ponte (Chair of the programme committee) and Jarmila Novotná, chair of the local organising committee. The activities were led by Paolo Boero, Rita Borromeo-Ferri, Uffe Thomas Jankvist, Esther Levenson, Despina Potari, Susanne Prediger, Cristina Sabena, Mario Sánchez, Susanne Schnell.

Without the deep involvement of its members, ERME and CERME will not exist. Each CERME conference represents a challenge that was successfully achieved by all involved people, but especially by Konrad Krainer, Chair of the IPC, and Nada Vondrová chair of the LOC. We address sincere thanks to both for the incredible work they have done in preparing and supervising the organisation of the conference, and the editing of the proceedings. We extend our thanks to Uffe Thomas Jankvist and Jarmila Novotná respectively co-chair of the IPC and of the LOC, to all the IPC members and to all the local organisers for their hard work, and to the Charles University in Prague, for making us so welcome.

We encourage interested researchers to meet us at the next CERME that will take place in February 2017, in Dublin (Ireland).

Viviane Durand-Guerrier,
ERME President

Susanne Prediger,
ERME Vice-President

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Editorial information

Konrad Krainer[1] and Naďa Vondrová[2]

1 Alpen-Adria-Universität Klagenfurt, Faculty of Interdisciplinary Studies, IUS, Klagenfurt am Wörthersee, Austria, Konrad.Krainer@aau.at

2 Charles University in Prague, Faculty of Education, Prague, Czech Republic, nada.vondrova@pedf.cuni.cz

The proceedings which the reader just opens is a shared product of many people, beginning with the authors of plenaries and papers from Thematic Working Groups (TWGs), the TWG leaders, and ending with the two editors.

So far, the process of editing CERME proceedings has not been openly discussed and the editors of previous proceedings approached the task differently. In order to take a step forward, we decided to describe the process here in a more detailed way, in particular highlighting innovations. This might allow the new editors for CERME10 to follow in our steps and improve the process further.

The work on the proceedings of CERME9 started immediately after the conference itself in February 2015. Already some weeks before CERME9, all TWGs presentations as well as abstracts of all plenary presentations had been available on the conference website as a kind of pre-conference proceedings. Then, the authors of TWG contributions were advised to elaborate their papers in order to meet the standard of proceedings papers of CERME, in particular, taking into account feedback they received during CERME9. In the next stage, the TWG leaders evaluated all the papers and posters and negotiated any necessary changes with the authors. This stage ended by sending the papers as complete bundles to the two editors, including letters of consent that the papers can be published in the Hall database (see below).

The proceedings consist of two main parts. The first part includes the texts of invited plenary activities, that is, the two plenary lectures and the plenary panel. There was no review process as such, but the editors commented on the papers (see below). The main part of the proceedings consists of 20 "chapters", each of them devoted to one TWG. Each chapter starts with a short introduction written by TWG leaders. In the cases of TWG09 and TWG20, the leaders responded to our call for a special kind of introduction, which is more than a short text about the papers in the TWG and a brief summary. Their introduction is rather a research paper on the advances in the topic of the TWG, however, relating it also to the developments in this field besides CERMEs.

The research reports and posters following the introduction are ordered alphabetically. All reports were reviewed twice. Prior to the conference, the papers were peer reviewed by authors from the same TWG and by at least one TWG leader. After the conference, the papers were reviewed again. TWG leaders and co-leaders acted, in fact, as chapter editors for the proceedings. They were responsible for checking the quality of the papers. Papers include accepted research papers (with a maximum of 10 pages in the original template) as well as summaries of accepted posters or short versions of not-accepted research papers (both with a maximum of 2 pages in the original template). The goal was to have all discussed papers and posters in the proceedings, if their quality allows. It was up to the TWG leaders to decide whether the final acceptance of the paper is done by them and co-leaders only, or whether people from the TWGs (e.g., former reviewers or new ones) are asked to help.[1] If the TWG leaders were unsure about the quality of the paper, they could ask the TWG liaisons in the IPC for help with the decision (before communication with the authors again), or the editors directly (this happened in two cases only).

1 For instance, when the modifications were major and the original reviewer should check them, or when the original reviewers did not agree on their decision, a different reviewer could see the modified version of the paper after the conference.

a) Editorial process for the plenary contributions

The texts for two plenaries and one panel were read and commented on by both editors. The comments were sent to the authors and they considered them. This process repeated until the editors and authors agreed that the papers were in good shape to be included in the proceedings.

b) Editorial process for the TWG introductory texts

The common CERME introductory texts were read by the two editors in order to check for any inconsistencies. The two special introductions were peer reviewed by both editors and by the leader of the second piloting TWG. The aim was to do a first systematic attempt to have all content-related texts in the proceedings (also introductions to TWGs) as research papers. This might help readers to better range the work of a TWG in the international context, might offer TWG leaders a second opportunity for a peer-reviewed paper, and, of course, contributes to raising the quality of CERME proceedings.

c) Editorial process for research papers and posters

All the files of research papers and posters were gone through by the two editors in the following way: The abstract and keywords were read in full and the rest of the text was scanned in order to see whether the language and content seemed to be on a good level. Only two papers had to be sent to their authors again to be proofread and several posters were sent to their authors so that they included an abstract and keywords. Then, the list of references were scanned in order to see whether the APA style was kept by the authors. We have to mention that there were several challenges in this area. Several papers had to be sent back to their authors to improve the literature as bibliography items missed a lot of information such as pages, editors, publisher, place of publishing. There were even problems with the name of the journal or the source document, etc. When it was possible, the editors made the changes themselves. However, the whole process was very time-consuming and finally, the editors had to give up on polishing the bibliography sections of papers and thus the APA style is not totally followed (but it is always clear what the bibliography item refers to).

After that, the approved texts were sent to the typographer (who had made a suggestion for the layout of the proceedings previously). This time we went, hopefully, for a more professional look of the proceedings, different from the mere saving of Word files into pdfs. Of course, it meant more work for the typographer, editors and also TWG leaders.

Each chapter of the proceedings related to TWGs underwent several rounds of proofreading. When the pdf files were made by the typographer, the two editors scanned the files for any typographic errors, inconsistencies (e.g., in the numbering of figures, fonts used for different parts of the text) and problems possibly caused by the transfer of the file into pdf. These changes were sent to the typographer who prepared the second version of the text which was sent to the TWG leader. His/her task was to check the files for the TWG chapter again: whether all papers were included and in the appropriate (alphabetical) order and none was omitted, whether, by mistake, some table or figure or mathematics formula disappeared and any other issue which might draw his/her attention. After their comments were taken into account by the typographer, the pre-final version was checked by the editors again.

The process was similar for the plenary texts.

The proceedings of CERME9 will be the first ones to be uploaded to Hal archive website https://hal.archives-ouvertes.fr. The ERME Board made this decision in order for proceedings to be openly available, easily searchable and downloadable and overall, more visible. It is assumed that all new proceedings will be put on the same website so that they are not in different places and elaborated in different ways. Some previous proceedings are downloadable as separate files only, some as a complete file. The Hall database enables both options. That is, the papers can be organised according to the TWGs (chapters) but can also be downloaded as separate files.

We wish the readers a pleasant journey through the ideas raised at CERME9. We thank all people mentioned above greatly contributing to these proceedings!

Plenary lectures

Cultural contexts for European research and design practices in Mathematics Education

Barbara Jaworski[1], Maria G. Bartolini Bussi[2], Susanne Prediger[3] and Edyta Nowinska[4]

1 Loughborough University, Loughborough, UK, b.jaworski@lboro.ac.uk

2 Università di Modena e Reggio Emilia, Modena, Italia, mariagiuseppina.bartolini@unimore.it

3 TU Dortmund University, Dortmund, Germany, prediger@math.uni-dortmund.de

4 Adam Mickiewicz University, Poznań, Poland, nowinska@amu.edu.pl

DEDICATION

Our young friends Mustafa Alpaslan and Zişan Güner Alpaslan passed away in a terrible car accident on July 31, 2015. Both were in CERME9: Mustafa in TWG12 (History of Mathematics Education) and his wife Zişan in TWG13 (Early Years Mathematics). They both were brilliant researchers and active members of the ERME community. In CERME9, Mustafa was elected as a representative of young researchers in the ERME Board. Mustafa took part in the preparation of this panel: We enjoyed his background in history of mathematics and his deep knowledge of the history of Turkish Mathematics Education, that made him very sensitive to the issue of cultural contexts. This text is dedicated to both of them.

The authors and guest-authors of this contribution worked together to prepare a plenary panel at CERME9. Starting from acknowledging the diversity of cultural contexts in which we work, the following questions are addressed:

- *What do we mean by cultural contexts in European Research in Mathematics Education?*
- *How do cultural influences challenge the universality of research and design practices and their outcomes?*
- *Which (hidden) values in different cultural contexts influence research and design practices?*
- *How could cultural awareness among Mathematics Education researchers be raised?*

The authors give concrete examples of cultural differences and their impact on research and design practices in Mathematics Education. They address various cultural contexts, covering mathematics, classrooms, research community contexts and international comparisons. The centrality and nature of theories in addressing mathematics educational context are also discussed. Young researchers as guest-authors contribute further experiences and reflections. The joint reflection offers multiple suggestions for raising cultural awareness in Mathematics Education research and design practices and policy issues.

Keywords: differences in cultural contexts, research practices, theoretical approaches as epistemic cultures, raising cultural awareness.

1. INTRODUCTION

While approaching the complex topic of cultural contexts and their impacts on research and design practices, we addressed four main questions:

- What do we mean by cultural contexts in European Research in Mathematics Education?
- How do cultural influences challenge the universality of research and design practices and their outcomes?
- Which (hidden) values in different cultural contexts influence research and design practices?
- How could cultural awareness among Mathematics Education researchers be raised?

We have settled on three themes as a focus for our input and subsequent discussion, presented respectively by Maria G. Bartolini Bussi, Barbara Jaworski and Susanne Prediger. These are:

1) Mathematics and Mathematics Education; how we analyse mathematical concepts; how these ways of explaining and analysing mathematical concepts can be developed for the curriculum and influence the curriculum.

2) Classrooms, teachers and students – how the cultures which underpin interaction and communication, and the use of language, enable or restrict attention to classroom approaches to mathematics, ways of conducting research and the ethical and moral principles in Mathematics Education.

3) Research approaches and theoretical perspectives, and ways in which they underpin research interpretations, the ways in which research findings emerge in research communities and are presented in published works.

We also include reflections from a group of young researchers: Annica Andersson, Mustafa Alpaslan, Edyta Nowinska and Marta Pytlak; represented in the panel and in this writing by Edyta Nowinska.

Section 2 addresses the three themes. In Section 3 we have the reflections of the young researchers. Section 4 presents a synthesis of ideas, looking backwards and then forwards towards taking up cultural issues and in Section 5 we offer questions for our ongoing practices as mathematics educators.

2. MEETING CULTURAL DIFFERENCES ON DIFFERENT LEVELS – THREE THEMES

2.1 Mathematics in cultural contexts (Maria G. Bartolini Bussi)

Every thought, when coming towards the other, questions itself about its own unthought.

(Jullien, 2006, p. vi)

2.1.1 Introduction

The mathematics developed in the West by professional mathematicians is, in some sense, *near-universal* (see Barton, 2009, who introduced the acronym NUC, that is Near-Universal Conventional mathematics, p. 10). This mathematics had become dominant all over the world, mainly for its century old effective applications to the development of science and technology. Nevertheless, the process of *mathematical enculturation* (Bishop, 1988) is, at least at the beginning, strongly dependent on the local context (often, although not always, identified with a country or a region where a language is spoken). Yet, sometimes people are so embedded in their own context as to ignore that in other contexts the "same" mathematical objects (yet, are these objects really the same?) might have had a different history and might convey even different meanings. It is only when, for some reasons, one is forced to exit her safe "niche" that she may become aware of that. There is a very positive feature in such a dialogue: "every thought, when coming towards the other, questions itself about its own unthought" as strongly claims Jullien (2006, p. vi) in his beautiful discussion of Chinese and European-Greek cultures. This awareness encourages the exploration of the geography and history of mathematical thinking (Bartolini Bussi, Baccaglini, & Ramploud, 2014).

2.1.2 First example: Whole numbers

My first example focuses on some aspects of whole numbers. This choice seems provocative: are there things more universal than numbers, at least small whole numbers, when we move from one context to another, from one language to another?

There is an extended classical literature on the history of numbers (e.g., Menninger, 1969; Ifrah, 1985) and on the use of numbers in far contexts, where, for instance, the body parts are used to represent whole numbers (Saxe, 2014) or spatial arrangements substitute the lack of number words in complex arithmetical calculations (Butterworth & Reeve, 2008).

Moreover, Barton (2009) tells the story of the verbal roles of number words in Maori.

> In Maori, prior to European contact, numbers in everyday talk were like actions. [...] Our awareness of this old Maori grammar of number suddenly sharpened when we tried to negate sentences that used numbers. [...] To negate a verb in Maori the word *kaore* is used. [...] Unlike English, where negating both verbs and adjectives requires the word 'not', in Maori, to negate an adjective a different word is used, *ehara*. (Barton, 2009, p. 5)

In Maori language to negate the sentences “there is a big house” and “there are four hills” two different wording of “not” are used. When this verbal feature of Maori numbers was ignored, the mathematics vocabulary process, translated from English, acted against the original ethos of the Maori language.

One might think that these examples are relevant only for historians or anthropologists or ethnomathematicians, but the pragmatic use of numbers in everyday communication offers some surprising evidence in familiar languages too.

There are examples (in many languages) where expressions with numbers are used to denote indefinite quantities. Bazzanella, Pugliese, and Strudsholm (2011) analyse translation problems between different languages. For instance, the Italian expression “Do you want two spaghetti?”, that does not mean exactly “two” but “a few”, might produce very funny episodes, in spoken communication between an Italian host and a not Italian guest, with the latter puzzled by the idea of eating exactly “two spaghetti” for dinner. In this case, the number is used in indeterminate or vague meaning. In other cases, what is focused is the starting point of the measuring process: for instance, an Italian speaker says “8 days” or “15 days” to mean one week or two weeks (e.g., “8 days from today” means “a week from today”). Actually in Italy a week is 7 days like everywhere, but, in similar expressions, it seems that today (the “zero” day) is counted.

Philosophers studied *vagueness* since antiquity. More recently, Black (1937) transferred the philosophical attention on vagueness to human language and Quine (1960) introduced his famous *principle of indeterminacy* of translation:

> There is no need to insist that the native word can be equated outright to any one English word or phrase. Certain contexts may be specified in which the word is to be translated one way and others in which the word is to be translated in another way (Quine, 1960, p. 69).

Humans do not really need to be precise in every situation and use vagueness and indeterminacy in thinking and communication, but are not always aware of that. It is worthwhile to reflect on the vague meanings of whole numbers that depend on the cultural context, for the importance they might have in the educational setting.

From the perspective of a mathematics teacher: is the everyday use of vague numbers consistent with the use of numbers in the mathematics classroom (e.g., just half a glass, please)? What about different utterances in the mathematics classroom, when the focus shifts from communication (“Be attentive for two minutes, please!”) to an arithmetic statement (“two times ten is twenty”)? What about multicultural classrooms where translation issues are interlaced with mathematical issues? Might vagueness foster or inhibit the construction of number meanings?

From the perspective of a researcher in Mathematics Education, is vagueness to be taken into account, as a tool, by researchers in studies on arithmetic teaching and learning in the mathematics classroom? Is the presence of either precise or vague meaning of numbers related to the development of the two core systems described by neuropsychologists for representing either small numbers of individual objects in a precise way or magnitudes in an approximate way (Feigenson et al., 2004)?

Some observation may be made also about curriculum development, when a cultural lens is used. We refer to two very recent “twin” papers prepared for the panel on “Traditions in Whole Numbers Arithmetic”, to be held on the occasion of the “Primary School Study on Whole Numbers” (the 23rd ICMI Study, Macau, China, June 3–7, 2015). They are authored by Bartolini Bussi (2015) and Sun (2015) and address very popular approaches to whole numbers in the West and in China.

Sun (2015) reconstructs the ancient Chinese tradition of whole number arithmetic and its strong connection with today’s curriculum. She emphasizes both linguistic and historic-epistemological perspectives. From the linguistic perspective, Sun’s paper reads (p. 141):

> Unlike English and most Indo-European languages, written Chinese is logographic rather than alphabetic, and uses the radical (“section header”) as the basic writing unit. Most (80–90%) of characters are phono-semantic compounds, combining a semantic radical with a phonetic radical. *Thus, the large majority of words have a compound, or part-part-whole structure. This differs from the phonetically based structure of writing in most*

> *Western languages, in which order is more important than the combination of parts* (my emphasis).

Then Sun mentions "classifiers" or measure units, which, as in many East Asian languages, are required when using numerals with nouns (spoken numbers). In Chinese, each type of object that is counted has a particular classifier associated with it. As a weakening of this rule, today it is often acceptable to use the generic classifier "ones" in place of a more specific classifier. As a character, "ones" (classifier) is pronounced *gè* and written 个 which represents a bamboo shoot. This feature highlights the focus on separate units since the ancient ages in China to present days.

Mathematics in ancient China was identified with arithmetic and numbers, and, in particular, with calculation[1]. To calculate with whole numbers in China, straight rods (also named counting sticks) were used 2500 years ago and influenced the early representation of digits (see Figure 1). The rods had square cross sections to prevent them from rolling and were carried in hexagonal bundles (Figure 2) consisting of 271 pieces with 9 rods on each edge (Lam & Ang, 2004, p. 44, see Figure 3): whenever calculation was needed, they were brought out and computation was performed on a flat surface.

Figure 2 shows the structure of the bundles of rod numerals and appears in ancient pottery. The photograph in Figure 3 was taken by the author in Banpo, a neolithic settlement close to Xi'an, dating back 5600–6700 years, and shows a fragment of a potsherd. The explanation reads: "This potsherd makes the concept of point and surface, number and shape together. It proves Banpo ancestors have certain knowledge of mathematics."

This focus on numbers as discrete quantities has remained in the tradition of teaching arithmetic. Even today, Chinese spoken numerals are the following:

> Chinese spoken numerals (and English literal translation):
>
> 一个、二个、…
> One ones, two ones, ...
>
> 一十一个、一十二个、…
> one tens & one ones, one tens & two ones, ...
>
> 二十个，二十一个…
> two tens, two tens & one...

The tools for calculation in ancient China were rods or, later, *suànpán*, the Chinese abacus. Drawing on both tools, spoken numerals are transparent for place value. Actually, place value was (and still is) considered an overarching principle for whole number arithmetic. No specific chapter on place value appears in Chinese textbooks, as place value representation is the only way to approach numbers in language, in written representation and calculation practice. Finally, Sun concludes:

> By comparison, if the number concept is represented by the number line, used with calculations by counting up or down, or skip counting, the associativity of addition is developed less naturally than with the composition/decomposition model incorporated into the *suànpán* [...] The *suànpán* makes place value explicit, and the calculation procedures of combining ones with ones, tens with tens, and so forth, are built into its structure. The model for numbers provided by the *suànpán* may be contrasted with the number line, which is a continuous, non-digital model for numbers, and is not naturally connected with place value. (Sun, 2015, pp. 151–154)

One might contend that discrete quantities and figured numbers such as the ones in Figure 2 and Figure 3 were known and used in ancient Greece since the age of Euclid and earlier. But these figural representations of numbers did not show any connection with verbal and written representation of numbers in calculation at that age. Hence, place value came later (through the Arabic mediation) and had to fight against the practical ways of calculating by means of abacus as a famous picture shows (Menninger, 1969, p. 350).

As far as the number line is concerned, Bartolini Bussi (2015) reconstructs the origin of this number representation, dating back to Euclid and reconsidered in the 17th century when scholars such as Descartes

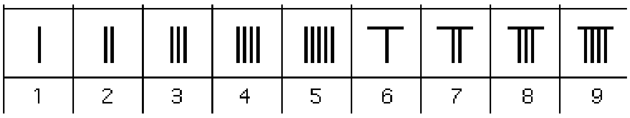

Figure 1: The first nine numerals from rods tradition

1 Actually, in Chinese 数学 (shùxué, i.e., mathematics) literally means "number study" whilst in Greek μάθημα (mathema) means "knowledge, what has to be learnt".

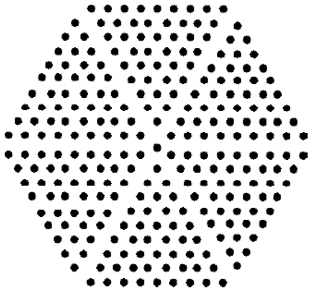

Figure 2: Cross-section of a hexagonal bundle of rods

Figure 3: A fragment in Banpo museum

and Wallis exploited the synergy between arithmetic and geometry. Euclid's use of straight line to represent numbers (in the Book 7 of the Elements, Heath, Vol. 2, p. 277 ff.) is interpreted by Netz (1999).

> Often the proof is about "any integer", a quantity floating freely through the entire space of integers, where it has no foothold, no barriers. [...] A dot representation implies a specific number, and therefore immediately gives rise to the problem of the generalisation from that particular to a general conclusion, from the finite to the infinite. Greek mathematicians need, therefore, a representation of a number which would come close to the modern variable. This variable [...] is the line itself. The line functions as a variable because nothing is known about the real size of the number it represents (Netz, 1999, p. 268).

This interpretation puts the number line into a Western cultural process, where the issue of variable, generality and proof are approached following Euclid's style.

We may conclude that rods and *suànpán* on the one hand and the number line on the other hand are cultural artefacts characterizing the Chinese and the Western curriculum. As cultural artefacts, they reveal valuable information about the society that made or used them and, when continuity between tradition and today's practices is maintained, foster the students' cultural awareness of the role mathematics played in their society. This was not always the case, in mathematics curricula, as the second example shows.

2.1.3 Second example: Fractions

The second example concerns the idea of fraction that appears in the language of everyday life, but is perceived as advanced and difficult in Mathematics Education. I have recently published with two colleagues (Bartolini Bussi et al., 2014) a short piece of speculation about the geography and the history of the idea of fraction. We observed that in most Eastern languages (Chinese, Japanese, Korean, Burmese and similar) fractions were written and are still read bottom up, i.e., reading first the denominator and then the numerator. This idea mirrors the genesis of fraction as a part of a whole: to know first in how many parts the whole has been broken and to tell later how many pieces one takes. This way of writing and reading fractions was presented also in the Liber abaci, the text, authored in Latin by Leonardo Fibonacci drawing on Arabic sources, that introduced into Europe the so-called Hindu-Arabic notation and written algorithms (Cajori, 1928, p. 269). It is still extant in some European languages developed in countries which were for centuries under the influence of Arabic and Persian culture (e.g., Turkey). Then the story of fraction names in the European languages diverged, going farther from the genesis and adopting a top-down writing and reading, with the additional puzzle of using ordinals for the denominator. A similar process happened in some Eastern countries (e.g., Myanmar) under the effects of colonialism that cut the roots with local traditions in schools.

From the perspective of a mathematics teacher: the awareness of the gap between the genesis and representation of fractions may possibly be used to support low achievers. Actually, the inversion of numerator and denominator and the use of cardinal numbers, based on the Chinese reading and writing ("three parts, take one!" "five parts, take two!") had an immediate positive effect on the performances of dyscalculic students in the task of quick positioning of a given fraction between 0 and 1 on a number line (Bartolini Bussi et al., 2014).

From the perspective of a curriculum developer: how do we consider the present trend of "imitating" Western curricula to innovate early schooling in many developing countries, when it cuts the roots with local languages and everyday experience, especially in those cases when the local language is closer to the genesis of the mathematical idea (see also Boero, 2013, pp. 25–26)?

2.1.4 Third example: Infinity and limit

The third example shortly refers to a recent study carried out by Kim, Ferrini-Mundy, and Sfard (2012) about students' colloquial and mathematical discourses on infinity and limit. The study involved large samples of USA and Korean undergraduate students. The findings of the study were interesting and the interpretation even more interesting. In a part of the study, the students were asked to create sentences with the words infinite and infinity (a separate sentence for each of these words). The two ethnic groups showed very different productions. In the USA group infinite was used in the context of real-life phenomena, whilst in the Korean group the context of the sentences was predominantly abstract and mathematical with a disconnection between colloquial and mathematical discourses on infinity. USA speakers produced processual sentences, whilst Korean speakers were more likely to produce structural sentences that are closer to formal mathematical discourse.

> In English, there is an obvious lexical continuity throughout all levels of infinity discourse. The principal link that keeps all kinds of English infinity talk together as a cohesive whole is the use of the single word infinite throughout all relevant themes and levels, and in both informal and formal versions of infinity discourse. [...] (Kim et al., 2012, p. 93).

In Korean, in contrast, there is a lexical rupture between levels. For instance, the Korean term for infinity (in everyday language) is taken from the Chinese *mu-su* (verbatim *numberlessness*) or *mu-gung* (verbatim *endlessness*), whilst the formal term (in mathematics) is taken from the Chinese *mu-han* (verbatim *boundlessness*). There is no emphasis on the shared part *mu* that means *none*. Hence, there is an evident break between everyday and mathematical terms (Kim et al., 2012).

It seems that English and, more generally, those European languages that were developed under the influence of the Greek thought, defined a path towards the mathematical idea of infinite and infinity (as it was constructed by mathematicians in the 19th century) that in some sense may model also the slow process of students from these cultures through the idea of potential and actual infinity. But this process is not likely to be the same for students coming from cultures where these ideas were developed only in schools (and not in everyday life), after and under the pressure of Western mathematics.

From the perspective of a researcher in Mathematics Education: have these findings the potential to question the faith in the "universal" validity of studies about the obstacles met by Western students in advanced mathematics?

2.1.5 Concluding remarks

In my contribution, I have considered only language to hint at the local culture and context. The examples contain arguments towards different and even opposite strategies for developing mathematics curricula: to exploit local languages and everyday experience with a slow transition from colloquial discourse to mathematical formal discourse versus to start from scratch ignoring the relationships between colloquial discourse and mathematical formal discourse. Is the difference related to the focus, i.e., elementary versus advanced mathematics? In all cases, however, it seems that considering the history and geography of mathematical thinking and the parallel development of language are essential for explaining and analysing the didactical phenomena to be considered in the design and implementation of mathematical curricula (Boero, 2013). Without this attention, it is likely that researchers from different contexts do not even understand each other and cannot exploit each other's findings (Bartolini Bussi & Martignone, 2013). This is one of the reasons why in the ongoing ICMI Study 23 (*Primary Study on Whole Numbers*, http://www.umac.mo/fed/ICMI23) a mandatory cover document about the context of each submitted paper was required in the Discussion Document. The papers which were submitted offered evidence not only of language differences (with strong effects on the arithmetic taught in primary school) but also of different societal norms, customs, institutional conditions, values and theoretical approaches, in one word, of different cultures.

In this section, I have offered mathematical examples, maybe strongly unexpected mathematical examples, as the audience's reaction showed in the panel and beyond. They question the idea of mathematics as a "universal language" or *lingua franca* of the modern world (Kim et al., 2012). In addition to general arguments about cultural relativism, mathematical examples offer a staggering evidence that cannot be ignored. They contribute to determine the contexts, the gap between them and the possibility/impossibility of easily transposing theories, findings and methodologies from one context to another, both in research and in classroom practice (Ramploud, 2015) against the naïve myth of "universality" of Mathematics Education research. In this case, it is the stunning difference that produces information.

The examples come from ethnomathematics, from the pragmatic use of numbers in everyday communication and from the comparison of mathematics curricula. In a more general way, examples may come from the curiosity about the history and geography of mathematical thinking. They concern both elementary and advanced mathematical thinking, with a potential conflict concerning continuity versus discontinuity between everyday and mathematical language. In particular, I challenge the presumed "universality" of number words and of mathematics language and, as a consequence, of theories, methodologies and findings in the studies on Mathematics Education.

2.2 Researchers, teachers, students in Mathematics Education and their values in cultural contexts (Barbara Jaworski)

The previous section (2.1) has pointed out that sometimes people are so embedded in their own context, the safe 'niche', as to ignore differences in other contexts. What seem like "the same" mathematical objects might have had a different history and might convey even different meanings. It is only when, for some reasons, someone is forced to exit her safe niche that she may become aware of that. The section focused on mathematics itself and the ways in which it is represented in writing and talking in differing cultures. Here, I take up this theme of the "safe niche" to look more broadly at context in mathematics learning, teaching and research in Mathematics Education to seek out cultural distinctions and anomalies. This will take us beyond mathematics itself into educational issues, particularly those that stem from the ways in which mathematics is regarded in education and society.

2.2.1 Introduction – who we are, and how this influences our work as Mathematics Education researchers

When we engage in research which involves human participants, we have *moral* and *ethical* responsibility towards our research participants (Pring, 2004). Pring writes:

> I shall argue that education itself is a moral practice ... Ideally the 'practice' should be in the hands of moral educators (who themselves should manifest the signs of moral development). (2004, p. 12)

As researchers in Mathematics Education, we are required to attend to ethical issues in our research. Pring (2004) goes further to suggest that we are tasked with a *moral* agenda where research in education is concerned. A question for us all is what such morality involves. For example, we need to be aware of the *values* we bring to interactions, decisions, interpretations and judgments (Bishop, 2001; Chin, Leu, & Lin, 2001), how they relate to mathematics itself, and how they fit with the cultures in which our research takes place. These cultures are manifested in our lives and work, the societies and systems of which we are a part.

As a *mathematics* educator, I have ways of seeing and arguing rooted in the mathematics which has formed a central part of my studies and professional life; this is likely to distinguish me from educators in other subjects or from scholars in the natural sciences or humanities. Mathematics itself has cultural resonances, related to moral questions and values within society, as I shall discuss further below. Indeed, Section 2.1 has drawn attention to many aspects of mathematics and how these vary across parts of the world. However, other cultures are also central to our activity. As a *researcher*, I belong to a different culture from that of a teacher I work with although we are both interested in the learning and teaching of mathematics – I have university and research values; the teacher has school and teaching values (Jaworski, 2008). Here, culture is related to where we work and the values associated with the job we do. My own values are theory-related, since an expectation of a university role is to engage with theory and research as well as the university as an institution; a teacher is concerned with school values, students' characteristics and needs, and societal and political demands such as examination results and league tables of 'effective' schools.

Being a teacher involves different expectations and values in different settings, particularly across national boundaries. Such differences are highlighted by a Finnish colleague, Kirsti Hemmi, who wrote to me as follows about her experiences as a teacher of mathematics in Finland and in Sweden:

> I was recruited to Sweden to teach Finnish speaking children in the beginning of [19]80s. Since then, over thirty years, I have worked in the cross-section of these two educational cultures and experienced and witnessed other teachers' similar experiences about the very different attitudes towards what it means to be a primary school teacher and towards what kind of skills and understanding we expect children to develop in reading, writing and mathematics during the first school years. In this work the different cultural-educational traditions really collided in various ways, not only through the Finnish and Swedish teachers' different educational backgrounds but also through the character of the teaching materials, especially in mathematics produced in these two countries. Sometimes the differences were concrete, sometimes they were hard to articulate. (Kirsti Hemmi, personal communication)

Kirsti Hemmi's words provide insight into differences between cultural settings where, more superficially, there might be expected to be common understandings and ways of interpreting educational issues. When we work as researchers across national boundaries, how we understand each other becomes central to the ways in which we undertake research. In a personal communication, Heidi Krzywacki (from Finland) wrote to me about her experiences of conducting professional development research in Peru, to provide new ideas about Mathematics Education and teacher education not common in the current educational reality in Peru. She writes:

> I have reflected on some issues related to language in international cooperation and development work that we have had with Peruvian partners for developing their education system. For example, it took a while to understand that we had no common apprehension of action research: for the Finnish partners it was used for referring to a methodological approach, but Peruvian partners interpreted it (after translation) as personal reflections.
> (Heidi Krzywacki, Personal Communication)

These words alert us to differences in perception that may be ignored, perhaps dangerously for the ensuing research, because the language of communication hides important subtleties of meaning. Since differences in practices and cultural values underpin what is possible in educational environments, researchers working in these environments must be alert to such differences and must factor them into a research study; not easy to achieve, and requiring awareness and sensitivity. In the next section, I offer further examples to highlight issues which arise from cultural norms, sensitivities and differing values.

2.2.2 Cultural contexts and their influence on how we think and behave

I start with examples from my own experience. I have worked as a teacher and as a researcher in several countries including Pakistan and Norway, where I come from a different culture (call it, rather superficially, a *British* culture) from the people in whose country I am working. In Norway, we share western ways of thinking and a Christian tradition, but there are differences, some subtle, but nevertheless important. One example, which I met very early in relationships with Norwegian colleagues, is the *Law of Jante*, created by the Dano-Norwegian author Aksel Sandemose (Sandemose, 1933/2005) – the idea that there is a pattern of group behaviour towards individuals within Scandinavian communities that negatively portrays and criticizes individual success and achievement as unworthy and inappropriate (Wikipedia; (http://en.wikipedia.org/wiki/Law_of_Jante accessed 20-4-15).

> Most Danes seem to [be] much more reserved and humble in everyday life. These rules refrain people from "judging a book by its cover," as they encourage assuming that they are no better than the person they are meeting. (Gratale, 2014).

In discussions with colleagues in Norway about research approaches and the teaching of mathematics, it became an issue for me to take a more modest, or 'humble' stance on my own perspectives. Thus, awareness of culture impacted on how I as a researcher interacted with colleagues and approached my research role.

In Pakistan, the cultural differences are more obvious, and religion plays an important role – the Muslim religion and associated social values permeate how people think and what is possible in schools and classrooms (Farah & Jaworski, 2005). For example, when working with teachers in a master's programme in Pakistan, I emphasised the *value* (in mathematics) of *asking questions* about the mathematics in which we engaged. My argument was that inquiry approaches to mathematics, involving questioning of relationships and procedures, encourage students to go beyond the procedural towards more conceptual understandings of the mathematics in focus (Jaworski, 1994). One teacher chose to write an essay about 'questioning as a pedagogic approach'. She drew attention to the fact that in Pakistani society questioning is largely discouraged because it shows a lack of respect for parents, teachers or anyone senior in the community (Jaworski, 2001, p. 312). This observation led to our addressing questioning not only as a pedagogic approach in mathematics, but also as an issue of *values* in the Pakistani society impinging on what is possible in the mathematics classroom. We see here issues related to pedagogic practice designed to improve the learning of mathematics and specific societal norms, alongside the communicative difficulties across cultural boundaries.

Anjum Halai addresses such issues from within her own and another cultural context, Mathematics Education in Pakistan and in East Africa, raising wider social issues that have a compelling need to be addressed.

> Recognition of learners who are marginalized due to socio-economic status, gender, language or other factors would mean questioning deep seated assumptions that underpin the organising structure and process of classrooms, in this case mathematics classrooms. For example, in patriarchal societies with roles defined on the basis of gender, teachers often subscribe to the dominant social and cultural views that boys are inherently better in mathematics thereby marginalizing girls in terms of participation in mathematics. (Halai, 2014, p. 69)

For researchers, not questioning those deeply held cultural views, which limit participation of both, boys and girls, is to take a moral position. Halai positively recommends *questioning*, claiming that it is through questioning that we challenge entrenched discriminatory practices. This clearly raises issues for researchers who wish to conduct research without offending their respondents/participants, but who nevertheless see a moral dimension to their questioning of values, both in teaching and learning mathematics and in the wider society. Halai writes further:

> For skills development, processes of teaching and learning in the mathematics classrooms would move away from routine memorization of procedures and algorithmic knowledge towards participatory learning involving application of mathematics knowledge to problems. Mathematics knowledge embedded in the history and culture of the learners would be a significant element of the cultural capital being re-distributed. This would socio-culturally embed mathematics learning and reduce alienation of learners with school mathematics. (Halai, 2014, p. 69)

We see here serious challenges for researchers cross-culturally: while respecting the cultures in which we work as researchers, and without alienating those with whom we work, we need to address what we know to be good didactic and pedagogic practices in mathematics for the good of the students whose lives depend on it. These words suggest that research and educational development go hand in hand to promote practices which theory and research support as more likely to promote mathematical learning.

Diverse perspectives on what constitutes good learning of mathematics and how this relates to cultural perspectives have permeated Mathematics Education's recent history, in both developed and developing worlds. Mathematics is seen in many countries as an essential ingredient of a good education, having "exchange value" for entry to diverse disciplines and work opportunities (e.g., Williams, 2011). However, for many people mathematics appears to be outside their comprehension, creating serious sociocultural antipathies, as the next section reflects.

2.2.3 Perceptions of mathematics and mathematical achievement in diverse cultures and systems

In Mathematics Education, teachers and educators have the task of promoting mathematical learning and understanding among the students with whom they work, and researchers study the processes, practices and outcomes of this work. In 1990, writing from an ICMI study focusing on the *Popularisation of Mathematics*, Howson and Kahane (1990, p. 2/3) wrote that, in most developed countries, the public image of mathematics is bad. They quote from their respondents: “All problems are already formulated”; “Mathematics is not creative”, “Mathematics is not part of human culture”, “The only purpose of mathematics is for sorting out students”. Moreover, “the image of mathematicians is still worse: arrogant, elitist, middle class, eccentric, male social misfits. They lack antennae, common sense, and a sense of humour”. We might ask *why* mathematics elicits such negative responses from a wide range of people. In more recent years, in a study of students’ views of mathematics in secondary classrooms in the UK, Nardi and Steward (2003) characterized students’ views as expressing *tedium*, *isolation*, *rote learning*, *elitism* and *depersonalization* (pp. 355–360) – students were T.I.R.E.D, of/with mathematics. These findings are an indictment on the students’ experiences of mathematics in their schooling. Such unwelcome messages challenge the educational *status quo* in the cultures to which they relate and impact on systems and practices. The challenges for researchers, and associated responsibilities, go beyond pointing out the failings towards a recognition of where persistent practices are failing learners.

For example, in the UK, long-standing practices which resist challenge involve the grouping students into ‘sets’ for mathematics based on achievement within the system. Such setting, based on achievement, is much less common in other disciplinary areas. Setting practices have discriminated against certain groups of students, leaving them to be defined as ‘low achieving’ or even ‘low ability’ (Boaler & Wiliam, 2001). In particular, setting regimes and associated forms of national assessment were found by Cooper and Dunne (2000) to discriminate again girls and students of lower social class. Why such practices are maintained, given the research evidence against them, is a cultural phenomenon, deeply embedded in the educational system and perceptions of policy-makers and teachers. In a study based on the ways in which committed teachers interpreted mathematics teaching in two schools in the UK, Boaler (1997) showed that differences in school organisation and teaching approach led to different ways in which students perceived and succeeded with mathematics, with differential effects for boys and girls. Such research findings at a national level beg questions about educational practices and learning outcomes in other countries which are addressed through international comparisons in mathematics.

In international comparisons of mathematics achievement, successive IEA studies have shown that several European countries perform relatively poorly in contrast with achievements in some eastern countries (Mullis, Martin, Gonzalez, & Chrostowski, 2004). We might ask whether ‘TIRED’ students are unlikely to achieve highly; or perhaps whether forms of ‘setting’ can be linked to national outcomes. It seems clear that the outcomes of testing students in international comparisons reflect cultural perspectives in mathematics and in education. As well as comparing learner outcomes in these countries, such studies beg many questions about the educational systems, classroom practices and education of teachers to which student learning outcomes relate. More recent studies, such as the TEDS-M study (Tatto et al., 2012), have taken up some of these questions.

For example, comparing national results in the TEDS-M study of teacher education, Kaiser and colleagues (2014) address the question: “What are the professional competencies of future mathematics teachers [in the countries to which the study relates]?” They write:

> In the secondary study, participants from Chinese Taipei outperformed all other participants, in relation to MCK [mathematical content knowledge] as well as MPCK [mathematical pedagogic content knowledge]. Participants from Russia, Singapore, Poland and Switzerland followed the Chinese Taipei prospective teachers with their achievements in MCK, German and US American prospective teachers achieved slightly above the average, … .
>
> These results point to interesting differences between prospective teachers for primary level and secondary level and confirm the superior performance of Eastern prospective teachers compared to their Western counterparts in most areas.
>
> In contrast, in Scandinavian countries, North and South America, and in countries shaped by US-American influence such as the Philippines or Singapore a so-called “progressive education” with child-centred approaches characterises

> school and teacher education. (Kaiser et al., 2014, pp. 42f)

By focusing on national achievements, these authors alert us to the differing systems and beliefs which underpin educational achievements. They suggest that links can be seen between these findings and those from surveys of student achievement in the same countries, and point towards the differing 'orientations' between countries as contributing to the findings of the study:

> The studies explore amongst others the extent to which a country's culture can be characterised by an individualistic versus a collectivistic orientation using the cultural-sociological theory of Hofstede (1986). The collectivism-individualism antagonism describes the extent to which the individuals of a society are perceived as autonomous, the role and the responsibility of the individual for knowledge acquisition plays an important role. (Kaiser et al., 2014, p. 44)

Our attention is drawn here to the differing beliefs which shape an education system, and which are rooted in the collectivist-individualist debate which underpins the values on which educational practice is based. It is far from simple to address such established positions, especially when they are supported by politics and legislation; nevertheless as moral educator-researchers we cannot ignore such challenges.

2.2.4 Values as a determinant of difference

Questions related to *values* have been addressed explicitly in research looking at the importance of values in relation to classroom mathematics. The values that a teacher holds are influential on the ways in which curriculum content is addressed in the classroom. For example, Alan Bishop has written:

> If a teacher continually chooses to present opportunities for investigation, discussion and debate in her class, we might surmise she values the ability for logical argument. (Bishop, 2001, p. 238)

Maybe further, we might surmise she believes that logical argument is important to mathematical conceptualization. Bishop argues that beliefs and values are reflexively linked for teachers deciding how to bring mathematics to the students in their classroom. For the TIRED students in the study reported by Nardi and Steward, we might wonder about the beliefs and values underpinning the teaching of these students. Although such issues are not addressed explicitly in their paper, Nardi and Steward write nevertheless: "The students seemed to resent mathematical learning as a *rote learning* activity that involves the manipulation of unquestionable rules and yields unique methods and answers to problems." (p. 362) Thus, *promoting rote learning* can be perceived by some as a legitimate way of teaching mathematics and is related to classroom values underpinning findings in this study.

In research in Taiwan, Chin, Leu, and Lin (2001) compared and contrasted the beliefs and values of two teachers and concluded that the process of making their values explicit had effects on their teaching related to the particular values they espoused. In one case, the teacher came to realize that using language that is more familiar to students, encouraging students to express mathematical ideas in their own language and only later moving to formal expressions, has positive outcomes for students' mathematical learning.

Language also formed a central issue for Lee (2006) in a study of her own teaching approaches in a UK comprehensive school. She pointed to the importance of the transition from students' own natural language to mathematical language, designing and exploring classroom approaches that promoted students' development of mathematical language. Students had to get used to using mathematical terms and expressing in their own ways the meanings of these terms. Lee, acting as teacher-researcher, stood out against the prevailing ethos in her school in relation to the wider educational system. Her research demonstrates possibilities for promoting the development of mathematical language in the classroom and stands as a beacon for other teachers within the system.

These examples point to morality issues at the classroom level, involving teachers working with their students. However, teachers have to work within the prevailing system which imposes values beyond their own activity and that of their students. The educational system, with its ways of organising schools, curriculum and examinations, is formed within the nation's societal and political forces which are culturally determined.

2.2.5 Ways of knowing and being
Cultural determination within any society might be seen in terms of 'ways of knowing and being'. Research by Belenky and colleagues (1986) pointed to "Women's Ways of Knowing", drawing attention to the ways in which women perceive their various 'worlds' differently from male perceptions. Such worlds are the focus of Holland and colleagues (1998) who have suggested that humans make sense within "figured worlds", or figurative, narrativized or dramatized worlds:

> [A figured world is] a socially and culturally constructed realm of interpretation in which particular characters and actors are recognized, significance is assigned to certain acts, and particular outcomes are valued over others. (Holland et al., 1998, p. 52)

Ways of knowing or *figured worlds* can be seen to underpin the ways in which Mathematics Education is approached and practiced in culturally different educational settings. The examples provided above are all situated with respect to such worlds which may be local national or cross-national. Researchers working in cross-national frames may be unaware of the worlds of their partners, of ways of interpreting constructs and concepts, of overpowering societal or political forces. Issues arising may present tensions and dilemmas which need to be exposed, discussed and deconstructed in reports from the research.

I have drawn attention to cultural values permeating mathematics learning and teaching in and beyond national boundaries raising moral and ethical questions for teachers, educators and researchers. Research activity and practices cannot be divorced from the educational values that permeate societies and are promoted by systems, politics and legislation. The researcher is more than a recorder of practices and issues and cannot avoid involvement. We therefore need much more cognisance of ways in which research questions cut across ways of knowing and being in the worlds in which we do our research. This in itself is a research agenda.

2.2.6 Key concepts
The following is a list of the key concepts addressed in this section:

1) The moral and ethical nature of Mathematics Education practices and associated responsibilities of practitioners and researchers;

2) The centrality of *values* to educational research and practice in mathematics;

3) Perceptions of mathematics and their relation to didactic and pedagogic practices;

4) Differences of perception of mathematics educational practice rooted in cultural contexts and issues for researchers in challenging established ways of being;

5) International studies with cross-national comparisons and the challenges they raise for culture-bound practices;

6) Figured worlds which narrativise the human collective in Mathematics Education and require recognition and acknowledgement in their power to condition beliefs and values, and hence teaching and learning.

2.3 Research approaches and research communities as and in cultural contexts (Susanne Prediger)
The previous two sections have discussed differences in cultural contexts concerning *the mathematics* itself (in Section 2.1) and the contexts of *teaching and learning mathematics* (in Section 2.2). For both cultural contexts, it was shown how the hidden assumption about the universality of our own practices and values (of writing and doing mathematics, ways of teaching, of educating teachers) must be challenged. Instead, the mathematical and educational practices and values are deeply shaped by the culture we live in. Problems of intercultural misunderstandings can appear when we are not aware of this cultural boundedness and assume that approaches or knowledge can easily be transferred between cultural contexts.

In both sections, the "culture" in "cultural contexts" mainly referred to countries or regions, where intercultural reflections can be triggered by international comparisons. However, Maria G. Bartolini Bussi has already mentioned the differences between everyday language and mathematics language about whole numbers and Barbara Jaworski has already mentioned cultural differences within a country, for example, the researchers' culture versus the teachers' culture. This widened use of the construct "culture" is in line with modern conceptualizations, not only as national culture but as *a system of shared meanings, values and practices* shared by a group of people, also within a country (Geertz, 1973; Knorr Cetina, 1999). Specifically, Knorr Cetina (1999) has coined the term epistemic cultures and showed how implicit values and practices can shape the work of research communities and their research approaches.

Within the Mathematics Education community, especially the diversity of different theoretical approaches and the underlying research practices have been discussed as different cultural contexts. This section reports briefly on the raising awareness on these more subtle cultural contexts and discusses how strategies invented for dealing with diversity of research approaches can be transferred to other cultural differences. Thus, in this section, the relation between research approaches and cultural contexts is twofold: on the one hand, the research approaches are themselves considered *as* epistemic cultures with their own values and practices, on the other hand, research approaches have emerged *in* different international contexts and different research communities, and their specific characteristics have always shaped the epistemic cultures.

On this meta-level, the mathematics is more implicit: although the theories and research approaches deal with mathematics and its epistemological specificity, the reflection on how to combine different approaches is not so specific to mathematics anymore.

2.3.1 Different research approaches – a further cultural context influencing research and design practices

Since 2005, the ERME community has gained an increasing awareness of the existence of different theoretical approaches. At CERME congresses, the methodological discourse was installed by establishing a Working Group which is ongoing now for 10 years (Artigue, Bartolini Bussi, Dreyfus, Gray, & Prediger, 2005 at CERME5; Arzarello, Bosch, Lenfant, & Prediger, 2007 at CERME6; and successors). Successively, the awareness increases that theoretical approaches are always connected to research practices, this section hence talks about research approaches at large, including the theories framing the research as well as the underlying aims and values.

No empirical finding exists independent from the way it is generated within a theoretical approach, even if this theoretical approach is not made explicit. However, the complexity of mathematics teaching and learning can be conceptualized in very different ways, depending on the chosen theory and research approach. Figure 4 sketches an example (presented and discussed in Prediger, 2010) of the empirical problem that immigrant students have difficulties with mathematical word problems. This problem can be conceptualized, described and explained by different lenses which are connected to different theoretical approaches, either in an individual cog-

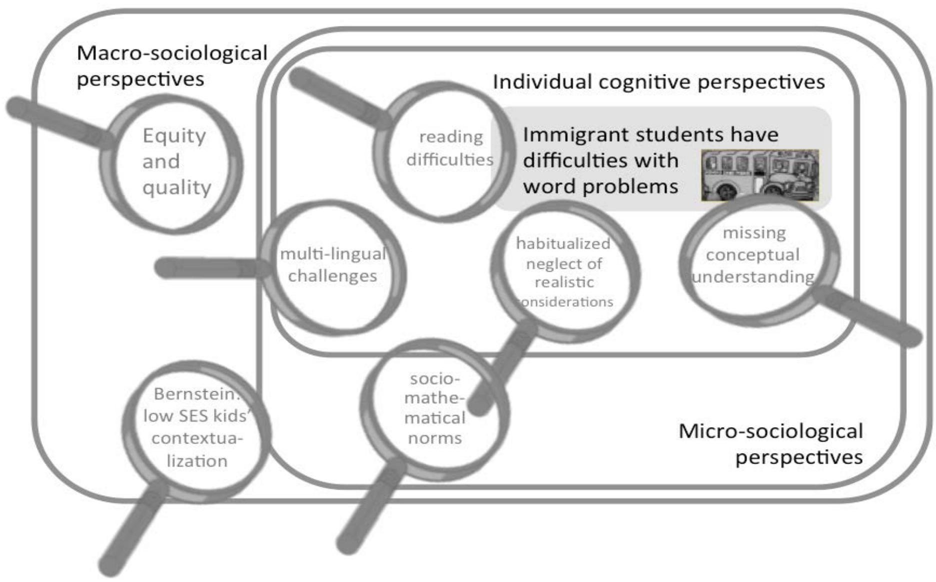

Figure 4: Different theoretical approaches to the same empirical problem (similar to Prediger, 2010, p. 183)

nitive perspective or a social perspective: Some focus (micro-sociologically) on the culture of mathematics classroom, and others also focus (macro-sociologically) on connections to students' social background and structures in society (cf., Sierpinska & Lerman, 1996, for the difference between cognitive and social perspectives at large).

In each case, the activated approach influences (but not determines) the way the problem is researched. The original problem changes its character, since the theoretical approach shapes the problem into a so-called conceptualized phenomenon (Bikner-Ahsbahs, Prediger, & the Networking Theories Group, 2014, p. 238). Additionally, the specific conceptualization of the problem influences the design consequences drawn for overcoming it. If the problem is conceptualized as students' missing conceptual understanding, the design might focus on fostering understanding. In contrast, the conceptualization as a habitualized neglect of realistic consideration due to inadequate sociomathematical norms in the classroom might lead to changing the sociomathematical norms about how to deal with word problems. If the immigrant students are mainly considered as students with underprivileged social background, a more explicit teaching might be claimed as consequences drawn from Bernstein's theoretical approach. In contrast, the conceptualization as multilingual students might result in fostering students' academic language or raising more general issues of equity (Prediger, 2010). Especially, considering challenges relating to a specific group of students (such as here immigrant students) allows for macro-sociological perspectives on issues of equity, but can also be treated as a purely cognitive problem within a specific mathematical topic.

Even this very rough outline of alternative approaches and design consequences shows how each theoretical approach influences research and design practices. This sketched example can raise the cultural awareness that *no empirical finding exists independent from its conceptualization within a (more or less explicit) theoretical approach.*

It also shows that the diversity of approaches is not only another example of cultural differences (here difference of epistemic cultures), but that it is also *necessary* in order to grasp different aspects of the complexity of mathematics teaching and learning. Having acknowledged the necessity of different epistemic cultures is however not enough to make use of the diversity, because the diversity can mainly become fruitful if the research approaches are connected (Artigue et al., 2005). As classroom reality is always complex and multi-facetted, connecting different theoretical approaches and research practices is promising in order to grasp a higher complexity at the same time.

2.3.2 Dealing with diverse research approaches and theories as an issue of methodological reflection

How can the discipline of Mathematics Education make use, more systematically, of the diversity of different epistemic cultures? As dealing with the diversity was identified as an important challenge for the international community, a subgroup of the CERME working group started to work more intensively in order to elaborate the methodological reflections, first on theories alone, later more widely on research approaches and the underlying epistemic cultures (Bikner-Ahsbahs, Prediger, & the Networking Theories Group, 2014).

Given the high complexity of mathematics teaching and learning, *one big unified theory* is not a realistic and adequate goal. Instead, the group developed the idea of aiming at connecting two or three approaches each. For this purpose, the group developed a landscape of so-called networking strategies by which these connections can be realized (Prediger, Bikner-Ahsbahs, & Arzarello, 2008).

Practical experiments with comparing and contrasting different theoretical approaches in different settings led to an increasing awareness that theoretical

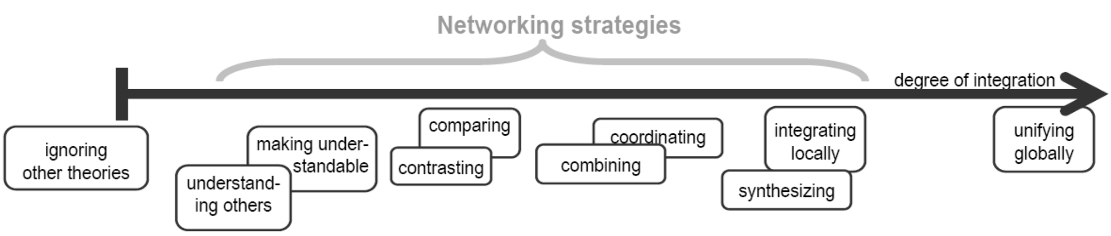

Figure 5: A landscape of strategies for connecting theoretical approaches (Prediger et al., 2008, p. 170)

approaches influence each step in a research process, not only the analysis of data as the initial example in Figure 4 might suggest:

- initial identification of a problem in classroom practice, loosely framed;
- conceptualisation of the classroom problem;
- transformation of the problem into more focused research questions;
- development of research design (including methodological choices on kinds of data, sample...);
- collecting data;
- choice and formulation of a research question;
- data analysis;
- consequences for a design of learning opportunities;
- evaluation of learning opportunities;
- ...

It was therefore an interesting experience to convert research problems from one research approach into another, because this required changes in every step of the research practice as well. These experiences showed the strong intertwinement between theories and research practices and hence the meaning of Knorr Cetina's (1999) construct "epistemic cultures".

Although there is no shared unique definition of theory or theoretical approach among Mathematics Education researchers (see Assude, Boero, Herbst, Lerman, & Radford, 2008), many authors emphasize the double role of theory being the outcome of research and, at the same time, the background theory guiding the research practice as a framework (Assude et al., 2008). Radford (2008) takes this intertwinement into account by describing theories as "way[s] of producing understandings and ways of action based on [...] basic principles, which include implicit views and explicit statements that delineate the frontier of what will be the universe of discourse and the adopted research perspective; a methodology [.... and] a set of paradigmatic research questions" (Radford, 2008, p. 320). Radford's triplet includes the so-called background theories (Mason & Waywood, 1996) with many hidden assumptions and general philosophical stances which often remain implicit.

For example, adopting a macro-sociological perspective on the example problem in Figure 4 immediately makes the researcher think about the students' background and how this is related to the mathematics learning. Relevant questions in this perspective must include societal questions. At the same time, the mathematics might be able to be treated in a more generic sense. In contrast, the cognitive perspective would focus on the specificity of the mathematical topic and its obstacles and might neglect the students' background as this question is not considered as important. Which approach is chosen might also depend on the national or regional contexts: where equity issues are prominently discussed, the macro-sociological perspective has entered Mathematics Education earlier than in other countries or regions.

Making crucial aspects of an approach explicit is therefore a major task when connecting theories and research approaches. That is why understanding another theory and making the own theory understandable was specified as the first networking strategy (cf. Figure 5), since these two strategies already require big efforts of the researchers for an "intercultural communication".

Before applying networking strategies with higher degrees of integration like combining, coordinating or synthesizing, the compatibility of the approaches in view must carefully been checked. This is necessary to avoid inconsistencies in the built network (cf. Bikner-Ahsbahs et al., 2014).

2.3.3 Theoretical approaches and research and design practices as embedded in different research communities and institutional backgrounds

Conceptualizing theories as ways of producing understandings and ways of action corresponds to culturalistic conceptualizations of research as being conducted in *communities of practice"* (Wenger, 1998). The construct *practice* is explained by Wenger as socially bound to its community:

> The concept of practice connotes doing, but not just doing in and of itself. It is doing in a historical and social context that gives structure and

> meaning to what we do. In this sense, practice is always social practice. Such a concept of practice includes both the explicit and the tacit. It includes what is said and what is left unsaid; what is represented and what is assumed. It includes language, tools, documents, images, symbols, well-defined roles, specified criteria, codified procedures, regulations and contracts that various practices make explicit for a variety of purposes. But it also includes all the implicit relations, tacit conventions, subtle cues, untold rules of thumb, recognizable intuitions, specific perceptions, well-tuned sensitivities, embodied understanding, underlying assumptions, and shared world views. Most of these may never be articulated, yet they are unmistakable signs of membership in communities of practice and are crucial to the success of their enterprises. (Wenger, 1998, p. 47)

This conceptualization of research as always being bound in epistemic cultures, or here more precisely in research communities of practice, makes it clear that the choice of a theoretical approach is not a free individual choice but always limited and fired by the specific research community and their institutional background.

The fact that Maria G. Bartolini Bussi, in Section 2.1, focuses on the mathematics itself and Barbara Jaworski in Section 2.2 on teachers, classrooms and values of the society, might also be traced back to the different research communities and institutional backgrounds in which the researchers work: Being located in the mathematics or more attached to the general education department might affect the choices about focus, perspective and theoretical approach. In this sense, every research is strongly influenced by the community of practice and its institutional background.

This point is also made by Marta Pytlak in Section 3.2: Different institutional conditions influence the research practices we can choose. For the individual young researcher, it might hence be challenging to switch between the communities.

2.3.4 Learning for cultural contexts on other levels?

After ten years of methodological reflection on the diversity of research approaches and strategies and issues for dealing with it, we can ask whether we can learn something from this cultural context on the research level for other cultural contexts (e.g., on the mathematical level or the classroom level, see Section 2.1 and 2.2 as well as 3). Are there aspects of cultural awareness that can be transferred from the level of research cultures to other levels of cultural contexts raised in other sections?

Here are some preliminary aspects which should be further discussed:

- Being aware of differences is the first step to avoid too hasty generalizations. It is worth *not to take the own assumptions for granted universally*.
- Differences do not only pose problems, but also offer *chances* since they provide a larger variety of options and increase the repertoire of research and design practices.
- Beyond openly visible differences, there are always substantial differences in more subtle, implicit layers which are more difficult to communicate. That is why *understanding* other contexts and *making* the own context *understandable* is a challenge in itself which should be taken seriously.
- *Comparing and contrasting* cultural contexts is a strategy that allows making implicit thought and aspects explicit.
- *Converting* from one cultural context to another requires very careful adaptations or even transpositions in order to adjust to the cultural context soundly.
- *Combining* different contexts requires very careful considerations of *checking compatibility* in order to avoid inconsistencies.

3. YOUNG RESEARCHERS' EXPERIENCES AND REFLECTIONS

The previous sections give concrete examples in which the *invisible culture* of mathematical thoughts (Section 2.1), classrooms, their norms and values (Section 2.2 and 2.3) had been made visible throughout a deep analysis of particular cultural contexts. These examples indicate cultural determinants of the mathematics taught and learned in mathematics classes. In the given cultural context (determined by its history, tradition, institutions, systems of values, ideologies), the invisible culture is so natural and obvious that its participants might take it for granted also for other contexts.

From the point of view of researchers it is necessary to be aware of these determinants. This awareness requires a holistic view of the learning process as a part of a certain context and its culture. The reflections worked out in Section 2.4 explain from a meta-level how researchers' awareness of the cultural dimension of learning processes might be affected by the tradition of dealing with particular theoretical perspectives being characteristic for their community. These reflections point out another level of an invisible culture – the invisible culture of scientific thought in Mathematics Education – and explain well-established strategies for raising the cultural awareness on this level.

The reflections presented by three experienced researchers motivated a group of four young researchers (Annica Andersson, Mustafa Alpaslan, Edyta Nowinska, and Marta Pytlak) to analyse and discuss their own experience in conducting and communicating research in Mathematics Education. Some results of these processes are presented in this section. They document

— the essential role of an intercultural discourse (Section 3.1 and 3.3) in raising the cultural awareness of research practices,

— the influence of the institutional context affecting the effectiveness of such a discourse (Section 3.2), and

— the need for researchers' awareness of the invisible culture underpinning ways of thinking and acting in mathematics classes in order to design effective methods for improvement of teaching and learning practices (Section 3.4).

Section 3.5 presents a discussion of the critical points raised by the young researchers and puts them in relation to the reflections presented in the previous sections. The discussion emphasizes the need for the *cultural* and *theoretical sensitivities* for better understanding of the surrounding invisible cultures in research practices in Mathematics Education.

3.1 International perspectives on local phenomena – A personal experience (guest-author Annica Andersson)

In my reflections, I focus on my experiences from Mathematics Education research and teacher education in the diverse cultural settings I have had the opportunity to work in, for example, Sweden, Denmark, Colombia, Australia, Papua New Guinea and Greenland. These cultural experiences have shown that *our own languages, contexts and cultures may become visible when we see them from the outsider's perspective, or when others confront us with questions motivating us to reflect on our own use of the languages, contexts and cultures we participate in or are familiar with.* This has been one of my richest learning experiences when communicating my cultural research with others.

For my thesis research, focusing on students' narratives about their hating/ disliking/worrying about/ Mathematics and Mathematics Education (cf. Andersson & Valero, 2015), I collected my data in Swedish upper secondary schools. The fact that I analysed my data while being outside Sweden, in an English-speaking environment (Australia), and communicated my research within an international research group in Aalborg (Denmark), facilitated me to explain and express the data to people of other languages and cultures. Consequently, I recognised some aspects influencing the ways of thinking and acting in school contexts in Sweden, and realized that languages, cultures and contexts fluctuate and are not stable. The opportunity to communicate this research within an international community raised my awareness on how one's own cultural context may be different from other contexts and how it influences the ways of acting as a teacher or researcher and, consequently, also teaching methods, research questions and theoretical approaches.

For example, Elin, a mathematics teacher I collaborated with, talked about herself as being a "Curling teacher" (Andersson, 2011). Curling is a culturally-bounded winter ice sport where competitors sweep the ice in front of a stone to get it in the best position. The metaphor of a *curling* teacher is transferred from the term "curling parent", which, in Sweden, refers to parents who "sweep the way", hence serve their children to get the right, or best, positions, solving possible problems and tensions beforehand and thus make children's lives as smooth and easy as possible. The idea of a "curling teacher" was culturally-bounded and not obvious for an international research community.

The international (or rather inter-cultural) discourse on my research contributed to make my research problems, approaches and results clear and understandable for external research communities. My experiences allow me to argue that there is value in raising discussions about understanding Mathematics Education research as culturally developed and situated. Here the question arises on how to value the "universality" of research results within an international community. It seems that this question has to be discussed from the background of the cultural contexts in which the research questions appear to be relevant.

3.2 Divergent expectations in different research communities – A personal experience (guest-author Marta Pytlak)

ERME conferences (CERMEs) are a great opportunity to get access to a research community in Mathematics Education, and get insight into new research problems, methods and theoretical approaches. Here, problems related to school mathematics, teaching and learning situations in real settings in mathematics classes are discussed from different theoretical perspectives. Theories developed in Mathematics Education are considered as important scientific achievements and research tools. CERME papers document research and development work of their authors and their quality must satisfy scientific criteria.

However, the institutional context of my work as a researcher in Mathematics Education in Poland – Faculty of Mathematics – has other criteria to evaluate my work and publications. Since Mathematics Education is not recognized here as a scientific discipline, my work is evaluated on the basis of the same criteria as the work of mathematicians. Papers published in CERME proceedings or in Mathematics Education journals are not considered as results of scientific work, regardless of the content. I am expected to focus in my work on problems relevant for a scientific work in mathematics and to deal with mathematical theories. Methods, approaches and theoretical constructs developed in Mathematics Education seem to be irrelevant in this context. This hinders the development of research communities in Mathematics Education in Poland and makes the communication within an international community very difficult.

In my PhD thesis, I focused on the development of algebraic thinking in elementary school students. In the whole process of my PhD project I used theoretical constructs, approaches and methods developed and used in international communities in Mathematics Education. While discussing my research within such communities, I received constructive responses regarding the novelty and importance of my research questions and results. Because of references to problems and literature known in Mathematics Education my work was understandable for others.

Due to my institutional context in Poland, it was nevertheless important to adapt my final version of the PhD to the institutional expectations and reduce the part related to theories in Mathematics Education. Instead, I wrote one chapter with elaboration on advanced mathematical theories relevant for my work. Addressing some historical aspects of the development of algebra allowed me to make some links between this chapter and other chapters in my thesis. The changes made to satisfy the institutional criteria for a PhD thesis brought into my work new aspects. But they also shifted the focus from theories which I had used to conceptualize particular problems in Mathematics Education to the more "universal" mathematical theories. Consequently, this changed the way that this work is embedded in the discourse of the European Mathematics Education community.

My experience indicates one of many challenges for researchers in Mathematics Education in Poland which sometimes hinder the development of research communities and the access of the small group of Polish researchers to an international community.

3.3 Cultural biases in review procedures – A personal experience (guest-author Mustafa Alpaslan)

My reflections are related to my experiences as a PhD student and graduate assistant working in teacher education for pre-service middle school (ages 11 to 14) mathematics teachers at the Middle East Technical University, in Ankara, Turkey.

My research interests focus on the integration of history of mathematics in the education of pre-service mathematics teachers. One component of my master thesis was to investigate Turkish pre-service middle school mathematics teachers' knowledge of history of mathematics and to develop a valid test for this investigation. Since the history of mathematics is a large area, the scope of the test was restricted to the historical and institutional context of mathematics taught and learned in Turkish middle schools. Mathematics curricula, textbooks and guidelines for mathematics teachers' competencies were used as a reference frame for decisions concerning this restriction.

Besides items reflecting various cultures' contributions to the historical development of mathematics, one item addressed the history of Turkish mathematical language. In it, Mustafa Kemal Atatürk's contribution to create a new mathematical language according to the new Turkish language, with Latin alphabet rather than the old Ottoman Turkish with Arabic alphabet, was captured. In Turkey, this contribution is valued as an element of our cultural identity and the awareness of it is seen as a part of mathematics teachers' professional knowledge. Atatürk's reason for preferring the new Turkish was that it was actually the spoken language in the public, thus it would provide easier and more meaningful understanding of geometrical concepts.

The context-bound nature of my research seemed to be very natural in discussions with researchers from Turkey. However, a discussion with researchers from *the International Study Group on the Relations between the History and Pedagogy of Mathematics* raised my awareness of the fact, that some items of the test designed in my research may not be understandable for this intercultural community. For example, one item in form of a multiple-choice question asking to mark Atatürk's contributions to the development of mathematics in Turkey was interpreted by the reviewers coming from other countries as an inadequate conceptualization of the investigated construct (teachers' knowledge of history of mathematics). This challenged me to provide additional information about the specific characteristic of my research context. By giving more context information, the reviewers decreased their doubts about the missing universality and significance of my research results. By my context-bound argumentation justifying my research questions, approach and results in the cultural context of mathematics teacher education in Turkey, I succeeded to publish an article from my master's thesis (see Alpaslan, Işıksal, & Haser, 2014). The cultural context was accepted by the reviewers as an essential contribution making my research understandable for them and for the potential international readers of this paper. Increased cultural awareness may help to avoid biases in review procedures.

3.4 Implementation of design research in new contexts – A personal experience (Edyta Nowinska)

In 2011, a group of researchers in Mathematics Education from Germany was assigned by the German foreign aid organization MISEREOR to support teacher education and the development of the quality of mathematics classes on the Indonesian island Sumba in order to educate the learners better for their career opportunities. For this aim, a long-term design research project had been conducted. The focus was on teaching and learning mathematics at the beginning of a secondary school, in particular on learners' cognitive habitus in learning mathematics.

Our first analysis revealed that the Sumbanese learners have difficulties with critical thinking and mathematical reasoning: On each level in the school system there, learners are used to learning by memorizing, answering collectively and waiting until the teacher tells them what is correct. They are not used to asking questions and practice monitoring (cf. Sembiring et al., 2008). Our further observations showed that there were some culture-bound variables influencing students' learning behaviour and hindering this kind of thinking and reasoning in mathematics classes. Critical thinking and rational reasoning are not essential characteristics of the Sumbanese culture, neither in the religion based on myths and legends, nor in everyday routines and system of values. This society exhibits a short-term point of view rather than a pragmatic future-oriented perspective based on critical thinking and precise planning.

It seems that the cultural context determines the ways of thinking and acting of teachers and learners. This determination results in culturally acquired epistemological obstacles, and beliefs that there are no alternative ways of acting and thinking. Thus, prior to the implementation of the teaching and learning concept developed on the basis of design principles worked out in the context of German secondary schools (cf. Cohors-Fresenborg & Kaune, 2005), a group of Indonesian pre-service mathematics teachers collaborated with the German educators to reflect on their (unconscious) ways of acting and change their own learning attitudes and teaching practice.

In this process of learning and reflecting, the pre-service teachers constructed mental models for mathematical reasoning, a system of metaphors enabling them to understand the core ideas of stepwise controlled mathematical argumentations and new beliefs about mathematics. They were astonished by the fact that the so-called mathematical "rules" can be derived and explained and that the meaning of symbols and signs can be negotiated in social interactions in class. The experience that Sumbanese learners are able to engage in such social interactions convinced the participating pre-service teachers that changes in the cognitive behaviour of the learners are possible to achieve, although the intended cognitive behaviour is not typical for the attitudes accumulated in the culture and everyday practices of the local society.

After two years of teacher professionalization courses and adapting the intervention developed in the context of German secondary schools to the Indonesian context, remarkable improvements in teaching and learning mathematics and in learners' competencies have been achieved (Nowinska, 2014). This was possible due to our holistic view of teaching and learning as a part of a certain cultural context and a culture itself. Neglecting this context and avoiding the interactive process with the participants of our project would change our intervention to a kind of indoctrination or anarchy and result in new forms of acting without understanding.

This cultural awareness and sensitivity motivated us to provide in our publications (cf. Nowinska, 2014) some explanations concerning the cultural context of our research, complementarily to the theoretical framework guiding our perception and conceptualization of "problems" in Mathematics Education. However, our experience suggests that it is a difficult task for reviewers to relate research problems and results from research and design practice in Mathematics Education to the context of this practice. It seems that in evaluating and reviewing design and research practices, the cultural aspect is often neglected and the major attention is paid to theoretical considerations and novelty of results. This may lead to trivialization of research problems and results in Mathematics Education.

3.5 Critical points raised by young researchers (Nowinska, Andersson, Pytlak, Alpaslan)

Internationalization of research in Mathematics Education (including international research conferences, publications, and collaborative and/or comparative cross-country research projects) challenges researchers to be aware of various contexts and their power to influence research and design practices in particular countries, cultures, societies, institutions and communities (cf. Atweh & Clarkson, 2001). The experiences described by four young researchers in this section indicate the essential role played by an international discourse in initiating reflections on one's own ways of thinking and acting as a researcher, educator or designer in a particular community, culture and society. Such a discourse challenges researchers to see their own practices from a broader perspective and may contribute to making them understandable for others.

The experience described by Annica Andersson indicates possible benefits that can result from participating and working in various contexts and from discourse within an international community of researchers for perception and better understanding of the unique characteristic of their own cultural context. Cultural differences and similarities become visible first as results of reflection and comparisons. Thus, an international discourse requires and facilitates cultural awareness.

Evidence of possible difficulties emerging in such a discourse is given in the reflections of Mustafa Alpaslan. His decisions while conducting his research project were affected by the historical context of the development of mathematical language in Turkey, yet this context was not made explicit when submitting a paper to an international community. Consequently, the novelty and importance of his research could not be understood by the reviewers coming from other cultural context until additional reflections on the cultural context of his research had been made explicit.

Similar challenges, yet related to the institutional context, are mentioned in the reflections provided by Marta Pytlak. The institutional criteria used to evaluate the work of many Polish researchers in Mathematics Education hinder the development of research communities and their work on problems related to teaching and learning of mathematics in schools.

Designing, justifying and evaluating indirect didactical actions is at the heart of research practices aiming at better understanding and improving teaching and learning processes (cf. Sierpinska, 1998). The experience described by Edyta Nowinska indicates that interventions and design principles resulting from design research cannot be seen as universal solutions for educational problems (cf. Plomp, 2013), even if some of these problems seem to be shared in various cultures.

From the perspective of researchers, seeing *similarities* in the nature of educational problems may contribute to a better understanding of these problems. However, complementarily, crucial *differences* between contexts where they appear should be considered. They provide some strategic guidelines for raising cultural awareness in our "daily" design and research practices: The challenge is to perceive the unique characteristic of the cultural context in which the design and research practices are conducted or have to be transferred into and to make them understandable for the participants of research as well as for readers and reviewers of research papers. Hofstede's dimensions of national culture (e.g., individualism versus collectivism, masculinity versus femininity, uncertainty avoidance, and long term orientation) (Hofstede, Hofstede, & Minkov, 2010) can be used to facilitate the *cultural sensitivity* needed to perceive such characteristics, complementarily to the *theoretical sensitivity* guiding researchers' perception and conceptualization of "problems" in Mathematics Education.

The reflections written by the four young researchers give insight into the complexity and variety of contexts that have to be taken into consideration to raise one's own cultural awareness. Some factors motivating and facilitating this kind of awareness can be identified. It seems that internalization of research and design in Mathematics Education, in particular the movement of young researchers within the international community, bring the importance of this theme again and again to light.

Critical points raised in reflections exposed by the young researchers:

— An international discourse initiates and facilitates reflections on one's own practices in a particular community of researchers, in a culture and society. It challenges researchers to raise cultural awareness and supports this awareness by providing new perspectives for contrasting one's own ways of acting and thinking with the ways of others.

— The challenge for raising cultural awareness of researchers is to make explicit the implicit decisions associated with their own design and research processes. It seems that a discursive approach within a community of researchers from different contexts may initiate, facilitate and raise cultural awareness of individuals.

— Writing about one's own research without justifying the choices of research questions and methods in the particular context of this research may not be understood by reviewers from another cultural context. Cultural awareness can help to avoid biases in review procedures.

— Internalization and globalization of design and research in Mathematics Education support transfer of knowledge and experience among researchers, in particular curricula and teaching interventions. However, teaching interventions cannot be implemented to a new context when the details of these interventions do not make sense in this context. Adaptation of design principles must take the local context into consideration.

Not only theoretical considerations but also various aspects of the particular social, historical, cultural and institutional context make design and research activities understandable.

4. LOOKING BACK AND LOOKING FORWARD

4.1 Looking back: Cultural differences in various contexts

The experiences and reflections presented in Sections 2 and 3 by experienced and young researchers all report on cultural differences which were only partly explicit and on how the differences or their implicitness affected the research or design practices.

However, these rich examples are located on different (of course overlapping) levels of cultural contexts (cf. Figure 6), covering the mathematics itself (Section 2.1), but also contexts of mathematics classrooms and the societies (Sections 2.2 and 3.1, 3.3 and 3.4), or the institutional contexts of the research communities (Section 2.3 and 3.2).

On the one hand, none of these differences are really surprising. In principle, we (should) know that these kinds of differences exist, and the international comparative studies on classroom cultures have shown such kind of differences systematically (cf. Stigler, Gonzales, Kawanka, Knoll, & Serrano, 1999; Clarke, Emanuelsson, Jablonka, & Mok, 2006). Additionally, the research on multicultural issues within one country has shown how differences in the societal context affect students in classrooms (e.g., Secada, 1992; and many others).

On the other hand, our principal awareness does often not reach our everyday research and design practices or it reaches them late, as a result of our practices instead as an input of them. The examples have shown that our daily research is often implicitly guided by the hidden assumption that contexts, problems, and outcomes are or should be universal and not shaped by the cultural relativity, briefly, the *hidden assumption of universality*. Instead, we plead for constantly questioning this hidden assumption of universality of research and design practices and outcomes and for raising cultural awareness. This seems necessary to establish the ERME community as an *epistemic community* using the multicultural diversity of its individual members and working groups to produce, widen and enrich our knowledge.

4.2 Looking forward: Raising cultural awareness

Being aware of differences and overcoming the hidden assumptions of the universality of research and development practices and outcomes, we and others can act in the directions outlined below:

— *when addressing mathematics,* we can try to be aware of cultural contingencies; we can challenge our own ways of perceiving and expressing mathematical constructs;

— *when reading other's papers,* we can avoid naïve transfers of constructs, approaches and outcomes from other cultural contexts; for example, what worked in Spain need not in Poland;

— *when reading other's papers,* we can systematically investigate the adequacy of transfers, not only for results, but also for theoretical constructs and approaches;

— *when writing papers*, we can describe explicitly our own cultural context, paying attention to the ways it affects what we write about research methodology and findings;

— *when conducting own research and development*, we can try to learn from other cultural contexts in order not to take for granted our own conditions.

Being aware of dominances and overcoming the hidden assumption of the universality of research and development practices and outcomes, we can attend to the following in our work with others:

— *when collaborating with colleagues from other cultural contexts,* we can take enough time to learn about other cultural contexts and consider differences; and we can exploit the gap between us, in order to become aware of our own unthoughts;

— *when importing research to other countries*, we can discuss and apply methodologies that allow us to be sensitive to the cultural contexts we join;

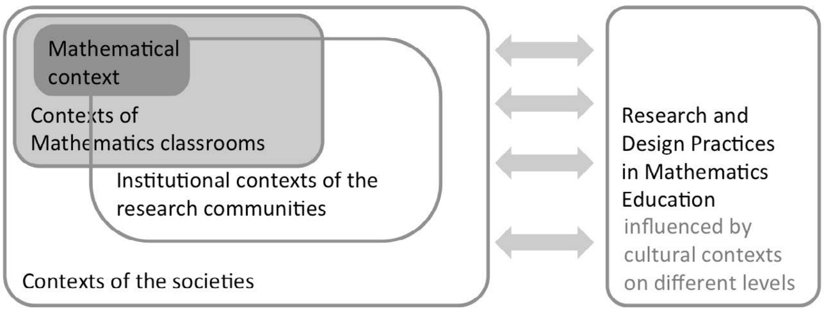

Figure 6: Nested cultural context on different levels which all influence the research and design practices in Mathematics Education

— *when acting as reviewers or editors for journals or conferences,* we can try to avoid the dominance of Western empirical research, where the sole adoption of the Western format of presentation (theoretical framework, research questions, methodology, findings) risk to hide new and fresh ideas, which have the potential to enrich our vision of the world;

— *when responsible for policy issues in international communities, we can follow good practices of* ICMI and ERME to strictly try to realize regional balances in all committees in order to alleviate cultural dominance;

— *when responsible for policy issues in international communities,* we can try to foster the standards for making explicit cultural contexts in writing and review guidelines (as, for example, in the practices seen in ERME guidelines, ICMI studies...);

— *when supervising PhD students from non-affluent countries,* we can reflect how far to impose own values on their research.

5. CONCLUSION: RAISING CULTURAL AWARENESS AS A COMMUNITY TASK

The focus here is both back onto mathematics and forward onto cultural awareness. We reflect on what we have written above and encourage questioning of our future practice. Our challenges in developing our own awarenesses place us in a position of responsibility to our academic and research communities in promoting more global values with respect to the cultural dimensions within which we work. Our intention here is to encourage discussion, and possibly debate, through which issues can be raised and addressed widely.

5.1 What makes mathematics central and different from other disciplinary areas?

In Section 2.1, we have raised issues relating to the ways in which mathematics is represented, symbolised and conceived in different parts of the world. Examples have shown how certain representational forms can foster or promote differing conceptions, some of which might be seen to limit how mathematics is understood. Perceptions of universality in mathematics can therefore be dangerous if they remain implicit and resist being challenged. Even the most experienced mathematics educators can learn from alternative cultural representations and build richer representational frames.

Section 2.2 has shown that teaching approaches and the ways in which mathematics is offered to learners carry a high responsibility with regard to learning outcomes. Narrow insistence on the internal consistency of mathematics as disseminated through representational forms and adherence to procedural rules, without corresponding attention to the underpinning concepts, have to bear a responsibility for public perceptions of mathematics and for mathematical achievement in diverse cultures. Richer cultural awarenesses can open up mathematical discourses that promote access and understanding and a broader willingness to engage and succeed with mathematics. However, these processes of innovation must themselves respect cultural differences.

5.2 How can a focus on mathematics take into account the moral and ethical issues of education for all?

The *moral* and the *ethical* are human constructs: being moral and ethical places responsibilities on mathematics educators as human beings. We have responsibilities to our disciplines of mathematics and education, and importantly to the people whose education in mathematics we promote. The intrinsic educational levels here present a complex weaving of responsibilities: educating students in mathematics; educating teachers in teaching mathematics; educating new researchers in theory and research; educating mathematics educators who educate at all of these levels. The awarenesses referred to in Section 5.1 with regard to mathematics underpin this edifice: we have to weigh the issues in deciding how best to interpret the role of educator. For example, the mathematics teacher who, with thoroughly good intentions, over-simplifies a mathematical concept to avert the struggles of the learner may not assist in the complexities in appreciating the concept; or a lecturer who emphasizes the fine details of a proof without attention to the sociohistorical origins of the proof may be true to mathematical rigor but leave a student mystified. Awareness of the choices we exercise as educators requires moral and ethical judgments in how we operate in our professional roles.

5.3 How can a focus on mathematics address the figured worlds of all who learn and teach?

The concept of *figured worlds* (Holland et al., 1998; cf. Section 2.2.5) recognizes that human beings are located within a complex synergy of cultures: societal, disciplinary, professional, familial, philosophical and personal, to name a few. In professional human relations, many of these 'worlds' are hidden; we have seen clear examples of this in the sections above. While any one or group cannot be expected to discern the extent of such complexity, we can be expected to be aware that it exists. This requires us to give overt attention to difference and meaning and their (potential) relationships to the concepts we address. It requires a willingness to be open, to encourage our learners to express their own conceptions and inform us of how things are done and seen in their contexts. As educators we have to do our best to make moral and ethical choices within our own knowledge and what we hear and learn by listening to others.

5.4 Which theories, research and design approaches can grasp the complexity and cultural differences?

As the reality of mathematics teaching and learning is very complex with all these different nuances and challenges, no single isolated theory, no single research or design approach can do justice to this complexity (cf. Section 2.3). Mathematics Education research that really contributes to relevant innovations in educational practices requires us to connect different approaches.

Raising cultural awareness also supports us to become aware of the strengths and limits of different epistemic cultures. This applies for Mathematics Education as well as for every other subject matter education and general education and requires not only the efforts of single researchers but the whole community.

5.5 What are the big issues that we should be addressing?

We wish to be true to mathematics as we know it. However, we can always learn more to enrich our own perceptions. As educators we are challenged continually to explore how best to bring mathematics and its learning and teaching to our learners. However, the biggest issue is not the question 'how can we bring mathematics and its learning and teaching to our learners?', but, 'how can we make *exploration* of such a question the basis of our professional activity'? We have to keep addressing this issue. For learners to see that their teachers, at any level of education, are also learners, questioning the very practices in which the teacher and learners are engaged, can be empowering and exhilarating, although it can also be frustrating for those who seek closure.

Closure is a philosophical position, related to seeking certainty and end points, and is something we have to address overtly. Rather than seeing closure in this concept, or this idea, or this issue, all participants can see themselves at their own stage of the educational journey, where the coming stages are open for participation. This very idea needs cultural reorientation in many contexts: for example, recognition that if we seek the 'right answer' to a mathematics problem, what is *right* may depend on a range of contextual factors; if we seek to *define* a mathematical entity, the very act of definition excludes other possibilities; if we present a Mathematics Education thesis in Poland, it will be judged differently from the same thesis in Spain. This is not to say there are no right answers, or to undermine the value of definitions, or to reject the judgments made in different communities; rather, while agreeing an answer or a definition, the limitations and exclusions of such acts need to be recognised and (insofar as we are able) addressed. Therefore, there are no end points, but many choices, challenges, and judgments. We have to embrace diversity and seek out alternative meanings and roots. Those more experienced are there to help all engage, to scaffold their growth of understanding, encourage progress and develop awareness, not to set limits or close off possibilities. This is the moral challenge for all of us!

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Understanding randomness: Challenges for research and teaching

Carmen Batanero

Universidad de Granada, Granada, Spain, batanero@ugr.es

The ubiquity of randomness and the consequent need to understand random phenomena in order to make adequate decisions led many countries to include probability throughout the curricula from primary education to University level. This need was also recognized by the mathematicians who developed the probability theory. This is a relatively young field, and is not free of controversies, which are also reflected in the lack of agreement on a common definition of randomness. Psychological and didactical research, suggest widespread misconceptions and misperceptions of randomness; however, these results have not always been taken into account in teaching, where randomness is considered a transparent concept.

Keywords: Randomness, epistemology, subjective views, teaching and learning.

The most decisive conceptual event of twentieth century physics has been the discovery that the world is not deterministic. Causality, long the bastion of metaphysics was toppled, or at least tilted… A space was cleared for chance (Hacking, 1990, p. 1).

When I was kindly asked to contribute with a plenary talk to this conference, I decided to select a topic that reflected a part of European research to stochastic education. Randomness is a good example, since it may be examined from the philosophical, psychological, mathematical and didactic perspectives, each of which has been dealt with by European researchers, and which globally reflect the European perspective for didactics. Furthermore, there is an increasing tendency to teach probability to very young children. However, as we will analyse in my presentation, this concept is far from elementary and we still have to find adequate ways to introduce it to students at different ages.

With this view in mind, I will first describe some of the meanings attributed to the idea of randomness since its emergence; secondly, I will summarize the main research dealing with the personal meanings that people attribute to randomness. I will finish with some personal suggestions for teaching and future stochastic education research that may help to increase our understanding and managing of random situations.

FROM CHANCE TO RANDOMNESS

From childhood, we are surrounded by uncertainty, in our personal lives, our social activities and professional work. The omnipresence of randomness implies our need to understand random phenomena in order to make adequate decisions when confronted with uncertainty. Mathematicians developed the field of probability as a set of models that can be applied to uncertain situations; however, progress in mathematical methods did not solve the philosophical debate around randomness.

Today, mathematics curricula for compulsory teaching levels increase the study of random phenomena. Expressions such as "random experiments", "random digit", "random variable", "random variation", "random even", "random sampling", "randomly", "randomization", "random variable" appear in curricular documents (e.g., CCSSI, 2010; Franklin et al., 2007), as well as in the school textbooks.

However, in these documents, the meaning of randomness is not always clear and unequivocal, because these expressions refer to an abstract entity, not entirely defined; thus, increasing potential difficulties for students or teachers arise. Randomness is a multifaceted object, as shown in the various interpretations received throughout history (Batanero, Green, & Serrano, 1998; Batanero, Henry, & Parzysz, 2005; Bennett, 1999; Saldanha & Liu, 2014). Even today, we find no simple definition that we can use unambiguously to classify a given event or process as being random or not. In the following, the reflections about the nature of randomness by eminent statisticians, philosophers, psychologists and researchers in mathematics education are summarised.

Primitive ideas

Early notions of chance were found in many ancient cultures. However, for centuries there was no theoretical speculation about the nature of randomness or systematic study of frequencies of results of these games. Possible reasons for the tardy development of probability, such as the connection of chance and divination to predict the future, were discussed by David (1962). Borovcnik and Kapadia (2014a) suggest that in Greek mathematics, data about variability was ignored as contrary to their ideal of scientific argument. Later, many different conceptions of chance arose, in particular (Batanero, Henry, & Parzysz, 2005, p. 27):

> Believing in a destiny predetermined by God or spirits; Assuming a personal chance factor, unequal for different individuals; Accepting natural necessity, ineluctably subjected to laws which still are partially unknown and which govern the world's origin and evolution; Arguing the inextricable complexity of the infinitesimal causes generating macroscopic phenomena, which we consider fortuitous as the only possible reasonable interpretation; Assuming the existence of a fundamental, chaotic and absolute natural randomness.

Bennet (1999) analysed different historical conceptions of chance that was later formalized in the mathematical concept of randomness (Saldanha & Liu, 2014). Some of these conceptions still appear in students and teachers (Batanero, Arteaga, Serrano, & Ruiz, 2014; Engel & Sedlmeier, 2005). Below I give a brief summary of these developments.

Chance and causality

We can find a first meaning of randomness in the Spanish dictionary by Moliner (2000) where "random" is defined as "Uncertain. It is said of what depends on luck or chance" (p. 123), and "chance" is defined as "the presumed cause of events that are neither explained by natural necessity nor by a human or divine intervention" (p. 320). In this description, random is something with unknown causes and chance is the assumed cause of random phenomena.

This meaning was prevalent in a first historical phase in the development of randomness according to Bennett (1999). According to David (1962), the astralagus was used in games of chance around 3500 B.C. Cubic dice were abundant in primitive civilizations like the Egyptian or Chinese civilizations, which used games of chance in an attempt to predict or control fate in decision-making. In spite of this wide use, there was no scientific idea of randomness until the Middle Ages. Whether it was attributed to supernatural forces or not, randomness suppressed the possibility that human will, intelligence or knowledge would influence decisions or destiny (Poincaré, 1909/2011).

Throughout this period, some philosophers related chance to causality (Bennet, 1999): Democritus suggested that everything is the combined fruit of chance and need. Leucippus believed that nothing happens at random; everything happens for a reason and out of necessity. Aristotle considered that chance results from the coincidence of several independent events whose interaction results in an unexpected result. Implicit in this meaning is to believe that every phenomenon has a cause. Randomness is only the measure of our ignorance. Random phenomena are, by definition, those whose laws are unknown (Poincaré, 1909/2011).

A deterministic vision of the world was common throughout the Renaissance as is visible in Bernoulli (1713/1987, p. 14):

> All which benefits under the sun from past, present or future, being or becoming, enjoys itself an objective and total certainty… since if all what is future would not arrive with certainty, we cannot see how the supreme Creator could preserve the whole glory of his omniscience and omnipotence.

This conception of chance as opposed to cause and due to our ignorance remained until the 19th century: "Present events are connected with preceding ones by a link based upon the evident principle that a thing cannot occur without a cause which produces it" (Laplace, 1814/1995, p. vi).

Modern concept of chance

This conception changed at the beginning of the 20th century. For example, Poincaré (1912/1987) noticed that some processes with unknown laws, such as death, are considered deterministic. Moreover, other phenomena, such as Brownian motion, are described by deterministic laws at a macroscopic level, while the behaviour of particles is random. Other situations are considered to be random because “A very small cause, which escapes us, determines a considerable effect that we cannot fail to see, and then we say that this effect is due to chance” (Poincaré, 1912/1987, p. 4).

Among the phenomena with unknown laws, Poincaré distinguished random phenomena that can be studied with probability calculus from other phenomena where probability is not applicable. Furthermore, probability will not lose its validity when we find out the rules governing the random phenomena. Thus, the director of a life insurance company is ignorant of the precise date when each person taking the insurance will die. Moreover, the distribution of the entire population´s lifetime does not change when we add knowledge about the precise death for each particular individual. Today, we accept the existence of fundamental chance around us and, in addition to the theory of probability, other theories, such as those of complexity or chaos, may be used to describe randomness.

The different philosophical conceptions of chance are compatible with the axiomatic mathematical theory of probability, which provides a system of concepts and procedures that serve to analyse uncertain situations (Batanero, Henry, & Parzysz, 2005). Mathematical probability does not enter philosophical debates and uses the ideas of random experiment and randomness as primitive (with no consideration of the nature of chance in each particular application). However, even today, the interpretations of randomness and probability continue to be subject of philosophical debates and the teacher of probability needs to be aware of these interpretations, because they influence students’ reasoning when confronted with chance situations.

CONCEPTUALIZING RANDOMNESS

According to Hacking (1975), probability was conceptualized from two complementary perspectives since its emergence: as a personal degree of belief in the likelihood of random events (epistemic view), and as method to find objective mathematical rules through data and experiments (statistical view). These two views unfolded in multiple perspectives that described what random events are, and how can we assign probabilities to them.

Randomness as Equiprobability

In the earlier applications of probability, randomness was related to equiprobability, which was a reasonable assumption in games such as flipping coins or drawing balls from an urn. Consequently, it was assumed that a member of a class was random (or was selected at random), when there was exactly the same probability to obtain this object or any other member of the same class. Thus, there is exactly the same probability to get the number 1 or any other number from 1 to 6, when throwing a dice. We can find, for example, this interpretation of randomness in the Liber de Ludo Aleae by Cardano (1663/1961, p. 189).

> The most fundamental principle of all in gambling is simply equal conditions…of money, of situation…and of the dice itself. To the extent to which you depart from that equality, if it is in your opponent’s favour, you are a fool, and if in your own, you are unjust.

Accordingly, in the classical definition of probability given by de Moivre (1718/ 1967) and refined by Laplace (1814/1995), probability is simply the number of favourable cases to a particular event divided by the number of all cases possible in that experiment, provided all the possible cases are equiprobable.

Kyburg (1974) criticised this definition of randomness since it imposes unnatural restrictions to its applications. We can only consider that an object is a random member of a class if the class is finite. If the class is infinite, then the probability for selecting each member is zero, and so (apparently) identical, even when the selection method is biased. Applying this definition in order to discriminate a random from non-random member in a given class is difficult, even in games of chance. How could we know, for example, that a given coin is not slightly biased?

Randomness as stability of frequencies

By the end of the 18th century, the study of random phenomena was extended beyond the world of games of chance to natural and social sciences. In these applications, for example, to the study of the blood type of a newborn, we cannot apply equiprobability. The concept of *independence* was essential to assure randomness in successive trials (Bennet, 1999). In this new view, we consider an object as a random member of a class if we could select it through a method providing a given a priori relative frequency in the long run to each member of this class.

In his attempt to extend the scope of probability to insurance and life-table problems, Jacques Bernoulli (1713/1987) gave the first proof of the Law of Large Numbers and justified the use of relative frequencies to estimate the value of probabilities. In the frequentist approach sustained later by von Mises (1928/1952) or Renyi (1966/1992), probability is defined as the hypothetical number towards which the relative frequency tends. Such a convergence had been observed in many natural phenomena so that the frequentist approach extended the range of applications enormously. A practical drawback of this conception is that we never get the exact value of probability; its estimation varies from one repetition of the experiment (called sample) to another. Moreover, this approach is not appropriate when it is impossible to repeat the experiment under exactly the same conditions. Another theoretical problem is that the number of experiments that provides sufficient evidence to prove the random nature of the object is undefined.

Subjective view of randomness

In the classical and in the frequentist approaches, randomness is an objective property of an event or an element of a class. Kyburg (1974) criticized this view and proposed a subjective interpretation of randomness composed of the following four elements:

- The object that is supposed to be a random member of a class;
- The set of which the object is a random member (population or collective);
- The property with respect to which the object is a random number of the given class;
- The knowledge of the person giving the judgement of randomness.

In this interpretation, randomness depends on the person's knowledge. Consequently, what is random for one person might be non-random for another; randomness is no longer a physical objective property, but a subjective judgement. We recognize here the parallelism with the subjective conception of probability, in which all probabilities are conditioned by information, and this is adequate in situations where some information may affect our judgement of randomness.

This view was reinforced by the Bayes's theorem, published in 1763, that proved that the probability for a hypothetical event could be revised in light of new available data. Following this new interpretation, some mathematicians like Keynes (1921), Ramsey (1931), or de Finetti (1937/1974) considered probability as a personal degree of belief that depends on a person's knowledge or experience. Via the Bayes' theorem, an initial (prior) distribution about an unknown probability changes by relative frequencies into a posterior distribution. However, the subjective character of the prior distribution in this approach was criticized; even if the impact of the prior diminishes by objective data and de Finetti (1934/1974) proposed a system of axioms to justify this view.

Axiomatization and formal mathematical views

Despite the fierce discussion on the foundations, the application of probability in all sciences and sectors of life was enormous. Throughout the 20th century, different mathematicians tried to formalize the mathematical theory of probability. Following Borel's work on set and measure theory, Kolmogorov (1933/1950) proposed an axiomatic theory that was accepted by the different probability schools because the different view of probability (no matter the classical, frequentist or subjectivist view) may be encoded by Kolmogorov's axioms. However, the particular interpretation of probability and the method used to assign probabilities to events differ according to the school one adheres to.

The development of statistical inference and the importance of assuring random sampling to apply inferential methods led to the practical interest to find procedures to produce sequences of "pseudo-random" digits. This need induced new discussion about theoretical models of randomness (Zabell, 1992). The need to distinguish two components in randomness was clear: the generation process (*random experiment*) and the pattern of the *random sequences* produced. We can generate random sequences with two different methods: one is using physical devices, such as coins or dice. Another is using deterministic algorithms; therefore, we can separate the generating process from the result (random sequence). More correctly, these results are called pseudo-random, because the generating process is a deterministic algorithm, although the sequence can pass some statistical tests for randomness. Most computer packages and calculators incorporate these algorithms, and thus we can easily obtain pseudo-random sequences of a given length with particular characteristics.

Different approaches served to describe the properties of a random sequence (Fine, 1971). Von Mises (1928/1952) defined a *collective* (population) as a mass phenomenon, a repetitive event or a long series of observations, for which we could accept the hypothesis of stabilization or the relative frequency towards a fixed limit. Starting from this idea, he defined a sequence of events to be random if, in any infinitely long series of outcomes, the relative frequencies of the various events have limiting values, and these values do not change in an infinite subsequence arbitrarily selected. Thus, contrary to the belief held by many players, there is no algorithm (at least theoretically) that serves to predict the behaviour of random sequences. However, since no statistical test can consider all potential pattern generators—because there are infinitely many—the possibility that a given sequence, in spite of having passed all our tests must always remain, and it should have some unnoticed pattern and so not really be random. Another problem is that we only produce finite sequences, so inevitably some tests will fail. In this sense, randomness is a theoretical concept and can only be applied to a process producing infinite sequences.

Kolmogorov (1965) defined the randomness of a sequence using the idea of *computational complexity*, taken from automata and computability theory. The complexity of a sequence is the difficulty in describing the sequence with a code that we can use later to reconstruct the sequence (or to store in a computer). The minimal number of signs needed to codify a sequence provides a scale of complexity. For example, 0101010101 can easily be coded with just a few symbols: 5{01}; while 0100110001 defies finding a shorter code, and then the first sequence is more complex than the second. In this approach, a sequence would be random if any coded description of the same is as long as the sequence itself. Therefore, a sequence would be random if the simplest way in which we could describe it is by listing all its components. Chaitin (1975) suggested that this definition establishes a hierarchy in the degree of randomness for different sequences and that perfect randomness is only theoretical.

Epistemic meanings of randomness

To sum up, we can use some ideas from the onto-semiotic approach to mathematics education. In this framework, mathematical knowledge has a socio-epistemic dimension, since it is linked to the activity in which the subject is involved and depends on the institutional and social context in which it is embedded. Mathematical activity is described in terms of practices or sequences of actions, regulated by institutionally established rules, oriented towards solving a problem (Drijvers, Godino, Font, & Trouche, 2013). In this framework, the meaning of mathematical objects is linked to the mathematical practices carried out by somebody (a person or an institution) to solve specific mathematical problems. Around the mathematical practices linked to these specific problems, different rules (concepts, propositions, procedures) emerge (Godino, Batanero, & Font, 2007); these rules are supported by mathematics language (terms and expressions, symbols, graphs, etc.), which, in turn is regulated by the rules. All these objects are linked to arguments that serve to communicate the problem solution properties and procedures, and to validate and generalize them to other contexts and problems.

An epistemic configuration (either institutional or personal) is the system of objects involved in the mathematical practices carried out to solve a specific problem (Figure 1). Each different epistemic meaning of randomness is linked to a specific type of problem whose solution involves particular mathematical objects, part of which are summarised in Table 1. Consequently, there is a specific onto-semiotic configuration linked to each of these meanings, which differ from each other not only in the philosophical aspects debated in the previous sections, but in the mathematical objects that characterize them. As a result, reducing the teaching of randomness to only one or a few of these views implies a reduction of the overall meaning of the concept.

Meaning of randomness	Problem	Concepts/Properties	Procedures
Intuitive	Divination Attempt to control chance	Luck, fate Opinion, belief	Physical devices (dice, coins...)
Classical	Establishing the fair betting in a game of chance	Equiprobability Proportionality Favourable/possible cases Expectation	Enumeration Combinatorial analysis Laplace's rule A priori analysis of the experiment
Frequentist	Estimating the tendency in the long run	Repeatable experiment Frequency Convergence	Collecting data Estimation Limit in the long run
Subjective	Updating a degree of belief	Subjective character Depends on information Non repeatable Conditional probability Prior distribution Posterior distribution Likelihood, risk	Bayes' theorem Decision theory and methods
Formal	Describing mathematical properties of randomness	Random experiment Sample space Events algebra Measure Complexity Random sequence	Abstract mathematics (e.g., set theory) Randomness tests Simulation Algorithms that produce pseudo-random sequences

Table 1: Example of mathematical objects linked to different epistemic meanings of randomness

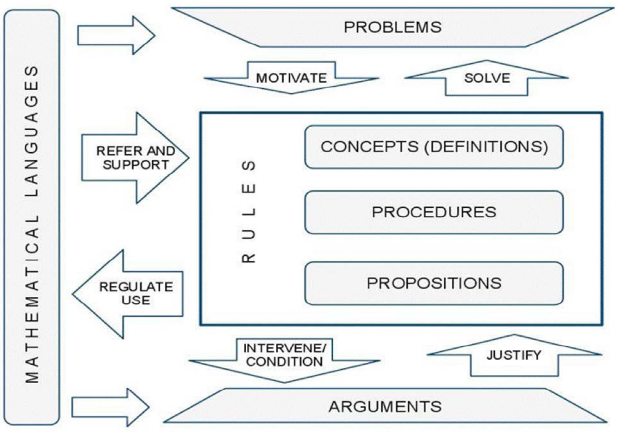

Figure 1: Onto-semiotics configurations involved in mathematical practices (Drijvers, Godino, Font, & Trouche, 2013, p. 28)

SUBJECTIVE CONCEPTUALIZATION OF RANDOMNESS

Research dealing with people's conceptualization of randomness has a long history, and different research paradigms are visible in this research.

Levels of acquisition of the concept of randomness

The pioneer work in probabilistic reasoning was due to Piaget and Inhelder (1951) who described levels (or stages) in children's understanding of chance and probability. These authors assumed that randomness is produced by the interference of independent causes, and then, children first have to understand deterministic cause-and-effect phenomena before they can grasp the nature of random events. Another prerequisite for understanding randomness, according to these authors is combinatorial reasoning, which is needed to describe the set of possibilities in random phenomena, and to accept that each isolated outcome is unpredictable.

Piaget and Inhelder (1951) investigated the understanding of patterns in random distributions by children.[1] They designed an experiment simulating the fall of raindrops on the tiles of a pavement. After situating a few counters (raindrops) on the pavement, they asked the children where the next raindrop would fall. In a first stage (6–9 year-old), the children assumed that the raindrops would approximately fall in equal numbers on each square of the pavement. When there was one drop in every square of the pavement, except for one empty square, the children invariably located the next drop in the empty square, so that a uniform distribution was achieved. With increasing

1 Piaget and Inhelder based their research on the classical view of probability.

age, Piaget and Inhelder assumed that the irregularity of the distribution would be accepted and that adolescents would understand randomness.

Later research, however, contradicted this assumption: Green (1989) investigated the probabilistic reasoning of 2 930 children in the United Kingdom and used some tasks related to perception of randomness at age 11–16; his findings suggest that the percentage of students recognizing random distributions does not improve with age (stagnation of children's perception of randomness during this period). Similar results were found by Green (1991) and Engel and Sedlmeier (2005) with different tasks (to students at age 10–15).

Intuitions and personal beliefs

While Inhelder and Piaget focused on the formal understanding of randomness, other authors tried to describe personal beliefs and intuitive understanding of this concept. This research suggests that the paradoxes and controversies about the meaning of randomness are reproduced in the intuitions people build when they face random situations; these intuitions often contradict the mathematical rules of probability (Borovcnik & Kapadia, 2014a).

Children use qualitative expressions (probable, unlikely, feasible, etc.) to express their degrees of belief in the occurrence of random events; however, their ideas are too imprecise and have difficulty in differentiating random and deterministic phenomena (Fischbein & Gazit, 1984). Young children may not see stable properties in random generators such as dice or marbles in urns and believe that such generators have a mind of their own or can be controlled by them (Fischbein, Nello, & Marino, 1991; Truran, 1994). Although older children may accept the need to assign numbers (probabilities) to events to compare their likelihood, a correct probabilistic reasoning rarely develops spontaneously without a specific instruction (Fischbein, 1975); for this reason, adults often have wrong intuitions about probability.

Fischbein's assumption has been confirmed by research in the field of decision making under uncertainty, where erroneous judgements in out-of-school settings are pervasive. The widely known studies by Kahneman and his collaborators (e.g., Kahneman, Slovic, & Tversky, 1982) support the idea that people violate probabilistic rules and use specific *heuristics*[2] to simplify uncertain decisions. According to these authors, heuristics such as *representativeness* or *availaibility* reduce the complexity of probability tasks and may be useful in many situations; however, under specific circumstances these heuristics cause systematic biases with serious consequences. Furthermore, some people do not understand the purpose of probabilistic methods, which allow us to predict the behaviour of a distribution, but are invalid to predict each specific outcome (Konold, 1989). A detailed survey of students' intuitions, strategies and learning at different ages may be found in Chernoff and Sriraman (2014), Jones (2005), Jones, Langrall, and Mooney (2007), and Shaughnessy (1992).

Generating and recognizing randomness

There is a wide research into adults' subjective perception of randomness (e.g., Bar-Hillel & Wagenaar, 1991; Batanero & Serrano, 1999; Chernoff, 2009; 2011; Engel & Sedlmeier, 2005; Falk, 1981; Kahneman & Tversky, 1972; Wagenaar, 1972), Two types of tasks have commonly been used: (a) In generation tasks subjects follow standard instructions to invent a series of outcomes from a typical random process, such as tossing a coin; (b) In comparative likelihood tasks (Chernoff, 2011), people are asked to select the most or least likely of several sequences of results that have been produced by a random device or to decide whether some given sequences were produced by a random mechanism. Related tasks have also been proposed using two-dimensional random distributions of points on a squared grill (e.g., Batanero & Serrano, 1999; Green, 1991; Engel & Sedlmeier, 2005; Toohey, 1995).

Generation tasks: Producing random distributions

In a longitudinal study on randomness with 7 to 11 year-old children, Green (1991) asked them to invent random sequences of heads and tails representing the results of flipping 50 times a fair coin. He first analysed whether the children produced approximately the same number of heads and tails in their sequences and found that they were very exact in reproducing equiprobability (the average number of heads was close to 25); furthermore, the children produced sequences with very consistent first and second parts (about 12 heads in each part). Green concluded that children were too consistent to reflect the random

2 The specific meaning of word heuristics in this research is a cognitive process that helps to solve a problem by reducing part of the data.

variability. Moreover, these children did not perceive the independence, as they produced sequences with too short runs (of heads or of tails), as compared to the length we expect in a random sequence. As suggested by Bryant and Nunes (2012), the independence of random events is hard to grasp and many adults believe that a head is more likely to appear on the sixth flipping of a coin after a run of five tails.

Comparative likelihood tasks: Properties attributed to randomness

Results from research asking people to distinguish random from non-random sequences of events suggest that our judgements about what random sequences are, is subjected to biases (e.g., Batanero & Serrano, 1999; Chernoff, 2009, 2011; Green, 1983, 1991; Kahneman & Tversky, 1972; Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Shaughnessy, 1977).

An example of this research is Green's (1983) study with 11–16 year-old children. In his questionnaire, he asked the children to discriminate between a random and a non-random sequence (both sequences consisted of results of flipping a coin 150 times). Most participants in the study chose the non-random sequence, regardless of their age. Some of them provided incorrect reasons to justify their choice; for example, they argued that the pattern of the sequence was too irregular (and then they did not accept the variability of the random sequence). Other participants expected exactly 50% of heads and tails in a random sequence or rejected the possibility of long runs. Very similar results were found in another study (Green, 1991) where the author asked the children to discriminate random and non-random sequences of heads and tails, as well as random and non-random bi-dimensional distribution of points.

Toohey (1995) used part of Green's tasks in a study with 75 12–16 year olds in Melbourne. He suggested that the understanding of randomness involves accepting the ideas of equal/unequal likelihood, multiple possibilities, model, causality and unpredictability. He also identified two different possibilities (local and global) in attributions of randomness. The local perspective of randomness is based on isolated results, while global perspective is reliant on the frequency distribution of the different outcomes.

Batanero and Serrano (1999) proposed some items taken from Green (1991) to 277 students aged 14 and 18 and analysed the reasons they gave to decide that a sequence or a distribution was random. The students' arguments were related to the observed frequencies of events (close or different from the expected value), the overall pattern of the distribution (uniform distribution or variability), the length of the runs (too short or too long runs), the existence of multiple possibilities and the unpredictability of results. Even when the authors found some widespread misconceptions, they also noticed that the students were able to perceive the characteristics of the random sequences presented to them and that this recognition improved by age. They also identified some partly correct conceptions that reproduced the conceptions of randomness described in the first sections of this paper, which were considered correct in different historical periods. Consistent results were reported by Engel and Sedlmeier (2005) in a cross sectional study that examined German students' understanding of variability in empirical data and by Batanero, Gómez, Gea, and Contreras (2014) in a study with Spanish prospective primary school teachers[3].

Facing the subjects with their own misconceptions: statistical analysis of their own data

As analysed in the previous sections, the perception of randomness is deduced with either generation or recognition tasks. Batanero and colleagues (2014) combined both tasks in a study with 208 Spanish prospective primary school teachers, using a formative activity with two parts. In the first part (a classroom session), the prospective teachers carried out an experiment to decide whether the group had good intuitions on randomness or not. The experiment consisted of trying to write down apparent random results of flipping a fair coin 20 times (without really throwing the coin, just inventing the results) in such a way that other people would think the coin was flipped at random (invented sequence). Participants recorded the invented sequences on a recording sheet (this is a typical generation tasks). These sequences were analysed by the researchers and were consistent with previous research reported on generation tasks.

Batanero and colleagues also asked the prospective teachers to analyse some variables deduced from

3 Prospective primary school teachers do not follow a specific course of probability. They study elementary probability in their first year of studies and along secondary school (as a part of mathematics).

their invented sequences (number of heads, number of runs, and length of the longer run) and compare them with the same variables in real coin-flipping sequences as a result of flipping a fair coin 20 times. The prospective teachers were asked to carry out the statistical analysis of differences between the same variables in the real coin-flipping and invented sequences for the whole group. They were also asked to prepare a report with their conclusions about how good their perception of randomness based on the statistical analysis, was. Each prospective teacher analysed the results in his or her group (30–40 students per group) using elementary graphs and statistics; they had freedom to use any method they wished. This second part of the activity is a sophisticated version of a comparative likelihood task, because the participants were not only asked to discriminate the random (real flipping) from non-random (invented) sequences. They were only asked to perform (intuitively) the type of analysis that researchers use to study people's perception of randomness. This activity was highly motivating for the prospective teachers and served to simultaneously increase their statistical and didactical knowledge.

Results from this second part firstly showed that these prospective teachers were able to use their statistical knowledge to solve a real world problem (deciding if the perception of randomness in the group was good). They secondly completed a modelling cycle: they started from a real problem (studying the intuitions on randomness), simplified the problem, and decided which aspects were relevant. They thirdly built some mathematical models to study the problem, worked with the models, and finally interpreted the results to answer the real world question. As regards their perception of randomness, many of the primitive conceptions described in Batanero and Serrano (1999) appeared and part of them were identified by the prospective teachers' themselves. Participants also recognised that the classroom showed a good perception of the expected value and a poor conception of both independence and variation. Some new results emerged; for example, some prospective teachers believed that it is not possible to apply mathematical methods (statistics) to study random phenomena, because of their unpredictability. A few participants also believed they could predict or control the outcomes in a random process (illusion of control described by Langer, 1975).

Other research paradigms

A different approach to evaluate people's perception was taken by Konold, Lohmeier, Pollatsek, and Well (1991) who concentrated on the random process (instead of concentrating on the random sequence). They asked the subjects in their study to decide whether different types of situations (processes) were or were not random and justify their responses. They used processes with equiprobable and non-equiprobable outcomes. While they found no differences in the subjects' categorization of the situations as random, novices tended to feel that the non-equiprobable situations were not random. The analysis of students' arguments served to describe the following conceptions of randomness:

— *Randomness as equiprobability:* Subjects that only consider randomness where all the possible results are equally probable.

— *Randomness as opposed to causality*, or as a special type of cause.

— *Randomness as uncertainty;* existence of multiple possibilities in the same conditions.

— *Randomness as a model* to represent some phenomenon, depending on our information about it.

Randomization is an important statistical procedure that assures the proper application of statistical methods, such as statistical tests. Pratt (2000) and Pratt and Noss (2002) investigated children's understanding of randomization when playing chance games and found 10-year olds that understood the connection between randomness and fairness, and the role of randomization in ensuring fairness (see also Johnston-Wilder & Pratt, 2007; Paparistodemou, Noss, & Pratt, 2008). Pratt (2000) suggests that children reason with two different meanings for randomness (very close to the description by Toohey, 1995): a local perception is related to the impossibility to predict the process behaviour in each trial, while a global perception involves the children´s understanding of patterns in the long run and in the distributions.

As it is apparent in our survey, research into people's perception of randomness has been faced with different paradigms that provided complementary results. Yet new questions remain open; in particular, it is not clear what model of randomness is better suited for children at different ages, or how we can help students acquire progressively more complete models of randomness as they become adult. We now analyse the way the topic has been taken into account in the curricula.

TEACHING AND LEARNING

Randomness in school curricula

The different views of probability have been reflected on the teaching of probability in schools, and on the way, randomness has been conceptualized in the curricula in Spain and other European countries; the concept itself is often only introduced via examples of random and non-random situations, or with indirect reference to isolated properties (e.g., unpredictability) (Azcárate, Cardeñoso, & Serradó, 2005), but is not formally defined.

According to Henry (2010), the classical view of probability based on combinatorial calculus dominated the French school curricula until the 80s, and this was also the case in Spain and other European countries. Since combinatorial reasoning is difficult, the teaching of probability was postponed until grades 8 or 9 (14 year-olds), an age where wrong intuitions difficult to eradicate are already acquired. Throughout the "modern mathematics" era, probability was used to illustrate set theory; there was little interest in modelling random phenomena from the real world. In these two approaches, the applications were restricted to games of chance; consequently, many school teachers considered probability as a part of recreational mathematics, with not much value for the education of children and tended to reduce its teaching.

Today, due to the technology available, we use the frequentist view to introduce probability as the limit of relative frequencies in a long series of trials. This change also involves a shift from a formula-based approach to an emphasis on providing probabilistic experience. Even very young children are encouraged to perform random experiments or simulations, formulate questions or predictions about the tendency of outcomes in a series of these experiments, collect and analyse data to test their conjectures, and justify their conclusions based on these data. This view also connects to the current interest for modelling in school mathematics (Henry, 2010), since simulation can also help students distinguish between model (the theoretical probability) and reality (frequencies of experimental results) (Girard, 1997; Engel & Vogel, 2004).

Randomness receives prominence today at high school level in relation to the introduction of inference (or "informal inference"). For example, in the CCSSI (2010) for grade 7 we find "use random sampling to draw inferences about a population" and "understand and evaluate random processes underlying statistical experiments". For high school (grades 9–12), this curriculum specifies "define a random variable for a quantity of interest by assigning a numerical value", "use random number generators", and "collect data from a random sample of a population". Many other curricula in Europe, as well as in the Australia, New Zealand and the United States approach probability and inference in a frequentist way, using simulation and resampling to estimate the probabilities of interest (e.g., Frischemeier & Biehler, 2013). The subjective view, that takes into account one-off decisions, which are frequent in everyday life, and where we cannot apply the frequentist view, is hardly considered in the curriculum. Moreover, the experiments we often simulate are atypical examples of random situations, in the sense that in few real-life applications of probability can we repeat a process many times in exactly the same conditions (Borovcnik & Kapadia, 2014a).

A didactic approach to randomness

The many perspectives and properties of randomness described in the previous sections suggest that a complete understanding of randomness is only achieved gradually. Moreover, probability models do not exactly fit reality and therefore should be viewed more as scenarios to explore reality than as images of this reality (Borovcnik, 2006). Since feedback in probability is only indirect (after a long series of trials), understanding of probability is not easy.

Throughout primary school, we can encourage children to discriminate certain, possible and impossible events in different context, and use the language of chance. Starting with specific materials with symmetrical properties, such as dice or coins, the children can compare their predictions from the a-priori analysis of the structure with frequency from data collected from repeated experiments to estimate probability.

In a second stage, we can progressively move to the study of materials lacking symmetry properties – spinners with unequal areas, thumbtacks, etc. –, where we only can estimate probability from frequencies. Once this phase is successful, we can turn to real life (e.g., sports, demographic, or social phenomena), using data available from the daily press, Internet, or other sources. Subjective situations (e.g., should the teacher ask me next time?) where only personal probabilities can be applied, can complete the field of application of probability.

By the end of primary school or in early middle school (10–11 year olds) children can start simulating simple situations using devices such as the box model simulator in the National Library of Virtual Manipulatives (http://nlvm.usu.edu/). Today, there are plenty of technological resources, including software specially designed to explore probability (see also Lee & Lee, 2009). With simulation, we introduce a modelling approach where the essential features of the situation are modelled by the simulator and irrelevant properties are disregarded. As shown by Pratt (2000), simulation of familiar objects like dice, can help 10–11 year-old children express their previous beliefs and articulate a more complete meaning for randomness in the light of their experiences with the simulator.

Towards the end of secondary school (15–16 year olds) a deeper analysis of the properties of the random numbers generated through a calculator or computer may be introduced. The experiments, recording and analysis of the sequences produced in these simulation activities will help to integrate study of probability and statistics. Eichler and Vogel (2014) propose a modelling approach for each of the main views of probability (classical, frequentist and subjective) and discuss the role of simulation in supporting students' understanding in each of these perspectives. The context of decision making, such as for example, taking insurance, is useful to introduce subjective views. When facing the uncertainty of a single decision, this decision could be made more transparent if we ask the students to weigh up the different possibilities, and compute the expected values of costs or prizes (Borovcnik, 2006).

The gradual introduction of concepts and notation will serve to mathematically explain the regularities observed in the data. Exploration of microworlds (e.g, Cerulli, Chioccariello, & Lemut, 2006) may serve to confront children´s intuitions to mathematical ideas. Johnston-Wilder and Pratt (2007) suggest that these tools help children see randomness as a dynamic process, since a printout of a random sequence loses the essence of what random is to be.

Through these activities, students will progressively acquire understanding of the following essential characteristics of random phenomena:

— In a random situation there is uncertainty; more than one result is possible.

— The actual result, which will occur, is unpredictable (local variability of random processes).

— We can analyse either the process (random generator) or the sequence of random results: these two aspects can be separated.

— In a few situations (e.g., games of chance) we can analyse the process before the experiment; this analysis will inform us of the likelihood of possible results

— Commonly, there is the possibility—at least in the imagination—of repeating the experiment (or observation) many times in (almost) similar conditions.

— In this case, the sequence of results obtained through repetition lacks a pattern; we cannot control or predict each result (local variability).

— In this apparent disorder, a multitude *of global regularities* can be discovered, the most obvious being the stabilization of the relative frequencies of each possible result. This global regularity is the basis that allows us to study random phenomena using the theory of probability.

— In one-off uncertain situations we still can apply probability if our initial degrees of beliefs are consistent (have reasonable properties).

— To conclude, randomness is a model we apply to some situations, because this model is useful to predict or control the situations.

As argued by Konold and colleagues (1991), it is preferable to consider randomness as a label with which we associate many concepts, such as experiment, event, sample space, probability, etc. In this sense, the word randomness refers to a collection of mathematical concepts and procedures, which we can apply to uncertain situations. We need to think about the orientation we take towards the phenomenon that we qualify as "random" rather than think of randomness as an objective quality of the phenomenon itself. We apply a mathematical model to the situation, because it is useful to describe it and to understand it; but we do not believe that the situation is identical to the model. Deciding when a probability model is more appropriate for the situation than other mathematical models is a part of the competence we want the students to develop.

FINAL REFLECTIONS

The complexity of the idea of randomness explains the counterintuitive results that abound even in basic probabilistic concepts (Székely, 1986; Borovcnik & Peard, 1996). This complexity is also reflected at higher levels in probability theorems (e.g., the Central Limit theorem) that are expressed in terms of probability. According to Borovcnik and Kapadia (2014b), our poor intuitions in this field may be explained by our desire for deterministic explanations, but they might also be attributed to an inadequate education.

In spite of this complexity, "Probability is the only reliable means we have to predict and plan for the future; it plays a huge role in our lives, so we cannot ignore it, and we must teach it to all future citizens" (Devlin, 2014, p. ix). It is then important to reinforce probability in the school curricula and to find appropriate conceptualizations of randomness for different ages.

One goal of probability education is to take advantage of children's intuitions from elementary school as a basis for the acquisition of probability reasoning. One important insight into this line of research is the power of representation formats, such as natural frequencies (Gigerenzer & Hoffrage, 1995) or "tinker cubes" and other manipulatives (Martignon, Laskey, & Kurz-Milcke, 2007). Experimental interaction with mathematical modelling in a co-operative setting can likewise help children develop secondary intuitions (Nilson, 2003). Besides, as suggested by Andrà and Stanja (2013) it is important to pay attention to the interpretation and use of signs, which is not self-evident in probability, and may be interfered with experience with the same signs in other mathematical domains.

It is also important to confront the students with their own misconceptions and erroneous beliefs. As discussed by Borovcnik and Kapadia (2014b), progress in the development of mathematical concepts is usually accompanied by ruptures and conflicts, but there is an opportunity for learning when one tries to solve the conflict and understand paradoxical results.

Eichler and Vogel (2014) analyse the role of simulation to explore a model that already exists, develop an unknown model approximately, and represent data generation. However, though simulation is vital to improve students' probabilistic intuitions and to materialize probabilistic problems, a genuine knowledge of probability can only be achieved through the study of some formal probability theory. Of course, the acquisition of such formal knowledge by students should be gradual and supported by experience with random experiments.

We should also complement the objective and subjective views of probability. Even when many people believe that events have a unique probability rather than considering probability as a measure of our knowledge (Devlin, 2014), the idea of updating previous information in the light of new data is very intuitive as it reflects the way how people think.

It is also important to empower teachers with a specific preparation to teach probability because teachers' beliefs influence their instructional planning, their classroom practices, and have an impact on their students' learning (Eichler, 2011). Even if prospective teachers have a major in mathematics, they may be unfamiliar with different meanings of randomness and probability, or with their students' most common misconceptions. Teachers should also be conscious that teaching principles valid for other areas of mathematics, are not always appropriate in the field of probability (Batanero & Díaz, 2012). As described in a teaching experiment reported by Brousseau, Brousseau, and Warfield (2002), the teacher may fail to produce a specific random result when needed (even if he/she manages to assure a good probability of happening for the given result). Thus, even a reasonable knowledge of probability would not suffice for the teacher to be able to reproduce the didactic situation exactly as he/she prefers, and this could be a source of challenge for the teacher.

The preparation of teachers requires the design of activities where teachers are first confronted with their previous ideas and then perform and discuss experiments (e.g., Batanero, Biehler, Engel, Maxara, & Vogel, 2005; Batanero et al., 2014) in order to simultaneously increase teachers' probabilistic and didactic knowledge.

ACKNOWLEDGMENT

Project EDU2013-41141-P (MEC) and group FQM126 (Junta de Andalucía).

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Research in teacher education and innovation at schools: Cooperation, competition or two separate worlds?

Jarmila Novotná

Charles University in Prague, Faculty of Education, Czech Republic, jarmila.novotna@pedf.cuni.cz

The plenary lecture builds on the plenary lecture from ICME10 (Survey team 3). The lecture defined areas that had attracted little attention of researchers but were crucial (not only) for teacher education. It also comes out of discussions on ICMI Study 15 and recommendations formulated in these discussions. The lecture is also significantly informed by the work of Working Groups at CERME conferences since CERME1 until CERME8, partially also taking into account information from CERME9. The field of research in mathematics teacher education has changed considerably over the years since ICME10, which asks for a new definition of issues and trends. The goal of the lecture is to point out some trends in this area of research, especially in the field of cooperation between teacher education and innovations at school.

The first part of the text focuses on trends in current research into teacher education and practice. The goal of this part is not an exhaustive overview but indication of the main trends in the research domain. The second part of the text presents a more detailed discussion of several current research areas, their theoretical backgrounds as well as applications of their findings in teacher education and everyday school practice.

Keywords: Teacher education, cooperation of teachers and researchers, changes in teachers' knowledge, beliefs and approaches, problem solving.

INTRODUCTION

Let me begin the text by my personal confession: When the Programme Committee of CERME9 offered me to give a plenary lecture on the topic "Research into teacher education and practice", I felt that this was great honour and I was excited, even thrilled by the ideas of starting work on the plenary lecture. However, my initial enthusiasm slowly lessened. The reason for this faltering were not any doubts on the relevance of the topic. However, the deeper I emerged into the issue, the more aware I grew of the immense scope of research I could get access to. I realized that my lecture would never be and could not be exhausting and that I would have to focus on selected aspects of the issue only. I decided to build the plenary lecture on three important resources to which I had personally contributed:

— The plenary lecture from ICME10 (Adler, Ball, Krainer, Lin, & Novotná, 2004) in which the areas that had attracted little attention of researchers but were crucial (not only) for teacher education were defined. However, the field of research in mathematics teacher education has changed considerably over the years since ICME10, which asks for a new definition of issues and trends.

— ICMI Study 15 "The Professional Education and Development of Teachers of Mathematics" (Ball & Even, 2009). This study confirmed the great variety of research in this area. All this research attracts a lot of attention worldwide and brings new and interesting results.

— CERME conferences, where teacher education has always been paid much attention to. The focal point has been shifting with respect to the development of research in the area. However, it has always been based on the interaction between practices in teacher education and requirements of everyday school practice.

The focus of the first part of the text is on developments in research into teacher education and practice until present. The goal of this part is to point out main trends in the research domain. It is important not only to list the topics addressed by research since CERME1 but also to show the used methodologies and posed research questions.

The aim of the second part of the text is to illustrate some of the trends in research in teacher education and innovations at schools, focusing mainly on the further development of teachers' knowledge, beliefs about and approaches to mathematics education resulting from cooperation with researchers. When selecting from the many, the attention was paid to those areas of research that the author is familiar with and in which she has been involved.

RESEARCH INTO TEACHER EDUCATION AND PRACTICE UNTIL PRESENT

Survey Team 3 at ICME10. Research on mathematics teacher education: Mirror images of an emerging field

The Survey Team 3 (ST3) consisted of the following members: Jill Adler, Deborah Ball, Konrad Krainer, Fou-Lai Lin and Jarmila Novotná.

As a member of ST3 at ICME10 in Copenhagen, I was involved in collecting information on research focusing on mathematics teacher education in the years 1999–2003 (Adler et al., 2005). This work clearly showed it was inevitable to delimit the areas and issues we would come out of. The survey included published research in international mathematics education journals, international handbooks of mathematics education and some international mathematics education conference proceedings. Some regional sources from various parts of the world were also included. The survey was restricted to 1999–2003, covering the period between ICME9 and ICME10. More than 200 papers were analysed.

The central question for the survey was: *Is research in the field contributing to the improvement of the education of teachers of mathematics?*

The work was framed by the following considerations: What is the state and status of research in mathematics teacher education, within and across contexts? Which problems have been constructed as central in this field in the recent past, and how have these been approached? What shifts – theoretical and methodological – can be discerned and how might they be explained? Who does the research? Where? What progress has been made, empirically, theoretically, methodologically? Are there evident gaps, and if so where? What kind?

We investigated "the *who* (who was writing/doing the research, and from where), the *how* (what methods were used) and the *what* (what was being studied, theoretical orientations, assumptions and outcomes)" (Adler et al., 2004). We also examined the range of findings and conclusions in these studies. These helped to identify four areas that asked for further investigation.

Where is the centre of the field? The investigated publications were divided into two groups. The first includes publications focusing on theorising and understanding teacher learning. The second concerns aspects of curriculum reform, the goals of teacher education initiatives, i.e., evaluation. We identified the shift from studies that tended to tell success stories about teacher education initiatives, and advocacy in the initial phases of curriculum reform, to deeper reflective research that is more convincing in the scholarly sense. Teacher educators' learning was paid much less attention to. It was noted that: "We do not understand well enough how mathematics and teaching, as inter-related objects, come to produce and constitute each other in teacher education practice. We lack adequate knowledge about what and how this happens inside a teacher education program, and then across ranging or contrasting programs, contexts and conditions."

What are the theories and methods in the field? We stated that the field is emerging and needs to increase rigour. The vast majority of this research is case study research, where at least one of the researchers is also a teacher educator, and often the educator(s) whose programme is under study. It certainly makes sense if we want to study teachers' learning and teaching practices. The emergence of theories of situated learning, and attempts to theorise learning of professional practice were identified. In many papers, theoretical frameworks are left implicit. Small-scale qualitative research predominates.

Contexts of mathematics teacher education research: Who, where, and with whom? Most teacher education research is conducted by teacher educators studying the teachers with whom they are working. Attention has mainly been paid to showing that particular programmes of teacher education 'work'; a large number of papers were dealing with reform processes, particularly in the USA, and with teachers in professional communities and in other institutional settings.

Dominance of English-speaking world. This dominance was remarkable (e.g., 80% of the papers published in JMTE have been written by authors from, and report research done in, English-speaking countries). Obviously, the situation is different if we focus on national or regionally focused conferences and journals. The influence of this situation on the orientation of research was not analysed but it certainly has a great impact.

ST3 also formulated domains of interest that were underrepresented in the analysed resources. It was noted that there were fewer studies on:

Teacher working outside of "reform" contexts: Many teachers make effort to develop their teaching skills in environments where reform is not the dominant issue but where they are assisting a wide range of learners in learning mathematics.

Teachers' learning from experience: We do not know enough what teachers learn from experience, whether they learn from experience at all, and what actually supports learning from experience. Teachers spend most of their time doing teaching; we do not understand enough about what helps some teachers to learn from their own teaching while others do not.

Teachers' learning to directly address inequality and diversity in their teaching of mathematics: We do not know enough about teachers' learning to directly address inequality and diversity within their teaching of mathematics (culture, gender, language, socio economic status and mathematical background).

Comparisons of different opportunities to learn: We lack comparisons in the field that compare different opportunities to learn. How does one approach to helping teachers to learn mathematics compare with another?

"*Scaling up*": We do not know enough about what happens when programmes spread to multiple sites, what it means to scale up or what it means to extend a programme that has worked in one setting to another setting – what works, what goes wrong, what designers need to know and think about.

Education of teacher educators: Despite their important role in the system of teacher education, educators' education, professional background, etc. was not studied in the analysed publications.

This was the situation in 2004 perceived through the analyses of ST3. Approximately at the same time, another important event focusing on teacher education, The Fifteenth ICMI Study, was launched. The study was designed to offer an opportunity to develop a cross-cultural conversation about mathematics teacher education in mathematics around the world. The Study Volume is described below (Even & Ball, 2009).

ICMI Study 15: The Professional Education and Development of Teachers of Mathematics

ICMI Study 15 focused on mathematics teacher education practice and policy around the world. As stated in (Even & Ball, 2009), its premise was that the education and continued development of teachers are keys to pupils' opportunities to learn mathematics. What teachers of mathematics know, care about, and do is a product of their experiences and socialization both prior to and after entering teaching, together with the impact of their professional education. It was claimed that systems of teacher education, both initial and continuing, are built on features that are embedded in culture, the organization and nature of schooling, and too rarely is there cross-cultural exchange of knowledge and information about the professional development of teachers of mathematics. Learning about practices and programmes around the world can provide important resources for research, practice, and policy in teacher education, locally and globally.

The contributions accepted to ICMI Study 15 were divided into two Themes: Theme 1 – *Initial mathematics teacher education*, and Theme 2 – *Learning in and from practice*. In several aspects, both Themes brought new ideas in the issues considered by ST3 as less studied in 2004. It is evidenced by the list of main questions discussed in the ICMI Study 15.

Theme 1 focused on the following main questions:

Structure of teacher preparation: How is the preparation of teachers organized – into what kinds of institutions, over what period of time, and with what connections with other post-secondary study? Who teaches teachers, and what qualifies them to do so? How long is teacher preparation, and how is it distributed between formal study and field or apprenticeship experience? How is the preparation of teachers for secondary schooling distinguished from that of teachers for primary and middle levels of schooling?

Curriculum of teacher preparation: What is the nature of the diversity most pressing within a particular context – for example, linguistic, cultural, socio-economic, religious, racial – and how are teachers prepared to teach the diversity of pupils whom they will face in their classes? How are teachers prepared to know mathematics for teaching? What are the special problems of content preparation in different settings, and how are they addressed?

Recruitment and retention: Who enters teaching, and what are the incentives or disincentives to choose teaching as a career in particular settings? What proportion of those who prepare to teach actually end up teaching, and for how long?

Most pressing problems of preparing teachers: Across the initial preparation and early years, what are the special problems of teaching mathematics within a particular context and how are beginning teachers prepared to deal with these problems?

The early years of teaching: What are the conditions for beginning teachers of mathematics in particular settings? What supports exist, and how effective are they, for what aspects of the early years of teaching? What are the special problems faced by beginning teachers, and how are these experienced, mediated, or solved? What is the retention rate of beginning teachers, and what factors seem to affect whether or not beginning teachers remain in teaching? What systems of evaluation of beginning teachers are used, and what are their effects?

Mathematics educators' activities and knowledge: It concerns one of the underrepresented domains mentioned by ST3. These contributions focused mainly on models of educators' development, their quality, national support, their own practice and research.

During the Theme 1 sessions at the ICMI Study 15 conference, additional important questions emerged that had not been included in the Study Volume: What is the role of didactics of mathematics (mathematics education) in teacher education? What is the place of ICT in teacher education? How is the practical part of this preparation (the teaching practicum) integrated? What do we know about the construction of professional knowledge of teachers in relation to teacher education programmes?

The collection of papers in Theme 2 provides a range of approaches to studying teachers' learning. The papers focused on four main domains:

Development of teaching in and from practice: What are the characteristics of the process of developing professional expertise in the teaching of mathematics in and from practice? What are the beliefs, experiences and structures that are significant as far as the development of mathematics teachers and teaching are concerned? What are the conceptual, institutional, cultural, etc. structures that enable and constrain research into teacher development?

Process of learning in and from practice: What are the changes and approaches to professional development? How is the new organization of professional development initiatives for teachers conceived and implemented?

Models, tools and strategies to support learning in and from practice: What are the tools, dynamics, tasks, contexts, and learning settings that can be mobilized for pre- and in-service mathematics teacher education? What are the tasks for mathematics teacher education that are offered to teachers for deepening their knowledge of what and how to teach their pupils? What can be learned from analysing instructional episodes? What is the role and advantages of forming teachers' learning communities where they can share experiences, meanings, knowledge, lessons, etc. from their school practice?

Balance of teachers' mathematical content and pedagogy knowledge: How can we overcome the difficulties in practising teacher education and professional development that are caused by the complexity of the knowledge required for teaching? What is the relationship between teachers' content knowledge and pedagogical practices, considering it from various perspectives?

The Study Volume contains one chapter summarising key issues for research in education and professional development of teachers of mathematics. It focuses on the goals of education, the role of mathematics education, understanding of practice-based professional development for mathematics teachers and the future of strengthening practice in and research on professional education and development of teachers of mathematics.

Examples of more recent work

Research in the area of teacher education and in the area of the potential and consequences of cooperation between teachers and researchers has undergone turbulent developments over the decade since ICME10 and ICMI Study 15. This can be documented by the variety of publications in the area – monographs, articles and special issues of renowned journals as well as various conferences focusing on research in this area. Let us present examples of some more recent work that do not focus narrowly on one aspect of research in the field but try to relate this area into a wider context of mathematics education. Considerable attention is paid to involvement of teachers in research, albeit in the form communities between teachers and teacher educators (see, e.g., Jaworski, 2005; Novotná et al., 2006) or in the form of independent research conducted by teachers themselves (see, e.g., Kincheloe, 2012). All these research studies stress the benefit of teachers' participation in them, despite some limitations.

Teacher education also attracts attention of the International Group for Psychology of Mathematics education. Every year the area is addressed by a significant number of research reports, short oral presentations, posters, working sessions, discussion groups and other components of the programme and is frequently addressed in plenary lectures, panels and research forums. The importance that IGPME pays to teacher education is highlighted also by publishing Handbook of Research on the Psychology of Mathematics Education (Gutiérrez & Boero, 2006) where the fifth section includes two chapters summarizing the PME research on teacher education and professional life of mathematics teachers (Llinares & Krainer, 2006; da Ponte & Chapman, 2006).

Third International Handbook of Mathematics Education (Clements, Bishop, Keitel, Kilpatrick, & Leung, 2013)

The book offers an overview of past, present and future aspects of all areas of mathematics education (social, political and cultural dimensions in mathematics education; mathematics education as a field of study; technology in the mathematics curriculum; and international perspectives on mathematics education). Four out of the eight chapters of the second section focus on mathematics teacher education; they present research methods in mathematics teacher education, teachers as researchers, teachers' learning from teachers and developing mathematics educators.

The chapter *Developing mathematics educators* discusses different types of mathematics educators including teacher educators. It addresses cooperation between teachers and researchers. The concept of teachers as researchers is discussed from different points of view. It contributes to the area described as underrepresented in research in the material prepared by Survey team on ICME10.

Note: The issue of teacher educators has been addressed increasingly in the last ten years. An important step was Volume 4 of the Handbook of Mathematics Teacher Education (Jaworski & Woods, 2008). Recently, the proceedings of the international conference on "Educating the Educators" were published (Maaß, Törner, Wernisch, Schäfer, & Reits-Koncebovski, 2015).

Encyclopedia of Mathematics Education (Lerman, 2014)

This reference work covers all topics in the area of mathematics education. The entries offer theoretical background, summary of important findings and results in the area and provide references to important publications where more detailed information can be found.

One section coordinated by Mellony Graven addresses research in teacher education. The entries cover both the areas of pre- and in-service teacher education and the area of teacher educators, i.e. an area described as underrepresented in research on ICME10. More than twenty entries address directly teacher education and teacher practice and many other are somehow connected to the areas. The consequence of this effort to describe fully and comprehensively all aspects of mathematics education is that also topics described as underrepresented in research on ICME10 were paid due attention. Very valuable are the references to other literature and publications dealing with the topics but also focusing on development of research in the area over years.

ZDM, Mathematics Education (special issue, 47(1), Rösken-Winter, Hoyles, & Blömeke, 2015)
This special issue of the journal focuses on scaling up sustainable interventions through evidence-based CPD. In the articles, four perspectives are considered: crucial aspects of teacher learning, different CPD frameworks and their influence on developments in CPD, the meaning of developing CPD in an evidence-based way and crucial aspects of spreading CPD on a large scale. As Roesken-Winter, Hoyles and Blömeke state in their introductory survey paper, they "draw on Coburn's four dimensions characterizing the process of scaling CPD interventions, depth, sustainability, spread, and shift in reform ownership to discuss how the challenge of scaling high-quality CPD might be successfully addressed". The articles help to fill in some gaps in areas identified by Adler, Ball, Krainer, Lin and Novotná (2004) as underrepresented in research.

CERME CONFERENCES

An immense amount of work on the topic of teacher education and professional development has been done during the CERME conferences, from their early beginnings in Osnabrück, Germany in 1998. Teacher education has always been paid much attention. One Thematic Working Group has always focused on the issue, both on pre-service and in-service levels. Table 1 contains a more detailed look at the development of the issue at CERME conferences. Proceedings from CERMEs are available online at http://www.mathematik.uni-dortmund.de/~erme/index.php?slab=proceedings. It shows that even if the programme components did not have the same focus and followed contemporary trends in research in the area of teacher education in the corresponding period, they always paid attention to interactions between practices in teacher education and requirements of everyday school practice.

The significance of the issue of teacher education since the beginnings of CERME conferences is confirmed by the publication of a separate third part of *CERME1* proceedings: *On Research in Mathematics Teacher Education. From a Study of Teaching Practices to Issues in Teacher education* (Krainer, Goffree, & Berger, 1999). The book builds on the work done by Working Group 3 *Theory and practice of teaching from pre-service to in-service teacher education.* It is divided into six parts with respect to the topic that is addressed: Teacher education and investigations into teacher education; Teacher education and investigations into teachers' beliefs; Teacher education and investigations into teachers' knowledge; Teacher education and investigations into teachers' practice(s); Teacher education through teachers' investigation into their own practice; Investigations into teacher education: Trends, future research, and collaboration.

CERME	WG	Other programme type
1	Theory and practice of teaching from pre-service to in-service teacher education	
2	Theory and practice of teaching from pre-service to in-service teacher education	
3	Inter-relating theory and practice in mathematics teacher education	Plenary panel *Theory and Practice: Facilitating teachers' investigation into their own teaching*
4	From a study of teaching practices to issues in teacher education	
5	From a study of teaching practices to issues in teacher education	
6	Mathematical curriculum and practice	
7	From a study of teaching practices to issues in teacher education	Plenary lecture *Research into Pre-service elementary teacher education courses*
8	From a study of teaching practices to issues in teacher education	
9	Mathematics teacher education and professional development	Plenary lecture *Research in teacher education and innovation at schools – Cooperation, competition or two separate worlds?*

Table 1: Development of the topic at CERME conferences

As far as the focus of this plenary lecture is concerned, the most interesting is the last of the above listed areas. The title of the plenary lecture speaks of collaboration, not competition or two separate worlds. Let us recapitulate here the main ideas presented in CERME1 proceedings. They remain topical for research in teacher education and innovation at schools despite being published in a book from 1999 (i.e., 16 years ago).

In the area of *Research in the perspective of teacher education*, the following questions, substantial for the area, are studied: To what extent do mathematics teachers' general beliefs relate to local beliefs (e.g., to specific topics as teaching algebra)? What are the conditions and constraints that influence teaching practice? How do teachers manage the connection between pupils' activities and the acquisition of mathematical knowledge? (The term "acquisition" used here is worth attention, it is broader than the term learning.) What is the interplay between mathematical knowledge and ability, self-confidence, personal history and conceptions of mathematics teachers? How do internal factors interplay with external factors concerning the professional development of teachers? How can problem solving be used as a tool to find out of mathematics teachers' beliefs in order to improve teachers' mathematical knowledge and mathematics teaching?

In the area of *Research in the context of teacher education,* authors study, for example, the following questions: Considering the professional development of teachers, what is the interplay between cognitive processes and cultural, social, affective processes? How do (student) teachers construct (what) knowledge? What is the role of discourse and collaboration? What kind of knowledge do teachers bring to in-service education and how does it grow? Is the gap between what teachers learn at the university (pre-service education) and their practice at schools evident and how could we explore it? How do student teachers develop their understanding of children's ways of thinking during school practice? Why and how do mathematics teachers from one school (want to) further develop their teaching practice using alternative learning and teaching methods?

In the contributions, serious attempts to find bridges between theories and practices of teacher education are present. In particular, the idea of viewing learning environments for (student) teachers at the same time as a meta-learning environment for teacher educators who investigate into (student) teachers' growth and at the same time reflect on their influence within the interaction process is obvious.

The following are the *major trends* sketched in the texts. A broader understanding of research in teacher education is needed; it covers investigations focusing on teachers including their beliefs, knowledge and practice, and engagement of (student) teachers in investigating their own practice. There is an increasing importance of action research as the systematic reflection of practitioners into their own practice. There is an increasing importance of "stories" (narratives, curricula vitae, cases, …). It seems more attention should be paid to cultural, situated, and organizational aspects of processes in classroom and teacher education courses. Moreover, looking for integration and interconnections is crucial.

Pupils' learning, (student) teachers' learning and researchers' and teacher educators' learning are considered as three domains of strongly interconnected learning. The attention is paid to learning from investigations (learning from research questions, from research methodologies, from elaborating the data and from presenting the research).

The Working Group continued its work also at *CERME2*. Its focus was on "teacher education between issues and practical realization". The contributions were based on teachers' knowledge, investigations into teachers' practices, their attitudes; research on the impact of the use of information technologies was also included.

The work in the WG was characterized as follows by its coordinators: "More than in other fields, the researcher in the field of teacher education subject has to balance what is suggested by the theoretical considerations and what is possible to realize in practice. The discussion reflected this position and the themes touched fluctuated between the two poles."

The WG formulated *perspectives for the future*: To investigate professional growth of pre-service teachers, qualified teachers and teacher educators, relationship between theory and practice, teacher development in the classroom, connection between pre-service and in-service education, development of teachers' subject knowledge.

At *CERME3*, two components of the programme were devoted to teacher education: The WG *Inter-relating theory and practice in mathematics teacher education* and the Plenary Panel *Theory and Practice: Facilitating teachers' investigation into their own teaching.*

The topic of the WG attracted an increasing number of authors. In order to keep the discussions efficient, the participants were divided into five subgroups: Teaching approaches in particular curricular areas; Teaching approaches and their development; Elements of reflection in teacher education; Role and nature of collaborative work in teacher education; Inter-relating theory and practice.

WG formulated *issues emerging from discussions*: Situations and problems in teaching are complex and need particular solutions that can only be developed in the specific context of their appearance. There are no general solutions that might be transferred from theory to practice; also at schools, improving and understanding one's own practice is important. More teachers who reflect critically on their teaching, exchange their experiences, and read theory-driven papers in order to broaden their understanding of educational processes are needed. More teacher educators who take their teacher education practice as an object of evaluation and research are needed. Also more collaboration between teacher educators and teachers in order to promote teacher education – as a field of practice and research is essential.

The Plenary Panel, chaired by B. Jaworski, focused on the relationship of theory and practice in mathematics education. Besides Jaworski's introductory and final thoughts, three panellists presented examples from their own country. Bartolini Bussi (2004) briefly presented one special national project for education in Science and Technology SeT Project (1999–2002). Krainer (2004) investigated the relationship theory-practice in the theoretical perspective of four dimensions of "learning systems": *Action*; *Reflection*; *Autonomy*; *Networking*. Bergsten (2004) showed the theory-practice relation in mathematics education as multifaceted.

At *CERME4 and 5*, WGs related to teacher education were focusing on the same main topic: *From a study of teaching practices to issues in teacher education.* In both cases, the work was organized in subgroups; see Table 2 where corresponding topics are in the same row.

It is also of interest to compare emerging issues from the discussion in WGs at both conferences, see Table 3.

Much attention was paid to communities of practice and collaborative work in them. Cooperation between teachers and researchers was evaluated as important. It attracted much more attention at CERME5. At CERME4, attention was also paid to the assessment in mathematics teaching and implementation of ICT. At CERME5, these topics became so common in the discussion that they needed no special emphasis. Moreover, they were also discussed in other WGs focusing on ICT in mathematics education or assessment.

At *CERME6*, teacher education and development were included in WG *Mathematical curriculum and practice* where teacher education was directly linked with school practices. Its subtitle *From study of teaching practices to issues in teacher education* evoked its close link with the corresponding WG at CERME4 and 5. The call for papers asked for theoretical, methodological, empirical or developmental papers on teachers' practices, professional knowledge and teacher education. The work of this WG was organized in the following subgroups: Mathematical curriculum and practice; Professional knowledge (similar but different terms used: knowledge base for teaching; pedagogical content knowledge; competence; subject didac-

CERME4	CERME5
Understanding practice, understanding and promoting the mathematics teacher's development	Models to analyse the practice
Process of becoming a mathematics teacher	Knowledge for teaching (or professional knowledge).
Means, resources and methodology to research on and promote the mathematics teachers' development	Tasks and resources in pre-service teacher education
	Approaching reflection in mathematics teachers' professional development

Table 2: Subgroups at CERME4 and 5

CERME4	CERME5
Demand for theories, perspectives and methods capturing or approaching the flavour and the essence of the classroom activity (various theoretical frameworks)	Discussion on theories, perspectives and methods to approach the flavour of classroom activity
Incompleteness of current models to give an account of the real teaching-learning process	Confrontations of frameworks and models by means of analysing some corpus of a classroom teacher practice observation
Relationship between researchers and teachers	The nature and conditions of collaborative work. Particularly the role of the experts, and the necessity of making it possible that teachers meet together in order to reflect on their practices
Knowledge, pedagogical content knowledge, teachers´ competence (including communities of practice and the socio-cultural theory)	Notion of community of practice and related notions
	Different notions about reflection
Assessment instruments as a tool to support learning	
The role of teacher when using ICT	

Table 3: Emerging issues formulated at CERME4 and 5

tical competence; practical knowledge – beliefs and knowledge); Professional development; Approaching reflection and collaboration in mathematics teachers' professional development (Reflection is a privileged way for professional enhancement. Collaboration is a means for professional development and for research strategy.); Models to analyse the practice (The practice of teachers includes classroom teaching, as well as education and other professional development contexts; How can we manage to make research results and instruments useful for teachers as means in their professional development, and for educators in education contexts?).

One of the *main conclusions* formulated in this WG was the following: "As for primary teachers, also for secondary teachers, mathematical content knowledge and pedagogical content knowledge must be interrelated in teacher education." (Durand-Guerrier, Soury-Lavergne, & Arzarello, 2010, p. 1690).

At *CERME7 and 8*, WGs returned back to the previous title *From a study of teaching practices to issues in teacher education*, which clearly expresses the main focus of the work. In Table 4, the subgroups at both conferences are summarized; again, the allied topics are in the same row.

At both conferences, *critical issues* instead of emerging issues were formulated. The descriptions of these issues at both conferences differ substantially in their number as well as details in their formulations.

CERME7	CERME8
Mathematical content knowledge for teaching	Resources for teaching: Teacher knowledge and teacher beliefs
Professional content knowledge for teaching	
Reflection in mathematics teachers' professional development	Teacher reflection
Professional development	Teacher education and professional development)
Collaboration in mathematics teachers' professional development	Teacher collaboration
Conceptions and practices	Studying mathematics teaching
Interaction in the classroom	

Table 4: Subgroups at CERME7 and 8

CERME7:

- Recognition of the value and complementarities of different approaches to the professional development of teachers

- Recognition that there are constraints and affordances for different approaches, which vary between cultural contexts; working across cultures on teacher development projects, which employ different strategies, was considered to be a useful way of moving forward our understanding of different approaches

- Considerable work to be done in understanding how different frameworks relate to one another and in supporting researchers in selecting elements of different frameworks that will enable them to answer specific research questions

CERME8

- Working with multiple frameworks

- Suitable model for teacher knowledge

- The purpose for developing new theoretical models or for modifying/revising the existing ones

- Analyses of the influence of different types of knowledge

- Ways for promoting teacher knowledge

- Study of the mutual relation between teachers´ knowledge and practice

- The role of context

- Role of teacher educators in helping students/teachers to develop different components of their knowledge – differences for prospective teachers and in-service teachers

Sierpińska's (2011) plenary lecture was devoted to research in teacher education and practices. In the talk, results of an ongoing research, focusing on a framework for analysing the "Teaching Mathematics" courses were presented. "Teaching Mathematics" courses were designed and implemented by the author in cooperation with her colleague. Sierpińska presented her research conducted within the frame of implementation of these courses. The framework of this research might be useful for other researchers wishing to contribute to professionalization of elementary mathematics teacher educators' work.

At *CERME9*, for the first time, teacher education was the theme of three TWGs: *Mathematics teacher education and professional development*, *Mathematics teacher and classroom practices* and *Mathematics teacher knowledge, beliefs and identity.* For the detailed information about all three TWGs discussions and results see the corresponding chapters in CERME9 proceedings.

However, research on teacher knowledge can be come across not only in the corresponding TWGs, it pervades all TWGs, whichever area of mathematics education they deal with. In every TWG, one can come across papers that focus on the topic of the working group but at the same time are related to teacher education. And this is why I decided to call my plenary lecture *Research in teacher education and innovation at schools – Cooperation, competition or two separate worlds*? Are teacher education and innovation at schools closely related areas or are they two separate worlds that have very little or nothing in common? Is teachers' attitude to innovation in mathematics education influenced predominantly by the environment of the school they work at, their own experience with pupils, or by what they have learnt in their teacher education? In other words: Is teacher education an obstacle in the introduction of innovation at schools or does it support the process? Or are they independent of each other? It is very easy to understand the questions, to formulate them. However, it is far from easy to find answer to them and it seems the answers will not allow to be generalized.

Research studies – Summary

Research studies in the field of teacher education and innovation at schools on international level can be divided into at least two main areas: I. Focus on curricula of teacher education; II. Focus on the knowledge a mathematics teacher needs to teach well.

Area I usually includes issues of pre-service teacher education (primary and secondary) and the first years of their teaching practice: for example, structure of teacher education; admission of students into teacher education and their prospective career in the field;

curricula for pre-service mathematics teachers; conditions for novice teachers; preparation of teachers for overcoming obstacles they will come across in their practice; history and development of systems of education in various countries; international comparative studies of teacher education.

The fundamental question related to life-long learning of mathematics teachers and primary teachers is how they can learn for, during and from their teaching practice.

The areas in the spotlight are: What can mathematics teachers learn from their own and other teachers' practice? How do they further develop their knowledge of mathematics and of the ways of teaching mathematics if they work with recordings from teaching practice? How do they learn important information about variety, sociocultural and economic background of their pupils? How is teachers' life-long learning organized? How difficult is it for a teacher to get access to materials such as video recordings, journals, to come to lessons and observe them, etc.?

Research in *Area II* focuses on the knowledge prerequisite to successful teaching of mathematics. International community distinguishes between several types of prerequisite knowledge related to mathematics: the most prominent ones are mathematical content knowledge, MCK, and pedagogical content knowledge, PCK (Shulman, 1986).

There is a lot of discussion on whether MCK and PCK should be regarded as independent of each other or interlinked: For example, should pre-service teachers be taught pedagogical knowledge separately from content knowledge in different courses and seminars or should this be taught simultaneously as pedagogical content knowledge? Much attention is also paid to comparison of experience of novice and experienced teachers.

The turbulent developments in ICT has brought fast development of research focusing on the impact of ICT on teaching mathematics. Knowledge of the potential, advantages and possible risks of using ICT in teaching has become an important part of a teacher's knowledge. ICT supported mathematics education is a complex activity that requires a teacher's deep insight into mathematics, knowledge of a suitable ICT tool and understanding of pupils' thinking processes. That is why the PCK model was amended by knowledge from the area of technology, the so called TPCK (technology pedagogical content knowledge) (Mishra & Koehler, 2006). Apart from the concept of TPCK, research also focuses on consequences of TPCK for teacher education programmes.

COOPERATION OF TEACHERS AND RESEARCHERS

In the second part of the text, we will focus on one aspect of the relationship between teachers' knowledge, approaches to teaching and beliefs on the one hand and innovation at schools on the other. It is connected to two areas: teachers as researchers and cooperation of teachers, teacher educators and researchers. This theme is not new, for example, a PME working group, *Teachers as Researchers*, first met in 1988, and then was meeting annually for nine years. Its work was based on the belief that classroom teachers could and should carry out research connected to the practice of teaching mathematics. The output of this work is the publication of a book (Zack, Mousley, & Breen, 1997). The attention paid at PME conferences to the topic did not end with this publication. The Plenary Panel at PME 27 focused on the issue "Teachers as researchers" (Novotná, Lebethe, Rosen, & Zack, 2003). The follow-up was organized in various forms: discussion groups, working groups and a research forum (Novotná et al., 2006).

In literature, a lot of attention is paid to the impact of teachers' contact with new educational trends in the development of their knowledge in a variety of ways: organisation of teacher education (pre- as well as in-service), opportunities to experience new approaches, access to appropriate resources, etc. Jaworski (2005) believes that one way to add to the body of knowledge is through 'co-learning partnerships':

> The action research movement has demonstrated that practitioners doing research into their own practice [...] learn *in* practice through inquiry and reflection. There is a growing body of research which provides evidence that *outsider* researchers, researching the practice of other practitioners in co-learning partnerships, contribute to knowledge *of* and *in* practice within the communities of which they are a part. (p. 2)

The important issue of teachers as researchers, either cooperating in communities with researchers or doing their own research, is frequently analysed from the perspective of what it adds to the body of knowledge on mathematics education. Less investigated is the issue of what impact this type of teachers' activities has on their beliefs, teaching approaches, their knowledge.

There is no doubt that the cooperation of teachers and researchers is influenced by their pedagogical beliefs and mainly by teachers' reactions to innovative approaches. Hofmannová, Novotná and Hadj-Moussová (2003) investigated how in-service and pre-service teachers react to them.[1] The authors are convinced that without deep changes in teachers' beliefs and attitudes, major changes in pupil learning cannot occur. This corresponds to (Rogers, 1996): "The introduction of learning changes into the area of attitudes is perhaps the most difficult task that faces the teacher educator." The results of the presented research into affective barriers showed prevailing negative attitudes of participating teachers towards new educational trends. The following scheme of categories based on Rogers (1996) was created:

Inner barriers: fear of failing, fear of not meeting the requirements, fear of uncertain success. The identified causes of inner barriers were: changes caused by aging, negative self-concept, too high self-requirements and too positive perception of the others, fatigue.

Outer barriers: lack of time, personal and family problems. The identified causes of outer barriers: inability in time management, too much stress.

These findings had a major impact on the teacher education course because it enabled inclusion of new incentives into the course curricula. These new elements focus on work with teachers' motivation and attitudes. Barriers could thus turn into resources (Moschkovich, 2002).

1 The innovative approach selected for this study was Content and Language Integrated Learning (CLIL): CLIL refers to any teaching of a non-language subject through the medium of a second or foreign language. CLIL suggests equilibrium between content and language learning.

In the following, one example of collaboration of teachers and researchers with an important input of participating teachers is described. The research project is presented from the point of view of the further development of teachers' beliefs and approaches to mathematics education resulting from the cooperation. It shows one form of research collaboration between university academics and teachers of mathematics. The question in the background is: What are the advantages and limitations of such cooperation?

Impact of teachers' participation in research (Eisenmann, Novotná, Přibyl, & Břehovský, 2015)

This study is a part of a three-year research project GAČR P407/12/1939 *Development of culture of problem solving in mathematics in Czech schools*. The goal of the research project is the development of a theory of mathematics problem solving with a focus on the role heuristic strategies play in the development of *pupils' culture of solving problems* (CSP). CSP is understood as a structure of internal factors that influence a pupil's performance and success in problem solving (Eisenmann, Novotná, & Přibyl, 2014).

In short-term (3 months) and long-term (18 month) experiments, lower and upper secondary pupils were introduced by their teachers to heuristic strategies that they rarely or never came across in usual lessons but that are very effective and useful in problem solving. The pupils were led systematically to the use of a suitable heuristic strategy when they come across a problem they cannot solve using "school solving algorithm" (Eisenmann, Novotná, & Přibyl, 2014; Novotná, Eisenmann, & Přibyl, 2015). The research focused on a number of research questions two of which are connected to the area of teacher education and teacher pedagogical beliefs: Will the experiments have impact on the teachers involved? And what will this impact be?

The research team developed sets of problems that can effectively be solved using one heuristic strategy. All these problems were carefully elaborated and commented upon and can be solved in several ways. Selected problems were also subject to a priori analysis (Nováková, 2013) and were piloted on a one-time basis in non-participating classes.

All participating teachers can be described as committed teachers who invest a lot of energy into their teaching and who had attended in-service teacher education courses. They were introducing their pupils to the use of heuristic strategies through solving problems for the period of the experiments.

During both types of experiments, the impact on the participating teachers was analysed. The following changes are reported based on interviews and observation data from the collaborative work with the teachers over the period of the whole experiment and on the basis of the analysis of the structured interviews. The teachers (Novotná, Eisenmann, & Přibyl, 2015)

— lowered their demands on accuracy and correctness in their pupils' communication and recording in favour of understanding the problem solving procedures (which does not entirely correspond with the commonly accepted characterization of mathematics as a domain where accuracy and correctness of communication is an important issue),

— showed more tolerance to variety in pupils' solutions,

— acknowledged a change in their teaching towards constructivist and inquiry-based approaches,

— grew more interested in pupils' solving processes while solving problems;

— one of them reported that she started to think how to eliminate the pervasive pupils' sense of failure (e.g., she decided to use group work more often).

One of the most important results is that most of the participating teachers started to pose their own problems with the aim of making the pupils understand the various strategies better.

How is this research study linked with the results presented in (Hofmannová, Novotná, & Hadj-Moussová, 2003)? It contributes to the discussion on the conditions of cooperation between researchers and teachers and on its benefits for the involved teachers. Deeper understanding of the conditions for successful participation of teachers in research offers

> teachers the scientific background by providing information about research results which is stimulating and which influences their work. Subsequently, the movement reversed and teams of researchers and teachers worked together, either in order to create and disseminate tools for improving education (curriculum, materials, recommendations) or to answer the ongoing needs of certain researchers. (Novotná, Brousseau, Bureš, & Nováková, 2012, p. 326)

The cooperation of teachers and researchers in mathematics education represents a broad and relevant topic. The focus is mostly on the improvement of the quality of mathematics teaching and learning (Brown & Coles, 2000). The above presented studies focus on another research area which is a change in behaviour and practices of teachers involved in research in the area of mathematics education.

CONCLUDING REMARKS

The plenary lecture only covers a small part of research on teacher education and its relationship to innovative teaching strategies. Its ambition was not to be (and considering the scope of the issue could never be) exhaustive. As it was already mentioned, there is an immense number of individual and collective monographs on the issue, a great number of national and international conferences, seminars, summer schools, there are journals specializing on mathematics teacher education, for example, the renowned Journal of Mathematics Teacher Education (JMTE).

Teacher education is the topic of a number of international projects, for example, the completed project Teacher Education and Development Study in Mathematics TEDS-M (Tatto et al., 2012; http://teds.educ.msu.edu/) focusing on pre-service teacher education or the currently running project FIRSTMATH – The First Five Years of Mathematics Teaching (firstmath.educ.msu.edu/) focusing on the first five years of teaching practice of novice teachers. One of the recent events, ICMI Study 23 Primary Mathematics Study on Whole Numbers, whose conference took place in June 2015 (Sun, Kaur, & Novotná, 2015) and the Volume is now under preparation, pays considerable attention to research in teacher education – two chapters in the volume focus on this topic.

The presented survey in various resources implies that some of the issues that seemed to be underrepresented at ICME10 in Copenhagen are now much more fully developed (e.g., the issues concerning teacher educators). However, there are still areas that deserve more attention and research work.

To conclude, let us recapitulate the main and most frequent areas of study in research into teacher education since ICME10: Much attention has been paid to the balance between mathematical content knowledge (MCK) and pedagogical content knowledge (PCK). This area also covers works on knowledge prerequisite to cross curricular teaching of mathematics (inside mathematics or between mathematics and other subjects). Another important issue is the field of professional seeing of (student) teachers of mathematics and the "ability to notice" as integral part of PCK.

Also technology pedagogical content knowledge (TCPK) is one of the subjects of research in mathematics education that have become more prominent. The use of e-learning and b-learning (blended learning) environments in teacher education and teaching practice belong to this field.

School mathematics is based on problem solving. Therefore it comes as no surprise that much attention of research is paid to a teacher's knowledge prerequisite to the efficient use of various solving strategies when solving mathematics problems (see e.g., Novotná, Brousseau, Bureš, & Nováková, 2012). This is closely related to a teacher's competence to pose problems.

Another research area that got more attention is the issue of mathematical literacy and numeracy. This research studies the relations between mathematical literacy and mathematics education and tries to define the requirements on the teacher and their knowledge. Important here is the teacher's mathematical culture and the potential for its development. Some researchers also focus on knowledge and skills prerequisite to the development of pupils' mathematical culture which has important consequences for research into assessment in mathematics and into didactic approaches to possible learners' difficulties. All this is in a narrow relationship with the cooperation between researchers and teachers and the further development of mathematics teachers' beliefs.

The paper started with the author's personal confession. Let it be concluded in a similar spirit: The three presented examples come from research studies in which the author was involved. They illustrate different views on the topic of this plenary lecture. I am convinced teacher education and innovation at schools are related to each other, they influence each other and if they are separate, the conditions for their development are much worse. Researchers in mathematics education should always bear in mind whom the research concerns and how to make the teaching community interested in the findings. Either by inviting teachers to participate in the research, its design and activities, or by communicating the findings well to practising teachers, giving them support and getting from them feedback on how the innovation works in real school conditions.

ACKNOWLEDGEMENT

The research was partly supported by Czech Science Foundation project P407/12/1939.

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TWG01
Argumentation and proof

Introduction to the papers of TWG01: Argumentation and proof

Samuele Antonini[1], Orly Buchbinder[2], Kirsten Pfeiffer[3] and Gabriel J. Stylianides[4]

1 University of Pavia, Department of Mathematics, Pavia, Italy, samuele.antonini@unipv.it

2 University of New Hampshire, Department of Mathematics and Statistics, Durham, USA

3 NUI Galway, School of Mathematics, Statistics and Applied Mathematics, Galway, Ireland

4 University of Oxford, Department of Education, Oxford, UK

INTRODUCTION

The role and importance assigned to argumentation and proof in the last decades has led to an enormous variety of approaches to research in this area. The 21 papers and 5 posters presented in TWG Argumentation and proof come from 15 countries, residing in 4 continents, and offering a wide spectrum of perspectives. These contributions intertwine educational issues with explicit references to mathematical, logical, historical, philosophical, epistemological, psychological, curricular, anthropological and sociological issues.

Taking into account this diversity, the contributions were presented and discussed in working sessions under seven themes: theoretical and philosophical issues; theoretical and philosophical issues including Habermas' rationality; argumentation and proof in teacher education; argumentation and proof in textbooks; argumentation and proof at the university level; proof representation; and performance, assessment and abilities.

THEORETICAL AND PHILOSOPHICAL ISSUES

The three papers presented in this session concerned various theoretical issues.

Knipping, Rott and Reid contrast three different perspectives when analyzing classroom argumentation (interactional, task analysis, and sociological). In particular, they state that multiple perspectives and levels of analysis are required in research on classroom argumentation, showing that each of these perspectives offers insights into students' argumentations but no single perspective is adequate to completely describe the nature of students' argumentations and ways to support their development.

In the framework of Husserl's transcendental phenomenology, Moutsios-Rentzos and Spyrou present a reading of the genesis of proof in ancient Greece. The philosophical and historical analysis aims to set up a didactical framework to foster students' need for proving.

The paper by Raman-Sunström and Öhman focuses on the notion of mathematical 'fit', with the goal of identifying some of its characteristics. In particular, their analysis leads to investigation of the relations between the 'mathematical fit', the notion of explanation and some issues related to the aesthetic aspects of proof.

Finally, two posters were presented. Pericleous discusses a study in a Cypriot classroom in the framework of Activity Theory. Vallejo and Ordoñez provide an example of proof-based teaching discussing knowledge construction in the field of natural numbers.

THEORETICAL AND PHILOSOPHICAL ISSUES, INCLUDING HABERMAS RATIONALITY

Three presentations were concerned with the 'scientific culture' in the classroom and in particular with analysis of different aspects based on Habermas' (1998) construct of rationality. Cramer investigates how Habermas' theory helps to explore obstacles and barriers to argumentation. Goizueta and Mariotti focus on the assessment of the validity of mathematical models in a problem-solving situation and underline the need for research to analyze epistemological aspects of the mathematical culture of the classroom.

Boero applies Habermas' theory to analyze a university student's attempt to prove an elementary theorem concerning continuous functions in epsilon-delta calculus.

The presentation of papers stimulated a rich discussion on Habermas' theory of rationality as a research tool which provides a 'dynamic vision' of mathematical activity, and as a general perspective for analyzing the epistemological dimension of classroom interactions and its socio-interactive roots.

The two posters presented in the session involve different frameworks that were discussed and compared with Habermas' rationality: the Toulmin (1958) model in the Ishii's poster, and a competence-based four step model set up by Süss-Stepancik and Götz.

ARGUMENTATION AND PROOF IN TEACHER EDUCATION

In recent years, the general interest in research in mathematics teacher education has stimulated many questions in research about argumentation and proof and teacher education. The presentations of papers in this session considered a variety of different tasks and activities.

Kempen and Biehler focus on perception of generic proofs in number theory and identify three different kinds of pre-service teachers' perceptions of proof: logical acceptance and psychological conviction, general acceptance of the concept and psychological uncertainty, and inappropriate understanding of the concept.

The paper by Buchbinder and Cook is concerned with learning opportunities for pre-service teachers. They suggest that proof construction can be fruitfully inspired by exploring unconventional computational algorithms presented through math-tricks.

From a different point of view, Erkek and Işıksal-Bostan's paper focuses on advantages and disadvantages of the use of GeoGebra in a study involving pre-service elementary mathematics teachers.

The poster by Modeste and Rojas discusses a research project that aims to build a model of mathematical activity that can be used in primary teacher education. The main issues discussed in the session were the design of proving tasks with the goal to avoid cultivating misconceptions in the teaching of proof, and to foster positive attitudes towards mathematics.

ARGUMENTATION AND PROOF IN TEXTBOOKS

Textbooks play a major role in everyday mathematics practice in many countries around the world and many teachers rely heavily on their textbooks that influence their decisions of which tasks to implement in the classroom, and how to implement them. Using different theoretical frameworks and adopting different approaches to the analysis, three papers investigate aspects related to argumentation and proof in textbooks in four different countries: Israel, Spain, Sweden and Finland.

Silverman and Even characterize justification and explanation for mathematical statements offered in 7th grade Israeli textbooks. The analysis revealed that the textbooks commonly used several modes of reasoning in explanations for each statement. Nearly every justification was deductive or empirical, yet different modes of reasoning were used for geometric and for algebraic statements.

Bergwall presents a framework for analyzing generality in proving tasks in calculus in Swedish and Finnish textbooks. The author discusses the usefulness of framework in analyzing and comparing textbooks and states that there is not necessarily a correlation between the number of general proving tasks and the opportunities for students to engage in reasoning about arbitrary functions.

Finally, Conejo, Arce and Ortega present the evolution of the proof schemes shown in grades 11 and 12 textbooks related to the theorems of limits. In particular, they develop a framework based on Harel and Sowder's (1998) notion of "proof schemes" and show a case study applying the framework to Spanish mathematical textbooks from the 70s until today.

The discussion focused on methodological aspects related to the unit of analysis (e.g. task, lesson, chapter, etc.) and on the difficulties identifying proof and argumentation tasks in textbooks (e.g. looking for keywords like "prove" or "show" might not be enough).

ARGUMENTATION AND PROOF AT THE UNIVERSITY LEVEL

Mathematics education at university level and, in particular, the teaching of proof and proving require specific methodological approaches and theoretical considerations that take into account the specific goals and the modality of teaching in this academic setting.

The theoretical paper presented by Annie Selden and John Selden suggests a perspective for understanding university students' proof constructions based on the ideas of conceptual and procedural knowledge, explicit and implicit learning, behavioral schemas, automaticity, working memory, consciousness, and two systems cognition.

One technique that future mathematicians should master is proof by reductio ad absurdum. Alvarado and González focus on it and present part of a research study in which college students performed a task which required application of this technique.

During the session, the discussion focused on the need for development of tasks, sequences of tasks and courses, as well as specific didactical approaches to support university students' proof production and comprehension. In this vein, Pfeiffer and Quinlan presented a paper on proof evaluation tasks in a university mathematics course. The responses to the task, in which students were asked to evaluate and rank different proposed proofs, provided rich opportunities for students to attend to the nature and functions of mathematical proofs; the task also revealed some interesting features of students' thinking. The authors argue that proof evaluation tasks can afford rich learning opportunities as well as enable novice students to participate in authentic mathematical practice.

PROOF REPRESENTATION

Contributions in this session discuss aspects of representation of proof, with particular attention to oral and written modes of representation, which involve different cognitive processes, and require careful consideration when one attempts to interpret research findings on students' conceptions of proof, and when comparing findings from different studies.

In particular, Andreas Stylianides focuses on the role of the mode of representation in students' argument constructions. He discusses findings from a classroom-based design experiment suggesting that the use of an oral mode of representation may be more likely, compared to a written mode, to support the construction of an argument that approximates or meets the standard of proof. This raises concern about the validity of research findings reported in the literature on students' conceptions of proof, and creates difficulties in comparing findings across different studies.

Azrou's paper deals with the writing of a proof text as the final step of the proving process. She describes university students' difficulties to get a satisfactory product, which frequently result in an unclear text in a disorganized form, in particular when students are asked to answer open questions.

The aim of Moulin's and Deloustal-Jorrand's work is to explore potential functions of stories in the learning of Science and Mathematics with the focus on potential connections between the mathematical space and the rhetorical space during problem solving activity. They characterize theoretically a processes-transferring space between the narrative activity and the problem solving activity. By analyzing oral and written products of children work, they show that the narration act supports students' mathematical reasoning.

PERFORMANCE, ASSESSMENT, ABILITIES

The main issues discussed in this session concerned the importance of a priori analysis of assessment tasks in order to understand their requirements and compare to students' mathematical histories, and the influence of the type of curriculum on students' proof performance.

In particular, Sears and Chávez examine students' performance on a proof task about corresponding parts of congruent triangles. Using data from 1936 students, they show that, regardless of curriculum type, students experience difficulty with constructing this type of proof.

Luz and Yerushalmy examine the design principles of e-assessment of understanding of geometric proofs. In particular, they review various proving task-design studies, looking for a template that incorporates interactive sketching that can be checked automatically.

Finally, the participants were involved in a debate on the question about the possibility, in mixed-ability lower secondary school classrooms, to engage all students in proof without compromising the development of the proof abilities of the most "talented" students. In particular, Moya, Gutiérrez and Jaime present a study on the ability to make proofs of mathematically talented secondary students attempting geometry proof problems.

CONCLUSIONS

We think that TWG on argumentation and proof has offered the participants the richness of diversity in this research domain and the opportunity of fruitful discussions. It also seemed to stimulate not only the interest of comparison but also the curiosity of undertaking a possible integration of different perspectives and the need of enhancing the development of international collaborations.

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TWG01

Research papers

Proof by reductio ad absurdum: An experience with university students

Angelina Alvarado Monroy[1] and María Teresa González Astudillo[2]

1 Juárez University of the State of Durango, Durango, Mexico, angelina.alvarado@gmail.com

2 University of Salamanca, Faculty of Education, Salamanca, Spain, maite@usal.es

One technique that future mathematicians should dominate is proof by reductio ad absurdum. This paper presents part of a research study in which college students performed a task in which they needed to apply the knowledge gained using this technique. Small group discussions and a discussion led by the teacher were the methodology used in the classroom to solve the task. Both types of interactions were analyzed using the RBC-C model (Schwarz, Dreyfus, & Hershkowitz, 2009) to document how the construction process took place. It was found through consolidation of epistemic actions that although the students had to deal with difficulties associated with the proof, they were able to use the newly acquired knowledge.

Keywords: Proof, interaction, construction, knowledge.

INTRODUCTION

One of the main activities in mathematics is to prove. However, in recent years, some countries, such as Spain and Mexico have almost banished the practice of proofs from the school curriculum: "The idea of proving has changed over time; it depends on the context and cultural environment. Since the development of modern mathematics, which put too much emphasis on formal proofs, there has been a decline in their use in high school, this has strong implications for the transition to college"(Gómez-Chacón, 2009).

Possible reasons for the difficulties students face when having to develop a proof are a poor ability to formulate [and identify] mathematical statements, inappropriate concept images, a lack of training to generate and use their own examples and only an intuitive understanding of the concepts involved in the proof (Moore, 1994).

Everyday language is an obstacle for the learning of a mathematical proof because of the differences between this language and mathematical language (Epp, 2003). A conditional statement in every day language often admits various connotations of causality and temporality that makes its meaning quite different from the mathematical sense. Sometimes, ordinary language gives a different meaning to the statement caused by the tendency to deduce what is not said. In this sense, Epp (2003) suggests that the difference between everyday language and mathematical language can lead to committing the "reciprocal error" in accepting that "p only if q" is logically equivalent to "if p then q"; to difficulty in the interpretation of quantified propositions; and to the mistakes made in trying to deny the implications.

A lack of knowledge of proof techniques, how to choose the facts and theorems to be applied or when to use or not knowledge based solely on symbolic manipulation and the use of mathematical procedures are also perceived as problematic (Weber, 2001). Logic and proof are conceived separately; to avoid this, instruction must show the proof as a form of validation and the usefulness of language in developing and communicating proofs (Weber, 2001).

In this paper the process of solving a task using proof by reductio ad absurdum is analyzed in a study with university students in Mexico. This activity is part of a broader research study about the design of instructional tasks for teaching proofs and recording the advances made in the process of proving followed by the students as they solved these tasks. The solving process performed both in small groups and through guided interaction with the teacher are analyzed.

However, observation and detailed analysis of the process of construction of proofs in context can be very complicated when the data are massive and confusing. Hershkowitz, Hadas, Dreyfus, & Schwarz (2007) provides an example of research in which the flow of knowledge from one student to another is analyzed until they arrive at a common knowledge base. This type of research focuses on the process of construction and on the constructs at a given point until consolidation is acquired. The authors consider knowledge to be shared if the common knowledge base allows the group of students to continue building and updating knowledge collaboratively on the same topic. The authors acknowledge that they have relied on the work by Cobb (1995) in regard to collaborative learning.

CONCEPTUAL FRAMEWORK

Since students are expected to develop abstract mathematical knowledge, we consider Abstraction in Context (AiC) by Schwarz, Dreyfus, & Hershkowitz (2009) as the a suitable framework for analysis of the interaction. In AiC, abstraction is defined as "a vertical activity for the reorganization of previous mathematical constructs within mathematics and by mathematical meanings so as to lead to a construct that is new to the learner" (Schwarz, Dreyfus, & Hershkowitz, 2009, p. 24).

An abstraction process has three stages: the need for a new construct, emergence and consolidation. The abstraction cannot occur without the need for a new contruct; this need may arise from an intrinsic motivation to overcome contradictions, surprises, or uncertainty. The second stage is central and is where the new construct emerges. Three epistemic actions can be observed in this stage: R-actions (Recognizing), in which the learner recognizes that a specific prior construct is relevant to the situation at hand, B-actions (Building-with), with which the learner constructs recognized acts to achieve understanding of a situation or solve a problem; C-actions (Constructing), using B-actions and integrating previous actions to produce a new construct. The C-action refers to the first time that the learner uses or mentions a construct. In this process, R-actions are nested within B-actions and B-actions are nested within C-actions. C-actions can be nested in C-actions at a higher level. Finally, the third step, consolidation, is a long-term process which occurs when the construct is mentioned, constructed or used after a C-action. This stage is characterized by personal evidence, trust, immediacy, flexibility and care when working with the construct (Dreyfus & Tsamir, 2004) and also when the language is becoming more precise (Hershkowitz, Schwarz, & Dreyfus, 2001), although Kidron (2008) and Gilboa, Dreyfus, & Kidron (2011) consider that the increase in language precision is characteristic of the construction stage itself and not just the consolidation.

In AiC the epistemic actions referred to are known the as RBC model (Recognizing, Building with, Constructing) and the RBC-C model, with the second C corresponding to the second stage of consolidation.

The aim of this paper is to describe the process of students' proof construction and how they transfer the knowledge already acquired to solve new situations. We analyze the interaction of a group of five students and with the teacher using the AiC model (Hershkowitz, Schwarz, & Dreyfus, 2001). Therefore the research questions to be answered are: what are the epistemic actions that arise in the course of an interaction in a group and with the teacher during the process of proving a statement by reductio ad absurdum? And, are the students capable of consolidation during a long process of teaching of the concept of proof through a collaborative work effort?

METHODOLOGY

The activity described below is part of a broader research study about introducing mathematical proofs to university students in their first semester of the Bachelor of Applied Mathematics program of Juarez University of Durango State, Mexico. The average age of the students was 18. Specifically, the task presented in this paper pertains to the eleventh session and their participation was voluntary. The students had previously worked with other tasks about proofs such as the generation of definitions (Alvarado & González, 2013a), which gave them some preliminary knowledge about proofs, the identification of the parts of a mathematical proposition (Alvarado & González, 2013b), the use of logical connectives, the generation of examples and counterexamples, the formulation of conjectures and the proof by direct demonstration (Alvarado & Gonzalez, 2013c). In this session, the students were introduced to the process of proof by contradiction. The teacher began by explaining this technique with the following example: *Show that prime numbers never finish, there is always one more.* Different tasks were proposed to the students to prove some statements in the way shown by the teacher and once they had proved them, a challenge task, described in the next section, was proposed. The challenge task was to prove the following statement: *It is impossible to write numbers using each of the ten digits once so that their sum is 100*.

The classroom activity was conducted in a group with five students working on their own. Once they obtained an answer, an interaction with the teacher took place. Through collaborative dynamics the students were given the opportunity to develop knowledge together and express their ideas verbally. Planas & Morera (2011) argue that interaction is a skill that must be practiced by students and teachers in math class. This kind of an interaction is based on every participant's right to express their opinion and try to convince the others of the validity of their ideas.

The interactions can be effective if they feature a real and true exchange or communication. That is to say, if the participants: 1) undertake social interactions voluntarily with peers and with the teacher; 2) actively participate in interactions and engage with the task; 3) have developed the basis for sharing and receiving (taken-as-shared) in equal conditions; and 4) do not represent a mathematical authority in the course of the interaction. Cobb (1995) and Steffe & Wiefel (1992) consider points 3) and 4) necessary for small group interactions, for good communication and genuine collaborative learning in mathematics. In this sense it is expected that effective interaction will result in a product (definition, rationale, conjecture, method, argument, demonstration, illustration, etc.) agreed on by all participants.

During small group work, the teacher can see if students spend a long time looking for a way to approach the task. For this case the researchers had previously suggested that the teacher should interact with the students and give adequate support to their thinking. The teacher must take into account the four "teacher movements" (Jacobs & Ambrose, 2008) while the students continued working on the task. These movements are: a) ensure that students understand the task (what do they know about the problem?) and if necessary change the context to a more familiar one, b) change the problem to a similar one with simpler values, c) ask students what they have tried until that moment, and d) suggest using another strategy.

Discussions in the small group and with the teacher were videotaped and transcribed in their entirety. The analysis of interventions was made by identifying different units of analysis determined by the discussion of one aspect of the student's task. For each unit of analysis the epistemic actions evidenced were identified, and this identification was then triangulated between the researchers. To identify the epistemic actions, in this analysis R-actions were considered to be the epistemic actions that the students used to recognize the information assumed to be true, as well as the definitions and concepts involved, the B-actions were those that emerged from the statement by extraction of its meaning, or when calculations were performed to obtain deductions and to understand the meaning of the statement, to finally build the requested proof and organize it (C-action). The C-action occurred once the statement of the proposition and its proof were considered as a unit.

DATA ANALYSIS

Below we describe, characterize and analyze the process of solving this task which took place in the classroom during the interaction of a group with five students. We have differentiated two parts: the first one describes the work in the group and the second one the corresponding interaction with the teacher.

Group discussion

In the following dialogues only the students participated. We have not differentiated which student made each contribution because we are interested in the whole process as the overall production of the group. Each contribution is numbered indicating that it belongs to one of the students. The comments in brackets are comments made by the researchers.

The first *R-actions* served to recognize the assumptions or information available [2, 4], the conclusion [1 and 12] and the extraction of meaning from the data [5, 6]. As discussed below, students misinterpreted the information. They were considering a 10-digit number (all possible combinations) in which the digits 0 through 9 appear only once.

1	We must see that [a number] with the given conditions is impossible.
2	We have to use each of the 10 digits.
3	Yeah. It is impossible.
4	In P [the hypothesis] we have 10 digits.

5 We can put that we form a total of however many numbers but the digits 0 through 9 always appear.
6 You are going to do all such combinations. One would be 1234567890.

Following this misinterpretation, we can identify some *B-actions* (advances in the proof from the *R-actions*) in their arguments [7–9 and 14] i.e. the commutative property was applied. The students showed flexible thinking [10] using another *B-action* linked to the process of deducting from the back to the front "The only way you have 100 would be ..." and as a result some [12 and 13] *B-actions* took place to extend their peers' understanding.

7 [...] The sum will always be the same because the digits change places but not their value.
8 Good. P is that. Then P1 [the first deduction] which does not change the sum.
9 And by the property of the sum 0 + 1 + 2 + ... + 9
10 The only way you have 100 is with 10 places where the numbers were 10 and that's not possible.
11 I don't understand.
12 If the sum of all digits must give 100...
13 You are going to have a big number with 10 digits. Which you want without repeating. But their sum is not going to give 100.
14 [They read again] Well, the sum will always be equal using the commutative property.
15 But not different. It will always be the same.
16 See. It says that it is impossible.
17 How to build a contradiction?
18 Now, what must we prove?
19 That it's impossible that the sum is 100. That is Q [the conclusion] //.
20 Here's something missing. The sum will always be 45.

Another *B-action* occurred when formulating the negation of Q [21]. Finally [22 and 23] they found a contradiction and that was the step used to conclude that this was sufficient for proving the statement so this was the final *C-action*, i.e. the construction of a contradiction and thus the proof of the statement. It is important to mention that the statement did not appear in the conventional form "if P then Q", and the students did not realize that the numbers could be formed with any number of digits, but each digit from 0 to 9 could only appear once. This led them to prove a different proposition from the one requested.

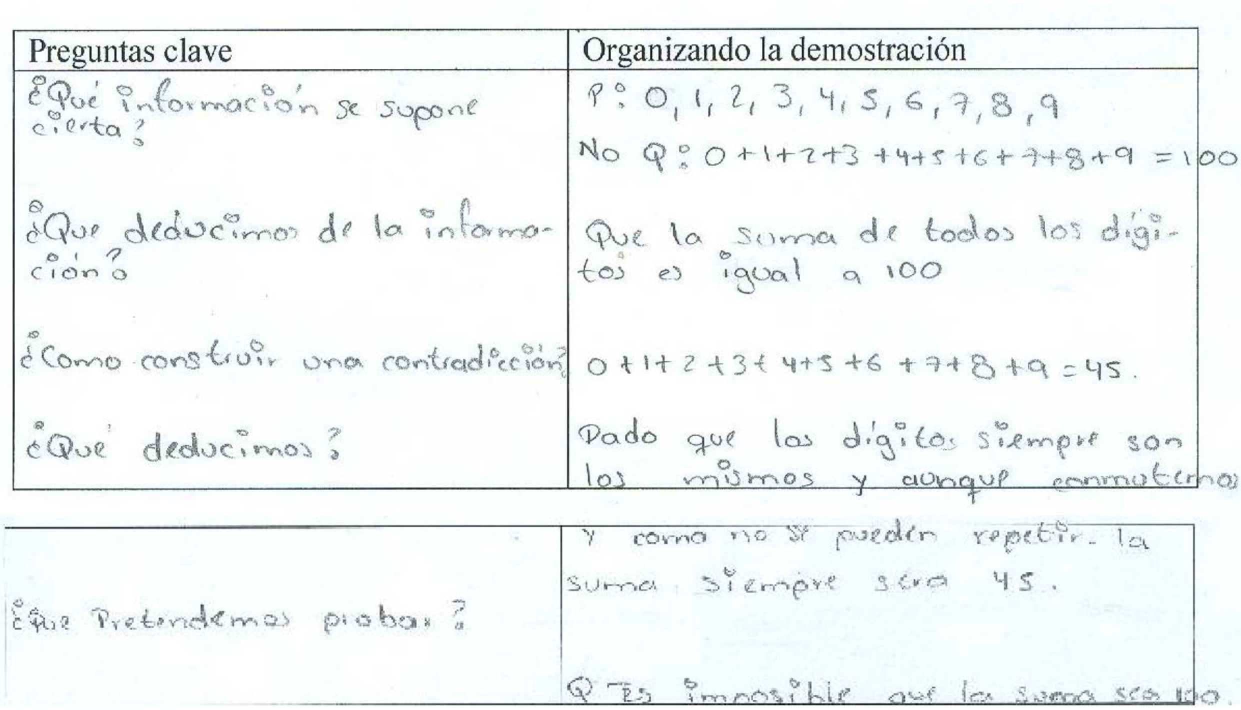
21 First there are 0, 1, ..., 9. Then assume that their sum is 100.
22 Now I do the sum 0 + 1 + ... + 9 and that gives me 45 different from 100.
23 Since the digits always add up to 45 although we change positions, then the sum can never be that, and therefore Q is impossible, that the sum would be 100.

The figure below shows the written production made by the students, well-organized by steps in order to clearly communicate their ideas, even though they have not proved the adequate proposition.

Interaction with the teacher

The teacher reread the proof written by the group. First, he clarified the misunderstanding of the proposition as students had thought they had to consider a single number with 10 digits and with each of the digits from 0 through 9 only once. In [24] the teacher presented an example involving two numbers [25]. The teacher [25] reframed the situation and students in *R-actions* [26 and 27] recognised the hypothesis and the negation of the conclusion [27] and implicitly the conclusion. They recognized [29] that they should move from hypothesis and NO Q to construct a contradiction. The teacher considered it important to verify that the interpretation of the task was suitable and asked for some examples [30]. They showed *B-actions* [31–35] to build examples with the aim of understanding the nature of numbers and the negation of the conclusion.

24 Teacher: [...] seems to be a confusion [...] Of course, if I write a number with the digits 0 to 9 each written once, their sum does not give me 100. But look. The exercise is the sum of numbers written with these digits used only once [He writes two numbers: 12345, 6789] what is the sum?
25 Teacher: Well I can write it this way [two numbers] but the digits can appear once. Let's see, what would be the way to reframe this situation? What do we know?
26 Students: The set of digits is 0, 1, 2, 3, 4, 5, 6, 7, 8, 9.



Preguntas clave	Organizando la demostración
¿Qué información se supone cierta?	P: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 No Q: 0+1+2+3+4+5+6+7+8+9 = 100
¿Que deducimos de la información?	Que la suma de todos los dígitos es igual a 100
¿Como construir una contradicción?	0+1+2+3+4+5+6+7+8+9 = 45.
¿Que deducimos?	Dado que los dígitos siempre son los mismos y aunque conmutemos y como no se pueden repetir la suma siempre será 45.
¿Que Pretendemos probar?	Q: Es imposible que la suma sea 100.

Figure 1: Students' written production

27 Students: The sum of them is equal to 100. That would be the negation [NO Q].

28 Teacher: How to build a contradiction?

29 Student: We must think of different numbers and their sum must be equal to 100.

30 Teacher: I have several numbers. I do not know how many. Their sum must be equal to 100. In addition, together they can contain no more than each digit once. See, what numbers would I use?

31 Student: So a_1, a_2 ... a_k should only have each digit once.

32 Teacher: That is, you may have one, two or more numbers, but if one digit is here in that number, you cannot use it in another number.

33 Student: Yes, [...] if I have 13, I cannot put 34 in another number [adding].

34 Student: But if in that case, we can obtain a sum even higher than 100.

35 Student: Yes, it may be higher, but we have assumed that it equals 100 [we must prove that it is impossible to add up to 100 and then its negation is that their sum is 100]

In the following excerpt, deductions were made from the hypothesis (they used numbers like 1245, 736 and 89) and the information derived from the negation of the conclusion. For the first deduction the teacher asked the students to analyze the digit of the units and think about what is required in this position so that the sum would be 100. They deduced with *B-actions* [37] that the sum of the digits of the units should give multiples of 10 or 20. They proposed examples of numbers to be in the position of the units [39–43] which is a *B-action* because it is a construction made by the students from the statement. Exploring with *B-actions*, they discovered that it was impossible to obtain the sum with 3-digit numbers [44–48]. When considering two-digit numbers [49] another *B-action* addressing the possibilities of the sum of the digits in the tens position [51] is given. They also gave an extreme example with the single case of a number with all digits. In this case the sum is itself [53 *B-action*] and greater than 100.

36 Teacher: Yes. It would be 100 [the sum], and what does that imply? For example, look at the numbers of units. What is needed to be 100?

37 Students: The units must give me a multiple of 10 [another student says]. Perhaps 10 or 20.

38 Teacher: What would it be? For example?

39–43 Students: 1 and 9 / 8 and 2 / 5, 2 and 3 / 7, 2 and 1 / 1, 4, 2 and 3.

44 Teacher: For example, in the case of 9 and 1, what happens? What about the other digits? Would they have a chance? If there were two numbers I would have

to accommodate the 8 digits left. Also if there were three-digit numbers the last digit of the third one would be 0.

45 Student: And I should accommodate 7 digits [2,3,4,5,6,7,8].

46 Teacher: So, at least on one of those three numbers we would have 3 locations to accommodate digits.

47 Students: Yes.

48 Teacher: But the sum is greater than 100, then that happens with numbers with more than three digits.

49 Students: One can do it if the number has two-digits, but I can't explain. Well, for example 5 two-digit numbers, using 0,1,2,3,4...

50 Teacher: For the units?

51 Student: Yeah, and then one must accommodate 5, 6, 7, 8, 9. We need to obtain in the tens, 9 [adding the units makes 10, therefore it accumulates 1 in the tens] and no ...

52 Teacher: Yes, well. I see.... Well, you can work on this idea. What matters to me is that you understand that the basic idea of reductio ad absurdum is to assume that the sum is 100 accompanied by the hypothesis. If I explore the possibilities and arrive at a contradiction, then we have proved the proposition.

53 Student: Well, if we think of a number with all digits the sum would not be 100 but rather greater.

Finally, the teacher stopped the discussion considering that the important thing is to understand the technique and the students must continue working later on the proof of the proposition on their own. As this took place in the last two days of school we were not able to collect the final evidence of the students' work on this proposition.

FINAL REMARKS

In this activity, a group discussion, the students' written production and the discussion led by the teacher to encourage knowledge construction from what they previously knew have allowed us to document the interactive generation of knowledge. The students recognized the hypothesis, the conclusion, and their role in the proof, they could construct the negation of the conclusion and from there, they proved the proposition although they made a misinterpretation of the statement. This is evidence of the consolidation of the knowledge acquired by the students.

They carefully managed the premises and constructed deductions from them, which is an indication of the consolidation of the new constructs as established in the RBC-C model and as shown by personal evidence. They also discussed the ideas they shared, showed confidence, immediacy, use of the construct when changing the context, working both in the group and with the teacher, seeking to clarify and refine their thinking, which may allow their language to increase in accuracy.

The RBC-C model allowed us to document and capture the complexity of group work for the construction of knowledge. Thanks to this model as a theoretical and methodological tool, we were able to identify the epistemic actions that occurred during the process. The group discussion with peers allowed students with the same level of knowledge to confront each other and identify some weaknesses, in this case comprehension of the task statement. The activity began with R-actions that progressed until a C-action. This activity sometimes required feedback to clarify to a classmate the reasoning performed, for example when a student says he does not understand. Throughout the task the students were persistent. They used examples in order to look at their structure, to understand their nature and to extract information in the construction of the contradiction.

Interaction with the teacher was essential in this case to clarify the meaning of the proposition. In relation to comprehension and application of proof by reductio ad absurdum, the students managed the technique, handled it properly and they understood the function of negation of the conclusion in this proof.

ACKNOWLEDGMENT

The authors acknowledge the support received by Spanish Ministry of Science and Innovation in the National Research Plan, reference EDU2011-29328.

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Proof writing at undergraduate level

Nadia Azrou

University of Medea, Medea, Algeria, nadiazrou@gmail.com

This paper deals with the writing of a proof text as the final step of the proving process. In particular, students' difficulties to get a satisfactory product, which frequently result in an unclear text in a disorganised form, are analysed. Differently from other difficulties related to proof and proving, we noticed this phenomenon particularly when third year university students were answering some open questions where the process has to be built up, differently from traditional questions of calculation or direct use of a mathematical result where the steps of the process are known, regardless the correctness of the proof. We'll try to identify the reasons behind writing an unclear, messy draft instead of a clear readable proof text; its consequences on students' making proofs at the university level will be considered.

Keywords: Proof, university, proof text, open tasks, meta-mathematics knowledge.

INTRODUCTION

The study reported in this paper is a part of a PhD research on proof and proving at the undergraduate level, focused on students' difficulties when making proofs within a course of Complex Analysis. Research concerns third year university students' productions while responding to tests during the academic year. These students are prepared to become engineers in a high level selective university in Algeria; the mathematics programme for them is the same as for mathematics majors in any other university, but with more time and more complete exercise activities. Each test contained three questions, chosen in order to investigate difficulties of students' proving. During the analysis of collected data (consisting of students' written productions) of the three tests, for a few students an expected difficulty emerged related to the production of the final written proof text; a fourth test was designed to explore that competence. In the first three tests, the questions were closed (like *'calculate', 'prove that'*), and most students' proof texts were quite clear, well organized and legible, regardless the correctness of the solution. In the fourth test, the questions were open (like *'why…', 'is it possible to have …'*) and students were required to justify their answers: we realized that in most cases the final proof text was written like a draft, in a messy form that makes it difficult to read, even though, most of the time the idea behind seemed to be correct.

We then decided to investigate the problem of writing a proof at the undergraduate level, because we think that it's a crucial skill and one of the factors that prevent students from making successful proofs or/and completing correctly a proof already began. When the first sentences are insufficiently developed and written like a draft in a disorganised style, they cannot be a strong base to develop a complete well organized proof and even to check the correctness of reasoning; moreover the students may misread what they wrote, thus they are misguided in their way through.

In a previous study (Azrou, 2013), performed with first year university students attending an algebra course, the findings indicated that most of the proofs given by the students in closed-type tasks, even the incorrect ones, were clearly written and could be read easily. The purpose of the study reported in this paper is to try to identify the reasons why many students produced those disorganised unreadable proof texts while answering open questions.

My hypothesis is that reasons might have been inherent: in the didactic contract that shapes the teacher-learners relationships; in the lack of meta-mathematical knowledge (in particular, as concerns proof); and in the cognitive difficulties inherent in moving from intuitions, knowledge and evidences elaborated at the inner or oral level, to a well-organized and clear written proof text, with an appropriate and correct use of mathematical language. Based on this working hypothesis, in the next section we will consider some theoretical elaborations concerning the above possible reasons, in order to address and deepen our analyses.

THEORETICAL BACKGROUND

The final proof text is a result of a process involving various components and is influenced by several factors. First of all, we will consider the relationship with the teaching and how proof texts are written by teachers; this will be explored using the construct of didactic contract (Brousseau, 1988) that concerns, in particular, the relationships between what teachers do and expect from the students, regarding knowledge that teachers have and intend to teach, and what students think that their teachers expect from them. Several teachers' expectations (concerning what students should do and learn) are not made explicit, or, when explicit, are not understood by students in the right way. On the contrary, the efficiency of a didactic relation assumes students having some intelligence about the intentions of the teacher (Mercier, 2010). In the case of proof, different teachers write different kinds of proofs at the blackboard according to their conceptions about what is relevant in a mathematical proof, and according to the needs of the moment. Some teachers write all the details, while others write only what seem to be important for them in that moment. Some teachers add special comments on proving, others write only a chain of symbolic statements and comment them orally. Students "by contract" might tend to provide proofs that are as close as possible to their teachers' ones.

At a meta-mathematical level, most students do not know what doing mathematics is, which means knowledge about mathematics as a science. By teaching only its contents, several teachers (especially school teachers) assume that over time, students will be able one day to acquire knowledge about that science and find out how it works. But it has been shown (see Morselli, 2007) that some university students' difficulties are still caused by lack of meta-mathematical knowledge, particularly as concerns proving, like: how to exploit a known theorem while proving another theorem; what is the difference between a definition and a theorem; what is a counter-example; thus, they ignore some important rules of the game. Moreover, as pointed out by Morselli (2007), frequently students confuse exploratory argumentation and proof; they produce written arguments that are different from a proof by writing their "proofs" when they are engaged in the exploratory phase. The problem is that teachers do not know that students ignore what exactly a proof and a complete proving process are, and thus they might tend to stick to models derived from their teachers' presentations of proof-models, which may be different from one teacher to another. We say briefly that, concerning proof, several students lack knowledge about it (which we consider as meta-knowledge about proof); and we hypothesize that this fact may result in a stronger influence of the didactic contract, which works as the only factor orienting their behaviour.

As concerns the writing of proof, it is also close to language and expression issues. The nature of mathematical language, with attention paid to the case of proof, was described in (Boero, Douek, & Ferrari, 2008), in particular as concerns the specific use of current verbal expressions in mathematics (with change of meanings, in comparison with ordinary language), and the integration of mathematical symbols within a verbal written text. In that paper, the authors also presented some results concerning university students engaged in problem-solving in an Algebra course; they found out that almost all those who, in an entrance test demanding verbal explanations, had produced answers with no verbal comment, or with rambling words and poorly organized sentences, failed the final exam. That paper is a general reference for our work: it offers a frame which provides a definition of mathematical language (pp. 265–266), a rich set of reflections and a wide perspective to deal with students' difficulties concerning mathematical language.

Finally, we'll also consider the logical structure of students' proof texts and how they link their written propositions together. For this, we'll make reference to Durand Guerrier's research work on the logical aspect of proving. In particular, students' use of '*then*' should be considered: it is currently used in mathematics to introduce a conclusion, to end a proof or set the desired result, and within the "*if…then…*" construct. Students may misuse it or not use it when they should, which makes a proof sound strange as there is no connection between the different propositions. In general students need to validate their different steps to carry on the proof process; if they are not able to do so because of difficulties related to logical symbols and connectives (cf. Boero et al., 2008; Durand Guerrier et al., 2012), their proof are likely to be vague and unclear, or mistaken.

METHODOLOGY

To study proof text, it's important to analyze students' proofs written individually. We have designed a test with open questions (thus not in the form: "prove that"), whose answers should not result from a standard procedure involving a mathematical result or a theorem, but rather engage students in a creative process. The test, administered to 98 students, proposes three simple short questions about a definition, a property or a result well known by the students, to be answered during thirty minutes. The degree of difficulty (complexity, connections to be established between known properties) of the questions was lower than in the case of some questions already tackled by the students during the exercises sessions. However, the questions were asked in a way unfamiliar to them, which put the students in a new situation that engaged them in gathering and linking their information and organizing them in order to build up a proof clearly written. This choice was aimed at identifying specific reasons influencing writing, and resulting in confused, unreadable proof texts.

The construct of didactic contract will be used to identify the relationships between students' productions, and how proofs were usually presented to the students by the two teachers (I'm one of them) during the course and the exercises sessions. Teaching the course consisted in two lecture sessions and one exercises session each week, all during a period of twelve weeks. There was no special focus on proofs even though all results are proved in lectures. The focus was rather on contents, statements of theorems, results and definitions and how students understand and use them; most of the exercises were about calculations, only some ones were about reasoning and links about concepts; time was not enough to develop in details every concept. There was no check of students' answers to questions (additional or not completely answered questions) posed to them, or of proofs written by them during the exercise sessions. Most of what students wrote was copying down what was already written by the teacher at the blackboard. The analysis will also consider students' oral expression ability during the academic year and how they managed to talk mathematically: as I know the students, I can tell about that for the students whose productions are analyzed.

The study will be focused on the organisation of the students' steps of reasoning in their proof texts; in order to better classify them, we will distinguish between three phases of proof production (Arzarello, 2006):

Step1: exploration and production of reasons for validating the statement.

Step2: organisation of reasoning into a cogent argumentation.

Step3: production of a standard deductive text.

Moreover, students' productions will be examined from a logical aspect, by exploring how they link the different steps and whether and how they deduce any written statement from the previous one, and what are the means used for that: are they, in particular, logical connectives, or symbols like the arrow implication, or other transition words like 'then', 'thus' or 'finally'?

A-PRIORI ANALYSIS OF THE FOURTH TEST

The questions of the fourth test were of open type and needed to be answered by producing the answer and building up a proof to validate it in an autonomous way (as concerns the whole process). The three questions were as follows:

1) *Is it possible to find a holomorphic function that admits 0 as a simple pole such that Residue* of f at *0* is *0 (Res (f, 0) =0)?*

2) *May the residue of a holomorphic function at the infinity be zero? Justify.*

3) *Why is the residue of a removable singularity zero?*

Like in the other tests, questions were related to properties and results that should have been well known by the students (and indeed other tests demonstrated that it was so). We aimed at ascertaining if the students were able to write down the argument, based on known definitions and theorems, in a well organised, clear mathematical form. Even if this kind of open questions was familiar for students in an oral form, it was quite new for them in a written form: they had never seen the written answers to such kind of question; they had never been engaged in producing written answers to them. Thus questions could have revealed the students' competence of *autono-*

mous proof writing (proof writing not induced by a stereotyped request) and the *consciousness* about its functions (related to the very nature of proof: shortly, as an unchaining of propositions aimed at validating a statement). The most interesting data have been collected through the first question.

The question is about some results concerning simple poles and residues (P ⇒ Q) and its negation (P and $\overline{Q}$) at the same time, which results in a contradiction: if a holomorphic function has a point 0 as a simple pole (P), it means that the residue of the function at this point (which is the coefficient of 1/*z* in its Laurent development) cannot be zero (Q); the reason is that, in the case at stake, the residue is calculated by the formula lim *zf(z)* when *z* → 0, which is exactly the coefficient of *1/z* already taken not zero. Logically speaking: the fact that the residue at a point is not 0 is a direct consequence of that point being a simple pole for the function. We have chosen to refer to the point 0 to simplify the formula. There was no doubt that students knew all these concepts because they had used them many times before, but always when performing calculations (to calculate the residue for a given point, to establish whether given points were simple poles, multiple poles or other singularities). However no request of identifying and exploring the links between concepts had been made especially in a written way.

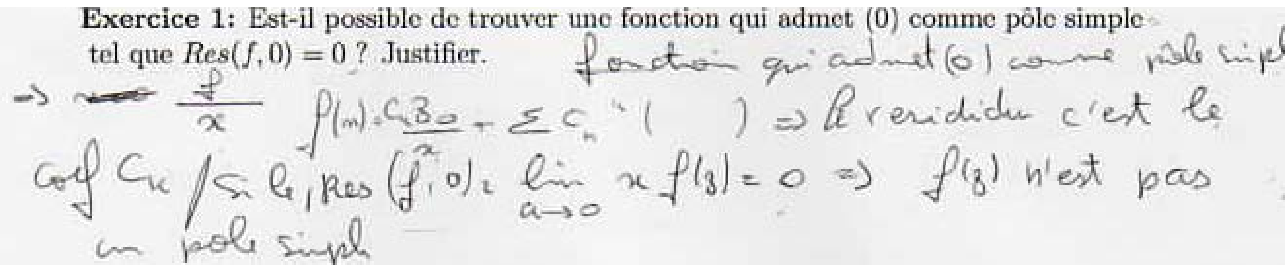
ANALYSIS OF SOME STUDENTS' PRODUCTIONS

We'll present and analyze two examples of students' productions (French is the ordinary used language) that consist of a disorganised, unreadable proof text. This will exemplify the work done on the collected productions, and how we got the conclusions reported in the next sections.

Student 1:

The student starts by rewriting the hypothesis of the question: '*a function admitting 0 as a simple pole*'. She continues at the beginning of the line by putting an implication arrow, followed by crossing out (*x=0);* then she writes *f/x*, adding the Laurent series of the function - not complete, with some blanks in it. Another implication arrow follows by stating the definition of the residue (that is the C_k coefficient). The student goes on by marking a slash, 'if the' and a comma; then she calculates the formula of the residue of *f* at *0* which is, according to the student, the limit when some non declared '*a*' goes to *0* of *xf(z)*, which is zero. With a last implication arrow, the student ends her proof by stating that *f(z)* is not a simple pole.

The proof in this production is not totally wrong but is not written correctly and not complete. This doesn't tell clearly if the student's reasoning is correct or is a result of a mere coincidence. The first thing to notice is that two variables are used: x and z. According to the student a function that admits *0* as a simple pole has a denominator with x, which is correct. She certainly meant to write f(x)=g(x)/x, but the way she expresses it is so odd, she writes *f/x*; she wanted probably to say that the principal part of the Laurent development contains only *1/x*. This shows how the student struggles to translate her intuitive vague idea about the formula into a mathematical expression with mathematical symbols. The writing of the student is confusing; she gives flashes of what seems important to her without taking into account how the proof should be presented to the reader and how the different steps and formulas should be linked, which indicates poor meta-knowledge about proof. Many variables like z_0, c_k, c_n and '*a*' are used without defining and specifying their meanings; we are not sure that their meanings are clear for her. At the end, she deduces, by an implication arrow, that *f(z)* cannot be a simple pole, and certainly she meant the point *0*. She writes down briefly how she remembers the related definitions and shows the main ingredients of the proof (one denominator with z, one coefficient is the residue and a non zero limit is for a simple pole).


Exercice 1: Est-il possible de trouver une fonction qui admet (0) comme pôle simple tel que $Res(f,0) = 0$? Justifier.

fonction qui admet (0) comme pôle simple

→ $\frac{f}{x}$ f(x) = C z₀ + Σ c_n ( ) ⇒ le residu c'est le coef C_k / si le, Res(f, 0) = $\lim_{a \to 0}$ x f(z) = 0 ⇒ f(z) n'est pas un pôle simple

Figure 1: Student 1

She works within the step 1, the exploratory phase and production of the reasons for validating the result.

We can see how the student' effort, influenced by the didactic contract, is aimed at showing to the teacher that she got the most important ideas related to the question, no matter if several details are mistaken or neglected. The text also allows to guess her own proof conception, which seems to be shaped by doing it for the teacher, who is supposed to understand the missing details and the unsaid words, according to the student. The logical links are not clear, particularly the implication used is not the logical one: it is more about shifting to another idea not necessarily and clearly connected to the previous one; moreover the student doesn't tell about her proof technique, which seems to be a proof by contradiction. From a language point of view, the student is one of those who showed real difficulty orally when expressing any question or a comment involving a mathematical idea. Her natural spoken language is confusing and poor, and lacks flexibility to express mathematical meanings (Boero et al., 2008).

Student 2

In this production, it's hard, first, to decipher the handwriting of the student and to follow the lines that are broken, so it's not clear where a line starts and where it continues and ends. He starts by explaining what means that a function admits *0* as a simple pole by putting one arrow, which is meant to be the implication arrow, oriented towards $f(z)$ expressed as $\varphi(z)/z$, φ being holomorphic. Then, the reader is puzzled where to continue reading, is it on the same line or beneath? At the same line, the student calculates the residue value by the limit formula, but it's indicated with z going to ∞ instead to *0*, even though the result is with z going to 0 which is $\varphi(0)$; this last result is declared to be different from *0,* without any justification, as an obvious fact. At the next line, he states that $\varphi(z)$ is $c_0 z$ with a hesitation on the power of z; at the next line, he adds that $f(z)$ is c_0/x plus a sum of c_n and something not clear; at the end, he concludes by writing 'then' *it is not pole*, without explicitly indicating what is '*it*'. With an unreadable handwriting presenting the proof like a sketch, the organisation fails to clarify what the student is exactly doing. However, after reading this text many times, the first part appears to be not totally incorrect. The student gives the definition of 0 being a simple pole and calculates the residue of the function, which is not 0, a valid step of reasoning, but this would need a conclusion: that is a contradiction to the question. The student gives the main idea: the formula of the function f is given by a holomorphic function ($\varphi(z)$) with z at the denominator in the case of having 0 as a simple pole, and then calculate the first element of the Laurent development of f, which should correspond to c_0, but fails to justify and make explicit the other details taken as obvious, like: why $c_0 \neq 0$, which is an important point in the proof. In doing so, the student is about showing to the teacher (or doing like the teacher) that he got the main idea of the proof, which seems more important than clarifying explicitly the "small details" which are clear for him. Leaving some blanks in the proving process makes inevitably the statements logically disconnected, especially because the student doesn't use logical connectors (except one 'then' at the end) to link a statement to another. I remember this student as one of the brilliant ones, but he is weak both orally and at writing. When he talks, no one understands him: he swallows his words, bubbles and repeats the same expression to say different meanings. On a cognitive level, he is very smart and most of the times he finds the good idea where all other students are stuck; it is rare that his answers are wrong, which makes his proof's idea possibly correct. For him, writing a proof text is writing a set of partial arguments presented in a disorganised way with incomplete formulas, far from how a proof should be.

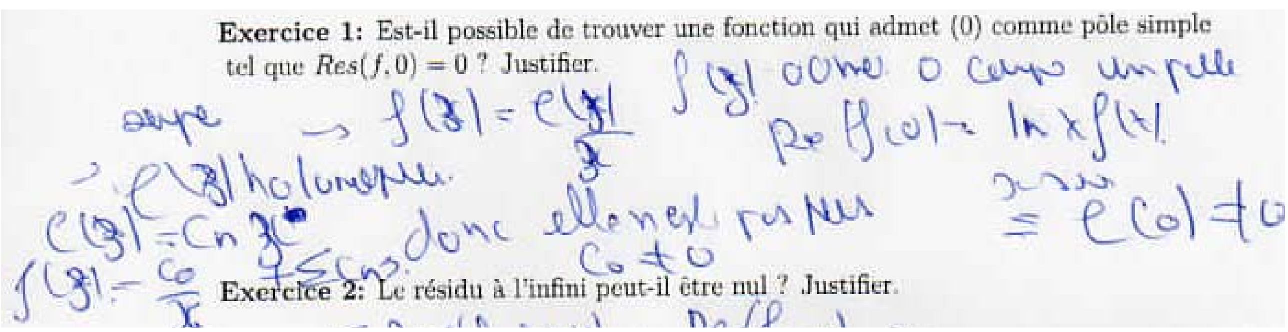


Figure 2: Student 2

RESULTS

Analysis indicates that students' difficulties concerning writing an organised and understandable proof text are originated in the didactic contract that substitutes the mathematically relevant aim of structuring and writing a clear proof text with the aim of imitating teacher' incomplete proofs. This results in a disorganised, poor text somehow similar to some sketches of proofs produced by teachers at the blackboard. Moreover, writing a proof is strongly influenced by students' engagement in showing to the teacher what they know (the relevant idea to achieve the proof), underestimating the importance of writing a complete, well organised proof text. The previous texts show that students fail to go through the three steps of the proving process; they only develop the first exploratory phase and write it down by presenting the main ideas and missing many details. Failing to engage in the second phase and reach the third phase of writing the proof text, logical links between statements remain unclear (student 1) and even missing (student 2); with misused logical connectors (the implication arrow for student 1 and 'then' for student 2), the logical structure of the proof text is totally disconnected. Such students' behaviors might be contrasted by a good meta-knowledge about proof, but as their meta-knowledge about proof is poor, what is suggested by the didactic contract becomes dominant. The poor proof texts show difficulties inherent in the mastery of natural language (for both students) in the mathematical register (Boero et al., 2008), particularly as concerns its logical features, which contributes to students' difficulties of logically connecting the steps of reasoning in an explicit and appropriate way, and to their using incomplete definitions and formulas (student 1 and 2). In some cases, disorganised proof texts have been produced by students with difficulties in mastering of natural language: students 1 and 2 are examples of them. Thus a flexible mastery of natural language (which cannot be achieved by means of every day-life experience alone, and requires specific interventions also concerning scientific communication) appears to be a necessary condition for mathematical proficiency (in agreement with (Boero et al., 2008), even for a course at the undergraduate level). The above presentation of the results of the analysis of students' productions suggests that the four components used to interpret the poor quality of the proof texts are not independent, but rather overlapping and intertwining.

DISCUSSION

Our hypothesis on difficulties of writing proofs to open tasks that require a creative construction of a proving process and writing clearly the logic structure of how links are created between different arguments and concepts were set on: lack of flexible mastery of natural language; lack of knowledge about proof at a meta-level; influence of the didactic contract; and weak consideration of logical proof structure. This led to the question of identifying the possible reasons for these results and suggesting some possible didactical implications to cope with these problems.

According to the didactic contract, we may hypothesize some links between the identified difficulties and our teaching of mathematics at the undergraduate level, which does not introduce learning of proofs and less writing a proof text. Moreover, during the grading of exams copies, we base our positive evaluation on correct ideas in students' proofs and give partial credit, even if they are not clearly and rigorously presented. As far as I'm concerned as a teacher, it happens that I write some of the proofs at the blackboard by giving the plan and the main idea, and devote the short available time to the full explanation of the concepts to be used, and the links between the ideas of the proof. I sometimes leave the final text to the initiative of the students, who feel satisfied by getting the main idea and the main details. Both students and I (as a teacher) have always held the assumption that when the main idea of the proof is clearly explained and understood, writing the proof details and organizing the proof text is a simple thing! However, the fourth test provides strong evidence that this is wrong. Most of the meta-mathematical knowledge about proof is absent in our teaching, in particular as concerns the relationships between the proving process and the proof text. The lack of this knowledge might explain why students write directly their final text at the same time when they explore the question: they have no mean to interpret and situate what their teachers do when they write a proof at the blackboard. We (teachers) generally write proofs in a direct linear way; knowing already the steps of the proof, we write them one after another, till the conclusion. We do not show the exploratory phase and how the partial arguments produced in that phase are re-considered and arranged to produce the final text (similar situation with mathematics textbooks). Students learn to do the same: when they first set some ideas about how to solve a problem, they write their first exploratory draft as a final text because they were never shown how to go further till the written proof text. Another component of the meta-knowledge about proof that would have helped to cope with the problem of writing the proof text is about the difference between argumentation and proof, which is not clear for the students especially from a structural perspective. The logical structure of an argumentation differs from the logical structure of a proof: while in a proof all steps are deductive, in an argumentation the steps may be of different nature: abductive steps or inductive steps (Peirce, 1960; Polya, 1962 cited in Pedemonte, 2007). In this case, the construction of a deductive proof requires a structural change: from abductive or inductive to deductive steps. This change is not always straightforward for students, but is usually necessary (Pedemonte, 2007). The results support the intrinsic cognitive difference between the open and the closed tasks, which implies different roles of verbal language. With closed tasks, proof text plays a narrative ritual function, while in the open ones, the text plays, in the first phase, a constructive creative function. With proofs for closed tasks, students go through a secure process, behaving in a conventional ritual way and showing a coherent reasoning guided by the didactic contract, according to teachers' and textbooks' models. The result is a clear, well organisation proof text (regardless the correctness of the proof). The use and control of meta-mathematics knowledge is needless in this case. On the contrary, in proofs for open tasks, a creative organisation of the arguments is needed, but lack of meta-mathematics knowledge about proof and some uncompleted proof text models offered by teachers lead students to write their proofs when reasoning (which is a nonlinear process). Failing to go through the two following phases, the text would look more like a draft. Furthermore, an in-depth analysis of students' mastery of their native language and of written French is needed to complete our study.

What instructional interventions can be effective in overcoming these difficulties? Teaching meta-mathematical knowledge and especially meta-knowledge about proof seems to be extremely important and effective for introducing the learning of proof. It should regard the differences between deductive, abductive and inductive reasoning, the logical structure of proof, and how to organize proof text. Introducing some open tasks, among the activities, would break the ritual pattern of proofs for closed tasks and encourage creativity. Explaining proofs and writing them directly fails to show students how really they are constructed in different phases; while some writing proof exercises proposed to students with hints and main idea of proof would enhance writing proof text.

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On a generality framework for proving tasks

Andreas Bergwall

Örebro University, Örebro and Mälardalen University, Västerås, Sweden, andreas.bergwall@oru.se

In this paper I present an analytic framework for generality in textbook proving tasks that involve functions. The framework is discussed in relation to results obtained when analysing tasks in integral calculus. The results show that the frameworks' categories are easily distinguishable if the functions are explicitly described. The results are also promising regarding the possibility to clarify differences between textbooks. The analysed sections exemplify that there is not necessarily a correlation between the number of general proving tasks and the opportunities for students to engage in reasoning about arbitrary functions. Limitations and possible refinements of the framework are also discussed.

Keywords: Mathematical proof, mathematics textbook, upper secondary school, undergraduate mathematics, integral calculus.

INTRODUCTION

Research on the teaching and learning of proof often involves the distinction between specific and general arguments and properties. For instance, students' tendency to take specific cases as sufficient justification of general properties is well documented (Harel & Sowder, 2007). The distinction between specific and general has also been used to study how mathematics textbooks support proof-related activities and learning (e.g., Stylianides, 2008; Thompson, Senk, & Johnson, 2012).

In an ongoing study, we use the analytical framework of Thompson and colleagues (2012) to investigate Swedish and Finnish upper secondary textbooks. One part of the analysis consists of determining whether or not textbook tasks provide opportunities for general reasoning. When tasks involve functions, this distinction is not always obvious. Various combinations of dependent variables, independent variables and other parameters mean significant differences in the 'degree of generality' between tasks. This suggests that such a textbook analysis would benefit from a more fine-grained classification of generality. In this paper I will address this issue by discussing a tentative generality framework for proving tasks based on the 'size' of the set of functions that the tasks call for a proof about. I will refer to this as the *function generality framework*. By 'proving task', I mean a textbook exercise explicitly asking the student to prove or show a mathematical property.

The function generality framework is an answer to the first of three questions stated below, around which this paper is focused. By applying it to proving tasks in Swedish and Finnish textbooks, some results relating to the other two questions will be obtained. The questions are: (1) How can 'degree of generality' in proving tasks involving functions be framed? (2) What analytical difficulties arise when proving tasks are classified according to function generality? (3) What can classification according to function generality reveal about textbooks that a 'specific-or-general' classification cannot?

Some initial results concerning the analysed textbooks will also be discussed.

BACKGROUND

One characteristic feature of a mathematical proof is that it usually provides a valid justification for a *general* property. However, numerous studies (many of which are referred to in Harel and Sowder (2007)) show that students on most educational levels, even at university (e.g., Hemmi, 2008; Weber, 2001), have limited understanding of this aspect of proof. Typically, students justify general statements with specific examples, view counter-examples as exceptions, believe that counter-examples might exist even if there is a general proof etc. In the literature, this has been referred to as *empirical response* (Bell, 1976), *pragmatic justification* (Balacheff, 1988) and *empirical proof scheme* (Harel & Sowder, 1998). Central to all these frameworks is some kind of distinction between the general and the specific.

Even though no curriculum program is self-enacting, research has stressed the wide use of textbooks in classrooms and how they are crucial links between national curricula and teaching practice (e.g., Stein, Remillard, & Smith, 2007). In line with this research, mathematics textbooks can be seen as potential sources for opportunities to learn. Hence, textbooks' treatment of reasoning and proving is an important object of study. Historically, such studies are rare (Hanna & de Bruyn, 1999), but in the past decade a number of studies with this focus have been published (Nordström & Löfwall, 2005; Stylianides, 2008; Thompson et al., 2012). In an analysis of an American reform-based curriculum for middle school, it was found that 40% of the textbook tasks were designed to engage students in reasoning and proving, but only 12% of these offered opportunities to provide general proofs (Stylianides, 2008). Thompson and colleagues (2012) report on an extensive study of US textbooks for upper secondary school, concerning opportunities offered for students to engage in proof-related reasoning within the topics of exponents, logarithms and polynomials. Their study showed that about 50% of the properties stated in narratives were given some kind of justification; 30% with a general argument (i.e. a proof) and 20% with a specific case. About 5% of all exercises were considered proof-related, half of them of a general kind and half a specific kind. Approximately 1% of all exercises urged the student to develop a general argument.

The frameworks used by Stylianides (2008) and Thompson and colleagues (2012) are similar to (or inspired by) those used by Bell (1976), Balacheff (1988) and Harel and Sowder (1998). Hence, they also distinguish between specific and general aspects of textbook content. In an ongoing study, we have used the framework of Thompson and colleagues (2012) to analyse Swedish and Finnish textbooks. Traditionally, deductive reasoning has primarily been studied in geometry courses, but more recently it has been suggested that reasoning and proof are important in all content areas (e.g. NCTM, 2000). Therefore, Thompson and colleagues (2012) choose to focus on algebraic topics instead of geometry. For the same reason, and to further complement their study, we have focused on calculus. In this broader study we analyse all parts of textbooks: expository sections, worked examples, exercise sets, review exercises etc. It is during this work that I have encountered differences in generality that I have found difficult to capture with the earlier frameworks, and which I therefore address in this paper.

When functions are involved in mathematical tasks, there are dependent as well as independent variables. A task like "Prove that $Dx^2 = 2x$" is *general* in the sense that the student is asked to prove something for all x, but *specific* in the sense that it only concerns one particular function. The inclusion of more or less arbitrary functions, parameter families of functions and other parameters also means (potential) differences in the 'degree of generality'. Tasks like "Prove that $De^{kx} = ke^{kx}$" and "Prove that $Df(kx) = kf'(x)$" are both general in the sense that the identities hold for all x and all k, but the second one is obviously more general than the first since it also holds for any differentiable function f. This difference in generality also implies different content focus; while the first focuses on properties of a certain function, the second focuses on a fundamental property of differentiation itself. These examples show a need for a more fine-grained framing of generality based on properties of the functions involved in the tasks.

I have also found several tasks formulated as "Show that ..." but that were not general in any sense and only required a routine calculation. Sometimes it was the other way around: theoretically and cognitively demanding tasks that from a mathematical point of view concerned proving but were formulated in words like "Why is it ..." or "Motivate why ...". This relates to findings regarding proofs being "invisible" in textbooks (Nordström & Löfwall, 2005). While other studies (e.g., Stylianides, 2008; Thompson et al., 2012) look for opportunities to engage in reasoning and proving in a broad sense, it is therefore important to also look specifically at proving tasks.

METHODOLOGY

Topic, context and textbooks

For the pilot study reported here, I have restricted the analysis to proving tasks in integral calculus. Like differential calculus, this topic is central in upper secondary school as well as in introductory courses at universities. However, the theory of integrals is more complicated and proofs are often omitted. There is a tendency that the underlying theory is not treated in detail in introductory courses at universities but rather postponed to intermediate courses, due to students' difficulties with a theoretical approach (Hemmi, 2006). Thus, it is a real challenge for upper secondary textbook authors to incorporate elements of reasoning and proving within this topic, and it is of research interest to study how this is done.

To get a reasonably rich set of data, four different textbook sources were chosen (see Table 1 below): two Swedish upper secondary textbook series (referred to as SW1 & SW2), one Swedish undergraduate textbook (SWU), and one Finnish textbook for upper secondary school (FI1). Publishers are unwilling to reveal their market share, but it is well-known that SW1 and its predecessors have long dominated the Swedish market. In 2011, more than 80% of those entering engineering programs at Örebro University reported having used these textbooks. The main reason for choosing SW2 was that, while SW1 is a traditional Swedish textbook, SW2 is a newer one with more reform-oriented intentions and a stated focus on reasoning. FI1 is the only Finnish textbook series available in Swedish for use in the Swedish-speaking parts of Finland. Its Finnish original is probably the most widely used textbook in Finland. Finally, for the sake of curiosity, and to get an indication of the usefulness of the function generality framework on introductory calculus texts for the university level, SWU was included; this is a Swedish single-variable calculus textbook that has been around for several decades.

Swedish upper secondary school is course-based. There are five mathematics courses, of which the first four are often a prerequisite for university studies in science and technology. Integral calculus is treated in Courses 3 and 4. The first three courses exist in different versions, depending on whether they are part of a vocational program (Track a), a program in the social sciences (Track b) or a program in science and technology (Track c). For this study, only textbooks for Track c were chosen. In Finland there is a short mathematics course serving as preparation for university studies in, for example, the humanities, and a long course serving as preparation for university education involving higher mathematics. The long course is divided into 13 parts, the first ten of which are mandatory. Part 10 is devoted to integral calculus only (but this topic is further developed in Parts 12 and 13). This study only includes Part 10.

Method

First, all textbook sections specifically dealing with integral calculus were identified. All exercises in these sections were included in the analysis, as were the review exercises on integral calculus (which were typically placed at the end of the book). The only exception was exercises on continuous distributions, which were all omitted since only one of the textbooks treated probability theory within the sections on integrals. Concerning the unit of analysis, whenever an exercise was divided into an enumerated list of subtasks, each subtask was regarded as one task. This resulted in a total number of 1,739 textbook tasks to be analysed (see Table 1). Since the function generality framework is meant to be a tool for analysing the opportunities offered to students to associate 'proving' with general justifications, I next looked for tasks explicitly asking the student to 'show' or 'prove' something. Such tasks are referred to as *proving tasks*. In total there were 80 proving tasks, all of which could be interpreted as concerning functions.

In SW1, SW2 and SWU all proving tasks were formulated as "Show that ..."; i.e., the word 'prove' was

	Publisher	Series	Book	Total no. of	
				tasks	proving tasks
SW1	Liber	Matematik 5000	3c & 4	371	21 (6%)
SW2	Sanoma utbildning	Origo	3c & 4	450	13 (3%)
FI1	Schildts	Ellips	10	529	30 (6%)
SWU	Matematik-centrum, Lund	Analys i en variabel	Exercises	379	16 (4%)
				1,739	80 (5%)

Table 1: Textbooks, tasks and proving tasks within sections on integral calculus

never used. In FI1 'prove' was used as often as 'show'. On three occasions, all in SW1, 'show' was used in a non-mathematical way, as in "Show in detail how you calculate the integral ..." (SW1, Book 3c, p. 185, ex. 3412a). I chose to include them in the analysis since they play a role in forming what students will associate with the word 'show'.

For every proving task, a detailed account was made of the function(s) it concerned. This included information on whether the task concerned specific functions, parameter families of functions (including the number of parameters) or more general non-parametric classes of functions. Notes were taken about the kind of elementary functions involved (polynomial, trigonometric function, exponential etc.) or, in the more general cases, what classes of functions were involved (continuous, periodic, odd etc.). It was also noted if a task contained additional parameters (not connected to the functions), or if it could be seen as general in some other sense.

Analytical framework

One way to determine whether a proving task offers opportunities for general reasoning is to determine whether the property to prove itself is general or specific. Therefore, all proving tasks were categorized as *case-specific* or *case-general* following the framework of Thompson and colleagues (2012). As mentioned earlier, the distinction between specific and general is sometimes difficult to make for tasks involving functions. The general principle used in this paper is that if one can think of a more specific case than what is stated in the task, without substituting the independent variable with a specific number, then the task is considered case-general. This means that the presence of an independent variable is not enough for a property concerning functions to be deemed case-general – either there need to be other parameters involved, or the property must concern a class of functions. Thus, for example, a proving task about e^{2x} is usually considered case-specific (unless other parameters are involved), whereas a proving task about a^x or e^{kx} is considered case-general (see the *Results* section for further examples). I believe this is in line with how Thompson and colleagues (2012) would have distinguished between specific and general cases. To further clarify this notion, consider the following properties, which were all found (explicitly or implicitly) among the analysed proving tasks:

a) $\int_1^4 \sqrt{x}dx = \frac{14}{3}$
(SW1, Book 3c, p. 197, ex. 12a)

b) $\int_1^a \frac{1}{x^2}dx$ never exceeds π
(SW2, Book 4, p. 158, ex. 4371c)

c) $F(x) = \frac{a^x}{\ln a}$ is a primitive to $f(x) = a^x$
(SW2, Book 3c, p. 159 ex. 5124)

d) $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$ if $f(x)$ is even
(FI1, p. 156, ex. F1)

Here (a) is case-specific, while (b)–(d) are all case-general due to the parameter . But, like the examples given in the *Background* section, there are differences in the 'degree of generality' between (b), (c) and (d). While (b) concerns one specific function, (c) holds for a one-parameter family of functions and (d) holds for all even functions, a class too large to be represented by use of a finite number of parameters. In order to capture these differences, I introduce the *function generality framework* with three subcategories: statements about a finite number of specific functions, like (b), will be called *non-general*; statements about parameter families of functions, like (c), will be called *finitely general* and measured by the number of parameters, as long as the number of parameters is finite; statements about more general sets of functions, like (d), which are too large to be represented by use of a finite number of parameters, will be called *infinitely general*.

From the student perspective, the difference between non-general and finitely general proving tasks is that in the latter case the student needs to distinguish the independent variable from other variables, and to be able to handle parameters when manipulating function expressions. But in both cases there are expressions available for algebraic manipulation. In the case of infinite generality, though, the student needs to find suitable ways to represent and use the relevant property (like the property 'being even' in (d)). Thus, I believe the three categories of function generality to be of educational relevance, even though it is sometimes easier to prove an infinitely general statement than a non-general one.

The classification of tasks according to function generality can be done independently of the classification of tasks as case-specific or case-general. But since all case-specific tasks will be non-general, nothing is gained by applying the function generality framework to case-specific tasks. I therefore only apply this framework to case-general tasks; i.e. I see function generality as a way to divide case-general tasks into subcategories.

During the analysis I soon discovered that proving tasks often express a relation between different classes of functions. In such cases, the task was classified according to the 'largest' of these classes. Examples are given in the *Results* section.

In our broader study, mentioned earlier, a second Finnish textbook series is included and parts of the analysis have been done independently by two researchers. During this work we have discussed and compared our coding and resolved all differences.

Finally, even though Examples (a)–(d) are from integral calculus, the ambition is for the framing of generality described here to be applicable to any topic involving functions. Aspects of generality that might be unique to integral calculus will be touched upon in the discussion.

RESULTS

In this section I will present a representative sample of the analysed proving tasks belonging to the different framework categories, as well as tasks that highlight the strengths and weaknesses of the framework as an analytical tool. A summary of the number of proving tasks, by textbook series and generality, is shown in Table 2. For example, SW2 had 13 proving tasks: three case-specific and ten case-general. When these ten were analysed according to function generality, three were found to be non-general, six finitely general, and one infinitely general.

In SW1 and SW2 (but not in FI1 or SWU) I found proving tasks that were case-general, even though they were non-general in the sense that they only concerned specific functions. This was always due to additional parameters, typically as limits of integration, as in Example (b) in the framework section.

Proving tasks of finite generality mostly concerned one- or two-parameter families of functions. SW2 also contained two tasks with three-parameter families. In FI1, six out of 13 finitely general tasks had the constant of integration as its only parameter, as in "Prove the integration formula $\int \sin x\, dx = -\cos x + C$" (FI1, p. 29, ex. 259a).

In SW1 there was no proving task of infinite generality, while in SW2 there was one: "Show that if $f(x)$ is continuous in $a \le x \le b$ then $\int_a^b f(x)dx = -\int_b^a f(x)dx$" (SW2, Book 4, p. 146, ex. 4340). About a third of the proving tasks in FI1 were infinitely general. Half of these were similar to "Prove the integration formula $\int f'(x)e^{f(x)}\, dx = e^{f(x)} + C$" (FI1, p. 35, ex. 271b); i.e., they were essentially related to the chain rule.

During the classification according to function generality, only three tasks proved somewhat difficult to categorize, all of them in FI1. The first reads as follows: "Show that all primitive functions to $g(x) = x^2(5 + 4x^3)^2$ are strictly increasing" (FI1, p. 25, ex. 250). In this task only one specific function is explicitly given, but the statement concerns the one-parametric class of its primitive functions. Therefore, I classified this task as finitely general. However, the proof need not take into account any parameters, since it mainly rests on the fact that $g(x) > 0$.

A similar difficulty concerns ex. 460, p. 112 in FI1, where the student is asked to prove a general formula for the area bounded by a parabola and a straight line. No explicit formulas are given for the two curves. However, since lines and parabolas are graphs of first- and second-degree polynomials, i.e. two- and three-parameter families of functions, I classified this task as finitely general.

	Proving tasks	Case-specific	Case-general			
			Total	Non-general	Finitely general	Infinitely general
SW1	21	15	6	1	5	0
SW2	13	3	10	3	6	1
FI1	30	8	22	0	13	9
SWU	16	13	3	0	1	2

Table 2: Numbers of proving tasks of different case and function generality

The third example connects to the uniqueness of primitive functions, but is actually a result of differential calculus, often referred to as *Rolle's theorem*: "If $f'(x) = 0$ everywhere then f is constant" (FI1, p. 156, ex. F3). The conclusion of this theorem means that the class of functions the theorem concerns is one-parametric. But in the proof, f must be handled as an *arbitrary* function with the property $f'(x) = 0$. I therefore considered this proving task to be infinitely general.

DISCUSSION

In the introduction I posed three questions. The first was how the 'degree of generality' in proving tasks involving functions could be framed. To answer this question, I have described a framework based on the 'size' of the class of functions under consideration. Applying this to proving tasks in integral calculus in four sets of textbooks has made it clear that proving tasks of all three kinds (non-general, finitely general and infinitely general) exist, and that the classification is straightforward as long as the proving tasks are explicit regarding which functions they concern. This indicates that for a generality analysis of upper secondary textbook proving tasks, the categories of the function generality framework are relevant and well-defined.

The second question concerned analytical difficulties. The three concluding examples in the *Results* section indicate that my framing of generality is less suitable when the functions under consideration are not explicitly given. In such tasks, the first step in providing a proof is often to find a suitable representation of the functions involved. One might therefore expect students to experience them as more general than our classification shows. This dimension of generality is not captured by my framework. The third example shows the difficulty in measuring generality in terms of the size of the class of functions when the statement itself is about this size. It is reasonable to believe that such difficulties arise more often when theoretically oriented textbooks are analysed, regardless of whether or not the topic is integral calculus.

The third question concerned the usefulness of the framework. Let us first look at the differences between the textbooks, shown in Table 2. A larger part of the proving tasks are case-general in SW2 than in SW1, and the same holds if we focus on function generality. But if we compare SW2 and FI1, the function generality framework reveals differences that cannot be seen simply by checking case generality. The proportion of case-specific to case-general proving tasks is approximately the same (1:3) for these textbooks. But while a third of the case-general proving tasks in SW2 turn out to be non-general when it comes to function generality, FI1 has no such non-general tasks. In addition, nine out of 30 proving tasks in FI1 are infinitely general (concerns 'any' function), while there is only one such proving task in SW2. The fact that SW1 has no infinitely general proving tasks and SW2 has only one also means that they provide few (if any) opportunities to associate the imperative 'prove' with the providing of a general argument valid for 'any' function. Since proving tasks concerning specific functions or parameter families of functions turn the attention to features specific to these functions and not to properties of integration in itself, the absence of proving tasks of infinite generality also means fewer opportunities for reification (Sfard, 1991) of the integral concept. Such information about textbooks may be of importance to teachers in planning and choosing complementary materials so they will be able to offer students sufficient opportunities to learn the generality aspects of reasoning and proof.

Since the analysis presented here only includes sections on integral calculus, we cannot draw any general conclusions about the analysed textbooks. What is said above only applies to the exercise sets in the integral sections. But the point here is that the results show that the function generality framework has the potential to reveal textbook properties of educational importance that a categorization of proving tasks as case-specific or case-general cannot. It is reasonable to believe that this holds true for other mathematical topics as well. As a first step, we plan to widen the analysis to differential calculus and to include tasks that are proof-related in a broader sense.

There are of course other aspects of proving that are not captured with this framework, and situations in which this framing of generality may be misleading. One topic-specific aspect concerns the constant of integration. As mentioned in the *Results* section, half of the finitely general proving tasks in FI1 had this constant as their only parameter. In such tasks, this parameter is seldom an essential part of the proof. Hence, the number of finitely general proving tasks can be misleading without further analysis of the parameters of the tasks. Another aspect relates to findings in the university textbook SWU. I was surprised to find so few general proving tasks in this book. On the other hand, my impression was that while proving tasks in the upper secondary textbooks often required only routine calculation (direct use of standard formulas for differentiation and integration), the proving tasks in SWU were more non-routine. They often concerned inequalities, and the proofs required that functions be estimated. One way to put it is that authors of upper secondary textbooks seem to want to acquaint students with the word 'show' by using it when one could just as well have asked them to calculate. This tendency is not evident in the university text. The extent to which proving tasks actually require reasoning and not simply standard symbolic manipulation is not covered by my framework, but would certainly be an important element of textbook proving tasks to investigate further.

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Analyzing the transition to epsilon-delta Calculus: A case study

Paolo Boero

Genoa University, School of Social Sciences and Department of Mathematics, Genoa, Italy, boero@dima.unige.it

The purpose of this paper is to analyze and discuss a think-aloud production of a first year university student trying to prove an elementary theorem, concerning continuous functions, within the frame of the epsilon-delta Calculus. I will shortly consider current studies on the relationships between a visual justification, based on the graphical representation of a continuous function, and such epsilon-delta proving. Then I will try to show how a comprehensive theoretical framework (integrating different tools) based on Habermas' construct of rationality may account for the complexity of the whole process, for the difficulties met by the student in the transition from the visual-graphical to the epsilon-delta proving, and for the relevance of the visual-graphical reasoning to overcome them.

Keywords: Calculus proof, visual-graphical rationality, epsilon-delta rationality.

INTRODUCTION

The transition, in the first half of the nineteenth century, from Calculus based on the ideas of continuity and derivability of a function as continuity and smoothness of its graphical representation, to epsilon-delta Calculus (i.e. Calculus based on the epsilon-delta definitions of limit, and subsequently of continuity and derivative, usually also called the Cauchy-Weierstrass Calculus) has been dealt with by several authors in different disciplines (History of Mathematics, Epistemology of Mathematics, Psychology, Mathematics Education). In particular, the historical-epistemological analyses (see Grabiner, 1981; Jahnke, 2003) have shown the need of considering different reasons for that transition, like: the emergence of *monsters* and contradictions within the intuitive-visual treatment of continuous functions, when new types of functions not represented by ordinary formulas were considered; the nineteenth century movement towards formal set-theoretic rigour, particularly when proving was concerned (see also Tall and Katz, 2014); the need of dealing in a rigorous way with Calculus in many variables (where reference to visual – graphical evidence may be lost, as concerns the validation of statements); and also the increasing needs arising from the applications of Calculus (particularly stressed in Jahnke, 2003). As concerns the cognitive side, the general agreement about the fact that transition to epsilon-delta Calculus results in many difficulties for students does not correspond to an unique interpretation of those difficulties. For most past Authors difficulties mainly originate in the passage from an intuitive, visual conceptualization to a formal treatment of the same notions. More recently, researchers belonging to the embodied cognition stream of research (cf. Nunez, Edwards, & Matos, 1999) put into evidence the fact that the *natural* notion of continuity of a function and the epsilon-delta notion of continuity refer to very different grounding metaphors: a dynamic metaphor, for *natural continuity*; and three static metaphors (among which the *Preservation of closeness*), in the second case, which result in a hard to overcome conflict with the notions developed according to the dynamic metaphor underlying *natural continuity*. Nowadays, by integrating historical- epistemological studies on the origins and development of Calculus and cognitive analyses, Tall and Katz (2014) provide us with reasons for

> a re-evaluation of the relationship between the natural geometry and algebra of elementary calculus that continues to be used in applied mathematics, and the formal set theory of mathematical analysis that develops in pure mathematics and evolves into the logical development of non-standard analysis using infinitesimal concepts (p. 97).

During the nineties I had the opportunity, as teacher of an experimental course of calculus for first year university students in Mathematics, of collecting some interesting data (written texts, transcripts of oral think-aloud solving processes and notes taken by the observers concerning related gestures and attitudes). Students had been presented the intuitive, graphical notions of continuity and derivative for a real function defined on real numbers (or on an interval of real numbers), then the epsilon-delta definition of continuity at a point had been introduced. The derivative at a point had been introduced through the prolongation by continuity of the incremental ratio. Main theorems of calculus in one variable (in particular, the intermediate value theorem, IVT [1] in the following) had been presented with their epsilon-delta proofs; and some tasks demanding the epsilon-delta proof of easy properties by using those theorems had been proposed. Finally, the usual epsilon- delta notions of limit had been introduced (both in the case of finite and in the case of infinite limits) and applied to solve some exercises and to put into evidence how both continuity and derivative at a point were particular cases of the notion of limit.

Such an experimental approach somehow differed from the usual teaching of the epsilon-delta Calculus in Italy and in many other countries, based on the epsilon-delta definition of limit, because in our experiment the approach to the epsilon-delta reasoning referred to the intuitive notion of *regular* functions (smooth, continuous functions), according to the historical treatment of functions before the epsilon-delta revolution, and then moved to a formal description of *regularity* with the epsilon-delta language, and only at the end considered the epsilon-delta definition of limits.

Students were aware of the experimental character of that teaching of Calculus; they knew very well (thanks to elder students' experience) the difficulty of the subject matter, thus they were willing to engage in an alternative experience of teaching and learning, which possibly might have diminished the difficulties met in the ordinary learning of epsilon-delta calculus. Students volunteered in furnishing documents like private writings, informal drawings, oral reasoning in think-aloud situations.

I will not present the (modest) results of that teaching experiment, and I will not discuss the limitations and the heavy consequences on students of the usual teaching of Calculus focusing on the epsilon-delta systematization (cf. Nunez et al, 1999; Tall & Katz, 2014) – in the reality, the teaching experiment did not represent a radical alternative to it, and probably it is not possible to do more at the university level. I only wanted to introduce the document analyzed in this paper by describing its context of production, in order to open the reflection on it. It is a written and oral document produced by a brilliant student (we will call him Ivan), during a think-aloud problem solving session; it concerns the epsilon-delta validation of a rather elementary statement of calculus. The interest of the document depends on the very detailed presentation offered in it of the mental path followed by Ivan to get the proof and on the fact that the difficulties met (and overcome) by him are very frequent (and frequently not overcome!) among students when they want to validate a statement according to the epsilon-delta criteria.

The aim of the research reported in this paper is to build a suitable theoretical framework to account for what happened during Ivan's problem solving, and to put into evidence the role played there by the visual – graphical treatment of the problem and its importance in the students' approach to epsilon-delta proving.

THE DOCUMENT

Myself as the teacher of the course and a Master Degree student of mine took notes about what we observed, and then compared and completed those field notes, finally integrating them in the transcript of the student's speech. The document is faithfully reported here. (...) for pauses of at least 5"; *italic* for sentences written by Ivan.

> f: R→R is a continuous function;
> $\lim_{x\to+\infty} f(x) = +\infty$; $\lim_{x\to-\infty} f(x) = -\infty$.
>
> The problem:
>
> Prove that there is at least one point c such that $f(c)=0$

Ivan reads the text and after a few seconds draws the x and y axes, and two arrows, upwards on the right, downwards on the left; then he makes a gesture joining with a finger the two arrows and slowly and repeatedly crossing the x-axis; finally Ivan joins the two arrows by drawing an "oscillating" line.

1 Well, it seems to me that (...) yes, it is a case of the IVT.

2 IVT says that I may find c such that $f(c)=0$ (...)

3 yes, but how to find a and b? First, I have to find a and b (...)

4 and to exploit continuity

Ivan comes back to look at the drawing; two index fingers are placed on the upwards and downward arrows.

5 Yes, the limit perhaps says something.

Ivan makes an horizontal movement with his right index finger and crosses the right upwards arrow.

6 If I take *a* value of y, the function must be over that value when x is enough big. But how to choose a value of x? (...)

7 Before I need to choose a value of y. But the value of y is whichever (...)

8 Whichever M, I take M and I may find T such that for every $x>T$ (...)

Ivan makes again a horizontal movement with his right index finger and crosses the right upwards arrow.

9 This might be the value of M, the value of y, then I find the value of x, the value of x which is b, such that if $x>b$ then $f(x)>M$

10 But I have $f(x)>M$, not $f(x)=M$

11 Is it $f(b)=M$? (...)

12 Not sure! (...)

13 I might take one point x' such that $f(x')>M$; that point is the extremity of the interval on the right!

14 But why $f(x')>0$?

15 Yes, I need to take $M>0$! M is a number, whichever number!

16 On the left it is the same: I find x'' such that $f(x'')<0$.

17 The interval for the IVT is (x'', x'). Let us write the proof:

Ivan speaks and writes at the same time, by dictating to himself what he says

18 *Thanks to continuity of f and IVT, $f(x'')f(x')<0$ implies that c exists in (x'',x') such that $f(c)=0$.*

19 Not, it does not work! First, I need to find x' and x''.

20 *Thanks to hypotheses on limits, given $x'>0$ I find $f(x')>0$.*

21 Not, still it does not work! I need to come back to how I could find $x'>o$ such that! No, not how, why I could find it! Because of limits! How to write it well?

22 *Thanks to the hypothesis on limits, given $M>0$ I may find $x'>0$ such that* (...)

23 Not: I may find x_M such that for every $x>x_M$ I have $f(x)>M$.

Ivan puts a strikethrough over all what he had already written, and starts again to dictate to himself and to write down:

24 *Thanks to the hypothesis on limits, given $M>0$ I may find x_M such that for every $x>x_M$ I have $f(x)>M$. I take one point $x'>x_M$. $f(x')>M>0$.*

25 Well, now it works.

26 *In the same way I find x'' such that $f(x'')<0$.* (...)

27 *There is a point c, $x''<c<x'$, such that $f(c)=0$* (...)

28 Not, this is the conclusion, before it I must write the hypothesis and then the conclusion.

Ivan puts a strikethrough over the last line, named 27 above, and dictates to himself and writes down:

29 *Now I may apply the IVT to the function f in the interval (x'', x'): $f(x'')f(x')<0$, thus c exists in the interval (x'', x'), such that $f(c)=0$*

When the document was analyzed, the complexity of the student's behaviour and the analytical tools available in that moment (1995) did not allow to perform an exhaustive, in-depth description and interpretation of the difficulties met by the student and of how he was able to overcome them. At that time we were only able to find some elements that could qualify as mature Ivan's mastery of the visual-graphical notion of continuity and infinite limits at the infinity, his identification of IVT as the theorem which might have allowed him to perform a rigorous validation of the statement, the immediate reference to the problem of finding the extremities of the interval where to apply IVT, and the identification of the epsilon-delta definition of infinite limits at infinity as the crucial tool to solve the problem. Then Ivan enters the epsilon-delta reasoning and the situation seems to become more and more confuse: the search for the extremities of the interval where to apply IVT results in an apparently messy sequence of steps of reasoning to get the abscissa of a point where the function is positive. When the interval to apply IVT is constructed, a further problem concerns the organization of the text in order to satisfy the textual and logical constraints of a proof text. This description is a pure narration of Ivan problem solving process. It does not allow to account for:

- the functional relationships between Ivan inner questioning (see steps 3, 6, 11, 14, 21) and the different kinds of answers (sometimes a decision concerning how to go on, sometimes the control of a proposition or a chain of propositions);

- the nature of the difficulty met in the central part of the process (from 6 to 17), and what allowed to overcome them;

- the nature of the difficulties met in the last part of the process (from 18 to 29).

THE NEED FOR A COMPREHENSIVE FRAMEWORK: THE HABERMAS CONSTRUCT OF RATIONALITY

The document does not speak by itself; we need theoretical tools to make Ivan's voice understandable by us, and possibly identify an intentionality that might drive Ivan's towards the final result, in spite of the superficial impression of "*a random walk luckily resulting in a good conclusion*" (according to the evaluation of the document by a colleague of mine who teaches Calculus for first year university students). We need also to understand why apparently so obvious steps (like finding a point x where f(x)>0, given the hypothesis of +∞ limit at +∞) become so difficult when the path moves through the epsilon-delta forest.

The first need suggested us to try and adapt Habermas' construct of rational behavior to Ivan's problem solving (as a process driven by intentionality to get a correct result by enchaining correct steps of reasoning, and to communicate it in an understandable way in a given community). The second need was satisfied by integrating, within the use of Habermas' construct, an analysis of some phases of Ivan problem solving process in terms of *mental dynamics* related to the treatment of propositions and the mastery of logical constraints, according to the Guala and Boero's elaboration on mental dynamics in problem solving (see Guala & Boero, 1999). They consider "mind times" that may be generated during the problem solving process: e.g. when imagining to move back from an hypothetically attained goal to a previous situation; or when going back and retrieving some information from memory, and projecting it in an imagined, future situation; etc.

Habermas' construct of rational behavior deals with the complexity of discursive practices according to three interrelated elements: knowledge at play (epistemic rationality); action and its goals (teleological rationality); communication and related choices (communicative rationality). Thus, it seems suitable for being applied to mathematical activities like proving and modeling that move along between epistemic validity, strategic choices and communicative requirements. The following aspects of Habermas' elaboration (1998, pp. 310–316) are relevant for us.

Concerning epistemic rationality

> We know facts and have knowledge of them only when simultaneously know why the corresponding judgments are true. (...) Someone is irrational if she puts forward her beliefs dogmatically, clinging to them although she sees that she cannot justify them. In order to qualify a belief as rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification (...) The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context. (p. 312)

The intentional character of rational behavior on the epistemic side emerges from these remarks in a perspective of progressive development of knowledge (the qualifying element being the tension towards knowing "*why the corresponding judgments are true*"). In Habermas' elaboration, the exercise of epistemic rationality is strictly intertwined with speech and with action (i.e. teleological rationality)— the latter resulting in the evolutionary character of knowledge:

> Of course, the reflexive character of true judgments would not be possible if we could not represent our knowledge, that is, if we could not express it in sentences, and if we could not correct it and expand it; and this means: if we were not able also to learn from our practical dealings with a reality that resists us. To this extent, epistemic rationality is entwined with action and the use of language. (p. 312)

We will see how this way of conceiving the interplay between knowledge, speech, and action will account for some relevant aspects of Ivan's proving.

Concerning teleological rationality

> Once again, the rationality of an action is proportionate not to whether the state actually occurring in the world as a result of the action coincides with the intended state and satisfies the corresponding conditions of success, but rather to whether the actor has achieved this result on the basis of the deliberately selected and implemented means (or, in accurately perceived circumstances, could normally have done so). (p. 313)

Let us consider problem solving in its widest meaning (including conjecturing, proving, modeling, finding counter-examples, generalizing, and so on): the above sentence highlights the quality of a process, which may be qualified as rational (on the teleological side) even if the original aim is not attained. The intentionality of action (including the choice and use of the means to achieve the goal) and the reflective attitude towards it are two relevant features of teleological rationality.

> A successful actor has acted rationally only if he (i) knows why he was successful (or why he could have realized the set goal in normal circumstances) and if (ii) this knowledge motivates the actor (at least in part) in such a way that he carries out his action for reasons that can at the same time explain its possible success. (pp. 313–314)

Concerning communicative rationality

> (...) communicative rationality is expressed in the unifying force of speech oriented toward reaching understanding, which secures for the participating speakers an intersubjectively shared lifeworld (...). (p. 314)

The above sentence illustrates an ideal practice of communicative rationality, and the related values. Then Habermas presents a condition that qualifies an actual individual behavior as rational on the communicative side:

> (...) The rationality of the use of language oriented toward reaching understanding then depends on whether the speech acts are sufficiently comprehensible and acceptable for the speaker to achieve illocutionary success with them (or for him to be able to do so in normal circumstances). (p. 314)

Even in the above sentence the intentional, reflective character is pointed out (for the specific case of communicative rationality).

Habermas' construct offers a model to deal (after adaptation) with important aspects of mathematical activity, without capturing *all* the aspects (see Boero & Planas, 2014, pp. 207–208 for a brief presentation of some of its intrinsic limitations). It has been initially used as a tool to analyse students' rational behavior in proving activities according to the researchers' (and teachers') expectations (see Boero, 2006; Morselli & Boero, 2011). Its application to analyses that also use other constructs gradually resulted in a rich toolkit with various applications (see Boero & Planas, 2014).

ANALYSIS OF THE DOCUMENT [2]

In the perspective of the Habermas' construct of rationality, integrated with Guala and Boero's elaboration on mental dynamics, the document provides us with the opportunity to analyze and interpret Ivan's proving as a *rational* enterprise. It also offers an occasion (based on that analysis) to reflect on some crucial aspects of a successful approach to what we may call the *epsilon-delta rationality*. Indeed in the reported document we may identify:

- the continuously renewing, conscious interplay, driven by an inner questioning (steps 3, 6, 11, 14, 21), between the need of performing strategic choices (*teleological rationality*) aimed at getting the elements to move forth, and the epistemic control on how they fit (or do not fit) the requirements of *epistemic rationality*;

- some mental dynamics (cf. Guala & Boero, 2009) related to Ivan's strategic choices (*teleological rational-*

ity) performed to meet *epistemic* requirements. We may observe how in the first part of Ivan's work he follows (as the movements of his hands show – see description before step 1) the time ordering of the text of the task, moving from the hypothesis on limits to the visual-physical search of the points of intersection with the x axis. Then an abductive shift is made to the IVT, which should guarantee the same result within the theory: this means a shift to a different *epistemic rationality*. At that point, a reverse mental movement is made, from focusing on the existence of the intersection point (step 2), to the search for the interval to which the IVT should be applied (step 3). This movement means moving back from the time of the solved problem to the time of the hypotheses that guarantee the solution. A similar movement will be replicated later (step 7), and expressed through a temporal adverb ("*before*", "*prima*" in Italian). In both cases an inner question related to how to validate (*epistemic rationality*) a partial result got during the development of the problem solving process suggests to go back to the condition that ensures its validity. Let us consider now what we might call the *quantifiers game,* played from step 7 to step 15 and then at least partially echoed in the writing phase (steps 18 to 24, where the *epistemic rationality* and the *communicative rationality* constraints are intertwined). From the satisfied condition of infinite limit at the infinity: *for every M there is* x_M *such that if* $x > x_M$ *then* $f(x)>M$ it is necessary to move to: *in particular, I choose M≥0 and then I get x' such that f(x')>M≥0*, with a change of logical status: from general quantification and consequent existential quantification and subsequent universal quantification, to particularization of the generality of M in order to get a point where the function is positive. Focus must move from the general condition of limit to a specific implication of it. The condition to be satisfied becomes f(x')>0, thus M must be chosen as M≥0. The Mind times toolkit, applied to a micro-analysis of this phase of the process, allows to interpret the mental weight inherent in this phase and its difficulty: a projection in the future time of the application of the IVT results in the choice of a *particular* value of M, at the beginning of the logical and mentally temporal chain of quantifiers, suitable to get the *appropriate* value of x';

- the full mastery of the logical and temporal structure of a theorem, interfaced with the above change of focus; this is evident in the steps 18 to 24;

- the need to satisfy the requirements of *communicative rationality*, consciously related to the *epistemic* requirements. *Communicative rationality* must account for satisfied *epistemic* requirements in the outer speech/writing, while epistemic requirements drive the inner dialogue towards consequent actions: this emerges in the steps 23–29, with communicative constraints particularly evident in the steps 27–29.

REFLECTIONS ON THE ANALYSIS OF THE DOCUMENT

The adaptation of the Habermas' construct to Mathematics Education offers the opportunity to qualify the visual-graphical Calculus as fully *rational*. Indeed that pre-Cauchy Calculus, when dealing with "ordinary" functions, has its own criteria of epistemic validity, its own strategies to solve problems, its own means of intentional communication. This is reflected in the first part of Ivan's elaboration: requirements of *epistemic validity* are of visual nature, and the chosen strategy to solve the problem consists in the translation of the verbal hypotheses of the statement into visual-graphical hypotheses, which will be connected to the thesis through the gesture of the finger that crosses the x axis in order to connect downwards and upwards arrows. Also inner and outer communication is mainly visual (through the signs on the sheet of paper and the gestures). This is not new (cf. some remarks by Nunez et al., 1999, and by Tall & Katz, 2014), but the use of Habermas' construct of rationality suggests us to reflect on how that system of thinking (with its specific epistemic, teleological and communicative characters), relevant in the history of mathematics, may work today not only as an intuitive first step when moving towards the epsilon-delta proving, but also as a resource during the epsilon-delta proving to support some of the most delicate student's actions (in the case of Ivan, this is particularly evident, thanks to his gestures, in the descriptions after the steps 4, 5 and 8).

In Ivan's elaboration, the statement of the IVT "represents" what is already clear in his first approach, and guaranteed by a gesture (see above); it also opens the way to its formal treatment by orienting the search for the interval in which it can be applied, based on the formal definition of infinite limit at infinity. We may interpret IVT as a pivot in Ivan's transition from the visual-graphical rationality to the epsilon-delta rationality in the treatment of the problem.

In terms of mental dynamics, we may better understand the complex relationships between the visual-graphical rationality and the epsilon-delta rationality, and the difficulties to manage the latter. On one side, the management of mind times in the epsilon-delta Calculus implies the necessity of changing the temporal and logical order of quantifiers, moving from $f(x) > M$ (whichever M) to the choice of a value of M such that we can get x' and $f(x') > M$. This suggests a perspective (alternative to Nunez, Edwards and Matos elaboration) to interpret why it is so difficult to move from visual-graphical to epsilon-delta calculus. On the other side, on the *teleological* dimension, the visual-graphical rationality provides Ivan with the visual and gestural support to bear the weight of the complex logical and temporal operations needed to construct the (x'', x') interval where to apply the IVT. Indeed the movement of his right index finger not only suggests the existence of a point where the function is positive, but also provides Ivan with the opportunity of "seeing" how to move from a general quantification on M, to the choice of a particular M in order to get a value x' such that $f(x') > 0$, thus orienting Ivan's mental dynamics.

CONCLUSIONS

The case of Ivan's think-aloud proving of a simple theorem of epsilon-delta Calculus was an occasion to search for a comprehensive framework to deal with the problem of analyzing the transition from visual-graphical proving, to epsilon-delta proving in Calculus. The Habermas' construct of rationality, integrated with an analysis of the proving process in terms of mental dynamics, suggests a solution, which in the case of Ivan accounts for his intentional work and his difficulties. It also suggests to reconsider the visual-graphical treatment of proving not only as a heuristic starting point, but also as a consistent *rationality* and, as such, a permanent, sure reference when students move within the forest of the epsilon-delta proving during the approach to the epsilon-delta Calculus as taught in most universities.

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ENDNOTES

1. Precisely, by IVT we mean the theorem whose statement in Ivan's textbook is: "If f is a continuous function in an interval (a,b) and $f(a) \cdot f(b) < 0$, then at least one point c exists in the interval (a,b) such that $f(c) = 0$."

2. In the Turin international symposium (November, 21, 2014) on "Mathematics Education as a transversal discipline", Ferdinando Arzarello presented an alternative analysis of the same protocol, based on an integration of the Habermas construct with analytical tools derived from Hintikka's Logic of inquiry, with some points of contact with the analysis presented in this paper as concerns the teleological dimension of rationality.

Pre-service teachers' construction of algebraic proof through exploration of math-tricks

Orly Buchbinder[1] and Alice Cook[2]

1 University of New Hampshire, Durham, USA, orly.buchbinder@unh.edu

2 University of Maryland, College Park, USA

This paper contributes to the ongoing effort to create rich learning opportunities for prospective teachers to engage with reasoning and proving. Twenty elementary and middle school pre-service teachers completed individual projects in which they explored "math-tricks" – unconventional computational algorithms – as a part of an undergraduate proof course. Our findings suggest that the task evoked uncertainty with respect to why the tricks work and motivation to resolve the uncertainty by means of algebraic proof. We discuss the potential of this task to create rich opportunities for prospective teachers to conduct explorations, construct algebraic proofs and reflect on their experience from learner's perspective.

Key words: Proof, pre-service teachers, uncertainty, algebraic reasoning.

INTRODUCTION

The central role of reasoning and proof in teaching and learning mathematics has been long recognized by mathematics education community and by policy makers. In recent years there has been a movement in the United States towards making proof an integral part of mathematics curriculum not just for students in high-school geometry but across all grade levels (NCTM, 2000; CCSSO, 2010). This trend is also associated with growing demand for deeper conceptual knowledge of number sense and algebraic reasoning (CCSSO, 2010).

In order to provide students with learning environments that emphasize reasoning and proof, teachers themselves need to have strong subject matter knowledge and a solid understanding of proof (e.g., Knuth, 2002, Stylianides & Ball, 2008). However, research studies consistently show that pre-service teachers (PSTs) have inadequate conceptions of proof, and experience difficulties in distinguishing between empirical and deductive arguments and in understanding the different roles of proof in mathematics (e.g., Martin & Harel, 1989; Ko, 2010).

In recognizing the impact of teachers' knowledge of proof on students' experiences with proof, many teacher education programs have designed courses and instructional activities oriented towards developing pre-service teachers' conceptions of proof, especially at the elementary and middle school levels (e.g. Stylianides & Stylianides, 2009). According to Ball, Hill and Bass (2005) teachers need to experience mathematics from the learner's perspective in order to gain appreciation of mathematical ideas, develop better understanding of the ways students interact with them and become aware of the difficulties students encounter. With respect to proving, this might entail engaging PSTs in mathematical exploration, conjecturing, and proof construction. However, designing such tasks for elementary and middle school teachers is a pedagogical challenge for teacher educators. Teachers' prior knowledge of mathematical content might hinder their ability to grasp the complexity of underlying mathematical ideas or their ability to consider students' perspective. It also might reinforce an inadequate view of proof as a routine exercise of justifying well-known and prior established facts (Knuth, 2002).

Several approaches to address this issue have been suggested. For example, Barkai and colleagues (2002) asked elementary PSTs to analyse students' arguments in elementary number theory. Stylianides and Stylianides (2009) suggested a "construction-evaluation" model in which elementary teacher candidates were asked to write a proof of a given statement and then to evaluate the validity and generality of their arguments. Despite the reported success of these approaches, Ko (2010) points to the need for more research on strategies for developing PSTs' conceptions of proof and adequate knowledge of proving.

With this paper we aim to contribute to the ongoing effort of the field to create learning opportunities for teacher candidates to engage with reasoning and proving. We report on one task that was developed and implemented in our undergraduate course titled "Reasoning, Justification and Proof for elementary and middle school teachers". The task aimed at promoting PSTs' understanding of algebraic proof through analysis of unconventional computational algorithms (math-tricks) and through reflection on their proving experiences. Qualitative methods were used to analyse the types of student-generated proofs, their spontaneous use of algebraic proof and/or of algebraic notation in it, and to explore cognitive and affective aspects of the ways in which prospective teachers' coped with the task from the students' perspective.

THEORETICAL CONSIDERATIONS

Our approach for designing the task "exploration of math-tricks" is grounded in several theoretical notions. First, we build on Harel's (2007) premise that instruction should appeal to and foster students' *intellectual need to prove*. Implementation of this principle entails creating learning situations in which the need for proof arises intrinsically. One possible way to achieve this is by creating uncertainty regarding whether a certain mathematical phenomenon is true or false. Research studies have shown that such tasks generated a need to resolve the evoked uncertainty by means of argumentation, explanation, convincing and proving (e.g. Buchbinder & Zaslavsky, 2011; Hadass, Hershkowitz, & Schwarz 2000).

Zazkis (1999) suggests that uncertainty can be evoked by exploration of "non-conventional" mathematical objects such as number systems other than base 10 or non-Cartesian coordinate systems. Building on this approach, we created a task that invited PSTs to evaluate non-conventional computational algorithms, or math-tricks, and either prove or refute them. Thus, prospective teachers could explore unfamiliar mathematical phenomena embedded within familiar content.

Finally, our work is grounded in the research that highlights the importance of teachers' reflection on their own thinking from the learner's perspective. According to Zaslavsky and Sullivan (2011) the process of resolving uncertainty combined with reflection on personal experience can lead teachers to revaluate and refine their understanding of mathematical content of the task, and promote teachers' awareness of difficulties students might experience while engaging with this content.

THE SETTING

Twenty elementary and middle school PSTs participated in the course on mathematical reasoning and proof which most students took during the 3rd or the 4th year of their program. The task "Exploration of math-tricks" was given to PSTs as individual project and asked them to (a) watch, understand and describe a math-trick presented in a video in their own words; (b) analyse the math-trick and either prove that it works or disprove it by a counterexample; (c) compare the math-trick to the corresponding convention algorithm and discuss similarities and differences between them; and finally (d) write a reflection on the exploration and proving process in this task. The students were given four weeks to complete the project. They were encouraged to cooperate with each other, but required to submit their original work.

Task analysis

The content to which we refer as "math-tricks" is a collection of short videos which we found on the web[1]. Each video presents an unconventional, albeit correct under limitations, algorithm for one of: multiplication, division, calculation of square or cube roots, solving systems of linear equations and many others. Due to space constrains we present here a short description of two such math-tricks (Figure 1).

These algorithms can be regarded as unconventional since they differ substantially from standard algorithms found in mathematical textbooks both in content and in presentation style. In the video, the presenter used several numeric examples to illustrate each algorithm, describing them as "faster, easier and smarter ways", "math-tricks", and "pure magic", without any expectation on the part of the

1 It is not our intent to either criticize or promote this online content for any purpose beyond described in this paper.

Trick #1: Special trick for multiplication
Suppose you want to multiply two two-digit numbers that have the same tens digits and the sum of their unit digits is 10. For example, consider: 36×34. First, multiply the tens digit of a multiplicand by next counting number: $3\times(3+1)=3\times 4=12$. These are the first two digits of the product. To obtain the last two digits of the product simply multiply the unit digits, e.g. $6\times 4=24$. Thus, the product is: $36\times 34=1224$.

Trick #2: *Trick for squaring numbers 10 – 19*
Suppose you want to square a number between 10 and 19, e.g. 17^2. To find the first two digits of the product add the unit digit to the original number: $(17+7)=24$. To find the last digit, square the unit digit of the original number $7^2=49$ and add it to the number obtained before, carrying forward the 4 and ensuring that 9 remains the units digit: $24\square + 49 = 289$. Thus, $17^2 = 289$.

Figure 1: Description of two "math-tricks" as presented in the video

viewer to understand why these algorithms produce correct results. However, as shown in Figure 2, when represented algebraically, the "tricks" appear to be just special cases of multiplication of two two-digit numbers. Thus, the math tricks could be analysed with algebraic techniques accessible to pre-service elementary teachers.

a proper justification. For example, students in this category merely restated the algorithm or wrote that it works because of its similarity to conventional algorithm. Two other categories were *Partial general argument* - unfinished general argument or missing details, and *Valid proof*. For the second criterion, use of algebraic notation, we identified the following categories: *No algebraic notation*– students used only numeric examples or described their thinking in words; *Semi-algebraic notation* – although some variables and algebraic symbols were present in student work, they were not used meaningfully or were not functional for proof construction; and *Correct algebraic notation*. The latter category was further divided into two sub-categories: *Standard algebraic notation*, such as using $(10a + b)$ to represent a two-digit number; and *Self-invented notation,* meaning that students came up with their own correct algebraic representations.

Conventional multiplication of two-digit numbers	Special trick for multiplication	Trick for squaring numbers 11-19
a, b, c, d digits	$a=c,\ b+d=10$	$a=c=1,\ b=d$
$(10a+b)(10c+d)=$ $100ac+10(ad+bc)+bd$	$(10a+b)(10a+d)=$ $100a^2+10a(b+d)\ +bd=$ $100a^2+100a+bd=$ $100a(a+1)+bd$	$(10+b)(10+b)=$ $100+10(b+b)\ +b^2=$ $10[(10+b)+b]+b^2$

Figure 2: Algebraic representation of the conventional multiplication and two "math-tricks"[2]

Data and analytic techniques

The data for this paper come from 20 student project papers on 13 different math-tricks. In order to explore the students' proof production, their mathematical arguments were analysed through open coding (Staruss, 1987). Two major criteria emerged, along with corresponding categories focusing on algebraic notation and proof justification. For the first criterion – proof production - the following categories emerged: *Invalid justification* – students either relied solely on empirical evidence or did not produce

When analysing students' written reflections we looked for evidence of uncertainty evoked by the task and students' approaches to resolve it. We also identified and examined instances of students reflecting on their proof experiences as learners, their mathemati-

2 Note that the constrains on variables in both tricks account for the place value of the digits in the product in such way that adding the unit digits can be described as "placing" them in the units place.

Representing the trick with algebraic notation: $\prod x \bullet y = a \bullet (a+1), b_1 \bullet b_2$

Where x is the multiplicand, y is the multiplier, a is the first digit of the multiplicand, b_1 is the last digit of the multiplicand and b_2 is the last digit of the multiplier. The comma is to separate portions of final product.

Why it works: The math trick works because the requirements are extremely narrow. Since both beginning portions of the multiplicand and multiplier must be the same, it allows for consistency between the parts of the algorithm. The last digits being multiplied by each other happens in both the conventional algorithm and the math trick. By adding the following consecutive integer to the first digit of the multiplicand, it covers the part of the conventional algorithm that incorporates the addition of the problem.

Figure 4: Melanie's analysis and justification of *Special trick for multiplication*

cal struggle, and connections to their future teaching. Due to the space constraints not all categories will be reported herein.

RESULTS

Students' mathematical arguments

First we report on the kinds of PSTs' mathematical arguments and their use of algebraic notation. This section illustrates some of the categories described above with examples of students' arguments and excerpts from their reflections. Figure 3 below summarizes distribution of students' arguments across all categories.

Our data suggests that success with proof production was strongly related to use of algebraic representation. PSTs who used correct algebraic notation were able to either produce valid proofs or partial general arguments that could potentially be turned into proofs. Interestingly, only 3 PSTs used standard algebraic notation, while 8 students invented their own representations (see examples below). Five students used, what we call, semi-algebraic notation, meaning that their use of variables was insufficient or not functional and did not result in production of valid proofs. Nevertheless, 3 students provided additional written explanations and produced partial general arguments. Four students whose arguments were categorized as invalid either relied solely on empirical evidence or used semi-algebraic notation. Only one of them seemed to be aware of limitations of such a line of reasoning.

Examples of students' work

Melanie's work on the *Special trick for multiplication* (Figure 4) illustrates the category of arguments that used semi-algebraic notation and produced invalid justification.

Melanie recognised the similarity between the conventional algorithm and the math-trick, and tried convey it in her explanation. She described her observations eloquently, but provided little insight on *why* the trick works. For example, it is not clear from her explanation *how* the similarity of "both beginning portions" of the two factors "allows for consistency between the parts of the algorithm". In her reflection, Melanie described the challenges she encountered in representing the trick algebraically and in explaining why it works. She also referred to her experiences as student and difficulties in understanding algorithms in general:

> *.... creating an algebraic notation was very challenging [...] I found it very challenging to under-*

Use of algebraic notation / Type of argument	No algebraic notation	Semi-algebraic notation	Correct algebraic notation	
			Self-invented	Standard
Invalid justification	2	2	--	--
Partial general argument	2	3	2	--
Valid proof	--	--	6	3

Figure 3: Distribution of teacher candidates' mathematical arguments (N=20)

Why it works?

Let's look at this algebraically: If we look at $ab \times ac$, we can see that a is the tens digit, so if your numbers were 22 and 28 then a would equal 20. Since we are adding 1 to the tens digit, we are essentially adding 10 to 20. This gives us the $a(a+10)$. I needed to incorporate the rule that the sum of b and c is 10, so I made $b = 10 - c$.

$$(a+b)(a+c)=a(a+10)+bc$$
$$(a+10-c)(a+c)=a(a+10)+(10-c)c$$
$$a^2+10a-ac+ac+10c-c^2=a^2+10a+10c-c^2$$
$$a^2+10a+10c-c^2=a^2+10a+10c-c^2$$

Figure 5: Thomas's analysis and justification of *Special trick for multiplication*

> *stand why the trick works as a whole, although I can understand how to do it and its limitations. I have personally never been one very strong in understanding the functioning of algorithms at the core so it did not surprise me that I had difficulty understanding how the trick worked.*

Another student, Thomas, who analysed the same trick, came up with his own way to represent two-digit numbers and used it to produce a valid proof (Figure 5).

Thomas represented a two-digit number using notation somewhat similar to the standard, as $(a+b)$, where a represents tens. He then used the distributive property to show that the product of two two-digit numbers yields the same algebraic expression as described in the math-trick. One disadvantage to Thomas's notation is the need to remember that the variables a and c in the final expression represent different things: tens and units respectively—a detail imperative to account for place value of the product's digits. Thomas's reflection is a detailed account of his thinking process, his initial frustration and feeling of excitement when the proof was completed:

> *I was impressed with this trick from the beginning, but I struggled expressing it algebraically. It seems simple now, but at the time it was very frustrating. During the process of writing my equation I kept treating a like it was a single digit number instead of treating it like the place value that it held. [...] I am not sure why it clicked, but I realized a was the tens digit, while b and c were the ones digits [...] It has been quite some time since I have been as excited as I was when all of this clicked. It took me analyzing and breaking down both the trick and the conventional algorithm to create my equation.*

Another student, Cindy, also came up with her own version of base 10 notation to analyse the *Trick for squaring numbers 10 – 19* (Figure 1) and was able to produce a valid proof for this trick. Cindy used n to denote the entire two-digit number between 10 and 19, and used $(n-10)$ to represent the units digit (Figure 5).

Cindy's reflection reveals both cognitive and affective sides of the process she went through in her exploration. Although the exploration of math-trick was not an easy task for Cindy she persevered and was able to solve it correctly. She wrote:

> *This task was a little time consuming and confusing at first. [...] I did this by trial and error, until I realized a key element I was missing. My "aha" moment was when writing the general rule. [...] I kept trying to manipulate the equation in different ways, and finally realized that I need to account that we are solving for the first 2 digits in a 3-digit number. Therefore, I need to multiply the first portion of the equation by 10 to account for the 10's and 100's digits.*

To find the first two digits of n^2: $10[n+(n-10)]$

To find the last digit of n^2: $(n-10)^2$

Conjecture: For all whole numbers from 10 to 19: $n^2 = 10[n+(n-10)]+(n-10)^2$

Check the conjecture using appropriate algebraic notation:

$$n^2 = 10[n+(n-10)]+(n-10)^2$$
$$n^2 = 20n-100+n^2-20n+100 \longrightarrow \boxed{n^2} = \cancel{20n} \cancel{-100} + \boxed{n^2} \cancel{-20n} \cancel{+100}$$
$$n^2 = n^2.$$

Figure 5: Cindy's analysis and justification of "*Trick for squaring numbers 10–19*"

Cindy and Thomas's reflection shows that the analysis of an unconventional algorithm led them to consider more carefully the standard multiplication algorithm and place value. This was a recurring theme in most of students' reflections, including students who eventually were not successful in producing valid proofs.

Students' reflections on the exploration process

In our analysis of PSTs' reflections we identified evidence for the uncertainty evoked by the task, expressions of students' interest in the math-trick project and their consideration of the task from the leaner's perspective. This analysis was overlaid onto an analysis of students' mathematical arguments. Not surprisingly, students who produced valid proofs or partial general arguments described feelings of accomplishment and satisfaction with the project (including the two students who produced invalid arguments but were unaware of this). Although the degree of uncertainty evoked by the task varied within the group, almost all PSTs indicated that they were surprised by the math-tricks. PSTs distinguished between challenges involved in understanding the steps of the trick, *why* it works, and constructing a proof. Some PSTs who understood the trick relatively easily, but struggled with proving, tended to describe the task as frustrating, or hard. For example, Lisa wrote:

> *It was easy for me to see how the trick was connected to the conventional method and to see that it would always work. The hard part was explaining it because I cannot think of a way to put this trick in algebraic notation and it is very hard to visualize the math using words.*

Other students felt initially perplexed by the algorithms, but felt more comfortable once they represented them algebraically. For instance, Helen wrote:

> *Understanding how to complete the trick wasn't too complicated, but understanding why it works was a difficult task. [...] for me, the "Aha" moment was once I thought of using variables. Seeing the steps in variable form, instead of using numbers made it very clear to me how the process worked.*

For the majority of PSTs, transitioning from empirical exploration to algebraic representation was not a straightforward process. Aside from 3 students who felt comfortable with it, the group reported on difficulties they encountered, the strategies employed to resolve the impasse and their "Aha!" moments. Students like Melanie or Lisa, who felt that they did not resolve the uncertainty, shared feelings of frustration and lesser competence in their mathematical ability. This might have been avoided or reduced by encouraging greater collaboration and sharing ideas among students.

Several students reflected on their exploration of math-tricks from both learner and future teacher perspectives. Following is an excerpt from Natalie's project paper, with the original emphasis:

> *This process was different for me because my role was to be both a student and a teacher. When I was a student in this process I had to figure out how the trick worked, but when I was a teacher in this process, I had to explain why the trick worked. As a student, this involved noticing patterns, and as a teacher, this involved synthesizing and explaining the real math behind the trick.*

CONCLUSION

In this paper we described the design and implementation of the task "Exploration of math-tricks" with elementary and middle school pre-service teachers. The task aimed to enhance PSTs' appreciation of proof and highlight its exploratory function through investigation of unconventional computational algorithms – "math-tricks". Our data show that, as anticipated, the task evoked uncertainty regarding how and why these algorithms function, which most of the students resolved through proving. The majority of students in our course used standard or self-invented algebraic representations to produce proofs or partial general arguments. Not surprisingly, prospective teachers who felt less proficient with algebra were less successful in producing proofs and in resolving uncertainty as evident from their comments. Nevertheless, exploration of unconventional algorithms allowed teacher candidates to review and refine their knowledge of standard computational algorithms and algebraic techniques. Our findings concur with theoretical notions underlying the role of uncertainty in fostering a need for proving and with empirical studies which utilized exploration of unconventional mathematical objects in instructional tasks with PSTs (e.g., Buchbinder & Zaslavsky, 2011; Zaslavsly, 2005; Zazkis, 1999).

Our analysis of PSTs' reflections revealed a complex account of cognitive and affective aspects of engagement with exploration of math-tricks; while some PSTs described difficulties in transitioning from informal argument to algebraic proof, others felt that algebraic notation aided in expressing their mathematical ideas. PSTs reported on mixed feelings of initial challenge and struggle with the task; but also on their excitement after successful (in their view) production of proof, or frustration when failed to produce one. We conclude that exploration of math-tricks combined with reflection on their own proving process allowed PSTs to evaluate this experience from learners' perspective and, in some cases, consider difficulties students might encounter with the concept of place value in multi-digit arithmetic. Furthermore, it seems that engaging in exploration and proving of math-tricks can provide pre-service teachers with the valuable opportunity to consider the role of both student and teacher.

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A case study: How textbooks of a Spanish publisher justify results related to limits from the 70's until today

Laura Conejo, Matías Arce and Tomás Ortega

University of Valladolid, Valladolid, Spain, lconejo@am.uva.es

In this paper we present the evolution of the proof schemes shown in grades 11 and 12 textbooks of a Spanish publisher related to the theorems of limits. In order to analyse this evolution, we use a framework developed from the definitions of proof schemes, preformal proofs and functions of proofs. Firstly, we describe our framework and then we show a case study applying the framework to textbooks (from the 70s until today) of a Spanish publisher. Some results and reflections about the analysis are described at the end as well as consequences for further studies.

Keywords: Proof schemes, preformal proofs, textbooks, limits.

INTRODUCTION

The main goal of a mathematical proof is to verify the correctness of mathematical statements. Under the perspective of mathematical education research we agree with Hanna (1995), who claims that mathematical proof promotes understanding. Moreover, Hanna & Barbeau (2010) think that proofs are bearers of mathematical knowledge in the classroom because proofs embody "methods, tools, strategies and concepts for solving problems" (Rav, 1999, p. 6) which is the essence of mathematics. Textbooks (that is, any book used by teachers and students, during a scholar year, in a teaching and learning process of a certain subject, González, 2002) are important elements in the teaching and learning process. Schubring (1987) claims that „teaching practice is not so much determined by ministerial decrees and official syllabuses as by the textbooks used for teaching" (p. 41). In addition, analysis of textbooks give us information about the mathematical knowledge that a society considers relevant in a particular historical moment (González, 2002) because they affect what and how students should learn (García-Rodeja, 1997). Spanish Educational System is an example of the effect described by Schubring (1987) because textbooks at pre-university education are usually the main reference material for teachers and students during the scholar year. Since the 70s, several changes of the Spanish Educational Law have occurred so a large amount of textbooks have been published and they have evolved in how they present mathematics.

The limit of a function is one of the most difficult and important concepts which are introduced at pre-university school in Spain. Research about this concept has shown that its learning involves a lot of difficulties. There are different studies which investigate how this concept could be taught at pre-university school: in an intuitive way (Henning & Hoffkamp, 2013), exploiting graphs of functions (Gunčaga, 2009),… Blázquez, Gatica & Ortega (2007) consider that students will be able to understand the ε-δ definition once they have understood the concept of limit as a tendency and approximation. We also think that proving results related to limits contributes to students' understanding of the concept, so we are interested in how textbooks present the concept of limit and justify the results related to them. Closer to our study, several researches have been developed about proving in secondary school textbooks in other countries but due to the limitation of space we do not include a detailed description of them but we will compare them with our own work in the future. For example, Nordströn & Löfwall (2005), who studied proof in Swedish textbooks, noticed that the frequency of proof items is low but they often exist invisible in the textbooks and this is a bit similar in Spanish textbooks. Ibañes & Ortega (2001) studied the student's proof schemes in the last courses of Secondary school and they conclude that students have difficulties in understanding proofs. Dos Santos (2010) noticed that mathematical proofs are disappearing in textbooks, that is, the newer a textbook is, the less number of mathematical proofs are in it. We claim that mathematical proofs is important in the understanding of mathematics, so it must be included in textbooks and for that we want to know how textbooks deal with this element of mathematics. The main goal of this study is to analyse the evolution of mathematical proof in textbooks and if they use other alternative justification procedures. To achieve this goal, we will try to answer some questions: are mathematical proofs replaced by other processes of justification? What kinds of justifications are used? Are all functions of mathematical proofs considered? Here we present a case study about the evolution of the treatment of mathematical proof in textbooks belonging to a Spanish publisher since the 70s until the present.

THEORETICAL FRAMEWORK

The aim of our study is to analyse the processes found in textbooks than could be used to convey students about the validity of the mathematical statements formulated in them. We use the concept of personal proof scheme (PPS) defined by Harel & Sowder (1998) because it includes other kinds of justification apart from mathematical formal proofs. Ibañes & Ortega (2001) studied the grade 11 students' proof schemes and they noticed that students accept any kind of proof schemes as a justification of a mathematical result, and some of these kinds of proof schemes could be found in textbooks. We are conscious that we do not know the intentionality of the publishers/editors. However, we want to classify the processes shown in textbooks according to the characteristics of personal proof schemes that they exhibit because these processes establish different levels of comprehension of the proofs. For that reason, we have adapted the definition of PPS to textbooks in the following way:

> *Proof scheme (PS) of textbooks: it consists of what is showed in the textbook which can constitute ascertaining and persuading for a generic reader of this textbook (here, a math student of the grade of the textbook), meaning by ascertaining the process showed in the textbook which could allow the reader to remove her or his own doubts about the truth of an assertion and meaning by persuading the process showed in the textbook that the reader could employ to remove other's doubts about the truth of an assertion.*

The processes of persuading and ascertaining are complementary, and both together constitute a PS. Regarding this definition, we have adapted the categories of classification developed by Harel & Sowder (1998) and Ibañes & Ortega (2001), including a new category using the concept of preformal proof (Van Asch, 1993).

> *PS0: there are no procedures of justification of the theorem.*
>
> *Inductive PS of 1 case (IPS1): we convey anyone about the validity of a conjecture by illustrating an example.*
>
> *Inductive PS of several cases (IPSs): as in the previous case, but now we verify several different examples.*
>
> *Inductive systematic PS (IsPS): as in the previous cases, but now examples are chosen in a systematic way, out different possible cases.*
>
> *Transformational PS (TPS): it is done by transformations of elements in a deductive way.*
>
> *Axiomatic PS (APS): the theorem is proved using axioms, meaning by axioms any primary results and other results which have been deduced previously.*
>
> *Preformal Proof (PP): a line of reasoning which can be formalised to a formal proof, by in which the essential idea is already present. It takes the same character of axiomatic and transformational PS.*

Figure 1 shows an example of inductive systematic PS. Something can be classified like an inductive PS if the textbook shows something which suggests that the example could be generalized to any other case. If not, it could be only considered like an application example of the theorem.

Transformational and axiomatic PS are the closest categories to a formal mathematical proof. Sometimes, we could find that a proof scheme has characteristics of both kinds of proof schemes, so they could be classified in both categories. Anyway, we classify the proof schemes found in textbooks only in the predom-

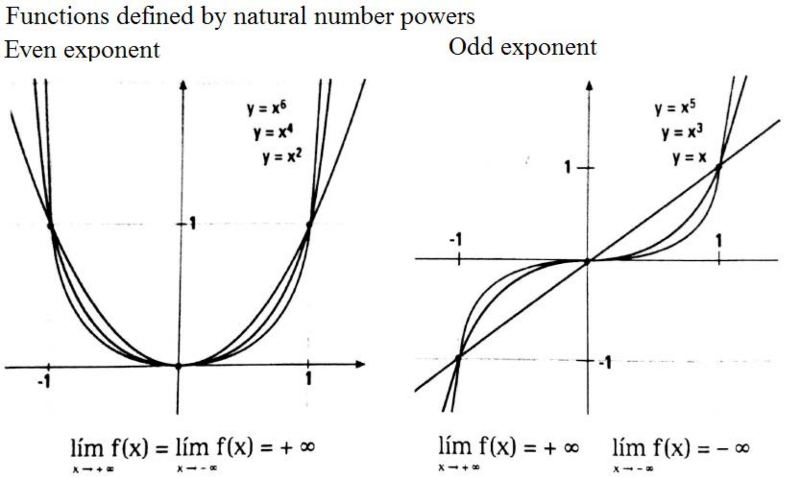

Figure 1: Example of inductive systematic PS to show the limits of functions defined by natural number powers (translated from the textbook LOGSE 11, 1998)

inant category to make their first count easier. Figure 2 shows some steps of an example of transformational proof scheme, although it uses some results which make it to be also considered as an axiomatic proof scheme. We classify it as a TPS because the author shows mainly transformations to justify the result. In the example, the justification ends by calculating the areas described below and comparing.

In the category of axiomatic PS we classify any formal mathematical proof. This kind of proof scheme uses always other theorems that were established before and they use deductive reasoning to prove the theorem. One well-known example of axiomatic PS could be the ε-δ proof to establish the uniqueness of the limit of a function if it exists.

Finally, we show an example of preformal proof. These kinds of justifications are rarely used but we think that they could be useful at these educational levels because they allow the students to understand a deductive reasoning without doing the abstraction of using a general case. In the example (LOGSE 11, 1998), the author justifies that the limit of a polynomial function at infinity is equal to the limit of the dominant term of the polynomial. The author shows the steps of the formal proof in a specific function instead in a general polynomial function, so the students only need to substitute by the general function to reach the formal proof.

> *Let be $P(x) = 2x^3 - 5x^2 + 8x - 12$. We study the ratio of $P(x)$ by its dominant term $2x^3$:*
>
> $$\frac{P(x)}{2x^3} = 1 - \frac{2}{2x} + \frac{8}{2x^2} - \frac{12}{2x^3} \xrightarrow{x\to\pm\infty} 1$$
>
> *Except number 1, when $x\to\pm\infty$ any addend approach to 0. For that reason, the limit is 1.*

Other important aspects that we consider are functions of mathematical proofs which are shown in the text. For that, we consider De Villiers (1990) model functions of proof (verification, explanation, systematization, discovery and communication). As Ibañes & Ortega (2001) defend, we also believe that the greatest worth of proofs is their function of explanation, although it is not their only value, and formal proofs could be exchanged for other kind of proofs. Apart from these classifications, we consider other aspects of proof in our broader study, but due to the limitation of space, we will not describe these items.

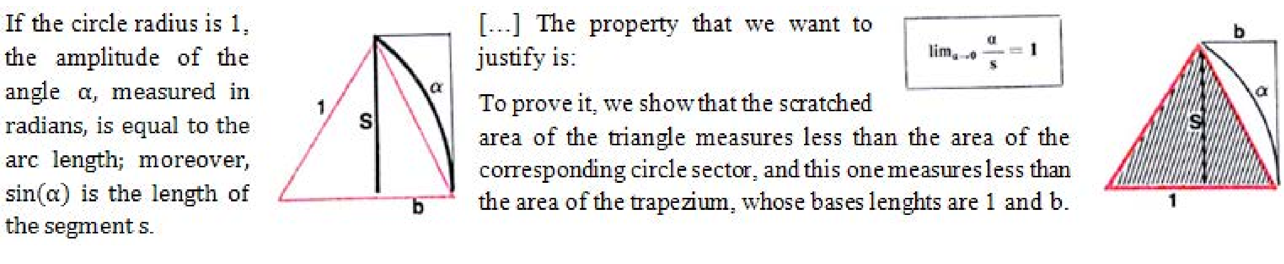

Figure 2: Example of transformational PS of "the limit at zero of $f(x) = x/\sin(x)$ is 1" (translated from the textbook LGE 10, 1980)

METHODOLOGY

Our research problem is related to the teaching and learning process of mathematical proof. We want to know how proofs have been treated in Spanish textbooks since the 70s. In the 70s it was enacted the first Spanish Educational Law which structures all the Spanish Educational System for the first time, they appeared the first common curricula and the government made control of the published textbooks. After that, there have been two changes in the Educational Law, and also in the curricula. As Dos Santos (2010) claims, we think that proofs are disappearing of Secondary School textbooks. In Spain, textbooks are used by almost every teacher and in almost every classroom of pre-university education, so the analysis of textbooks gives us information about the teaching of mathematics of a special moment. Due to the historical character of our work, we have combined two methods which are appropriate to this kind of work, the method of historical research on education (Ruiz-Berrio, 1976) and the research process on education described by Fox (1969). The combination of both methods gives us a method but here we only show a part of the general study, considering the analysis items of proof schemes and preformal proofs. By using these analysis items, we have classified the justifications which appear in textbooks according to them. Then, we have compared the different PS or PP used in each textbooks and their evolution along the time.

Although we are analysing textbooks of four different Spanish publishers (selected according to two criteria: publishers that have been widely used by Spanish mathematics teachers and those which exist since the 70's), here we show the analysis of one of them as a case study. We focus on proofs of theorems related to limits which are taught at pre-university levels, that is, the last courses of Secondary School (grades 11 and 12, 16–17 and 17–18-year-old students). The textbooks of this publisher correspond to three different education laws: the first one, LGE, was promulgated in 1970, the second one, LOGSE, in 1990, and the third one, LOE, in 2006, and it is still in force for the considered courses. The sample is composed by seven textbooks: one corresponding to LGE (grade 10; limits do not appear in grades 11 and 12), four corresponding to LOGSE (we have found two different collections corresponding to different years which differ on the treatment given to limits, and we have the textbooks of grades 11 and 12 of both collections) and two related to LOE (grade 11 and grade 12). We will use the code *Law G (Year)* for each textbook to simplify the notation, where "*Law*" denotes the Law of the textbook, "*G*", the grade, and "*Year*", the year of edition. In the Table 1 we show a summary of our sample.

ANALYSIS RESULTS

Each textbook considers a different treatment of the concept of limit and different ways of justifying the results that they show. We have only found some similitude in the presentation of the concept and its properties between textbooks of the same collection. For example, LGE 10 (1980) considers an unusual way to present limits: it firstly defines the continuity of a function and it presents limits in the chapter of derivability (in fact, it uses a description of limit in the definition of continuity). In the study of limits, the textbook shows the behaviour of functions in points of discontinuity and then, it defines the concept of limit of a function.

Textbooks corresponding to second period (LOGSE) present limits in a different way depending on the collection: the older collection (textbooks from 1998 and 1999) presents limits like a tool to study functions and their graphics; for example, they give a description of limit using an example and then, they give examples and describe the properties of the limit of a function. The newer collection (textbooks from 2003 and 2004) makes a more traditional presentation of limits: they do not present limits like a tool to study functions but they give formal definitions, the one in terms of sequences and the one in terms of absolute value (but they don't justify the equivalence between different definitions). Textbooks corresponding to the third period (LOE) give a simpler treatment of limits: firstly

Education Law	*LGE (1970–90)*	*LOGSE (1990–2006)*	*LOE (2006–14)*
Textbook	LGE 10 (1980)	LOGSE 11 (1998) LOGSE 12 (1999) LOGSE 11 (2003) LOGSE 12 (2004)	LOE 11 (2008) LOE 12 (2009)

Table 1: Sample of textbooks analysed

Textbook	Proof schemes							*Total results*
	PS0	*IPS1*	*IPSs*	*IsPS*	*TPS*	*APS*	*PP*	
LGE 10 (1980)	21	0	0	0	2	3	0	26
LOGSE 11 (1998)	16	4	3	0	1	0	1	25
LOGSE 12 (1999)	35	3	1	1	7	1	1	49
LOGSE 11 (2003)	9	0	2	0	0	0	0	11
LOGSE 12 (2004)	23	0	0	0	0	0	0	23
LOE 11 (2008)	9	2	0	0	0	0	0	11
LOE 12 (2009)	9	2	0	0	1	0	0	12

Table 2: Proof schemes used by textbooks

they consider all the definitions of limit of a function (finite limits, infinite limits, limits at a point or limits at the infinite) and then they formulate the arithmetic properties of limits. The differences between grade 11 and 12 are that in grade 12 definitions are formal (it gives the ε-δ definition) instead the intuitive definition of grade 11 (*we say that the limit of a function f*(*x*) *at the point* a *is* L *if the function f*(*x*) *approach to* L *as much as we want, whenever we take a value for x sufficiently close to the value*, LOE 11, 2008) and it goes deeper in the calculus of limits than the textbook corresponding to grade 11.

As we have seen before, there are significant differences in how textbooks deal with the concept of limit and the definitions that they use (formal, intuitive, no definition,...) These differences in how to present and define the concept affect the kind of proofs which can be used (informal definitions do not allow the editor to make formal proofs). Regarding the properties of limits, the uniqueness is only formulated in LGE 10 (1980); one-sided limits always appeared in these textbooks (they are necessary in the study of functions) but they are not always used to characterise the limit of a function: LOGSE 11 (1998) and LOGSE 12 (1999) are textbooks that do not link the existence and equality of one-sided limits to the existence of the limit of a function at a point. The arithmetic of limits is formulated in all textbooks but it is never justify except in one textbook (LGE 10, 1980). Indeterminate forms are always presented in the textbooks so, that let us think that textbooks are more oriented to promote mechanical calculus of limits than understanding of the concept. However, some textbooks give justifications of some cases of indeterminate forms, so we think that this indicates the consideration of the explanation function of proof in some cases.

Table 2 shows a summary of the proof schemes used in textbooks to justify the results corresponding to limits. We notice a big difference on the number of results considered in each textbook. It is due to the different ways in which the textbooks present the properties of limits: some of them consider separately the arithmetic of finite limits at a point, of finite limits at the infinity, of infinity limits at a point and of infinity limits at infinity; other ones consider only the difference between finite and infinity limits; there are textbooks which study the limits of families of basic functions, and consider them like theorems... This diversity in what a textbook considers as a result affects the amount of results presented in textbooks. Regarding the kind of proof schemes used, we notice that there is a significant change after 1999: the amount of results is smaller than in the previous years, but the number of justifications too. The predominant behaviour in all textbooks is not to justify the results (77.7% of results are classified as PS0). Among the different categories of justifications, it depends on the textbooks: LGE 10 (1980) only uses transformational and axiomatic PS; the textbooks of the older collection of LOGSE use all kind of justifications, but the predominant PS are inductive PS of 1 case in grade 11 (16%) and transformational PS in grade 12 (14.3%); on the contrary, the newer collection does not justify except two inductive PS of several cases in grade 11; finally, textbooks of LOE mainly use inductive PS of 1 case (18.2% in grade 11 and 16.7% in grade 12).

The results justified by textbooks are generally related to limits of any kind of functions (potential, polynomial, rational, logarithmic or exponential functions). Textbooks never justify the properties related to arithmetic of limits (except the addition in LGE 10 (1980)). Moreover, this one is the textbook which uses more axiomatic PS, although it is not the one which shows more results. We specify the results

Textbooks	*Proof schemes*
LGE 10 (1980)	TPS: equivalent infinitesimals, limit at zero of $f(x) = x/\sin(x)$ APS: uniqueness of limit, addition of finite limits at a point, limit at infinity of logarithmic function
LOGSE 11 (1998)	IPS1: limits of types $\to k/\to\infty$ and $\to k/\to 0$, limits at infinity of polynomial functions, limit of exponential functions IPSs: limits at infinity of $f(x) = x^n$, $g(x) = x^{-n}$, $h(x) = x^{1/n}$ TPS: limit at zero of $f(x) = x/\sin(x)$ PP: limits at infinity of rational functions
LOGSE 12 (1999)	IPS1: limit at infinity of $f(x) = x^2$+k, limits of exponential and logarithmic functions IPSs: limit at infinity of $f(x) = 1/x^n$ IsPS: limit at infinity of $f(x) = x^n$ TPS: limits at a point of rational functions ($\to 0/\to 0$), limit at zero of $f(x) = \sin(x)/x$, addition of infinities, equivalent infinities, equivalent infinitesimals, limits of $(1 + f(x))^{1/f(x)}$ and $f(x)^{g(x)}$ (when they contain indeterminate forms, $\to 1^{\to\infty}$) APS: limits at infinity of rational functions PP: limits at infinity of polynomial functions
LOGSE 11 (2003)	IPSs: limits at infinity of polynomial and exponential functions
LOE 11 (2008)	IPS1: equivalence of existence and equality of one-sided limits and the definition of limit, limits at infinity of rational functions
LOE 12 (2009)	IPS1: equivalence of existence and equality of one-sided limits and the definition of limit, limits at infinity of rational functions TPS: limit at zero of $f(x) = \sin(x)/x$

Table 3: Results related to the proof schemes used

which are proved in each textbook in Table 3. Due to the number of inductive PS found in textbooks, we think that authors try to convince students about the validity of the results from intuition and not from mathematical reasoning.

Regarding the functions of proofs, the kind of results that are justified and the kind of PS used let us think that the predominant function that is showed in textbooks is communication. We also consider the function of verification in the justifications classified like axiomatic or transformational PS. The function of explanation could be noticed in some textbooks which select examples that contribute to explain why the result is true, but we do not notice this function especially in proofs but in the application examples of theorems. Indeed, this function appears in results related to calculus of limits (indeterminate forms or the arithmetic of limits).

FINAL REMARKS AND QUESTIONS FOR FUTURE RESEARCH

As we have seen, there is not a significant fall of the number of mathematical proofs in textbooks since the 70's in this publisher. That is because there are only three axiomatic PS and two transformational PS in the first book and there are four axiomatic PS and eleven transformational PS in all of them. However, we have noticed that the number of justifications have dropped along the time. It can be due to diversity in how textbooks deal with limits: for example, the newer collection of the LOGSE period does not show formal definitions, so they do not justify. Textbooks make few references to the justifications procedures used, only in some cases they specify that they are justifying. We think that more justifications must be included in textbooks and it should be indicated that this is a justification and what kind it is. We conjecture that all textbooks (except LGE 10, 1980) have the intention of teaching students to calculate limits but they are not so interested in the understanding of the concept. We will go deeply in this aspect in future research.

The analysis shown in this paper gives a first insight in this topic, but there are a lot of open questions. For example, we have said that the differences of justifications founded in textbooks probably depend on the way that these textbooks introduce the concept of limit, so it is necessary to study how it affects to proofs. Other future questions to answer are: Which is the best way (from a didactical point of view) to present and develop this concept? Is it better to consider intuitive definitions of the concept than a formal one? We think that a good way to present limits is to adopt a rigorous (but not formal) definition of limit in the sense defined by Blázquez, Gatica & Ortega (2007) and then, introduce the formal definitions. An example of these authors' definition is: *the limit of a function f(x) at a point* a *is* L *if, for every approximation* K *of* L, K≠L, *there exist a punctured neighborhood of* a *such that the images of all its points are closer to* L *than* K. We also recommend using preformal proofs, as a way to introduce formal reasoning, using specific functions that allow students to understand the mathematical reasoning without abstraction. This kind of reasoning is also preferred by students as it is claimed in González (2012). Finally, this case study is not enough to know how textbooks deal with the concept of limit and the justifications, but it let us realize of the difficulties of this kind of analysis (for example, how to compare among proofs when results are organizing or formulated in different ways), and that will help us in our future work.

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Argumentation below expectation: A double-threefold Habermas explanation

Jenny Christine Cramer

University of Bremen, Bremen, Germany, cramerj@math.uni-bremen.de

Teaching mathematical argumentation is a challenging task, and how to teach argumentation to students from all backgrounds remains an open question. It is especially difficult to say why some situations evoke the vivid exchange of arguments, while other situations completely fail to engage students in argumentation. This paper points out how two related concepts from Habermas' theory of communicative action may help to explore possible barriers and obstacles to argumentation: The rules of discourse ethics for argumentation processes, procedures and products and the consideration of rationality from the epistemic, the teleological and the communicational perspective. Both approaches help to recognize that mathematical argumentation requires more than mathematical content knowledge.

Keywords: Argumentation, Habermas, rationality, discourse ethics.

INTRODUCTION

Argumentation is highly relevant for the learning of mathematics, and may even be seen as a prerequisite for learning (Krummheuer, 1995). Besides argumentative learning, Knipping and Reid (2015) point to learning argumentation as an important research interest. In order to learn argumentation and proving, Boero states that (2011, p. 120, italics in original) "the best didactical choice is to exploit *suitable mathematical activities* of argumentation and proof". Learning argumentation is a good starting point in school, because as Douek (1999) highlights: if an emphasis is put on proving this can be a restraint to argumentation. But beyond the question of how to find suitable mathematical activities for fostering mathematical argumentation there remains the open question of how to ensure that all students may participate in argumentation, regardless of their socioeconomic or linguistic background. In the light of research results for problem solving and modelling activities, Knipping (2012, p.1) warns that: "classroom argumentation could become a social filter" if this question does not remain in focus. Gaining a better understanding of argumentation in general is a good first step towards identifying possible filtering effects.

Knipping and Reid (2015) point out that analysing argumentation in the mathematics classroom is important in order to gain a better understanding of its characteristics. Situations in which argumentation remains below expectation in that it either does not happen at all or not in the way anticipated by the teacher are equally interesting to consider if we want to shed a light on the difficulties argumentation entails for students. In this paper I consider a lesson from my research work in which argumentation did not happen in the intended way. Two related parts of Habermas' theory of communicative action are used to identify obstacles and barriers to argumentation: Habermas' view on argumentation as a process, procedure and product from his work on discourse ethics as introduced in Cramer (2014a) and the threefold perspective on communicative, teleological and epistemic rationality as introduced to mathematics education by Boero (2006). In the first part of the analysis, I consider moments in which a student does not engage in argumentation. By means of Habermas' discourse ethics I analyse the situational conditions and present a possible explanation by identifying subjectively perceived barriers. In a second analysis I focus on rudimentary arguments, which either broke off or did not provide a contribution to the solution of the task. Communicative, teleological and epistemic aspects are considered to identify obstacles within the argumentation process. Benefits and limits of the double approach are discussed in the end.

THE IMPORTANCE OF (LEARNING) ARGUMENTATION

Proof lies at the heart of mathematics as an academic discipline, and argumentation is vitally important for the development of mathematical understanding. Krummheuer (1995) even considers participation in collective argumentation processes as the social precondition for learning. Argumentation, proof and their "complex, productive and unavoidable relationship" (Boero, 1999) have been central to mathematics education research for several decades. In my research, I adopt Habermas' (1981, p.38) definition of argumentation as a type of speech in which participants discuss controversial claims and support or criticise these by arguments containing reasons which are rationally connected to the claim. Proving processes lie within this definition of argumentation. Proof as the goal of classroom activity can be a restraint for the development of argumentation, according to Douek (1999). Therefore, argumentation in general is the focus of my research, including, but not limited to, deductive reasoning as the path towards proving.

The learning of mathematics is tightly interwoven with, if not even dependent on, student participation in argumentation processes. Taking this into consideration, the emphasis on mathematical arguments in NCTM and other national standards documents is a welcome development. It is, however, unclear how to achieve involvement of all students in argumentation. Lubienski (2000) has shown that children from lower socioeconomic backgrounds benefit less from the reform-based emphasis on problem solving. Concerning the achievement of children in high-stakes tests, Prediger and colleagues (2013) have pointed out that language proficiency is the main predictor for success. Knipping (2012) highlights the necessity of decontextualized language in argumentation. To avoid the looming filtering effect of argumentation she envisages that a thorough analysis of unsuccessful attempts at including argumentation in class could be helpful. In the following, I choose the term "barrier" for occasions in which argumentation did not develop, whereas "obstacle" is used when argumentation is begun but breaks off. I distinguish these two terms because they are different in nature. I will clarify the distinction by exploring two different threefold concepts from Habermas' theory of communicative action for the analysis of situations when argumentation remained below the teacher's expectations. By the analysis, I hope to better understand what makes access to argumentation difficult.

A TWOFOLD HABERMAS RESEARCH TOOL

The foundation for my analyses is laid in Habermas' theory of communicative action (1981) and related works. Within this framework, Habermas (1983) describes communicative action and strategic action as two opposed forms of social conduct. Whereas in strategic action the speaker's intention is to enforce a claim regardless of means, speakers who act communicatively seek consent by supporting their claim with suitable arguments. These arguments are exchanged in discourse. To foster vivid discussions and active argumentation in the mathematics classroom, discourse opportunities must be created.

Barriers arising from argumentation as process, procedure and product

Discourse is described by Habermas (1983) as a communicative situation shaped by argumentation. New knowledge is inferred from shared common knowledge. Habermas (1983) describes rules for argumentation from the interwoven perspectives of processes, procedures and products. These rules need to be subjectively fulfilled for speakers to engage in argumentation. In the following I present a translated adaptation of these rules for the mathematics classroom (cf. Cramer 2014a).

The first of the three interwoven perspectives Habermas presents is the view on argumentation as a *process*, traditionally considered by the science of *rhetoric*. Characterizing features of argumentation processes are the exclusion of force and the reliance on nothing but the best argument. The process rules are:

(R1) Everyone may participate in discussions.

(R2) The topics to be discussed are conjointly determined.

(R3) There are equal rights and no compulsion to participate in communication.

Argumentation can also be seen as a *procedure* of hypothetically checking claims by giving reasons, free from any immediate pressure to action. This view on argumentation is traditionally rooted in the science of *dialectics*. Its rules are:

(D1) A speaker is only allowed to claim what he or she believes to be true.

(D2) Shared knowledge may not be attacked without reasons.

Arguments are the *products* of argumentation. Their structure is governed by the rules of *logic* and Habermas gives three rules from this perspective:

(L1) No speaker may contradict himself.

(L2) Who uses a warrant in one situation must be willing to use it in analogous cases.

(L3) Expressions need to have shared meanings.

These three perspectives must not be regarded as separate entities. According to Habermas, they are all equally important preconditions for argumentation. For participation, all of the rules have to be subjectively fulfilled. Of course, perfectly equal positions and the conjoint determination of topics in school are virtually impossible to achieve. However, I have explored before (Cramer, 2014b) how the *subjective* fulfilment of these rules can be shown in the situation of a logical game in which argumentation was successfully evoked. The focus of this paper is on communicative situations where argumentation unexpectedly did not develop. My interpretations of the students' subjective interpretations of the situational preconditions are based on their contributions and the detailed analysis of the situation as a whole. Within these, I try to identify barriers resulting from the subjective non-fulfilment of the rules for argumentation processes, procedures or products, as Habermas claims that subjective fulfilment of these criteria is a prerequisite for engaging in argumentation.

Obstacles arising from epistemic, teleologic and communicative rationality

Boero (2006) introduced Habermas' threefold perspective on rationality into mathematics education research to account for students' rational behaviour in proving activities. As Boero and Planas (2014) remark, epistemic, teleological and communicative perspectives on rationality gradually became a toolkit for many different aspects of research in mathematics education. I use Habermas' elaborations on rationality to account for obstacles that students may encounter in argumentation processes, leading to breakdowns or failures in reaching the pursued goal.

According to Habermas (2009), a person's behaviour can be described as rational if that person can account for his or her behaviour from three different but interwoven perspectives. *Epistemic rationality* is concerned with the propositional structure of knowledge. In order to act rationally from the epistemic point of view, an interlocutor needs to be aware not only of whether he or she holds certain statements to be true, but also of the reasons he or she has to justify this belief. In discourse, epistemic rationality leads to the possibility of negotiating and transferring knowledge, as the speakers are not only capable of sharing their convictions and beliefs, but also their justifications. Rational action is furthermore characterized by the actors' conscious choices of strategies and tools to help them arrive at mutual or even shared understanding. This conscious choice of strategies according to the aim the actor is following is called *teleologic rationality*. It implies that there are reasons for the actor why he prefers one tool to another, and that these reasons justify the actor's belief that he can achieve his aim under certain preconditions. In mathematical argumentation, actions are usually expressed with words. Language serves as the link between knowledge, aims and the communicative situation. Language is not, however, rational in itself (Habermas, 1996). *Communicative rationality* is located within the use of speech in discourse to develop a common understanding. This uniting power of speech secures the continuity of shared knowledge and the frame within which all interlocutors can refer to it.

The practice of argumentation unifies and requires all three forms of rationality (Habermas, 2009, p. 17). Obstacles may be encountered on the epistemic, teleologic or communicative rationality layer. In contrast to the aforementioned barriers, these obstacles occur after a student has begun to engage in argumentation. They can cause the argumentation to break off, which makes them an interesting analysis approach.

GREY AND WHITE BOXES

The analyses in this paper are based on data from my dissertation project in which I worked as a teacher-researcher with a group of five 15-year-old girls, all of whom are non-native German speakers from different schools. The lesson took place one week into the project; it was the first content lesson after an introductory interview in the preceding week. Before the project, the girls knew neither me as their teacher, nor each other. In the lesson considered, four girls were present. They worked in pairs with very little teacher guidance in order to foster the creation of their own arguments. The following problem was given to the students:

> In a square of grey boxes you are supposed to place white boxes. The grey boxes must no longer touch afterwards. Example for 16 squares:

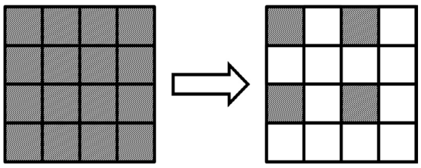

Figure 1

> How many white boxes do you need if the square consists of 36, 100, 1024 grey boxes?

The tasks allows for many different solutions (splitting the square into 2 × 2 squares with three white boxes each, recognizing that every second column and half of every second row are filled with white boxes, etc.). The desired outcome was that the students find a pattern that allows them to deduce that for even side lengths, 0.75 times the number of grey boxes is the solution. In the following, I describe episodes from Ayla and Jawahir's pair work process to illustrate how the two approaches from Habermas' work may help to identify barriers and obstacles. A combination of descriptions and transcript excerpts is used for clarity.

Getting started: argumentation barriers

After distribution of the problem, some introductory teacher comments and further clarification of the task, the girls start to work on the task individually and in silence. Only few comments are exchanged; both girls are visibly engaged in drawing and counting activities. Some minutes after Ayla instructed Jawahir to draw a 6x6 square for the 36 boxes task, they exchange results. Both arrived at 27 white squares as the correct answer by drawing and counting. Hardly any other verbal exchange takes place. Four minutes after their comparison and 17 minutes after the problem was given, the first longer conversation takes place. The transcript of this episode about the number of white boxes for 100 grey boxes is given in the following.

3	Ayla:	Did you just do one hundred? (...)
4	Jawahir:	I have six rows here, here.
5	Ayla:	But why six?
6	Jawahir:	I wanted five, I did f/, I did six. (..)
7	Ayla:	But look, (..) you did six here, didn't you? Do //four//
8	Jawahir:	//Mh// (affirmatory).
9	Ayla:	f/, no.
10	Jawahir:	It is supposed to be hundred, ehm
11	Ayla:	Yes yes but look, you can just do it like this (takes piece of paper) simply ten here (points in one direction with a pen) and ten here (points to another direction with a pen). Then you have hundred in here.
12	Jawahir:	(....) I hate maths.

In this exchange, Ayla inquires about Jawahir's approach to the task. She wants to know why Jawahir chose six as the side length of her "square". Jawahir does not respond with an argument for her approach, but states that she had originally planned to work with a side length of five (6). Ayla questions her approach by pointing to the preceding task where Jawahir had used six as the side length (7). It is not clear if she names four as the side length of the example on the worksheet or if she wants to tell Jawahir to enlarge the number of rows and columns by four. Jawahir responds by recalling the task (10). Finally, Ayla takes over and gives the argument that a ten by ten square contains 100 grey boxes (11). Jawahir neither reacts to justify her solution, nor does she articulate agreement with Ayla's justification. Instead, after a short pause, she claims to hate maths (12). Two minutes after this exchange, Jawahir says, "I am stupid" in the middle of her work.

In the situation at hand, Ayla is trying to enter into a social process of argumentation with Jawahir, while Jawahir does not enter into the discourse. Regarding Habermas' preconditions we can deduce from Ayla's behaviour that she does not meet any barriers that prevent her from requiring or giving arguments. She justifies her choice of ten as the side length of the square, she questions Jawahir's approach to the task and substantiates her doubts by referring to the former task where six already served as the side length for the 36 boxes square (7). Jawahir, on the other hand, does not justify her solutions nor does she ask for reasons when Ayla presents her approach.

In the following, I speculate on possible barriers for Jawahir's participation in argumentation. From the *process* perspective on argumentation, there are several possible subjective limiting factors. Jawahir does not participate in the discussion. It is therefore possible that she does not feel welcome to contribute (R1). Her statement that she "hates math" makes mathematical content an unlikely choice of discussion topic for her (R2). Furthermore, her claim "I am stupid" suggests that she does not perceive herself as an equal discussion partner (R3). The same claim also shows a lack of confidence in her mathematical abilities. It is well possible that this presents a barrier inherent in the *procedure* precondition that speakers are only allowed to claim what they believe to be true (D1). No knowledge is exchanged, so nothing can be deduced about shared knowledge (D2). As no arguments are produced, the barriers arising from the *product* perspective can hardly be assessed. Speculations are that mathematical insecurity might lead to self-contradiction (L1), deficiencies in the knowledge of mathematical structures and concepts may lead to problems with identifying analogous situations where warrants can be transferred (L2), and a lack of conceptual knowledge could lead to differing usages of expressions (L3).

Although it is difficult to nail down exactly what it is that kept her from entering into discourse, it is clear that Jawahir in contrast to Ayla did not engage in argumentation in the situation at hand. A lack of mathematical content knowledge probably contributes to the problem, but self-perception seems to be equally relevant. A deficiency in deductive reasoning skills can be excluded as the cause of her silence, as Jawahir produced highly sophisticated arguments in other situations (cf. Cramer 2014b). Within the complex interplay of situational constraints, Habermas' discourse ethics rules provide a tool that can account for some barriers for participation.

Spontaneous breakdowns: argumentation obstacles.
Several minutes later, the girls are still engaged in drawing boxes and counting. No exchange about patterns or generalities has been observed so far. The teacher directs the attention towards the side length of the 1024 square. Following some hints, Ayla finds out that the solution is connected to the calculation of square roots. She is used to working with a table of square roots from her text book (she calls it "clever book"). After she has looked up the square roots of 36 and 100, this exchange takes place:

40	Teacher:	So why does that match the side lengths?
41	Ayla:	Because it is the same, isn't it, this way (hand from bottom to top) and this way (hand from left to right).

The table in her book only contains numbers below 1000, so the teacher uses a calculator and shows 32, the square root of 1024 to Ayla and Jawahir. Afterwards, Jawahir starts conversation by asking the teacher for a way to solve the task:

60	Jawahir:	How does the thousand work, I mean this (points at worksheet) here?
61	Teacher:	Think about that together. (...) Ayla already found the side length of that thing. (..) And maybe you see something HERE (points to the worksheet) that you can (.) carry on (.) somehow. Something this and this here (.) have in common. If you look at them all next to each other, this and this and this.
62	Ayla:	Yes, there always is one, and then not, and then one, and then not.
63	Teacher:	Yes, exactly. And then there for example is a row where there is nothing.
64	Ayla:	Mhm. (Affirmatory)
65	Teacher:	And that is actually kind of the same everywhere. And maybe you find something general (.) how you can find it out WITHOUT COUNTING.

The teacher leaves the table. 50 seconds later, Ayla starts talking:

66	Ayla:	So if thirty (..) times thirty is nine hundred (..) nine hundred thirty, (..) nine hundred thirty, thirt/ (5 sec), nine hundred sixty. It has to be nine hundred sixty. See (.) write that down. (..) I will now draw these boxes, if nine hundred sixty comes out I was (..) right.

After this monologue, Ayla and Jawahir start to create a 32x32 square by gluing together various pieces of paper. Some interesting comments during this activity:

67 Ayla: Wow, how many boxes ARE THERE? (..) I think this (.) is this one thousand-thingy, isn't it, isn't it, isn't it?

Due to the limited time, their drawing remains incomplete. During the whole-group comparison, Ayla presents 960 as their solution. She justifies her answer as follows:

81 Ayla: We first (.) calculated the square root with you. And then ehm, we arrived at thirty-two. Then we tried (.) to do thirty-two boxes, top and bottom. We didn't manage to. Yes, and we got to nine hundred sixty, because we calculated it.
82 Teacher: So what did you calculate to get to the nine hundred sixty?
83 Ayla: Yes well, because the book said ehm (.) thirty by thirty is nine hundred, and then we calculated a bit.

This longer episode is very interesting from the perspective of rationality. From the *epistemic* perspective, it is observable that Ayla has understood the concept of square root as possibility to calculate the side length of a square (41). She does, however, appear surprised at discovering the huge number of boxes in the 32x32 square they create (67). A possible explanation for her surprise is that she does not see squaring as a reverse operation to calculating the square root. This can be seen an epistemic obstacle. It leads both to difficulties in her calculation attempt for 32 squared (66) and to her disbelief faced with the enormous number of boxes in the 32 by 32 square.

Her abovementioned lack of understanding square roots can also be considered a *teleologic* obstacle, as it goes hand in hand with her strategy of multiplying seemingly random numbers. Ayla multiplies 30, the highest number in her multiplication table, by 32, the result for the square root of 1024 given by the calculator (66). Even when asked (82), Ayla does not give any reasons for the appropriateness of her strategy. Ayla's identification of the pattern (62) sheds light on another obstacle in her teleologic rationality. Although she recognizes an overarching structure for the differently sized squares, she cannot transfer this insight into a useable tool towards finding a mathematical solution. Finally, Ayla's justifications "because the book said" and "we calculated a bit" (83) show obstacles in her *communicative* rationality. Ayla does not seem to know how to justify her solutions in a mathematically acceptable manner. Furthermore, she does not or cannot use algebra as a tool to represent the pattern she discovered (62), "one, and then not, and then one, and then not".

Possible obstacles can be identified on all three levels of rationality and provide a possible explanation for why Ayla did not arrive at a mathematically correct solution to the task despite her willingness to engage in argumentation. Identifying these obstacles for argumentation can however provide teachers with a tool to work on better support for their students. In Ayla's case, working on calculating square roots and squaring as reversing each other could be beneficial, but also a fostering of using algebra to express patterns might prove helpful.

OUTLOOK AND DISCUSSION

In this paper, I have elaborated on two different but related concepts from Habermas' theory of communicative action that can help to cast a new light on barriers and obstacles for argumentation. The subjective fulfilment of Habermas' rules for argumentation as process, procedure and product can help to identify barriers leading to non-participation in argumentation. The threefold perspective on epistemic, teleologic and communicative rationality on the other hand provides a tool for the detailed analysis of obstacles leading to unsuccessful argumentation.

Both approaches are united in that they take a perspective that simultaneously highlights and limits the importance of mathematical content knowledge for argumentation. A lack of knowledge can be both a barrier to engaging in, and an obstacle for being successful at argumentation. However, among other influences, the social situation, feelings of inequality, availability of suitable strategies and knowledge about communicative practices are equally important to consider if we want all students to engage in argumentation, especially in the light of looming effects of social and linguistic disparities. Habermas' considerations underline the importance of perceived equality, shared meanings and the absence of force for engaging in argumentation. He also shows that the awareness of justifications, suitable tools and means of communication is essential both to start and to continue argumentation. More research on barriers and obstacles for argumentation may help to finally overcome them and enable participation in argumentation to all students.

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Is the use of GeoGebra advantageous in the process of argumentation?

Özlem Erkek and Mine Işıksal-Bostan

Middle East Technical University in Ankara, Faculty of Education, Ankara, Turkey, ozdursun@metu.edu.tr

The purpose of this study was to investigate the advantages and disadvantages of using GeoGebra in an argumentative application. For this purpose, the data were collected from eight prospective elementary mathematics teachers via video recordings, reflection papers and interviews. Data were analyzed through qualitative data analysis methods. According to the results, making accurate drawings, creating confident justifications based on exact measurements, dragging objects to see relationships and saving time were the main advantages of GeoGebra in argumentation. On the other hand, not attending class discussion and not reasoning the relationships after measuring with GeoGebra were the disadvantages for argumentation.

Keywords: Argumentation, GeoGebra, geometry, advantageous, disadvantageous.

INTRODUCTION

Argumentation is defined as "a process of establishing or validating a conclusion on the basis of reasons or the act of proposing, supporting, evaluating and refining the process, context, or products of an inquiry (Sampson & Clark, 2011, p. 66). Toulmin (1958), the pioneer of this multidisciplinary trend in literature, proposed the Argumentation Model, which is used to determine arguments in numerous scientific studies. In this model, an argument comprises three elements which are claim, data and warrant (Toulmin, 1958). Claim is a statement to be supported. The other element data corresponds to evidence presented for supporting the claim. Warrant is an inference rule enabling data to be connected to the claim. In addition to these elements, Toulmin (1958) also asserted that qualifier, rebuttal and backing may also be needed to describe an argument. While qualifier refers to the strength of the argument, the rebuttal expresses a counter-argument. Backing corresponds to additional support for the warrant of an argument. Although, Inglis, Mejia-Ramos and Simpson (2007) asserted that the full version of the Toulmin model need to be used in mathematics, backings, modal qualifiers and rebuttals are less often used in the studies related to mathematics education corresponding to the nature of mathematics. Toulmin's (1958) argumentation scheme was represented in the Figure 1.

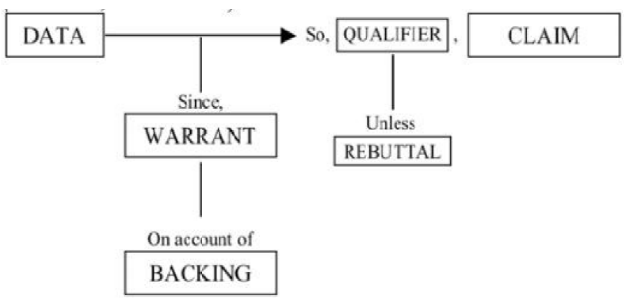


Figure 1: Toulmin's (1958) Model of Argumentation (p. 104)

The importance of argumentation has increased gradually in learning and education and many researchers implied that the students should participate in discussions for mathematics learning (Krummheuer, 1995; Stein, Engle, Smith, & Hughes, 2008). Inglis and colleagues (2007) pointed out that the first researcher who integrated Toulmin's argumentation scheme into the mathematics field was Krummheuer (1995) who analyzed classroom-based mathematical arguments. Krummheuer (1995) used the term collective argumentation and defined it as "a social phenomenon when cooperating individuals tried to adjust their intentions and interpretations by verbally presenting the rationale of their actions" (p. 229). After this first attempt, the number of studies related to the applications of argumentation into mathematics has increased (Inglis et al., 2007; Pedemonte, 2007; Wentworth, 2009). For instance, Wentworth (2009)'s study include technology allowing online synchronous and asynchronous discussion to explore argumentation of prospective teachers and secondary school students. The results indicated that it is difficult for participants to write all their arguments and it is difficult to detect and follow their reasoning process only from their writings (Wentworth, 2009). In the same way, mathematics educators are aware of the fact that logical inferences and formal justifications are not always convincing and comprehensible so they propose that additional informal justification ways, like dynamic geometry tools, are necessary (Prusak, Hershkowitz, & Schwarz, 2012). Thus, mathematics educators have developed dynamic geometry tools allowing inquiry-based environment and encourage argumentation in geometry (Prusak et al., 2012). Furthermore, researchers emphasized the importance of dragging in conjecturing in many studies (Arzarello, Olivero, Paolo, & Robutti, 2002; Baccaglini-Frank & Mariotti, 2010). For instance, Arzarello and colleagues (2002) stated that dragging encourages conjecturing and exploring since individuals have the chance of observing the invariant properties after changing the shapes. Obtaining immediate feedbacks was claimed to be helpful for discovering and proving invariant properties of drawings (Arzarello et al., 2002). Dynamic geometry software allows students to construct and experiment with geometrical objects to make conjectures and interpretations (Healy & Hoyles, 2001). Since conjecturing is a crucial action in argumentation, it is worth to study the advantages and disadvantages of dynamic geometry program (GeoGebra) in argumentation process. It is claimed that it is not only the technology that brings an educational change but also the teacher (Arzarello et al., 2002). However, few studies analyzed the prospective elementary mathematics teachers' argumentation in geometry. The prospective teachers' argumentation skills are important to be searched because they are the people who will facilitate and manage the whole class discussions, ask questions, listen to students who will develop mathematical understanding in the future (Stigler & Hiebert, 1999). Their own proficiency in developing arguments has also great importance for their future performances in argumentation applications. Therefore, the investigation of the argumentation process with dynamic geometry program, GeoGebra, will be beneficial to determine the critical issues to be considered by prospective teachers.

This study is significant since the results would provide insight to prospective teachers and teachers who want to integrate GeoGebra and argumentation into their geometry lessons. In addition, the results would provide a clear picture of possible benefits and problems that prospective elementary mathematics teachers may confront while experiencing argumentation with GeoGebra. Based on the rationale mentioned above, the purpose of the present study was to investigate advantages and disadvantages of using GeoGebra in argumentation while solving geometry tasks. The following research question has guided the study: What are the advantages and disadvantages of using GeoGebra in argumentation process?

METHODOLOGY

This study is a qualitative case study. Purposeful sampling was used to identify the participants. The researchers selected 8 prospective teachers enrolled in Elementary Mathematics Education program of a public university in Ankara, Turkey. All participants were 4th grade students who have completed almost all courses related to elementary mathematics education program. The four participants in GeoGebra group were voluntary students who have taken the elective course called "Exploring geometry with dynamic geometry applications". In this way, there is no need to teach them how to use GeoGebra program. The other four participants in Paper-pencil group were the communicative students who have not taken the course. The application was done in one session for each group and it lasted 2 hours to solve three geometry tasks.

To triangulate data, we utilized multiple data sources. Data collection took place in the spring semester of 2012–2013 education year through video and audio recordings of the two applications, reflection papers and interviews. Interviews were also recorded with a camera.

The researcher who was the participant observer in the classroom settings started the application by giving some information about argumentation in each group. The purpose of this was to give an idea to the participants about the application process and to encourage them to justify their claims as much as possible. Also, the researcher asked them to think aloud in pair-work to share their ideas with their partners. Moreover, the researcher asked participants to concentrate on the discussion on the board in order to encourage all students' contribution to the classroom discussion. There were 3 geometry tasks in the application. The first geometry task was like a warm-up

Geometry Task 1

1. A recreational park has a rectangular shape. At each vertex of the rectangular park there is an attraction. The manager of the park decided to locate the ticket booth at an equal distance from the four attractions. Find the point in the rectangle which fits for the ticket booth.
2. Another park has the shape of an equilateral triangle. At each vertex of this park there is an attraction. The manager of the park decided to locate the ticket booth at an equal distance from the three attractions. Find the point in the triangle which fits for the ticket booth.
3. What if the recreational park has the shape of a scalene triangle? Find the point (if any) in this triangle which fits for the ticket booth.

Geometry Task 2

In any ABC triangle D and E points are midpoints of the sides |AB| and |AC|. F and G points are placed on side |BC| so as to be |BG|=|CF|. |DG| and |EF| intersects at point H.

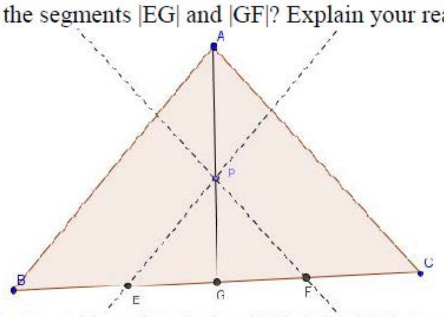
When does |AH| become angle bisector of the angle A? (Think about all triangle types)

Explain your reasoning and justify your solutions.

Geometry Task 3

Let P be any point on the median of |AG| of a triangle ABC. Let m and n parallel lines through P to the sides |AB| and |AC| of the triangle.

1. What relation is there between the segments |EG| and |GF|? Explain your reasoning.
2. What if the triangle ABC is equilateral or isosceles triangle? Can any generalization be made for the relation between the segments |EG| and |GF|? Explain your reasoning.



3. Where must be the point P positioned such that |BE|=|EF|=|FC|. Justify your solution.

Figure 2: Geometry tasks

task which was discussed in pairs and then the class discussion began. After getting all different solutions, the researcher moved to the second task's pair discussion. In the same way, the third task was discussed in pairs and later with the whole class. The researcher reworded the claims of students in order to provide a collective argumentation environment and to make it clear for other participants. In addition, the researcher asked probing questions such as "Why do you think so? Are you sure about that? What is your justification for that? Do you all agree with your friend's claim?". That is, the researcher motivated students to justify why a conjecture is true or false and convince others. The subsequent class discussions focused on student reasoning and argumentation.

In GeoGebra group there were two pairs of students and each pair had one computer and one worksheet to study together. In Paper-pencil group there were two pairs of students and they had a ruler, a compass and a protractor to be able to draw accurate shapes and one worksheet to study collaboratively. After the participants solved the tasks in each group, one voluntary pair was interviewed; the other pair wrote a reflection paper in order to collect more information about their experiences regarding the benefits and the drawbacks of using GeoGebra in argumentation.

For the data analysis, the recorded dialogues of the participants were transcribed. Toulmin (1958)'s Argumentation Model was used to determine the arguments of the participants. By using the Toulmin's (1958) predefined argument elements, the researcher had the chance of organizing the arguments of the GeoGebra and the Paper-pencil groups in such a way that they are comparable because they discussed the same geometry tasks. Two experts assisted with the researcher determining the argument schemas from video recordings. An inter-rater reliability agreement was 90%. In addition, the reflection papers and interviews were also open coded for triangulation of the data to detect advantages and disadvantages of using GeoGebra in argumentation.

Data collection tools

Three geometry tasks about triangles (see Figure 2) were selected for the argumentation application based on the criteria such that the tasks should be suitable for the discussion and argumentation, have alternative solutions, and be able to be solved via GeoGebra and paper-pencil. After getting permissions from the authors, the tasks were translated into Turkish. Expert opinions were taken for the statements to be clear for the participants. The first task was about the circumcircle and taken from the study of Prusak and colleagues (2012, p. 28). The second task was prepared by Ceylan (2012) and it was adapted by the researcher to be able to be suitable for argumentation. The third task was prepared by Domenech (2009) for high school students.

The researcher prepared interview questions for the voluntary pair just after watching the video of the application. The day after the implementations the interviews were conducted. If the participants did not remember some details about their arguments, the video record of that part was shown to make them remember what they did in the application. The aims of implementing interview were to obtain the ideas of participants about the advantages and disadvantages of GeoGebra while solving geometry tasks through argumentation and to clarify their arguments. Some components of arguments may be missing in discussion when the discussion flows to different directions. In some situations, the participants may not explain their reasoning but imply it somehow. The researchers asked those parts to the participants through interviews in order to make correct interpretations.

RESULTS AND DISCUSSION

After the applications, the schemas of all the arguments of the participants were drawn and then the arguments of GeoGebra group and Paper-pencil group were compared in terms of the number of arguments, the existence of argument elements and the contents of argument elements. Considering Inglis and colleagues' (2007) suggestion the elements of the full model of Toulmin were searched but it was detected that few modal qualifiers and rebuttals and no backings were expressed by the participants. In addition to this, some arguments were generated by more than one student during the class discussion. That is, claim was stated by one student while the warrant of that claim was stated by another student. By comparing these arguments, the researcher concluded that GeoGebra was both advantageous and disadvantageous in argumentation depending on the educational goals of teachers and the nature of tasks.

Advantages of using geogebra in argumentation

One of the advantages was that the participants in GeoGebra group were able to made accurate and dynamic drawings for the tasks. In this way, they could see the relationships easily, became more confident in their arguments, and went on forming justifications and inferences. However, in Paper-pencil group, they were not sure about their conjectures or made wrong conjectures because of their inaccurate drawings. The conversation and argument schema below in paper-pencil group exemplifies such a situation.

Teacher: How can we find the circumcenter of any triangle?

S2: We could not remember the rule so we could not determine the lines which intersect and generate circumcenter.

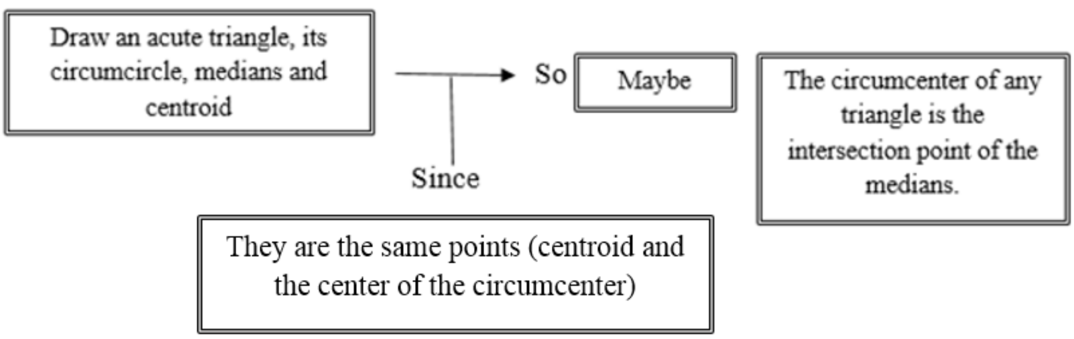


Figure 3: A Sample Argument Schema for the conversation

Maybe it can be the intersection point of medians. Let me try it.

(S2 drew an acute triangle, a circumcircle and medians of the triangle on the board and saw that the centroid approximately seems to be the center of the circumcircle)

S2: Oh they are the same points. I think we can find circumcenter by intersecting medians of a triangle.

In the conversation above, S2 concluded that the circumcenter of a triangle may be the intersection point of the medians, centroid. This was a wrong argument and the reasons for the mistake were approximate drawing and not thinking all triangle types. S2 thought that centroid and circumcenter are the same point but in fact they were very close to each other in an acute triangle that she draw. The other students were also satisfied with the drawing and they did not discuss the solution further. Then the instructor asked whether that was valid for an obtuse triangle or not? This question helped participants to think other triangle types. However, in GeoGebra group, the students drew the medians and found the centroid and compared it with the circumcenter in different triangle types. In this way they did not generate wrong arguments. Therefore, they formed further arguments and found different solutions.

Another advantage of the use of GeoGebra in argumentation can be the dragging option which enables maintaining some geometrical properties of a figure and exploring the relationships. The students in GeoGebra group used dragging option effectively and made generalizations easily which support conceptual understanding. Moreover, they were more motivated to discuss and to find alternative solutions to the problem. For instance, when they saw any relationship in an equilateral triangle, they could check whether the relationship was valid for other triangles by dragging the vertices of the triangle and changing its properties. However, in Paper-pencil group, participants did not have such an opportunity. They got stuck while they were drawing the new shape again and again on the paper with materials such as ruler, protractor and compass. Therefore, the participants had difficulties in focusing on the relationships between the unchanging properties of different shapes, because of repetitive drawings and their motivation for discussion decreased. For instance, in GeoGebra group while participants were discussing the geometry task 2, they noticed that |AH| is angle bisector when |DE| is perpendicular to |AH| (see Figure 4). In addition, they investigated that |AH| is angle bisector when ADHE is deltoid. Then they analyzed whether |AH| is angle bisector when ADHE is any other type of quadrilateral. They used dragging option and see the relationships in a short time. In the end, they concluded that |AH| is angle bisector when ADHE is deltoid and rhombus. However, in Paper-pencil group, participants concluded that |AH| is angle bisector when ADHE is only deltoid since they could not move F and G points to obtain different quadrilaterals on the paper.

It can be said that dragging option in GeoGebra was beneficial in argumentation because the participants could see the relations that they could not see on fixed shapes drawn on the paper and they justified willingly

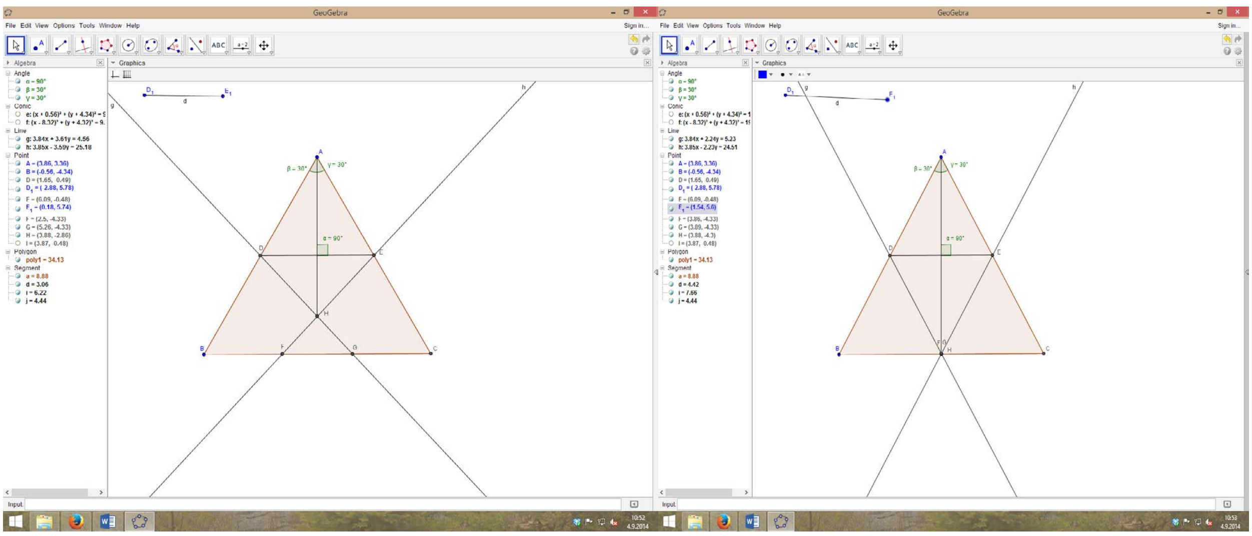


Figure 4: A drawing for task 2 in GeoGebra group class discussion

their reasoning based on the properties of the different shapes composed by dragging.

Based on the findings, the other advantage of GeoGebra is time issue. The participants drew shapes easily and quickly with GeoGebra options such as measuring angle, measuring side, drawing circle, drawing perpendicular or parallel line etc. In GeoGebra group, the participants did not spend too much time on drawings but they used most of their time for reasoning and argumentation. However, it was time consuming for Paper-pencil group to measure angle, to draw angle bisector, to draw circle with compass… etc. They sometimes had to draw the shape again because of wrong conjectures and the argumentation paused very often. In a similar situation, the GeoGebra group members had the chance of going back on the screen with "Ctrl+Z" and continue to draw on the same shape. Thus, GeoGebra was useful in order to save time and to retain the argumentation.

Dynamic geometry software were developed in order to engage inquiry based strategies and argumentation in geometry (Prusak et al., 2012). In argumentation, the participation of all students is crucial because all students should evaluate others' arguments and contribute to their solutions. At the same time, they should convince others by justifying their claims. As for the advantages, Geogebra allowed the participants to use dragging option, to construct accurate drawings such as segments, angles, angle bisectors, parallel lines, perpendicular lines. In line with the results of Prusak and colleagues' (2012) study, dragging option gave students the opportunity to see relationships visually (Healy & Hoyles, 2001) and check whether their conjectures are valid. Thus, they could obtain immediate feedback and enable rethinking and making new conjectures and geometrical analysis to solve problems (Prusak et al., 2012) which promoted their argumentation. In this way, the reasoning process of participants could be followed by the instructor clearly with the help of their arguments in classroom interactions and their products with GeoGebra. Moreover, GeoGebra was time saving for drawings and motivating for students to discuss the task. When the teachers do not want to spend much time for drawing shapes but to promote reasoning on the task, they can integrate Geogebra into their lessons so that students will draw the accurate shapes quickly and discuss on the properties of the shapes more.

Disadvantages of using GeoGebra in argumentation

The results indicated that there were some disadvantages of the use of GeoGebra in argumentation. One of them was that all students were not at the same pace in using GeoGebra and interpreting the drawings. Therefore, it was difficult to control all participants and to make them participate in the class discussion. When one of the students came to the board to show her solution, some of the students who were thinking on their own drawings did not follow the discussion and did not make comments for the solution. However, all participants should have followed and contributed to the arguments of others in collective argumentation process in order to reach a conclusion. When all students could not follow the discussion, there may be some missing points in their understanding. In the collective argumentation, the arguments are developed through two or more students' participation (Krummheuer, 1995). In order to critically make sense of the arguments developed, the interaction with other students was stated to be crucial in collective argumentation (Yackel, 2002). Thus, teachers should promote student interactions by rewording their claims and asking probing "Why" questions during the argumentation.

Another disadvantage was that the task may not be compatible with the GeoGebra usage so students who obtain feedbacks from GeoGebra immediately may generalize easily without reasoning. De Villiers (2003) asserted continuous transformations of objects with dynamic geometry software made students convinced easily with general validity of a conjecture. For instance, in the second task, they said that "I measured with GeoGebra and dragged point A to check for other triangles. The lengths are equal for all triangles." In such a situation the teacher asked the reason why these lengths are equal and made them justify their solution. In the class discussion, the teacher directed participants to think about the reasons for this equality to obtain their justifications. However, in Paper-pencil group, participants started to reason whenever they see the worksheet and their discussion was more productive and they formed more arguments. This is because the geometry task 2 was not suitable for GeoGebra use since the measurement tool in GeoGebra program stopped the argumentation of the participants. It is stated that conjecturing and justifying has a great importance in argumentation (AEC, 1991). However, the tasks should be suitable for conjecturing and the use of Geogebra to find different solutions. Otherwise, the participants will solve the task on their worksheet and use GeoGebra only for checking the solution by dragging or measuring. That is, the tasks that can be solved with some measurements via GeoGebra may terminate the argumentation. This may not promote their argumentation and they may not justify their claims so that they do not state warrants. When the geometry task requires conjecturing and dragging to recognize the relationships and teacher needs to hear students' justifications to decide whether they conceptually understand the topic or not, the GeoGebra might be advantageous for students. Otherwise, the students might not think critically on the task so the argumentation might not be effective. Thus, it was suggested that the teachers should select GeoGebra tasks carefully and determine whether Geogebra should be used or not in their argumentation based geometry lessons considering the objectives of the lessons and the needs of their students.

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Constructing validity in classroom conversations

Manuel Goizueta[1] and Maria Alessandra Mariotti[2]

1 Universitat Autònoma de Barcelona, Barcelona, Spain, mgoizueta@gmail.com

2 Università degli Studi di Siena, Siena, Italy, mariotti21@unisi.it

We account for different strategies used by a group of students to talk about and assess the validity of mathematical models while working in a problem-solving task. Two main competing strategies are described, one centred in ritualized uses of well-known mathematical constructs as a means to cope with perceived didactical expectations and a second strategy centred in the assessment of the representativeness of mathematical models when accounting for the proposed empirical situation. The interactions analyzed exemplify the difficulties students and teachers experience when dealing with epistemological aspects of knowledge being constructed in classroom conversations. Our findings point to the need for research to focus on epistemological aspects of the mathematical culture of the classroom.

Keywords: Validity, argumentation, mathematics classroom epistemology, classroom culture.

INTRODUCTION

It is widely accepted that argumentative competencies should be developed within the mathematical activity in the classroom, both as a product of this activity and as a means to support it. We share Boero's (2011) idea that a 'culture of argumentation' is to be developed in the classroom and that it should include practices on the production of conjectures, meta-mathematical knowledge about the acceptability of references advanced for the validation (acceptance/rejection) of claims and knowledge about the role of counter-examples and generality. It should include elements for evaluation of mathematical productions and general ideas about the use of all this knowledge within argumentative practices, along with the needed awareness to allow deliberate and autonomous control of the process.

In order to promote such a culture, the processes by which students actually construct and validate mathematical knowledge in classroom activities, as well as the meta-mathematical knowledge underpinning this processes, should be better understood. This report is part of a series in which we argue that it is worth characterizing this processes in terms of validity conditions emergence and fulfilment in order to account for the epistemological dimension of classroom interaction and its socio-interactive roots (see Goizueta, Mariotti, & Planas, 2014). Our main interest is to investigate the epistemological basis of argumentative practices in the mathematics classroom and, particularly, how is validity interactively negotiated in rich problem-solving mathematical activities. In this occasion, we compare different strategies for validation of mathematical models observed in a secondary mathematics classroom. We account for the epistemological complexity of these strategies and explain, from this point of view, the difficulties students face to discuss the validity of mathematical models.

THEORETICAL FRAMEWORK

According to Habermas (1998), accepting a validity claim is tantamount to accepting that its legitimacy may be adequately justified, that is, that conditions for validity may be fulfilled. Validity conditions are not restricted to absolute standards; instead, they are contingent constraints that emerge to accommodate elements considered suitable for validity appraisal in a given context (Goizueta, Mariotti, & Planas, 2014). They embody what are intersubjectively considered good reasons in a particular context of justification. Thus, validity relates to acceptance based on contingent validity conditions fulfilment. This implies that validity is not a property of claims themselves, but emerges from the mode they are contextually dealt with. It should be stressed that whatever is considered as a good reason, must not necessarily be explicit or even stateable, nor the individuals must be aware of it in a conscious way, it might be enacted implicitly in successful social participation (Ernest, 1998).

According to Habermas' construct of rational behavior (Habermas, 1998) and its adaptation by Boero and colleagues (2010), in the students' argumentative practices we can distinguish an epistemic dimension (inherent in the epistemologically constrained construction and control of propositions, justifications and validations), a teleological dimension (inherent in the strategic decision-making processes embedded in the goal-oriented classroom environment) and a communicative dimension (inherent in the selection of suitable registers and semiotic means to communicate within the given mathematical culture). Such a distinction reveals useful to reconstruct the origin of specific validity conditions. For instance, focusing on the epistemological dimension and following Steinbring (2005), we assume that a "specific social epistemology of mathematical knowledge is constituted in classroom interaction and this assumption influences the possibilities and the manner of how to analyse and interpret mathematical communication" (p. 35). Within this socially constituted mathematics classroom epistemology a criterion of mathematical validity is interactively negotiated between the participants. A central consequence of these assumptions is the basic necessity for interpretative research to reconstruct the situated conditions in which (and from which) mathematical validity emerges as part of the interactive development of mathematical knowledge. Although 'conditions' might be considered in a broader sense, we are particularly interested in the epistemological assumptions at stake, the references (mathematical and not) that might be considered as relevant and the social environment in which the process is embedded.

In the context of the mathematics classroom, the general relation between classroom epistemology, mathematical activity and social environment must be considered under the light of a specific, content-related didactical contract (Brousseau, 1997). Mathematical acceptability of students' explanations is linked to the reciprocal expectations and obligations perceived within the didactical situation by the teacher and the students and to the mathematical contents at stake (or perceived as being at stake). Thus, when faced with a problem, the didactical contract may indicate relevant mathematical knowledge and references to the students, according to the proposed didactical situation. However, not all the emerging references are linkable to well established and intersubjectively shared mathematical knowledge. We might also need to consider other references (statements, visual and experimental evidence, physical constraints, etc.) that are not part of institutionalized corpora, such as scholar mathematics, but (nevertheless) are used de facto as taken-as-shared, unquestionable knowledge (Douek, 2007). This corpus of references might be tacitly and operatively used by students to make sense of the task, semantically ground their mathematical activity and back their arguments. Accounting for the reference corpus at stake might be particularly relevant when considering problem-solving settings in which empirical references are to be considered as part of the proposed milieu.

PARTICIPANTS, TASK AND DATA COLLECTION

The participants in our design experiment were thirty 14/15-year-old students and their teacher in two lessons in a secondary school mathematics classroom in Barcelona, Spain. It was a problem-solving setting, with time for small group work and whole-class discussion. The researchers suggested the following task:

Two players are flipping a coin in such a way that the first one wins a point with every head and the other wins a point with every tail. Each is betting €3 and they agree that the first to reach 8 points gets the €6. Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. How should they split the bet? Justify your answer.

As we ascertained in a pilot experiment, this task can be approached and solved using arithmetical tools, without having been taught formal probability contents, which was the case of this group. The teacher was explicitly asked to avoid showing approval or disapproval to the students' numerical answers and proposed models. Instead of hint-guiding the students towards a correct answer, she was asked to foster the emergence of competing models and their discussion. By avoiding directive guidance, teacher's interventions were meant to foster autonomous processes of validation within the students' mathematical activity. Models observed during the pilot were discussed with the teacher and numerical variations of the problem were designed in order to help her problematize these models when and if necessary.

We were aware of the high complexity of the problem as a modelling task. The construction by the students of a situation model (in the sense of Blum & Borromeo-

Ferri, 2009) is related to some well-known epistemological obstacles regarding probability-thinking development (García Cruz, 2000; Wilensky, 1997). This circumstance, along with reports on the use of this problem in the classroom (García Cruz, 2000), suggested it might be a good candidate to promote a rich argumentative environment and to foster the discussion of mathematical, but also meta-mathematical issues. The novelty of the task was expected to prevent students from using mechanical approaches based on well-established solving strategies. Different situation models were expected to emerge and, consequently, a variety of mathematization processes. The need to compare competing mathematical models was expected to foster the emergence of different arguments to validate them.

For data collection, three small groups were videotaped and written protocols were collected. The data were analysed and coded with the aid of qualitative data analysis software. By constant comparison of similarly interpreted situations and triangulation with other research team members' perspective, key aspects of the classroom mathematical work were inductively inferred. This process was iterated for analysis refinement and confirmation.

DATA ANALYSIS AND RESULTS

Constructing a situation model: Initial validity conditions

Whether explicitly or implicitly students bring to the discussion taken as shared references about the proposed empirical situation. In so doing, they often describe it as a 'random game' and use words associated with this notion in informal ways. An example is the following intervention by a student, Anna, during the first session:

Anna Obviously, because it's random, the game (...) A does have more chances of winning but B could win as well (...) From what we've got so far, A would have to get more money, because he's got more points.

These taken as shared references, a cluster of empirical information, beliefs and words to talk about the situation, conform part of the reference corpus the students resort to in order to semantically and empirically ground the task, understand the problem and conform (often in a tacit way) an associated situation model. Observed in all groups and illustrated by Anna, a validity condition emerges expressing the need for the winning player to get more money. Any possibly valid mathematical model, within which an acceptable answer can be constructed, should fulfil this requirement, so we categorize it as a validity condition. Students' choices also indicate adherence to common clauses of the didactical contract: they tend to disregard and discard solutions that appear spontaneously but are not considered mathematics-related (e.g., 'each one gets his money back because the game did not come to an end'). A common clause of the didactical contract constrains them to actually do some mathematics in order to solve the problem. The need for the solution to be mathematics-related acts as a validity condition that students take into account to decide on the model's validity.

Often, the first numerical answer students propose roots on models associated to typical school problems about proportional costs, which tend to be solved by manipulating the numerical data appearing in the wording. A ritualized way of presenting this solution may be paraphrased as: 'if by winning 8 points a player gets €6, for each point won a player should get €0.75'. The students quickly realize that the amounts of money distributed according to this model, namely €5,25 and €3,75, do not add up to six euros, what, according to them, falsifies the model. The necessity for 'adding up to six' emerges as a validity condition that any numerical answer, and the model within which it is constructed, should fulfil.

The emergence of these validity conditions (namely the need for the winning player to get more money, the need for the distributed amounts of money to add up to six euros and the need for the solution to be mathematics-related) was observed in all groups, as well as the resorting to them as means for validity appraisal. While the need for the solution to be mathematics-related derives from expectations related to the didactical contract, the other validity conditions relate to the need for any possible valid model to account for the empirical references considered by the students. This stance accounts for both the epistemic and the teleological dimensions of the students' activity: a mathematical model that satisfies the validity conditions considered must be constructed; it should be acceptable in the context of the mathematics classroom and lead to an acceptable solution for the game.

The proportional solution

In the following we focus on the case of the group of Lyn, Ely, Tim and Lucy.

After ascertaining that the initial model does not fulfil the validity conditions, Tim considers the twelve points won by both players and proposes a second model: give away the bet proportionally to the points won: €3.50 for the winning player and €2.50 for the other; what corresponds to 7/12 and 5/12 of the bet respectively. Let us call it, from now on, 'the proportional solution'[1].

41 Tim Look, we add the points, right? We take the money and divide it by the number of points, then we get how much is one point worth out of these twelve points that we have. Then we divide it and we get zero point five.

42 Lyn No, because seven plus five is not eight!

43 Tim Three point five plus two point five is six! One gets three point five and the other two point five.

44 Lyn OK, but listen: seven plus five is not eight. OK?

45 Tim It is twelve.

46 Lyn Right, and the six euros were for eight points.

47 Tim Yeah, but, because that was not right [referring to the '€0,75 per point' model], we distribute it this way. 'Cos they couldn't finish, we must distribute it this way!

By focusing in a well-known procedure [41] Tim is suggesting that proportionality might be an adequate model to solve the problem. He infers from the numerical information the "correct numbers" to introduce in the model: it is not eight but twelve what they should use. Lyn [42] does not address Tim's proposal in a straightforward manner; it is just by looking at her three utterances that we may try to understand what she is actually talking about. Lyn focuses [46] on the rules of the game: a player wins by winning eight points and by no other circumstances. We conjecture that Lyn objects taking twelve points into consideration because this action does not represent these rules. This interpretation will be confirmed later (see [63] below). Tim responds [43] by focusing on a validity condition: the amounts of money distributed according to this model 'add up to six'. We infer that, for him, this indicates some degree of certainty about the validity of the model and the numerical solution, although the epistemic status is not addressed in an intelligible way. We observe the emergence and fulfilment of validity conditions as a means for validation. The shift of foci in Tim's utterances [41] and [43] and the disconnected answer in [45] might indicate that he is simply not understanding Lyn's objection. It is just after Lyn's clarification [47] that Tim seems to address it, but rejecting it in a rather authoritarian way. Noticeably, on the one hand Tim does not take care of Lyn's objection about the representativeness of his model, on the other hand Lyn seems not be able to make her objection explicit or even clear. It is not possible to infer from these utterances whether Tim does not understand the objection or does not find it relevant.

The teacher's intervention

The teacher comes and the following conversation takes place:

54 Lyn Ok, if there are just six euros to distribute... and six euros are for eight points but there are twelve points in total... Do we have to forget that six euros are for eight points?

55 Teacher No! You should never forget. You are confusing things; six euros are for eight points of one single person. When you say twelve points, the twelve points were obtained by two persons. All right?

56 Lyn I know, but distributing means giving to both...

57 Teacher No. Because the first arriving to eight gets everything; the whole six euros (...) The problem is that, because the game couldn't be finished, we have to distribute.

58 Tim I have given to the twelve points, the six euros, and then one gets three point five and the other two point five. And it would be fair.

59 Lyn But that doesn't make sense, 'cos...

60 Teacher Why it doesn't make sense? That's a possible solution. I agree. A criterion to divide the money is what Tim just

1 For the case of Anna's group we describe the emergence of this model in (Goizueta, Mariotti, & Planas, 2014).

said. Now Lyn says it doesn't make sense. You should give me an argument why it doesn't.

61 Lyn Because… I mean… No, no, no. Because the six euros are for eight points. And what he is doing is picking the six euros and dividing by all the points.

62 Teacher He's done that exactly.

63 Lyn No, no, no. I mean… That is not like the rules we have.

64 Teacher To see if our solution is consistent, what we can do is to change how the game ended up. Let's imagine six-two. They are six-two when the game is interrupted. Let's check how to split the money and check if it's fair or not.

Lyn refers in a highly condensed way [54] to what they have done to this point and to their current dispute. If we consider this intervention as continuing the students' conversation, it provides the group with new elements to understand Lyn's objection: dividing the money by twelve entails "forgetting" that it corresponds to eight points. The teacher, who is interacting with this group for the first time, seems not capable of inferring the precedents and thus of understanding the epistemological complexity of the discussion. She interprets [55] Lyn's intervention as a mere confusion. The following utterances [56] [57] confirm the ineffectiveness of the exchange. Tim interrupts with his explanation [58], focusing on a procedure and the numerical answer it yields, which surely reveals to the teacher the proportional solution. Lyn addresses [59] the problem of the representativeness of the model again, but still in a rather unintelligible way for the teacher, who reacts to Lyn's objection [60] by apparently supporting Tim and asking Lyn for a justificatory argument. Although it is not clear how the students might interpret the teacher's utterance, it is in line with the didactical planning of the session: fostering the emergence of competing models and students' discussion of them. By saying "a possible solution" and "a criterion" she is implying that other models and criteria could be considered as well. The following utterances by Lyn [61] [63] seem to reveal that the teacher interventions do not help her better frame the problem of the representativeness of the model. Even when she tries to reformulate the issue [63], the teacher does not address it directly. Instead, she sticks again to the planning and proposes [64] to consider a numerical variation of the problem along with a hint. It is difficult to decide how the students might interpret the words 'consistent' and 'fair', but the teacher seems to suggest that the model's validity can be assessed by assessing the 'fairness' of the solutions it yields. The epistemological status of such a link is not made clear, but the teacher is tacitly indicating a new objective to the students: to explore this validity criterion for the situation model that they intend to exploit.

Lyn and Tim's actions suggest different epistemological and teleological stances underlining and shaping their arguments. Lyn seems to understand their immediate task as assessing the relation between the proposed mathematical model and the implicit situation model; thus her intention (as we interpret it) is to put in evidence the lack of representativeness of Tim's mathematical model. Tim seems to focus the solution of the task on the identification and skilful execution of well-known, suitable mathematical procedures; by evoking proportionality in a rather ritualistic way and proposing a numerical result, he copes with that demand. While Lyn's activity suggests the need for a reflective approach, Tim's suggests a reproductive one.

A shift of attention: Towards a probabilistic model

After working for some minutes alone, Lyn proposes a new idea:

87 Lyn Look, I don't know where I want to get to with this, but to arrive to where you get six, this needs just one and this needs three. So, based on that… let's see how to distribute the eight. But I don't know how.

Lyn shifts the focus of attention from the actual score, the base of the proportional solution, to the potential scores, the base of a probability related solution. Although we have observed this shift in other groups, Lyn's case is exceptional in that it is not the response to teacher's regulative interventions (in fact it involves disregarding them); hence it is not a case of perceived expectations fulfilment. It seems to be a genuine, autonomous attempt to investigate elements of the empirical situation in order to construct a situation model and look for a representative mathematical one.

95 Lyn If we divide six euro by this, that is what is missing [writes down '3 + 1 = 4' and '6/4 = 1,5']. We get one point five. [writes down

'3 x 1,5 = 4,5' and '1 x 1,5 = 1,5'] But we should give more to this [winning player], 'cos he's closer than this [losing player]. So, what I would do is to give this ['4,5'] crossed to this [winning player] and this ['1,5'] crossed to this [losing player]. I don't know why!

96	Ely	That looks OK!
97	Lyn	But, why?
98	Ely	'Cos it's proportional. I think it's OK.
104	Lyn	I think it makes sense. But I don't know why!
105	Ely	If you add them up you get six.

Lyn's argument [95] develops through relating the data '3' and '1' (points needed to win) to the loosing and wining players respectively. By simple extension, the product '4,5' ends up to be associated to the losing player while the product '1,5' does to the winning one, what does not conform to what she considers a validity condition. She proposes a 'crossed association', but recognizes it as arbitrary by stressing that she does not find a reason to do so. Instead, Ely supports this solution [96] [98] [105] by referring it to well-known validity conditions: the soundness of proportionality as an adequate mathematical construct to solve the task and the two distributed amounts of money adding up to six. However, such support does not fit Lyn's fundamental need for validating the mathematical model according to its representativeness of the empirical situation [95] [97] [104]. We observe again different epistemological and teleological stances underlying students' arguments. Lyn subordinates the validity of the mathematical model, she proposes establishing a relation between it and the implicit situation model. Instead, Ely positively assesses the validity of the mathematical model on the basis of the soundness of proportionality, to which she associates it, and the fulfilment of previously considered validity conditions.

CONCLUDING DISCUSSION

As observed throughout the analysis, the students seem to be unable of explaining to each other the base on which they claim or challenge the validity of the models they propose. This may be interpreted referring to the difference between the teleological and epistemological stances they adopt. Tim and Ely seem to understand the task as the need for identifying some known, relevant mathematical construct to produce a sound answer that is consistent with shared expectation of the didactical contract. Accordingly, the goal they are pursuing is that of producing a numerical solution through a proportional model, tacitly linked to the problem through its ritualistic use in situations of distribution. It is this link and the fulfilment of the related validity conditions what seems to confer validity to the model. However, the evaluation of the model always remains implicit. Lyn's activity reveals a different stance: the mathematical model must represent the empirical situation if it is to be considered valid, thus her immediate goal when discussing Tim's model and her own is assessing their representativeness.

Both cases illustrate how different epistemological perspectives may influence students' teleological stances and thus students' attitudes towards validating the models they produce, but also how undeveloped is this attitude and far from a desirable culture of argumentation. In the case of Lyn, although she clearly focuses on validating the model with respect to how accurately it represents the proposed situation, she seems unable to share her ideas with her peers in an intelligible way. Meanwhile, Tim and Ely prove incapable of interpreting her stance and seemingly unaware of the necessity of linking the validity of the model to its representativeness. Even the teacher seems unable to perceive the tension between the different stances and help the students fruitfully discuss it.

From our point of view, this experience illustrates a more general situation, namely that students and teachers do not always have the tools to deal with, or talk about, the epistemological complexity of mathematical knowledge in the classroom. If a desirable culture of argumentation is to be fostered in the classroom to allow students to autonomously construct and validate mathematical knowledge, some awareness about one's and other's epistemological stances should be developed as a condition for a critical, reflexive account of one's and other's actions. It should not only be about producing valid arguments that satisfy conventional communicative requirements, but also about being aware of and able to discuss the grounds for their validity in their context of justification. Suitably fostering such discussions in the classroom should be the teleological counterpart of such epistemological stance. We suggest that more attention should be paid to the mechanisms that regulate the emergence of the mathematics classroom epistemology in order to provide teachers with adequate means to plan and control its development as part of a desirable culture of argumentation. To this regard, we claim that the interpretation of classroom mathematical activity in terms of validity conditions emergence and fulfilment might be worthwhile in order to reveal epistemological aspects of classroom interaction and their socio-interactive roots. This interpretation might help researchers describe and model the advocated development of a culture of argumentation.

ACKNOWLEDGEMENT

This report is part of the research funded by Project EDU2012-31464, Ministry of Science and Competitiveness, Spain; co-funded by AGAUR, Generalitat de Catalunya, through the grant 2014-SGR-972 granted to GIPEAM-Grupo de Investigación en Práctica Educativa y Actividad Matemática (Grup de Recerca Consolidat de la Generalitat de Catalunya). It has been possible due to a PIF-Grant granted by Universitat Autònoma de Barcelona to the first author.

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Pre-service teachers' perceptions of generic proofs in elementary number theory

Leander Kempen and Rolf Biehler

University of Paderborn, Paderborn, Germany, kempen@khdm.de

In this paper, we present our concept of the usage of four different types of proofs to engage students in the proving process: the generic proof (with numbers), the generic proof in the context of figurate numbers, the so-called "formal proof", and the proof in the context of figurate numbers using "geometric variables". Further, we report from our case study, where 12 pre-service teachers were interviewed after attending our bridging-course, which had a focus on argumentation and proof. We investigated students' perceptions of the generic proof with numbers. Our findings suggest the classification of perceptions in three categories: "Logical acceptance and psychological conviction", "general acceptance of the concept and psychological uncertainty", and "inappropriate understanding of the concept".

Keywords: Generic proof, conviction, transition, figurate numbers.

INTRODUCTION

Constructing mathematical proofs is said to be a major hurdle for many university freshmen. It still remains a challenge of tertiary education to impart knowledge about the notion of proof. Selden (2012) stresses: "Understanding and constructing such proofs entails a major transition for students but one that is often supported by relatively little explicit instruction" (Ibid., 392). To tackle this problem, the University of Paderborn offers the course "Introduction into the culture of mathematics" as an obligatory course for the first year secondary pre-service teachers (non grammar schools). This course has been developed and taught by the second author in collaboration with the first author. Its content was selected in order to help students to successfully get to know the way of mathematical proving that is prevalent at university. During this course, the students are to investigate mathematical problems (e.g., concerning figurate numbers) and to construct generic proofs and formal proofs. Refining and evaluating the course are a main focus of the first author's dissertation. In this context, students' perceptions of the different kinds of proofs were investigated.

THEORETICAL FRAMEWORK AND RELATED RESEARCH

In their article concerning generic examples, Mason and Pimm (1984) describe the basic feature of a generic proof: "The generic proof, although given in terms of a particular number, nowhere relies on any specific properties of that number" (Mason & Pimm, 1984, 284). This concept of a general verification in a concrete context has mainly been substantiated in Leron and Zaslavsky (2009) and Rowland (1998). But still there seems to be no overarching consensus regarding the notion of generic proofs as valid mathematical proofs. However, generic proofs are said to be especially useful in the learning of reasoning and proving in the field of number theory (e.g., Karunakaran et al., 2014; Rowland 2002; Stylianides, 2012). The advantages are mainly seen in the accessibility even for low-performing students because of the absence of algebraic variables. But as shown by Biehler and Kempen (2013), the concept and the construction of a generic proof remain problematic for many pre-service teachers.

It is very common to give examples of generic proofs with the use of arrays or patterns of dots. This is a valid mathematical approach, since these patterns of dots can be considered as a notation system (Dörfler, 2008). In a semiotic view, concrete numbers, or patterns of dots, with their different properties and rules for operations, can build the notation system to perform diagrammatic reasoning and to construct valid mathematical proofs (e.g., Stjernfelt, 2000).

In contrast to the recommended use of generic proofs, it seems surprising, that students' perception of generic proofs has not been investigated in detail yet. The research of Tabach and colleagues (2010) indicates that teachers often reject correct verbal justifications (including generic proofs), because of a claimed lack of generality and an assumed overemphasis of concrete examples. But as was shown by Malek and Movshovitz-Hadar (2011), students can benefit from being presented to generic proofs concerning proof comprehension and proof construction.

THE CONCEPT OF OUR BRIDGING COURSE

The course is meant to introduce students to the culture of the science mathematics, as it is practiced at university. We want to illustrate the procedural aspects and also give considerations to the ready-made knowledge of mathematics. Therefore, the students are to investigate assertions, build up hypotheses, test conjectures and form lines of arguments to finally achieve a mathematical proof. We also want to deal with the topics, which are known to be a barrier in the transition to tertiary level, so its content covers (1) discovery and proving in arithmetic, (2) figurate numbers, (3) proof by induction, (4) assertions, reasoning, types of proofs, (5) equations, and (6) functions.

The first chapter starts with the question "Someone claims: The sum of three consecutive natural numbers is always divisible by three. Is this correct?". Here, the students are engaged in testing concrete examples and in discussing their informative value. In this context of discovery and justification, we distinguish the significance of examples in a logical and in a psychological way: Logically, it is not important how many concrete examples one has tested, as the assumption is made for all natural numbers. Psychologically, the testing of several concrete examples can be seen positively to understand the assertion, to strengthen one's presumption on the validity of the assertion and maybe to get an idea why the statement is true (in all cases). Following these considerations, we introduce a generic proof of the statement as a possible student's answer (see below).

The concept of the generic proof is then exposed with an emphasis of its general argumentation. As a generic proof, we consider the combination of the following three parts: (1) there are operations on concrete examples that can be generalized, (2) one gives a (generic) argumentation, why the assumption is true in these specific cases and finally, (3) one has to point out, why this argumentation also fits all possible cases. In doing so, it becomes possible to highlight the difference between purely empirical verifications and general valid arguments. And in using generic proofs, the phase of examples-based exploration becomes an intuitive part of the proving process. Referring to the generic proof, we formulate a correspondent formal proof using algebraic variables (see below). Although this proof production seems to be an almost trivial task for a mathematician, our experience has shown, that our freshmen are not used to this kind of argumentation and are not familiar with this usage of algebraic variables.

We also construct the correspondent proofs in the notation system of figurate numbers (see below). Afterwards, the assertion with three consecutive numbers gets generalized: "Is the sum of four consecutive numbers always divisible by four" and so on. At the end of the chapter, we attain the statement that the sum of k consecutive natural numbers is divisible by k if and only if k is odd.

In this first chapter, arithmetic is the area for doing research and proving conjectures. Here, the notation system of figurate numbers is a tool for constructing alternative types of proofs (see below). But in the second chapter, the figurate numbers themselves become the object of investigation, where arithmetic and algebra can be treated as tools for proving. The field of figurate numbers (e.g., triangular numbers and square numbers) offers excellent possibilities for exploration, forming conjectures and proving. Here, it seems very natural to argue with the arrangement and the number of dots.

FOUR TYPES OF PROOFS

As we pointed out above, the generic proof is firstly presented in the context of the value of testing several examples. (See the following generic proof to the claim: The sum of three consecutive natural numbers is always divisible by three).

Generic proof with numbers:

$1 + 2 + 3 = (2 - 1) + 2 + (2 + 1) = 3 \times 2$

$4 + 5 + 6 = (5 - 1) + 5 + (5 + 1) = 3 \times 5$

You can always write the sum of three consecutive numbers as: ("number in the middle"-1) + "number in the middle" + ("number in the middle"+1). Since this sum equals three times the „number in the middle", the sum is always divisible by three.

It is the narrative reasoning that follows the generic examples, which makes a generic proof a valid general argument. So it gets possible to stress the differences between purely empirical examples and valid general arguments. In this way, we work against the misconception that examples on their own can form a valid proof. But the use of generic proofs is also meant to pick up a form of argumentation that is said to be used at school (e.g., Leiß & Blum, 2006). So our students could be somehow familiar with this type of proof and if not, they get equipped with this appropriate way of proving for their future work. In the following transition to the so-called formal proof (see below), it gets possible to introduce and promote the mathematical symbolic language: (1) to express generality, (2) to communicate general incidents, (3) to explore further a supposed relationship, (4) to fulfill arguing and proving and (5) to provide a complete verification of a given statement (as recommended by Malle (1993) and Mason and colleagues (2005)).

Formal proof:
For all $n \in \mathbb{N} \setminus \{1\}$: $(n - 1) + n + (n + 1) = n + n + n = 3 \times n$. *This sum is divisible by three, because* $n \in \mathbb{N} \setminus \{1\}$.

In the whole course we establish figurate numbers (geometrical representations and operations with arrays or patterns of dots or squares) as another notation system. So the students are also asked to construct generic proofs using figurate numbers (see below) and the proof with geometric variables (see below).

Generic proof in the context of figurate numbers:
In the example, one can see the sum 3 + 4 + 5 and 5 + 6 + 7. In every sum of three consecutive numbers, one obtains the same steps, independent from the starting number. After the transposition of the square at the far right, one always obtains three equal lines of squares. So, the sum is always divisible by three.

Proof with geometric variables:

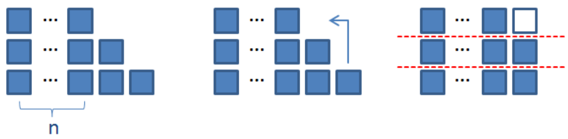

Table 2: A proof with "geometric variables" and figurate numbers

We defined the representation of an arbitrary number by inserting three little dots as a "geometrical variable" to have an analogy to the algebraic variables in the notation system of figurate numbers. This parallel treatment of arithmetic/algebra and geometric representation is said to be useful to ease the transition from arithmetic to algebra. Flores (2002) suggests a similar approach. Moreover, by using these alternative notation systems, it gets possible to stress the immanent quality of an argument, independent from its representation. While in the generic proof, the argument is given in a concrete context, which has to be generalized, the variables in the formal proof imply generality. So, it gets possible to emphasize the notion of variables in algebra and also in the geometrical context.

RESEARCH QUESTIONS

We will address the following research questions in this paper:

1) With which type of proof do the students start their proving process?
2) How can students' different perceptions of generic proofs with numbers be characterized?
3) How can the different perceptions of generic proofs with numbers be distinguished with regard to the logical and psychological aspects mentioned above?

The research presented in this paper is part of a wider research project, which comprises the following further research questions: Are the students able to construct the four different kinds of proofs? Are there common pitfalls in their proof productions and if so, what are these? How can students' different perceptions of proofs in the context of figural numbers be

Table 1: The sum of three consecutive numbers represented by figurate numbers

characterized? How can these perceptions be distinguished in regard to the logical and the psychological aspects? How do the students judge the different proofs in comparison to each other? By knowing our entire research interest, the following research design becomes coherent.

RESEARCH-DESIGN

Our study took place in the last week of the semester. We conducted sessions including proof construction and interviews with six different pairs of students, which were randomly selected from a group of volunteers. In the beginning of each session, the students were asked to proof a theorem (see "task analysis and expected solutions") and afterwards, to construct the remaining three types of proof that they had not used spontaneously. For every proof production, the students were given two sheets of papers, the "draft paper" and the "clean copy". The first sheet was meant to be used for their individual work, e.g., tests, explorations and proofs. After having finished the task individually, the two students were asked to develop a joint solution on the basis of their draft solutions and to write it correctly on the "clean copy". After having constructed all proofs, the students and the interviewer looked through the proofs, correcting gaps or inaccuracies. This phase was included in order to ensure the reference to correct proofs in the following interview phase. At the beginning of the interview, the students had to answer a questionnaire and to rate the four proofs with respect to their persuasiveness, their validity, their quality of explanation and their appropriateness for school mathematics on a six-level Likert scale. The last part of the session was an interview. Firstly, the students' were asked about the reasons for their spontaneously chosen type of proof at the beginning. Secondly, the interviewer asked questions based on the students' responses in the questionnaire to get to know their perceptions of the different types of proof. All sessions were recorded with two cameras, one in front, in order to capture gestures as well as motions in particular, and one in the back, filming students' writings.

TASK AND EXPECTED SOLUTIONS

At the beginning of each session, the following task was given to the students:

Prove or disprove: If one takes a natural number and adds its square, the result will always be divisible by 2.

Since we only discuss students' different perceptions of generic proofs with numbers in this paper, we will only give one possible solution for the generic proof with numbers:

Generic proof (with numbers):

$3 + 3^2 = 3 \times (1 + 3) = 3 \times 4 = 12$

$4 + 4^2 = 4 \times (1 + 4) = 4 \times 5 = 20$

The sum of a natural number and its square always equals the product of the initial number and its successor. One of two consecutive natural numbers is always even and the other one is odd. Since the product of an odd and an even number is always even, the sum is even, i.e. divisible by two.

DATA-ANALYSIS

We transcribed each session and analyzed the transcripts and students' proof productions. For this case study investigating students' perceptions of the generic proof with numbers, we followed the quasi-judicial procedure developed by Bromley (1986). We focused on common and characteristic patterns in students' comments to ultimately categorize them as cases of a certain type. The findings suggest three different types of students' perceptions of generic proofs with numbers.

RESULTS

Students' spontaneous choice of types of proof at the beginning of the session

Out of the 12 students participating in the study, nine started immediately to construct a formal proof. In one group, the two students started with testing the statement with two concrete examples. Afterwards, one student explained, referring to their examples ($3+9=12$ and $4+16=20$), that the sum must be even, because the square of an odd number is always odd. And if you add any two odd numbers, the sum is always an even number. So, she discovered a part of a general argument, she could use for a generic proof with numbers. But afterwards, both students started to construct a formal proof with algebraic variables to verify the statement.

All participants started to use algebraic variables when writing down the assertion. For their choice of the formal proof with algebraic variables, they named different reasons: their socialization in school and/or university, that the formal proof is easier, because one does not have to have an "idea", or they thought that the construction of the formal proof would be the intended task.

Only one student immediately wrote "generic proof" and started to investigate concrete examples. She gave the following reasons for her choice in the interview:

> *Yes, for me it is a support. [...] When I'm writing these things down, I recognise how it works. And afterwards, I insert variables. [...] I'm looking for regularity or something similar.*

Students' perceptions of generic proof with numbers

Analyzing the data, we could identify three different kinds of perceptions of generic proofs with numbers, which are described briefly and illustrated by examples from the transcripts. (The following transcripts were translated and linguistically smoothed.)

(1) Logical acceptance and psychological conviction

A student with this perception fully understands the concept of the generic proof and accepts the immanent general verification. The generic proof convinces the student without any doubt that the statement holds in any case. Moreover, he gets an insight, why the statement is true.

Michael reports his perception of generic proofs:

Michael: So, I marked "strongly agree" [for validity]. Because ... step by step, one can immediately follow the idea, e.g., you add this one and then you add the next one and so one. It is not the same as in a formal proof: It is just there and you have to prove it and that's it. This [the generic proof] is not [as] illustrative [as a proof] with pictures, but one can clearly see what happens. [...] That's what I like about a generic proof. If you see this, you can easily understand the way it has been done. There are always some examples given which are used for the following argumentation you can easily understand.

(2) General acceptance of the concept and psychological uncertainty

In this case a student understands the concept of the generic proof and is willing to accept the immanent general verification. In contrast to this conviction, a subjective, intuitive doubt remains, but this emotional uncertainty is considered as unnecessary. Sarah mentions her perception of generic proof with regard to the persuasiveness:

Sarah: I do understand its [the generic proof's] general validity and overall meaning. But with regard to the persuasiveness: if one submits a generic proof to me, I would say: "Can I also have a formal one?". For me, it is more convincing.

Christin describes her intuitive need to test the assumption for all natural numbers:

Christin: But this one [the formal proof] – for me – is somehow a more correct and coherent proof. For me, I would have to test it for all "n" - but it is nonsense, because one recognises the scheme, but...

(3) Inappropriate understanding of the concept

A student with this perception does not (fully) understand the concept of the generic proof. The student focuses on the concrete examples, without noticing the whole wider scheme and its general argument. The generic proof gets misinterpreted as a purely empirical verification.

We cite Paul and Amy as examples of perception (3). Paul quotes, that in a generic proof you are only testing specific examples:

Paul: It [the formal proof] is just – let me say – more correct.

Interviewer: What does it mean "more correct"?

Paul: Yes, correct in the sense that it shows its [the statement's] accuracy, so the validity, yes. And that the proposition is valid. We also show this in the generic proof, but this only applies to the numbers, we have tested.

Amy mentions her perception of generic proofs, which illustrates her misconception:

Amy: Yes. If it [the generic proof] is sufficient for me? I'm not sure. So I wouldn't say that there is 100%-validity. Since there has been a homework, where we had to refute or to prove a statement. And then we had to test examples. I found two examples that worked, but maybe, there is a third one that doesn't. So for me, it [the generic proof] is not sufficient for I have not proven it. I did only prove it for these two examples and not for all numbers. So there is no validity.

Considering all interviews, we only found one student holding perception (1), identified four participants with perception (2) and five with perception (3). In two cases we could not categorize students' answers, as they did not state clear positions.

DISCUSSION AND FINAL REMARKS

Nearly all students in our study started to work on the given task with formalisation and the construction of a formal proof using algebraic variables. Their socialization during their time in school and the reputation of advanced mathematics may be considered as reasons for this result. But one has to stress, that we explicitly did not try to convey an overemphasis of the formal proof in our course, but tried to highlight the validity of all four types of proofs. However, in this simple task the formal proof can be considered the easiest one. Another possible explanation could be that students think that "to provide a proof" implies the use of algebraic variables as we did not ask them to provide a "generic proof". Students may have misinterpreted this as a semiotic norm in the sense of Dimmel and Herbst (2014). Although this is incoherent to the semiotic norm the course has intended to establish.

Our findings concerning students' perceptions of generic proofs with numbers indicate that even after having passed our course, the notion of generic proof still remains problematic for a substantial proportion of our students. Only about half of the students in our study hold a perception, where the generic proof is a general and logical valid argument [perception (1) and (2)]. Five students did not realize the generic aspect of the investigated examples and still hold the view of generic proofs as purely empirical verifications [perception (3)]. One can identify an important pitfall in the usage of generic proofs: When students do not understand the important difference between testing several examples and generic examples combined with valid narrative reasoning, they might be confirmed in their understanding that checking several examples may constitute a proof. Here, the question "why is it true" in combination with the explanatory power of proofs seem to be a promising way to address this misconception. One has to point out, that perceptions (1) and (2) are desired effects of our course. While there are students, who are completely convinced by a generic proof, others accept its logic, but still feel an intuitive doubt. For the latter group, the meaning of variables and the usage of the mathematical language can be pointed out even clearer. While we argue, that our proposed use of generic proofs and formal proof gives a meaningful introduction into the process of proving and the mathematical language, it becomes clear, that generic proofs are not "generic" by themselves.

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Disparate arguments in mathematics classrooms

Christine Knipping, Daniela Rott and David A Reid

University of Bremen, Bremen, Germany, knipping@math.uni-bremen.de

Understanding students' classroom argumentation requires the analysis at multiple levels and from multiple perspectives. Using analyses of students' arguments when solving the Problem of Points we illustrate three perspectives: a classroom interactional perspective, a Stoffdidaktik task analysis (at the individual level), and a sociological theoretical perspective (at the community level). Each of these perspectives offers insights into students' argumentation in mathematical contexts, but no single one is adequate to completely describe the nature of students' argumentations, their underlying influences, and ways to support their development. Multiple perspectives and levels of analysis are required when researching classroom argumentation in particular and mathematics learning in general.

Keywords: Argumentation, classroom interaction, task design, social context.

In this paper we argue that understanding students' classroom argumentation requires the analysis at multiple levels and from multiple perspectives. Levels of analysis touched on here include individual, small group, whole class, school and community, but more are possible. Perspectives considered here include a classroom interactional perspective, a *Stoffdidaktik* task analysis (at the individual level), and a sociological theoretical perspective (at the community level).

We begin with a short description of a task which was used in our research and a brief description of some of our results. We then compare these results to those of Goizueta (2014; Goizueta, Mariotti, & Planas 2014) and offer his framework for analysis as the first perspective considered: an interactional perspective at the classroom level. Goizueta proposed the task in Spanish and we proposed it in German, and we will consider changes to the task that may have occurred in translation, using a second perspective from the work of Schupp (1986): a *Stoffdidaktik* task analysis at the individual level. We will then describe additional results from our research that suggest that additional perspectives are needed. An interesting parallel between our results and those obtained by Holland (1981) suggests a third perspective that applies to our results: a sociological theoretical perspective at the community level. We close with some final comments on the need for multiple perspectives and levels of analysis in researching classroom argumentation in particular and mathematics learning in general.

THE PROBLEM OF POINTS

Rott (2014) asked 14/15-year-old students at Gymnasium (grammar school/college prep school) in Bremen (Germany) to solve the classic Problem of Points in the context of their regular classroom mathematics lessons.

> Silke and Acun take turns flipping a coin. Silke scores a point if the coin comes up heads. Acun scores a point if the coin comes up tails. At the beginning of the game Silke and Acun each bet 3€. The first to score 8 points receives the 6€. When the score is 7 to 5 in Silke's favour, they have to interrupt their game. How can they now divide the money? (Our translation from Rott, 2014, p. 25)

Rott classifies the students' answers into five types:

> Proportional A total of 12 rounds were played (7 + 5) and there is 6€ to be won. So each round played scores obtained 0.50€. Accordingly, Silke wins 7 · 0.50€ = 3.50€. Acun receives 2.50€.

> Point Difference Silke requires only one throw to win. The difference between Silke and Acun is 2 points. So Silke gets 2€ more than Acun.

> Mistaken Reasoning Silke receives 7/8 of the 6€, Acun 5/8. However, that totals too much: 5.25€ + 3.75€ = 9€. So Silke gets 3.75€ and Acun 6€ – 3.75€ = 2.25€.

> Point-Ratio The ratio of the scores so far is 7:5 = 1.4 So Silke gets 1.4 · 3€ = 4.20€. That leaves 1.80€ for Acun.
>
> Winner Silke has more points.
>
> (p. 40, our translation)

INTERACTION AT THE CLASSROOM LEVEL: RATIONAL BEHAVIOUR AND THE DIDACTIC CONTRACT

Some of these types of answers correspond to answers obtained by Goizueta (as reported in Goizueta, Mariotti, & Planas, 2014) and they can be accounted for by an analysis from the same theoretical perspective. Goizueta et al. analyse Goizueta's data using Habermas' construct of rational behaviour, in which accepting a validity claim amounts to accepting that certain conditions for validity have been fulfilled. This means that the criteria for validity have also been accepted. They describe three dimensions in students' argumentations:

> According to Habermas' construct of rational behavior and its adaptation by Boero et al. (2010), in the students' argumentative practices we can distinguish an epistemic dimension (inherent in the epistemologically constrained construction and control of propositions, justifications and validations), a teleological dimension (inherent in the strategic decision-making processes embedded in the goal-oriented classroom environment) and a communicative dimension (inherent in the selection of suitable registers and semiotic means to communicate within the given mathematical culture). (pp. 169–170)

They also make use of Brousseau's (1997) concept of the didactic contract, the implicit rules of acceptable classroom behaviour that govern teachers' and students' actions, and Douek's (2007) notion of a reference corpus, knowledge which is assumed to be unquestionable and shared and hence available as a basis for argumentation.

The five types of answers observed by Rott and listed above share a common characteristic which is of special interest to us here. They are all *mathematical*. Goizueta et al. account for this characteristic in the answers Goizueta recorded by reference to the didactic contract.

> When Vasi reminds the group of the need to resort to calculation, we recognize a constraint imposed by the didactic contract: any possible correct answer must be mathematics-related. (p. 172)

> According to the teleological and communicational dimensions, the didactic contract-related need to provide a mathematics-related answer, acting as a necessary normative validity condition, is what drives the students' efforts towards the construction of a first mathematical model. (pp. 174–175)

In short, from the perspective adopted by Goizueta et al. the students' choice to give mathematical answers to the question is accounted for by the didactic contract, which operates at the level of the classroom.

STOFFDIDAKTIK TASK ANALYSIS: LANGUAGE AT THE INDIVIDUAL LEVEL

The five types of answers listed above leave out one type that Rott observed: Money Back, of which an example is "The game was not completed. Both have a chance to win. The game was interrupted, without either reaching 8 points. The game is a tie." (2014, p. 40). This type of answer differs from the others in that it is not mathematical, instead making reference to everyday practices in game playing. Only two answers of this type were observed in the Gymnasium class, but its occurrence requires an explanation as it is not accounted for by Goizueta and colleagues' analysis.

The work of Schupp (1986) provides a different perspective that suggests a reason for the occurrence of such an answer. Schupp considers different wordings of the Problem of Points and suggests how students' answers might vary in relation to different wordings. He distinguishes three fundamentally different views of the problem: 1) situative; 2) quantitative; 3) stochastic and proposes different formulations of the problem to communicate these different views. The formulations all begin in the same way: "Two players are flipping a coin ... Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5. ..."

Situative view: What now?

One possibly ending to the task formulation is very open, asking “What now?” or “What happens next?” Schupp suggests that such a formulation invites a “situative view” of the problem resulting in answers drawing on everyday experiences of game playing. For example, students might answer: “Why not continue at another time?”, “Why not annul the game and give both players their money back?” “Toss the coin one more time to decide the winner.” All these answers seem plausible but they are not mathematical, as Schupp points out (p. 217).

Quantitative view: Fair division; how can this be accomplished?

Another possible ending is “The two players decide to split the money fairly. How can this be done?” (p. 218, our translation). Such a formulation invites a mathematical answer based on ratios. Interestingly, such answers were among the first given to the problem by mathematicians such as Pacioli. Many students in Goizueta’s and Rott’s research also seem to take this view.

Stochastic view: Dividing the pot according to probability

A third formulation considered by Schupp makes it clear that a “fair” division must take into account the players chances of winning when the game is interrupted, for example, by stating “Before dividing they agree that a ‘fair’ division must be done according to the probability each have at that moment of winning the game. How then must the pot be split?” (p. 218) This formulation of the problem motivates solutions like those given historically by Pascal and Fermat. In contrast to other approaches the focus is on the rounds that have not been played.

The tasks as given

How do the formulations of the task used by Goizueta and Rott fit into Schupp’s categories? Both Goizueta’s Spanish formulation and Rott’s German formulation, like Schupp’s, begin “Two players are flipping a coin … Unexpectedly, they are asked to interrupt the game when one of them has 7 points and the other 5.” It is in the way the final question is asked that they differ.

Goizueta’s formulation ends, “¿Cómo deben repartirse el dinero? Justifiquen su respuesta.” (Goizueta, personal communication). The phrase “¿Cómo deben repartirse el dinero?” (How should they split the money?) refers only to dividing the money, and not to the concepts of fairness and probability. However, the word “deben” (should) suggests that there is a single correct answer to this question. Also, the requirement to “Justifiquen su respuesta.” (Justify your answer) in the context of a mathematics class could suggest that mathematical methods of justification are expected. Hence, while this formulation is not as clearly promoting a quantitative view as Schupp’s example of invoking “fairness”, it is not surprising that a quantitative view was adopted by all the students in Goizueta’s study.

Rott’s formulation ends, “Wie können sie nun das Geld aufteilen? Begründe deine Antwort!” (Rott, 2014, p. 25). The phrase “Wie können sie nun das Geld aufteilen?” refers only to dividing the money, and not to the concepts of fairness and probability. And the word “können” (can) is more open than the alternative “sollen” (should). However, Rott’s formulation is not so open as Schupp’s examples of situational view formulations as there is an explicit mention of dividing: “aufteilen” and a requirement to justify the answer “Begründe deine Antwort!”. This places the formulation somewhere in between the situational and the quantitative which would account for answers of both kinds occurring in the Gymnasium class. The task formulation gave students the option of choosing between taking a situational or a quantitative view, and while most took a quantitative view some did not, perhaps reflecting personal preferences at the individual level.

A SOCIOLOGICAL PERSPECTIVE AT THE COMMUNITY LEVEL: RESULTS FROM THE OBERSCHULE

Rott did not only propose the Problem of Points in the Gymnasium class. She also proposed it in another class, in an Oberschule (comprehensive or mixed school). There the results were strikingly different. No students gave mathematical answers of the five types listed above. Most answered that Silke and Acun should get their money back. The only other type of answer Rott observed was a group that discussed extensively a variation on the Point-Difference argument, in which the difference of each players’ winnings from their original bet of 3€ should be equal to the point difference between the players. As one member of the group, Melanie, put it:

> If the score were six five, would then, they would then — ah, if it were six five, I would say that she gets two Euro, because that, he would get two Euro and she four Euro, but it is actually seven five. One point less. Then I would make it five Euro and one Euro. (Rott, 2014, p. 56, our translation)

If the score were 6:5 the point difference would be one, so Silke would win 1€ more than she bet, that is 4€, and Acun would win 1€ less than he bet, 2€. As the score is actually 7:5, the point difference is two, so Silke wins 2€ more than she bet, 5€, and Acun wins 2€ less than he bet, 1€. Melanie's group did not accept this argument, and at the very end of their work together, they almost persuaded her to accept the answer given by the standard Point-Difference argument, but she objected, and their final answer is a sort of compromise without any justification.

125 Melanie: Let's make it Silke four and Acun two.
126 Ines: [???]
127 Melanie: But then, look, just think, then Acun has two Euro, no? And she has, she has three
128 Euro, only one euro of Acun, although she has ||two|| points more.
129 Anne: ||let's make it|| one fifty for him and four-fifty for her. -
130 Ines: That's also an idea. Then both are somehow on the same wavelength, so then the two are
131 actually equal.
132 Melanie: Okay, do that. (p. 55)

Earlier Rott asked Melanie how she arrived at her answer of 5€ from Silke and 1€ for Acun.

92 D. Rott: How did you get that Silke gets five euros and Acun one Euro?
93 Melanie: (Points to the worksheet) So there it was, yes seven to five.
94 D. Rott: Yes.
95 Melanie: And if the score would be seven to six, Silke would get four and Acun
96 two, but it is actually seven to five, so Silke gets more and Acun one Euro.
97 D. Rott: And what did you calculate exactly, to get that?
98 Melanie: Nothing
99 D. Rott: Just by intuition?
100 Melanie: Yes. (p. 57)

Rott calls this groups argument "Intuition" in light of Melanie's comment about not calculating, and the group's quick conclusion to divide the money 4.50€ to 1.50€. While this group's arguments included mathematical elements, which is unusual for the Oberschule groups, their final answer does not have a mathematical basis.

Recall that the answers from the Gymnasium groups were mostly justified in mathematical ways, except for a few who said to give the money back. In the Oberschule, however, the situation is the opposite. Almost all the groups said to give the money back, with only one group providing an intuitive argument with some mathematical elements. We can be sure the formulation of the task was the same in the Gymnasium and the Oberschule as it was provided to the teachers by Rott. How then to account for the very different types of answers? To do so we will need further results from the two classes, as well as another perspective, at another level.

A second difference between the two classes is in the number of different answers students gave. In the Gymnasium it was expected and happened that students gave alternative answers, including that the players should get their money back. The seven groups offered a total of 11 answers, of six different types, while in the Oberschule the five groups gave one answer each, four answering that the players should get their money back. One could account for this by saying the didactic contract is different in the two classrooms. However, even when pushed to give alternative answers in the interview, the Oberschule students only gave answers based on everyday experience. The different preferences of the students, for mathematical answers in the Gymnasium and answers based on everyday experience in the Oberschule, combined with the ability of the Gymnasium students to give answers based on everyday experience as well as mathematical answers, is reminiscent of a study done by Holland (1981).

Holland gave 8 year old children the task of sorting photographs of familiar foods. She found that children from middle-class backgrounds tended to sort the photographs in terms of abstract properties (e.g., "animal/vegetable/dairy/cereal" or "from the sea/farm"). On the other hand, working-class children tended to sort the photographs in terms of their personal experiences (e.g., “things I eat for breakfast/lunch/supper”). Furthermore, when asked to re-sort the photographs, the children from middle class backgrounds could do so, sorting them in terms of their personal experiences, while the working class children could not offer additional sortings. As this result is very similar to the differences between the Gymnasium students and the Oberschule students, it is worth considering the theoretical framework used by Holland and looking at Rott’s results at a sociological level.

A sociological perspective

The neighbourhoods of Bremen are classified by the government into different types. The Oberschule is located in a Group A neighbourhood, in which the proportion of families with immigrant backgrounds as well as the proportion of people on social assistance is above average. Specifically in the neighbourhood around the school, the proportion of families with immigrant backgrounds is 65–86% and many parents have low levels of education (Die Senatorin für Bildung, Wissenschaft und Gesundheit, 2012a, p. 55). The Gymnasium, on the other hand, is a private religious school located in a Group B neighbourhood in which the proportion of families with immigrant backgrounds is below average (15–30%). This does not mean that all the students in the school come from upper-middle-class homes, but it does mean that the majority do.

This difference suggests that a sociological perspective might be useful in understanding the different answers given by students in the two schools to the Problem of Points.

Sociological approaches fall roughly into two groups. One approach collects data on very large groups of people and uses statistical techniques to draw conclusions about the relative weight of various social factors in determining, for example, success in school mathematics. Such an approach is clearly not suitable in this case. A second approach involves applying well developed sociological theories to describe and analyse the actions of smaller groups of people. We have chosen one such theory, the work of Bernstein, to analyse the data presented here.

Holland makes use of several concepts from Bernstein’s work: restricted and elaborated orientations to meaning, realisation and recognition rules, and re-contextualisation.

> For Bernstein an orientation to meaning is created by inter-actional practices which act selectively on what is to be meant, and what form the realization of meaning takes in which contexts. The inter-actional practices in the family and school transmit recognition rules which mark contexts as requiring a specific text, and realization rules which regulate what meanings are to be offered and how these are to be made public. Bernstein argues that families in different social class locations are typified by different inter-actional practices which regulate different recognition and realization rules and generate an elaborated or restricted code ... (Bernstein, 1977) or ... an orientation to context independent meanings or to context dependent meanings. For some children then the re-contextualizing principle of the school will entail recognition and realization rules very different from those acquired in the family. (Holland, 1981, p. 2)

Children learn at home how to *recognise* contexts that require certain ways of making meaning, and how to *realise* those ways of making meaning. School is a context in which meaning is context independent. Children’s experiences are re-contextualised in school into abstractions (Bernstein, 1977). The middle class children in Holland’s study had both the recognition rules to see Holland’s task as calling for abstract categories, as well as the realisation rules needed to use abstract categories in classifying. The working class children did not. Bernstein (1977) characterises an orientation to abstract, context independent meanings as an elaborated orientation, and an orientation to context dependent meanings as a restricted orientation. From this perspective we can account for the differences between the argumentations of the Gymnasium students and the Oberschule students by suggesting that the out of school experiences of the students led some to develop elaborated orientations to meaning, including both recognition and realisation rules related to using mathematical arguments in school contexts, while others developed restricted orientations to meaning, lacking either the recognition or the realisation rules needed to produce mathematical argumentations in school contexts.

Cooper and Dunne (2000) describe difficulty recognising the border between everyday and mathematical contexts as the "boundary problem". They researched sixth grade and ninth grade students' solutions to test items of two types: "realistic" items in which a mathematical task is embedded in an everyday context, and "esoteric" items in which the task was decontextualised. Working class students performed less well than middle class students, especially on realistic items. However, in an interview situation in which the students were explicitly told to disregard the context, they were successful on items they had incorrectly answered in the test situation. This suggests that their difficulty did not stem from an inability to act appropriately. Rather, they faced the boundary problem; they lacked the recognition rules to distinguish between everyday and mathematical contexts. If this was also the difficulty faced by the Oberschule students, then a formulation of the task that more clearly invokes a mathematical or stochastic view of the problem might help them to recognise the type of argument expected. In addition, a reframing of the didactic contract to more explicitly require multiple solutions and mathematical solutions might also help to overcome the boundary problem.

Knipping (2012) discusses the value of sociological approaches to research on argumentation, and identifies decontextualised language as a key issue. She cites the work of Hasan (2001), who studied the opportunities children have to engage in this kind of discourse in their homes before they begin schooling. In some homes the talk of adults to small children remains tied to the context, related directly to the activities and objects that are present to the senses. In others there is already at this early age a fluid shifting between decontextualised and contextualised language, which Hasan refers to as a "con/textual shift". There is reason to believe that the division Hasan found, between home situations in which con/textual shifts occur often and those where they occur rarely, is influenced by social class. This could provide a mechanism to account for the occurrence of the boundary problem among the Oberschule students.

CONCLUSIONS

In this paper we have offered three different perspectives at three levels that help us account for students' responses to the Problem of Points: a classroom interactional perspective, a *Stoffdidaktik* task analysis at the individual level, and a sociological theoretical perspective at the community level. We have made no effort to integrate these perspectives into a single all-encompassing model. This is likely to be impossible given the different fundamental assumptions of the theories involved. Moreover, as Reid (1996) points out, multiple perspectives offering different interpretations are valuable, even if contradictory, as no single perspective can capture the complexity of human learning in social contexts. And as we take a closer or wider view of phenomena of interest, different theories become applicable. Learning can be viewed at many different levels, from the neurological to the ecological. Different theories and methodologies are appropriate for these different levels. "Discourses concerned with different phenomena (such as radical or social constructivism—or neurology, ecology, or biological evolution) can be simultaneously incommensurate with one another and appropriate to their particular research foci." (Davis & Simmt, 2003, p. 143).

Researching classroom argumentation must involve theories at at least three levels. Argumentation is conducted by human beings, and influences the thinking of human beings, and so perspectives at the level of individual human cognition are needed. But argumentation is also a necessarily social phenomenon which takes place among groups of people using language. Perspectives that focus on communication, language, and interactions between people in small groups are therefore needed. People and communication develop in larger social and cultural contexts that shape them and are shaped by them. Argumentation in mathematics classrooms cannot be considered independently of argumentation in mathematics and people in mathematics classrooms cannot be considered independently of their social and cultural backgrounds. Multiple perspectives at every level are needed.

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E-assessment of understanding of geometric proofs using interactive diagrams

Yael Luz and Michal Yerushalmy

University of Haifa, Faculty of Education, Haifa, Israel, yael.luz.halevi@gmail.com

The present study is part of a larger one examining the design principles of e-assessment of understanding of geometric proofs. The interactive assessment tasks are checked automatically and feedback is generated. We reviewed various proving task design studies, looking for a template that incorporates interactive sketching and that can be checked automatically. Our findings suggest that examples sketched in a dynamic geometry environment in order to complete the assessment task are a challenging target for e-assessment and are relevant for eliciting the students' understanding of geometric terms, geometric statements, and their validity.

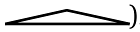
Keywords: Assessment, DGE, examples, validity.

BACKGROUND

The present paper focuses on the challenge of automated assessment of knowledge of geometric proofs. Specifically, we propose to assess processes of conjecturing and argumentation. To this end, we study the roles of examples (designed to be checked automatically) demonstrating the logical validity of geometric statements.

Assessing understanding of geometric proofs

For decades an extensive efforts has been made to automate the process of checking proofs. But a deductive proof is a complex process, and most of the proofs produced by students are subject to their teachers' evaluation. We are interested in a facet that is not often checked in a school setting (neither by humans nor by machines), although it is an inherent part of mathematicians' work on proofs: conjectures and argumentation. In the process of conjecturing, mathematicians try to generate different examples, including extreme and boundary examples, in order to create a rich example space from which a pattern can emerge. When mathematicians investigate the validity of a new conjecture, they usually do not only look for the proof but try to construct counter-examples by means of quasi-empirical testing, because such testing can expose hidden contradictions, errors, or unstated assumptions (De Villiers, 1990). Proving often involves understanding how the proof relates to specific examples and how these examples can illustrate it. Being able to follow a sequence of inferences in a proof based on a specific example has been considered by mathematicians to be an indispensable tool for understanding a proof. Mejia-Ramos and colleagues (2012) constructed a model for assessing the comprehension of aspects of a proof. They discovered four main facets: summarizing the main idea of the proof, understanding the components or modules of the proof, applying the method of the proof in other contexts, and illustrating the proof with examples or diagrams. Kuzniak (2013) presented a new framework for the didactics of geometry work. The framework encapsulates instrumental processes. It transforms artefacts, such as dynamic geometry software, into tools in the construction process and into a discursive process of the proof that confers meaning on the properties used within the mathematical reasoning. The assessment template we designed follows Ramos's (2012) assessment model and is inspired by the framework suggested by Kuzniak (2013).

Buchbinder and Zaslavsky (2009) offered a mathematical framework for designing tasks that question the students' understanding of the role of examples in determining the validity of mathematical statements. This framework provides a basis for constructing tasks that assess and support students' understanding of the logical connections between examples and statements. Any mathematical statement can be reduced to two sets of mathematical objects: the "*if*" part of the statement, which is the domain, the set of all mathematical objects to which the statement refers (e.g., isosceles triangles), and the "*then*" part of the statement, the proposition that defines the set of all mathematical objects that exhibit a certain property (e.g., right angle triangles). In this context, the status of the example object can be defined as follows: a supporting example is an object in domain D that exhibits P (e.g., an isosceles triangle that is a right-angle triangle ). A counter-example is an object in domain D that does not exhibit P (e.g., an isosceles triangle that is not a right-angle triangle ).

Interactive diagrams

Herbst and Arbor (2004) found that building reasoned conjectures or using deductive reasoning to find out what could or should be true can be supported by tasks that engage students in generative interactions with diagrams. This study uses interactive diagrams as a tool to create examples that either support or refute conjectures. The interactive diagrams we use are designed to describe the domain and context of the task (an example of a geometric figure or drawing) and to support autonomous guided inquiry, by enabling direct manipulations of the example and by providing feedback that reflects the process of inquiry (Yerushalmy, 2005). Students interact with the diagram while exploring the conditions under which a version of the problem can be solved (Herbst & Arbor, 2004). By dragging a dynamic diagram students generate an example that demonstrates specific properties. The use of dragging allows students to experience kinematic dependence that can be interpreted as logical dependence within the dynamic environment, but also within the geometric context (Mariotti, 2006). According to Mariotti, the dragging interaction can be compared with formal arguments used in mathematical proofs (Mariotti, 2006).

e-Assessment

Whereas traditional assessment focused primarily on testing factual knowledge, new technologies gave rise to the need to assess new skills, such as problem solving, creativity, critical thinking, and risk-taking. To nurture and develop these skills, the assessment strategy should aim beyond testing factual knowledge and capture the less tangible themes that underlie these skills. Although technology offers a rich learning experience, studies show limitations when it comes to assessing solutions for complex mathematical problems. First-generation e-assessment was limited to multiple choice questions, subsequently enhanced by short verbal or numeric answers (Scalise & Gifford, 2006). Studies show that assessments based exclusively on questions of this sort lead to limited learning and incorrect inferences of purpose of the assessment, such as "there is only one right answer," " the right answer resides in the head of the teacher or test maker," and "the role of the student is to get the answer by guessing" (Bennett, 1993). To assess the complex processes involved in proofs we must understand the role that examples play in proofs and the links between these examples (in our case, dynamic figures) and logical argumentation.

THE STUDY

Objectives

The present paper presents results derived from a broader research study for which we developed a tool to assess students' skills in geometry proofs. The tool provides immediate multiple representation feedback and analysis at the level of the single student, group, or class, based on the teacher's choice. The challenge of the innovative development is to study strategies followed by students in investigating the validity of a geometric statement and in generating examples. We proposed to determine which aspects of the comprehension of a geometry proof are demonstrated by the automatically generated evaluation presented either as a personal solution or as a visually comparative collection of answers, and which aspects are demonstrated in the accumulated visual feedback.

Study setup

Sixty three middle school students participated in the study; 33 were in the 8th and 30 in 9th grade. The geometry classrooms featured mixed abilities, including gifted students and students with special needs. We use the terms "high-, medium-, and low-skill students" for students who showed high, medium, and low achievement in the subject of the examination before the experiment.

The experiment included a practice session, an examination session, and a whole-class discussion period. Before the examination, students had a practice session aimed at presenting the instructions and the technological interface (personal tablets) using a similar template task. The examination included three problem-solving tasks with a repeated five-item template [1]. In this paper we report the results of the work on two items of the template appearing in three tasks. We also report on the class discussion that took place a few days after the examination and on task-

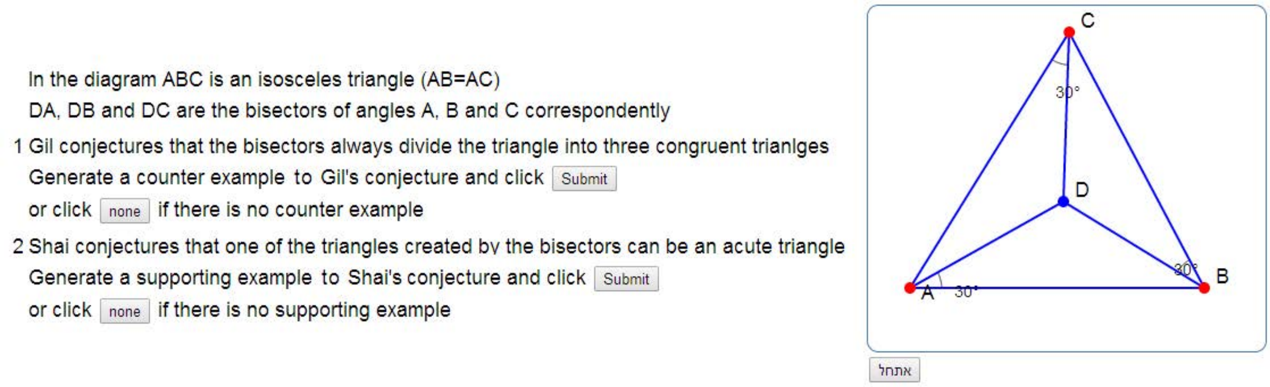


Figure 1: Page setup of paired items B1 and B2 (points A, B, C are draggable)

based interviews with three pairs of students (two high-skilled and one low-skilled) solving the same examination items a few weeks after the exam. The group feedback used in the whole-class discussion and the task-based interviews were conducted for the purpose of triangulation.

Design of the research tools

Task: The examination included three tasks that are instances of the same five-item template. Three items focused on the comprehension of terms and the reading of the proof. Here we discuss the other two items, which focused on assessing students' understanding of the validity of a geometric statement and generating supporting or counter-examples. The students were asked to provide a counter-example for the universal statement and a supporting example for the existential statement (Figure 1). The students could also choose "none," indicating that such an example does not exist. The domain of both items was identical in each task. The interactive diagram was a *figure*, therefore robustly in the class of objects described by the domain, and the properties defining the domain were invariant under dragging. The initial orientation of the *figure* does not represent the correct answer, and students must actively drag the diagram to generate the correct answer. The items of the three tasks are described in Table 1.

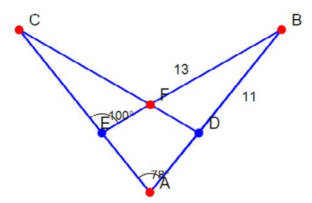
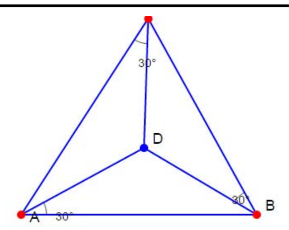
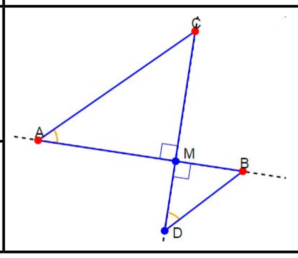
Objectives	Domain (if)	Proposition (then)	Diagram
A1. Counter- example	$AB = AC$ $BD = CE$	*AEFD* cannot be a square	
A2. Supporting example		ΔFDB can be an isosceles triangle	
B1. Counter- example	$AB = AC$ *DA*, *DB* and *DC* are bisectors	The bisectors always divide the triangle into three congruent triangles	
B2. Supporting example		One of the triangles created by the bisectors can be an acute triangle	
C1. Counter-example	$AB \perp CD$ $\angle A = \angle D$	$\Delta AMC \sim \Delta DMB$	
C2. Supporting example		$\Delta AMC \cong \Delta DMB$	

Table 1: The logical structure of the three pairs items

In each diagram, some measures of lengths and angles were displayed. Other measures that were required for constructing the example were such that could be derived from the displayed measures and the properties of the domain (e.g., the measure of only one of the equal sides of an isosceles triangle is displayed). The use of measurements in interactive diagrams confronts students with the issue of the relationship between measurement and proof (Chazan, 1988).

Rubric: A rubric was developed for each task based on a literature review of relevant misconceptions and on pilot trials of the tasks conducted with students (mostly pre-service mathematics teachers).

Feedback: We designed two feedback sheets. A personal feedback sheet displayed the student's examination answers and personal evaluation feedback [2]. The feedback was not provided to the students immediately after completion of the examination in order to support an unbiased class discussion. The group feedback sheet (Figure 6) was designed to display the collective answers and results of all students for a single item.

Data sources and analysis

Data analysis included the personal and group feedback sheets, the class discussion videotape, and videotapes and transcripts of the interviews. We verified the automatic check by manually checking the results. Next, we used the group feedback to look for patterns of mistakes. Expected pre-configured misconceptions were captured by the automatic check, and other mistakes were identified manually. We conjectured about the source of the mistake and later verified our conjectures during the interviews.

FINDINGS AND RESULTS

The findings are classified into three categories: strategies, misinterpretation of terms, and understanding statement validity according to the three questions formulated in the objectives of the study. Each category includes the triangulation of the results obtained by analysis of the three research tools.

Strategies used to create examples

During the examination we found that highly skilled students demonstrated active and fearless engagement with the interactive diagrams, dragging points to the extreme and collapsing them to a single point.

This impression was verified during the interviews, when the two high-skill pairs intensively dragged the diagram from an initial state to extremes, constantly expanding the example space (Figure 2), whereas the low-skill pair showed hesitating and careful dragging, making small changes to the initial diagram (Figure 3).

Some students (mostly high-skill) submitted extreme examples even when standard examples were sufficient to provide a counter-example, as shown in Figure 4.

Another strategy was the use of symmetrical diagrams, with as regular as possible shapes. This strategy helped students to avoid common error such as matching non-corresponding sides, trying to create congruent triangles (see Figure 5) or examining isosceles triangle as a possible counter example to the statement "the median divides the triangle into two equal area triangles".

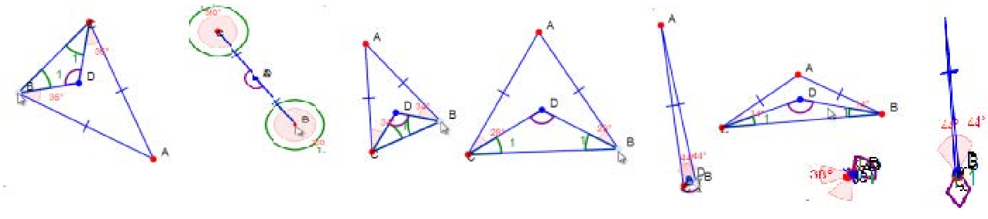


Figure 2: Example space of item B2 created by the high-skill pair

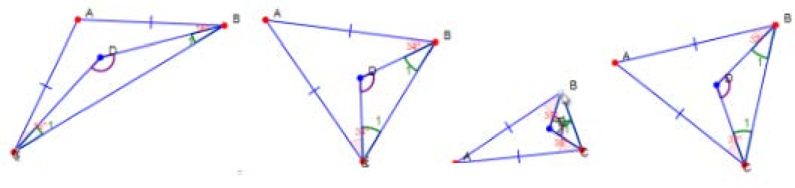


Figure 3: Example space of item B2 created by the low–skill pair

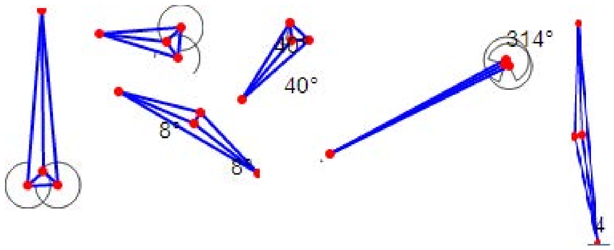


Figure 4: Counter-examples for item B1

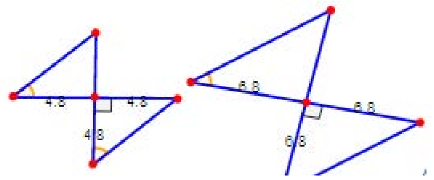


Figure 5: Examples of common strategy in item C2 (using isosceles triangles)

Analyzing terms in the given statement

The study uses the group feedback sheet to demonstrate the affordance of the assessment tool to inform the teacher in regard to students' interpretation of the terms in the given geometric statement. The examination feedback provided a view of the interpretation of terms and concepts in class. In the feedback sheet for the task "if AB ⊥ CD and ∠A = ∠D, and M is the intersection point of AB and CD, then ΔACM ≅ ΔDBM" (Figure 6), we were able to identify misinterpretations of the concept of "corresponding parts in congruent triangles." Although most students generated triangles with equal angles and one pair of equal sides, a substantial number of them (marked in orange) matched non-corresponding sides (AM = CM, instead of AM = DM).

Another common misinterpretation was found on the group feedback sheet of another task (B2), where students were asked to provide a supporting example for the incorrect existential statement: "the bisectors of an isosceles triangle create an acute triangle" (Figure 7). Although the example on the left is an extreme one and the closest to an acute triangle, the bisectors in the other two diagrams create clearly obtuse triangles. We conjectured that students misinterpreted the term "acute triangle." This assumption was verified in the class discussion following the ex-

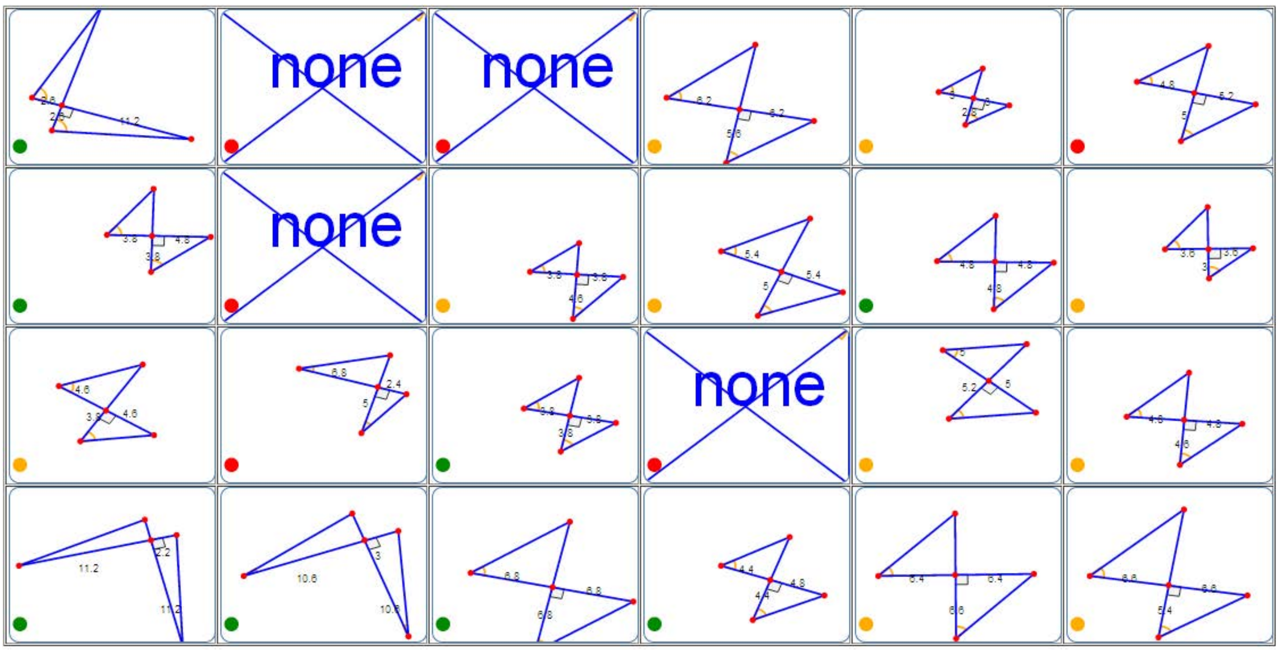


Figure 6: Group feedback for item C2

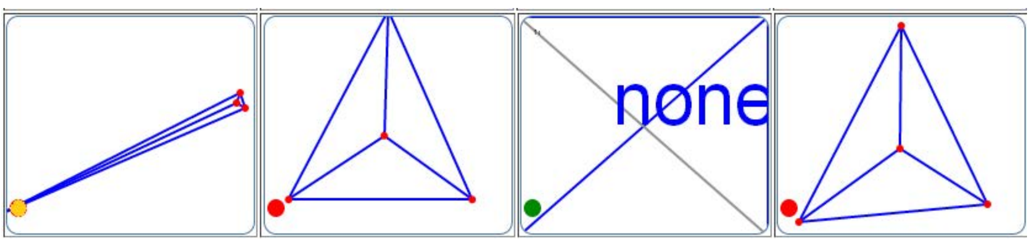


Figure 7: Partial feedback sheet for item B2 supporting example

	Counter-example		Supporting example	
	Validity	% Correct	Validity	% Correct
A	Invalid	84%	Valid	83%
B	Valid	79%	Invalid	59%
C	Valid	68%	Invalid	76%

Table 2: Correct reponses in regard to validity of statements

amination with students who defined an acute triangle as one that includes a single acute angle.

Determining the validity of the statement

In general, students were able to determine the validity of the statement, as can be seen in Table 2. The lower scores were in task B2, where the term “acute triangle” was misinterpreted by many.

Although most students correctly identified the validity of the statement, they did not necessarily provide a correct example figure (e.g. correspondence in congruent triangles). In a class discussion following the examination, students were asked to justify the “none” answers presented in the aggregated feedback sheet. All students based their justifications on empirical evidence, which means that they were not able to find any examples. This phenomenon was confirmed in the interviews. But during the interviews the high-skill pairs were able to produce a deductive explanation relatively quickly.

DISCUSSION

Based on the partial findings presented above we can draw several conclusions with regard to the objectives of the study: (a) we were able to learn what strategies students used, and take pedagogical action, such as encouraging low-skill students to use more active and fearless dragging in order to investigate extreme examples; (b) we were able to use the group feedback sheet as a tool to identify misinterpretations of terms; (c) the design of the items and the automatic checking provided immediate quantitative and personal data about the validity of the understanding of the geometric statement, but not about the presence of a justification when examples were not available; students simply based their answers on empirical evidence.

The visual feedback sheet helped us quickly identify central patterns of knowledge in class. In a discussion that took place in class after the examination, we displayed the group feedback sheet and found that it encouraged active discussion, as students were eager to participate in a conversation based on their generated examples. Successful brain-storming, involving many members of the class, was conducted about the existence of multiple correct answers, the importance of a large-scale example space and the use of active and fearless dragging strategy for expanding it, the role of extreme examples, and the need for students to justify their “no such example” answer. Future research on the role, use, and format of personal and group visual feedback should be a fruitful field for studying the use of the ample and immediate data produced by e-assessment.

ACKNOWLEDGEMENT

This research was supported by the Israel Science Foundation (grant 522/13)

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ENDNOTES

1. Third test task can be viewed at: http://geo.gigaclass.com/tasks/en/task2bEn.html

2. Third task feedback is available at: http://geo.gigaclass.com/tasks/en/fb2b2En.html

Building stories in order to reason and prove in mathematics class in primary school

Marianne Moulin and Virginie Deloustal-Jorrand

University Claude Bernard Lyon 1, EA 4148 S2HEP, Villeurbanne, France,
marianne.moulin@ens-lyon.fr, virginie.deloustal-jorrand@univ-lyon1.fr

As a part of a multidisciplinary research lead by a team from the S2HEP Laboratory, the aim of our work is to explore the potential functions of stories in Scientifics and Mathematics learning. In this paper we focus on the potential connections between the mathematical space and rhetoric space during problem solving activity. We first characterized theoretically, and then tested experimentally, thanks to a didactical engineering, a didactical environment. We characterise a processes-transferring space between the narrative activity and the problem solving activity. Our results show that the narration act supports the student's mathematical reasoning.

Keywords: Problem solving, stories, narration act, processes-transferring space.

INTRODUCTION

This paper aims to study the contributions of the narrative act in the proving process. There is indeed a natural inclination of the Humans for stories (Bruner, 2003) with a valuable heuristic potential. Our research group has shown, for instance, how the stories' plot and possible worlds brought by fiction can lead children to question their knowledge and build new scientific knowledge (Bruguière & Triquet, 2012). This heuristic prospective leads us to imagine that a reasoning[1] can be built on narrative structures even in mathematics. The reasoning and the narrative acts are both structured process and thanks to Bruner's and Fayol's works, we can assume that the development of a reasoning can rely, from a structural point of view, on narrative structures. Six years old children, who are as able as adults to build complex narratives structures (Fayol, 1985), have to develop mathematical and logical structures. Maybe theses structures and the proof skills related can grow for a part on those already mastered abilities. Following this idea, we developed a model allowing us to anticipate and study connections between the reasoning and the narrative act (Moulin, 2013). In the first part of this paper, we share some theoretical elements about relations between problem solving and story writing activity. We focus on the processes involved and characterize a *processes-transferring space*. Then, we present a didactical environment shaped in order to allow a joint development between narrative and reasoning. We show on chosen examples how the narration-act provides structured guidelines and take part in the proving process.

THEORETICAL ELEMENTS: PROCESSES-TRANSFERING SPACE

The aim of this theoretical part is to define what we call a *processes-transferring space* between the reasoning act in problem solving activity and the narrative act in story building. We rest on Scardamalia & Beireiter's framework (1987) that postulates that during the drafting of a text, the interactions between the rhetoric space (inherent to the construction of the text) and the content's space (concerning disciplinary knowledge) lead to the application of high cognitive functions and to the transformation of the relevant knowledge in both fields (Figure 1).

Outlooks opened by this model lead us to the assumption that during problem solving activity, the commitment to a task related to story building can allow children to initiate, build, structure and/or prove their

1 In this paper, we call "reasoning" the cognitive process that consists in drawing conclusions from facts, evidence, etc. In school, a reasoning is expected form the children when solving a problem.

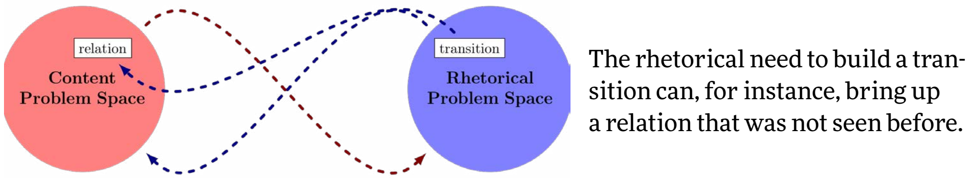

Figure 1: Problem spaces and interactions extract from *Knowledge Transforming Strategy* (Scardamalia & Beireiter, 1987)

reasoning. This assumption brings us to consider a double theoretical framework.

Characteristic elements of mathematical problems2 and solving process

This first point, about problem solving activity, settles down in the field of the didactics of mathematics. Most European curricula consider that the main objective of problem solving is to develop the reasoning and logic skills and to give meaning to mathematical objects. In our work, we study this activity by getting interested in the (cross-disciplinary) skills involvements linked to a heuristic activity (Polya, 1945). To study the processes involved in problem solving, we choose to consider as one object the problem and the solution in a three-block framework (Figure 2).

Determine the whole data's structure with structuring and modelling processes; or build valid conjectures about the nature and the value of the solution involving conception and reflexive processes; or calculate the value of the solution and prove the validity of the calculations using among other explanation and argumentative processes.

The proving process depends on each of theses cognitive processes, which are dependant of the quality of the problem environment. However, school problems are not always enough complex to impose to the pupil the construction of reasoning. Supported by various linguistic tools, we realized an analysis of school problem statement (French textbooks) as if they where stories. As a matter of fact, school problems are often

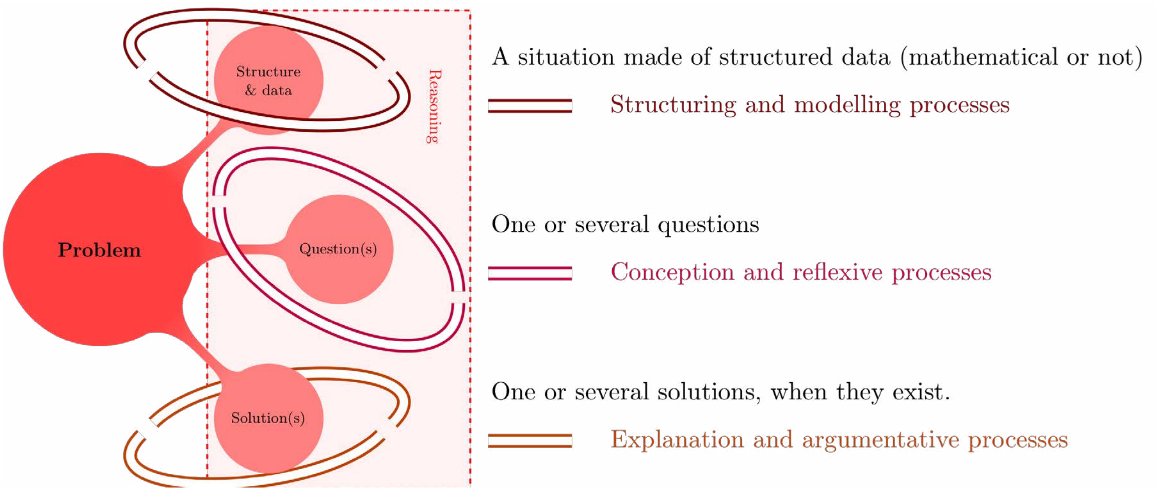

Figure 2: Mathematical problem's components and reasoning process

Problem solving is then a combination of processes developing the necessary actions to determine the solution(s) taking care of the mathematical object(s). Each one of the three problem's components must be handled. The complete processing of each component is equivalent to the problem's resolution. The problem can bee seen as solved when you have either:

(if not always) presented as little stories. But, none of them included neither problematic elements nor triggering factors as intended in the context of stories (Moulin et al., 2012). We could raise here the question of the relevance to present a problem in the form of a story if we remove the element from it that indeed "poses problem". But our main focus here is to grant a larger place to the story dimension already given in the problem's statements. The purpose of this approach is to supplement problem-solving process with *stories functions* that will be presented in the following point.

2 According to us, the name *mathematical problem* group together all the situations involving a mathematical object and asking one (or several) question(s) to which it is possible to answer only after the elaboration of a reasoning.

Characteristic functions of stories and narrative act process

We move our focus off didactics to the heuristic and structuring functions of stories linked to the narrative act. With this opening we want to highlight similarities in the cognitive activities and processes related to both activities (structuring, explanation, problematization, argumentation). When it comes to stories, one first thinks of a linguistic object, telling a story and showing some characteristics of singular shape. However, one can also approaches stories from the angle of their elaboration, as a mindset, with regard to the heuristic, and structuring functions put forward by several researchers such as Bruner and Ricœur. This is the way we chose.

To involve theses functions in the problem solving process, we have to consider in the same kind of way in one hand problems and stories and in the other hand reasoning and narrative act process. Stories have structural characteristics (Reuter, 2009) of their own organized around a plot. The plot may be considered as a question to which it is necessary to try to answer. From then on, the reception of a story, just like its production, involves cognitive processes of recognition and reproduction of this structuring. The solving of the plot therefore corresponds to a (more or less complex) sequence of movements of actions proposed by the story allowing to find a state of balance. Combining Genette's (1972) and Bruner's (2003) work, we came up with a three-block design for stories (Figure 3).

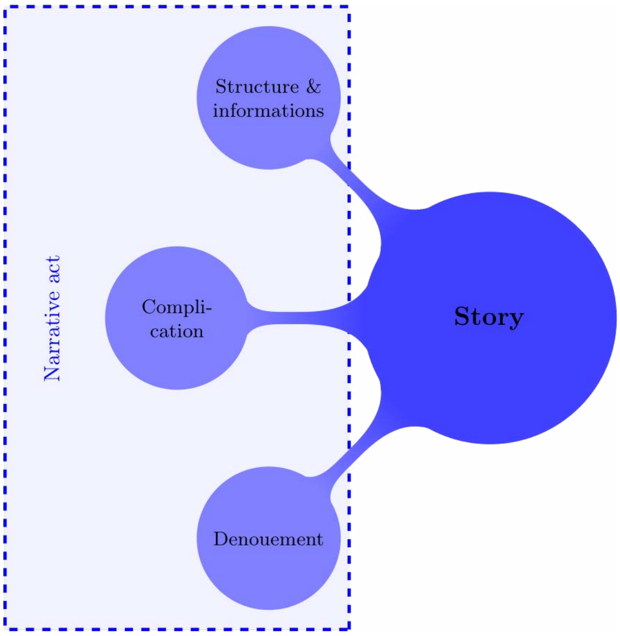

Figure 3: Stories' components and narrative act process

The characteristics of the story object express themselves in the processes of thought allowing to understand it, to build it and conversely. The solver/reader has to imagine from the information different explanatory possibilities. Fiction, as a characteristic of stories, brings a whole space to do it. For Ricœur (1983), the story has moreover this capacity to organize what is disparate into a coherent whole (*holos*). The reception, like the production of a story, therefore imposes to (re)build the temporal, spatial and causal relations of the presented events. These heuristic and structuring functions of stories are strongly connected to problematization and are submitted to an internal logic. So they play a part in similar processes to those that we have described for problem solving. There are three main functions considered in our work:

- The structuring function through situation, structure and information;
- The problematization function induces by the complication and the plot's study;
- The explanation function linked to the solving of the plot and the resolution.

Process transferring space in problem solving and story writing activity

We can now identify a potential *process transferring space* between the content problem space (including the mathematical problem) and the rhetorical problem space within with the story is build (Figure 4). Considering, as Scardamalia & Beireiter (1987) did, that a process is an answer to a local problem we can assume that each local problem (content or rhetoric) and the related process can be either realised in the content problem space or in the rhetoric one. In the context of a mathematic problem, we can make a "transfer hypothesis" assuming that *every processes related to the proving process (explanation, conjecturing, argumentation, etc.) can be handled with the help of tools available in the rhetoric space and vice versa.* In other words, as part of a problem solving activity, the narrative act and the narrative functions going with it can under certain conditions come in the solving and proving process.

In our work, we develop the story scope in integrating it as an essential component of the didactical environment (Brousseau, 1998). We position stories as an antagonistic object of the pupil in problem solving

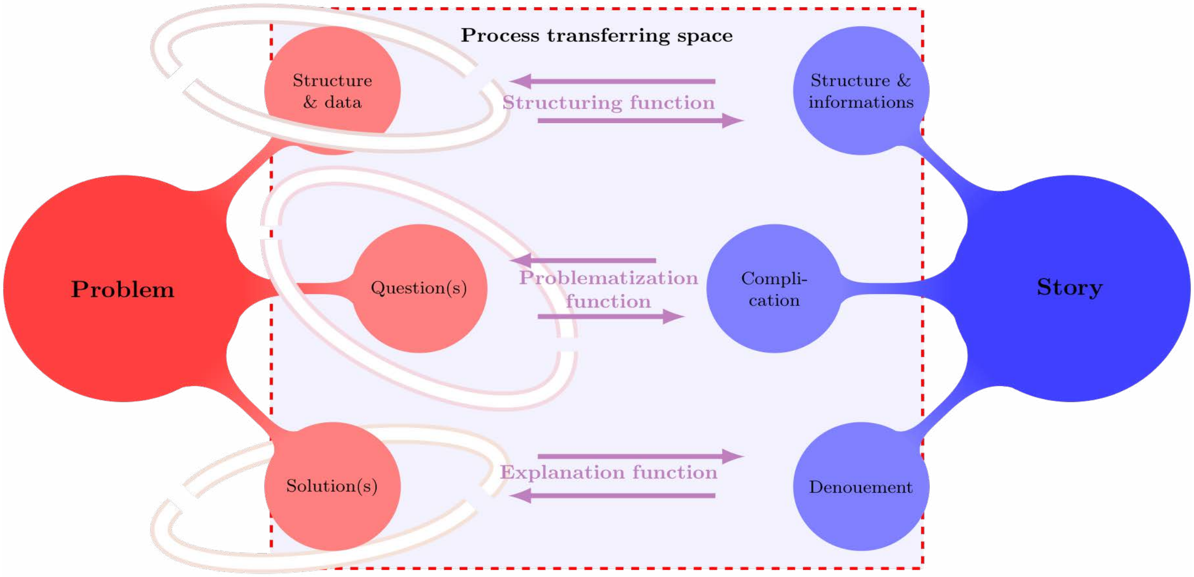

Figure 4: Process transferring space

activity. Indeed, with a structure and logic of its own, story complements the didactical environment. The story structures a space of thought in accordance with the situation at stake in the problem. According to us, this structured space allows the pupil to build and to validate his/her reasoning. We set up an experiment to test various aspects of the interactions between the story and mathematical reasoning. We present the original situation in the following point.

SITUATION AND DIDACTICAL ENVIRONMENT

With the aim of testing our transfer hypothesis, we shaped a didactical environment (Brousseau, 1998) with the objective of ensuring a joint development between narrative and reasoning. The situation of problem solving we offer is built around a game of spinning tops in which two players are in confrontation according to the rules given in Figure 5.

Due to its progress – a sequence of rounds bringing gains and losses of points – this game has a structure that is possible to determine completely through the application of mathematical rules. For instance, the score of the winner of a game is always between 7 and 9 and it is necessary to play a minimum of three rounds to end a game. These properties form, among others, the mathematical and logical structure fixing the possibilities and the impossibilities of the situation. It is on this structure, constituting a local axiom (Tarski, 1969), that we built our mathematical problems.

In a first time, we asked children to produce descriptive narratives based on games actually made. These descriptive stories were meant to be a support for the children to move from a material situation to a more objective one. At this point, the events of the game are the

The game is divided in rounds. To the signal, both players throw their spinning top in a stadium at the same time. It contains a play area and two areas of penalty. The round ends when: One of the spinning tops does not spin any more; One of the spinning tops is in the penalty area; One of the spinning tops is not any more in the stadium; A player touches the stadium. When the round is over, the points are distributed (the rules are applied in order, as soon as a point is given or removed, we move to the following round):
- 1 for the player who throws his spinning top outside the stadium;
- 3 for the player who touches the stadium during the round;
+3 for the player who sends the spinning top of his/her opponent outside the stadium;
+2 for the player who corners the spinning top of his/her opponent in the penalty area;
+ 1 for the player whose spinning top is the last one to stop.
The first player who gets 7 points (or more) wins the game.

Figure 5: Rules

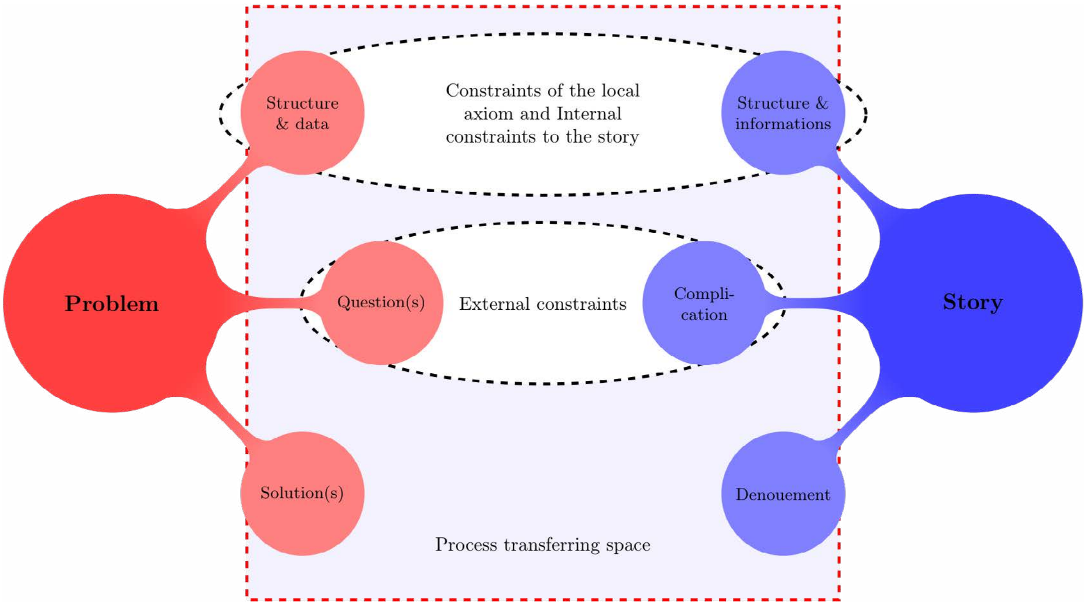

Figure 6: Constraints and didactical environment

events of the story; the scores are included in the story and the game's rules are automatically taking cared of.

Then, in a second time, we confronted them to various mathematical problems about the mathematical structure of the game and the properties presented above (about possible scores, number of rounds, etc.) To address these problematic situations, children could use anticipated-games narratives (based on imaginary games). At this point, the (imaginary) events of the game are still the events of the stories; however, points became mathematical objects and the game's rules are a mathematical structure. The problem's question reflects on the story's complication. There is a room for co-building between the reasoning and the story. In the built situation, the resolution of the problem and the proof process is subject to three types of constraints (Figure 6):

— *Constraints of the local axiom:* These are the constraints carried by the situation (Durand-Guerrier & Diaz, 2005). In our context, they define what can be done or not in the situation and are related to the structure and the data. For instance, the winner always ends the game with a 7, 8 or 9 points.

— *External constraints:* These are the constraints imposed by instructions. They are one or several additional constraint(s), which add up to those of the local axiom. They take place in the question. For example, to impose that the winner wins with 8 points instead of leaving the possibility that he ends up with 7 or 9.

— *Internal constraints to the story:* These are the constraints imposed by the style of the story. As the latter is a structured production, it has to respect an internal logic of time, characters and place.

The interaction between the two structures is the essence of our didactical environment. While solving various problems, children can rely on an environment that includes all the feasible games in accordance with the *local axiom*. We made the local hypothesis that children can produce their results through their stories and validate them thanks to the mathematical and logical rules that govern the situation. The construction through the interaction between the structure of the game and the production of the story constitutes the originality of our experiment. The outcomes presented in the following point are part of a larger research (Moulin, 2014).

RESULTS

We put ourselves in the methodological framework of *didactical engineering* (Artigue, 1988). This one allows us to confront the potentialities of our environment, by an analysis of the choices made determining the "possibilities of action, choice, decision, control and validation that [the pupil] has at his disposal" (p. 258), to the effective productions of the pupils. Therefore, we can validate, in an internal way, our hypothesis thanks to the confrontation between *a priori* and *a* *posteriori* analysis. We conducted our experiment in three primary schools (six class of 10–11 years old children, 138 children). We collected and analysed oral interactions in the class and children written productions. In this part of this article we present some chosen extracts to highlight two significant results regarding argumentation and proof.

Result 1: Children's natural tendency to stories in conjecturing and arguing

The first meaningful result we want to highlight is that, even in a mathematical context, children have a natural tendency to stories.

For instance, after playing and write some descriptive stories children were asked to establish to conjectures about the mathematical structure of the situation:

1. In your opinion, what is/are the score/s that can be obtain by the game's winner?
2. In your opinion, what is the minimum number of rounds needed to end a game.

The only constraint they had in some cases was to justify all of their answer (using the shape of a story or not). They indeed used stories to justify and/or prove. For the most part, 61 over 113, children use an explanatory possible story to justify their answer[3]. Only ten answers are based on mathematics using numbers and calculations. It seems easier for children to work in the narrative space than in the mathematical one (which concerns directly the game's rules). Moreover, the justifications build in the narrative space produce more correct conjectures than the one build on the mathematical space. 85% of the thirty-two complete conjectures came along narrative justification.

When we orally asked children to conjecture about the structure of the situation, they build and used stories in various ways:

— Anticipate potentially feasible games to put forward a conjecture;

— Propose an example (or a counter-example) to validate (or invalidate) a conjecture; the validation (or non-validation) is made by the confrontation between the stories, the mathematical structure of the situation and the constraints of the question.

While producing stories, students embrace the mathematical constraints of the situations and step back from the sensible world. No more restrained by material thinking, the students were more inclined to mathematical approach. Therefore, they develop or enhance their mathematical proving skills (example and counter-example arguments, mathematical conjectures and demonstrations, etc.). Some of them even produced mathematical proof of the impossibility to reach ten points.

Result 2: Children easily travel between the two spaces to solve problems

During our experiment, we asked children to solve various mathematical problems. For the most part, according to Vergnaud's typology for additive structures, they were problem with a composition of transformations (Vergnaud, 1986). The one we present now had this structure and the following wording:

> *Laura plays with spinning tops. During the beginning of the game, she wins 5 points. In total, she gains 3 points. What happened during the end of the game?*

With this formulation, we didn't constraint the children to use a narrative answer. We could have asked them to "tell" like we did in other exercises. However, a lot of them use narrative ways in order to solve this problem and the same type's other problems. We got a good rate of success comparing to the usual rate with this kind of problem4. But, most of all, we want to focus here on the effective practice use by the pupils to solve these problems. The oral correction sessions reveals that children easily travel between two problem spaces: the mathematical one and the rhetorical one. In this specific problem, the difficulty was to identify that you need two transformations (instead of one) to get the lost of two points. In the following example you can notice that the child goes from the narrative point of view, to the mathematical one, then again to the narrative one and finish with the mathematical one.

At the beginning, I though that it needed just one round, just that ... because it's said that she gain [loose]

3 The other answers were based on the games played during the previous session or on the game's rules. For instance, "you can have 8 points because when I played I won with 8 points" or "you can have 7 points because when you have 7 you won".

4 More than 50% against 25% according to Vergnaud (1986) in similar situations.

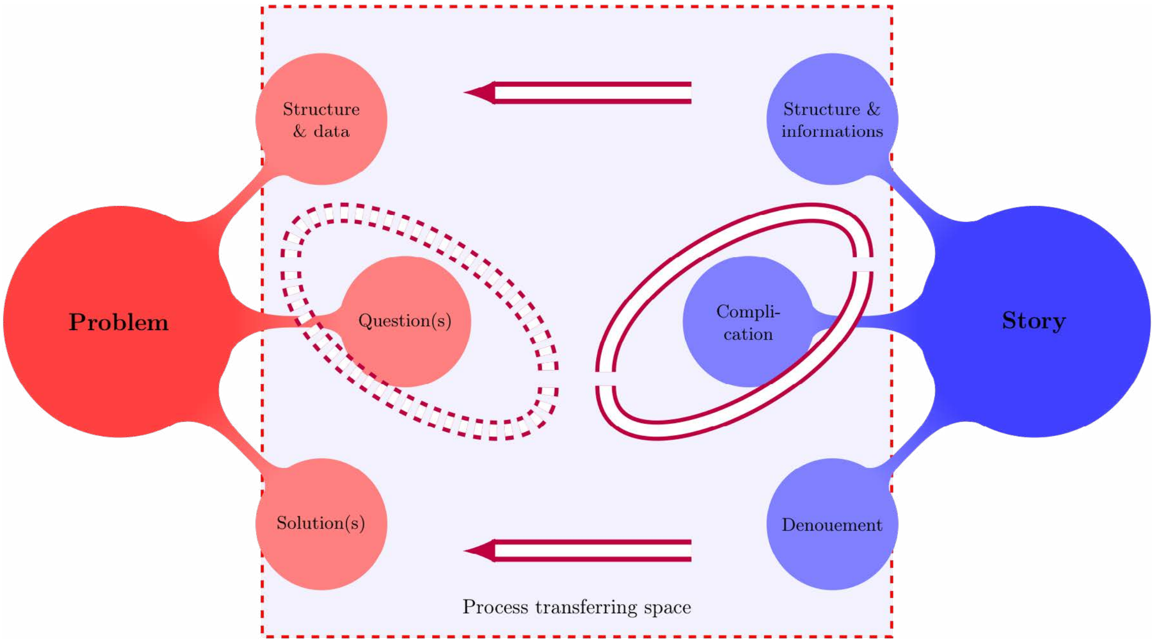

Figure 7: Problematization processes transfer

three points. She had five points in the beginning so she lost two points. And then, because there is no rules bringing the lost of two points, there is only minus one or minus three (...) so I said that I needed two rounds.

It seems that the combination of the narrative and the mathematical space offers a rich environment for children to work in. Instead of being considered only in the mathematical space, the problem can be part-treated in the story space (Figure 7).

The management of the problem in the narrative space brings children to improve their study of the situation. Because of that, their solving and proving are more accurate. Due to their heuristic and structural functions (Bruner, 2003), stories can play a part in problem solving. The act of narration supports the student's mathematical reasoning and justification. The story enriches, with the meaning of Hersant (2010), the didactical environment: there is more possibilities to explore the empirical part of the environment thanks to the fiction brings by stories; there is a need of proof brings by the structural aspects of stories. In accordance with our theoretical framework, story building together with a problem solving activity produces an environment allowing pupils to commit themselves, to structure and to justify the followed reasoning.

CONCLUSION

Our analysis of oral and written children productions reveals that stories are a powerful asset in problem solving activity. By integrating the story into the didactical environment, we offer children a structured space to reason and argue about problematic situations. Taking charge of an explanatory possible or impossible, via story building, allows them to argue and so to conjecture and to get into a proof approach. In a more general way, all the functions of the story can be mobilized as part of problem solving. Structuring, explanation, argumentation are inherent processes for both story writing and problem solving. The possibilities of interaction between these two activities let imagine a joint development, in the pupils, of the capacities needed in the proving process.

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The genesis of proof in ancient Greece: The pedagogical implications of a Husserlian reading

Andreas Moutsios-Rentzos and Panagiotis Spyrou

University of Athens, Department of Mathematics, Athens, Greece, moutsiosrent@math.uoa.gr

In this essay, we present a reading of the genesis of proof in ancient Greece through the lenses of Husserl's transcendental phenomenology. We argue that the Husserlian perspective acts as the epistemological bedrock upon which a didactical framework that fosters the students' need for proof may be built. Importantly, we posit that this framework allows for the students' developing internal need for organising the corpus of mathematical knowledge within a deductively derived structure.

Keywords: Proof, geometry, Husserl, phenomenology.

THE STUDENTS' NEED FOR PROOF

The notion of proof is at the crux of modern mathematics, constituting the backbone of the axiomatic system implied by Euclid. Mathematics educators have investigated the phenomena related to proof, considering amongst others different protagonists (including students, teachers, mathematicians), their conceptions of proof and its functions, their cognitive and affective proving products and processes (Boero, 2007; Moutsios-Rentzos & Kalozoumi-Paizi, 2014; Reid & Knipping, 2010).

Though researchers have documented various functions of proof (including verification, explanation, systemisation; Hanna, 2000), the students seem not to share these conceptions. For example, high-school students appear to consider proof as means for establishing verification and to a lesser extent for explaining and communicating (Healy & Hoyles, 2000). Moreover, mathematics undergraduates would employ mathematical proof as an exam-appropriate answer, but they may choose a 'softer' argument (example, figure etc) to convince themselves (Moutsios-Rentzos & Simpson, 2011): they produce a proof to meet the externally-set requirements of a task, but their internal need for proof seems not to necessarily be in line with a fully-fledged conception of proof. The students need a reason to produce a proof (Balacheff, 1991), which may be externally or internally referenced (Moutsios-Rentzos & Simpson, 2011).

Zaslavsky, Nickerson, Stylianides, Kidron and Winicki-Landman (2012) discussed the mathematical and pedagogical aspects about the need for proof, differentiating internal needs amongst: *certainty* (verification of the truth of a statement), *causality* (why a statement is true), *computation* (quantification of definitions, properties or relationships through algebraic symbolism), *communication* (formulation and formalisation in conveying ideas), *structure* (logical re-organisation of knowledge).

Everyday activities utilising the notion of inquiry are suggested as possible means for fostering the students' developing these aspects of internal need for proof. Though existing didactical frameworks may be employed to help the students to develop internal need for proof, we argue that a Husserlian reading of the genesis of proof in ancient Greece may provide the epistemological backbone of a didactical framework that would foster the students' developing all aspects of internal need for proof, notably 'structure'. The realistic mathematics education research paradigm (Streefland, 1991) appears to be a suitable framework, since a problematic situation that is perceived as 'real' for the students is actively re-organised by the students with the teachers' guidance. The re-organisation of the situation results in the 're-invention' of the required mathematical tools that, constructed as a response to a 'real' need, are meaningful for the students. The process of mathematisation of the 'real' situation allows the incorporation of the constructed mathematical ideas within the existing mathematical world, but it does not explicitly address 'structure'. The new mathematical constructs need to derive from existing mathematical knowledge, but this necessarily implies (at best) only a local mathematical structure and certainly there is no 'real' need for attempting to re-organise the re-invented mathematical tools within a global mathematical structure (such as an axiomatic system). Additional requirements have to be activated for a student to develop the internal need for the logical re-organisation of the re-invented mathematical tools. From a different perspective, Radford (2003) emphasised the sociocultural aspects of mathematical thinking, suggesting a semiotic-cultural approach to highlight the subjective nature of the meaning constructed through semiotic activities. Meaning is constructed by subjects within specific sociocultural context and, thus, proof is meaningful for a student who experiences a specific sociocultural reality. Though we acknowledge the importance of the socio-semiotic dimension, Radford's research was not focused on the students' development of an internal need for proof.

Overall, in this essay we address the fundamental question: *What are the didactical principles constituting an epistemologically coherent framework that may foster the students' developing a fully-fledged need for proof?*

THE GENESIS OF PROOF IN ANCIENT GREECE

Katz (2009) notes that the notion of proof appeared in ancient Greece. Many of the mathematical results were already known, in the same way that something is known in the sensory-perceived world: as rules that held true for all the till then considered cases. With Greek mathematics things changed, including "objects whose existence cannot be visualised and which cannot be physically realised" (Grabiner, 2012, p. 152). Moreover, the mathematical ideas were re-organised to form a primitive proof-based version of an axiomatic system. A multiplicity of factors formed a complexity within which proof appeared to be 'natural'. But which were those factors?

The sociocultural context of the ancient Greek city (*polis*) appears to be the crucial factor that enabled the change of perspective about the issues that proof addresses in mathematics. Polis was the result of a transformation from monarchy to democracy. Employing the case of ancient Athens as an exemplar, we find that the legislation of Solon and crucially of Cleisthenes changed the social structure of Athens, resulting to a radically transformed lived social reality. The Athenians were administratively organised in ten geographical regions that purposefully did not correspond to the traditional *phyla* ('families'), in order to shuffle the traditional, blood-centred, immediate social circle of the individual. Thus, the new immediate social family was based not only on blood relations, but also on a *purposefully arbitrary* geographical proximity. Moreover, each of the ten new regions was the ruling region for a tenth of the solar year. This meant that each ruling month was not a lunar month. The time that a region was in power was *not* measured with reference to nature, but according to a *purposefully arbitrary* chosen fraction of the solar year. 'Arbitrary' is emphasised, because the number of the new regions could be anything that would ensure the *un-settlement* of the old structure. Furthermore, Solon's changes produced a hierarchy of citizens, according to specific analogies forming a harmony (2/1, 3/2, 4/3). Cleisthenes' reform reduced all these relationships to a single analogy, the simplest possible: 1/1. All the citizens were equal with respect to access and power within the polis, regardless of their profession, family name or wealth.

Within the polis all the important aspects of life assumed a public character. The 'significant' private obtains its 'significant' status by becoming object of the community. For example, murder was not a private matter to be resolved amongst individuals. It is a public matter open to the actions of the community which focus on the 'objective', verbally described characteristics of the *situation*, rather than on *who* was involved in the incident. In order for 'justice' to be reached, the community had to be convinced of what happened, to construct a shared logos. Note that *logos* in Greek has a multiplicity of meanings including oral speech, reasoning and ratio (and relationship in general). The common logos emerged as the ruling power of the city, forming a differentiated from ethics law; the ethically acceptable may or may not be lawful. Justice became a matter of a social, non-metaphysical, construction. The citizens of the polis were characterised as such by actively participating in the common matters. The Athenian *idiot* ('private') was the person who either lacked the reasoning skills or chose not to contribute in the public affairs. The citizen was a 'subject' to the logos, to the verbal communication and co-construction of the common, argued meaning. Language and the arguments employed were at the crux of this process. Through language the private meanings were communicated and through convergences and divergences the shared public meaning emerged.

Cleisthenes' changes towards the equality of the citizens within the public affairs allowed the transcendental notion of power to obtain an anthropological character: the numerical majority was right, true and responsible and the minority had to accept it. The ruling power was not divinely-given, nor inherited, but lied within the *countable community*. The shared logos, the thesis voted by the citizens was within the reach of every citizen-subject, as long as it was accepted as such by the majority. This conceptual lift from the subjectively described to the objectively defined by a simple number, by a numerical relationship, allowed for the city itself to obtain a transcendental aspect, to exist regardless of who were its citizens. Its infrastructure transcended the people who represented it. In this way, the polis achieved its supertemporal continuity. Thus, the subject was at the same time unique and the same, one and many, important and insignificant. Heraclitus stressed that "although *logos* is common to all, most people live as if they had a wisdom of their own" and that "having listened not to me but to the logos it is wise to agree that one is all". It should be clarified that Heraclitus wording for 'agree' is *homo-logo* (common logos), indicating that agreement is a result of a shared logos. Hence, common logos implies all private understandings and reasonings are in agreement with (*homo-logia*), in a relationship with, the public *logos*. Notice that the shared logos does not imply the disappearing of personal identity (Vernant, 1983), as the self becomes a multiplicity of higher mental internalised social relationships. Vygotsky (1978) notes that the external social processes are closely linked with the internal psychological processes so that in "their own private sphere, human beings retain the functions of social interaction" (p. 164). Thus, the argument became a dominant social instrument.

Within this sociocultural framework, the requirement of producing a proof for a mathematical statement seemed to naturally fit in. The mathematical community as part of the general community requires arguments that cannot be logically disputed. Such an argument could not be based on perception, which was philosophically treated at the time as false, changeable or unreliable. Nor could it be based on authority or affective linguistic tricks. The Sophists, the Eleates (notably Zeno) and the philosophy of Plato and Aristotle crucially determined Euclid's decision to organise old and new mathematical ideas in a deductive structure, within which each proposition derives from already proved or accepted as true ideas.

Moreover, within a social framework that the public is appreciated and the private is frowned upon, mathematical ideas had to be open to the community and not to be only for a certain social cast (the clergy or other). This required resorting to commonly lived experiences, which were inescapably bodily experiences masked as 'semi-abstract' ideas. This is reflected in the 'pseudo-axiomatic' character of Euclid's elements. The definitions, the common ideas, the axioms derived from the shared lived perceptual reality, which ensures the wider acceptance of the logos that draws upon such a structure, but clearly limits the breadth and depth of the mathematical structure. Nevertheless, Euclid's organisation enabled the synthesis of seemingly unrelated ideas, deriving from the same underlying ideas and reasoning (for example, the study of incommensurable magnitudes and the irrational numbers). Though Szabó (1978) claims that the notion of deductive proof did not meet any practical needs, we argue that it met the lived needs within the broader ancient Greek sociocultural context when transposed in the abstract-like Euclidean world. In this conceptual extension of the perceived reality, the logos and the argument are the only means for establishing the truth of a proposition.

Overall, we agree with Vernant (1975) who argued that the formation of the *polis* was the decisive event that allowed the shared logos to become the backbone of the social structure. We briefly discussed some of the factors that may have constituted this event: the *shared logos*; a *purposefully arbitrary administrative structure*; the *1/1 citizen relationship*; the *countable decisive power*; the *convincing the majority verbal argument*; the *reign of the public over the private*; the *quantification of power*; the *argument based on commonly experienced notions and ideas*; the inescapable *reign of the deductive over the inductive within an axiomatic-like system*. All these elements are some of the crucial events that posed the need for a deductive proof, rather than settling for an inductive or other argument.

ELEMENTS OF HUSSERL'S PHENOMENOLOGY

Husserl's phenomenology may be summarised in the phrase "back to 'the things themselves'" (Husserl, 2001, p. 168), implying the attempt to 'unearth' the sedimented relationships and the decisive factors, in order to mobilise the mental processes that constitute an ideality. Husserl's idealities crucially differ from the platonic ideas in that they are intentionally subjectively *constructed once within history*. Once objectified, they become atemporal, in the sense that every subsequent subjective knowing requires only the *reactivation* of this objectification. Language (oral or written) constitutes the means for the objectification of the subjective experiences, allowing their subsequent transcendental existence. The reactivation of objectification requires the subject to develop suitable intentionality, suitable "conscious relationship [...] to an object", (Sokolowski, 2000, p. 8). Such intentionality requires the suspension of the subjects' natural attitude, their "straightforward involvement of things and the world" (Audi, 1999, p. 405), implying that the objectification is not merely a psychological process, as it explicitly incorporates the relationship between the subject and the community.

Husserl contrasts the intersubjective experience of the communicated shared meaning with the transcendental subjectivity in which there is an awareness of a phenomenon that transcends the subjective perceptual experience: "a possible communicative subjectivity [...] through possible intersubjective acts of consciousness, it encloses together into a possible allness a multiplicity of individual transcendental subjects" (Husserl, 1974, p. 31). In order for such processes to be activated, Husserl's phenomenological reduction (*epoché*) is required. By bracketing out, suspending, natural attitude and by investigating the sedimented intentional history of the object, the phenomenological attitude is activated in order for the subject to "seek for its "constitutive origins" and its "intentional genesis" (Klein, 1940, p. 150). During epoché, the subjects' thinking is characterised by the subjects' intentionality and immanence to bring to the surface the sedimented already constructed and existing within the community knowledge.

TOWARDS A DIDACTICAL FRAMEWORK

In what way can the aforementioned genesis of proof be read through a Husserlian perspective in order to inform a didactical framework that fosters a fully-fledged need for proof? Though 'replicating' history in the classroom is clearly not possible, an ancient idea "through an adaptive didactic work, may probably be redesigned and made compatible with modern curricula in the context of the elaboration of teaching sequences" (Radford, 1997, p. 32). We shall argue that the Husserlian perspective may help in determining the principles of the 'adaptive' work required.

In order to identify the ways that Husserl's views may inform a didactical framework, we should first consider the following: What is the students' natural attitude towards mathematics and learning in general? What is the role of language? Of technology? What is the perceived by the students' natural form of argumentation in mathematics? In everyday life? We do not claim that there are universally applicable answers to these questions. Each country, city, school unit, class have their special characteristics constituting a unique system (Moutsios-Rentzos, Kalavasis, & Sofos, 2013). Nevertheless, we shall describe some elements that we think characterise the lived reality in Greece. With respect to the students' natural attitude to mathematics, it appears that many students' consider mathematics to be beyond their lived reality, to be hard, boring or unnecessary (Brown, Brown, & Bibby, 2008). Healy (1999) argues that the current technologies prevent the students' minds from developing deductive reasoning, while it has a negative effect on their "ability to remain actively focussed on a task" (p. 201). Though such claims may sound too strong, the current sociocultural context is fast, based on inductive arguments and decisions, while the virtual social networking sites produce a multiplicity of realities within which the students act and interact (Moutsios-Rentzos et al., 2013). The role of language in this complex context appears to have radically transformed. The need for fast, usually factual, communication developed shorter versions of words, sentences, meanings. Such abbreviated forms hardly suffice when discussing mathematical objects. Thus, the verbal, logically complete argument identified as the main vehicle for establishing the need for proof appears to be in stark contrast with the contemporary linguistic habits. A further consequence of the steep rate of change is that even a local 'logos' or connotations may suffice for the arguer to accomplish his/her purpose. The shared memory is short and the lived present is even shorter; there is no real need for immanence. This fragmented, disjunctive, sociotemporarily fragile common 'logos' and practices do not bear any resemblance to the common logos experienced in ancient Athens. Moreover, the contemporary way of living is characterised by connectedness, by the existence of non-linear networks. The seemingly simplistic linear, deductive argument makes sense to be considered by the students as incompatible or even useless in such a connected, seemingly non-hierarchical reality, favouring other forms of argumentation (including inductive or abductive). Consequently, the shared 'logos' of the contemporary sociocultural students' reality, their natural attitude, seems to be far from the phenomenological attitude that lead to the genesis of proof. Which pedagogies may facilitate the students' experiencing the reactivation of the need for proof? We argue that an appropriate *epoché* should be cultivated to suspend the students' 'natural attitude' (for example, not to prove something obvious), allowing for the students' phenomenological attitude to reactivate their need for proof.

Drawing upon these and upon the realistic mathematics research paradigm (the epistemological bedrock of which is close to a Husserlian perspective), we provide a sketch of a didactical framework (with examples deriving from Moutsios-Rentzos, Spyrou, & Peteinara, 2014). First, the students should be familiar with the practical, everyday uses of mathematical ideas. Mathematics should be 'real' for the students, it should be 'discerned' in the lived world, as it can be practical, useful. This may require the teachers' drawing the students' attention to everyday situations that incorporate sedimented mathematical ideas. For this purpose, the starting point may be a problematic 'real' situation for the students that requires the re-invention of mathematical tools to be resolved. For example, the construction of a table requires a perpendicularity identification physical tool and the construction of such a tool may facilitate the students' re-invention of a mathematical tool (e.g. the Pythagorean Theorem). It should be stressed that the materials employed in the students' investigations are at the crux of the proposed framework, since they constitute the physical shared reference of each communication (see Moutsios-Rentzos, in press).

In line with our reading of the genesis of proof, the mathematical ideas should derive from some common (at least in the beginning) to human principles. For this purpose, the common to the human body sensory experiences of the world may be the bedrock upon which the shared logos may be professed. Though perceptually born, those common principles can, by the necessity of obtaining a shared meaning, be potentially stripped of their subjective nature. For example, the human body is evolutionally designed to identify verticality, which enables us to survive in a perceived as perpendicular to verticality (horizontal) world. The sensory experience of perpendicularity –in order to be potentially infinitely communicated– is required to be linguistically described with appropriate signs. The aforementioned perpendicularity identification physical tool may be initially constructed with reference to an independent from human activity, naturally existing, perpendicularity (e.g., the angle between the surface of the liquid and the string of the 'plumb-bob').

Furthermore, appropriate interventions may facilitate the students' conceptual shift in the semiotic registries employed in their *communicating* their embodied experiences. For this purpose, it is crucial for the students to realise the need for employing more symbolic and abstract semiotic registries in order to successfully resolve the situation *and* to communicate (and to convince) their argument about the validity of their solution to their classmates, to their teacher, to whomever whenever may face such a situation. For example, the students may construct a wooden triangular frame that visually fits the natural perpendicularity, but the teacher's guidance towards revealing what are the properties that the frame has that renders such a fit feasible may foster the employment of mathematical symbolism. For this purpose, the students may be guided to realise the constraints of the physical material in conveying the 'general' (rule, case, etc.) to a large (potentially infinite) audience.

Mathematical symbolism may help in realising that the mathematical ideas logico-deductively derive (through mathematisation processes) from the communicated, shared experiences, but they no longer (need to) exist within the experience. They (may) have a pragmatic reference, but only ideal essence. For example, the triangle the lengths of the sides of which are 3, 4 and 5 *units* is right-angled regardless of the physical magnitude of the unit, since $5^2 = 3^2 + 4^2$ holds true under the usual algebra.

Establishing a common linguistic expression (homo-logia) of the shared sedimented axiomatic system of some common ideal, yet anthropological, principles is a crucial step in transforming this system to an object upon which mental processes may be acted. In the proposed didactical framework, the students realise that the backbone of the axiomatic system derives from the physical constraints of the human body and as such cannot be absolute or 'given'. Hence, once the axiomatic framework has been objectified, it can itself be subjected to metacognitive investigations. For example, "What if ... we perceptually experience the surface we walk as the surface of a sphere?". Or, "What if ... the $5^2 = 3^2 + 4^2$ is not true?". Our reading of Husserl's phenomenology allows the students' questioning the very fabric upon which the situation is perceived, because the central role of language and communication allows the learners to realise (re-reveal) that the mathematics they experience everyday are only an instance of the infinite potential mathematics the mind can create. Within this potential, the students may come to realise that the mind games with the constituting common principles can be played only with conceptual tools, with reason (for example, algebraic geometry). The need for proof in these strange (to perception) worlds appear to be natural, since proof is the only means for evaluating the validity of a statement. At the same time, the lack of a means for establishing some perceptually derived intuition of the new structure facilitates the students' developing the need for proof as the gatekeeper of the structure itself.

Overall, we argue that the Husserlian reading of the genesis of proof in ancient Greece helped in identifying pedagogical principles – *a 'real'; problematic situation, embodied experiences, pre-scientific materials, language (oral or written), communication (argumentation) to self and others through different semiotic registries* – that form an epistemologically coherent didactical framework. Within this framework, the "divergences of the different levels of communication and experience are constantly re-negotiated in order to converge to a shared logos of condensed meanings and experiences" (Moutsios-Rentzos, in press), thus fostering the students' developing a fully-fledged need for proof (including 'structure').

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Discriminating proof abilities of secondary school students with different mathematical talent

Juan Antonio Moya, Angel Gutiérrez and Adela Jaime

Universidad de Valencia, Departamento de Didáctica de la Matemática, Valencia, Spain,
juan.moya@uv.es, angel.gutierrez@uv.es, adela.jaime@uv.es

A characteristic differentiating mathematically talented students from average students is their ability to solve problems, in particular proof problems. Many publications analyzed mathematically talented students' ways to solve problems, but there is a lack of data about the ways those students learn to make proofs. We present results from a study were we posed some geometry proof problems to secondary school students having different degrees of mathematical ability. We have classified their answers into categories of proofs. Our results suggest that the ability to make proofs of the mathematically talented secondary school students is better than that of the average students in their grade and also that mathematically talented students could be ready to begin learning to make deductive proofs even at secondary school grade 1.

Keywords: Mathematically talented and gifted students, learning proof, secondary school, geometry education.

INTRODUCTION

During last decades there is being an increasing number of publications informing on research on several components of teaching and learning proof (Hanna & de Villiers, 2012; Reid & Knipping, 2010; Harel & Sowder, 2007; Mariotti, 2006). Another important area of research in mathematics education is related to mathematically talented students (MT students hereafter), including, in particular, mathematically gifted students. The literature pays attention to their identification and to several aspects of those students' learning, reasoning, problem solving styles, behaviour, affectivity, etc. (Greenes, 1981; Leikin, 2010; Sriraman, 2008). A frequent methodology to identify MT students' characteristics is to compare the ways they and average students solve the same tasks (Heinze, 2005).

We are interested in the link between both research programs: The ways MT secondary school students learn to make mathematical proofs. There is a general agreement that MT students' learning processes are different from their age peers' ones (Sriraman, 2004). To better understand MT students and to design adequate ways to teach them proof, teachers and researchers work on identifying those differences. In particular, two research questions still needing an answer are:

Are secondary school MT students different from their classmates when solving geometry proof problems?

Are secondary school MT students in various school grades different when solving geometry proof problems?

To get information on these questions, we carried out a research experiment aimed to analyze proofs produced by secondary school students and to find possible differences among students with different levels of mathematical ability or in different school grades. We posed several geometry proof problems to a sample of 1st and 4th grade students and we classified their answers according to the categories of proofs in Marrades & Gutiérrez (2000). Due to the limited number of MT students in our experiment, we do not pretend to get general conclusions, but to bring data from this case study that might point at some differences.

LITERATURE REVIEW

Most of the literature on teaching and learning proof pays attention to whole class groups or, when they are based on case studies, to ordinary students. On the other side, many publications on MT or gifted students pay attention to their ways to solve problems, in particular proof problems, but they do not inform on characteristics of proofs produced by MT students nor on MT students' progress in learning to prove.

Leikin (2010) reflects on some aspects of classes for groups of MT students. She suggests that classes must be challenging for the students and a way to get it is by means of problem solving. Then, Leikin analyzes different types of challenging problem solving tasks, namely inquiry-based, multiple-solution, and proof tasks. The paper also presents an example of a lesson for 9th grade MT students, including examples of students solving proof problems, but Leikin's analysis does not enter into the characteristics of the proofs produced.

Koichu & Berman (2005) present a study based on MT students trained to solve olympiad-style problems. They analyze students' preferences for algebraic or geometric ways to solve geometry problems. Their conclusions show that MT students choose or value proofs taking into consideration effectiveness and elegance, and also that, when students are deciding how to solve problems, they may experience a conflict due to their preference for effectiveness or for elegance.

Housman & Porter (2003) analyzed the types of proof schemes (Harel & Sowder, 1998) produced by undergraduate mathematics majors who had received none, one or two proof-oriented courses, and had earned only A's and B's in their university mathematics subjects. The authors considered that these students were above-average mathematically talented. The students were presented 7 conjectures, and they were asked to state whether the conjectures are true or false and write the proof. Some conjectures are false, so one would expect empirical proofs (show a counter-example) for them. Housman and Porter's results demonstrate that we should expect a variety of types of proofs in any group of MT students, but they did not inform on students' processes to learn to prove nor included average students solving the same tasks.

Sriraman (2003) compared average and MT 9th grade students' solutions to non-routine combinatorial problems. His results show that the MT students were able to get generalized solutions, while the average students used particular cases. Although Sriraman's problems are not proof problems, the ability of his MT students to generalize is an indicator that they could give deductive answers to proof problems.

Sriraman (2004) analyzed the answers to a proof problem on triangles by 9th grade gifted students with no previous contact with proof. Students' processes of solution began with a checking of examples; they concluded that the conjecture is true only for equilateral triangles; then the students looked for counter-examples for non-equilateral triangles and, finally, they tried to prove the conjecture for the equilateral triangle. Sriraman did not give information about the types of proof produced by the students, but it seems that their proofs were empirical although near to deductive.

All the authors referenced in this section have used proof problems as part of their experiments, but none of them has analyzed students' processes of learning to make proofs. Only Housman & Porter (2003) paid attention to the types of proofs produced by MT students, although they did not compare MT and average students.

THEORETICAL FRAMEWORK

The literature shows a diversity of definitions of (mathematically) able, talented or gifted students, where some authors consider the terms as equivalent while others consider them as different (Leung, 1994; Sriraman, 2004), but entering into the analysis of different definitions is not our objective. For us, MT students are those who, when doing mathematics, show certain traits of mathematical ability higher or more developed than other students with the same age, experience, or school grade. Behaviour traits of MT students have been identified, among others, by Freiman (2006), Greenes (1981), Krutetskii (1976) and Miller (1990). In this context, we consider mathematically gifted students as an extreme case of MT students (that, in many countries, got more than 130 points in an IQ test).

In the context of secondary school mathematics, a mathematics education research line focuses on students' process of learning to prove, including both the ability to make a proof and the understanding of the characteristics of mathematical proofs. There is, among teachers and researchers, a diversity of positions respect to the concept of mathematical proof (Cabassut et al., 2012). For some of them, the term proof refers to the logic-formal proofs, and they use terms like justification, argument or explanation to refer to non-formal ways to warrant the truthfulness of a mathematical conjecture. Others define a proof as any mathematical argument created to convince somebody (oneself or an interlocutor) of the truthfulness of a conjecture. We agree with the later, since it has proved to be very fruitful to consider as proofs both authoritarian or ritual arguments, empirical arguments, informal deductive arguments, and logic-formal arguments (Harel & Sowder, 2007). In this framework, the process of learning to prove can be seen as a continuous progress, along the years, starting with authoritarian proofs and ending with logic-formal proofs.

Marrades & Gutiérrez (2000) presented a structure of categories of empirical and deductive proofs. We have used a variation of that structure (Figure 1) to organize the proofs made by our students. This consists of: i) Remove the "intellectual" category, since it really can only be matched to generic example proofs, so it is unnecessary. ii) Add the category of "informal deductive proofs" to discriminate deductive proofs lacking the formal style of language from formal proofs; those proofs are typical of students beginning to understand the need of deductive proofs (3rd Van Hiele level). We also add a "No answer" category, addressed to those outcomes that were either blank or providing no information at all about students' reasoning.

The categories cannot be linearly ordered according to their quality, but there is a perception that empirical categories are more elaborated from top (failed) to bottom (generic example). The same can be said for deductive categories of proofs.

METHODOLOGY

The research experiment took place in a secondary school in a big city in Spain. The students were a convenience sample consisting on several class groups of pupils of a mathematics teacher willing to collaborate. Table 1 shows the number of students in each group. We are reporting here results from students in grades 1 (aged 12–13) and 4 (aged 15–16). A part of the students participating in the experiment were in average class groups (without MT students), while the others were MT students attending a workshop devoted, mainly, to problem solving. The students participating in the experiment had not worked before on the stated problems nor on other similar ones.

To inform on the two questions stated in the introduction, we present data to compare the answers i) by grade 1 average and MT students and ii) by grade 1 and grade 4 MT students. We do not use here data from the grade 4 average students.

We selected several paper-and-pencil geometry proof problems to pose two problems to each group of students participating in the experiment. We did not use the term "prove" in the statements to avoid the possibility of a misunderstanding by some students that could not be habituated to it. To select the problems we had to take into account i) that the students should know the geometric contents necessary to solve them

	Grade 1	Grade 4
Average students	13	41
MT students	3	4

Table 1: Distribution of the students in the sample

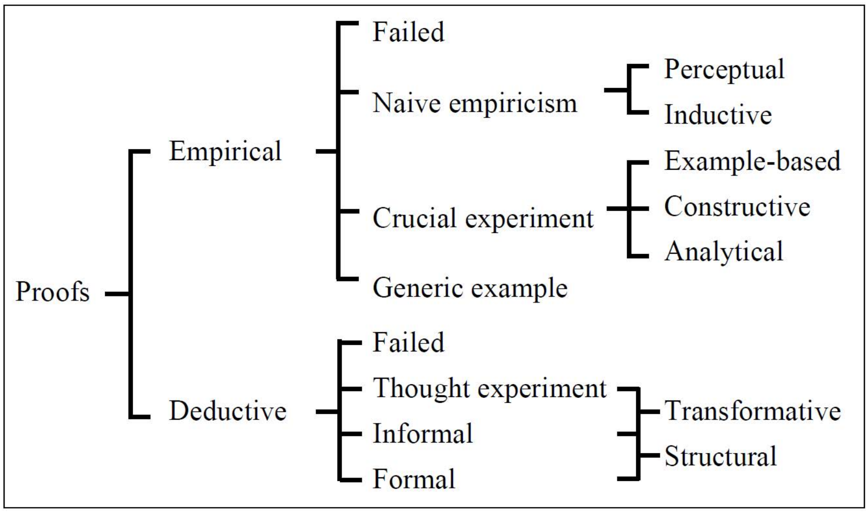


Figure 1: Structure of categories of mathematical proofs

but they had not solved the problems previously, and ii) that the topics of the problems should be related to the one being studied by the average groups at the time of the administration. The second condition to be fulfilled impeded us to pose the same problems to all students. Our aim is to identify the types of proofs produced by the students so data may suggest differences among average and MT students in the same grade or among MT students in different grades.

The problems were posed to each group of students in an ordinary class session of about 50 minutes. They worked alone and the teacher answered questions about the meaning of the statements but he did not give clues for the solution. Our data are the students' written answers.

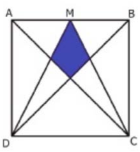
In grade 1, the two problems posed both to the average and the MT groups where:

1) How many diagonals does an n-sided polygon have? Justify your answer.

2) How much is the sum of the internal angles of an *n*-sided polygon? Justify your answer.

The two problems posed to the grade 4 MT group were problem 1 and:

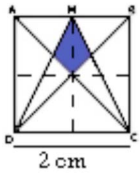
3) In square *ABCD* (figure on the right), *M* is the midpoint of side *AB*. We draw the two diagonals and segments *CM* and *DM*. Which fraction of the total area is the area of the shaded shape? Justify your answer.



To facilitate the answers to less able students and to students who did not come up with a way to solve a problem, each problem had two parts, labelled A and B. Part A was the statements given above. Part B included a clue aimed to help students to start solving the problem or to find a way to the answer. Problem 1B was:

1B) How many diagonals can be drawn from a vertex of a pentagon? How many diagonals can be drawn from a vertex of an *n*-sided polygon? How many diagonals does an *n*-sided polygon have? Justify your answer.

Problem 2B asked for the sum of the internal angles of a quadrilateral, a pentagon and an n-sided polygon. Problem 3B had the same statement as problem 3A but the figure was the one on the right.



To prevent the possibility that students could answer part A of a problem after having read the statement of part B, students were given part B of the problems only after they had completed part A and had given it back to the teacher.

RESULTS

We have classified students' answers according to the categories of proofs displayed in Figure 1. In the following paragraphs, we include information about

	No answer		Empir. failed		Naive empiricism				Crucial experiment						Generic ex.	
					Percep.		Induct.		Examp. -based		Constr.		Analyt.			
	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT
Problem 1A	3		3		1	2	1		4					1	1	
Problem 1B	6		1			1	1		2		1		1		1	1
Problem 2A	6		2		1	2	2								1	1
Problem 2B	7		2				1		1					1	1	1

	Deduct. failed		Thought experiment				Informal				Formal			
			Transf.		Struct.		Transf.		Struct.		Transf.		Struct.	
	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT	AV	MT
Problem 1A														
Problem 1B						1								
Problem 2A	1													
Problem 2B	1					1								

Table 2: Proofs made by grade 1 average (AV, 13 students) and MT students (3 students)

	No answer		Empir. failed		Naive empiricism				Crucial experiment						Generic ex.	
					Percep.		Induct.		Examp. - based		Constr.		Analyt.			
	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4
Problem 1A					2					1			1	1		1
Problem 1B					1									1	1	1

	Deduct. failed		Thought experiment				Informal				Formal			
			Transf.		Struct.		Transf.		Struct.		Transf.		Struct.	
	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4	G 1	G 4
Problem 1A						1								
Problem 1B					1	1				1				

Table 3: Proofs made by grade 1 (3 students) and grade 4 (4 students) MT students in problem 1

the results of the experiment respect to the question stated in the introduction.

Do average and MT students in the same grade make different proofs?

Table 2 presents the types of proofs made by the grade 1 students in the sample.

All 12 possible MT students' answers (100%) were meaningful proofs. Respect to the categories of proofs produced, 5 out of 12 possible MT students' proofs (41,7% of the answers) were basic empirical proofs (naive empiricism perceptual) and 7 proofs (58,3%) were in the most elaborated types of empirical proofs and in the basic type of deductive proofs (2 crucial experiment analytic, 3 generic example, and 2 thought experiment structural proofs).

On the other hand, 20 out of 52 possible average students' answers (38,5%) were meaningful proofs. All these proofs were in the different empirical types, with more presence in the basic empirical categories (70% of the answers; 7 naive empiricism and 7 crucial experiment example-based proofs) than in the more elaborated categories (30%; 1 crucial experiment constructive, 1 crucial experiment analytic, and 4 generic example proofs). The average students did not produce any deductive proof.

Do MT students in grade 1 and grade 4 make different proofs?

Both MT students in grades 1 and 4 solved problem 1, so we can give an answer to this question based on their proofs in this problem. Table 3 presents the types of proofs made in problem 1 by the MT students in grades 1 and 4.

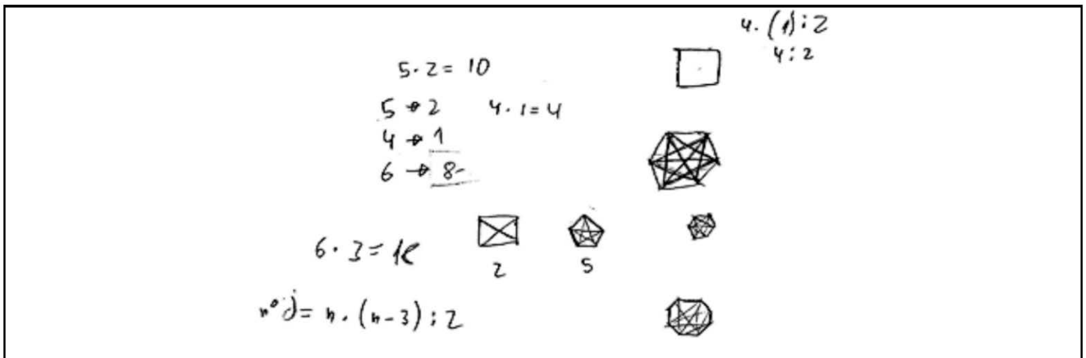
In problem 1A, all MT students in grade 1 (100%) and 3 MT students in grade 4 (75%) made empirical proofs. In problem 1B, 2 MT students in grade 1 (67%) and 2 MT students in grade 4 (50%) made empirical proofs. All but one deductive proofs made were the type thought experiment structural and one MT student in grade 4 made an informal structural proof for problem 1B. Figure 2 shows the answer of this student to problem 1A. He started by checking some specific polygons (4, 5, 6, 8 sides); this let him identify a relationship that he succeeded in expressing as a (correct) formula, although he was not able to prove its truthfulness. This incomplete proof has the characteristics of a thought experiment structural, since the student uses some examples to get an abstract relationship.

This student demonstrates the usefulness of parts A and B of the problems, since the help in problem 1B allowed him to write the deductive proof that he was not able to imagine when he was working in problem 1A.

This complete proof has the characteristics of a deductive informal structural proof, since the student does not use examples to write the deductive proof of the formula.

CONCLUSIONS

Related to the first research question (about differences in proof abilities among MT and average students in the same grade), our data show a clear difference in the ability to write mathematical proofs in favour of secondary school grade 1 MT students (100% of their answers) respect to their average peers (38,5% of their answers). Although the number of MT students does not allow a valid statistical comparison with the average students, the data from our experiment suggest a



You have to multiply the number of vertices by itself minus 3 *n·(n-3)* *because*

Figure 2: Answer by a grade 4 MT student to problem 1A

1. [From each vertex] *ever can be drawn 3* [diagonals] *less that the number* [of sides] *of the polygon because it is not possible to join* [a vertex] *with itself nor with the two* [vertices] *next to it.*
In the previous page I wrote the formula n·(n-3):2 because you multiply the number of vertices times the [the number of] *other non-consecutive vertices, and then you divide by* [2] *because there are repeated diagonals.*

Figure 3: Answer by the same grade 4 MT student to problem 1B

clear difference in the capabilities of MT and average students to produce mathematical proofs, in favour of the former.

It is not surprising that very few deductive proofs were produced by the 1st graders, nor that all of them were in the basic deductive category (thought experiment). The deductive proofs were made by one MT student in part B of the problems. This suggests that it would be worth study more in detail whether, as early as in grade 1, MT students could be introduced into deductive reasoning and, with some help from the teacher, some of them could write simple deductive proofs.

Related to the second research question (about differences in proof abilities among MT students in several grades), the data show that the empirical proofs made by the MT 4th graders were better than those made by the MT 1st graders, since 4th graders did not made naive empiricism proofs and most of their empirical proofs were in the types crucial experiment analytic and generic example.

MT 4th graders made more deductive proofs than MT 1st graders, and the proofs made by MT students in grade 4 were in the same type or better than those made by MT grade 1 students. These data suggest that secondary school MT students are able to advance in the learning of mathematical proof when they are allowed to gain experience in solving proof problems and they have guidance by their teachers. In our case, the ordinary classes provided such experience and, mainly, the workshop they were attending.

As a final summary, we can conclude that MT secondary school students seem to be more capable of producing mathematical proofs (either empirical or deductive) than the average students in the same grades, and also that MT students are able to begin learning mathematical proofs from grade 1, and they can learn to express their justifications in an organized way, progressing along the grades in their ability to make more elaborated proofs, even deductive proofs.

ACKNOWLEDGEMENT

The results reported are part of the research project *Analysis of learning processes by primary and middle school mathematically gifted students in contexts of rich mathematical activities*, funded by the R+D+i Program of the Spanish Ministry of Economy and Competitivity (EDU2012-37259).

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Proof evaluation tasks as tools for teaching?

Kirsten Pfeiffer and Rachel Quinlan

National University of Ireland, Galway, Ireland, kirsten.pfeiffer@nuigalway.ie

This article reports on our experience, arising from an earlier research study, of incorporating proof evaluation tasks into a university mathematics curriculum. In particular, we discuss a task in which students were asked to evaluate and rank five different proposed proofs of a statement from elementary linear algebra. The students' responses to this task prompted rich learning opportunities on the nature and functions of mathematical proofs, as well as revealing some interesting features of their thinking. We argue that proof evaluation tasks can afford rich learning opportunities as well as enabling novice students to participate in authentic mathematical practice.

Keywords: Proof, proof evaluation, curriculum.

BACKGROUND

A CERME 7 article by Kirsten Pfeiffer (2011a) presents a conceptual schema that provides a frame of reference for consideration of what needs attention in a proof evaluation exercise. In accordance with Hemmi (2008), Pfeiffer regards proof and proofs as *artefacts* of mathematical practice. She adapts ideas of Hilpinen (2004) on evaluation of artefacts in general and specializes them to the case of mathematical proofs. In this context an *artefact* is a (physical or conceptual) object that is designed and made by an author (or authors) in order to fulfil a specific purpose (or purposes). Thus the quality of an artefact can only be judged in terms of its success at achieving its purpose(s). In the case of a proof of a mathematical statement, a primary and non-negotiable purpose is that the argument establishes the truth of the statement. Other purposes might include provision of a satisfying explanation, enhancing understanding of the concepts involved, and so on. Motivated by Hilpinen, Pfeiffer suggests that a proof can be evaluated in relating the three features of an artefact, its *intended character*, its *actual character*, and its *purposes*. Therefore evaluating a proposed proof might involve three considerations: that the author's intention or "proof design" is appropriately matched to the purpose of the proof, that the author's intention is appropriately realized in the actual written proof, and that the written proof appropriately achieves its purpose. The point of Pfeiffer's schema is to provide some context and terminology for discussion of what proof evaluation entails and for discussion of specific evaluations of particular proofs. It is intended not as a rigid framework but as a helpful theoretical construction.

The outcomes of Pfeiffer's research study (Pfeiffer, 2011b) included strong indications that proof evaluation tasks, including those involving more than one "proof" of the same statement, have the potential to prompt students to consider the mechanism and fitness-for-purpose of a proof in a serious way. Some students in the study even recognised a change in their own thinking stimulated by the task of comparing different proofs of the same statement. These observations encouraged us to include proof evaluation tasks in the curriculum alongside learning activities of other types.

Over the last decade several investigations into students' performances when validating or reading mathematical proofs have shown that students have difficulties in determining whether a proof is valid (Selden & Selden, 2003; Alcock & Weber 2005). Other studies describe the behaviour of experienced mathematicians when validating proofs (Weber & Mejia-Ramos, 2011) or the differences between novice and experienced readers (Inglis & Alcock, 2013). Techniques or teaching methods to improve students' proof comprehension have been suggested, for example *unpacking proofs* or *proof frameworks* (Selden & Selden, 1995), inclusion of instructional sequences in mathematics courses (Stylianides & Stylianides, 2008), *e-proofs* (Alcock & Wilkinson, 2011) or *self-explanation* (Hodds, Alcock, & Inglis, 2014). We consider *proof evaluation exercises* as another possible teaching practice to accomplish proof reading skills. In our experiences proof validation activities provide a rich teaching and learning tool provoking fruitful discussions and ultimately making a wide range of features and purposes of mathematical proof *visible* to learners. We aim to test the efficiency of proof evaluation exercises incorporated into a University level mathematics course and also to prepare resources for use by teachers.

In this paper we report on one particular proof evaluation exercise performed in a linear algebra course for first year students run by the second author of this article, who is a research mathematician and a lecturer in a university mathematics department. We will show that the students' responses to proposed proofs potentially stimulate a considerable variety of themes to discuss in a teaching/learning environment. As students have engaged with the relevant mathematical context and the suggested proofs in advance, and as they are encouraged to discuss their own feedback rather than experts' proofs and evaluations, students are inclined to participate actively and appreciate these discussions. We will also report on our experiences with the construction of suitable partly flawed 'proofs' and show how Pfeiffer's schema is useful to assure opportunities to highlight various aspects of proof.

EXAMPLE OF A PROOF EVALUATION EXERCISE

The task described below was included in the first written homework assignment in an introductory course on Linear Algebra for first year students. The students were familiar with the concept of a linear transformation of R^2 as a function that respects addition and multiplication by scalars, and they were familiar with the matrix representation of a linear transformation and with the procedure of using matrix-vector multiplication to evaluate a transformation at a particular point.

The proof evaluation task

Students were presented with the following text.

> Alison, Bob, Charlie, Deirdre and Ed are thinking about proving the following statement.
>
> *If the function T:$R^2 \rightarrow R^2$ is a linear transformation, then T fixes the origin, i.e. $T(0,0)=(0,0)$.*
>
> **Alison's Proof**
> Suppose that $T(1,1)=(a,b)$. Then

$T[(1,1)+(0,0)]=T(1+0,1+0)=T(1,1)=(a,b)$.

But on the other hand since T respects addition

$T[(1,1)+(0,0)]=T(1,1)+T(0,0)=(a,b)+T(0,0)=(a,b)$ from above.

So $T(0,0)=(a,b)-(a,b)=(0,0)$.

Bob's Proof
We know that for any element u of R^2 and for any real number k we have $T(ku)=kT(u)$.

Then applying T to $(0,0)$ and multiplying the result by any real number k must give the same result as multiplying $(0,0)$ by k first and then applying T. But multiplying $(0,0)$ by k always results in $(0,0)$ no matter what the value of k is. So it must be that the image under T of $(0,0)$ is a point in R^2 which does not change when it is multiplied by a scalar. The only such point is $(0,0)$. So it must be that $T(0,0)=(0,0)$.

Charlie's Proof
Think of T as the function that moves every point one unit to the right. So T moves the point $(0,0)$ to the point $(1,0)$. Then T is a linear transformation but T does not fix the origin. This example proves that the statement is *not* true.

Deirdre's Proof
Suppose that (a,b) is a point in R^2 for which $T(a,b)=(0,0)$. Then

$T[2(a,b)]=T(2a,2b)=2T(a,b)=2(0,0)=(0,0)$.

Thus $T(2a,2b)=T(a,b)$, so $(2a,2b)=(a,b)$, so $2a=a$ and $2b=b$. Thus $a=0$, $b=0$ and $T(0,0)=(0,0)$.

Ed's Proof
Since T is a linear transformation it can be represented by a matrix. Suppose that the matrix of T is

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

Then the image of $(0,0)$ under T can be calculated as follows:

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} a0+c0 \\ b0+d0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So $T(0,0)=(0,0)$.

The students were asked the following questions about the text above.

(a) Does Alison's answer prove that the statement is true? If not, why not?

(b) Does Bob's answer prove that the statement is true? If not, why not?

(c) Does Charlie's answer prove that the statement is *not* true? If not, why not?

(d) Does Deirdre's answer prove that the statement is true? If not, why not?

(e) Does Ed's answer prove that the statement is true? If not, why not?

(f) Please rank the five answers in order of your preference (according to your own opinion). Include some comments to explain your ranking.

The five proposed proofs provide a sufficient variety of different approaches to provoke learning and discussion about the nature and features of mathematical proof and about the process of proof evaluation. Alison's proof is sufficient to prove the statement, it actually proves a more general statement. An evaluator might question the unexplained introduction of the point *(1,1)* and whether there is a reason for this choice. Bob's proof is written in text and also proves a wider statement than required. Charlie mistakenly proposes a counterexample to prove that the statement is incorrect. Deirdre's approach is written in a style which is familiar to students in the context of mathematical proof. However, her proof contains significant logical errors and does not establish the truth of the statement. Using Pfeiffer's terminology, the evaluator may find mismatches between the intended character and purpose of the proof, and between the actual and intended characters. Ed's proof establishes the truth the statement, but other purposes of proof such as enhancing understanding of the content and context of the statement are not met, i.e. the intended character of Ed's proof does not match these wider purposes.

DISCUSSION OF THE RESPONSES AND OPPORTUNITIES FOR DISCUSSION AND LEARNING

The 28 students whose responses are discussed here are those who fully answered all six parts of the question and included comments (many other students answered only some parts or gave "yes/no" answers without explanation). This account is intended to highlight some features of these students' thinking about proof and some opportunities for learning (for both the students and instructor) that arise. We followed the task with a lecture-based discussion session focussing on the themes mentioned below and prompted by the students' work. This session, though conducted with a large group, was notable for the students' interested attention and for an unusually high level of interaction. This may be due to the fact that many of the themes of the discussion arose directly from the students' written comments.

Alison's proof – responses and learning opportunities

Of the 28 respondents, 17 expressed the view that Alison proves that the statement is true. One was non-commital, and the other 10 stated that Alison's answer does not prove that the statement is true.

Five students objected to the introduction of the point *(1,1)*, apparently believing that focussing attention on this chosen point constituted a restriction of the statement to a particular example. Another accepted Alison's proof as correct, but commented:

Student: I would prefer if she used a point *(c,d)* in R^2 instead of *(1,1)*.

This last comment prompted a discussion about purposes of proofs. The student approves the proof but suggests altering it to avoid the choice of a particular vector. This alteration may make the argument more accessible for some readers, for example for the five of our students who were misled by the introduction of *(1,1)* to the extent that they rejected Alison's proof. On the other hand, some readers might see Alison's specification of a particular vector as simplifying the presentation and might prefer this to the alternative of cluttering the text with "general" notation that is not strictly necessary. The comment quoted above gave us the opportunity to highlight the fact that readers may have different mathematical tastes and that alternative presentations of essentially the same argument may appeal differently to different readers.

Bob's proof – responses and learning opportunities

21 students accepted Bob's proof as correct. One student described it as partially correct, and 6 considered it to be incorrect. It was the second most popular of all the proposed proofs, being ranked first (or joint first) by 8 students.

The six students who rejected Bob's proof stated two reasons for doing so. Two students objected to the assertion in Bob's proof that *(0,0)* is the only element of R^2 that "does not change when multiplied by a scalar" pointing out that (for example) "*(2,3)* does not change when multiplied by the scalar 1". This misinterpretation of Bob's intention highlights the importance of precision in mathematical proof.

The other four students who rejected Bob's proof (as well as three who accepted it) complained that it only used part of the definition of a linear transformation, namely the property of respecting scalar multiplication. All seven of these students criticized Alison's proof on the same grounds; the following is a representative comment.

> Student: Bob supplies the other half of Alison's proof, he proves the statement for scalar multiplication. He is also half right.

The students who reject or criticize Bob's and Alison's proofs on these grounds appear to recognize the strategy of reasoning from a definition towards a desired conclusion, but their verdict that the argument cannot be complete if it uses only part of the definition seems to be automatic. Their written comments do not indicate attempts to assess the significance to the argument of the "missing" part of the definition; their conclusions appear to be founded purely on an inspection of features of the proof without consideration of its logical structure.

No student cited as a reason to favour the proof of either Bob or Alison that both of these arguments prove a more general statement that they are directly concerned with. Alison's argument proves that every additive function fixes the origin, and Bob's proves that every function that respects scalar multiplication fixes the origin.

In the ensuing discussion, attention was drawn to the logical structure of Bob's proof and to the more general statement that it establishes. Students were reminded that a proof evaluator must consider the full content of what is achieved or omitted in a line of reasoning, and not hastily accept or dismiss an argument on the basis of superficial inspection. From the instructor's point of view, the student responses to Bob's proof highlight the important point that novice students are sometimes more attentive to the internal details of an argument than to its deductive quality.

Charlie's proof – responses and learning opportunities

Charlie's proof was considered incorrect by 25 students, and correct by three. It was ranked last by 21 students.

Of the 25 students who rejected Charlie's proof, only 11 did so on the grounds that the proposed counterexample is not or may not be a linear transformation.

Not all of the remaining 14 students who rejected Charlie's proof gave clear reasons. It is possible that the conflict between Charlie's conclusion and those of the other authors prompted some to object, but only two students cited this as a reason. Six students objected to the restriction of attention to a particular function. There is no sign in the work of these six students of acknowledgement that Charlie's goal is exceptional amongst the five, that he is trying to disprove the statement by exhibiting a counterexample. In the context of Pfeiffer's schema, their evaluations of Charlie's proof do not appear to include consideration of the relationship between the content of the proof and its main purpose.

A possibly surprising feature of the responses to Charlie's argument is that of the three students who considered it to be correct, each also accepted at least one of other four proofs. For example, one commented as follows on Charlie's proof:

> Student: As he notes, the linear transformation could possibly move every point one unit to the right. Therefore *T* does not fix the origin.

The same student accepted (for example) Bob's proof, and recognized the conflict between Bob's and

Charlie's positions, commenting on ranking Charlie's proof 3rd:

> Student: Even though Charlie disproves the statement, it's a very valid reason to disprove it.

The three students who approved Charlie's proof did not appear to be troubled by the inconsistency of their own positions, and apparently believed that the statement could simultaneously be validated by a correct proof and contradicted by a counterexample. The intriguing phenomenon of such beliefs is investigated and thoughtfully discussed by Stylianides and Al-Murani (2010).

Our in-class discussion of the responses to Charlie's proof focussed on the exceptional character of his argument among the five, on the roles of examples and counterexamples in mathematical reasoning, and on the inappropriateness of rejecting an argument solely on the grounds that it consists of a single example, without considering what it claims to establish. The opportunity arises also to discuss the question of whether a statement which has a valid proof can ever admit "exceptions", a question whose answer seems to be less clear to inexperienced students than to practising mathematicians.

Deirdre's proof – responses and learning opportunities

Deirdre's proof was considered correct by 19 students, and ranked 1st or 2nd by 11 of these. It was considered incorrect by 8 students, with one reporting no verdict.

Deirdre's argument begins with a linear transformation T and a hypothesized point (a,b) whose image under T is the origin. (There is no *a priori* guarantee that such a point exists.) From the fact that T respects multiplication by scalars, it is established that (a,b) and $(2a,2b)$ have the same image under T. It is then erroneously deduced that these two points must be the same and hence that $a=b=0$. This is a specific error in the line of reasoning documented in Deirdre's argument. The argument also suffers from a structural error in its logic: what it attempts to establish is not that $T(0,0)=(0,0)$ for every linear transformation T, but that if a point is mapped by a linear transformation to $(0,0)$, then that point must be $(0,0)$. In the context of the schema of Pfeiffer, this error corresponds to an opposition between the author's proof strategy (the *intended character* of her proof) and her purpose (establishing that $T(0,0)=(0,0)$ for a linear transformation T). The "internal" error in Deirdre's proof, (that $T(a,b)=(0,0)=T(2a,2b)$ means $(a,b)=(2a,2b)$) corresponds to a failure in the author's implementation of her strategy, a mismatch between the *intended character* and *actual character* of her proof. That such an error must exist is inevitable in this instance, since the author's intention is to prove an untrue statement.

Obviously notable is the fact that two-thirds of the students accepted an argument that has (at least) two serious flaws, one in its overall logical structure and one in its internal deductions. Many identified similarities between Deirdre's proof and Bob's, which may partly explain their readiness to accept this fundamentally flawed proof.

The 8 students who rejected Deirdre's proof did so for a variety of reasons. Two of them (as well as two who accepted the proof) criticized the use of the scalar 2 instead of a general k. Two of these students suggested that this specialization amounted to restriction to a special case and constituted a reason to reject the proof, the other two only that it compromised the quality of the argument (as opposed to its correctness).

Two students rejected Deirdre's proof on the basis of the erroneous deduction that $T(a,b)=T(2a,2b)$ means $(a,b)=(2a,2b)$. For example,

> Student: Her second line contains a mistake, when she states $T(2a,2b)=T(a,b)$ $(2a,2b)=(a,b)$. This is not necessarily true. She is incorrect.

For us, the most remarkable feature of the data on our students' responses to Deirdre's proof is that not one student noted its significant logical flaw. The only possible reference to the unexpected structure of Deirdre's argument is an oblique one from a student who accepted the proof and commented:

> Student: she works backwards to reach her conclusion.

From their comments it is not evident that any of the students gave careful critical attention to the question of "fitness-for-purpose" of Deirdre's strategy. In the context of Pfeiffer's schema, none of the students' written comments indicate consideration of the relationship between the intended or actual characters of this argument and its purpose. A key learning outcome for the instructor here is that the mathematician's practice of constantly testing the connection between the text of an argument and the statement that it purports to prove is not automatically adopted by students. Our discussion on Deirdre's proof focussed on this mental discipline and its essential role in mathematical practice and in the development and validation of mathematical knowledge. The validity of a mathematical argument cannot be assessed without analysis of the deductive process from the hypotheses to the conclusion. To conduct such analysis, a reader of proofs needs to have a measure of confidence in her ability to extract the logical thread from a passage of text, and to assess whether it does what it claims. As students progress through mathematical education at university, we expect their sense of their own reliable mathematical authority to evolve. Alertness to the possibility of logical failures in an argument is a habit of mind whose development may need explicit attention from both teachers and students. It is a key feature of mathematical practice, which might plausibly be encouraged by critical study of proofs, including some that are incorrect or inadequate in different ways.

Ed's proof – responses and learning opportunities

Ed's proof was accepted as correct by all 28 students, and was by far the most popular of the five proofs, being ranked 1st by 18 students and 2nd by 5 students.

Few students commented in detail on Ed's proof. Typical remarks included that it was clear, simple, short and easy to understand. Overall the group demonstrated a clear preference for Ed's translation of the problem into an easy matrix calculation over Alison and Bob's processes of reasoning from the defining properties of a linear transformation. The matrix representation of a linear transformation had been discussed in detail in lectures, and manipulations with matrices featured in several other tasks on the homework assignment that included this proof evaluation exercise.

Our discussion of Ed's proof and the responses to it focussed on the wider purposes of proof and on the reasons that a reader might have for preferring Alison or Bob's proof to Ed's, despite the fact that more effort is required to understand them. Students were invited to consider whether any of these proofs enhanced their appreciation of the significance of the defining properties of a linear transformation, or their understanding of *why* the statement is true.

CONSTRUCTION OF PROOF EVALUATION TASKS

Composing a suitable collection of "proofs" for a proof evaluation task can be an absorbing but time-consuming challenge for an instructor. It is not essential that such a task involves multiple proposed proofs, but our experience suggests that the invitation to compare different attempts to prove the same statement can stimulate meaningful learning opportunities. A first step in constructing a task of the type described here is to identify a statement is relevant to the disciplinary learning context, and admits different proofs that students have the requisite knowledge to understand. In the preparation of "proofs", there are at least two areas of potential scope for variability. One is the manner in which the proof is presented – whether it primarily consists of text or of algebraic formulation; whether it includes diagrams, either as a support or as the main content; whether the style of text content is formal and technical or more conversational. The presentation style can often be varied independently of considerations of the correctness of the proofs, and we have found it useful to give different styles of writing and presentation to our fictitious authors.

Another important dimension of variability is in the nature of errors or imperfections that appear in the range of proposed proofs. In this context the schema of Pfeiffer provides a useful framework for preparation of variously erroneous proof attempts. A task designer might decide to include one or more "proofs" in which the author's intention is mismatched to the stated aim, for example because of inappropriate logical structure (as in Deirdre's proof), inadvertent restriction to special cases, or unjustifiable deductions. A proof whose intended character is appropriate for the purpose but poorly conveyed in the actual character might also be included. Such a mismatch might be manifested through an insufficiently explained (but justifiable) line of reasoning, through the omission of some routine but necessary ingredient, or through imprecise or unclear statements. Proof evaluation tasks are flexible and adaptable and a number of degrees of freedom exist for their design. Instructors wishing to include such tasks in curricula will find opportunities to highlight essential points relating both to the nature and purposes of proofs and to relevant disciplinary knowledge.

A shared repository of adaptable proof evaluation tasks relating to different subject areas and levels would be a very useful resource.

CONCLUDING REMARKS

As anticipated by the research study of Pfeiffer (2011b), our incorporation of the activity of proof evaluation into a linear algebra course led to positive learning opportunities for our students as well as giving us some insights into their thinking about proof. We were surprised by the range of discussion opportunities that arose from students' responses to the task. Examples include the importance of precision, the role of counterexamples, and attention in proof-reading to overall structure as well as internal features. Proof evaluation activities with an advanced course in group theory have been similarly encouraging. All of these experiences motivate us to extend our resources for the use of proof evaluation tasks and to conduct comprehensive tests of their effectiveness for the development of proof-reading skills.

As a further argument for the incorporation of tasks of this nature in learning activities, we remark that a great deal of the professional activity of research mathematicians is concerned, directly or indirectly, with the validation and evaluation of proofs. However, in our experience it is rarely the subject of explicit attention in curricula. We propose that proof evaluation tasks, as well as providing meaningful opportunities in teaching and learning, also provide opportunities for students at all stages of expertise to participate in authentic mathematical practice.

Finally, we remark that our classroom experience with proof evaluation tasks demonstrates the potential of collaboration between researchers in mathematics education and academic mathematicians to deliver rich learning opportunities for students. Cooperation and mutual support of this nature is essential if insights arising from research in mathematics education are to have a significant impact on curricula and on the learning of mathematics at university level.

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Mathematical fit: A first approximation

Manya Raman-Sundström[1] and Lars-Daniel Öhman[2]

1 Umeå University, Department of Science and Mathematics Education, Umeå, Sweden, manya.sundstrom@umu.se
2 Umeå University, Department of Mathematics, Umeå, Sweden

We discuss here the notion of mathematical fit, a concept that might relate to mathematical explanation and mathematical beauty. We specify two kinds of fit a proof can have, intrinsic and extrinsic, and provide characteristics that help distinguish different proofs of the same theorem.

Keywords: Proof, fit, explanation, beauty, aesthetics.

INTRODUCTION

Mathematics, as a subject, stands out from other fields in several ways. Even if proofs derive from axioms, there is a sense – which is grounded in the rigorous process of proving – that certain claims can be shown to be true or false. This is part of what makes mathematics satisfying. There are answers that can be shown to be, beyond a shadow of a doubt, true. Similar to the feeling one gets from establishing the truth of a claim, one can often experience a feeling in mathematics that a claim is right, that a certain proof fits a theorem, or that a particular argument is exactly the one needed. This is a stronger and somewhat more mysterious requirement than that a particular claim is true. There might be many arguments that establish the truth of a claim, but what is that makes us feel that some arguments are right? Is this just a subjective feeling or is there some objective grounding for this feeling? The purpose of this paper is to sketch some criteria for what it might mean for the (somewhat more limited and hence manageable) question of what it means for a proof to fit a theorem. While the criteria grew originally from empirical data (polling data and discussions with mathematicians) we present the paper as a theoretical one, with the idea of building, or at least starting to build, a framework that could be tested more broadly.

FIT IN THE LITERATURE

The notion of fit has been discussed in both science and mathematics communities. For instance, Wechsler (1978) compares the experience of "fit" to more aesthetic experiences often considered to be more artistic (and less scientific):

> Scientists talking about their own work and that of other scientists use the terms "beauty," "elegance," and "economy" with the euphoria of praise more characteristically applied to painting, music, and poetry. Or there is the exclamation of recognition — the "Aha" that accompanies the discovery of a connection or an unexpected but utterly right realization in art and science. These are epithets of the sense of "fit" — of finding the most appropriate, evocative and correspondent expression for a reality heretofore unarticulated and unperceived, but strongly sensed and actively probed.

Sinclair (2002) has discussed the role of fit in the process of mathematical discovery. She discusses several different kinds of fit. One kind of fit has to do with recognizing a particular theorem in a larger class of theorems (a case of a square was a specific case of the more general category of polygon.) Another kind of fit has to do with a corporeal sensation of physically putting together a vertex and an edge in a diagram to get a desired result. Sinclair connects these experiences of fit to a more general aesthetic sensibility. She quotes Beardsley who said that the first feature of aesthetic experience was "a feeling that things are working or have worked themselves out fittingly" (Sinclair, 2002, p. 288).

Fit, or the related notion of fitting (more on this distinction in the discussion), are natural, but somewhat vague terms to describe an aesthetic experience in mathematics, of something being appropriate or right, or something sharing some sort of family membership or having some kind of inner coherence. In this paper we try to clarify what some of the characteristics of fit might be (we do not claim that the list of characteristics is exhaustive, but it is at least a start). We illustrate our analysis with two contrasting proofs of the Pythagorean theorem.

THEORETICAL MODEL

Here we discuss two ways that a proof can be fitting in mathematics, which we will refer to as intrinsic and extrinsic fit. Proofs are not the only mathematical objects that can possess fit. Definitions, diagrams, even theories, might be fitting, but in this paper we will limit the discussion to proofs. In this section we list some criteria for determining if a proof has intrinsic or extrinsic fit.

Criteria for intrinsic fit

Intrinsic fit refers to the relationship between a theorem and the underlying ideas in a proof of the theorem. It is what gives one a sense of what is going on in a proof, and how accessible the underlying ideas are. There are (at least) three criteria for intrinsic fit.

> **I_1**: Economy. The underlying ideas are represented as concisely as possible. We say a proof is economic (or not economic).
>
> The number of words is not really what determines economy. Sometimes a proof can be too terse to see what is going on. The proof should be as short as possible, but a knowledgeable reader should still be able to follow it, filling in the missing details as appropriate.
>
> **I_2**: Transparency. The proof allows the underlying idea to be easily grasped. The structure of the argument is clear. We say a proof is transparent (or not transparent).
>
> This criterion deals with the question of how easy is it to see what is going on in a proof. This has two components: that the general logical structure of the proof is clearly presented, and that the underlying idea that makes the proof work is clearly stated.
>
> **I_3**: Coherence. The proof is stated in the same terms as the theorem. We say that a proof is coherent (or not).
>
> A proof that coheres has a clear underlying idea that is explicitly put to use in proving the theorem. The terms with which one would naturally state the proof idea (such as areas or graph cycles or eigenvalues of matrices) are the same terms which are stated in the theorem, allowing one to easily see why that particular idea is essential to the theorem.

Criteria for extrinsic fit

Extrinsic fit refers to the relationship between a particular proof and a family of proofs. The single case and the family might be related via an idea, but what stands out is not as much the idea as the family membership. When you realize that a proof is in the family you think, "Oh it is one of those!" There are (at least) three criteria for extrinsic fit:

> **E_1**: Generality. The idea of the proof generalizes to a larger class of theorems. We say that a proof is general (or not), though we often mean that the idea of the proof is general.
>
> This criterion deals with how well an underlying idea generalizes to prove a class of theorems. The proof at hand is seen as a specific instance of a more general claim. Generality is not the same as explanation. For instance in category theory, theorems might be perfectly general but not at all explanatory. See Steiner (1978) for other examples.
>
> **E_2**: Specificity. The proof requires a specific tool, or a particular technical approach, to make it tractable. We say that a proof is specific (or not).
>
> This criterion also deals with how well a proof fits into a class of theorems, but not through the idea, as with the criterion of generality above, but through the specific choice of technical tool that makes the proof work. Whereas with the criterion of generality the focus is on the family membership (This is one of those kinds of proofs!), here the focus is on the appropriateness of the specific technical tool for the job (We found a tool that works!). The tool itself is not as much of interest as is the fact that the proof is now within reach.
>
> **E_3**: Connectedness. The proof idea connects to proof ideas of other theorems. We say a proof is connected (or not).

We see this particular proof as one of family of proofs. This criterion is at the heart of what is meant by family membership. The idea in one proof is the same as the idea in a family of proofs, and the common idea is what makes the proofs hang together as a family.

AN EXAMPLE

We take as an example a classic, and much discussed theorem — the Pythagorean theorem. We will show two proofs of the theorem and use the model described above to discuss the extent to which each proof fits the theorem. The first proof comes from Euclid (300 B.C./2008, VI. 31) and has been discussed, for instance, in Polya (1954), as a particularly nice proof of the theorem. We suggest this proof is a clear example of a proof that really fits the theorem. It has been suggested that this proof captures exactly what the theorem is about (e.g., Steiner, 1978). The second proof, which uses a clever argument based on trigonometry, does not fit. Our model helps clarify why this is the case.

Theorem: In a right triangle with sides a and b and hypotenuse c, $a^2 + b^2 = c^2$.

Proof 1: We are given a triangle with sides a and b and hypotenuse c. Also line h ^ line c.

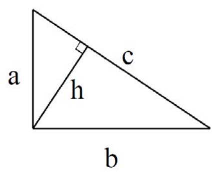

Figure 1

We can see that the sum of the areas of the smaller triangles is the same as area of the large triangle (by construction). We can also see (you can imagine folding out each of the three triangles over its longest side), that the sum of the areas of the triangles on sides a and b is equal the area of the triangle on side c. This relationship will hold for any similar figures on those sides, in particular squares, so $a^2 + b^2 = c^2$. QED

Why does this proof exhibit fit? Let us consider the criteria:

$\mathbf{I}_1$: Economy. The proof is economic. It gives relevant information simply and concisely. It could also be made more concise assuming more knowledge of the reader. (In this paper we have added some details for ease of reading.)

$\mathbf{I}_2$: Transparency. This proof is transparent. The proof consists of two main ideas, clearly presented, namely the dissection of the triangle into similar triangles, and that the equality of the areas carries over to arbitrary shapes.

$\mathbf{I}_3$: Coherence. The proof is coherent because the proof and the theorem are in the same terms, namely area. The theorem is a statement about the relationship of certain areas, and the proof directly relates these areas using properties of the triangles.

We also note that the idea of preserving areas is in line with the more famous proof in Euclid, where the areas of the squares constructed on either side are shown to be equal by area-preserving steps. A “Greek” proof that two areas are equal should ideally show that the one shape can be transformed into the other shape, using only area-preserving steps. The area-preserving step in the present proof is the reflection of either small triangle in the corresponding side of the large triangle. The scaling argument can also be traced to Euclid, and the present proof is also (less famously) given in Euclid.

$\mathbf{E}_1$: Generality. The proof is general. Steiner (1978) gives an account for this generality. This proof happens to be, according to Steiner, the proof that is most explanatory and most general. The generality comes from the fact that the proof works for arbitrary similar shapes constructed on the sides of the triangle.

$\mathbf{E}_2$: Specificity. The proof fulfils the criterion of specificity. The technical tool that works in this case is dividing the original triangle into similar triangles. Put more generally, this could be described as dissection. This division allows us to see the crucial relationship, namely that all three triangles are similar and their areas add up. The move to arbitrary similar figures, including squares, is a relatively small one.

$\mathbf{E}_3$: Connectedness. This criterion is not as easy to apply here as the other criteria, but we are inclined to say that this proof is connected. The class of proofs to which the proof can be seen to belong (other classes may be possible) might be taken to be proofs by area preservation, for instance the one referred to as "Greek" above.

Now consider the second proof, from Zimba (2009), which uses trigonometry.

Proof 2: Suppose we are given the subtraction formulas for sine and cosine:

$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, and
$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$

Let α be the angle opposite to side a, and β be the angle opposite to side b, and without loss of generality, assume that $0 < \alpha \leq \beta < 90°$.

We now have:

$$\cos(\beta) = \cos(\alpha - (\alpha - \beta)) = \cos(\alpha)\cos(\alpha - \beta) + \\ + \sin(\alpha)\sin(\alpha - \beta) = \cos(\alpha)(\cos(\alpha)\cos(\beta) + \\ + \sin(\alpha)\sin(\beta)) + \sin(\alpha)(\sin(\alpha)\cos(\beta) - \\ - \cos(\alpha)\sin(\beta)) = (\cos^2(\alpha) + \sin^2(\alpha))\cos(\beta)$$

from which it follows that $\cos^2(\alpha) + \sin^2(\alpha) = 1$, since $\cos(\beta)$ is the ratio between one leg and the hypotenuse of a right triangle, and as such is never zero. The theorem now follows from the definitions of sine and cosine and scaling. QED

To what extent does this proof fulfill the criteria for intrinsic and extrinsic fit?

I_1: Economy. The proof is economic. It gives relevant information simply and concisely, and here some details are left out.

I_2: Transparency. This proof is not transparent. There is no clear sense of direction in the calculations performed. The structure of the proof is clear enough, and each step can easily be verified, but it seems that there is little in the way of a natural sequence of ideas, and the introduction of trigonometric quantities seems extraneous. Also, it is hard to see, for instance, why one would initially want to rewrite $\cos(\beta)$ as $\cos(\alpha - (\alpha - \beta))$.

I_3: Coherence. The proof is not coherent. The trigonometry used in this proof is not in the same terms as the theorem, which is about areas. The work of the proof, that is to say the algebra, takes place in the language of trigonometry. We translate in and out of that language to see that the trigonometric manipulations establish the theorem.

E_1: Generality. The proof as it stands is not general. It is true that once $\cos^2(\alpha) + \sin^2(\alpha) = 1$ is established, one can add the scaling argument to show that the result holds for arbitrary similar shapes, but the scaling argument is not an integral part of the proof.

E_2: Specificity. The proof exhibits specificity, in that the tool used (the subtraction formulas), surprisingly works out to be adequate for the conclusion to be drawn.

E_3: Connectedness. The proof is not connected. It is, as far as we know, the only trigonometric proof of the Pythagorean theorem, so there is no obvious family of proofs it would belong to.

Comparison

Here is how the two proofs compare in terms of fit, according to the six criteria. An X indicates that the proof satisfies the given criterion.

Criterion	I_1	I_2	I_3	E_1	E_2	E_3
Proof 1	X	X	X	X	X	X
Proof 2	X				X	

We conclude from this analysis that Proof 1 fits the Pythagorean theorem better than Proof 2 does. Notice that our notion of fit appears to be gradable. It seems natural to say that one proof fits better than another without all the criteria being fulfilled (or not fulfilled). What is less clear is how many criteria must be fulfilled to say that a proof exhibits fit at all. Further, the criteria are not equally weighted. It seems more central to the notion of fit to be coherent than to be economic.

DISCUSSION

We will now take up a few issues relating to fit that have a more general nature than those discussed above. First, we will discuss the relation between the terms 'fit' and 'fitting'. Next we will discuss the relation between fit and two other concepts, explanation and beauty. Finally we will discuss the applicability of the model given here.

Fit, Fitting, Fitness

There are several words related to fit which differ in meaning and use. With the examples above in mind of what it means for a proof to fit a theorem, or to fit into a class of theorems, we will explore the relation between these words. First, fit and fitting: Fit appears, commonly, to be a relation between two objects. A glove fits a hand. A model fits the data. The objects may be abstract, such as: The experience of going to Rome fits my expectations. The term can also be used metaphorically: Anna is a good fit for Roberto. In all of these cases, the objects that fit work like puzzle pieces. One set of objects has features that complement the features of the other object. When the match is found, we get a sense of satisfaction from having made and accomplished that match. However the fit might be more or less good, as in the case of a glove fitting a hand, or might be a perfect match, as in the case of a key fitting a lock.

Fitting, which has similar meaning to 'fit' has a slightly different connotation. Fitting often means 'appropriate', such as "that behavior was fitting for a man of his stature". Unlike fit, fitting often has a connotation of being socially appropriate. One would not say that a square is a fitting choice for this particular tessellation. 'Fit' refers more broadly to patterns found in nature, mathematics, etc. while 'fitting' is more restricted to the human sphere.

Fitness might not seem as obviously related to fit, but we mention it here to raise a question about whether the notion of 'fit' in mathematics might be at all related to the notion of 'fitness', say, in natural selection. One use of the term fitness has to do with physical aptitude. One trains to stay fit. When one is fit, one has achieved some level of fitness. In Darwinian terms, fitness is related to adaptability. The more adapted a species is for the environment the better it will fit. This reading of 'fit' or 'fitness' is not so different from what we call extrinsic fit above. The features that make a proof fit into a family of other proofs might be the ones that make it 'survive' in some sense, that it is more likely to be remembered, cited, and/or developed in mathematics. (This discussion bears resemblance to Gopnik (2000) who links understanding with sexual reproduction.)

Relation to explanation

Intrinsic fit and extrinsic fit have some parallels with contemporary treatments of mathematical explanation. We make no attempt here to summarize the literature in this area, and we do not try to spell out in detail any of the main models of mathematical explanation, but simply point to a few places in some of the more prominent theories where there are similarities with our account of mathematical fit.

Steiner (1978) provides an account of mathematical explanation in terms of 'characterizing property'. He described this as "a property unique to a given entity or structure within a family or domain of such entities or structures". This description of characterizing property has an obvious parallel with our notion of 'coherence'. Familial membership is central, both in identifying an entity as one that could explain, as well as in finding the grounds for the explanation. As Steiner continues, "an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property." Similar, but not identical, with our notion of coherence, the relationship between the entity or structure and the result is central for determining if a proof explains (or has fit). A proof that has the same terms as a theorem, which is how we have characterized coherence, seems similar to a proof that evidently gives rise to a particular result.

We will mention very briefly Steiner's notion of 'deformation' because it is essential to his view of explanation (though it has been criticized elsewhere, e.g., Hafner & Mancosu, 2005). To Steiner, a proof that explains can be modified for members of a particular family (e.g. the set of all polygons) while keeping the proof idea the same. An explanatory proof can be deformed to produce a new theorem [3]. While problematic, the idea behind 'deformation', that an explanatory proof contains an idea that is invariant to certain inter-family sorts of transformations, is not completely counter-intuitive. The focus on characterizing properties and deformation seems to build on an intuition similar to that which underlies our distinction between intrinsic and extrinsic fit. In both Steiner's account of explanation and our account of fit there is some aspect that is internal and related to proof idea (and its accessibility, in the case of fit), and there is some aspect that is external and related to situating a proof as a member of a larger family.

Contemporary philosophy of mathematics has reached no consensus on mathematical explanation, but we will consider two more theories, one of which is more aligned with our notion of connectedness, and the other with our notion of coherence. These two aspects of fit correlate most closely related with current accounts of explanation.

Kitcher (1989) offers a view of explanation that is considered to be a counter proposal to Steiner's view, based on the notion of unification. Kitcher says that explanation arises from the use of arguments that have the same form (see Lange (in press) for a summary of this view). These explanations can be found in what Kitcher calls "the explanatory store" and the main task of a theory of explanation is to "specify conditions on the explanatory store" (Kitcher, 1989, p. 80). While the details of what gives rise to an explanation differ greatly in Kitcher's and Steiner's account, one similarity seems to be the emphasis on familial membership, or what we would call connectedness. In Steiner's account the membership comes about via characterizing properties, and in Kitcher's account it comes about via the explanatory store. The fact that there is some kind of unification or some sort of family traits that naturally carry over to similar entities or structures seems central both in these two accounts of mathematical explanation and in our account of mathematical fit.

In contrast to Steiner and Kitcher, whose views of explanation seem to have some component similar to that of connectedness, Lange (in press) suggests a view with three components (unity, salience, and symmetry), the first of which seems to be related to coherence. To Lange, "A proof is unified when it exploits a property that all of the cases covered by the theorem have in common and treats all of those cases in the same way" (personal communication). Unlike Kitcher and Steiner, whose unification and characterizing property ideas involve family membership, Lange's notion of unity is one that is intrinsic to the proof. Lange's concept of salience might also overlap with our criteria for intrinsic fit. Salience is a feature that is "worthy of attention" (Lange, p. 27). Transparency includes a feature that makes a certain idea accessible. While not exactly the same concept, the idea that certain features of a proof make it more readily processed by the mind seem to be a commonality between our account of mathematical fit and Lange's account of mathematical explanation.

Relation to beauty

Less clear than the relation between fit and explanation is the relation between fit and beauty. The opening quote by Wechsler hints at a connection, as does the quote by Beardsley (that the aesthetic experience arises from things working themselves out *fittingly*). Could there be any motivation in mathematics for aesthetic experiences to arise from some sense of fit or fittingness?

We have gathered a small amount of pilot data related to this question. The proofs of the Pythagorean theorem, given above, were shown to a group of six mathematicians and three mathematics educators, along with several other proofs of the theorems. Participants were asked to rank the proofs according to which was most aesthetically pleasing. All of the mathematicians ranked the first proof higher than the second. The two math educators who chose the second proof stated that they did so because they felt they did not fully understand the first proof, but they could follow the steps of the second proof. The words given by the mathematicians to describe the first proof included "simple", "beautiful" and "conceptually correct". The words given for the second proof included "ugly", "clever" and "unnatural". Note that a proof can be clever without it being beautiful.

While far from conclusive, this data seems to suggest that there might be reasons to believe that fit has something to do with beauty. The criteria of economy and coherence seem to be similar what the mathematicians meant by simplicity and conceptual correctness.

Some applications of the model

The model proposed here for mathematical fit is only a sketch, but it may be a small step toward clarifying a few open questions related to the philosophy and aesthetics of mathematics. First, it helps identify some ways that current theories of explanation are at odds with each other. Steiner's account of explanation, which overlaps with our category of coherence, and Kitcher's account, which is more related to our category of connectedness, might be, like the parable of the blind men and the elephant, in which the men describe different parts of the beast, characterising different, but not necessarily contradictory, aspects of fit.

Second, our model takes a modest step towards clarifying what beauty in mathematics might have to do with explanation. If Wechsler is right that beauty involve some sort of fit, we have sketched two different kinds of fit that might play a role in our aesthetic judgments. This may also be a way to identify whether beauty is an objective quality.

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ENDNOTES

1. Algebraic details: Let A, B, and C be the areas on the sides a, b, and c, respectively. Then $A/a^2 = B/b^2 = C/c^2$, which implies $A+B=a^2C/c^2+b^2C/c^2$. Since $A + B = C$, it follows that $(a^2+b^2)/c^2 = 1$ which implies $a^2+b^2= c^2$.

2. Note that for angles between 0 and 180 degrees, the subtraction formulas can be proven without recourse to the notion of distance, and are hence not dependent on the Pythagorean theorem. This proof is therefore not circular.

3. Counterexamples to this claim have been suggested, e.g. in Lange (in press).

Students of two-curriculum types Performance on a proof for congruent triangles

Ruthmae Sears[1] and Óscar Chávez[2]

1 University of South Florida, Tampa, USA, ruthmaesears@usf.edu

2 University of Texas at San Antonio, San Antonio, USA

This descriptive study examines students' performance on a proof task about corresponding parts of congruent triangles. We collected data from 1936 students, 59.8% used a subject-specific curriculum and 40.2 % used an integrated curriculum. Our findings indicate that, regardless of curriculum type, students experience difficulty with constructing the proof. Additionally, we observed that although students from the integrated curriculum were more likely to obtain partial credit when compared to their subject-specific counterparts, only 28 students in the sample (21 from subject-specific curriculum and 7 from the integrated curriculum) were able to obtain full marks.

Keywords: Proof, geometry, US textbooks, high school.

OVERVIEW

Although there is a consensus that students should have experiences with proof in all areas of mathematics (NCTM, 2000), proof has traditionally been part of only the geometry curriculum (Zaslavsky et al., 2012). Proof plays a central role in mathematics, and yet for many high school students it remains an alien concept, as suggested by weak performance of secondary students in proving (Harel & Sowder, 2007). For example, Senk (1985), studied 1520 students doing geometrical proofs and found that only 30% of students enrolled in a geometry course were able to demonstrate mastery of proof. Despite the low performance across the various proof tasks, Senk noted 47% of students were able to prove that a pair of congruent triangles was congruent. Similarly, Healy and Hoyles (1999) noted 19% of students were able to write a complete proof for familiar geometry statements, and 4.8% of students were able to write complete a proof for an unfamiliar geometry statements. Moore (1994) noted that students who are challenged to write proofs, may have difficulty with language and notation, may lack understanding of the concept, cannot state appropriate definitions, have difficulty structuring the proof, or may not know how to begin the proof.

Furthermore, large-scale assessments have documented that student performance in geometry is relatively poor. In the 1996 NAEP mathematics assessment, geometry was identified as a strand where student performance was low, particularly for 12th grade students (Martin & Strutchens, 2000). Moreover, extended constructed response items in the NAEP assessment had a much lower rate of satisfactory responses than multiple-choice items or short constructed responses for grades 8 and 12 (Silver, Alacaci, & Stylianou, 2000). Furthermore, in its report on evaluation of curriculum effectiveness, the National Research Council (2004) analyzed nine evaluation studies of NSF-supported curricula, and geometry was one of the strands where these curricula failed to show strong favorable results. On the other hand, two of four studies of commercial materials showed favorable results in geometry. Nevertheless, neither the analysis of NAEP data nor the NRC report specifically included results regarding proof in geometry.

Textbooks convey a mathematical progression for curriculum objectives and cognitive developmental structures for learners (Van Dormolen, 1986). The majority of secondary schools in U.S. follow a curriculum built around a sequence of three full-year courses, Algebra 1, Geometry, and Algebra 2 or Algebra 1, Algebra 2, and Geometry (Dossey, Halvorsen, & McCrone, 2008). Integrated curriculum materials developed since 1990 have been adopted in some high schools. These materials integrate algebra and geometry content, together with functions, data analysis, and discrete mathematics each year of the secondary mathematics curriculum (Hirsch, 2007).

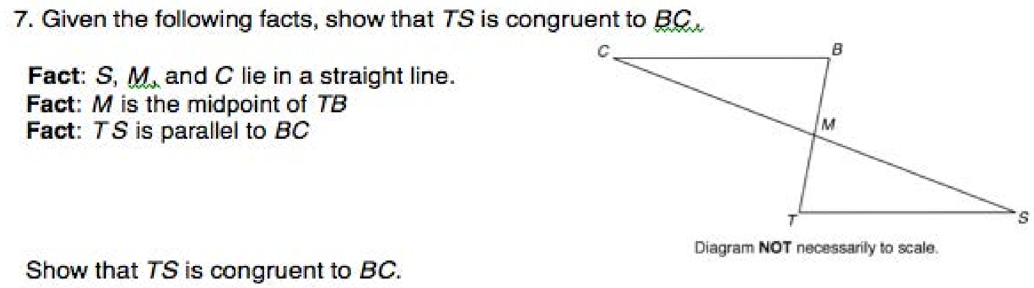

Figure 1: Proof task relative to corresponding parts of congruent triangles

Researchers have been interested in the different student outcomes that can be attributed to the use of these different types of textbooks. For example, at the fifth and sixth grade levels, Carroll (1998) found that students who used the UCSMP (an integrated curriculum that is designed to align with NCTM *Standards* (NCTM, 2000) outperformed on geometrical reasoning activities their counterparts that used a subject-specific curriculum, which are publisher-developed (Stein, Remillard, & Smith, 2007). Otten, Males, & Gilbertson (2014) documented that reasoning and proof tasks are visible in high school geometry subject-specific textbooks and integrated textbooks; however, few studies have documented high school students performance of proof in respect to the organizational structure of the curriculum used. Chávez and colleagues (2013) found that students using an integrated curriculum scored significantly higher than those using a subject-specific curriculum on a common objectives test. In this paper we examine students' performance on a geometrical proof task from this common objectives test with consideration to the curriculum type used. We investigated the following research questions:

— How successful are students on a proof about congruent parts of congruent triangles?

— What differences are there between curriculum types for students' performance on a geometrical proof task about congruent parts of congruent triangles?

METHOD

The mathematical task used and students' data were drawn from Test E in the Comparing Options in Secondary Mathematics: Investigating Curriculum (COSMIC) study. Test E is one of 5 tests developed for the COSMIC study (Chávez et al., 2011). This longitudinal study examined the impact of different content organizations on students' learning. The mathematics emphasized on Test E was *functions* and other topics that were common to both types of textbooks and were considered important, namely: symbolic algebra, Pythagorean Theorem and a proof for congruent triangles. Test E had 10 items: 1 multiple-choice item, 1 matching item, and the 8 remaining items were constructed response. For this paper, we will discuss students' performance on the proof task.

Mathematical task

In Test E, question 7 (Figure 1) required students to prove that two sides of two triangles were congruent. The given information was enough to prove that the triangles were congruent and therefore the corresponding sides were congruent. Figure 1 includes the task, as presented to students in the assessment instrument. The purpose of this task was to assess students' ability to write a proof for a very typical geometry problem. Our curriculum analysis had shown that tasks similar to this were found on both sets of textbooks.

This item was scored on a 9-point scale. The scoring rubric is shown in Figure 2. According to this rubric, the item was divided into four subitems, corresponding to the four expected components of a correct answer. Students were awarded points for identifying a pair of congruent triangles, pointing out at least two pairs of congruent angles, stating the correct theorem as to why the triangles were congruent, and for providing a concluding reasoning that the sides of congruent triangles were congruent. These four subitems are referred to as 7.1, 7.2, 7.3 and 7.4. As described elsewhere (Chávez et al., 2011), the rubric and the problem itself were submitted to external reviewers. The problem was also piloted before being use with the study participants. From these rounds of review, revision, piloting, and revision, we were confident that the problem was appropriate and that the scoring rubric was suitable. In particular, the scoring rubric enabled us to document student understanding, and also was

Student argues that one pair of sides is congruent:	
Student states that $\overline{TM} \cong \overline{MB}$ because M is the midpoint of $\overline{TB}$ (or equivalent statement)	2 points
Partial Credit — 1 point	
Student states that $\overline{TM} \cong \overline{MB}$ without a valid supporting reason	
Student argues that <u>two</u> of the three following pairs of angles are congruent:	
1. Student states that $\angle TMS \cong \angle BMC$ because they are vertical angles (or equivalent statement)	4 points
2. Student states that $\angle STM \cong \angle CBM$ because $\overline{TS} \parallel \overline{BC}$ or alternate interior angles	
3. Student states that $\angle TSM \cong \angle BCM$ because $\overline{TS} \parallel \overline{BC}$ or alternate interior angles	
Partial Credit — 3 points	
Student provides two of the above statements, but only one appropriate supporting reason	
Partial Credit — 2 points	
Student provides two of the above statements, but no valid supporting reasons	
OR	
Student provides one of the above statements, and one appropriate supporting reason	
Partial Credit — 1 point	
Student provides one of the above statements, and no valid supporting reasons	
Student argues that the two triangles are congruent:	
Student states that $\triangle TMS \cong \triangle BMC$ by ASA	2 points
OR	
Student states that $\triangle TMS \cong \triangle BMC$ by AAS	
Partial Credit — 1 point	
Student states that $\triangle TMS \cong \triangle BMC$ with no valid supporting reason	
Student argues that the required two sides are congruent:	
Student states that $\overline{TS} \cong \overline{BC}$ because $\triangle TMS \cong \triangle BMC$	1 point
OR	
Student states that $\overline{TS} \cong \overline{BC}$ because corresponding sides of congruent triangles are congruent (or equivalent statement)	
Total points	9 points

Figure 2: Rubric for proof task relative to corresponding parts of congruent triangles

sensible enough to allow us to measure variation in student's responses.

Participants

As described in previous papers (Grouws et al., 2013; Tarr et al., 2013), the schools in our study were high schools that offered both an integrated mathematics sequence (Course 1, Course 2, Course 3, Course 4) and a subject-specific sequence (Algebra 1, Geometry, Algebra 2, Pre-calculus); and that gave students the choice between these two alternatives without tracking them by ability. The students in the study ages ranged from 15–18 years old. Our sample included 10 high schools in six United States (US) school districts located in five geographically diverse states. *Core-Plus Course 3* (Coxford et al., 2003) was the textbook series used by all of the teachers teaching an integrated curriculum. Teachers of subject-specific curriculum used Algebra 2 textbooks produced by several different publishers.

The textbooks had similar content (Chávez, Papick, Ross, & Grouws, 2011; Chávez, Tarr, Grouws, & Soria, 2013). For example, in most subject-specific geometry textbooks Chapter 4 focuses on Congruent Triangles (Sears & Chávez, 2014). Teachers that used subject-specific geometry curriculum acknowledged that the chapter on congruent triangles is used to develop students' conceptions about proof (Sears & Chavez, 2014).

We collected data from 1936 students from these 10 high schools. Of these, 1157 students (59.8 %) used a subject-specific textbook; 779 (40.2 %) used an integrated textbook. The students in the study were in the third course for their respective curriculum sequence (Course 3 and Algebra 2, respectively). There is a slight variation among the subject-specific textbooks on their attention to proof (Sears & Chávez, 2014; Otten, Males, & Gilbertson, 2014). Given our focus on the impact of content organization on students learning, in this

Score	Frequency	%
Did not attempt	223	11.5
0	1092	56.4
1	190	9.8
2	166	8.6
3	89	4.6
4	51	2.6
5	36	1.9
6	28	1.4
7	20	1.0
8	13	0.7
9	28	1.4
Total	1936	100.0

Table 1: Scores for Question 7, Test E

paper we make no distinction between the different subject-specific curricula.

Results

Score	Frequency	%
0	1329	68.6
1	227	11.7
2	157	8.1
Did not attempt	223	11.5
Total	1936	100.0

Table 2: Scores on Test E Question 7.1

Most students did not attempt the task or obtained no credit for the task if they did attempt it. Table 1 shows the frequency of students who were awarded the different possible scores in this task. As indicated in Table 2, 8.1% of the students stated that segments *TM* and *MB* were congruent and offered a justification.

Score	Frequency	%
0	1331	68.8
1	119	6.1
2	150	7.7
3	36	1.9
4	77	4.0
Did not attempt	223	11.5
Total	1936	100.0

Table 3: Scores on Test E Question 7.2

Score	Frequency	%
0	1503	77.6
1	110	5.7
2	100	5.2
Did not attempt	223	11.5
Total	1936	100.0

Table 4: Scores on Test E Question 7.3

In Table 3 we can see that, although partial credit was given for correct statements (without justification), about pairs of angles that are congruent, very few students received partial or full credit. Furthermore, few students stated that the two triangles were congruent (Table 4). Only half of those who did offered any justification.

Score	Frequency	%
0	1562	80.7
1	151	7.8
Did not attempt	223	11.5
Total	1936	100.0

Table 5: Scores on Test E Question 7.4

The vast majority of students did not conclude that congruent sides of congruent triangles were congruent (Table 5).

Frequency by curriculum type

The following tables (Tables 6–10) show the scores by curriculum type. A larger percentage of students using subject-specific curriculum did not attempt to solve the problem. Among those students using subject-specific curriculum who did attempt the problem, a larger percentage received no points compared to those using integrated curriculum that attempted the problem. On the other hand, comparatively fewer students using the integrated curriculum received full credit for the problem. The mean score for question 7

	SS		INT	
Score	Frequency	%	Frequency	%
Did not attempt	139	12.0	84	10.8
0	680	58.8	412	52.9
1	116	10.0	74	9.5
2	83	7.2	83	10.7
3	39	3.4	50	6.4
4	26	2.2	25	3.2
5	20	1.7	16	2.1
6	16	1.4	12	1.5
7	11	1.0	9	1.2
8	6	.5	7	.9
9	21	1.8	7	.9
Total	1157	100.0	779	100.0

Table 6: Scores for Question 7 Test E, by curriculum type

	SS		INT	
	Frequency	Percent	Frequency	%
0	801	69.2	528	67.8
1	132	11.4	95	12.2
2	85	7.3	72	9.2
Did not attempt	139	12.0	84	10.8
Total	1157	100.0	779	100.0

Table 7: Scores on Test E Question 7.1 by curriculum type

	SS		INT	
	Frequency	Percent	Frequency	Percent
0	821	71.0	510	65.5
1	62	5.4	57	7.3
2	76	6.6	74	9.5
3	14	1.2	22	2.8
4	45	3.9	32	4.1
Did not attempt	139	12.0	84	10.8
Total	1157	100.0	779	100.0

Table 8: Scores on Test E Question 7.2, by curriculum type

	SS		INT	
	Frequency	%	Frequency	%
0	891	77.0	612	78.6
1	67	5.8	43	5.5
2	60	5.2	40	5.1
Did not attempt	139	12.0	84	10.8
Total	1157	100.0	779	100.0

Table 9: Scores on Test E Question 7.3, by curriculum type

	SS		INT	
	Frequency	%	Frequency	%
0	930	80.4	632	81.1
1	88	7.6	63	8.1
Did not attempt	139	12.0	84	10.8
Total	1157	100.0	779	100.0
0	930	80.4	632	81.1

Table 10: Scores on Test E Question 7.4, by curriculum type

was significantly different in these two groups, after controlling for prior achievement ($F = 6.355$, $p = 0.012$). Nevertheless, it is important to consider these differences in the context of the full test and taking into account all the relevant variables (Chávez et al., 2013). The results in these tables are simple summaries of the points scored by students in one assessment item.

DISCUSSION

Our results are consistent with others in showing that students, regardless of curriculum type, have difficulties writing proofs. There is evidence, however, that the organizational structure of curriculum can have implications on students' opportunities to prove. In particular, students in subject specific curricula were less likely to complete the proof task or to receive no points than students using integrated textbooks. Most students were not able to obtain full credit on the task. According to the results, only 1.8% of students in subject curricula, and 0.9% of students in integrated curricula were able to obtain full credit, which suggests that students continue to struggle with writing proofs. Otten et al (2014) noted that, although both subject-specific and integrated curriculum provides opportunities for students to differentiate between deductive and inductive reasoning, students were seldom required to construct complete proof arguments. This can be problematic, because it deprives students from viewing proofs in their entirety and can potentially contribute to students experiencing difficulty in starting proofs, as well as conceptualizing the structure of proofs (Moore, 1994).

Students should experience proofs more often. They should learn to expect that posing conjectures and proving statements is central to mathematics. Classroom observations for the COSMIC study indicated that teachers using integrated curriculum placed a greater focus on reasoning than teachers using subject-specific curriculum, "although it was not a strong focus for either group of teachers" (Grouws et al., 2013, p. 442). Complete proofs are seldom included in assessments, and more often they appear as fill-in-the-blank tasks. Emphasis on ritualistic aspects of proof may explain why students have difficulties writing proofs on their own.

CONCLUSION

Without a doubt, the results summarized here show that students in US high schools, regardless of the curriculum used in their schools, have difficulties writing proofs in geometry. Work done by Sharon Senk in the 1980s, among others, shows that this is an old problem. It was beyond the scope of the COSMIC study to conduct a fine-grained analysis of this specific topic. Nevertheless, it is clear that the evidence presented here points to grave deficiencies in how geometry is taught in schools. The case of proof is a particularly delicate one, because teachers attempt to teach proof as a topic that must be taught among other topics in the curriculum, rather than as a way to communicate mathematics. Therefore, these results may be interpreted as a lack of certain skills. It would be more appropriate, however, to consider them as an indication that when proof is taught as a set of rules that should be applied to specific problems in geometry rather than as a way of communicating mathematics, students learn neither. Hence, curriculum developers, and educators, must find ways to introduce experiences with proof in their materials and lessons, so that students develop sound habits of justification and proof.

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A theoretical perspective for proof construction

Annie Selden and John Selden

New Mexico State University, Las Cruces, USA, aselden @math.nmsu.edu

This theoretical paper suggests a perspective for understanding university students' proof construction based on the ideas of conceptual and procedural knowledge, explicit and implicit learning, behavioral schemas, automaticity, working memory, consciousness, and System 1 and System 2 cognition. In particular, we will discuss proving actions, such as the construction of proof frameworks, that could be automated, thereby reducing the burden on working memory and enabling university students to devote more resources to the truly hard parts of proofs.

Keywords: University level, proving actions, behavioral schemas, System 1 and System 2 cognition, proof framework.

INTRODUCTION

This theoretical paper suggests a perspective for understanding university mathematics students' proof constructions and how the ability and skill to construct proofs might be learned and taught. We are interested both in how various types of knowledge (e.g., implicit, explicit, procedural, conceptual) are used during proof construction, and also in how such knowledge can be acquired. If that were better understood, then it might be possible to facilitate university students' learning through doing, that is, through proof construction experiences. Although one can learn some things from lectures, this is almost certainly not the most effective, or efficient, way to learn proof construction. Indeed, inquiry-based transition-to-proof courses seem more effective than lecture-based courses (e.g., Smith, 2006). Here we are referring just to inquiry into proof construction, not into theorem or definition generation. These ideas emerged from an ongoing sequence of design experiment courses meant to teach proof construction in a medium-sized U.S. PhD-granting university.

The Courses

There were two kinds of courses. One kind was for mid-level undergraduate mathematics students and was similar (in purpose) to transition-to-proof courses found in many U.S. university mathematics departments (Moore, 1994). In the U.S., such courses are often prerequisite for 3rd and 4th year courses in abstract algebra and real analysis. The other, somewhat unusual, kind of course was for beginning mathematics graduate students who felt that they needed help with writing proofs. The undergraduate course had from about 15 to about 30 students and the graduate course had between 4 and 10 students. Both kinds of course were taught from notes and devoted entirely to students attempting to construct proofs and to receiving feedback and advice on their work. Both courses included a little sets, functions, real analysis, and algebra. The graduate course also included some topology.

Psychological considerations

Much has been written in the psychological, neuropsychological, and neuroscience literature about ideas of conceptual and procedural knowledge, explicit and implicit learning, automaticity, working memory, consciousness, and System 1 (S1) and System 2 (S2) cognition (e.g., Bargh & Chartrand, 2000; Bargh & Morsella, 2008; Bor, 2012; Cleeremans, 1993; Hassin, Bargh, Engell, & McCulloch, 2009; Stanovich & West, 2000). In trying to relate these ideas to proof construction, we have discussed procedural knowledge, situation-action links, and behavioral schemas (Selden, McKee, & Selden, 2010; Selden & Selden, 2011). However, more remains to be done in order to weave these ideas into a coherent perspective. In doing this, two key ideas are working memory and the roles that S1 and S2 cognition can play in proof construction. Working memory includes the central executive and makes cognition possible. It is related to learning and attention and has a limited capacity which varies across individuals. When working memory capacity is exceeded, errors and oversights can occur. The idea behind S1 and S2 cognition is that there are two kinds of cognition that operate in parallel. S1 cognition is fast, unconscious, automatic, effortless, evolutionarily ancient, and places little burden on working memory. In contrast, S2 cognition is slow, conscious, effortful, evolutionarily recent, and puts considerable call on working memory (Stanovich & West, 2000). Of the several kinds of consciousness, we are referring to phenomenal consciousness—approximately, reportable experience. We turn now to the first of the two components of the proposed perspective.

THE PERSPECTIVE: MATHEMATICAL COMPONTENT

The genre of proofs

There are a number of characteristics that appear to commonly occur in published proofs. They tend to reduce unnecessary distractions to validation (reading for correctness) and raise the probability that any errors will be found, thereby increasing the reliability of the corresponding theorems. Proofs are not reports of the proving process, contain little redundancy, and contain minimal explanations of inferences. They contain only very short overviews or advance organizers and do not quote entire definitions that are available outside the proof. Symbols are generally introduced in one-to-one correspondence with objects. Finally, proofs are "logically concrete" in the sense that they avoid quantifiers, especially universal quantifiers, and their validity is independent of time, place, and author. (Selden & Selden, 2013).

Structure in proofs

A proof can be divided into a formal-rhetorical part and a problem-centered part. The *formal-rhetorical* part is the part of a proof that depends only on unpacking and using the logical structure of the statement of the theorem, associated definitions, and earlier results. In general, this part does not depend on a deep understanding of, or intuition about, the concepts involved or on genuine problem solving in the sense of Schoenfeld (1985, p. 74). Instead it depends on a kind of "technical skill". We call the remaining part of a proof the *problem-centered* part. It is the part that *does* depend on genuine problem solving, intuition, and a deeper understanding of the concepts involved (Selden & Selden, 2009, 2011).

A major feature of the formal-rhetorical part is what we have called a *proof framework*, of which there are several kinds, and in most cases, both a first-level and a second-level framework. For example, given a theorem of the form "For all real numbers x, if $P(x)$ then $Q(x)$", a proof framework would be "Let x be a real number. Suppose $P(x)$. ... Therefore $Q(x)$." A second-level framework would be obtained by "unpacking" the meaning of $Q(x)$ and putting the second-level framework for that between the beginning and end of the first-level framework. Thus, the proof would "grow" from both ends toward the middle, instead of being written from the top down. In case there are subproofs, these can be handled in a similar way. A more detailed explanation with examples can be found in (Selden, Benkhalti, & Selden, 2014).

THE PERSPECTIVE: PSYCHOLOGICAL COMPONTENT

In this second, psychological component of the perspective, we view the proof construction process as a sequence of actions which can be physical (e.g., writing a line of the proof or drawing a sketch) or mental (e.g., changing one's focus from the hypothesis to the conclusion or trying to recall a relevant theorem). The sequence of all of the actions that eventually lead to a proof is usually considerably longer than the final written proof itself. This fine-grained approach appears to facilitate noticing which actions should be taken to write various parts of a proof correctly and how to encourage such actions on the part of students.

Situations and actions

We mean by an (inner) situation in proving, a portion of a partly completed proof construction, perhaps including an interpretation, drawn from long-term memory, that can suggest a further action. The interpretation is likely to depend on recognition of the situation, which is easier than recall, perhaps because fewer brain areas are involved (Cabeza, et al., 1997). An inner situation is unobservable. However, a teacher can often infer an inner situation from the corresponding outer situation, that is, from the, usually written, portion of a student's partly completed proof.

Here we are using the term, action, broadly, as a response to a situation. We include not only physical actions (e.g., writing a line of a proof), but also mental actions. The latter can include trying to recall something or bringing up a feeling, such as a feeling of caution or of self-efficacy (Selden & Selden, 2014). We also include "meta-actions" meant to alter one's own thinking, such as focusing on another part of a developing proof construction.

Situation-action links and behavioral schemas

If, in several proof constructions in the past, similar situations have corresponded to similar actions, then, just as in traditional associative learning, a link may be learned between them, so that another similar situation yields the corresponding action in future proof constructions without the earlier need for deliberate cognition. Using *situation-action links* strengthens them and after sufficient practice/experience, they can become overlearned, and thus, automated. A person executing an automated action tends to (1) be unaware of any needed mental processes, (2) be unaware of intentionally initiating the action, (3) put little load on working memory, and (4) find it difficult to stop or alter the action. We call automated situation-action links *behavioral schemas*. Morsella (2009) has pointed out

> Regarding skill learning and automaticity, it is known that the neural correlates of novel actions are distinct from those of actions that are overlearned, such as driving or tying one's shoes. Regions [of the brain] primarily responsible for the control of movements during the early stages of skill acquisition are different from the regions that are activated by overlearned actions. In essence, when an action becomes automatized, there is a 'gradual shift from cortical to subcortical involvement ...' (p. 13).

Because cognition often involves inner speech, which in turn is connected with the physical control of speech production, the above information on the brain regions involved in physical skill acquisition is at least a hint that forming behavioral schemas not only converts S2 cognition into S1 cognition, but also suggests that different parts of the brain are involved in access and retrieval. Something very similar to the above ideas on automaticity in proof construction has been investigated by social psychologists examining everyday life (e.g., Bargh & Chartrand, 2000).

We see behavioral schemas as partly conceptual knowledge (recognizing the situation) and partly procedural knowledge (the action), and as related to Mason and Spence's (1999) idea of "knowing-to-act in the moment". We suggest that, in the use of a situation-action link or a behavioral schema, almost always both the situation and the action (or its result) will be at least partly conscious.

Here is an example of one such possible behavioral schema. One might be starting to prove a statement having a conclusion of the form p or q. This would be the situation at the beginning of the proof construction. If one had encountered this situation a number of times before, one might readily take an appropriate action, namely, in the written proof assume not p and prove q or vice versa. While this action can be warranted by logic (if not p then q, is equivalent to, p or q), there would no longer be a need to bring the warrant to mind.

It is our contention that large parts of proof construction skill can be automated, that is, that one can facilitate mid-level university students in turning parts of S2 cognition into S1 cognition, and that doing so would make more resources, such as working memory, available for the truly hard problems that need to be solved to complete many proofs.

The idea that much of the deductive reasoning that occurs during proof construction could become automated may be counterintuitive because many psychologists, and (given the terminology) probably many mathematicians, assume that deductive reasoning is largely S2.

Sequences of behavioral schemas

Behavioral schemas were once actions arising from situations through warrants, but that no longer need to be brought to mind. So one might reasonably ask, can several behavioral schemas be "chained together" outside of consciousness? For most persons, this seems not to be possible. If it were so, one would expect that a person familiar with solving linear equations could start with $3x + 5 = 14$, and *without bringing anything else to mind*, immediately say $x = 3$. We expect that very few (or no) people can do this, that is, consciousness is required.

Implicit learning of behavioral schemas

It appears that the entire process of learning a behavioral schema, as described above, can be implicit. That is, a person can acquire a behavioral schema without being aware that this is happening. Indeed, such unintentional, or implicit, learning happens frequently and has been studied by psychologists and neuroscientists (e.g., Cleeremans, 1993). In the case of proof construction, we suggest that with the experience of proving a considerable number of theorems in which similar situations occur, an individual might implicitly acquire a number of relevant behavioral schemas, and as a result, simply not have to think quite so deeply as before about certain portions of the proving process and might, as a consequence of having more working memory available, take fewer "wrong turns".

Something similar has been described in the psychology literature regarding the automated actions of everyday life. For example, an experienced driver can reliably stop at a traffic light while carrying on a conversation. But not all automated actions are positive. For example, a person can develop a prejudice without being aware of the acquisition process and can even be unaware of its triggering situations. This suggests that we should consider the possibility of mathematics students developing similarly unintended negative situation-action links, and behavioral schemas, implicitly during mathematics learning, and in particular, during proof construction.

Detrimental behavioral schemas

We begin with a simple and perhaps very familiar algebraic error. Many teachers can recall having a student write $\sqrt{(a^2 + b^2)} = a + b$, giving a counterexample to the student, and then having the student make the same error somewhat later. Rather than being a misconception (i.e., believing something that is false), this may well be the result of an implicitly learned detrimental behavioral schema. If so, the student would not be thinking very deeply about this calculation when writing it. Furthermore, having previously understood the counterexample would also have little effect in the moment. It seems that to weaken/remove this particular detrimental schema, the triggering situation of the form $\sqrt{(a^2 + b^2)}$ should occur a number of times when the student can be prevented from automatically writing "$= a + b$" in response. However, this might require working with the student individually on a number of examples, mixed with nonexamples.

For another example of an apparently implicitly learned detrimental behavioral schema, we turn to Sofia, a first-year graduate student in one of the above mentioned graduate courses. Sofia was a diligent student, but as the course progressed what we came to call an "unreflective guess" schema emerged (Selden, McKee, & Selden, 2010, pp. 211–212). After completing just the formal-rhetorical part of a proof (essentially a proof framework) and realizing there was more to do, Sofia often offered a suggestion that we could not see as being remotely helpful. At first we thought she might be panicking, but on reviewing the videos there was no evidence of that. A first unreflective guess tended to lead to another, and another, and after a while, the proof would not be completed.

In tutoring sessions, instead of trying to understand, and work with, Sofia's unreflective guesses, we tried to prevent them. At what appeared to be the appropriate time, we offered an alternative suggestion, such as looking up a definition or reviewing the notes. Such positive suggestions eventually stopped the unreflective guesses, and Sofia was observed to have considerably improved in her proving ability by the end of the course (Selden, McKee, & Selden, 2010, p. 212).

USING THIS PERSPECTIVE

Decomposing the proving process

In order to facilitate students' automation of certain parts of the proving process by developing helpful behavioral schemas, we have been decomposing the reasoning parts of the proving process, and focusing on those that occur frequently. Such decompositions of parts of the proving process can be mainly mathematical in nature or mainly psychological in nature. We find the psychological decompositions to be more surprising because they include things one might expect university students to be able to do without instruction. Some more mathematical possibilities are: (1) writing the first- and second-level proof frameworks which themselves can have parts (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995); (2) noting when a conclusion is negatively phrased (e.g., a set is empty or a number is irrational) and early in the proving process attempting a proof by contradiction; and (3) noticing when the conclusion asserts the equivalence of two statements, "knowing" there are two implications to prove, and actually originating the two subproofs.

Here are some decompositions that may be more psychological in nature. One can change one's focus, for example by deciding to unpack the conclusion of a theorem, by finding or recalling a relevant definition, or by applying a definition. Such actions are sometimes part of constructing a second-level proof framework (Selden, Benkhalti, & Selden, 2014; Selden & Selden, 1995). Also developing a feeling of knowing or of self-efficacy can have a major effect (Selden & Selden, 2014). A student may develop and have for a time a feeling of not knowing what to do next, that is, the student might be at an impasse. Upon reaching such an impasse, the student might decide to do something else for a while, and coming back later, might hope to get a new idea. Many mathematicians have benefitted from this kind of "incubation". Nonemotional cognitive feelings (Selden, McKee, & Selden, 2010), such as those mentioned above can play a considerable role in proof construction, but we do not have space to elaborate on them here.

Proving activities that we have tried to help students automate include converting formal mathematical definitions into *operable interpretations*, which are similar to Bills and Tall's (1998) idea of operable definitions. For example, given $f: X \to Y$ and $A \subseteq Y$, we define $f^{-1}(A) = \{ x \in X \mid f(x) \in A\}$. An operable interpretation would say, "If you have $b \in f^{-1}(A)$, then you can write $f(b) \in A$ and vice versa". One might think that this sort of translation into an operable form would be easy, but we have found that for some students it is not, even when the definition can be consulted. We have also noted instances in which students have had available both a definition and an operable interpretation, but still did not act appropriately. Thus, *actually* implementing the action is separate from *knowing* that one *can* implement it. We are not sure whether, in implementing such actions, automaticity is difficult to achieve, but not acting appropriately can clearly prevent a student from proving a theorem.

Seeing similarities, searching, and exploring

How does one recognize situations as similar? Different people see situations as similar depending both upon their past experiences and upon what they choose to, or happen to, focus on. While similarities can sometimes be extracted implicitly, teachers may occasionally need to direct students' attention to relevant proving similarities. On the other hand, such direction should probably be as little as possible because the ability to *autonomously* see similarities can be learned.

For example, it would be good to have general suggestions for helping students to "see", without being told, that the situations of a set being empty (i.e., having no elements), of a number being irrational (i.e., not rational), and of the primes being infinite (i.e., not finite) are similar. That is, the three situations—empty, irrational, and infinite—may not seem similar until one rephrases them to expose the existence of a negatively worded definition. Unless students rephrase these situations, it seems unlikely that they would see this similarity and link these situations (when they occur as conclusions of theorems to prove) to the action of beginning a proof by contradiction.

In addition to automating small portions of the proving process, we would also like to enhance students' searching skills (i.e., their tendency to look for helpful previously proved results) and to enhance students' tendency to "explore" possibilities when they don't know what to do next. In a previous paper (Selden & Selden, 2014, p. 250), we discussed the kind of exploring entailed in proving the rather difficult (for students) *Theorem: If S is a commutative semigroup with no proper ideals, then S is a group.* Well before such a theorem appears in course notes, one can provide students with advice/experience showing the value of exploring what is not obviously useful (e.g., starting with $abba = e$ to show a semigroup with identity e, where for all $s \in S$, $s^2 = e$, is commutative, as discussed in Selden, Benkhalti, and Selden, 2014).

Understanding students' proof attempts

Here is a sample student's incorrect proof attempt of the *Theorem: Let S be a semigroup with identity e. If, for all s in S, ss = e, then S is commutative.* The student's accompanying scratchwork consisted of the definitions of identity and commutative. Here, the line numbers are for reference only.

1 Let S be a semigroup with an identity element, e.

2 Let $s \in S$ such that $ss = e$.

3 Because e is an identity element, $es = se = s$.

4 Now, $s = se = s(ss)$.

5 Since S is a semigroup, $(ss)s = es = s$.

6 Thus $es = se$.

7 Therefore, S is commutative. QED.

Line 2 only hypothesizes a single s and should have been, "Suppose for all $s \in S$, $ss = e$." With this change, Lines 1, 2, and 7 are the correct first-level framework.

There is no second-level framework between Lines 2 and 7. This was a beneficial action *not* taken and should have been: "Let $a \in S$ and $b \in S$. ... Then $ab = ba$." between Lines 2 and 7. Line 3 violates the genre of proof by including a definition easily available outside the proof. Lines 3, 4, 5, and 6 are not wrong, but do not move the proof forward. Writing these lines may have been detrimental actions that subconsciously primed the student's feeling that something useful had been accomplished, and thus, may have brought the proving process to a premature close.

TEACHING AND RESEARCH CONSIDERATIONS

The above considerations can lead to many possible teaching interventions. This then brings up the question of priorities. Which proving actions, of the kinds discussed above, are most useful for mid-level university mathematics students to automate, when they are learning how to construct proofs? Since such students are often asked to prove relatively easy theorems—ones that follow directly from definitions recently provided—it would seem that noting the kinds of structures that occur most often might be a place to start. Indeed, since every proof can be constructed using a proof framework, we consider constructing proof frameworks as a reasonable place to start.

Also, helping students interpret formal mathematical definitions so that these become operable might be another place to start. This would be helpful because one often needs to convert a definition into an operable form in order to use it to construct a second-level framework. However, eventually students should learn to make such interpretations themselves.

Finally, we believe this particular perspective on proving, using situation-action links and behavioral schemas, together with information from psychology and neuroscience, is mostly new to the field and is likely to lead to additional insights.

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Textbook explanations: Modes of reasoning in 7th grade Israeli mathematics textbooks

Boaz Silverman and Ruhama Even

Weizmann Institute of Science, Rehovot, Israel, boaz.silverman@weizmann.ac.il

The goal of this study is to characterize the justifications and explanations offered in 7th grade Israeli textbooks for mathematical statements. The justifications and explanations offered in eight 7th grade Israeli textbooks for ten selected mathematical statements were analysed, using the modes of reasoning framework (Stacey & Vincent, 2009). The analysis revealed that the textbooks commonly used several modes of reasoning in explanations for each statement. Nearly every justification was deductive or empirical, yet different modes of reasoning were used for geometric and for algebraic statements. It was also found that empirical justifications were more prevalent in textbooks of limited scope, whereas deductive justifications were typically offered in textbooks of regular/ extended scope.

KEYWORDS: Modes of reasoning, textbook analysis, justifications, explanations, empirical and deductive justifications.

INTRODUCTION

Proving, justifying, and explaining are important components of doing and learning mathematics. However, the extensive research on students' conceptions of proof and ways of justifying mathematical claims reveals students' difficulties in understanding the need for justification and in distinguishing between deductive and other types of justification (e.g., Harel & Sowder, 1998). One factor that considerably influences classroom instruction and students' opportunities to learn mathematics is the textbook used in class (Eisenmann & Even, 2011; Haggarty & Pepin, 2002).

Studies of justification and proof in mathematics textbooks have examined various aspects, such as reasoning and proving activities (e.g., Davis, Smith, Roy, & Bilgic, 2014; Fujita & Jones, 2014), the nature of proof (e.g., Miyakawa, 2012), and the opportunities provided in the textbook for students to learn reasoning and proof (e.g., Dolev & Even, 2012; Stacey & Vincent, 2009). To better understand students' opportunities to develop the habit of justifying, and to learn how to justify in mathematics, this study examines the justifications and explanations to key mathematical statements offered in mathematics textbooks, specifically 7th grade Israeli textbooks.

BACKGROUND

Justifications of mathematical statements vary in their nature, from informal and intuitive explanations to rigorous deductive proofs (e.g., Blum & Kirsch, 1991; Harel & Sowder, 2007; Sierpinska, 1994). Research on the issue of justifications in school mathematics attends to a wide range of aspects. Some researchers focus on the formality of justifications (Blum & Kirsch, 1991); others consider the community addressed (Sierpinska, 1994); and yet others focus on the proof scheme of justifications (Harel & Sowder, 2007).

Studies of the opportunities for students to read justifications and explanations in textbooks show that textbooks justify mathematical statements in several ways, and that valid proofs are rare. Building on Harel and Sowder's (2007) framework, Stacey and Vincent (2009) developed the modes of reasoning framework and used it to analyse Australian textbook explanations. Stacey and Vincent identified seven modes of reasoning in textbook explanations:

— *Appeal to authority*: null explanation or reliance on an external source of authority.

— *Qualitative analogy*: reliance on a surface similarity to non-mathematical situations.

- *Experimental demonstration*: identifying a pattern after checking selected examples.
- *Concordance of a rule with a model*: comparing specific results of a rule and a model.
- *Deduction using a model*: a model that serves to illustrate a mathematical structure.
- *Deduction using a specific case*: an inference process conducted using a special case.
- *Deduction using a general case*: an inference process conducted using a general case.

These seven modes can be generally divided into three categories: External sources (*appeal to authority* and *qualitative analogy*); Empirical justifications (*experimental demonstration* and *concordance of a rule with a model*); and Deductive justifications (*deduction using a model*, *a specific case*, or *a general case*). Stacey and Vincent found that justifications offered in the analysed textbooks used several modes of reasoning, yet students were given no indication regarding which can be classified as deductive proofs and which can only serve as supportive empirical evidence at best.

Drawing on Stacey and Vincent's (2009) conceptual framework, Dolev (2011) analysed the modes of reasoning in justifications offered for three mathematical claims in six 7th grade Israeli textbooks (experimental version). She found that all the textbooks offered justifications for the sampled claims, at times using several modes of reasoning. Additionally, Dolev found a difference between the modes of reasoning used in algebra and in geometry – geometric claims were often justified by *deduction using a general case*, whereas algebraic claims were rarely justified that way. A similar pattern was noted in other studies – an abundance of proofs in geometry (e.g., Fujita & Jones, 2014; Hanna & de Bruyn, 1999), and a small number of formal proofs in algebra (e.g., Davis et al., 2014; Hanna & de Bruyn, 1999).

This study builds on these studies, and expands their scope. The research objective is to examine the justifications and explanations for key mathematical statements offered in mathematics textbooks, centring on 7th grade Israeli textbooks (approved version). We examine three aspects: (1) the modes of reasoning offered, (2) the nature of justifications of algebraic vs. geometric statements, and (3) the nature of justifications in textbooks of limited vs. regular/extended scope, designed for students with different achievement levels.

METHODOLOGY

Ten key mathematical statements were selected for analysis from the Israeli 7th grade mathematics national curriculum (Ministry of Education, 2009), across several curricular topics, similar to those in Stacey and Vincent's (2009) study on Australian textbooks. The selection criteria for each analysed statement were: (1) it contains a mathematical idea or concept that requires justification in the national mathematics curriculum, and (2) it is considered to be an important result in the curriculum and in mathematics education literature. The statements are listed in the following:

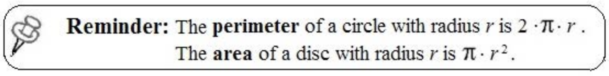
Algebra:

- The distributive property: $a \cdot (b + c) = ab + ac$ for every three numbers a, b, c.
- The product of two negative numbers is a positive number.
- Division by zero is undefined.
- Performing a basic operation on both sides of an equation maintains their balance.
- Two algebraic expressions are equivalent if for arbitrary values of the symbols in them the equality holds.

Geometry:

- Vertically opposite angles are congruent.
- The area formula for a trapezium with bases a, b and altitude h is $(a + b) \cdot h/2$.
- The area formula for a circle with radius r is πr^2.
- Angle sum of a triangle is 180^o.
- The corresponding angles between parallel lines are equal.

Analysis included all eight approved 7th grade textbooks for Hebrew speakers and their accompanying



Reminder: The **perimeter** of a circle with radius r is $2 \cdot \pi \cdot r$.
The **area** of a disc with radius r is $\pi \cdot r^2$.

Figure 1: A justification using an appeal to authority

teacher guides and supplementary materials. Six textbooks are of regular/extended scope, designed for the general student population (labelled A-F); two are of limited scope, designed for students with low achievements (labelled G-H). Data analysis included the textbook chapters introducing the statements – a total of 677 textbook pages; 57–110 pages (9–15%) from each textbook.

We analysed the explanations and justifications in the explanatory texts and those embedded in tasks and problems in the related task pools. Over 70% of the selected sections in the textbooks were analysed and discussed by 2–6 members of our research team until a consensus was achieved, and the remaining sections were analysed by the first author alone. First we identified distinct justifications of the statements in each section of the textbooks (i.e., explanatory texts and task pools). We then classified each justification for its mode of reasoning (following Stacey and Vincent, 2009) and compared frequencies relevant to the examined aspects: (1) the modes of reasoning offered; (2) algebra vs. geometry; and (3) limited vs. regular/extended scope.

FINDINGS

A total of 200 distinct justifications of statements were found in the textbooks. Comparison of the justifications by textbook section revealed that justifications were typically included in the explanatory texts, and seldom in tasks intended for student individual or small-group work, as shown in Table 1. This pattern was found in all textbooks (except textbook A) – regardless of the target student population, and in all statements – both in geometry and in algebra.

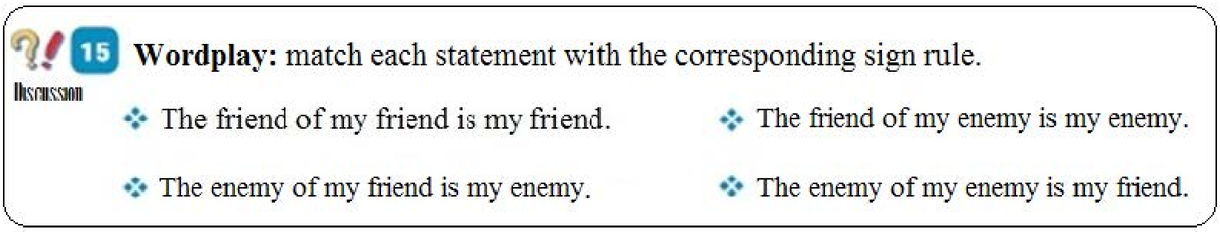
The modes of reasoning offered

Six of the seven modes of reasoning included in Stacey and Vincent's framework (2009) were identified in the Israeli textbook justifications, all but *concordance of a rule with a model*. Figures 1–6 exemplify justifications representing each of the identified modes of reasoning (translated from Hebrew).

Figure 1 illustrates a justification using an *appeal to authority* – a null explanation. The textbook merely presented a reminder for the area formula of a disc.

Figure 2 illustrates a justification using a *qualitative analogy*. The textbook included – among justifications using other modes of reasoning for the law of signs – an analogy that relies on a superficial similarity to the proverb "the enemy of my enemy is my friend". This justification does not reflect the mathematical structure of multiplication.

Figure 3 illustrates a justification using an *experimental demonstration*. The students were asked to tear paper triangles and rearrange the three angles, in order to convince themselves that the angle sum in a triangle is a straight angle.



15 **Wordplay:** match each statement with the corresponding sign rule.

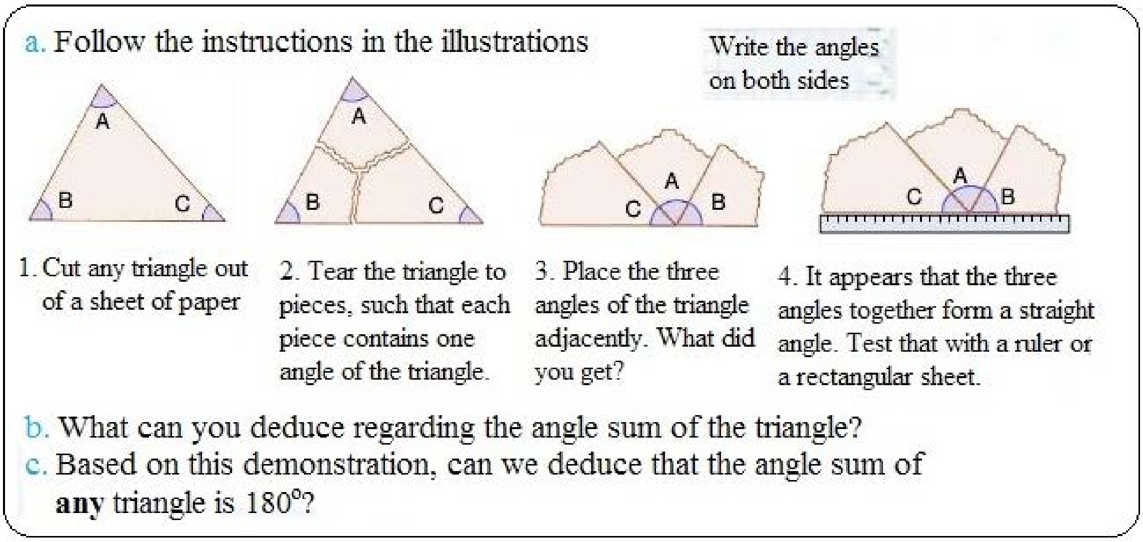
Discussion

- The friend of my friend is my friend.
- The friend of my enemy is my enemy.
- The enemy of my friend is my enemy.
- The enemy of my enemy is my friend.

Figure 2: A justification using a qualitative analogy

Section	Textbook								Total
	A	B	C	D	E	F	G	H	
Explanatory texts	16	22	26	17	25	28	20	23	177
Task pools	14	3	0	5	0	0	1	0	23
Total	30	25	26	22	25	28	21	23	200

Table 1: Number of distinct justifications by textbook section for each textbook



a. Follow the instructions in the illustrations

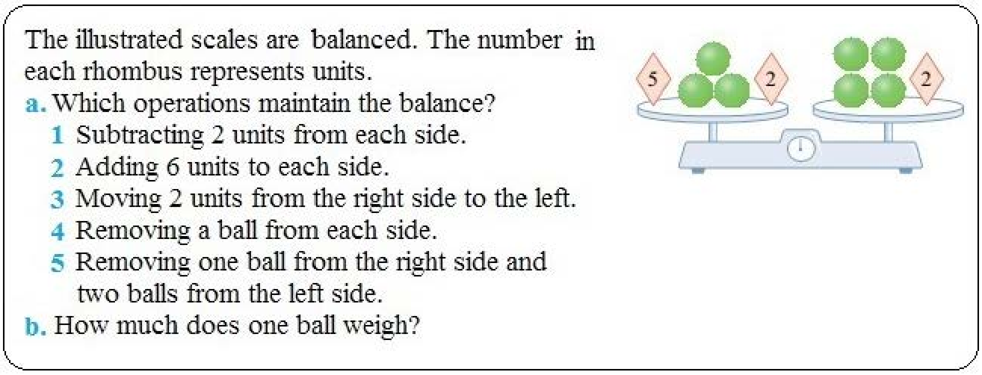
Write the angles on both sides

1. Cut any triangle out of a sheet of paper
2. Tear the triangle to pieces, such that each piece contains one angle of the triangle.
3. Place the three angles of the triangle adjacently. What did you get?
4. It appears that the three angles together form a straight angle. Test that with a ruler or a rectangular sheet.

b. What can you deduce regarding the angle sum of the triangle?

c. Based on this demonstration, can we deduce that the angle sum of **any** triangle is 180°?

Figure 3: A justification using an experimental demonstration



The illustrated scales are balanced. The number in each rhombus represents units.

a. Which operations maintain the balance?

1 Subtracting 2 units from each side.
2 Adding 6 units to each side.
3 Moving 2 units from the right side to the left.
4 Removing a ball from each side.
5 Removing one ball from the right side and two balls from the left side.

b. How much does one ball weigh?

Figure 4: A justification using a deduction using a model

Figure 4 illustrates a justification using a *deduction using a model*. The textbook relied on a structural similarity between the scales (the model) and balancing equations (the mathematics involved) to justify the statement.

Figure 5 illustrates a justification using a *deduction using a special case*. The textbook justified the area formula of a trapezium by splitting the area of a specific trapezium and forming a chain of reasoning, in which each step is logically deduced from previous steps. The given lengths are intended as a generic case (i.e., the specific values can be replaced without loss of generality).

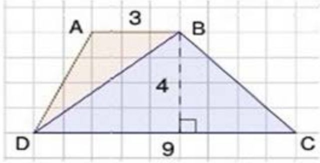
Figure 6 illustrates a justification using a *deduction using a general case*. The textbook justified the area formula of a trapezium by splitting the area of a general trapezium and forming a chain of reasoning, in which each step is logically deduced from previous steps. Pronumerals are used to note the lengths of the bases and of the altitude in the trapezium.

The textbooks provided justifications for all analysed statements (with one exception in Textbook E), commonly using several modes of reasoning in justifications for each statement, as shown in Table 2. Most of the justifications were of empirical (in blue) or deductive (in green, dark green, and olive green) modes, and only three instances of external sources (in red and dark red) were found. As Table 2 depicts, textbooks often justified a statement by using a certain mode of reasoning multiple times.

Algebra vs. geometry

Analysis of the modes of reasoning in the explanations offered in the textbooks revealed that different modes of reasoning were used for statements in algebra and in geometry. As Table 2 shows, algebraic statements (five top rows) were typically justified by deductive modes of reasoning, whereas geometric statements (five bottom rows) were usually justified by both deductive and empirical modes of reasoning. This analysis further shows that *deduction using a general case* appeared more in justifications of geometric

Draw the diagonal BD and split the area of the trapezium to two triangles: ΔABD and ΔDBC. The two triangles share an altitude (4 cm). The area of the trapezium ABCD is the sum of the areas of the two triangles:



$$S_{\Delta ABD} = \frac{3 \cdot 4}{2} \qquad S_{\Delta DBC} = \frac{9 \cdot 4}{2}$$

$$S_{ABCD} = S_{\Delta ADB} + S_{\Delta DBC} = \frac{3 \cdot 4 + 9 \cdot 4}{2} = \frac{(3+9) \cdot 4}{2} = 24 \text{ cm}^2$$

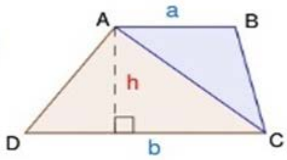
Let's observe the last step and identify the components of the formula. We see that we can find the area of a trapezium by finding the sum of lengths of the bases, times the length of the altitude, and dividing the product by 2.

$$S_{ABCD} = \frac{(\underset{\text{short base}}{3} + \underset{\text{long base}}{9}) \cdot \underset{\text{altitude}}{4}}{2}$$

Figure 5: A justification using a deduction using a specific case

Let's show that the area formula of a trapezium is true in all cases.
Fill in the missing steps in the proof.

- Draw the diagonal AC in the trapezium. It divides the trapezium ABCD into two triangles ΔABC and ΔADC.
- Therefore: $S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC}$
- The triangles ΔABC and ΔADC share an altitude h (why?)
- Let's write the area formula. Fill in the missing steps:



$$S_{ABCD} = S_{\Delta ABC} + S_{\Delta ADC} = \frac{a \cdot h}{2} + \frac{}{2} = \frac{ah +}{2} = \frac{(a+b) \cdot h}{2}$$

In other words: **The area of a trapezium is the sum of the lengths of the bases, times the length of the altitude, divided by 2.**

Figure 6: A justification using a deduction using a general case

Statement	Textbook A	B	C	D	E	F	G	H
Distributive law	m,m	e,m	m,s	m,m ,s	e,m	m,m,s,g	e,m	m,s
Multiplication of negative integers	q,m,m, m,s,g	s,s	s,g	s,g	s,g	g	a,e	g
Division by zero	m,s,s	s	s,g	s,g	s,g	s,g	s	s,g
Balancing equations	e,m	s	e,m,s	s	e,m,g	m,s	e	m,s
Equivalent expressions	e,m,m,s	e,m,m,s	e,e,m,m	e,s	e,m,m,m,s	e,s	e,m,m,s	e,e,m,m
Opposite angles	e,g	e,g	s,g	g	s,g	e,g	e,s	s,g
Area of a trapezium	s,s,s,g	s,s, g,g,g	e,s,s, s,g	e,e,e, s,s	e,e,e,s	e,e,s,g	e,e,e,g	e,e,e, s
Area of a disc	g	g	g	g	a	e,g	g	g
Angle sum of a triangle	e,g,g	e,e,e, g,g	e,e,g	e,g,g	e,e,g	e,e,e,g, g,g,g	e,e,g	e,e,e
Corresponding angles	e,s,g	e,g	e,e	e,g	e	e	e	e,e

Legend: a= appeal to authority; q= qualitative analogy; e= experimental demonstration; m= deduction using a model; s= deduction using a specific case; g= deduction using a general case.

Table 2: Distribution of modes of reasoning in textbook explanations (n=200)

statements than in justifications of algebraic statements. Figure 7 presents the frequencies of modes of reasoning in textbook explanations in each textbook by content topic – algebra and geometry.

Limited scope vs. regular/extended Scope

Comparison of the explanations offered in textbooks of limited scope and of regular/extended scope was conducted on seven of the ten statements due to the textbooks' structure, a total of 136 distinct justifications. Figure 8 presents the distribution of the modes of reasoning in textbook explanations for these seven statements. A chi-square test of independence was used to test the association between textbook scope and empirical justifications. Textbooks of limited scope offered significantly more empirical justifi-

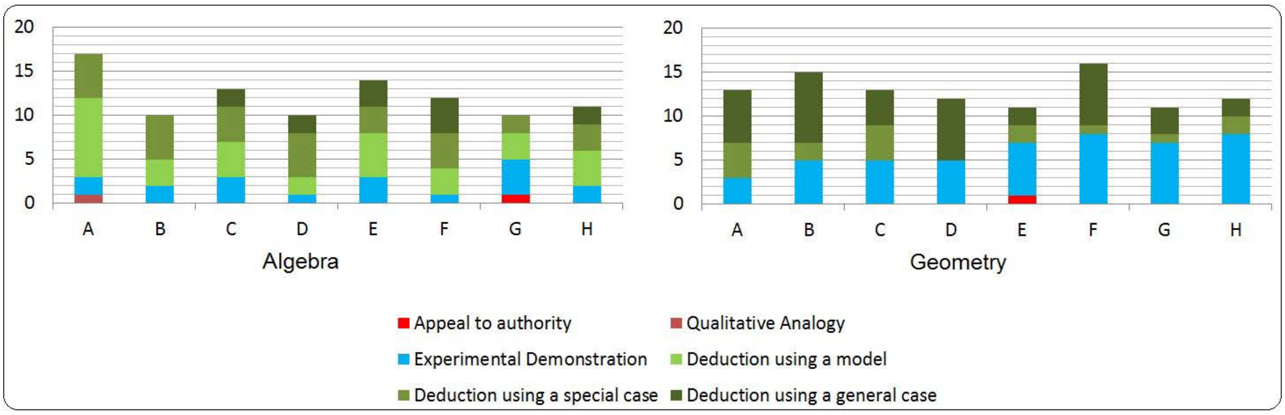


Figure 7: Frequencies of modes of reasoning by textbook and topic: algebra / geometry

cations than textbooks of regular/extended scope – roughly twice as much, $\chi^2_1 = 6.49$ (p = 0.01). An additional chi-square test was used to test the association between textbook scope and deductive justifications. Textbooks of regular/extended scope offered significantly more deductive justifications than textbooks of limited scope – roughly twice as much, $\chi^2_1 = 7.16$ ($p < 0.01$).

DISCUSSION

The findings reveal that Israeli 7th grade mathematics textbooks provide justifications for all the analysed key statements (but one statement in one textbook), commonly using several modes of reasoning in justifications for each statement. The inclusion of multiple modes of reasoning in textbooks might indicate an attentiveness of the textbook authors to the complexity of developing mathematical understanding, as the richness and diversity of textbook justifications might provide opportunities for students with different abilities, strengths, and backgrounds to learn and understand mathematics. Our results are in line with Stacey and Vincent's (2009) results on similar topics in Australian textbooks and with Dolev's (2011) results in Israeli textbooks (experimental version).

Our study shows that while most of the justifications for the analysed mathematical statements in 7th grade Israeli textbooks were included in the explanatory texts, some were embedded in the task pools intended for student individual or small-group work. However, our findings show that inclusion of the latter type of justifications in our analysis did not change the emerging patterns.

We found that 197 out of the 200 justifications analysed in the Israeli textbooks were deductive or empirical, implying that the textbooks typically explain each statement rather than present rules without reason. This finding does not comply with the findings reported in Stacey and Vincent's (2009) study, where 17% of the explanations for similar topics were neither deductive nor empirical.

Despite their emphasis on mathematical reasoning, our analysis shows that Israeli textbooks generally

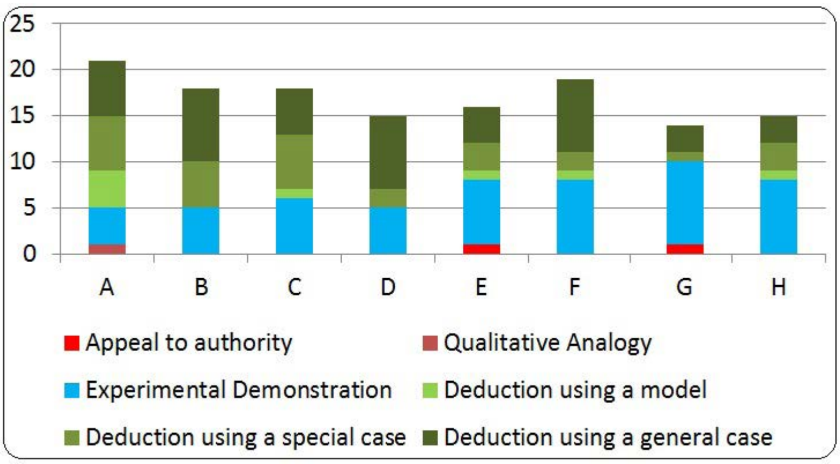


Figure 8: Frequency of modes of reasoning by textbook

give no indication regarding which justification can be considered a mathematically valid deductive proof and which serves only as a didactical tool to assist students' learning.

Comparison of the justifications for geometric statements and for algebraic statements in the Israeli textbooks showed that the textbooks typically use different modes of reasoning: deductive for algebraic statements, and both deductive and empirical for geometric statements. This finding is surprising at first glance due to the historic bias toward geometry as a subject suitable for teaching proof. However, careful analysis shows that the mode of reasoning closest to formal proof – *deduction using a general case* – appeared more in justifications of geometry statements than in justifications of algebra statements. Similar results were reported in Dolev (2011).

This study also shows that empirical justifications were more prevalent in textbooks of limited scope, whereas deductive justifications were predominant in textbooks of regular/extended scope. This pattern implies that student with different achievement levels are exposed to different types of justifications, and specifically that students with low achievements have fewer opportunities to deal with higher-order thinking and reasoning. These differences may have the potential to considerably limit the opportunities of students with low achievements to learn how to justify in mathematics, because teachers often follow teaching sequences suggested by textbooks (Eisenmann & Even, 2011; Haggarty & Pepin, 2002).

Our study focused on 7th grade textbooks. As Thompson (2014) notes, the similarities and differences we identified in this particular grade level – between textbook sections, between target student populations, and between curricular topics – might change over a textbook series. Additional research is needed to characterize the modes of reasoning in textbooks intended for higher grades.

ACKNOWLEDGEMENT

This research was supported by the Israel Science Foundation (grant No. 221/12).

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The role of mode of representation in students' argument constructions

Andreas J. Stylianides

University of Cambridge, Faculty of Education, Cambridge, UK, as899@cam.ac.uk

Research into students' understanding of proof has generally considered few of the factors that can mediate the relation between students' argument constructions or their evaluations of given arguments and conclusions about students' understanding of proof. This raises concern about the validity of research findings and creates difficulties in comparing findings from different studies. I summarize some of these factors and explore the role played by another one: the mode of representation used in students' argument constructions. In particular, I report and discuss findings from a classroom-based design experiment suggesting that the use of an oral mode of representation may be more likely, compared to a written mode, to support the construction of an argument that approximates or meets the standard of proof.

Keywords: Proof, argument construction, oral representation, written representation.

INTRODUCTION

An assumption (often tacit) underpinning the findings of many studies on students' understanding of proof is that students' argument constructions or students' evaluations of given (researcher-generated) arguments are indicative of their understanding of proof. While this is a sensible assumption to make, there exist some factors, often uncontrolled for in relevant studies, that may mediate the relation between students' argument constructions or evaluations and researchers' conclusions about students' understanding of proof based on that work. For example, there is evidence to suggest the following: it is easier for students to evaluate given arguments than it is for them to construct their own arguments (Reiss, Hellmich, & Reiss, 2002); it is easier for students to identify invalid arguments as invalid than it is for them to identify valid arguments as valid (ibid); students' constructions can be poor indicators of their understanding of proof, as students can be well aware of limitations of their non-proof constructions (Stylianides & Stylianides, 2009); and students can evaluate given arguments in different ways based on different perspectives, such as what would satisfy them personally or what would satisfy their teachers (Healy & Hoyles, 2000).

Studies in this area considered at most one or two of these factors that may mediate the relation between students' argument constructions or evaluations and conclusions about students' understanding of proof. Thus a concern is raised about validity of research findings regarding students' understanding of proof. Take, for example, a study that draws conclusions about students' understanding of proof based on students' argument constructions in response to a number of proving tasks. This study is likely to report a poorer picture of students' understanding of proof than another study that considered also students' evaluations of their own constructions, for relevant research (Stylianides & Stylianides, 2009) suggests some students are fully aware of the reasons for which their non-proof constructions are not proofs. In addition to the issue of validity of research findings, there is also the difficulty in comparing findings from different studies; this creates in turn an obstacle to the development of a cumulative and coherent body of research knowledge in this area.

In this paper, I explore another factor that is worth attention by future research in this area. The factor relates to the *mode of representation* (Stylianides, 2007) used in students' argument constructions, that this, the forms of expression (written, oral, pictorial, etc.) with which an argument is communicated (ibid). I focus on two main modes of representation – written and oral – and I address the following research question: How does the mathematical sophistication of a student's arguments, for the same claim, compare when the bulk of each argument is communicated with a different mode of representation – written versus oral?[1]

There is some evidence to suggest that the verbal mode of representation may be associated with arguments of higher level of mathematical sophistication than the written mode (Schoenfeld, 1985). Schoenfeld described an episode where two students produced a lucid verbal argument, essentially a proof, for a geometrical construction problem, but then the students put down their ideas in writing by producing a contorted argument following the strictly prescribed 'two-column' presentational form (statement; reason). If further evidence was found that oral modes of representation were generally associated with mathematical arguments of higher level of mathematical sophistication than written modes, an important methodological implication would follow: An interview study that examined orally students' argument constructions would likely report a better picture of students' understanding of proof than a survey study that examined in writing the argument constructions of the same group of students and using the same proving tasks.

RESEARCH CONTEXT

The data for the paper are derived from a design experiment, which examined what may be involved in engineering classroom instruction to support secondary students to learn about proof. The design experiment was carried out in an English state school with 165 Year 10 students (14–15 year olds) who were set in seven classes according to their performance in a national assessment at the end of Year 9. All 61 students from the two highest attaining Year 10 classes, and the two mathematics teachers of these classes, participated in the research over a period of two years.

The focus of the study on high-attaining students was partly motivated by the findings of a prior large-scale longitudinal study in England (Küchemann & Hoyles, 2001–03) that showed (1) weak knowledge about proof amongst a national sample of high-attaining Year 8–10 students and (2) modest (if any) improvements in students' knowledge from Year 8 to Year 10. These findings raised concerns about English high-attaining secondary students' learning about proof, and suggested an even more pessimistic prospect for lower attaining or younger students.

The design experiment involved the development, implementation, and analysis of the effectiveness of six lesson sequences, each ranging from one to five 45-minute periods. Lesson sequences 1–4 were implemented when the students were in Year 10, while the rest in Year 11. At the beginning of the study I took the lead role in planning the lesson sequences, but over time the teachers felt more confident to take responsibility for planning the lesson sequences and this allowed me to assume more of a supportive role. All lessons were taught by the regular teacher of each class.

In this paper, I focus on lesson sequence 2, which lasted three 45-minute periods in one class and two in the other. It was implemented three months into Year 10 and capitalized on lesson sequence 1. Lesson sequence 1 lasted two 45-minute periods in each class, was implemented one month into Year 10, and had two main goals: (1) to help students begin to realize the limitations of empirical arguments as methods for validating mathematical generalizations and see a need to learn about more secure validation methods (i.e., proofs); and (2) to introduce students to the notion of proof in mathematics, including a list of criteria for deciding whether a mathematical argument met the standard of proof. The criteria were as follows.

> An argument that counts as *proof* [in our class] should satisfy the following criteria:
>
> 1. It can be used to convince not only myself or a friend but also a *sceptic*.
>
> — It should not require someone to make a leap of faith (e.g., "This is how it is" or "You need to believe me that this [pattern] will go on forever.")
>
> 2. It should help someone *understand why* a statement is true (e.g., why a pattern works the way it does).
>
> 3. It should use *ideas that our class knows already or is able to understand* (e.g., equations, pictures, diagrams).
>
> 4. It should contain *no errors* (e.g., in calculations).
>
> 5. It should be *clearly presented*. (PowerPoint slide used during Lesson Sequence 1)

The criteria were consistent with the following definition of proof, with care taken so that the phrasing of the criteria was suitable for secondary students.

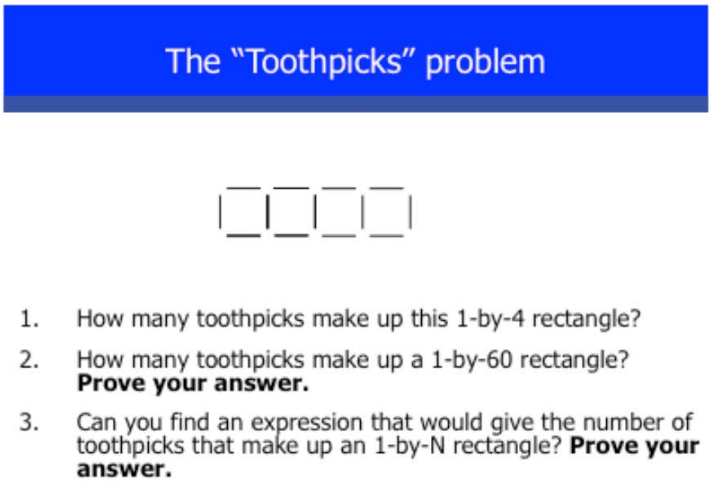

Figure 1: The first proving task

> *Proof* is a *mathematical argument*, a connected sequence of assertions for or against a mathematical claim, with the following characteristics:
>
> 1. It uses statements accepted by the classroom community (*set of accepted statements*) that are true and available without further justification;
>
> 2. It employs forms of reasoning (*modes of argumentation*) that are valid and known to, or within the conceptual reach of, the classroom community; and
>
> 3. It is communicated with forms of expression (*modes of argument representation*) that are appropriate and known to, or within the conceptual reach of, the classroom community. (Stylianides, 2007, p. 291)

Criteria 1 and 4 correspond to the requirement in the definition for valid modes of argumentation; criterion 5 to the requirement for appropriate modes of argument representation; and criterion 3 to the requirement that all components of a mathematical argument (set of accepted statements, modes of argumentation, and modes of argument representation) be readily accepted, known to, or within the conceptual reach of the class. Furthermore, criteria 1 and 2 reflect, respectively, two important functions that the development of arguments and proofs intended to serve in the two classes: to promote conviction at both the individual and social levels (e.g., Mason, 1982) and understanding (e.g., Hanna, 1995).

The goals of lesson sequence 2 were (1) to help students further understand the proof criteria and (2) to offer students opportunities to apply these criteria in three proving tasks. The tasks were mathematically similar though not necessarily of the same level of difficulty; they all involved making and proving a generalization by reference to an underlying structure. Lesson sequence 2 started with review of the proof criteria, and then the teacher introduced task 1 (Figure 1). There was individual or small group work on the task, during which the teacher asked the students to write down their 'best' arguments. The students were given ample time to do that and were free to work in pairs or larger groups. While students were instructed to write down individually their arguments, few of them wrote arguments in pairs. Finally, there was a whole class discussion during which several students presented individually or in pairs their arguments. Similar procedure was followed for tasks 2 and 3.

METHOD

Data

The data for the paper are the written arguments of, and transcripts of the subsequent oral presentations of these arguments by, 17 students in the two classes. These were all the students who, in response to an open call by the teachers, offered to present their arguments for any of the three proving tasks during whole class discussions. Thirteen of these students presented arguments for only one task while two students presented arguments for two tasks. The distribution of student-presenters across the three tasks was 10 students for the first task, 4 for the second, and 3 for the third.

Analysis

A research assistant and myself coded independently all written and oral (transcribed) arguments of the 17 students. First, we used the coding scheme developed by Stylianides and Stylianides (2009) to code each argument into one of the following five categories according to the argument's level of mathematical sophistication. The codes are presented in decreasing level of mathematical sophistication. We compared our codes and discussed disagreements to reach consensus.

- code M1: proof
- code M2: valid general argument but not a proof
- code M3: unsuccessful attempt for a valid general argument
- code M4: empirical argument
- code M5: non-genuine argument

All three tasks required proving the truth of a generalization. The definition of code M1 was consistent with Stylianides' (2007) definition of proof, which underpinned in turn the criteria for proof used in the two classes. Specifically, code M1 was defined to be an argument that was *general* (i.e., it referred to all cases in the domain of the generalization), used *valid* modes of argumentation (i.e., it offered conclusive evidence for the truth of the generalization), and was accessible to the students in the class (i.e., it used statements that were readily acceptable by the class as well as modes of argumentation and modes of argument representation that were known to the students or within their conceptual reach at the particular time). Code M2 was used for arguments that approximated but not quite met the standard of proof, because, for example, of missing or inadequate justification of an assertion that could not be considered readily acceptable by the class. Code M3 was used for arguments that reflected an attempt to justify the generalization for all cases in its domain, but were either incomplete or used *invalid* modes of argumentation (i.e., they included a logical flaw). Code M4 was used for arguments that verified the truth of the generalization only in a proper subset of the cases in its domain but concluded it was true for all cases. Finally, code M5 was used for responses to the proving tasks that showed minimal engagement, were irrelevant to what was being asked, or were potentially relevant but the relevance was not made evident to the coder.

In addition to the above, for each argument we coded the following:

- Who wrote or orally presented the argument: an *individual student* or a *pair of students*;
- The kind of input from the teacher or the rest of the class during the oral presentation of the argument: *no input*, *some but not substantial input* (i.e., input that simply reiterated or briefly clarified a point mentioned by the student without influencing the presented argument), or *substantial input* (i.e., input that influenced the presented argument and possibly altered its level of mathematical sophistication).

Furthermore, we examined whether there was any evidence to suggest that preceding oral presentations of arguments for a proving task influenced subsequent presentations for the same task. We found no evidence of such influence: students' oral presentations were rather distinct from one another; students' orally presented arguments matched closely their written arguments (e.g., oral presentations tended to be based on the same figures or drawings as in students' written work); and students looked at or referred to their written work during their oral presentations.

Yet, the temporal sequencing of students' arguments (first written, then oral) was a factor we could not account for. It is possible that a student's original efforts to write an argument for a proving task helped the student build familiarity with the task and underlying concepts, thus placing the student in a position to orally present later on an argument of higher level of mathematical sophistication. Another factor we could not account for was whether the teacher offered any substantial input during students' written work in small groups. According to the plan that I had agreed with the teachers prior to the lessons, the teachers would ask students probing questions, but they would not directly influence students' argument constructions. There was no concrete evidence that the teachers deviated from the agreed plan. But even if they had done that, the result would have been better written arguments and, presumably, better oral presentations of those arguments, too. Thus there would likely be limited if any impact on the *comparison* between the levels of mathematical sophistication of the written and oral arguments, which is the issue examined in this article.

RESULTS

The results are summarized in Figures 2–4, which show the relationship between the level of mathematical sophistication of the written arguments produced

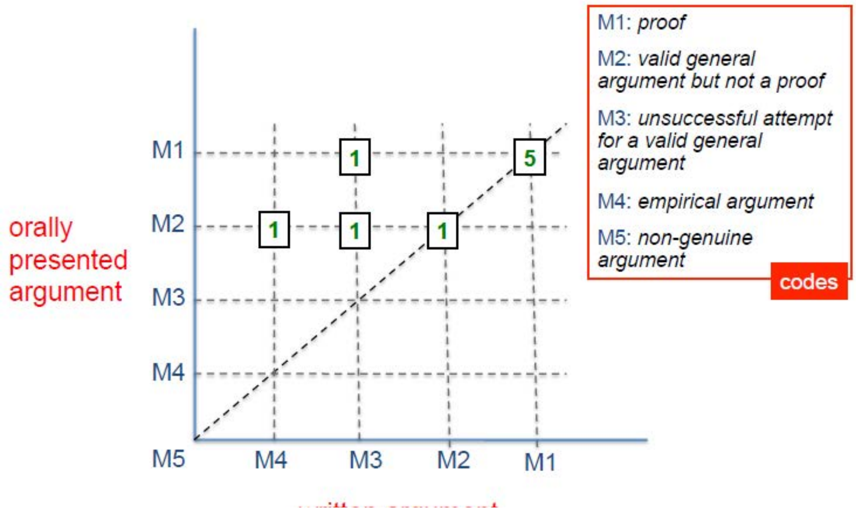

Figure 2: Students who wrote and presented their arguments individually, with no input (N=9)

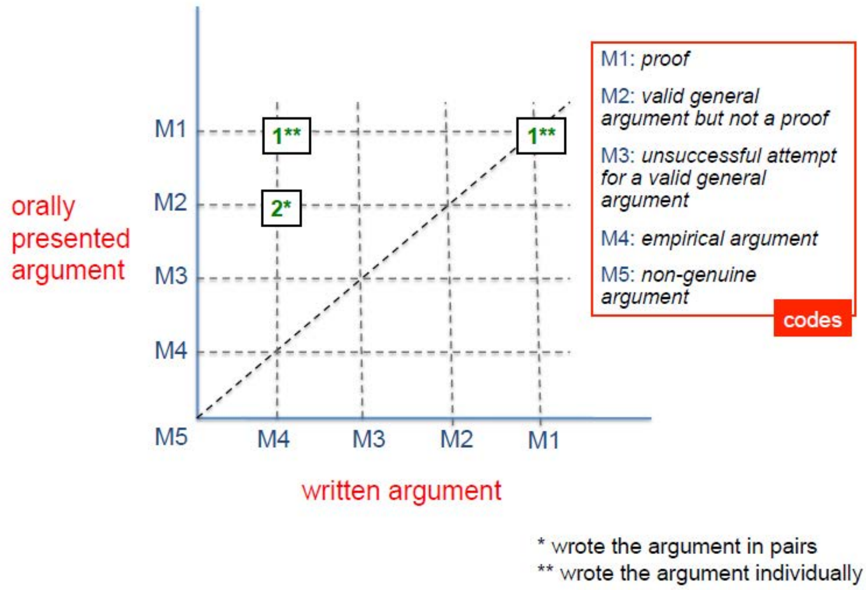

Figure 3: Students who presented their arguments in pairs, with no input (N=4)

by certain groups of students and the corresponding level of orally presented arguments by the same groups. The numbers in the figures represent frequencies of students.

Nine of the 17 students wrote and presented their arguments individually, with no input from the teacher or the rest of the class during those presentations (Figure 2). Five of these students wrote and presented proofs (M1), one wrote and presented a valid general argument but not a proof (M2), and the other three presented more mathematically sophisticated arguments than they had written earlier: two wrote arguments that reflected an unsuccessful attempt for a valid general argument (M3) but presented M1 and M2 arguments respectively, while one wrote an empirical argument (M4) but presented an M2 argument.

Four other students presented their arguments in pairs and received no input from the teacher or the test of the class during those presentations (Figure 3). The students in one of the pairs wrote together an M4 argument but presented (again together) an M2 argument. The students in the other pair wrote different arguments but made a joint presentation, which was coded as M1; one student had written an M1 argument while the other had written an M4 argument.

The remaining four students wrote and presented their arguments individually, but during their presentations they received input from the teacher (Figure 4).

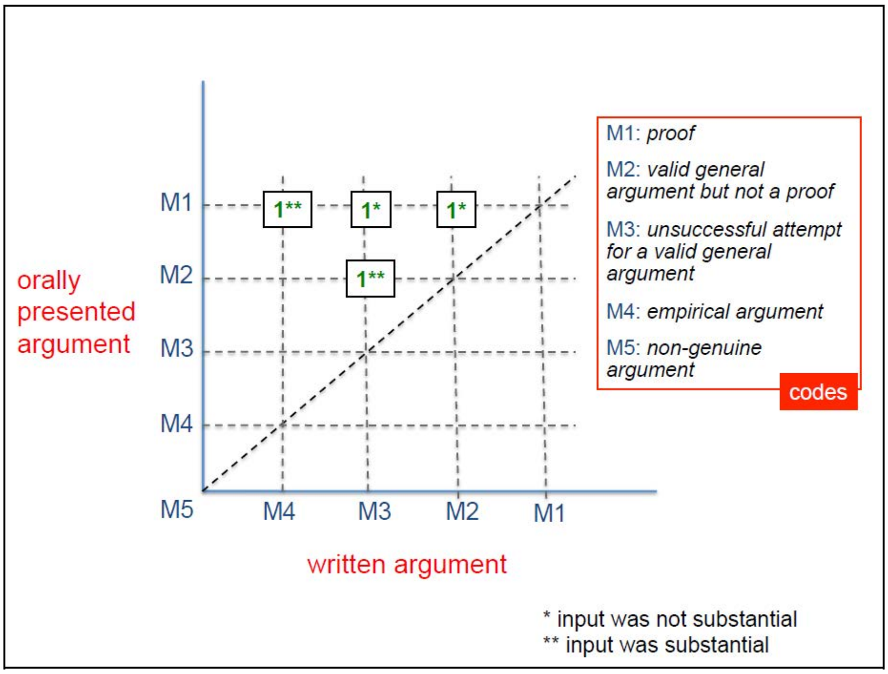

Figure 4: Students who wrote and presented their arguments individually, with input (N=4)

Yet, in two of the cases the input was not substantial: the teacher simply reiterated or clarified briefly a point mentioned by the students during their presentations, without influencing the presented argument. One student wrote an M3 argument and presented an M1 argument, and the other wrote an M2 argument and presented an M1 argument. In the remaining two cases the input from the teacher was substantial. For example, in one of the cases the teacher made a comment at a point during the student's oral presentation when the student paused and seemed to have difficulty articulating the general case; this comment might have helped the student articulate an argument of a higher level of mathematical sophistication than she would have presented otherwise. In both cases the presented arguments were of higher level of mathematical sophistication than the arguments the students had written in their papers: one was M4 and became M1, while the other was M3 and became M2.

To sum up, the proportion of arguments for which the teacher's input was substantial was small (2 out of 17), while in all of the cases the orally presented arguments were of the same or of higher level of mathematical sophistication compared to the corresponding written arguments. All orally presented arguments were either proofs (M1) or valid general arguments but not proofs (M2). Eight of the written arguments were already at the M1 or M2 levels, but the remaining nine were either empirical (M4) or unsuccessful attempts for a valid general argument (M3) and yet, during the oral presentations, all of these arguments were elevated to the M1 or M2 levels.

DISCUSSION

The findings offer support to the hypothesis that the level of mathematical sophistication of students' arguments for the same claims may depend on the mode of representation (oral vs. written) students use to communicate these arguments. The students whose arguments I examined in this study tended to omit few essential steps or explanations in their written work, but they addressed most of these omissions during their oral presentations. All orally presented arguments were of the same or higher level of mathematical sophistication than their written counterparts. A methodological issue stands out from these findings: If a study had analyzed students' oral arguments only, it would have reported a better picture of students' ability to construct arguments than another study that analyzed students' written arguments only. Yet, limitations of the research design, notably the lack of control over the temporal sequencing of students' arguments (first written, then oral), do not warrant any definite statement that the oral mode of representation is generally advantageous over the written mode in the construction of arguments that approximate or meet the standard of proof.

Below I present four other possible and not necessarily competing reasons for which students' oral arguments tended to rank higher than their written arguments.[2] Reasons 1 and 2 reinforce, while reasons 3 and 4 weaken, the presumed role played by the mode of representation in the observed differences between students' written and oral arguments, thus highlighting the need for more research in this area.

1. *Relative difficulty of written versus oral arguments.* Writing mathematical arguments may be genuinely more difficult than presenting orally mathematical arguments, especially for students who were recently introduced to the concept of proof as were the students in this study.

2. *Relative preference for oral versus written expression.* Students may tend to prefer oral over written expression of their mathematical ideas, and so students in the study might have responded in a minimalistic way to the teacher's expectation to produce written arguments for the three tasks (indeed, a didactical contract regarding proof work was not yet established in the class).

3. *Possible role of the specific nature of proving tasks.* The three proving tasks all belonged to a special family of tasks and involved making and proving a generalization by reference to an underlying structure. High linguistic demands are imposed on solvers as they try to express a general argument by reference to a specific diagram that nevertheless exemplifies the underlying mathematical structure; students may be better able to cope with these demands when they express their ideas orally rather than in writing.

4. *Possible influence of the broader social context on students' oral presentations.* Even though almost all of the oral presentations were carried out with no verbal input from others in the class, extra-linguistic forms of expression, notably gestures, might have given to presenters some non-verbal cues (cf. Roth, 2001) that encouraged them to also evaluate or elaborate more on their arguments, thus addressing some of their limitations and elevating their status (cf. Stylianides & Stylianides, 2009).

To conclude, in this paper I have called, and reinforced the need, for more research into factors mediating the relation between students' argument constructions or evaluations and conclusions about students' understanding of proof. Do we, as a field, document an accurate picture of students' understanding of proof? Are findings from different studies in this area comparable with each other?

ACKNOWLEDGEMENT

The research reported in this paper was supported by a grant from the Economic and Social Research Council in England (RES-000-22-2536). The opinions are those of the author and do not necessarily reflect the position or policy of the Council.

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ENDNOTES

1. While a written or an oral mode of representation may be used to communicate the bulk of a student's

argument, other modes of representation can also be used alongside this mode. For example, a student who presents orally an argument in front of a class can draw on the board a picture, or write on the board an algebraic expression, in order to supplement the verbal expression of the argument.

2. I do not mention time as a possible reason, for students were given ample time to produce their written arguments. Yet, one cannot completely exclude the possibility that few students recorded in writing their 'exploratory' work and then, during whole class discussion, shifted to a more 'deductive' form of presentation of their finished products.

TWG01

Posters

Considerations on teaching methods to deepen student argumentation through problem solving activities

Tsutomu Ishii

Bunkyo University, Faculty of Education, Koshigaya, Japan, benishii@koshigaya.bunkyo.ac.jp

In this research project we evaluate the Toulmin model to analyse students' mathematical arguments (Toulmin, 2003). We use this method to analyse a teaching episode in which the sum of the interior angles in a five-cornered polygon was discussed. This episode took part in a mathematics lesson in junior high school, Japan. The analysis highlights students' difficulties to understand ideas of their peers, and how these led to an improved level of discussion.

Keywords: Argument, problem solving, Toulmin's model, argumentation, 180(n-2).

THE AIM OF RESEARCH

Teachers promote problem solving in a mathematical lesson. In the beginning of the lesson the teacher introduced the topic to initiate a discussion. In order to achieve the purpose of the lesson, it is necessary for the teacher to evaluate students' arguments correctly. However, it can be difficult to understand students' arguments. The purpose of this research is to examine if the Toulmin model (Toulmin, 2003) helps to gain a better understanding of students' mathematical arguments.

THE METHOD OF RESEARCH

I chose the Toulmin Model as a tool to analyse students' arguments. Toulmin claims that arguments typically consist of six parts: claim(C), data (D), warrants (W), qualifiers (Q), rebuttals (R), and backing (B) (Hitchcock & Verheij, 2006). A typical Toulim diagram to describe an argument may look as follows:

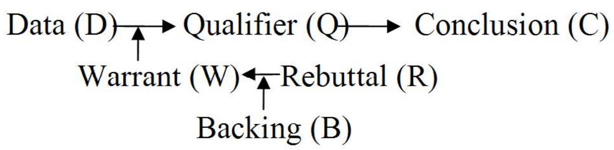

Figure 1: The Toulmin Model

EPISODE

The episode which I describe here took part in a mathematics lesson in the second grade of junior high school in Japan. The topic of this lesson was the sum of the interior angles of a polygon. The teacher's aim was for students to develop the fact that the sum of the inside of an n-cornered polygon is *180(n-2)*.

Individual solutions and preparation of the argumentation are summarized in the following diagrams or comments from students D-G (Figure 2).

D E F

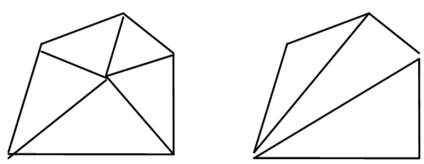

Figure	Triangle	quadrangle	equal form
The sum of angle	180	360	540

G If the number of angles is set to $\chi 180(\chi - 2)$

Figure 2

The discussion leading to this result began with an explanation given by student H. Then student G explained his view which is opposed to H's. His argument showed that H was wrong. From this result the following points can be deduced why the mistake occurred and how correct evidence can be concluded. Student D's figure illustrates a misconception caused by the fact that an additional angle has been constructed in the centre of the pentagon. As student E's figure shows, all the angles are included in the polygon. Therefore this figure illustrates the general formula.

THE CONCLUSION OF RESEARCH

We use the Toulmin Model to describe the arguments of students H and G, as illustrated in Fig 3 and Fig 4 below.

(D) D,E,F,G → (Q) D is right → (C) E is wrong
(W) F and G had the same ideas as E ← (R) 180(x-2)
(B) 360 about 180(x-2) at the time of x=4

Figure 3: Student H's argument described by the Toulmin Model

(D) E is wrong → (Q) E is right → (C) H is a wrong
(W) 180x-360=180(x-2) ← (R) 180-360
(B) 360=180×2

Figure 3: Student G's argument described by the Toulmin Model

H was not able to understand the solution of D correctly. The difficulty to understand the other students' ideas was emphasized in our analysis. And since H misunderstood, G felt excited. Although feeling controlled the understanding to become a new driving force was decided in the restorative. As a result, the quality of arguments improved remarkably.

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Unpack and repack mathematical activity with pre-service teachers: A research project

Simon Modeste[1] and Francisco Rojas[2]

1 Université de Montpellier, Montpellier, France

2 Pontificia Universidad Católica de Chile, Santiago, Chile

We present a research project that aims to build a model of mathematical activity that can be used in primary teacher education. This model must be useful both for teacher training and be able to support teachers' practices. We propose the idea of Unpacking and Repacking mathematical activity to achieve this goal.

Keywords: Unpacking, repacking, teacher training, mathematical activity.

CONTEXT, HYPOTHESIS AND RESEARCH QUESTIONS

Assuming that experiencing mathematical activity (MA) is important in learning mathematics, a corollary for mathematics teacher education is that teachers have to be confronted with an authentic MA in order to learn mathematics, to be conscious of its nature and to be able to deal with it at school. By authentic, we mean an activity that is *open* in the meaning that in it, the didactical contract leaves more mathematical responsibilities to the students than classical tasks in the classroom and is closer to the experts' practice of mathematics (mathematicians).

In order to make effective and efficient the opportunities to experience and think about MA in primary teacher training, conceptual and didactical tools are needed. We aim at building such tools, by tackling the following questions: Can a model of MA and a way to experience and think about it can be proposed in primary teacher education? How primary teacher education can include opportunities using this model in order to develop knowledge of the activity? Do such a model and opportunities make primary teachers use MA in their classrooms and help them to deal with it? Does it impact teachers' conceptions about MA or their conceptions about teaching mathematics?

THEORETICAL APPROACHES

Genres

MA involves different genres of activity, that is to say a set of prototypical practices which configure this activity. Research in mathematics education often distinguish mathematizing and modelling; defining; specifying, generalizing and extending; proving (reasoning, demonstrating), among others. Particularly, the MTSK model (Carrillo et al., 2013) develop the knowledge about mathematics as a dimension that is considered as a part of the mathematics teacher specialized knowledge. This dimension includes knowledge of ways of knowing and creating or producing in Mathematics, aspects of mathematical communication, reasoning and testing, knowing how to define and use definitions, establishing relations, correspondences and equivalences, selecting representations, arguing, generalizing and exploring.

Unpacking and repacking

The model we want to develop must be designed to be used by teachers and teacher educators to think about MA when they solve problems or give opportunities to their student do it. We consider as important to deal with deep epistemological aspects of the MA, in the sense that it is fundamental to understand, through a reflexive MA, what are the objects, the objectives, the ways of validating, etc., in mathematics.

This is why we propose, inspired by Martin (2013), the concept of unpacking the MA, which is the work of separating and analysing each part of the mathematical practice and its role and place in the global process of problem solving. Genres would be a first level of unpacking. Besides, repacking is the part of the work that permits to underline the links between the different parts and levels unpacked and understand how they interplay.

We claim that a teacher who can unpack and repack his/her (and others') mathematical work would be better equipped to deal with MA in the classroom.

METHODOLOGICAL PLAN

In order to tackle the research questions, we plan to:

— Build a model of MA and its genres, based on specialized research results on each genre, but also on literature on problem solving, advanced mathematical thinking, Theory of Didactical Situations, or other theories dealing with MA;

— Propose problems that permit primary teachers to experience a MA that can be unpacked and repacked, involving different genres – here, using Research Situations for Classroom (Gravier & Ouvrier-Buffet, 2009) may be pertinent;

— Experiment with mathematicians, mathematics educators, and then with primary teachers to analyse the potential for unpacking and repacking;

— Propose a model of unpacking and repacking and experiment it in primary teacher education with selected and well-tried problems;

— Evaluate the impact on teachers' conceptions about mathematics and its teaching, how they feel prepared and how they manage with MA in school.

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How is proving constituted in Cypriot classroom?

Maria Pericleous

University of London, UCL Institute of Education, London, UK, mariapericleous19@gmail.com

This report focuses on a research study the aim of which is to investigate the activity of proving as constituted in a Cypriot classroom for 12 year old students. By drawing on Activity Theory, this study explores the way the teacher is working with the students to foreground mathematical argumentation. Analysis from classroom discussions points toward a teacher-directed mathematical argumentation as an approach to establish justification as a socio-mathematical norm in the classroom

Keywords: Activity theory, proving, socio-mathematical norms.

BACKGROUND OF THE STUDY

In recent years, it has been established that proof and proving should be integrated across all levels of schooling (Hanna, 2000; Stylianides, 2007). In this context, explanation, justification and argumentation are aspects of proof that provide a foundation for further work on developing deductive reasoning and the transition to a more formal mathematical study. In the social environment of the classroom, where hypothesising, explaining and justifying geometric conjectures is encouraged, the tools and tasks used, the rules of the classroom, the way the students work together, the way the teacher negotiates meanings and other external factors all interact, interrelate and influence each other in forming classroom activity. The purpose of the present study is to explore the way the structural resources of the classroom setting shape students' argumentation. Research has responded to the need to conceptualise proof and proving in such a way that is can be applied not only to older students but also to those in elementary school (Stylianides, 2007). The question remains however to understand how proof is constituted in such classrooms.

METHOD

Cultural Historical Activity Theory (CHAT) provides this study with a theoretical basis to steer the identification of forces that interact to shape pre-proving activity in a complex environment. That is, CHAT is used both as a framework for conceptualising the research and formulating the research design. This study was conducted in a year 6 classroom in a public primary school in Cyprus. The content of the curriculum covered during the classroom observations was the area of triangles, and the circumference and area of circles. The overall process of analysis of the collected data was one of progressive focusing. The systematisation of the data led to the evolution of two activities: (i) activity of exploration which is concerned with the degree of exploring mathematics in the classroom. It includes the exploration of mathematical situations and exploration for supporting mathematical connections and (ii) activity of explanation which is concerned with instances of classroom discussion that are related with explaining the purpose of which is to clarify aspects of one's mathematical thinking that might not be apparent to others and, explaining why, that is justification the aim of which is to establish for somebody else the validity of a statement.

FINDINGS AND DISCUSSION

Analysis of the classroom episodes show the teacher frequently using the word 'play'. Two contrasting values have emerged through the teacher's ambiguous use of the term 'play'; play/learn. When providing opportunities for exploration and investigation, the teacher was presenting the exploration constructively as 'play.' The word 'play' had a negative value when the exploration was interpreted as 'play' instead of learning. Closing down a task clashed with the object of the activity of exploration. The use of play highlights a tension between the two activities of exploration and explanation. Exploration was understood by the teacher as worthwhile in order for the students to seek out explanation but at the same time exploration in her eyes might have led to loss of focus, which might have resulted in different activity from explaining. By closing down the investigation, the students did not have the opportunity either to initiate a solution, or to test the hypothesis made, thus limiting their explaining and justifying. The 'play' contradiction relates to the notion of the play paradox (Hoyles & Noss, 1992) and the notion of the planning paradox (Ainley, Pratt, & Hansen, 2006). While a play-like exploration can facilitate learning, it is not automatically clear that it is the teacher who decides what counts as meaningful. Thus, the teacher may find it difficult to take advantage of such opportunities. The teacher had concerns about focus and discipline, which seemed to lead to such closing down. The intention therefore in leading the discussion around justification to establish this as a socio-mathematical norm is comfortably in line with maintaining focus. We might therefore conjecture that the students will have few opportunities in the near future to engage with proving related activity in a more independent way.

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Four steps on the way to create argumentation competence supported by technology

Evelyn Süss-Stepancik[1] *and Stefan Götz*[2]

1 University College of Teacher Education Lower Austria, Baden, Austria, evelyn.stepancik@ph-noe.ac.at

2 University of Vienna, Vienna, Austria

Extending the spiral curriculum which is originally focused on content matter areas dealing with typical mathematical activities we create a competence-based four step model to develop arguing. According to the four levels, the use of technology has different purposes: from exploring at the beginning to explaining at the end of the search for mathematical reasons. This cannot only prove but also enlighten certain problem areas.

Keywords: Argumentation competence, use of technology, reasoning, proof.

ARGUMENTATION AND PROOF

The Austrian educational standards for Mathematics at grade 8 define competences such as "*cognitive abilities which are available for a longer time (and the readiness to make use of them)*". Moreover, they define standards descriptors within four content areas and four types of mathematical activities/processes that students should make use of when making use of their content related competence (IDM, 2007). One of these activities is, as with many other mathematical standards *Reasoning and Proof*. To be able to reason and prove requires continuing development. For that we propose to extend the spiral curriculum with its main ideas (revisiting topics, increasing complexity) to typical mathematical activities. Accounting this we generated a four level model to develop arguing on mathematical topics from grade 5 to grade 12.

A competence based four level model for arguing

Based on the forgoing our model includes several levels. All levels can occur in a specific field of arguing. Each level should be dealt with in one particular grade or even across grades.

Level 1 (Disposition): Disposition to engage in mathematical tasks that demand (simple) reasoning.

Level 2 (Comprehension): Ability to understand, comprehend and explain prescribed reasons. An important aspect in this context is to identify the frame of reference.

Level 3 (Communication): Ability to explain and argue completed mathematical proof in communication situations.

Level 4 (Autonomy): Finding a reason for a (mathematical) statement/conjecture, including the choice of the frame of reference, autonomously.

The four levels in a specific field

The Pythagorean Theorem is an example for a broad field for reasoning and proof with various possibilities to use a paper and pencil approach as well as technology. Applets like the one shown in Figure 1 (left) can engage students in reasoning tasks (Level 1) as well as explaining and arguing a proof (Level 3). Figure 1 (right) addresses Level 2.

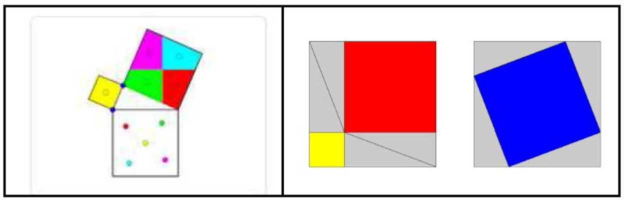

Figure 1: http://tube.geogebra.org/student/b615817#; Proof by Zhou bi suan jing

Here technology fosters the exploration and generation of conjectures.

The four levels across several grades

In grade 6 students explore triangles and construct particular points of a triangle. These points have in-

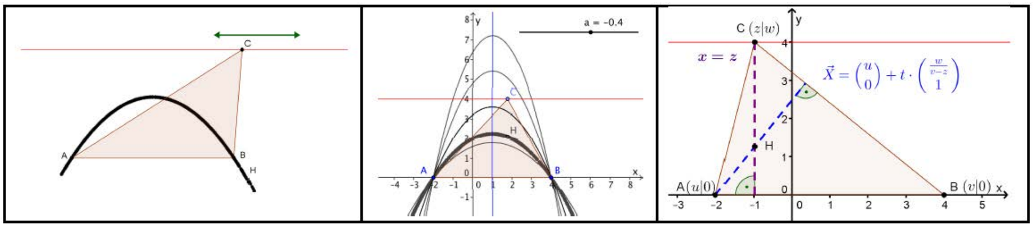

Figure 2: The trace of the orthocentre ***H*** and the embedded triangle ***ABC***

teresting traces under certain conditions. The exploration of the shapes of these traces leads from Level 1 to Level 4 if one starts with the following task (Figure 2): Assume vertex C of triangle ABC moves on a line parallel to AB and observe the trace of the orthocentre H (Figure 2, left). In grade 6 on Level 1 students should explain why the trace of H always intersects the vertices A and B. Here technology helps to explore the phenomenon. In grade 9 and on Level 3 students should find a quadratic function that describes the trace of the orthocentre H, a parabola. Here a slider helps to find the fitting parameter of the function (Figure 2, middle). Also in grade 9 on Level 2 students should understand the analytic description of the trace of the orthocentre H. For that purpose the triangle is embedded in a coordinate system. Here technology helps to see that the movement of C is just a numerical variation of the coordinate z (Figure 2, right). After finishing these levels it is possible to reach Level 4. At this level students should find an analogous result for the circumcenter (low requirement) or the incircle (high requirement).

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An example of Proof-Based Teaching: 3rd graders constructing knowledge by proving

Estela Vallejo Vargas and Candy Ordoñez Montañez

Pontificia Universidad Católica del Perú, Lima, e.vallejo@pucp.pe

This study provides an example of what proof-based teaching is and how students of elementary school level can construct their own knowledge about division and divisibility of natural numbers by following this approach.

Keywords: Proof-based teaching, constructing mathematical knowledge, elementary school level mathematics.

BACKGROUND

In the last decades research on the teaching and learning of mathematical proof has substantially increased (Blanton et al., 2011; Reid & Knipping, 2010; among others). In addition to this, there is worldwide a growing tendency to include mathematical proofs in school programs, including at the elementary level, as exemplified by the Principles and Standards for School Mathematics from the National Council of Teachers of Mathematics (NCTM, 2000).

Despite all of this interest, most of the time it is still quite common to find researches focused on mathematical proof as a subject of study and not as a means to contribute to constructing mathematical knowledge. On that subject, Reid (2011) has proposed that proving could be the vehicle for learning new mathematics through what he calls “proof-based teaching”. He tells us:

> We must ensure that we see proof as fundamental to mathematics as a way to develop understanding of mathematical concepts, and as a way to discover new and significant mathematical knowledge. Proof cannot be limited to the format of proofs, and to the role of verification of knowledge (for which there is probably good empirical or other evidence already). (p. 28)

THE EXAMPLE

The work of Ordoñez (2014) was developed with students around 7–8 years old who did not have prior knowledge about division when this research began. This study provides a clear example of what Reid (2011) calls proof-based teaching. In this work, for which Estela Vallejo was the supervisor, Ordoñez shows how third graders are capable of constructing their own knowledge of division and divisibility of natural numbers from the key notion of equitable and maximum distribution, which is understood by students in a natural way. The knowledge construction becomes evident when students are capable of answering problems that demand justifications of their answers. In the process of knowledge construction, it can be seen that students not only participate actively, but are also encouraged to correct their classmates’ or their own answers, refine ideas, suggest conjectures, etc. All of this shows us that it is possible to develop a classroom environment rich in knowledge construction, in which the students experience similar processes to those experienced by professional mathematicians, including especially the process of proving to discover and establish new knowledge.

In this research study two important elements of proof-based teaching are combined: establishing a framework of established knowledge from which to prove, and establishing an expectation that answers should be justified within this framework.

This transcript from the class shows these two elements:

Tutor: Can we have 3 marbles left after a distribution of certain number of marbles among 3 people?

Student 1: No, because you have to distribute the maximum number of marbles.

Tutor: So, does it mean I have not distributed the maximum number of marbles?

Student 2: We can still distribute these 3 marbles! One more for each person!

The tutor's question could be answered with a 'yes' or 'no', but the student provides a justification as well, in keeping with the expectation that answers should be justified. It refers explicitly to the basic notion of maximum distribution, which is part of the framework of established knowledge. The tutor questions whether this basic notion applies in this case, and the second student provides an additional justification, a backing for the use of the basic notion in this case. This way of constructing division knowledge helped these students to realize why they cannot have 3, or a number greater than 3, as a remainder when they are dividing by 3.

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TWG02
Arithmetic and number systems

Introduction to the papers of TWG02: Arithmetic and number systems

Sebastian Rezat[1], Lisser Rye Ejersbo[2], Darina Jirotkova[3] and Elisabeth Rathgeb-Schnierer[4]

1 University of Paderborn, Institute for Mathematics, Paderborn, Germany, srezat@math.upb.de

2 Aarhus University, Department of Education, Aarhus, Denmark

3 Charles University in Prague, Faculty of Education, Prague, Czech Republic

4 University of Education Weingarten, Weingarten, Germany

THE FOCUS AND SCOPE OF TWG02

Thematic Working Group 2 focused on six questions related to teaching and learning of arithmetic and number systems from grades 1 to 12:

1) What is the interplay between conceptual understanding and procedural skills for number operations, and how should these two aspects be balanced in the design of learning environments?

2) What does it mean to operate flexibly with numbers, and what knowledge and skills are required therefore?

3) What are the roles of models in teaching and learning of arithmetic and number systems, and how do they support flexibility and conceptual understanding?

4) What are effective approaches for teaching and learning of arithmetic and number systems in inclusive education? How can this content area be taught for students with special needs?

5) What aspects of number theory (including specific reasoning) are supposed to be taught in primary and secondary school, and how can this be done?

6) How is it possible to support and analyze long-term learning processes from grade 1 to grade 12? How can different transitions that come with a long-term learning process be taken into account, especially the primary-secondary transition, but also the secondary-tertiary transition.

NUMBER SENSE, CONCEPTUAL UNDERSTANDING AND FLEXIBILITY

Although it is not mentioned explicitly in the focus of TWG02, the notion of number sense included in most of these questions.

In their paper, Sayers and Andrews remind us that there are very different definitions of number sense. Not only different notions of number sense are used in different domains, but in fact, there are no two researchers who use the same definition of number sense. Sayers and Andrews distinguish three distinct perspectives on number sense, which they label *preverbal, applied* and *foundational*. In their paper, they aim to offer a concise conceptualisation of what they call foundational number sense in a way that would support a range of activities, including developments in curriculum, teacher education or assessment, as well as cross-cultural classroom analyses.

An important issue related to the first question is the role of counting strategies in the development of conceptual understanding of number and operations with numbers. Counting strategies are usually regarded as an intermediate stage in the development of conceptual understanding of number. On the one hand, it is assumed that counting strategies are necessary, but on the other hand, they need to be substituted by other strategies that rely on knowledge about number facts and relationships between numbers. At a certain point counting strategies are even regarded as an indication for mathematics disabilities.

Gaidoschik, Fellmann and Guggenbichler challenge the view that counting strategies are actually a necessary intermediate stage in the development. They investigate how teaching of derivation strategies affects strategy use by first graders. They are able to provide empirical support for the claim that counting-on is not a necessary strategy in the development of conceptual understanding of number and operations with numbers.

Häsel-Weide is also interested in possible ways of substituting counting strategies. Her study focuses on the role of peer interaction in cooperative learning situations that encourage children to pay attention and use number relations as well as relations between tasks and problems. She concludes that it is rewarding for both children in heterogeneous pairs to work together. Additionally, she observed that children rather focus on number relations than on relations between tasks and problems.

The first question also addresses the design of learning environments as an important issue. Nührenbörger and Schwarzkopf tackle this issue related to the conceptual understanding of equations. They investigate the development of a rich learning environment that enable students to access the algebraic meaning of equalities and the equal sign. Furthermore, they argue that a rich learning environment itself does not suffice, and highlight the importance of argumentation in order to foster the learning process.

Veldhuis and van den Heuvel-Panhuizen also emphasize the fact that it is not only the design of learning environments and related activities that foster students' learning of mathematics, but also a matter of classroom assessment. In a large-scale experimental study they are able to provide evidence that teachers' use of classroom assessment techniques in mathematics has a positive effect on students' achievement.

The notion of flexibility in terms of operating flexibly and adaptively with numbers is taken up in the second question. Explicitly or implicitly flexibility is seen as an indicator of what Sayers and Andrews call applied number sense. The papers in this TWG focus mainly on two aspects of flexibility:

— conceptualizing flexibility and adaptivity in mental calculation,

— promoting flexibility in mental calculation.

Rathgeb-Schnierer and Green target to identify degrees of flexibility in students' mental arithmetic by revealing the cognitive elements that sustain the solution process (learned procedures or problem characteristics, number patterns, and relationships). They are able to identify three forms of reasoning: flexible (multiple reasons predominantly referring to characteristics), rigid (one reason referring to a solution procedure) and mixed (multiple reasons when referring to characteristics, one reason when referring to a solution procedure). Furthermore, the results show the tendency that students who refer to problem characteristics in their reasoning seem to be more cognitively flexible.

Serrazina and Rodrigues are also concerned with the notion of flexibility in terms of adaptive thinking. However, their approach is different since they do not regard strategies as the unit of analysis, but quantitative reasoning. They argue that quantitative reasoning underlies the development of flexible calculation because it focuses on the description and modeling of situations and the involved comparative relationships. In a qualitative approach they attempt to understand how children establish a network of connections through their reasoning about different representations of the numbers, and about relationships between numbers and quantities.

Given the importance of the fundamental arithmetic properties for flexible mental calculations on the one hand, and for algebraic thinking on the other hand, Larson aims to better understand how students make sense of arithmetical properties in particular of the distributive property in multiplicative calculations. She analyses students' arguments when they have to evaluate the validity of (wrong) strategies to carry out a multiplication of two two-digit numbers. Finally, she makes inferences from students' arguments to students' understanding of the distributive property.

Besides developing a better theoretical understanding of the notion of flexibility, the issue of fostering the development of flexibility is another important issue related to the second question. Rechtsteiner-Merz and Rathgeb-Schnierer pay attention to the development of flexible mental calculation in less advanced students. In this regard, they investigate the contribution of a specific approach (called "Zahlenblickschulung") to foster the recognition of problem characteristics, number patterns and numerical relationships.

Within a qualitative design they identify different types based on three dimensions: (1) the amount of correct solutions, (2) solution procedures, and (3) related reasoning. They conclude that knowledge of basic facts and strategies seem to be insufficient for the development of a deep understanding of calculation that goes beyond counting. Therefore, the focus on numerical relationships and structures is essential in order to develop flexibility in mental calculation. Their approach of "Zahlenblickschulung" seems to be promising in this regard.

Lübke tackles the issue of estimation as another aspect related to number sense. She investigates fourth graders' conceptual understanding of computational estimation using indirect estimation questions. She argues that the interrelation between an estimate and the exact calculation is not only a very important aspect of understanding computational estimation, but also proved to be a useful tool to analyse students' concept of computational estimation. Her findings indicate that even students who are able to carry out estimations do not necessarily understand the concept of estimation.

Carvalho and da Ponte as well as Almeida and Bruno address aspects of number sense in the domain of rational numbers. Carvalho and da Ponte analyze mental computation strategies and errors of 6th grade students particularly focusing on how they use relational thinking. Based on the theory of mental models they are able to better understand the relation between mental representations and mental calculation strategies and errors respectively.

Almeida and Bruno analyze the effects of an intervention on three abilities of grade 8 students related to number sense in the domain of fractions: the use of benchmarks, the use of graphical representations of numbers and operations, and the recognition of the reasonableness of a result.

THE ROLE OF MODELS

The third question relates to the role of models in teaching and learning arithmetic. The notion of model consciously has been left open. The papers in the TWG mainly address four different kinds of models: models in the sense of manipulatives, models in the sense of visual representations, models in the sense of mental models, and finally, models in the sense of role models.

Hejný, Jirotková and Slezáková offer a theoretical approach to the role of models in the learning process. A process that they call "desemantization" lies at the heart of their Theory of generic models. It describes stages of learning processes in mathematics in terms of two different kinds of mental models, which differ in their semantic embedding: isolated models denote the set of pupils' experiences related to a certain mathematical concept, relationship, or situation, and generic models refer to generalizations of this previous experience. Through an abstraction process the generic model is transferred into abstract knowledge, which does not include the semantic embedding. The ability of pupils to work with models is thus an indicator of their level of understanding of mathematical phenomena.

Finesilver analyzes strategies and errors of S.E.N. students when asked to determine the number of cubes in cuboid blocks made up of multilink cubes. It was intended to evoke the use of different multiplicative strategies by highlighting different structures using different colors. The tasks proved to be a useful tool to understand students' multiplicative thinking in terms of structuring, enumeration and errors.

Hattermann and vom Hofe analyze the effects of a game in the domain of negative numbers on students' performance in tasks with addition and subtraction of negative numbers and students' argumentation schemes. The intention of the game is to foster metaphorical reasoning, which is regarded as crucial for the understanding of negative numbers and the construction of mental models by the authors. First results that are based on a small number of students indicate positive effects on both, performance and metaphorical reasoning.

Pöhler, Prediger and Weinert investigate the influence of different representations (numerical, visual, verbal) in percent problems on students' performance in relation to their language proficiency. Their findings indicate that the difficulties with verbal test items cannot be explained by students' restricted reading proficiency, but rather seem to be a consequence of their lack in conceptual understanding of percentages.

Papadopoulos focuses on the teacher as a role model. He analyses the influence of a teacher's persistent misconceptions on students' performance. The results highlight once more the importance of content knowledge in pre- and in-service teacher education.

SUBJECT-MATTER ANALYSIS

Three papers take the subject-matter and related content knowledge as their starting point. Real and Figueras present a framework of notions, concepts and processes for fractions and rational numbers, which was developed based on Freudenthal's Didactical Phenomenology. The framework is organized according to five classes of phenomena / processes related to fractions: describing, comparing, dividing, distributing, and measuring. It has proved to be a useful analytical tool.

Nicolaou and Pitta-Pantazi evaluate the impact of an intervention, which was based on a framework comprising seven abilities that are supposed to be important for conceptual understanding of fractions in elementary school. They conclude that the intervention had a positive impact on students' understanding of fractions.

Gómez and García carry out a rational analysis of problems with unequal ratios. In the first step, their analysis aims to work out critical components of the problems in order to evaluate them empirically in the second step. Their findings indicate that students do not apply different strategies flexibly when solving the problems, but rather stick to a standard strategy.

OPEN QUESTIONS AND FUTURE DIRECTIONS

Many of the papers in TWG relate to the notion of number sense. Consequently, number sense can be regarded as one of the focal concepts of this TWG. The papers in this TWG have contributed significantly to improve the understanding and applicability of this concept. However, many questions are still open.

First of all, very different conceptualizations of number sense are used in different domains. Whereas number sense in cognitive psychology refers to the innate arithmetic of the human brain, in mathematics didactics it relates to the ability to perceive number relations and to make use of them when solving problems. It seems promising to relate these different perspectives and strive for a comprehensive definition of number sense. Main questions in this context are:

- How is number sense in the didactical meaning related to the innate number sense of the brain?
- How can insights into the innate number sense contribute to the development of number sense in the didactical meaning of the term?

Secondly, the relation between number sense on the one hand and flexibility and adaptivity on the other hand needs to be further clarified. Is number sense conceptualized via flexibility and adaptivity or is it a prerequisite for flexibility and adaptivity?

Finally, it is tempting to use the term number sense related to other number domains than the natural numbers – and some papers in this TWG actually do so. However, it is not clear yet, what number sense means related to these number domains and how it relates to the number sense in the domain of natural numbers.

Important questions related to number sense in every number domain are:

- What influences the development of number sense?
- What is the role of metaphors, manipulatives, models, and mental models in the development of (didactical) number sense?
- What is the role of reasoning and argumentation in the development of (didactical) number sense?
- What is the role of counting in the development of (didactical) number sense?

ACKNOWLEDGEMENT

We thank Véronique Battie (France) for all her work in the preparation of this TWG before the conference. She started as an official group leader, but unfortunately was not able to participate at the conference.

TWG02

Research papers

A study on the changes in the use of number sense in secondary students

Rut Almeida and Alicia Bruno

Universidad de La Laguna, San Cristóbal de La Laguna. Spain, rutalca@gmail.com

Number sense involves the understanding and the ability to use numbers and operations in a reasonable and flexible way. This paper refers to part of a qualitative study that is being conducted to analyse the development of number sense through an intervention designed for this purpose. Our work aims to analyse the strategies used by 8th grade students when solving numerical tasks before and after an intervention. An initial and a final interview were conducted on 11 students to analyse the effect of the intervention on their tendency to use different types of strategies. The results show improvements in the understanding of fractions and on the use of number sense, but also underscore the difficulties still encountered by some students.

Keywords: Number sense, rational numbers, secondary education, strategies.

RATIONALE AND PURPOSE

In recent decades mathematics education has focused on the development of skills that make students competent to use this knowledge in a flexible and justified way. These kinds of mathematical skills include an understanding of numbers and operations and the ability to use them flexibly, exhibiting different strategies for handling numbers and operations, and the ability to evaluate the reasonableness of results, which is commonly called *number sense* (McIntosh, Reys, & Reys, 1992). This term appears in the curricula of several countries and is an important topic to be studied. Research intended to assess number sense in students and teachers in primary and secondary education showed a lack of these abilities (Reys & Yang, 1998; Veloo, 2010; Yang, Reys, & Reys, 2009). Although research indicates this is a problem for many students, there is a minority who made use of number sense. At this point some questions arise: Which of these strategies do they know? Can students learn number sense strategies through a suitable intervention? What changes occur after a classroom intervention regarding the use of number sense strategies?

BACKGROUND

The term number sense refers to a broad set of skills and knowledges, and as such it does not have a closed definition. However, some researchers agree that number sense is recognised in the action itself and that it includes the knowledge and flexible use of numbers, operations and their properties, the use of different strategies to solve numerical problems and being able to recognise the reasonableness of the problem statement and data, the way it is solved and the result obtained (McIntosh et al., 1992; Sowder, 1992). The term number sense is included in the curriculum of different countries, such as Australia, the United States, Canada and Spain, but it seems that this ability is not being sufficiently developed in every country, despite being featured in the teaching objectives.

In an effort to devise a more operational term, there have been attempts to describe it by components (McIntos et al., 1992; NCTM, 1989; Reys & Yang, 1998; Yang, 2003). They all agree on the aspects involved in number sense, though some are more detailed. For this study the framework considered to characterise number sense is composed by the following components: (1) understand the meaning of numbers; (2) recognise the relative and absolute size of numbers and magnitudes using estimates or numerical properties to make comparisons; (3) use benchmarks to estimate a number or magnitude when comparing or doing calculations; (4) use graphical, manipulative or pictorial representations of numbers and operations; (5) understand operations and their properties; (6) understand the relationship between the problem's context and the operation required; (7) realise that there are multiple strategies; (8) recognise the reasonableness of the problem (data, strategies and results).

Research on number sense has focused on evaluating primary education students and teachers (Veloo, 2010; Yang, 2003; Yang et al., 2009), with less research being conducted on secondary education (Reys & Yang, 1998; Veloo, 2010). All of these studies have found that there is a lack of number sense in these groups when solving numerical tasks. A tendency to use algorithms and written computation, and a lack of relationship between good written computations and good number sense have been shown in several studies on high grades of primary education and lower secondary education (Reys & Yang, 1998; Veloo, 2010). In contrast, it was revealed that students with higher academic marks in mathematics performed better in number sense tests (Veloo, 2010) as did those with good mental computation and estimation skills (Sowder, 1992). In an effort to address this situation, interventions to develop number sense have been designed. Markovits and Sowder (1994) established that number sense can be developed, but over a long period of time, and that students, in addition to gaining new knowledge, reorganised prior knowledge. Those studies that included an intervention to develop number sense agree on the idea that activities must contain a process of numerical exploration, the search of number and operation properties and a methodology based on the discussion (Veloo, 2010; Yang, 2003). All of them have shown that this type of learning is more significant than traditional learning and that it is possible to develop number sense and the activities they used for it. These studies only show pre-test and post-test results that enable us to see that the activities worked, but qualitative data concerning the changes in their use of strategies is still needed.

OBJECTIVES AND METHODOLOGY

This work is part of a broader study whose objective is to identify the strategies used by students before and after an intervention to analyse the possible changes in their use of number sense. Since this term encompasses multiple concepts that are developed over the course of mandatory schooling, in this case we have opted to focus our study on three of the lesser studied components in research into this topic involving secondary students. The intervention focused on the development of three components from the framework involving whole, decimal and rational numbers: (3) use of benchmarks; (4) use of graphical representations of numbers and operations; (8) recognise when the result is reasonable. The design was intended to foster these three components, although the use of other components was expected since they are not independent.

The sample consisted of two groups (25 and 22) of 8th grade students (12–13 years old) from a public secondary school in Spain. These students were involved in an intervention designed to develop and encourage the use of number sense. The students' existing knowledge of number sense was in keeping with Spain's curriculum in traditional numerical learning, which at this level involves consolidating students' knowledge of whole, decimal and rational numbers.

Initial test

The students were given an initial written test that included 12 items designed to evaluate the strategies used when facing numerical tasks focused on the use of the three components mentioned above. This initial test revealed the students' shortcomings in the use of benchmarks and in making graphs to aid in estimating numbers and in operations, their preferences for using rules or algorithms, and conceptual errors, especially with fractions (arranging them considering differences between the numerator and denominator, making incorrect graphical representations of fractions and operations) (Almeida & Bruno, 2014). The test yielded background information that was used to design the classroom intervention tasks.

Classroom intervention

The classroom intervention took place over the course of eleven 50-minute sessions and was based on a collaborative environment and on discussions between students of each task. They solved tasks working individually, in pairs, in small groups or as a whole, though always showing the strategies they used so as to discuss them with the rest of their classmates in a final discussion. Since, generally, students tended to use memorized rules, at the end of each task during discussion they were asked to find different strategies in order to explore numbers and find number sense strategies. This methodology allows students to share their knowledge and to discover number sense strategies without the intervention of the teacher/researcher. The students were given written tasks designed by the researchers that were intended to show that an exact answer or calculation is not always necessary to resolve certain questions, to highlight the importance of questioning one's results by evaluating if an answer is reasonable or not, and to develop strategies that involve the use or graphical representations and show the usefulness of taking benchmarks. The tasks themselves involved estimating quantities, sorting numbers and doing operations with whole numbers, decimals and fractions while encouraging debates among the students to find the most suitable strategy.

Interviews before and after the classroom intervention (case studies)

The initial test was also used to select 11 students who were interviewed before and after the classroom intervention with the aim of making a case study of the possible improvement. Both interviews, while designed by the researchers, also relied on previous research (Yang & Huang, 2004; Reys & Barger, 1991). Their aim was to assess the use of the three components being studied. The interviews consisted of two phases. In the first one the students solved all of the tasks without the researcher's intervention. Once they had finished, they were asked to explain each answer by trying to find new strategies without using a written computation method. If they had not used a graphical representation, they were encouraged to find a way to use one. The initial interview had six items and the final one had five. Regarding the analysis of the data from interview, the answers of the eleven students were analysed to identify the correctness (1 for correct answers and 0 otherwise) and the type of strategy used. A category system adapted from Yang and colleagues (2009) was used to classify their reasoning: number sense based (NS), when they used exclusively one or several components of the number sense framework; partially number sense based (PNS), if they combined the use of number sense components by using memorised rules and/or algorithms; not number sense (NNS), when they only made use of algorithms or memorised rules; other (Oth), when students do not provide sufficient grounds to identify the reasoning that led them to the final answer(s); blank (B), when the question is unanswered. The NS and PNS categories also considered the type of component used.

Due to space limitations, in this paper we present the results of one item that was similar in the initial test and in both interviews. This allows us to compare the answers of the eleven students before and after the intervention. Focusing on just one item, we wish to exemplify the type of analysis that we are undertaking. These items were the following: *Initial interview*, "Sort from smallest to largest the following numbers: 2/5, 7/8 and 4/3"; *Final interview*, "Sort from smallest to largest the following numbers: 7/8, 0.3, 4/3, 0.55 and 2/9". We opted for this problem type involving sorting fractions because of the poor results obtained on the initial test, along with the variety of answers given based on rules. It was also presented in the initial and final interviews, and despite having a standard problem statement, it can be solved using the components of number sense that are being studied, which allows us to observe how the students put them into practice. Specifically, the objective of these items concerning the number sense was to encourage students to use benchmarks (component 3) and/or graphical representations (component 4) to compare fractions without the need to use written computation. All of the tasks in the initial and final interviews were designed such that they could be solved using the components in this study. The initial and final interviews were separated by a period of three months. Over this time the students were involved in a classroom intervention. Of the eleven sessions, two were devoted to working on tasks related to the items analysed, i.e. sorting rational numbers. An example of one intervention task is shown below.

The students were given a problem to solve individually for later discussion with the whole group. The statement was: "Sort from smallest to largest the following numbers: 9/20, 8/5 and 3/10". After solving the problem individually, they were asked to present their strategies on the blackboard. In this case the instructor's intervention was not needed and the students found all of the strategies expected: (1) compute the least common multiple to apply an algorithm and express all the fractions with a common denominator; (2) apply the division algorithm to express fractions as decimal numbers; (3) graphically represent the fractions; (4) use the benchmarks ½ and 1 to compare the fractions. The idea was to show every possible way they knew to solve the problem so that the students could see the different possible strategies and be able to choose the one they preferred. In those cases with insufficient variety, they were encouraged to explore numbers in different ways such as understanding the meaning of numbers, the magnitude, use of benchmarks, graphical represention, etc.

RESULTS AND DISCUSSION

The eleven students interviewed answered the two items mentioned above, using different methods when asked to do so. Table 1 shows a summary of the results by the category of the strategy the students used for each task. The number in parentheses shows the component from the number sense framework that they primarily used. The first problem (1st) was the one the students did on their own without the interviewer's intervention, and the rest resulted from asking them to use new strategies to solve the task.

In the answers we find four out of five different categories: *NS*, *NNS*, *Oth* and *B*. A description of the strategies included in each category is presented below. The PSN category did not appear in this item, as the students made exclusive use of number sense components or rules independently, though there were strategies that combined the use of both in other items in the interviews.

Number sense (NS)

Component 1: Using the properties of numbers to express fractions as equivalent fractions or decimals as fractions indicates an understanding of the meaning of numbers. An example of this is provided by student S4, who states for the task in the final interview "(...) we can express decimals as fractions to compare them, for example, 0.3 is equal 3/10 and 0.55 is around ½ (...)".

Component 3: Using benchmarks to facilitate comparisons. The students used different benchmarks, some using more than others to compare against. One example of the use of benchmarks is provided by student S1 who, after the intervention, used four different benchmarks (1, ½, 1/3 and ¼): "7/8 is almost 1, 0.3 is a third, 4/3 is greater than 1, 0.55 is a half and 2/9 is close to ¼".

Component 4: Using graphical representations to ascertain a number's magnitude for comparison purposes, especially for fractions. Figure 1 shows an example from the final interview.

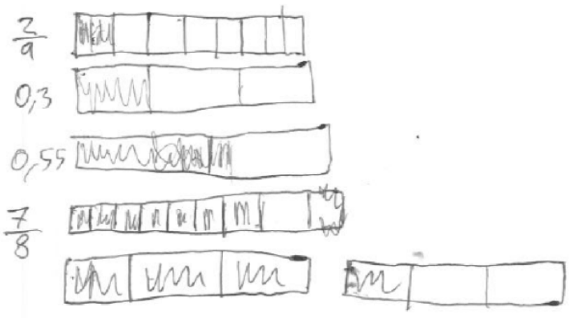

Figure 1: Answer from S2 in the final interview

Not number sense (NNS)

We found two kinds of strategies that made use of rules and/or algorithms. The most common in the initial interview was to compute the least common multiple to apply an algorithm and express the fractions with a common denominator. The other strategy

Student	Initial interview's strategies			Final interview's strategies			
	1st	2nd	3th	1st	2nd	3th	4th
S1	0NS(1)	1NS(3)	0NS(4)	1NS(3)			
S2	1NS(3)	1NS(4)		1NS(3)	1NS(4)		
S3	1NNS	1NS(3)	1NS(4)	1NS(3)	1NS(4)		
S4	1NNS	1NS(3)	1NS(4)	1NNS	1NS(3)	1NS(1)	1NS(4)
S5	0NNS	1NS (3)	0NS(4)	0NS(3)	0NS (4)		
S6	0B	1NS(3)	1NS(4)	1NS(3)	0NS(4)		
S7	1NNS	1NS(3)	0NS(4)	1NNS	1NS(3)	0NS(4)	
S8	0NNS	0NS(3)	1NS(4)	1NNS	1NS(4)		
S9	0NNS	0NS (4)		0NS(3)	0NS(4)		
S10	0Oth	0NS (4)		0NS (4)			
S11	0B	1NS(3)	0NS(4)	1NS(3)	1NS(4)		
0: Correct final answer; 1: Incorrect final answer. NS: Number sense; NNS: Not number sense; Oth: Other; B: Blank. (1) Understand the meaning of numbers; (3) Use of benchmarks; (4) Use of graphical representations;							

Table 1: Strategies classification for the item selected in both interview by student

found in this category was a memorised rule in which they established that a fraction was greater than other if the difference between the numerator and the denominator was smaller, without making sense of the fraction concept and the incorrectness of the rule for cases with different denominators. In the final interview, the most common strategy was to apply the division algorithm to express fractions as decimals.

Other and blank (Oth and B)
Common to the answers in these two categories is the fact that they did not have a justification for the reasoning or, in the case of "Blank", even an answer.

Analysis of the answers
An individual analysis of the students' answers reveals the improvements and/or differences in the use of strategies (Table 1) when solving this item.

NS answers before and after the intervention
Students S1, S2 and S6 used number sense strategies as their initial answers in both interviews, though we observed more confidence in the use of number sense strategies after the intervention, since in the initial interview students argued that they were used to use exact written computation so they were not allowed to use what they named as "logical reasoning" referring to estimates or the use of benchmarks, but in the final interview they made use of number sense strategies without this statement, a hint of changes in the didactic contract during the intervention. S1 used benchmarks and graphical representations correctly in both interviews, whereas students S1 and S6 had problems with the graphical representations of fractions and decimals before and after the instruction. We regard Student 1 as being representative of this group.

In the initial interview, *Student 1* used three different strategies to answer the task, but only one of them was completely correct. In his first answer his reasoning was correct, making use of the fraction's properties to find equivalent fractions, but he made a mistake when estimating the magnitude of 2/5. The student was unable to graphically represent the fractions although he was able to correctly compare them with 1 and ½. After the intervention, the student decided to use this last strategy but in this case using more benchmarks (1, 1/2, 1/3 and 1/4) before stating that he did not know how to represent fractions. This student, although he knew some number sense strategies, did not show any apparent improvement involving the use of graphical representation. He demonstrated a mastery of the use of appropriate benchmarks, even more so after the intervention.

NNS answers before and after the intervention
Students S4, S7 and S8 were characterised by their preference to use rules for the first answer despite knowing other procedures involving either graphical representation or benchmarks. We regard Student 7 as being representative of this group.

Student 7 had similar results in both interviews; in his first answers he gave rule-based reasoning, although in the initial interview this involved computing the least common multiple and in the final one it was related to the division algorithm. For the other strategies used, he followed the same reasoning in each interview, including the use of benchmarks and graphical representations, but although it was correct, he was unable to interpret it. Therefore, the student made no progress in this task after the intervention.

Improvement in NS after the intervention
Students S3, S5 and S11 were characterised by exhibiting a change in strategy between the two interviews toward an improved use of number sense. The case of student 11 is described in greater detail below.

Student 11 used the same strategies in both interviews but with a clear improvement in the second case. Both times she used benchmarks and graphical representations; regarding benchmarks, she exhibited a richer justification in the final interview, choosing two benchmarks (1 and ½), while in the initial interview she said she was not able to do so, and giving more accurate explanations of their comparison; as concerns graphical representations, she demonstrated an improvement in this aspect as well, being able to represent all the numbers. This last improvement is evident since in the initial interview she was not able to represent fractions correctly, dividing the units into unequal portions.

No improvement in NS after the intervention
Students S9 and S10 used incorrect strategies when they attempted to employ improperly memorised rules before the instruction. However, they exhibited a change in intention after the final interview by trying to make sense of fractions and their magnitude instead of using memorized and non-argued

rules. They applied number sense strategies, although conceptual errors related to fractions that remained after the intervention impeded them from obtaining the correct answer. As an example of this group, let us consider the answers given by student 9.

In the initial interview, *Student 9*'s answers demonstrated a lack of understanding of fractions; she applied a memorised rule in which she decided the magnitude of the fractions based on the difference between the numerator and denominator, which led her to an incorrect answer. She also tried to use a graphical representation in which she divided the units without realising that the pieces had to be equal for each fraction. After the intervention, she demonstrated an improvement by attempting to make sense of the strategy used and not applying a memorised rule. She tried to apply the use of benchmarks and graphical representations but the improvement was insufficient, given her misconception of the meanings of the numerator and the denominator: namely, she reversed their meanings (Figure 2).

tions of decimal numbers kept them from obtaining the correct answers (S3, S5 and S11).

As other interventions that aim to encourage the use of number sense have shown, developing these strategies is a long-term process (Markovits & Sowder, 1992). Evidence of this statement is the insufficient improvement of S9 and S10 related with their mathematical misconceptions concerning rational numbers. The use of number sense requires the use of conceptual ideas that, in this case, involve rational numbers, their meaning and properties. Therefore, in cases where misconceptions arise, more time might be needed to develop number sense strategies. These mistakes are particularly evident as they relate to the concept of fraction and to the graphical representations of fractions as a means for obtaining an answer to a problem. Even though not every student showed optimal improvement, some changes were noted as a consequence of the intervention. The results of the interviews led us to delve into the different options students used to obtain an answer: in some cases students used a rule-based strategy, but they were able to use number sense strategies as a second or third option. This flexibility in the use of strategies is one limitation of a written test, since students only show one strategy that conceals the number sense they may possess, perhaps because they think that is what teacher expected or because they feel more confident using written computations.

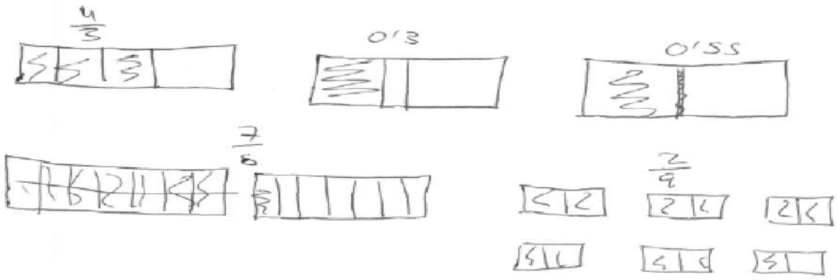

Figure 2: Answer from S9 in the final interview

CONCLUSION

Students who used number sense strategies in the initial interview as their first answer continued to use these strategies, but were more confident using them, knowing that they were allowed to use any strategy they knew (S1, S2 and S6). In contrast, after the intervention some students preferred to use strategies classified as *not number sense* as their first option (S4, S7 and S8). These students expressed a knowledge of other number sense strategies and yet opted to continue with rule-based methods, as they were more at ease with them and because they felt their teachers expected them to use these procedures. There were also students who changed their strategies towards the use of number sense, although in some cases their misconceptions concerning graphical representa-

ACKNOWLEDGEMENT

This research was partially funded by the project EDU2011-29324, Formal and Cognitive Skills Models for Numerical and Algebraic Thinking in Primary and Secondary Education Students and in Pre-service Teachers. Funding was also granted thanks to the ULL by the Canarian Agency for Research, Innovation and Information Society, financed 85% by the European Social Fund.

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Student strategies and errors in mental computation with rational numbers in open number sentences

Renata Carvalho and João Pedro da Ponte

Universidade de Lisboa, Instituto de Educação, Lisboa, Portugal, renatacarvalho@campus.ul.pt; jpponte@ie.ulisboa.pt

Mental computation is an important skill for mathematics learning. This paper analyzes mental computation strategies and errors in grade 6 students with positive rational numbers in a teaching experiment based on tasks. The tasks involve the four basic operations and are done in conjunction with a discussion of strategies with the entire class. We focussed on student relational thinking and our findings suggest that open number sentences highlight the meaning of the equal sign and the importance of using properties of operations in mental computation. This is reflected in student's strategies when they use numerical relationships to compute mentally. Mental representations such as propositional representations are strongly linked to strategies based on numerical relationships.

Keywords: Mental computation, rational numbers, student errors, student strategies, relational thinking.

INTRODUCTION

Carrying out mental computation tasks in the classroom helps in the development of important mathematics skills. As Taton (1969) indicates, mental computation develops notions of order, logic, reflection, concentration and memory. It also prepares students for daily life, providing them with tools to perform simple computations without the help of external resources. Former studies (e.g., Caney & Watson, 2003; McIntosh, 2006) highlight the importance of understanding student strategies in mental computation, particularly the numerical relationships they use in solving problems involving rational numbers. This understanding may support teacher planning and teaching activity. This paper is drawn from a teaching experiment based on mental computation tasks involving open number sentences with four operations and the discussion of strategies analyzing mental computation strategies and errors in grade 6 student, focusing on how they use relational thinking.

MENTAL COMPUTATION WITH RATIONAL NUMBERS

Reys, Reys, Nohda and Emori (1995) refer to mental computation as the process of computing exact arithmetic results without external support. In line with this perspective this paper takes mental computation to be the process of computing arithmetic results in a quick and efficient way using number facts (e.g., multiplication tables), memorized rules (e.g., how to multiply by powers of ten) and numerical relationships (e.g., converting between different representations) supported by mental representations (e.g., mental images of fractions or relational thinking based on propositional representations).

The main elements of our framework are the use of (i) number facts; (ii) memorized rules, (iii) numerical relationships and (iv) mental representations (Figure 1).

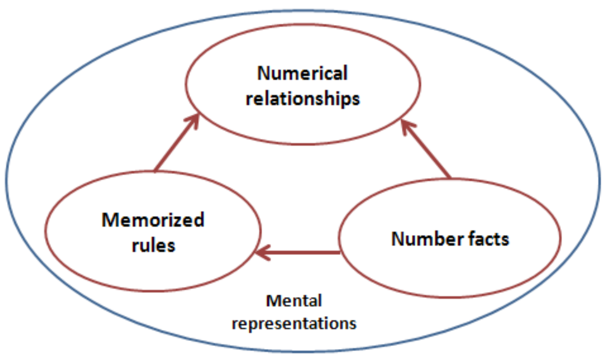

Figure 1: Conceptual framework

To develop mental computation, Heirdsfield (2011) indicates that students need to know number facts but also to understand the size and the value of numbers, the effect of an operation on a number, and to be able to make estimates to check if the solution is reasonable. As computing mentally with rational numbers involves more complex reasoning than computing with whole numbers (Barnett-Clarke, Fisher, Marks, & Ross, 2010), we assume that the use of memorized rules may also support student computation and relational thinking (Empson, Levi, & Carpenter, 2010).

The use of mathematical relationships is evidence of relational thinking. This is related to the capacity to use fundamental properties of operations and the notion of equality to analyse and solve problems. As Carpenter, Franke and Levi (2003) highlight, "many fundamental mathematical ideas involve relations between different representations of numbers and operations on them" (p. 38). From their perspective, the use of open number sentences (e.g., 1/2 − _ = 1/4) may be a viable way to draw a student's attention to relations between numbers and operations and, if students are experienced in looking at these kind of relations, they will be able to use them to compute quickly. For example, if students understand that 3/4 may be split into 3 × 1/4 they quickly compute 3/4 ÷ _ = 1/4. Also, if they understand 10% as a benchmark and know how to multiply/divide by powers of ten, they easily compute 5% by halving or 20% by doubling.

Another important idea to consider in mental computation is the fact that we construct mental representations from the world that surrounds us, which we use in sense making and making inferences. Understanding student mental representations may help us to understand their relational thinking because they use mental representations to relate numbers and operations to their real world knowledge, including knowledge about mathematics. In the theory of mental models (Johnson-Laird, 1990), these mental representations may be mental models, mental images or propositional representations: (i) mental models are generalized perceptions of the world (e.g., using a generalized context of sales to compute 20% of 25); (ii) images involve a more specific perception of a real world where some characteristics are considered (e.g., relating the symbolic representations 1/2 to one part of a pizza divided in two parts); (iii) propositional representations represent true/false propositions that play an important role in the inference process (e.g., to compute 40% of ? = 48, using a sequence of propositions leading to the solution: 'if 48 are 40%, 48 ÷ 4 are 10% which is 12. So, 10% × 10 is 100% and 12 × 10 is 120 which is the missing value').

When students compute mentally, they often make errors that arise from a lack of understanding of rational numbers. This is incorrect usage of the elements presented in Figure 1. Carpenter and colleagues (2003) suggest that such errors may be based on generalizing properties in other situations. For example, knowing that multiplication and addition are commutative may lead students to apply this knowledge to subtraction and division. In adding fractions students often regard them not as a single number but as two separate numbers (Lamon, 2006), thus adding numerators and denominators. As McIntosh (2006) indicates students make conceptual and procedural errors in mental computation. A conceptual error arises when the student fails to understand the nature of the numbers or of the operation involved and a procedural error takes place when the student knows what strategy to use but makes a computation error when putting it into practice.

RESEARCH METHODOLOGY

The study is qualitative and interpretative with a design research approach (Cobb, Confrey, diSessa, Lehere, & Schauble, 2003). This is a developmental study that aims to solve problems identified from practice—the difficulties in learning rational numbers and near absence of mental computation with this number set. It is based on a teaching experiment with mental computation tasks that provide opportunities for discussing student strategies and errors. It was carried out in three phases: preparation; experimentation and analysis. In the preparation phase (2010) a preliminary study in grade 5 was undertaken (conducted by the first author in their classes) in order to collect information on strategies that students use and errors that they make when computing mentally with rational numbers. This showed that practical aspects of students' mental computation practice were potentially important for planning the teaching experiment. Such planning also took into account research on rational numbers and mental computation with rational numbers. In the second phase (2012–13), two experimental cycles were implemented involving two teachers and two grade 6 classes (39 students) from two different schools with the first author as a participant observer. During this phase some refinements were made in the teaching experiment (e.g., changing the sequence of tasks). Data was collected through video and audio recordings of classwork with mental computation and researcher notes. Finally, in the analysis phase, audio and video records of student strategies and errors were transcribed. In this paper we focus on the collective discussions in order to understand student strategies and errors, particularly student relational thinking in questions involving open number sentences. The analysis of student mental computation strategies is carried out using categories taken from previous studies (e.g., Caney & Watson, 2003) and from the concept of mental representations (mental models, images and propositional representations), number facts, memorized rules, and numerical relationships. From these concepts and taking into account the data, three main categories of strategies were defined (numerical relationships, number facts and memorized rules) and in each of these, several subcategories (Table 1). The categorization was made according to the strongest concept involved in student strategy. For example, if there is strong usage of numerical relationships, for example, the part-whole relationship a strategy from the category "numerical relationships" and subcategory "comparison part-whole" was named Mental representations were not considered as a fourth category since they are common to all categories. Thus mental representations were analyzed as a support to student strategies. To analyze student errors McIntosh's (2006) notion of conceptual and procedural error was used. In each of these categories some subcategories are identified as shown in Table 2.

THE TEACHING EXPERIMENT

The teaching experiment relies on the conjecture that a systematic work with mental computation tasks with rational numbers represented as fractions, decimals and percentages, and whole class discussions may contribute to the development of students' mental computation strategies and understanding of their errors. Before the teaching experiment the students had already worked with rational numbers in different representations and operations, with an emphasis on algorithms. The teaching experiment involves ten mental computation tasks (with several number sentences or word problems each) with rational numbers to carry out weekly, all prepared by the first author and discussed in detail with the participating teachers. These tasks (Table 3) were presented at the beginning of a mathematics class, using a timed PowerPoint. The students had 15 seconds to solve each number sentence and 20 seconds to solve each word problem individually, in both cases recording the results on paper.

	Categories		
	Numerical relationships	Number facts	Memorized rules
Subcategories	Equivalence Part-part comparison Part-whole comparison Inverse operation	Two halves make a unit A half of a half is a quarter Two quarters is a half 10% of...	Rule to multiply/divide by powers of ten Rule to add/subtract fractions with different denominators

Table 1: Categories of strategies in mental computation with rational numbers

	Categories	
	Conceptual error	Procedural error
Subcategories	Strong use of additive reasoning in multiplicative problems Ignore the place value in decimal representation Add/subtract numerators/denominator when operating with fractions	Calculation error Solve a part of an expression when a valid strategy is possible to identify

Table 2: Categories of errors in mental computation with rational numbers

Each task has two parts, with 5 numbers sentences or 4 word problems in each part. On finishing each part, there was a collective discussion of student strategies and errors. These mental computation lessons lasted between 30 and 90 minutes. Some examples of the content of the tasks used in the teaching experiment are presented in Table 3. The timed tasks were seen as a way to challenge students to compute mentally. The discussion moments were regarded as the most important part of the lesson, and allowed students to show how they think, the strategies that they use, and the errors that they make. They were important for students to think, reflect, analyze, make connections, share, and extend mental computation strategies, as well as to identify skills that they could develop in numbers and operations.

In the teaching experiment the students began to compute mentally with rational numbers represented as fractions (addition/subtraction in task 1 and multiplication/ division in task 2), then with decimal and fraction representations with the four basic operations (task 3), and then solely with decimal representation (addition/ subtraction in task 4 and multiplication/ division in task 5). Then they solved word problems in measurement and comparison contexts involving fractions and decimals (task 6). The percent representation was used in task 7, as the teacher began working with statistics. Then, students used the three representations (decimal, fractions and percent) in tasks 8, 9 and 10. In task 10, they solved word problems.

The design of tasks is based on three principles, taking into account previous research on mental computation and rational numbers.

Principle 1. Use contexts to help students to give meaning to numbers. Structured knowledge is associated with the context in which it was learned and most of the time it is difficult for a student to bridge this knowledge to new situations. Therefore, number sentences and word problems were used to provide students with a variety of contexts and help them to establish connections between them.

Principle 2. Use multiple representations of rational numbers. Fractions, decimals, and percent representations were used in the same task and in several tasks across the teaching experiment (e.g., , question from task 3). We used even numbers and multiples of 5 and 10, benchmarks such as 25% or 1/2 to facilitate equivalence between decimals, fractions and percent, and to stress numerical relationships (e.g., multiplying by 1/5 is the same as dividing by 5) and part-whole relationships (e.g., if 10% corresponds to 5, then 100% is 10 times bigger). The rational number representation used was related to the topics that students were working on in class with their teacher because the tasks had been integrated into the teachers overall planning. This option provided students with a further opportunity to learn rational numbers on different mathematical topics throughout the mathematics curriculum.

Principle 3. Use tasks with different cognitive demands. For example, taking into account mental computation levels (Callingham & Watson, 2004) tasks in which the students have to use the concept of half to compute (e.g. 50% of 20 or 1/2 + 1/2=_) or need to use more complex numerical relationships (e.g., 20% of _ = 8) to do the computation were designed.

In planning the lessons, we tried to anticipate possible student strategies and errors, to prepare better collective discussions. All classroom activities were led by the teachers with the first author making occasional interventions to clarify aspects related to student strategy presentation.

STUDENT MENTAL COMPUTATION STRATEGIES AND ERRORS

Student relational thinking through their strategies and errors, supported by mental representations in

Think fast! What is the exact value?	
Task 1 :	3/4 - 1/2 = _; 1/2 - _ = 1/4
Task 6:	The area of the base of a cylinder is 4.2 m^2 and the volume 12.6 m^3 Calculate the height of the cylinder.
Task 7	90% of 30 = _
Task 8:	_% of 20 = 18

Table 3: Examples of tasks

open number sentences was analysed. Examples of related questions were chosen (regarding numbers or the numerical relations between them).

In task 1, students were challenged to mentally compute "3/4 – 1/2 = _" and "1/2 – _ = 1/4". These sentences involve two possibilities to reach 1/4. To solve "3/4 – 1/2 = _" Ivo explained: "I made the rule. 2 multiply by 2 gives 4 and then following this I multiplied by 2 also. Then 3 minus 2 is 1. 1/4." Ivo used a strategy based on memorized rules where he applied the procedure to subtract two fractions with different denominators, which may have been supported by a mental image of a written algorithm. Concerning "1/2 – _ = 1/4", we present a dialogue where it is possible to understand Marta's and Rogério's relational thinking and Eva's error:

Eva: I put two quarters.
Teacher: First, Eva will explain to us why she put two quarters.
Eva: Because if we multiply by 2 ... We need to have the same denominator and if we multiply the 2 by 2 it gives us 4. It gives us 4 and then 2 multiplied by 1 gives 2. One quarter ... gives two quarters.
Teacher: Now, you are going to listen to your colleagues and then I will talk to you.
Ivo: Eh, I had a quarter and I crossed it out!
Teacher: Do you think that a quarter is correct Ivo?
Ivo: Yes.
Teacher: Why?
Ivo: Because 2 minus 2 is not one . . . Because following this in the numerator, 2 minus 2 is not 1. If you use two quarters you have to multiply the numerator by 2 and the denominator also, and it is not 2 minus 2.
Teacher: Marta.
Marta: It gives a quarter. I saw that 1/4 + 1/4 is 1/2 then, if we take a quarter from a half, it gives us a quarter.
Teacher: Rogério, did you hear Marta?
Rogério: I put a quarter.
Teacher: Explain your reasoning. Let's see if it's the same as Marta or not.
Rogério: We have a cake. A half of a cake. We ate a half [from this cake] and left only a quarter. So, it is half less a quarter.

Marta and Rogério presented a number fact strategy, but supported by different mental representations. Marta used the number fact "1/4 + 1/4 = 1/2" and the propositional representation: (if 1/4 + 1/4 is 1/2 then, 1/2 – 1/4 is 1/4) to relate this knowledge with the open number sentences that she needed to solve. She applies a property of subtraction (to get the subtractive, we take the difference from the additive). Rogério also used a number fact (a half of a half is a quarter) but he used a mental image ("A half of a cake"; "We ate a half"; "and left only a quarter"). Eva had a strategy to solve the question but she only focused on the fraction equivalent to 1/2 and did not make the subtraction between 2/4 and 1/4 so she made a procedural error. It seems that she was trying to apply the same rule that Ivo used above and a similar mental representation (mental image of a written algorithm). The explanations of Marta and Rogério show that a strategy based on number facts may be supported by different mental representations, probably according to the student's strongest experience. Marta used mathematical knowledge in her propositional representation and Rogério used a real world experience in his mental image, which shows their relational thinking. A student's experience is crucial in defining the mental image that supports their strategies. Rogério used a different mental image from those used by Ivo and Eva. A strong mental image of a written algorithm (like Ivo and Eva used) does not emphasize student relational thinking and may lead them to make some errors (like Eva did) because they are focused on the procedure and not on the relationships between numbers and operations.

In the second part of task 6 students were challenged to compute word problems mentally. To solve the problem shown in Table 3 involving the concept of volume the students could use open number sentences like "12.6 = 4.2 _". To solve the problem Pedro, Maria and Acácio used different strategies supported by propositional representations. They explained: "It gives 3. I divided 12.6 by 4.2" (Pedro), "I didn't divide. I multiplied. I tried to find a number. 4.2 times a number to get 12.6 and I found 3" (Maria) and "It gives 3.3. I divide 12.6 by 4.2. 4 plus 4 is 8, plus 4 is 12. 2 plus 2 is 4, plus 2 is 6" (Acácio). They probably had in mind the open number sentence that we had indicated. Pedro and Maria used relational thinking, taking advantage of the properties of operations (Pedro) and relations between the numbers on each side of the equal sign (Maria). Pedro used the inverse operation strategy and a propositional representation based on the proposition "if 4.2, then ," showing knowledge about the relation between multiplication and division. Maria used an equivalence strategy supported by a propositional representation based on a sequences of true/false propositions where she "tried to find a number" that could make the equivalence true (4.21 = 12.6 (false); 4.22 = 12.6 (false) and 4.23 = 12.6 (true)). Acácio made a conceptual error because he was not able to understand how to give meaning to his result. He identified the correct operation (12.6 4.2 like Pedro did) and used a splitting strategy (separating the integer and the decimal points of the number in order to operate/compare them) but was not able to understand that the relation concerning the whole number must be seen and not each part separately. His strategy may be supported by a propositional representation based on a false proposition: 12 is and 6 is. If 12.6 is "a twelve plus a six" and 4.2 is "a four plus a two" then the result is 3.3, "a three plus a three"). The origin of this error could lie in the additive reasoning used by Acácio. The three students used different strategies but the same type of mental representation in their relational thinking. This suggests that propositional representations play an important role in this process, where different mathematical knowledge must be analyzed and related. Acácio could have achieved success in his relational thinking if his reasoning were more multiplicative and not as strongly additive.

In task 7, students had to compute "90% of 30 = _", which is related to the open number sentence "_% of 20 = 18" from task 8. To solve these sentences, Dina and João used different strategies based on propositional representations. Dina explained: "100% of 30 is 30. 10% of 30 is 3. Then, from 100% to 10% this gives 27". She used a part-whole comparison strategy supported by a propositional representation: (if 100% is 30 and 10% is 3, then 100% - 10% is 90% and 30 - 3 is 27). She used 10% as a benchmark, compared 90% with the whole and removed 10% from 100%. To solve "__% of 20 = 18", João used a part-part comparison strategy and explained: "I saw that 10% was 2. So I divided 18 by 2 which is the same as dividing the total [result 18] by 10% which gives me 9. Then I multiplied 9 by 10 to get 90." He used 10% as a benchmark (like Dina did) and probably, 10% of 20 as a number fact (he did not explain how he computed 10% of 20) to relate part to part based on a propositional representation: (If 10% of 20 is 2 and % of 20 is 18, then ?% / 10% = 18/ 2), He divided 18 by 2 not to "divide the total by 10%" but to relate part to part. He gets 9 and multiplies it by 10 to get 90 because the whole is 10 times bigger than 10%. Dina's and João's strategies show that they understood that the equal sign expresses a relation, an important aspect in relational thinking. Once more, propositional representations seem to play an important role in student relational thinking. The part-whole (Dina) or the part-part strategy (João) used by students requires a systematic analysis of the relation between the whole and its parts which is supported by true propositions. These propositions help students to make inferences about the relationships between numbers and operations on each side of the equal sign.

CONCLUSION

Students' mental computation strategies are mostly based on using equivalence, number facts, operation properties, and part-part or part-whole comparison, supported by mental representations. Their relational thinking is more evident when they use strategies based on numerical relationships. We also identified conceptual and procedural errors (McIntosh, 2006). The source of some of these errors may be a lack of some mental representations as well as a lack of understanding of multiplicative structures, especially in operating with rational numbers. The theory of mental models (Johnson-Laird, 1990) helps us to understand and interpret the mental representations used by students as a support of their mental computation strategies. For example, a strong use by students of mental images of written algorithms suggests a lack of knowledge of relationships between numbers and operations and a poor repertoire of mental representations, obstacles to relational thinking and conceptual understanding. In contrast, as has been shown, the use of propositional representations based on true propositions is evidence of student relational thinking. It highlights student understanding of properties of operations and how they use them to establish numerical relationships and use the equal sign to express relationships between numbers and operations. This was more explicit in sentences like "_% of 20 = 18". Such open number sentences lead students to analyze numbers and operations in a systematic way on both sides of the equal sign as they did in the part-part comparison strategy.

We understand that in analyzing procedural errors students have a viable strategy to compute mentally but a computation error has led them to an incorrect solution. An emphasis on applying memorized rules and procedures based on mental images of written algorithms may also contribute to such an error. Students that make strong use of additive reasoning (as Acácio) make conceptual errors easily as they fail to understand the multiplicative relation that exists in the rational numbers set.

This study provides suggestions for the teachers practice in the classroom in developing student mental computation and relational thinking. It provides information to teachers about the kinds of knowledge that students need and use to compute mentally such as number facts, memorized rules, numerical relationships, and mental representations. It also shows the kind of errors that need to be clarified and discussed in the classroom to improve student mathematics learning.

ACKNOWLEDGEMENT

This study is supported by national funds through a grant provided by to the first author by FCT – Fundação para a Ciência e a Tecnologia (reference FRH/BD/ 9413/2010).

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Spatial structuring, enumeration and errors of S.E.N. students working with 3D arrays

Carla Finesilver

King's College London, London, UK, carla.finesilver@kcl.ac.uk

The move from understanding and working with arithmetical structures in one dimension (i.e. additive) to two or more dimensions (i.e. multiplicative) requires a significant change in children's thinking. This paper investigates the varied and developing strategies and understandings of young people struggling with that change, through a series of 3D array enumeration tasks. Participants relied heavily on counting-based strategies, and a new analytical framework is proposed with which to diagnose initial (mis-)conceptions and observe microprogressions on the path towards multiplicative understanding.

Keywords: Numeracy, counting, arithmetical strategies, multiplicative thinking, low attainment.

THEORETICAL BACKGROUND

2D and 3D arrays

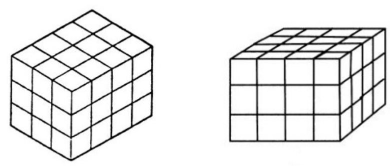
The 2D rectangular array is a standard visuospatial representation for working with multiplicative structures, such as solving simple multiplication and division problems, or connecting replicating spatial patterns with repeated addition and multiplication. By age 11, all students in UK mainstream education will have encountered rectangular arrays (both dots and grids), and these representations linked to multiplication. The 2D array is thought to have particularly good potential for supporting reasoning in multiplication, and is one of the best for demonstrating the commutative and distributive laws (Harries & Barmby, 2007).

While clearly a powerful tool, the 2D array is limited in terms of enumeration. With a 3D array, the options are more complex: when all dimensions are >2 units, simply counting the visible cubes will not work, as there are non-visible interior cubes; successful enumeration must rely on conceptualising the organisational structure of the array as a space-filling object.

While the expected final, formal strategy for students would be a three-dimensional multiplicative formula, on the way to this stage, there are various potential concrete, visuospatial strategies in which the cuboid structure is deconstructed into manageable parts, such as a stack of 2D layers.

An influential series of writings on 3D array tasks (Battista & Clements, 1996, 1998; Battista, 1999) introduced the concept of *spatial structuring,* which I adopt.

> We define spatial structuring as the mental act of constructing an organization or form for an object or set of objects. The process . . . includes establishing units, establishing relationships between units . . . and recognizing that a subset of the objects, if repeated properly, can generate the whole set (the repeating subset forming a composite unit). (Battista & Clements, 1996, 282)



Figures 1a-b: Isometric and perspective cuboid images

Ben-Haim's work during the 1980s on 3D arrays involved students interpreting isometric drawings of cuboids (e.g. Figure 1a), so requiring participants to interpret a pattern of identical rhombuses as a solid object. Thus, his set of proposed error types reflects students' tendency to interact with the presented image as a flat object ("1. counting the actual number of faces showing" (Ben-Haim, Lappan, & Houang, 1985), or to have difficulty picturing the cubes not shown. During the 1990s Battista's research on 3D arrays used line drawings with perspective projection (e.g. Figure 1b). His expansion of the set of error categories

(Battista & Clements, 1996) reflects similar difficulties. Thus, there is impetus to observe the strategies and errors when participants are, instead, simply presented with the solid shape itself.

Counting in multiplicative tasks

The use of counting 'in ones' as a major strategy in additive and multiplicative situations is generally associated with younger children, but persists through adolescence and indeed, adulthood as a supplementary or back-up strategy. However, there is a distinction to be made between retaining counting as a backup, and relying on it as a primary enumeration strategy. Studies (e.g., Siegler, 1988; Geary, Bow-Thomas, & Yao, 1992; Gray & Tall, 1994) indicate that children with arithmetical difficulties are more likely than their typically-attaining peers to rely on counting-based strategies (compared with, e.g. retrieval or derived fact strategies).

Anghileri (1995, 1997) describes a progression in children's counting in multiplicative scenarios, beginning with 'unitary counting', through 'rhythmic counting', to 'skip counting' (or 'step counting'). I propose a refinement: that 'rhythmic counting' is actually made up of two sub-stages, (a) grouping of the numbers and (b) regular rhythmic emphasis of vocalisation or gesture. I use 'grouped counting' for the former, reserving 'rhythmic counting' to describe the specific phenomenon of the musical 'drive' that results from temporally equally-spaced sounds/movements and emphases.

METHOD

The data derive from tasks set during a series of individual or paired problem-solving interviews (with the author), which took place as part of a larger project using microgenetic methods to study emerging and developing multiplicative structure in low-attaining students' use of visuospatial representations. The thirteen participants were aged 11–15, attending mainstream schools in inner London, and identified by their mathematics teachers as numerically weak (compared with peers). All names have been changed.

In each interview, students were presented with a cuboid block formed of multilink cubes, and informed that the blocks were solid, not hollow. The blocks were:

(1) One 3×4×5 cuboid (colours mixed randomly);

(2) One 3×3×5 cuboid (as above);

(3) Two 2×3×6 cuboids, one constructed in three differently-coloured 2×6 layers, the other in six differently-coloured 2×3 layers; students were given the choice which of the two to enumerate;

(4) Two identical 2×2×3 cuboids, both coloured in 2×3 layers; students were asked for the total number of cubes.

The intention was that through these tasks students should come to perceive and use the multiplicative structures inherent in the objects, and their initial and changing conceptualisations could be diagnosed. Specifics of each block were based on the previous observed performances. No time constraint was imposed (actual time varied from 1–15 minutes). If necessary, I provided a series of minimal prompts (described below). Students were allowed to handle the blocks but not pull them apart.

Documentation was via audio recording, photographs, scans of students' papers and observation notes made during and immediately following each interview.

FINDINGS

Task 1: Initial responses

When presented with the first block, all students used counting-based strategies, and all 13 gave incorrect answers. Battista and Clements's analytical categories, while intended for drawn array images, include descriptors also applicable to solids (e.g. "counts outside cubes on all six faces" (1996, 263). However, my students not only made errors in which cubes to count, but in the counting process. Thus one must distinguish an erroneous strategy from errors made in carrying out a correct strategy.

Two students independently made perceptive, effective use of one deconstruction of the array structure, and would have been successful had they not made minor counting errors delivering answers of 59 and 61 rather than 60. With the block on the table, they placed a finger on one of the cubes in the top (4×5) layer and said "1, 2, 3", referring to the touched cube and the two vertically beneath it, then moved the finger along, continuing to group-count threes for every cube in the top layer of 20. It is notable that neither appeared to recognise the set of multiples of three.

Ten of the remaining students began by counting the top layer, then moved onto the other faces of the block, turning it around and attempting to count all the external cubes. Although some students asked for confirmation that the shape was solid as opposed to hollow, their face-based counting strategies nevertheless ignored non-visible interior cubes. Meanwhile, the lack of clear points at which to start and stop counting, and of an obvious 'route' around the six faces, also led to some cubes and/or whole faces being counted more than once, while others were missed. Close observation of gestures and comments indicated that four of the students were attempting to avoid double- or triple-counting cubes, but the other six gave no sign of noticing. The last student unsuccessfully tried to count cubes one colour at a time.

Prompts

If a student was consistently trying to enumerate the squares making up the surface area rather than the cubes making up the volume, I used two prompts: (a) picking up a single loose cube, reminding that these were the items to count; and (b) pointing to a vertex cube, demonstrating how it might be double- or triple-counted. After one or both of these, all students were observed attempting not to over-count edge and vertex cubes. Further prompts drew attention to the layered structure, i.e. the vertical replication aspect of the cuboid shape. The 'layers' prompts were:

> Enquiring how many cubes made up the top layer;
>
> Enquiring how many were in the layer underneath (and, if necessary, the next layer);
>
> Commenting explicitly that all layers contained the same numbers of cubes;
>
> Stating the numbers in each layer in the form of an addition (and, if necessary, supporting or performing that calculation).

Six students responded to one of the first three prompts by stating the number of cubes in each layer and calculating a total of 60, while others heard a full demonstration and gave verbal indications that they understood. One (Paula) gave no such indication that she understood either the addition procedure or its relevance.

One student's response indicates the potential effectiveness of a single prompt.

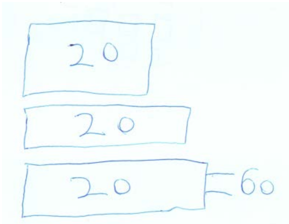


Figure 2: 'Drawers'

CF: How many are in just the top layer?
Leo: *[Step-counts 5, 10, 15, 20]* Ah!
CF: Does that help at all with getting the total number?
Leo: Well now I think I have a solution to this! If you were able to split this, if you chop the layers off, it'll be 20 there – underneath that is another 20, and underneath that is <u>another</u> 20! *[draws Figure 2]* That's 20 there and 20 there. You could just pull it out like a drawer, then pull that out like a drawer. It would be 20, 20, 20.

Task 2

The second interview, I presented a slightly smaller/easier 3×3×5 cuboid. Two students used a full layers strategy correctly, and three others (including Leo) began by stating the number in the top layer, but then needed one or more prompts. Of the two who had used the columns strategy the first time, one reused it, while the other tried horizontal rows instead. Four students reverted to counting around the faces, but switched to layers when prompted. Paula again gave no sign of understanding.

Task 3

So far only four students could carry out effective strategies without prompts; one appeared not to follow even complete demonstrations, and all others were at a stage of partial understanding and operationalisation. Hence, in the third interview I highlighted the physical structure by constructing blocks with each layer a different colour. Rather than force students into a single colour structure (and thus numerical structure), I gave them the choice of two equal-sized blocks: a 3-colour block in horizontal 2×6 layers or a 6-colour block in vertical 2×3 layers. This time 11 students used the layers structure; of these, nine were independently successful, while two required prompts. Only two students' initial response was still face-based, and Paula was for the first time able to comprehend and work through the task (with support). The coloured layers were also indicated to be helpful by student comments, such as "[if there are] 6 cubes there [i.e. in one layer] then you know there's six in the rest".

Task 4

The final interview introduced an additional structural aspect: I presented students with two identical colour-layered blocks, and asked for the total number. With a numerical structure of 2(2×3×3), this task extension allowed increased calculation possibilities for the more confident students. Ten produced a correct answer without any arithmetical or strategic support, and the other three succeeded with prompts. All made clear use of the cuboid structure – in particular referring to the coloured layers.

Regarding the duplicate blocks, five students used some form of layer-organised counting for the first, then continued the count similarly for the second. Two students pushed the two blocks together to make them a single mass. Five worked out there were 18 cubes in the first block and doubled (or added another 18) for the total; one more thought of doing this, but was unsure the two blocks were really the same, and insisted on counting the second as well.

ANALYSIS

The data are considered under three distinct but connected analytical aspects: *structuring* (i.e. how the physical structure of the blocks is used by students, and the corresponding numerical structures drawn from them); *enumeration* (i.e. how students used the numbers they derived from the physical blocks), and *errors* (what went wrong in their invention, selection and application of enumeration strategies).

Structuring

The classification for spatial structuring is based on that of Battista and Clements (1996), adapting their descriptors to apply to actual physical objects, re-ordering them into a loose hierarchy, and expanding the category structure.

Apart from the two C3-strategy students, initial responses to the task lacked awareness of the array structure. Students interacted with one face at a time,

M	**The student conceptualises the set of cubes as a 3D multiplicative structure**	
		Student finds the length, width and height of the block, and multiplies.
L	**The student conceptualises the set of cubes as a stack of 2D layers**	
	1	*Layer multiplication*: Student computes or counts the number of cubes in one layer, counts the number of layers, and multiplies the two.
	2	*Layer addition*: Student computes or counts the number of cubes in one layer and uses addition or step-counting to get total.
	3	*Counting subunits of layers*: Student's counting of cubes is organised by layers, but the student unit-counts or step-counts by a number smaller than the number of cubes in a layer
C	**The student conceptualises the set of cubes as a 2D array of columns**	
	1	*Column multiplication*: Student counts the number of cubes in one column, counts the number of columns, and multiplies the two.
	2	*Column addition*: Student counts the number of cubes in one column and uses addition or step-counting to get total.
	3	*Counting subunits of columns*: Student's counting of cubes is organized by columns, but the student unit-counts or step-counts by a number smaller than the number of cubes in a column.
F	**The student conceptualises the set of cubes in terms of its faces**	
		Student counts one or more faces of the cuboid. They may be counting cubes (partial volume) or counting squares (surface area).
O	**Other**	Student uses a conceptualisation other than those described above.

Table 1: Spatial structurings of a 3D array

failed to coordinate orthogonal views from different perspectives, and in many cases did not even have a complete faces-based conceptualisation (i.e. surface area). All showed increased awareness of structure following prompts, but the amount of prompting required and strategic change observed varied widely. There was a general move from F towards L strategies, as would be expected. Only Paula and one other attempted an F strategy on all occasions, and both could identify and use layers (with colours and prompts) eventually. However, there was no clear trend within conceptualisation types, i.e. from L3 to L2 to L1 (or equivalent).

On finding a successful strategy, some students repeated it, while others tried alternatives. Battista and Clements consider layers strategies an indication of "see[ing] the array as space-filling" and having "completed a global restructuring of the array" (1998, 234), while "Those in transition, whose restructuring was local rather than global, utilized [column-based] strategies . . . They had not yet formed an integrated conception of the whole array" (ibid). It is unclear why a columns-based spatial structuring should be considered any less sophisticated than a layers-based one. The former deconstructs a 3D array into a 2-dimensional array of 1-dimensional stacks, the latter a 1-dimensional stack of 2-dimensional arrays; both are equally valid (and complementary) space-filling conceptions.

Enumeration

The *enumeration* classification is based on that of Anghileri (1997), and may be used in combination with the spatial structuring categories (producing, e.g., C3R).

All students began with some form of counting-based strategy, and overall these were by far the most popular. Four students clearly used multiplication in Tasks 3–4, and there were other instances where language implied multiplicative thinking. However, between unitary counting and multiplication was observed a spectrum of ad-hoc grouped-, rhythmic-, step-counting, and addition, and mixed methods.

1	**Multiplication**	
		Student calculates a total without any interim step-counting.
2	**Step-counting/Addition**	
		Student counts in steps formed of the cardinal number of each layer or column, without any interim numbers (i.e. using a number pattern).
3	**Counting**	
	S	Step-counting (within a layer or face)
	R	Rhythmic counting: Student counts each cube individually, in rhythmically consistent sequence, with clear emphases on cardinal numbers of subgroups.
	G	Grouped counting: Student counts each cube individually, but with the count sequence organised into subgroups.
	U	Unitary counting: Student counts each cube individually, with no grouping.

Table 2: Enumeration strategies for a 3D array

Spatial structuring (SS) Student uses an incomplete or incorrect conceptualisation of the array structure, e.g. double-counting edge cubes, not accounting for interior cubes.
Numeric calculation or retrieval (NC) Student makes an error in calculating or retrieving a number fact while multiplying, adding or step-counting, e.g. "three twelves... 12, 24, 38".
Verbal count sequence (VC) Student makes an error in their counting, e.g. "26, 27, 29, 30".
Visuospatial/kinaesthetic (VK) Student makes an error relating to the physical aspect of counting, e.g. desynchronisation of verbal count and gesture, repeating a layer/column, etc.

Table 3: Types of error in enumerating 3D arrays

Errors

Under the analytical aspect *error* are proposed the four types below, between which are covered all errors observed in this dataset.

SS: Issues of spatial structuring have already been covered. While most students' initial responses to Task 1 involved mis-structuring, there were only nine subsequent SS errors.

NC: On nine occasions, students mis-recalled addition facts and number patterns, or unsuccessfully attempted formal 'vertical' addition notation for the layers; however, the predominant preference of low-attaining students for counting-based strategies meant that recall of arithmetical facts or procedures was not often required.

VC: Students were all confident in their ability to unit-count individual cubes, yet there were examples of missing and repeating a single number, and missing out a decade.

VK: The most common error type, 22 instances were observed. Some appeared related to fine motor skills, e.g. 'jumping over' a cube. Students also skipped rows, layers and faces, lost track of their start point, etc. Spatial structuring affected this error type: end point and rotation issues happen when working on faces, but with a columns or layers conceptualisation, the block can remain immobile throughout.

Issues of classification

The framework above is useful for identifying individual trajectories and group trends. However, there may be issues in identifying strategies used, e.g. with a student who works silently or with minimal gestures, and does not have the verbal skills to explain coherently how answers were obtained. With students who do verbalise their work, there may be inconsistencies between what they report and what is observed. On several occasions, students used language of multiplication (e.g. "it's three twelves"), but employed a counting strategy; they perceived the multiplicative structure yet were unable to carry out the multiplication operation in any other way.

CONCLUSIONS

Students' understanding of multiplicative structures

Contemplating the visuospatial patterns within physical structures can reasonably be expected to increase awareness of the numeric structures embodied within, at both more advanced stages and the most fundamental stages. E.g. the motion required to move a pointing finger to the next row (etc.) causes a pause in the verbal count, naturally grouping the counting sequence and emphasising the last number spoken. Thus even an incorrect faces-based spatial structuring of a 3D array contains enough structure to serve a useful purpose for the very weakest. While one might assume that students' enumeration of arrays stems directly from their spatial structuring, the relationship is bidirectional; enumeration can also guide structuring. E.g., a student better at step-counting long sequences of small steps than adding a short sequence of larger numbers may (sensibly) opt for a C2 strategy, despite perceiving the layers. Students may seize on familiar number patterns; e.g. noticing there were five units in a row, column or stack could make that the salient grouping of the physical/numeric structure. If struggling students have access to more than one potential structuring, they can choose the one that best suits their capabilities and preferences.

Development of strategies over time and in response to prompts

On finding their initial solutions incorrect, one might expect the kind of cognitive conflict which results in reflection and adaptation; this did not happen. Some students immediately started to re-count in the same way as previously, i.e. they believed in the efficacy of their strategy, but mistrusted their ability to have carried it out properly. Some acquiesced to failure, while others were engaged enough to argue and insist their answer was correct. However, none independently responded by thinking critically about the strategy they had used and improving it or attempting an alternative. Strategic progression in every case required external input. I suggest individuals' willingness (or otherwise) to try alternatives is linked to their relationship with mathematics (or school); on finding a successful strategy, arithmetically insecure students cling to it, while security allows for experimentation.

Further reflections

This simple task proved extremely rich in information about the nature of individuals' current multiplicative thinking, the 'gaps' in their multiplicative understanding, and the variety of enumerative strategies in use. Presented with a task that could be solved by counting, but where that counting was non-trivial and non-routine, adolescents had to reconsider this most basic of numerical skills, and how to apply it. Use of task variants with the same students on four occasions allowed tracking of their progression in terms of spatial structuring, enumeration and error patterns.

Although the layered spatial structure of a cuboid seems obvious to a teacher, and indeed, seemed obvious to some students once given a minimal prompt, others struggled significantly to conceptualise the array as a coordinated, space-filling structure. The use of minimal, sequential prompts, along with the introduction of colour-defined structure, demonstrated the variation in how much input and effort it can take for a student to 'see it'. Furthermore, individual students took their own paths from an essentially 2D, faces-based conceptualisation to a coordinated space-filling structure, with paths through layers, columns, rows, stacks, and combinations of these. While the ability to perceive multiple structurings is unnecessary in the short term (i.e. for solution of this particular task), I assert that in the wider scheme, it is mathematically advantageous and to be encouraged.

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Computing by counting in first grade: It ain't necessarily so

Michael Gaidoschik[1,2], Anne Fellmann[1] and Silvia Guggenbichler[2]

1 Alpen-Adria-University Klagenfurt, Institute for Mathematics Education, Austria, michael.gaidoschik@aau.at

2 University College of Teacher Education Carinthia, Klagenfurt am Wörthersee, Austria

Regarding first-grade strategy use, counting-on is widely promoted as an alternative to counting-all. However, there are concepts of initial arithmetic education that aim at developing derivation strategies systematically from the outset, skipping counting-on. This paper refers to an ongoing study that provides some empirical evidence in support of the latter approach. Students of four Austrian classes whose teachers participated in a professional development programme designed to promote derivation strategies were interviewed to ascertain their computation strategies at the end of first grade. We present first results comparing them to those of a previous study that, using the same tasks, examined a random sample of children whose arithmetic education followed a distinctly different pathway.

Key words: Counting-on, derivation strategies, difficulties in learning mathematics, in-service teacher education.

INTRODUCTION

In the German-language literature on mathematics education it is widely held that the use of counting-based strategies constitutes "a *necessary* stage in each child's learning process" that "*none* can skip" (cf. Lorenz, 2003, p. 105; our emphasis). Less frequent, but still to be found, are explicit recommendations such as by Schipper, Wartha, and von Schroeders (2011, p. 14) that children first be encouraged to use counting-on rather than counting-all. In the respective English-language literature – as far as we oversee it [1] – similar recommendations are the rule. According to van de Walle (2004, p. 164), counting-on is "the most widely promoted strategy" in the USA. In the UK, Australia and New Zealand the working out and consolidation of counting-on constitutes a central element of national programmes to promote numeracy (cf. e.g. New Zealand Ministry of Education [NZME], 2010). Nota bene: Counting-on is promoted merely as a *transitional strategy*; also, there is broad international consensus that children should *eventually* overcome the use of whatever kind of counting-based strategy (cf. Schipper et al., 2011; NZME, 2010). Reliance on counting to add or subtract in higher grades is considered to be a main characteristic of so-called mathematics disabilities (cf. Häsel-Weide & Nührenbörger, 2013).

It is mainly in view of mathematics disabilities which prompts us to call for a revision of notions concerning counting-on like the ones cited above – knowing that we are not the first to do so. Referring to the state of research we shall set forth that promoting counting-on is by no means *necessary* nor, as we argue, is it *conducive*. We shall then discuss first interim results of an ongoing study that promises to provide valuable empirical material for the further exploration of potential alternatives.

EMPIRICAL AND THEORETICAL FRAMEWORK

The recommendation that early mathematics education should *first* systematically foster counting-on is, as a rule, embedded in developmental models suggesting that numeracy evolves through stages of counting-all and counting-on to finally arrive at fact retrieval (cf. NZME, 2010). Usually such models draw upon, although not always explicitly, Siegler's (cf. 2001) "Model of Children's Strategic Choices".

As for the notion of a quasi-natural sequence of stages it has already been pointed out that at least the last stage, fact retrieval, is obviously *not* achieved by *all* children. What is empirically validated is the first step: Most children are able to solve one-digit additions and subtractions already at preschool age, mainly by counting-all (cf. Verschaffel, Greer, & De Corte, 2007). However, also at an early stage they use other strategies as well, e.g., counting-on, which they often, and prior to any classroom instruction, discover as a short-cut (cf. Baroody & Tiilikainen, 2003). In parallel, or even earlier, many children come to devise strategies using their fingers. Some of these strategies do *not* include counting (see below). Many children do soon retrieve from memory at least single tasks. Already at preschool age, some draw on tasks committed to memory to derive other tasks (cf. Gaidoschik, 2010, p. 332–340). Thus, not only is the *usual development* too varied to be captured by a stage model. It also differs from child to child and, most important, does not remain unaffected by exterior influences (cf. Verschaffel et al., 2007, p. 565). In this light, there is no reason to claim that each step is *necessary*: With influences changing, a different development is at least conceivable. That the strategies pursued by children *usually* also include counting-on does not provide sufficient justification to direct the influence of classroom-instruction towards the elaboration or consolidation thereof.

For such an approach to be justified, counting-on would have to prove *conducive* to overcoming counting strategies. This is what Siegler's model of strategic development suggests. According to Siegler (2001, pp. 377–383), each time a child provides a correct solution by counting, the "bonds of association" between task and solution in long-term memory will get strengthened. As soon as the "associative strength" between a task and the resulting number exceeds a certain value, the child would abandon counting and retrieve the result from memory instead. Counting-on is considered to be instrumental for this process to succeed in that it increases both the probability and speed with which correct solutions are obtained (cf. Schipper et al., 2011, p. 16).

What speaks against this model are empirical findings like those by Gray (1991). Gray's theory of the "proceptual divide" distinguishes between children who come to abandon counting following their recognition of relationships between numbers and numerical operations, and those for which the use of counting-based strategies is rather an obstacle in developing viable concepts. The "schema-based view" proposed by Baroody and Tiilikainen (2003) argues in a similar way, maintaining that the decisive factor in abandoning counting strategies is conceptual knowledge of numbers and operations, i.e., the ability to recognise relationships as a basis upon which to derive tasks from other tasks. Intercultural studies also add evidence in support of the targeted promotion of derivation rather than of counting-on. Geary, Bow-Thomas, Fan, and Siegler (1996), e.g., comparing children in the US with their Chinese counterparts found that on problems up to 20 the latter had by and large abandoned counting strategies already by the end of first grade, whereas the former still used counting on 40 % of tasks by the end of third grade. Those differences certainly ensue, at least in parts, from different teaching traditions – unlike in the US, in China it is common to promote derivation as an alternative to counting from early on. As intervention studies show the learning of derivation strategies based on insight facilitates the abandonment of counting strategies (cf., e.g., Steinberg, 1985). To the same direction points a recent study by Rechtsteiner-Merz (2013) which evaluated a similar teaching conception.

This is further corroborated by a longitudinal interview-based study conducted by Gaidoschik (2010; 2012) which investigated into the calculation strategies of 139 randomly selected children at the beginning, in the middle, and at the end of first grade. A significantly higher share of children who by the middle of the school year had solved a task by derivation did retrieve the same task from memory by the end of the year compared to those that formerly had relied on counting-on. Teacher interviews indicated that in all of the children's 22 classes arithmetic lessons followed a rather uniform pattern in two central respects: derivation strategies were widely neglected, while counting-based strategies were fostered at least during the first term. Against this backdrop, by the end of the year some 27 % of children would work out solutions to problems up to 10 mainly by counting whereas some 33 % would use non-counting strategies (cf. Table 2). These children, except for two, had repeatedly used derivations also in the course of the interviews. Children who predominantly resorted to counting were not observed making use of derivation strategies (cf. Gaidoschik, 2010, p. 438).

Interim conclusion: Studies conducted to date do not allow inferring a *necessity* to pursue counting-on for a considerable period of time. Nor do findings corroborate the thesis that fostering the consolidation of counting-on is *conducive* to overcoming the use of counting strategies. Studies on the relationships between the development of strategies and conceptual knowledge as well as theories derived thereof, in fact, suggest promoting derivation as an alternative to counting-on early in first grade.

PRESENT STUDY

The recent study on which we elaborate in the following may be characterised as an ad-hoc field study designed to investigate a number of questions arising from the findings by Gaidoschik (2010, p. 519–521) as set out above. "Ad hoc" means that the four classes covered by the study had actually been selected to serve another end, notably the conduct of a design study on second-grade teaching of multiplication. Decisive for the selection of classes was their teachers' participation in a professional development programme which is offered (so far without any participation of the authors of this study) in the Austrian province of Carinthia under the name "EVEU – Ein Veränderter Elementar-Unterricht" [2]. As one of its main elements, EVEU recommends the systematic working out of derivation strategies on additive basic tasks in first, and on multiplications in second grade. With a view to optimise the cooperation with the teachers, visits to the four classes were made already at the end of first grade. These included sitting in individual arithmetic lessons and interviewing teachers about their teaching derivation strategies over the school year. Inquiries were made also into children's ways of working out solutions to addition and subtraction tasks up to 20. Both the lists of tasks and the procedure were the same as those administered in the interviews conducted at the end of first grade in the framework of the previous study (Gaidoschik, 2010, p. 237–245). This allows us to compare strategies as applied by two different groups of children at the end of first grade drawing on identical tasks. On the basis of guideline-based interviews with their teachers tentative conclusions may be drawn concerning these children's arithmetic lessons. The overall analysis will particularly be devoted to answering the following questions:

A) During first grade, did EVEU teachers – unlike the teachers of the 2010 study who had not participated in any specific professional development programme – actually work out and consolidate derivation strategies? If so, how, with what intensity and consistency?

B) By the end of first grade, are there any important differences between EVEU children and the sample surveyed by Gaidoschik (2010) regarding the use of calculation strategies? If so, may these differences (also) be attributed to differing teaching concepts?

Sample and design

The sample surveyed for the present study was comprised of teachers and pupils (six and seven-year-old) of four first-grade classes (A-D) from four public elementary schools in Carinthia. The teachers had completed the first year of the EVEU programme (8 half-days) outlined above. The interviews covered *all* children of each class provided firstly that 2013/2014 was actually their *first* school year and secondly they had a *command of the teaching language* – qualifications that had applied also to the 2010 study. In addition in eight cases parents refused to give their consent. Due to these restrictions, the sample covered 11 children (out of 23) from school A, 16 (out of 20) from school B, 19 (out of 22) from school C, and 25 (out of 25) from school D.

The interviews were conducted by the authors themselves in June 2014, towards the end of the school year, in some extra rooms near the classrooms. The children were presented with 22 tasks up to 10 (like 3+4, 3+7, 4+6 or10-9, 7-4, 10-7), and 14 tasks up to 20 (like 6+6, 5+8 or 12-6, 14-9; for more details cf. Gaidoschik, 2010, p. 239–241). Each task was read aloud by the interviewer, at the same time the child was shown the task written on a DIN A7 card. The children were asked to solve the task mentally in the same way as they would usually do and state the result verbally. Immediately thereafter, in case the solution was not provided spontaneously or by using a strategy that could be perceived by observation without any doubt (see below), the child was asked to explain or show how it had obtained the solution.

The video-based evaluation was carried out analogously to Gaidoschik (2010, p. 243–245), i.e. the children's strategies were coded on the basis of the children's utterances, their gestures and facial expressions, and the time needed to produce a solution, as well. The videotapes, without using transcriptions, were analysed repeatedly by one, respectively (in randomly selected 10 % of cases), two members of the interviewing team; disagreements on single codings were resolved through discussion. The main categories that were applied are as follows: *Fact retrieval*, if a correct solution was produced spontaneously (within two seconds); *erroneous retrieval,* if a spontaneously uttered solution was incorrect; *derivation*, if the solution followed an at least short, recognizable reflection and the child described a fitting derivation strategy as his or her solution path thereafter; *counting*, differentiated into counting-all, counting-on and partial finger counting, if fingers were obviously used as a counting-device, or if there were signs of intrinsic counting (nodding of the head etc.) or attempts at using fingers surreptitiously. In the latter case the children were asked to explicitly demonstrate their problem-solving pathway, what they usually did without any further concealing.

We are aware of the flaws inherent in the chosen method (cf. Verschaffel et al., 2007); however, in the absence of a more valid alternative we did our best to classify the children's strategies as appropriately as possible.

First results with regard to participating teachers (cf. question A)

The four teachers stated unanimously that enabling children to compute without counting had been one of their priorities and that they had strived to work out derivation strategies based upon a solid conception of numbers as composed of other numbers. As the main element of classroom practices they referred to classroom discussions, which set the stage for children to put forward proposals as to the most appropriate solution to a task. The teachers also noted that efforts had ever been directed at enabling possibly all children to use non-counting strategies, whereas counting-strategies were not encouraged at any stage of arithmetic lessons.

A content-analytical evaluation of the semi-structured interviews yielded clear evidence of differing intensities and consistencies of the instruction in mainly two aspects. Firstly, with regard to "helping facts" such as doubles (e.g., 4+4=8), ten facts (e.g., 3+7=10), or partitions by the power of five (e.g., 8=5+3), teachers A, B and C stated that, once elaborated, these facts were continuously exercised as an important derivation basis with a view to automatisation. Teacher D, on the other hand, conceded that she might have failed to make sure that these facts were thoroughly known by all children. Secondly, teachers A, B and C made a point of emphasising how essential it was to push children tenaciously, virtually in every arithmetic unit, to explain their solving strategies and again and again put single derivation strategies centre stage. Teacher D reported self-critically, that the effortlessness with which many children went about derivations during classroom discussions, misled her into believing that other children would solve problems in the same way, i.e., by non-counting strategies. Not least, she confessed, did she often feel overtaxed given the rather large size of her classroom (25 children).

First results with regard to the participating children (cf. question B)

Table 1 shows the frequency at which counting strategies were used on problems by the children of the four EVEU classrooms and those surveyed in 2010. In the

	Number of instances in which tasks were solved by counting out of 14 nontrivial tasks with sums/minuends up to 10				Number of instances in which tasks were solved by counting out of 8 problems with one digit-numbers and totals greater than 10			
	mean/ median	standard deviation	min	max	mean/ median	standard deviation	min	max
2010 overall	5.5 / 5	4.6	0	14	3.7 / 4.0	2.6	0	7
2014 class A	0.1 / 0.0	0.3	0	1	0.0 / 0.0	0.0	0	0
2014 class B	0.0 / 0.0	0.0	0	0	0.0 / 0.0	0.0	0	0
2014 class C	1.0 / 0.5	1.6	0	5	0.9 / 0.5	1.5	0	5
2014 class D	1.9 / 1.0	2.6	0	10	1.7 / 1.0	2.0	0	6

Table 1: Problems solved through counting in different groups of students

framework of the 2010 study, a total of 14 problems up to 10 had proved nontrivial, i.e., were known by rote by less than two thirds of children. Table 1 compares means, medians, standard deviations, as well as the minimums and maximums of solutions to these 14 problems as well as to 8 additions and subtractions with one-digit numbers and totals greater than 10 that were achieved by counting-strategies.

As can be seen from Table 1, in the course of interviews conducted in classrooms A and B there was only one single instance of counting-on. In classrooms C and, more significantly, D, besides the vast majority who mostly or entirely used non-counting strategies, there were single children who were still making relatively abundant use of counting. The Kruskal-Wallis test reveals significant differences ($p < .001$) between medians across the five groups of students with regard to both sorts of tasks. Post-hoc pairwise comparisons based upon the Mann-Whitney U test show that in each of the four EVEU classrooms problems from both sorts of tasks were solved by counting to a significantly less extent ($p < .001$) than in the previous sample. Taking into consideration teachers' statements cited above it appears legitimate to draw a distinction in terms of the quality of mathematics education between classrooms A, B and C as a subgroup on the one hand and classroom D on the other. Differences are significant ($p < .001$) also between these subgroups within EVEU classrooms.

With regard to the children's strategy preferences during their first year of school, within the random sample surveyed by Gaidoschik (2010, p. 425–461) six types of strategy development could be distinguished (empirically grounded construction of types, cf. Kelle & Kluge 1999). Given the fact that EVEU children were interviewed only once, assignment to a certain type must be done with caution. Still, it is instructive. Table 2 shows frequencies only of the two types representing the poles in strategy preference at the end of the school year on problems up to 10.

The type "Counting, no derived facts", while not occurring at all in EVEU classrooms A and B, is rare also in classrooms C and D comprising only 5, and 8 %, respectively. In the previous sample, the percentage of children who could be assigned to this type, with solving more than two thirds of problems up to 10 by counting and not a single one by derivation, was about 27 %. In EVEU classrooms A and B, conversely, *all* children belonged to the type "Retrieval and derived facts", i.e., they displayed a high level of retrieval complemented by a flexible use of several derivation strategies. In the 2010 sample, only one third corresponded to this type. In EVEU classrooms, two subtypes could be distinguished within this type – the first being comprised (mostly) of children with a clear prevalence of direct fact retrieval even on tasks involving going-through-ten. Derivation, therefore, was not needed any more in most cases; if, however, it was done quickly and could be clearly explained. On the other hand, there were a few children who frequently resorted to derivation even on tasks up to 10. For a few, this obviously was an arduous process with single derivations taking 30 seconds and longer. Still they would not regress to a counting strategy. These children too were able to provide plausible explanations of their derivation pathways.

In classrooms C (3 children) and D (4 children) a type could be identified that the random sample surveyed in the previous study did not display so distinctly: children who, although not resorting to counting, used fact retrieval or derivation on less than two thirds of tasks up to 10. On the rest they would use non-counting finger strategies – e.g., to figure out the solution to 9-8 they would, without counting, put up nine fingers *in one move* and subsequently, again in one move and with obvious routine, put down eight fingers (four of each hand) to "read off" the result from their finger pattern.

DISCUSSION AND OUTLOOK

The study provides some further empirical support for van de Walle's (2004, p. 164) dictum according to which counting-on "is not necessary if other strategies are used". The EVEU children, according to teachers' statements, had not been encouraged in the classroom

	2010 overall	2014 class A	2014 class B	2014 class C	2014 class D
Counting, no derived facts	27 %	0 %	0 %	5 %	8 %
Retrieval and derived facts	33 %	100 %	100 %	63 %	44 %

Table 2: Distribution of strategy preferences on tasks up to 10 at the end of first grade

to use counting-on as a strategy at any point of time. Rather, and deliberately, computation was addressed only after it had been attempted to consolidate children's conceiving of numbers as being composed of other numbers. From the very beginning, tasks were dealt with in relation to other tasks; relationships were used to derive other tasks. In two out of four classrooms students had almost entirely abandoned counting even on problems with totals greater than 10; in the other two classrooms only one and two children, respectively, relied predominantly on counting strategies. Such level of achievement by the end of first grade is not at all a matter of course – it is at least *this* which can be derived from the comparison with the random sample surveyed by Gaidoschik (2010).

Given the ad-hoc character of the present study, conclusions on the EVEU children's levels of achievement at the *beginning* of first grade must be drawn mainly based on teachers' statements. Teacher A noted that her students were "high performers" in comparison to previous classrooms. Teachers B, C and D described their classes as "average". All the four schools are characterised by a mixed catchment area; socioeconomic backgrounds, however, could not be established for all children. Actually many spheres of influence on children's learning remain largely obscure. While we are far from attributing the children's differing uses of counting/non-counting strategies *exclusively* to the respective classroom practices (which were, moreover, established with limited methods), still we find it plausible that these may *also* have been important.

Evidence of didactically important differences regarding classroom concepts can be found also between the four EVEU classes. Thus, in classroom D both the preparation and reinforcement of derivation was obviously done less consistently than in the other classrooms. This may at least partially account for the significantly higher share of counting strategies use in classroom D. The frequent occurrence of non-counting finger strategies in classrooms C and D, on the other hand, corresponds with teachers' C and D statements that this kind of strategy was explicitly encouraged any time children were observed using their fingers for counting. Whether children who, by the end of first grade, still rely heavily on non-counting finger strategies will in second grade move on to fact retrieval and derived facts strategies; whether this requires targeted support, and what kind of support, are just a few of the many questions we intend to address in a follow-up study.

Implementing the classroom concept set out above was perceived as a great challenge by each of the four teachers surveyed. It is indeed plausible that the larger the classroom and the greater its heterogeneity, the more demanding the teacher's role (cf. teacher D quoted above). This is why we consider it all the more important that teachers in implementing innovative concepts regarding *central* contents of elementary school mathematics be given *long-term* support within the framework of design studies. Professional development programmes, while providing didactic stimuli, cannot, as a matter of principle, translate into a technology. An analysis of teacher interviews against the backdrop of the theory of recontextualisation (cf. Fend, 2006) reveals a mismatch, particularly in case of teacher D, between the knowledge explicated in the EVEU programme and the teacher's implicit knowledge [3]. Such kind of difficulties should not come as a surprise, so this is why expert teachers' visits on a weekly basis form an integral part of the EVEU approach. As to our discipline of mathematics education, we would essentially have to provide scientific expertise to such kinds of measures seeking to work out, together with teachers, solutions to concrete questions arising in the classroom day-by-day and to evaluate the impact of relevant decisions in order to create the basis for the design to be developed further.

Particular attention should be paid to children with learning difficulties. In our study, there were three students in classroom C, and five in classroom D, who had considerable difficulties especially with sums greater than 10. Unable to apply any of the non-counting strategies taught in the classroom, they – as some explicitly admitted – regarded counting strategies as something they were not supposed to use. As a result, they seemingly did not know what to do at all. All teachers convincingly stated that solving tasks by counting had not been *forbidden* at any point of time. Yet, with classroom practices persistently pursuing alternatives to counting, it may be difficult for some children not to think of counting as something they are simply not allowed to do. This might be disregarded if at the same time non-counting strategies were available for all students, which in classrooms C and D was not the case. That for these students counting is a necessity which must not be withheld from them is a conclusion we think is premature – especially in view of the encouraging findings of this study. We do see, however, the need for further development of designs that promote alternatives to counting as a computation strategy from the very beginning.

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ENDNOTES

1. We admit that our knowledge is incomplete with regard to English-language literature and completely lacking as regards literature in languages other than German or English.

2. "A changed way of instruction in elementary school"; for details see (Benke, Kittner, & Krainer, 2014).

3. Going into greater detail as to the possible implications for the implementation of innovative forms of teaching and learning is beyond the scope of this paper. For a theoretical embedding see Fellmann, 2013.

What is a better buy? Rationale and empirical analysis of unequal ratios tasks in commercial offers contexts

Bernardo Gómez and Amparo García

University of Valencia, Department of didactics of mathematics, València, Spain,
bernardo.gomez@uv.es, garcianadal.amparo@gmail.com

In this work, the comparison of unequal ratios tasks in commercial offers contexts is studied. A rational and empirical analysis of tasks helps to identify the critical components and students responses to each task. The results confirm the deficits in the relative thought of pre-service teachers, and also that their difficulties are not in the algorithmic aspects and "norming" techniques, but in conceptual aspects and ratio referents.

Keywords: Ratio and proportion, relatively, norming, didactic phenomenology.

INTRODUCTION AND RESEARCH OBJECTIVES

"The box of Bites (net weight 16 oz.) costs \$3.36 and the box of Bits (net weight 12 oz.) costs \$2.64. Which cereal is the better buy?". This example is used by Lamon (2012, p. 106) to encourage thinking flexibly in unitizing. Examples of this type involve the comparison of relative quantities, which are ratios, though the explicit formulation of the problem needs a relational term, the word "relatively", to specify the price comparison must be in relation to the weight of the product. The ability to compare the large amounts in this way widens the range of applicability of certain words, such as the word "more" which has two meanings, one absolute or additive and one relative or multiplicative, both of them are correct.

As Streefland (1985, p. 75) says: "In mathematics programs for elementary instruction as far as ratio is concerned, one is often struck by the poverty and brevity of the approach chosen by their authors". The poverty of this approach to ratio can be more generally characterized as follows: the concept building is exercised with mathematical objects unrelated to reality; the lack of real applications, and isolation of the subject "ratio", which is not connected with any other subject.

In this way, Freudenthal (1983) in his didactic phenomenology, highlights the importance of considering ratios in situations in which the idea of "relatively" (or comparatively) and the complex of techniques designated by *norming* are required.

Understanding "relatively" in the sense of "in relation to..." involves the use of the term ratio as Smith (2002, p.14) proposes: "I will use the term ratio to describe a relational number that has two properties: (1) it relates two quantities in one situation, and (2) it projects that relationship onto a second situation in which the relative amounts of the two quantities remain the same". This use of ratio is in accordance with the very meaning of ratio: "to speak about equality (and inequality) of ratios, without knowing how large the ratio is" (Freudenthal, 1983, p. 180).

All of this situates our interest in problems of quantitative comparison of ratios. In particular, in commercial offers comparison, which offer discounts that are given as relative amounts. As a standard norming percentages are usually used to express discounts.

In order to provide relevant tasks for a didactic phenomenology of ratio involving these ideas, a test has been designed. The tasks, which are typical of commercial offers, have been analyzed in a rational and empirical way. This allowed for a better understanding of critical components and their relationship with the response patterns of students.

According to Cramer, Post & Currier (1993, p. 2), "the critical component of proportional situations is the multiplicative relationship that exists among quantities that represent the situation". In the quantitative comparison of ratios problems, the multiplicative relationship that exists between the quantities represented in the situation can be equal or unequal. These multiplicative relationships express relative quantities, that is, quantities put in multiplicative relationship with other quantity of reference. This is usually called "the referent". So, we consider that the critical components (c.c.) in these situations are: not only the multiplicative relationships, but their equality or inequality and their referents.

Note that tasks that have been experimented can be used not only with the intention of assessing knowledge, but also in teaching situations and for teacher training. It allows them to promote the metacognitive reflection about their own cognitive processes and the didactic task complexity and the mathematical contents involved. So, the research questions are: which are the critical components of tasks?, what strategies do the students use?, and what difficulties do the students show?

CONCEPTUAL FRAMEWORK

As we have just said before, comparing relatively is to put something in relation to, and norming is a process of reconceptualization of a system in relation to some fixed unit or standard (Lamon, 1994, p.94). One of the common forms of norming is the unification of the antecedent (numerator) or consequent (denominator) of the ratios to favor the comparison. This can be done by an algorithmic process that links them to the unit (e.g. the unit rate obtained by quotient), the decimal numbering system (percentage or decimal) or equivalent fractions. These techniques connect the various forms of ratio: fraction, decimal, percentage or quotient, and are linked to the flexibility of thinking in order to choose convenience.

The norming techniques are used with the intention to make more visible the comparison of pairs of phenomena that Freudenthal (1983) calls "expositions" or "compositions". When two distinct defined expositions are compared on the same set, e.g., Ω is a set of countries, each with its assigned inhabitants and its area by the ω_1 and ω_2 functions. Then ratio ω_1/ω_2 expresses the population density. Comparison of density couples allows to state whether a country has in proportion to its area the same number of inhabitants or a higher or lower number than another country. In the comparison of two compositions on the same set, e.g., Ω is an alloy composed by copper and zinc to form bronze and each component is assigned a different mass in each alloy by the ω_1 and ω_2 functions. Comparison of the pairs of internal ratios, copper mass/zinc mass, allows one to know which alloy has relatively more, less or the same amount of copper to zinc (Freudenthal, 1983, p. 186).

The "best buy problems" can be interpreted as a pair of expositions or compositions. Under this interpretation, we define the objectives of this study. The first one is to determine, through the rational analysis, the critical components of the tasks such as: the multiplicative relationships, the equality or inequality of ratios and their referents. The second one is to determine, through the empirical analysis, the relationship between these components and students' performance.

METHODOLOGY

The work is based on the methodology of the empirical and rational analysis of tasks. According to Lamon (2007, p. 641), the distinction between empirical and rational analysis is adopted to distinguish between children's mathematics (children's actual performance on tasks); and the exam of content from a mature mathematical perspective, making assumptions about the ways of thinking that are necessary to solve problems. The rational analysis begins at the theoretical level, in order to identify the critical components of tasks and their procedural cognitive and conceptual objectives, to support theoretical inferences from the data obtained in the empirical analysis. The empirical analysis begins with the implementation of tasks given to students in order to interpret their responses. This is taken as criteria for analyzing the critical components identified in the rational analysis.

We choose 4 tasks for a pencil and paper test. They are called Pizza, Beer, Softener and Mosquito repellent. They are realistic tasks taken from offers in current commercial brochures. Due to their typology they are characterized as quantitative comparison of ratios tasks, where one has to judge which of two ratios is higher, lower, or perhaps the same, so you can do it roughly or precisely. Moreover, they may be characterized by their phenomenology as pairs of expositions or compositions involving comparing relatively and applying norming techniques, as Freudenthal says. So, we choose these tasks because they are real applications of ratio and one has to judge equality or inequality of ratios, as Streefland and Freudenthal require. The test was implemented using individual worksheets given to 9 groups of students working in a normal mathematics class time in their second year of the teaching degree, at the beginning of the course (341 students). The study was conducted during 2013–2014 at the University of Valencia.

We select these participants because, according to Ben-Chaim, Ilany & Keret (2002), we think that the pre-service teachers need to improve their knowledge and their attitudes toward mathematics, in general, and all of the components and aspects of ratio and proportion, in particular. And we think that realistic tasks, such as the commercial offers, are suitable for teaching the topic of ratio and proportion in pre-service elementary teacher education.

TASKS

In Pizza, one asks: What is better, two regular pizzas 30cm in diameter for 14.95€ each or a large pizza 50cm diameter for 27.95€? Justify your answer. In Softener, the question is: Which of the two options is more expensive, the concentrate on the left or the non concentrate on the right? Justify your answer.

In Beer, the text says: Usually beer cans are 1/3 liter or what is the same, 33'3cc. One option offers a 15% discount on the price and the other option has 14% more beer. What is more expensive? And, in Mosquito repellent: Fogo and Bloom sell for the same price and have the same volume without any promotion. What discount is better? Explain your answer. Softener and Mosquito are easier than the other two because the text provides all the data needed for doing a correct comparison.

Rational analysis and critical components of tasks

In Pizza and Softener the critical component is the equality or inequality relationship of the two relative quantities. These quantities are given by composed ratios formed by the pair (€, washes) or (€, cm^2). It is required to apply norming by quotient for making the comparison visible. In Pizza, the solution process requires finding the areas and the cost of the pizzas, and comparing prices with areas or vice versa. Alternatively, one can compare areas, prices, and then

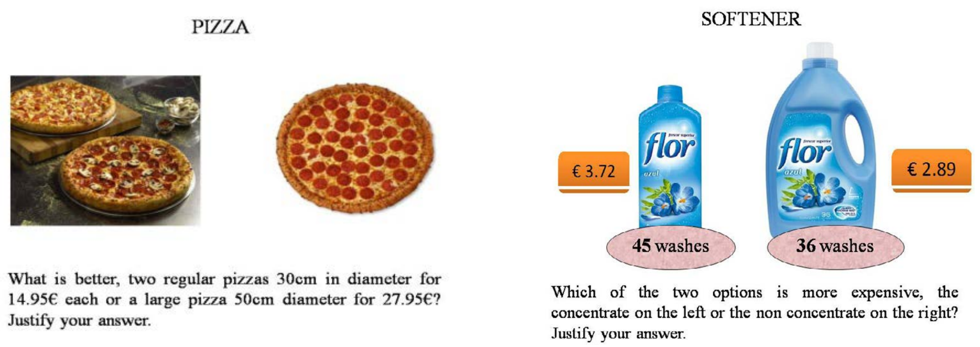

Figure 1: Pizza and Softener

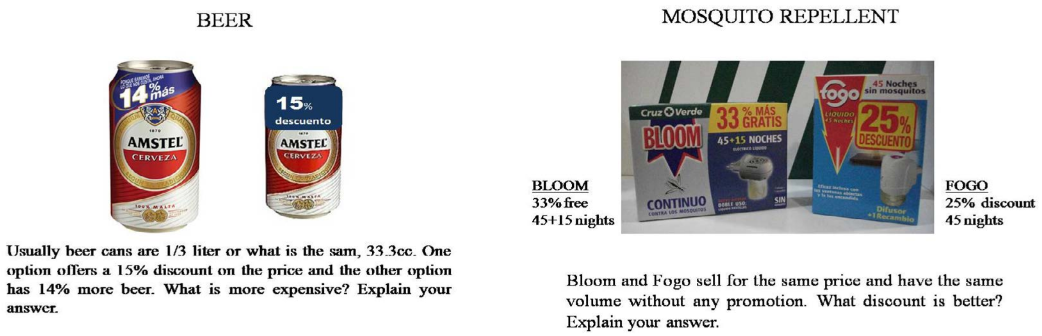

Figure 2: Beer and Mosquito repellent

compare both comparisons. In Softener, the process is reduced to compare the relative quantities €/washes. In Beer and Mosquito, it is needed to convert the gift in a discount or vice versa, i.e., to change one of the two referents. Beer task requires finding the volume of the larger can and calculating what percentage discount is equivalent to the extra volume. Once found, we can compare two discounts. Likewise, the mosquito repellent task requires finding what discount percentage is equivalent to the 33% extra free nights. Once found, you can compare the discount percentage in Bloom with the 25% discounts in Fogo. In both tasks, the reciprocal process (to transform a discount percentage into a gift percentage) is similar.

The processes that account for these transformations, calculations and norming techniques, are displayed in tables from 1 to 6, as pairs of expositions or compositions. In the case of pair of expositions, Ω is formed by the set of offers; and, in the case of pair of compositions, Ω is composed by the parts that form each offer.

As we can see in tables 1 and 2, in comparing norming ratios by quotient, $\omega_1(\Omega)/\omega_2(\Omega)$=cost/cm² or cost/washes, the unit rate (u.r.) or its reciprocal (r.u.r.) are obtained. These comparisons are the main c.c., because they show which offer pays more, less or the same compared to what is acquired (comparing how many € per cm² or per wash). Alternatively, in Pizza, while comparing the internal norming ratios, $\omega_i(\Omega)/\omega_i(\Omega)$=cost

PIZZA \ Ω	Large Pizza	Regular pizzas	Internal comp. →
ω_1: Ω → Cost	$C_L = 27.95€$	$C_R = 2·14,95 = 29.90€$	$\frac{C_L}{C_R} = \frac{27.95}{2 \cdot 14.95} = 0.94$
ω_2: Ω → Area	$A_L = \pi·25^2 cm^2$	$A_R = 2·\pi·15^2 cm^2$	$\frac{A_L}{A_R} = \frac{\pi \cdot (50/2)^2}{2 \cdot \pi \cdot (30/2)^2} = 1.39$
External comp. ↓	$\frac{C_L}{A_L} = \frac{27,95}{\pi \cdot 25^2} = 0,014 \frac{€}{cm^2}$	$\frac{C_R}{A_R} = \frac{2 \cdot 14,95}{2 \cdot \pi \cdot 15^2} = 0,021 \frac{€}{cm^2}$	

Table 1: Couple of expositions. Pizza task

SOFTENER \ Ω	Concentrate softener	Non concentrate softener
ω_1: Ω → Cost	$C_C = 3.72$ €	$C_{NC} = 2.89$ €
ω_2: Ω → Washes	$W_C = 45$	$W_{NC} = 36$
External comp. ↓	$\frac{C_C}{W_C} = \frac{3.72}{45} = 0.082 \frac{€}{wash}$	$\frac{C_{NC}}{W_{NC}} = \frac{2.89}{36} = 0,080 \frac{€}{wash}$

Table 2: Couple of expositions. Softener task

BEER \ Ω	Part free	Part paid	Internal comp. → (increase percentage)
ω_1: Ω → C_L (large can)	$F_L = 14$	$P_L = 100$	$\frac{F_L}{P_L} = \frac{14}{100} = 14\%$
ω_2: Ω → C_S (small can)	$F_S = 15$	$P_S = 85$	$\frac{F_S}{P_S} = \frac{15}{85} = 0,176 = 17\%$

Table 3: Couple of compositions. Beer task

MOSQUITO \ Ω	Part free	Part paid	Internal comp. → (increase percentage)
ω_1: Ω → Bloom	$F_{Bloom} \cong 15$	$P_{Bloom} = 45$	$\frac{F_{Bloom}}{P_{Bloom}} = \frac{15}{45} = 33\%$
ω_2: Ω → Fogo	$F_{Fogo} \cong 11$ (45-25% · 45)	$P_{Fogo} = 34$ (45-11)	$\frac{F_{Fogo}}{P_{Fogo}} = \frac{11}{34} = 33\%$

Table 4: Couple of compositions. Mosquito repellent task

BEER \ Ω	Large can	Small can
ω_1: $\Omega \to$ free part	$F_L = 14$	$F_S = 15$
ω_2: $\Omega \to$ total product	$T_L = 114$cc.	$T_S = 100$cc.
External comp. ↓ (discounts)	$\frac{F_L}{T_L} = \frac{14}{114} = 12\%$	$\frac{F_S}{T_S} = \frac{15}{100} = 15\%$

Table 5: Couple of expositions. Beer task

MOSQUITO \ Ω	Bloom	Fogo
ω_1: $\Omega \to$ free part	$F_{Bloom} = 15$	$F_{Fogo} = 11$
ω_2: $\Omega \to$ total product	$T_{Bloom} = 60$	$T_{Fogo} = 45$
External comp. ↓ (discounts)	$\frac{R_{Bloom}}{A_{Bloom}} = \frac{15}{60} = 25\%$	$\frac{R_{Fogo}}{A_{Fogo}} = \frac{11}{45} = 25\%$

Table 6: Couple of expositions. Mosquito repellent task

regular pizza/cost large pizza, or regular pizzas area/large pizza area, we see that for almost the same price, the large pizza has got more area.

When comparing the norming ratios by quotient, $\omega_1(\Omega)/\omega_2(\Omega)$=free part/total product or $\omega_i(\Omega)/\omega_i(\Omega)$=free part/part paid, after homogenizing the referent with respect to the part paid (h.p., tables 3 and 4) or with respect to the product acquired (h.a., tables 5 and 6) decimals or percentages are obtained. All of these processes are the c.c., because they show which discount or increase percentage are higher, lower or the same.

EMPIRICAL ANALYSIS OF TASKS

The empirical analysis takes into account the response patterns of students in relation to their strategies and their difficulties. Apart from the difficulties related to the c.c. other conceptual difficulties have been observed such as: the linearity (specific of Pizza) and the misinterpretation of the r.u.r. We highlight also two strategies that have appeared in the tasks and they are different from the strategies pointed in tables from 1 to 6.

Most significant alternative strategies and difficulties: Examples

Difficulty in linearity of the unit rate in Pizza task. Students adopt a relative approach, comparing external ratios and applying norming by quotient to obtain a u.r. Rather than comparing prices with pizza areas, they compare prices with pizza diameters (example 1). In this case, the difficulty is in the referent of the relative quantities that they have to compare.

> (14.95/30)·2 = 0.996; 27.95/50 = 0.559. A cm of the large pizza is cheaper. That is, the large pizza is more economical.

Example 1: Student's response to Pizza task

The student compares the relative amounts given by the ratios between the diameters and the prices, i.e. the u.r. of each offer. This strategy could be valid if there was only one item in each offer because more diameter implies more area. Note that the student multiplies the u.r. of a regular pizza by 2 [Regular: (14.95/30)·2=0.996]. This calculation suggests us that he is not aware of the invariance of the ratio.

Difficulty in interpreting the reciprocal of the unit rate. Students adopt a relative approach, comparing external ratios and applying norming to obtain a unit rate, but they interpret the unit rate in the reverse way that corresponds to the stated ratio. In this answer, the difficulty is the loss of meaning of the referent when they apply norming techniques.

> b) 100% - 15% = 85% price → 33cl; a) 14% of 33.3 = 4.662 cl more. 33.3 + 4.662 = 37.962 cl in can A; 33/85 = 0.388; 37.962/100 = 0.3796. In offer B you pay 0.388€ per cl, while in offer A you pay 0.3796€ per cl so that, offer A is cheaper.

Example 2: Student's response to Beer task

The student compares the ratios cc-acquired/percentage-paid and applies norming by quotient: 33/85=0.388 and 37.96/100=0.3796, but interprets these unit rates as what is paid per cc, which is the reciprocal of the ratio: cc acquired per unit paid. This leads to giving the opposite answer expected.

Comparison of quantities of the same nature. This strategy consists of comparing the areas, the volumes, the number of washes, the costs and the percentages among them, and if it is necessary compare the results of these comparisons. They can adopt an absolute approach (not using ratios) or a relative approach (see Table 1, internal comparison). In the first one, the comparison is rough and uses the reasoning: there is more, less or equal in one than in the other. It includes also the usual additive calculations (no example included). The answer may be insufficient (examples 3 and 4), or not (example 5). The second one has been explained above in Pizza.

This student assigns the arbitrary price of 1€ to a 33cc can. Then, the student calculates the larger can volume, 37.96cc, which will cost 1€, and calculates the cost of the smaller can after the discount, 0.85€. Finally, the costs are compared. This data is insufficient to determine which one is the most expensive can.

The student sets an arbitrary price, 10€, then, calculates 33% of 10 and adds it to the price, 13.30€. The student calculates 25% of 10 and deducts it from the price, 7.50€, and compares it with the discount given.

> A large pizza is better because the sum of the area of two regular pizzas is lower and, moreover, is more expensive: $2 \times 30 = 706.5 \times 2 = 1413$ cm^2; $\pi \cdot r^2 = \pi \cdot 15^2 = 706.5$ cm^2; $\pi \cdot r^2 = \pi \cdot 25^2 = 1962.5$ cm^2

Example 5: Student's response to Pizza task. Adequate

The student compares the difference between the areas with the difference between the prices. It is sufficient only in this case because the data favor it.

Cost comparison increasing or decreasing the matching amounts. Students establish the total cost or the cost of a unit of a product. After that, they determine if by increasing or decreasing the amount of the product to match the other you get the same total price. While in the intermediate process students can use relative quantities (example 7), they finally compare absolute amounts. It seems to be that this strategy has not been identified in the previous research.

This student fixes an arbitrary price of 40€ to the non promotional products. He finds the cost of Fogo, 30€ after the discount. Then, he calculates what the Bloom reducing to 45 nights would cost and if it keeps proportionate to their offer price. It is concluded "it is the same because in both of them, 45 nights cost 30€".

A) is more expensive because although there is 14% extra beer, in the other one costs less although there is less beer.

A) 100%→33.3; 14% → x; x=4.662cc 33.3cc + 4.662cc = 37.96cc; 1€→37.96cc

B) 100% →1; 15% → x; x=0.15€ 1 - 0.15 = 0.85€

A) 37.96cc→1€; B) 33.33cc→0.85€

Example 3: Student's response to Beer task. Inadequate

Bloom would have the best discount because the free added percentage implies paying 33% more in the total cost.

(1)10 → 100; x → 33; x = 33·/100 = 333/100 = $\underbrace{\underline{3.30}}_{\text{€ discounted}}$+10 = $\frac{13.30€}{10}$

(2)10 → 10; x → 25; x = 250/100 = $\underbrace{\underline{2.5}}_{\text{€ discounted}}$ 10-2.5 = $\frac{7.50€}{10}$

Example 4: Student's response to Mosquito repellent task. Inadequate

BLOOM	← 40€ →	FOGO	
60 nights		45 nights	40 → 60 → 40·45 = 1800:60 = 30€ x →45
		25% of 40 = 10 40-10 = 30€	Is the same because 45 nights cost 30€ in both of them.
Price 40€		Price 30€	

Example 6: Student's response to Mosquito repellent task

At first glance, it is obvious that the concentrate is more expensive, as it costs 3.72€, which is 0.082cents per cup. On the other hand, the 36 wash product has an added cost of 0.080 for each of the 9 cup difference and results in a price of 3.61€, making it cheaper than the concentrate and therefore, less expensive than the one on the right.2.89/36 = 0.080 2.89 + 0.72 = 3.61; 0.080·9 = 0.72→cup difference from the concentrate.

Left: 45 cups = 3.72. Right: 45 cups = 3.61.

Example 7: Student's response to Softener task

This student finds the difference between the 2 softeners: 9=45-36; calculating both unit rates: 3.72/45=0.082€/wash the concentrate softener, and 2.89/36=0.080€/wash the non concentrate softener, and uses one of them to calculate the total cost that the non concentrate softener would have if there were a 9 wash increase. If he had 36+9 washes, it would cost 2.89+9·0.080=3.60€, which is cheaper than the 3.72€ concentrate.

RESULTS

The absolute frequencies are displayed in Table 7. The columns (from left to right) show: the number of students that use the strategy of u.r., those who have difficulties in this strategy, students who homogenized the referent, the participants who compare the quantities of the same nature, those who use the strategy of cost comparison after matching amounts, and the group of qualitative answers, random, blank, etc (others).

Regarding the critical components of the tasks, we observed that in Beer and Mosquito there are more students who compare absolute quantities instead of relative quantities. Nevertheless, in Pizza and Softener tasks the contrary occurs. Moreover, there are very few students who homogenize the referents (4 in Beer, 2 in Mosquito). They may not interpret the gift like a discount or vice versa. It highlights how students are inclined to calculate the u.r. although it is not needed in this case. If we focus on the strategies and difficulties identified, in Pizza there are few students who calculate the u.r. or its reciprocal without difficulties because of the linearity. In fact, 175 students calculate the u.r. with the diameter instead of the area. In contrast, in Softener the u.r. strategy dominates as we expected. Only 23 students show difficulties calculating the r.u.r. Finally, there are students who give blank, random, qualitative or incomplete responses, especially in Beer (82 students) and Mosquito (44 students).

CONCLUSIONS

Predominance of the unit rate strategy in Beer and Mosquito tasks can be due to the lack of flexibility and the application of a mechanical rule learnt at school. The use of this strategy implies to assign an arbitrary price although it is not needed. It does not imply that their responses are wrong, but it is an indicative of a price-dependence attitude. Note that, there are other more efficient strategies. In addition, resistance is also observed in accepting that an increase percentage can be interpreted as a discount and vice versa. Moreover, there are students who misunderstand the r.u.r., suggesting a mechanical knowledge of the rule. Finally, the use of linearity is widespread in Pizza, and very few students perceive the invariance of the ratio calculating the u.r. of two regular pizzas instead of the u.r. of one regular pizza. They do not realize that both of them are equivalent.

These results confirm the deficits in the relative thought of pre-service teachers, and that, according to Ben-Chaim, Ilany & Keret (2002, p. 81), their knowledge is frequently technical, unrelated and incoherent. Moreover, the difficulties shown by the

Strat. / Task	Unit rate		Difficulties with u.r.		h.p. o h.a.	Same nature		Matching amounts	Others
	u.r.	r.u.r.	Recip.	Linearity		Absolute	Relative		
Pizza	13	1	2	175	-	87	1(lin.)	42	20
Softener	215	5	23	-	-	41	1	33	23
Beer	103	3	16	-	4	113	-	20	82
Mosquito	107	2	11	-	2	168	-	7	44

Table 7: Tasks' results

students are not the algorithmic aspects and norming techniques, but in conceptual aspects, ratio referents and the "price-dependence" when they are comparing discounts in commercial offers. The next step is to design a teaching sequence that helps the students to widen their knowledge of ratio and, according to Lamon (2012, p. 107), to help them to develop flexibility in situations like the best buy problems, to encourage multiple correct strategies and to discuss which strategies are easier, faster or more reasonable.

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Replacing persistent counting strategies with cooperative learning

Uta Häsel-Weide

University of Siegen, Siegen, Germany, haeselweide@mathematik.uni-siegen.de

Persistent counting strategies often come along with a weakness in arithmetic. Due to this, the central objective is to replace counting strategies by teaching how to realize, recognize and use structures. Within the context of the project ZebrA1 (Zusammenhänge erkennen und besprechen – Rechnen ohne Abzählen) cooperative learning settings were developed to encourage children to use different interpretations of patterns and structures instead of counting. Results show that working in pairs of children with heterogeneous competences is productive for both partners. Especially activities like "comparing" and "sorting" are suitable to make children interact about mathematics and help them become aware of mathematical structures.

Keywords: Counting, cooperative learning, difficulties in learning mathematic, design research.

INTRODUCTION

Meta-analysis verified that forms of cooperative and peer-assisted learning at elementary schools go hand in hand with stronger effects in content knowledge than in traditional forms of teaching (Rohrbeck, Ginsburg-Block, Fantuzzo, & Miller, 2003). The learning performance of children with learning difficulties improves especially in well-structured, heterogenous and tutorial learning environments (Gillies & Ashman, 2000). Research also seems to show that children with difficulties in learning mathematics can profit from cooperative learning. For that reason cooperative learning is chosen for an intervention intended for children who use counting strategies as a persistent strategy. It is well known that persistent counting is one of the central symptoms of weakness in arithmetic. It is thus necessary for further success in mathematics that children replace counting strategies with mental computation and the use of adaptive strategies.

This paper intends to show how learning environments could be of such design that all children profit from cooperative situations. On the one hand children with a weakness in arithmetic get a first understanding of mathematical structures and relations which are important to replace counting strategies (Häsel-Weide & Nührenbörger, 2013). On the other hand children who already use mental computation and adaptive strategies can deepen their knowledge and improve their competences in verbalising and reasoning. Working together in pairs of children with heterogeneous competences may be a chance for both of them.

THEORETICAL UND EMPIRICAL STARTING POINTS

Replacing persistent counting strategies

The central symptom for a weakness in arithmetic is the persistent use of counting while problem-solving. Children who use counting as a main strategy in grade 2 and beyond often develop wide problems in mathematics since counting is not a calculation strategy that can be built up to work in higher number spaces. Besides, persistent counting often comes along with a mechanical, non-reflected procedure as well as isolated problem solving. There is a risk that the missing insights evolve to comprehension problems in mathematics education.

To replace persistent counting strategies children have to develop alternative strategies like mental computation or the use of derived-fact strategies. Most importantly an understanding of numbers and operations has to be built up which than can be followed by exercises to memorize the central problems. Fostering

1 Recognizing and speaking about connections – calculating without counting on

children in replacing persistent counting strategies should enable them to represent numbers and operations and imagine them as well as to recognize and use the relationship between numbers and problems. The central objective is also a comprehensive structural view on numbers and operations. This means in effect that those children who have problems in recognizing structures need to be enabled to see them. In practice children with persistent counting strategies should be enabled to realize, decompose, represent and describe numbers as structured quantities. The following four aspects characterise the comprehensive objective: (1) The ordinal conception of numbers – which often comes along with counting procedures – needs to be complemented by a conception of quantities; especially the part-whole-concept needs to be understood. (2) Counting in ones needs to be extended with counting in steps and furthermore be used to identify quantities. (3) Children need to figure out that addition and subtraction come along with changes of quantities. In a fourth step based on this, (4) basic problems can be memorized and first relations could be used in calculation (Häsel-Weide, Nührenbörger, Moser Opitz, & Wittich, 2014). All mentioned aspects are not "extraordinary" ones as they contain competences that are essential for all pupils to be achieved during the first years in elementary school. The competences are so fundamental that they are taught not only in grade 1 but are continuously taught and extended on in the next grades of primary school. The respective lessons are suitable for cooperative learning because children with heterogeneous competences can work on the same contents but on different levels of understanding. In this setting, children with persistent counting strategies can profit from the work of their partner that might probably be different, more elaborated and structure-focused.

Therefore the learning environments need to be designed in a way which allows working at various levels. In addition the children should be motivated to recognize and talk about the mathematical structures.

Cooperation and interaction in mathematics education

As mentioned above, cooperative learning seems to be a successful teaching method. This is true especially for forms of cooperative learning which are strongly structured and use clear and common methods (Tarim & Akdenzi, 2008). Nonetheless the way children interact in cooperative situations is vitally important. The success of cooperative learning seems to depend on interaction activities like verbalising, defending, asking and arguing (Pijls, Dekker, & van Hout-Wolters, 2007). Learning opportunities occur for children if they interpret actions and comments of others.

Children need to learn to communicate about mathematics, to describe their solution process, to defend their ideas or to question the idea of another child. In classroom interaction as well as in cooperative learning (young) pupils may need the moderation of the teacher. Teachers are invited to ask open questions and give children the opportunity to show their insights and perceptions. The best way to support the cooperation process of students is to give them help in structuring the cooperative process instead of supporting the finding of the (right) solution (Dekker & Elshout-Mohr, 2004). Teacher activities should lead children to stay and participate in discussions and support them to disagree with each other (Wood, 1999).

All in all for the designing of cooperative learning environments it is also important that the tasks and the methodical setting initiate different perceptions which will be communicated between the children. The methodical setting of cooperative learning needs to be complemented by a content structure in the tasks encouraging different views. In order to replace counting strategies and focus on relations between numbers and operations cooperative learning environments need to initiate different perspectives of numbers, number representations and operations.

DESIGN OF THE STUDY

The present study is a part of the project ZebrA (Zusammenhänge erkennen und besprechen – Rechnen ohne Abzählen). It combines the design of learning arrangements and the empirical research of the interaction and learning processes that can be reconstructed when children work in their environments. In that way the design of the study follows the idea of mathematics research as design science (Wittmann, 1995) and comes along with many elements of design research (Gravemeijer & Cobb, 2006).

According to the idea of design science 20 lessons – cooperative learning environments – have been constructed to foster children in replacing persistent counting strategies (Häsel-Weide et al., 2014). The learning environments focus on an understanding, demonstration and imagination of numbers and operations as well as on realizing the relations between them. The children learn the cooperative methods by working in the learning environments.

The environments are used at the beginning of grade 2 in primary school (ages 7 and 8) or grade 4 in special education schools. This period seems to be a proper time for two reasons: (1) In Germany the children in grade 1 work with numbers up to 20, figure out relations between those numbers and add and subtract with numbers up to 20. Obviously, children are allowed and encouraged to make experience with higher numbers, but according to the German curriculum the number space is opening up to one hundred at the beginning of grade 2. Here children orientate themselves in the new number space, they work with representations of numbers and focus on the relations between them. These contents run in parallel to a deeper understanding of numbers and operations in the lower number space. (2) The limits of counting strategies become very obvious when children operate with numbers up to one hundred. If the familiar counting strategies become exhausting, children can easily be encouraged to try an alternative strategy (Gaidoschik, 2012).

The study was realised from September to December 2010. All children of the class are taking part in the lessons. Each child who uses counting as its main computational strategy works with a partner who uses different strategies. The pairs are put together by the teachers considering their own estimation of children`s competences and strategy utilization, as well as the results from a test that was developed, realized and analysed in the scope of further researches of the project ZebrA.

The project is accompanied by two empirical studies which allow focusing the replacement of persistent counting strategies from different empirical points of view. Whereas the quantitative study researches the effects of cooperative fostering the study presented in this paper focused on the interpretations of children dealing with the problems and discussing with the partner. We are interested to explore if and how children using counting strategies can be encouraged to consider and use mathematical structures. Therefore the work of five children using counting strategies as persistent computation (three boys and two girls) and their partners – belonging to four different classes and three schools – was video-graphed in ten lessons of the ZebrA-project. Corresponding transcripts have been made and interpreted by a group of researchers. The analyses has been compared in an interactive way with empirical findings of other studies and theoretical approaches (Häsel-Weide & Nührenbörger, 2013). Only the results are presented in this paper.

SELECTED RESULTS

Learning environments

In the tradition of mathematics education as design science the designed arrangements are central part of research output (Wittmann, 1995). So the design principles are pointed out below and it is shown exemplarily in which way they become apparent in the learning environments.

Structure-focused view on numbers and operations

To replace counting strategies children need – among other things – to figure out that operations change quantities. Subtraction needs to be understood as taking away or determining the difference. To help children build up this idea a transparency is used (Figure 1). The transparency allows covering an amount of dots at one go. So subtraction in the model of taking away is linked with the action of "covering" and not by "counting" (back).

The transparency can be used to cover concrete dots or dotfields which allows finding the difference by a quasi-simultaneous determination of the uncov-

Figure 1a & b: Representation of 15-5=10 and 35-5=30 ered dots. The children are taught in two learning environments to handle the transparency. First, in a dotfield only the ones are covered, so that problems, e. g. 25 – 5 = 10, are represented and solved. In this environment called "subtraction to tens" we focus on the decimal structure of the number systems. The children may realize that these types of problems are easy to solve and that they can manage them without counting – even in higher number spaces. In a second learning environment other "easy" subtraction problems, e.g. "minus 1", "minus 10", "minus tens" and "minus ones" are focused on.

Sophisticated tasks

The contents of the developed learning environments for replacing counting strategies are designed for all children of a class. Children with persistent counting strategies have the possibility to work on a central understanding while other children deepen their competences. But therefore the given task has to be so complex that different levels of understanding and working are possible. Learning environments according to this so-called "natural differentiation" (Krauthausen & Scherer, 2013) are characterised by tasks that are (partly) open and/or include descriptions, explanations and arguments. Open problems allow the children to pick up on their own mathematical ideas, to deepen their understanding of contents or problems and sometimes to extend their own limits.

In the learning environment "subtraction to tens" each pair of children gets a couple of cards. There are some cards with quantities until 20 and some with quantities up to hundred (Fig 2). The children are free to choose which cards and how many. They are asked to cover the ones and note the corresponding subtraction problem. If children manage already without the transparency, they are free to do so.

Cooperative settings and discursive tasks

According to the research result that successful cooperation is based on structured settings, two cooperative are developed. The cooperative setting "path fork"[2] starts with a period of individual work which is followed by a period of working together on a further task. This task is build upon the individual work and focused on interaction about the insights and strategies. In the second cooperative method "seesaw"[3] the children work together on a problem from the beginning. They undertake different activities which are referred to each other. In the consequence they pick up the interpretation of their partner and carry on. The tasks for the cooperative method need to allow different interpretations so that a discussion can follow.

The learning environment "subtraction to tens" follows the cooperative setting "path fork". First, the children work with the cards on their own as described above. Then they collect their problems with the difference 10 on a worksheet (Figure 3). In the cards not all numbers from 10 until 19 are represented as

2 Weggabelung

3 Wippe

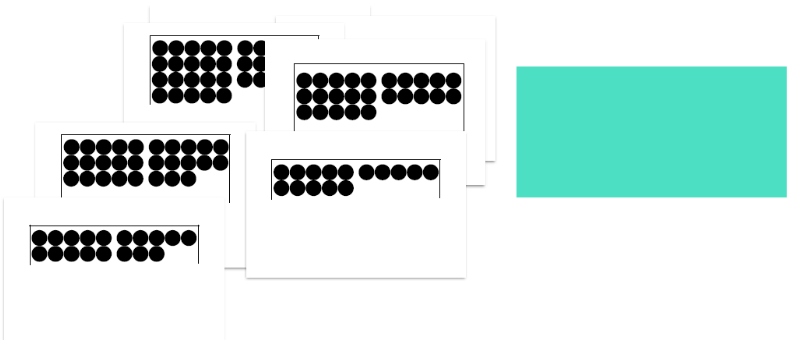
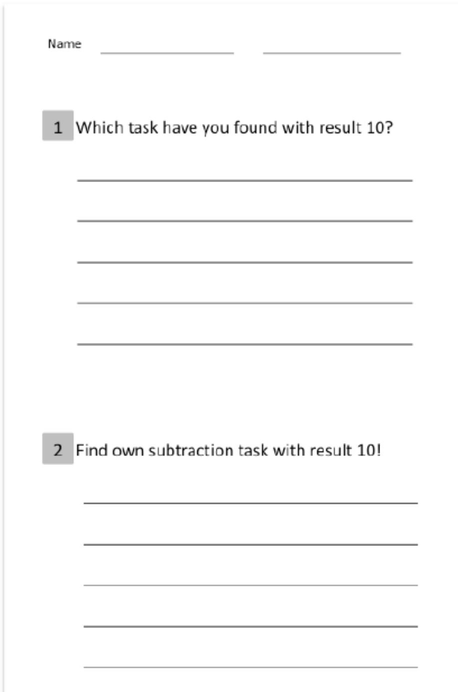


Figure 2: Material for sophisticated learning



Name

1 Which task have you found with result 10?

2 Find own subtraction task with result 10!

Figure 3: Cooperative worksheet

quantities, so the children are challenged to find the missing ones. They could use the discovered structure or find other problems such as 30 − 20 = 10 or 45 − 35 = 10. The freedom in the interpretation of the formulation on the worksheet allows and probably stimulates interaction.

In some environments "discursive tasks" are used to strengthen the difference and the discussion. Children of a pair are not given the same tasks but tasks which refer to each other (Figure 5). After each of them has solved their card they should compare and describe common and different features. Moreover "sorting" is used as an alternative method to induce different interpretations and an interaction about them (Häsel-Weide & Nührenbörger, 2013).

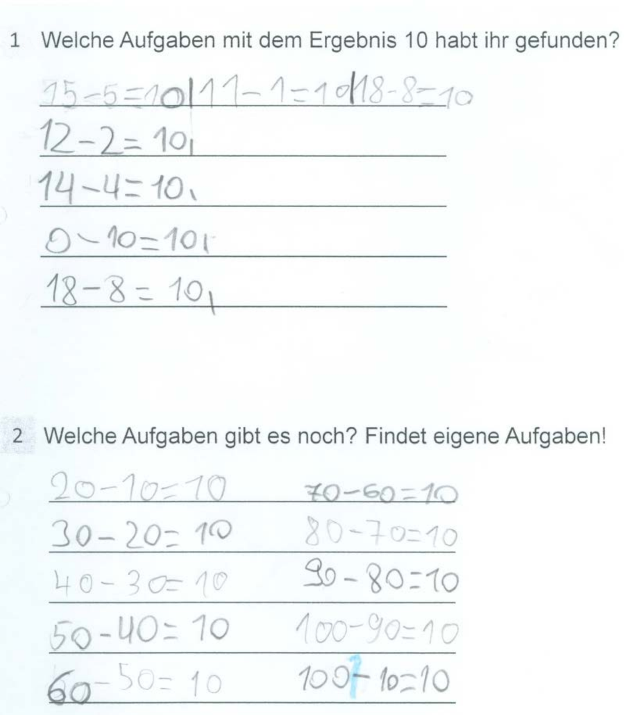


Figure 4: Document of Kolja and Medima

Results of the analyses of cooperative situations

The interaction of the children is analyzed to figure out in which way the interpretations of the children are affected by the cooperative interaction with their partners. Especially the interpretation of relations between numbers and problems in the interaction are analyzed. As pointed out above, it was of interest if and how the interpretations of the children with counting strategies differ from the interpretation of their partner and if these children modify the interpretations in the course of interaction. In this paper selected *results* of the analyses are presented and *illustrated with examples*. For a deeper insight into the interpretation process see Häsel-Weide and Nührenbörger (2013).

All children of the study were able to work and to cooperate in the learning environments. *Children with persistent counting strategies bring in own ideas and act as real partners.*

Example 1: After Kolja and Medima have noticed the problems according to the cards in the learning environment "subtraction to ten" they are now asked to find new problems. Kolja, a child with persistent counting strategies, suggest the problems 20 − 10 = 10 and 30 − 20 = 10. It is not clear if he wants to create a new, different pattern of problems or if he expresses common problems with the result ten. In the interaction his partner Medima notes the problems without questioning and the children find a structured column of problems in turns (Figure 4). On the basis of Kolja's idea they find a structured sequence of problems. Even if Kolja was not aware of the new structure he suggested, he could pick it up and find the next problem. Since Medima seems to realize the structure immediately nobody feels a need to para-

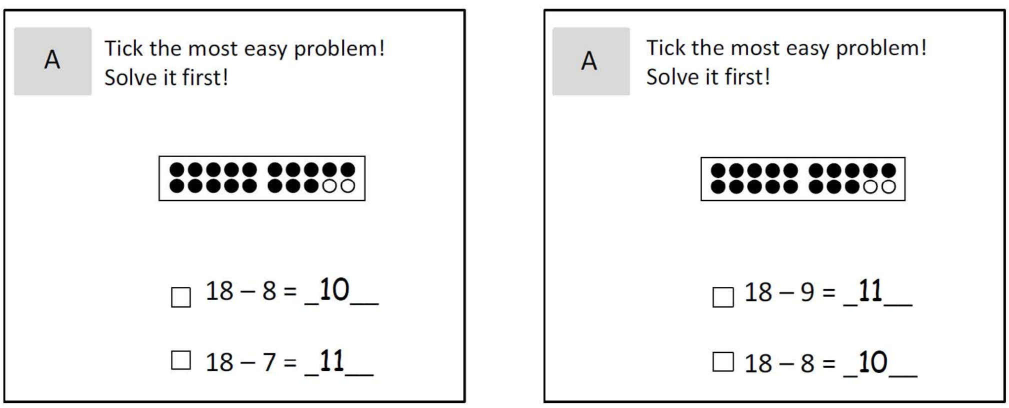


Figure 5: Discursive tasks with reconstructed results of Thomas and Max

phrase the action. Both children are pleased to note the problems and also do not notice the mistake.

The example as well as the results of the analyses made clear that children need reasons to negotiate different interpretations (Nührenbörger & Steinbring, 2009). *First of all comparing and sorting as formal cooperative settings function as reason to interact.* In informal situations some interactions come up by mistakes.

Example 2: Asked to compare the cards (Figure 5) in the second period of the cooperative setting "path fork", Thomas said:

Thomas: Hey Max, Max, I have noticed something. Here I have ten above [*points at the 10 on the left card*] and you have ten here below [*points at the 10 on the right card*].
Max: Oh. I have notice something, too. I have something different, too. There must be nine [*rubs out the result 11 and notes 9*]. Nine. Look, because nine plus nine equals eighteen.

Thomas notices that the difference ten can be found on both cards on different positions. He focuses on a concrete mathematical sign and describes its physical position. But it seems as if his interpretation causes a process in Max's thinking. Perhaps Max realizes that there are two other equal results, but not two other equal problems. His statement is not clear at that point. However, it seems to be a consequence of Thomas's comment that Max now realizes that he has made a mistake. He corrects it and explains to his partner why the new result should be correct now. Here Max uses the inverse relation between the addition problem 9 + 9 = 18 and the given problem 18 − 9 = 9.

In this episode, relations between problems are realized and described in the cooperative setting, probably initiated through the discursive tasks. It is open how far Thomas, the child with persistent counting strategies, understands Max's argumentation and realizes the relations and differences for himself, because the interaction ends at this point. But there is a chance that he picks up this argumentation later on.

Thomas's interpretation which focuses on corresponding signs was one of the typical interpretations of children using persistent counting strategies. According to results from Gray, Pitta and Tall (1999) the analysis show, that *children with persistent counting strategies focus often on similar signs*. Furthermore the *children focus on relations between numbers instead of relations between problems* as Thomas did in comparing the cards as well.

Thomas: Hey, here is nine [*points on the "9" in the problem 18 − 9 = on the right card*] and there it is eight [*point on the "8" in the problem 18 − 8 =*]. It is one more and there [*point on the problem 18 − 7 =*] it is one less.

Here, Thomas describes the relations between the subtrahends. He seems to formulate their relation based on a cardinal concept of numbers.

It becomes clear that the children see correct aspects, however they do not seem to realize the relations between problems as whole. As a consequence it is difficult for them to use the relations for deriving. Realizing corresponding signs and first relations between numbers is essential but not sufficient for developing derived fact strategies. It needs to be pointed out that children who already solve problems *without* counting, formulate relations between problems only in few episodes. Mostly they also focus on concrete objects and relations between numbers (Häsel-Weide, 2013). Analyses shows that even teachers are pleased if children do so and do not ask them to figure out the consequence of the relation between numbers for the relations between problems. Probably children with counting strategies benefit less than possible from the cooperation because the interpretation of their partner did not differ essentially from their own.

CONCLUSION AND OUTLOOK

The presented study aims at developing and researching cooperative learning environments which enables a structure-focused view on relation between numbers and problems for all children in grade 2. The design results and the results of qualitative research show that it is possible and productive for children with heterogeneous competences to work together. Children with counting as persistent strategy act as real partners in the cooperation and bring in own interpretations and solutions. Although informal cooperation processes are observed the formal cooperative settings, especially the activities "comparing" and "sorting", are suitable to make children interact about mathematics.

The children's interpretations show that the children focus on relations between numbers. In the next step children need to focus on relations between problems. This could only be observed in very few episodes of the study. Since even children without problems in mathematics focus mainly on the relation between numbers, further research has to show if another combination of pairs (perhaps with partners of higher age) or a more direct moderation by the teacher make all children recognize and describe structures between problems. Besides, it has to be figured out how far an explicit focus on relations between problems goes along with non-counting procedures (Gaidoschik, 2012).

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Students’ argumentation schemes in terms of solving tasks with negative numbers

Mathias Hattermann and Rudolf vom Hofe

Bielefeld University, Faculty of Mathematics, Bielefeld, Germany, mathias.hattermann@uni-bielefeld.de

In the context of an experimental project targeted at improving the teaching and learning of negative numbers, we explored the argumentation schemes of students in terms of solving respective tasks. The test items used to record the learning progress of the students were taken from the large-scale PALMA-study. While learning gains were above average, the argumentation schemes of our experimental students show both, expected and unexpected patterns. Interestingly, we find well performing students with a preference for metaphorical reasoning instead of a mixed argumentation including formal reasoning.

Keywords: Negative numbers, lesson studies, metaphorical reasoning.

THEORETICAL FRAMEWORK

The Theory of Grundvorstellungen (GVs)

The theory of Grundvorstellungen (GVs) has a long tradition in Germany (e.g., vom Hofe, 1995). It reflects on the fact that the intuitive level of thinking is often responsible for the understanding and building up of mathematical knowledge as well as for problems in mathematical thinking in the sense of Fishbein:

> It is very well known that concepts and formal statements are very often associated, in a person’s mind, with some particular instances. What is usually neglected is the fact that such particular instances may become, for that person, universal representatives of the respective concepts and statements and then acquire the heuristic attributes of models. (Fishbein, 1987, p. 149f.)

In contrast to the opinion that mathematics is pure logic, GVs take social aspects into account as well as the environment in which mathematical concepts are built up. The theory of GVs has its anglo-american correlate in the term concept image which forms a fundamental part of Tall and Vinner’s theory of mental models:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures. (Tall & Vinner, 1981, p. 2)

The concept image must be differentiated from the concept definition that consists of formal definitions dealing with mathematics. The theory of GVs (e.g., Kleine, Jordan, & Harvey, 2005) exemplifies the given idea of concept image in more detail and widens the perspective to descriptive and prescriptive utilization.

> GVs as a prescriptive notion describe adequate interpretations of the core of the respective mathematical contents which are intended by the teacher in order to combine the level of formal calculating with corresponding real life situations. In contrast, the term GV in descriptive empirical studies is used also as a descriptive notion to describe ideas and images which students actually have and which usually more or less differ from the GVs intended by mathematical instruction. (vom Hofe et al., 2008, p. 49)

GVs can be formulated in different contexts, such as fractions (part of a whole, operator, ratio), subtraction (taking away, supplementing, comparing) or functions. On the one hand the descriptive and prescriptive characters of GVs allow on the one hand the formulation of desirable GVs in different contexts and on the other hand the testing of individuals’ GVs .

GRUNDVORSTELLUNGEN AND METAPHORICAL REASONING

Metaphorical reasoning is fundamental to human thinking and can lead to the construction of mental models (English, 1997). According to Lakoff and Nunez (1997) metaphors can serve as a tool to understand difficult and new concepts in mathematics. They see mathematics on the one hand as a product of human imagination and on the other hand as a product of the embodied mind. Sfard (1994) emphasizes that metaphors allow the translation of bodily experiences into abstract mathematical ideas. This means that a metaphor maps from a source domain (experiences) to a target domain (mathematics) and theses experiences help to understand the abstract ideas in mathematics. Lakoff and Nunez (1997) propose four grounding metaphors to understand basic arithmetic. These four grounding metaphors (*Motion along a Path* (numbers as point locations or movements), *Object Collection* (bringing together, taking away), *Object Construction* (combining, decomposing), *Measuring stick* (comparing) link structures from every-day life to mathematics. But these grounding metaphors are insufficient to handle operations with negative numbers, so Lakoff and Nunez (1997) propose to stretch these metaphors to be useful for the explanation of the enlarged number domain including zero and negative numbers, aware of the fact they get more contrived the more they are stretched. In this respect metaphors can be used to build up GVs for different contexts in mathematics. However, it is well known that the use of metaphorical reasoning also has disadvantages since it is less efficient compared to facts or algorithms. Thus, it is important to investigate the relation between prescriptive GVs of an individual that is based on metaphorical reasoning (in our case a card game called Plus-Minus-Game) and the competence of the individual to solve computational tasks.

THE PALMA-PROJECT

The Project for the Analysis of Learning and Achievement in Mathematics (PALMA) focused on students' development in mathematics throughout grades five to ten in Bavaria, Germany (vom Hofe et al., 2008), with probands of eleven to sixteen years of age. The theoretical framework stressed the differences between the performance of algorithmic operations and the activation of basic concept images/GVs. Because of its longitudinal design, the PALMA study allowed new insights into the achievement development of students as well as into the impact of modelling competences that require the utilization of basic concept images/GVs. PALMA included annual assessments between 2002 and 2007. About 2000 students from 83 classes and 42 schools participated in the study so that the development of achievement was observed over a period of 6 years (vom Hofe et al., 2008).

CURRENT RESEARCH ON NEGATIVE NUMBERS

The genesis of negative numbers in mathematical history is of historical-cultural significance (Schubring, 1986). Since decades negative numbers have been part of the curriculum in many countries and for Germany it was Freudenthal who pointed out that dealing with them must be studied in detail (Freudenthal, 1973). He talks about didactic models of negative numbers, a problem that is still relevant today. Nearly all existing models dealing with everyday life and negative numbers contain specific insufficiencies, so that an ideal model for their teaching and instruction does not exist. Fishbein takes reference to the fundamental article by Glaeser (1981) and refers to this issue as follows:

> The difficulty of accepting the negative numbers as meaningful mathematical entities derives from the difficulty of identifying a good intuitive, familiar model which would consistently satisfy all the algebraic properties of these numbers, says Glaeser. As a matter of fact, such a model does not exist. One may create some models, but only by using a system of artificial conventions. (Fishbein, 1987, p. 100)

Current research in Germany concerning negative numbers is insufficient. Nevertheless, schoolbooks around the world (with only a few exceptions) promote the utilization of didactic models such as temperatures, balance-debt-models or elevator-models. With regard to our focus of interest, two publications are of peculiar interest. Chiu (2001) analyzes the utilization of metaphors (in the sense of Lakoff and Nunez 1997 standing in line with the discussed theories of GVs and concept images) for the explanation of computations with negative numbers in a quantitative study and with the help of additional clinical interviews. He concludes

> ...that both novices and experts have the same arithmetic metaphors but use them differently. [...]. Experts used metaphors less often in favor of more efficient methods. Both used metaphors when they faced difficulties. However, novices had more difficulties and used metaphors more often. (Chiu, 2001, p. 113)

In her work with 99 students Kilhamn (2008) also examines forms of metaphorical reasoning. 23 probands justified their answers to the task "-3 – (-8) = " by metaphorical reasoning using the thermometer, money debts or movements along a number line in their explanations. The result of the study is surprising because all students who solely referred to metaphorical reasoning failed (n = 14), whereas every student who used both metaphorical reasoning and arithmetic rules succeeded (n = 9) in solving the task -3 – (-8) = (Kilhamn, 2008, p. 6). In spite of the small amount of probands, it seems as if both, metaphorical reasoning and formal computations are substantial elements when dealing with tasks related to negative numbers.

RESEARCH PROJECT: DESIGN, QUESTIONS AND METHODOLOGY

Design

Our current project focuses on the generation of specific GVs and argumentation schemes based partly on metaphorical reasoning with regard to negative numbers in a long term-study. As the cited studies show, GVs as well as metaphorical reasoning are very relevant for the effective learning of negative numbers, even though a formalized understanding must complete the GVs. It is known that the stretching of the grounding metaphors (see the theoretical part) to handle negative numbers is problematic and yields not always to the desired results, so we will use a card game to build up GVs and to research students' argumentation schemes for the addition/subtraction of integers. Our project, lasting from 2012 to 2015, is based on a collaboration between Bielefeld University and the Laboratory School Bielefeld. The latter is innovative with respect to both, its educational profile as well as its Teacher-as-Researcher Model (Hollenbach & Tillmann, 2009). This model allows teachers to easily perform educational or subject-related research that compensates for parts of their teaching obligations. Three teachers and three researchers have collaborated to plan a 12-weeks-unit of instruction for negative numbers in grade 7 (n = 21 students), in which GVs play a fundamental role. The unit is divided into three parts: 1. introduction, 2. addition/subtraction, 3. multiplication/division of negative numbers. In the second part we introduced the Plus-Minus-Game which is explained in the next paragraph. It is designed to build up GVs and to foster metaphorical reasoning for the addition and subtraction of negative numbers. Teachers and researchers worked together on the methodological basis of Lesson Studies (e.g., Hart, Alston, & Murata, 2011). This collaborative method has proven to be effective with regard to teachers' professional development.

The Plus-Minus-Game

The game consists of a dice with a green, a blue, a red, a yellow, a black and a white face, 11 green and 11 red cards. The 11 green cards are labelled with the numbers from 0 to 10. The 11 red cards are labelled with the numbers from -10 to 0. The game is played by three to four players. After having shuffled the cards, you put a deck of green cards and a deck of red cards on the table. Every player draws a card and puts it on the table without covering it. The youngest player begins to throw the die. For each colour the die shows, there are specific instructions what to do. Green: Take a green card from the deck. Blue: Give any green card to your left neighbor. Red: Take a red card from the deck. Yellow: Give any red card to your left neighbor. Black: Give the red card with the highest absolute value to your left neighbor. White: Give the red card with the lowest absolute value to your left neighbor. The aim of the game is to get the highest score in total. The game is over when all cards are collected. After playing the game, we introduced the notation presented in Figure 1 to motivate the utilization of brackets as claimed by Malle (2007). We used brackets to differentiate between a card value and a score. In an equation, we interpret the first summand as a score (without brackets), the second summand as a card value (with brackets) and the result as a score (without brackets). Figure 1 focuses on the notation in addition and subtraction tasks. The interpretation of the minuend as a score in a subtraction task is crucial. Example: 3 – (-6) = 9 Interpretation: The player has several cards (e.g. (5), (-6) and (4) which are added up to 3). He has to give away the card (-6), so the player's new score is 9. If someone interprets the minuend as a card value instead of a score, the interpretation will fail because it is not possible to give away the card (-6) while holding only the card with value (3). Problems resulting from such misinterpretations are analyzed in detail in Hattermann (2013). For more information concerning teaching experiences see Hattermann, vom Hofe and Viehmeister (2014).

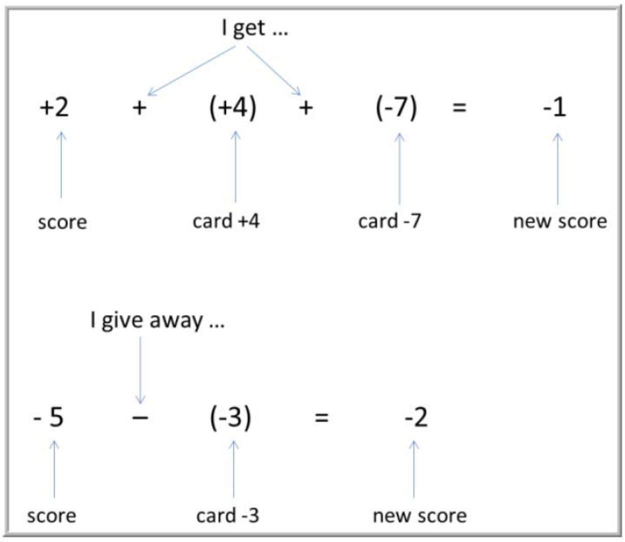


Figure 1: Notation and interpretation for addition and subtraction tasks with integers

Research questions

In this paper we focus on three selected research questions of the project. Since the students who took part in the PALMA study did not participate in lessons focusing on the development of GVs, as it was the case with our experimental group, we formulate the first question: 1. How does the experimental class from the Laboratory School perform in answering the PALMA-items on negative numbers in comparison to the PALMA-students after the instruction unit on negative numbers? 2. What argumentation schemes (GVs) can be identified in the students' explanations for the tasks "(-5) + (-7) =" and "(-9) − (-4) =" after having finished the course of addition/subtraction of negative numbers? 3. Are there any particular interrelations between both, the argumentation schemes identified and the students' success by solving computational tasks?

Methodology

After the second and the third part of the course the students took part in an evaluation in which they computed different tasks and answered questions that sought to identify underlying conceptions and GVs. In the first evaluation students had to answer the following question:

> Imagine your classmate Jacob was ill last week and he was not able to attend class. Write a short letter in which you explain to him what has to be done in the following tasks and explain why: a) (-5) + (-7) = and b) (-9) − (-4) =. You may use all devices for an explanation that we used in class.

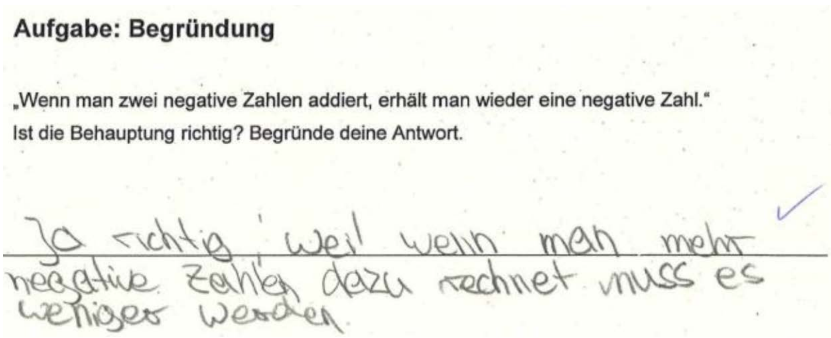
The written letters were analyzed with regard to the argumentation schemes used by the students by means of a qualitative text analysis. In a first analysis the research group analyzed together five letters to create categories of argumentation schemes. In the following two researchers analyzed independently students' letters to Jacob and decided for one of these categories. In a last step inconsistent classifications were discussed and the categories were revised. In a second step the identified individual argumentation schemes were confirmed with semi-standardized interviews. In these interviews tasks as "(-3) + (-7) =" occurred (as in the letter to Jacob) and students' were motivated to solve these tasks and to explain their argumentation explicitly. They could use a number line, the plus-minus-game or other explanations. The video analysis was carried out by two researchers. They assigned independently argumentation schemes to students' statements in the interview. Finally, the argumentation patterns of the students were compared to their performance in solving 14 computational tasks e.g. "(-12) + (-7) =". Our students worked on the PALMA items on negative numbers at the end of the school year (like the students in PALMA).

RESULTS

Exemplary results from the experimental class in comparison to PALMA

To answer our first research question, we compared the solution rates with regard to PALMA-items (grade 7) on negative numbers of the representative PALMA-sample and with our experimental group at the Laboratory School Bielefeld. For example, the task termed *Justifying* from PALMA had a solution frequency of only 19%, the lowest solution rate of all items on negative numbers in the PALMA-sample. It read: "If you add two negative numbers, you get a negative number again. Is this statement correct? Justify your answer." (Figure 2)

Figure 2 displays the task in German together with the solution of Leo (pseudonym) from our experimental group. Leo writes: "Yes, it is right, because if you add more negative numbers it becomes less." The overall solution rate of 43% of our experimental group for



Aufgabe: Begründung

„Wenn man zwei negative Zahlen addiert, erhält man wieder eine negative Zahl.“
Ist die Behauptung richtig? Begründe deine Antwort.

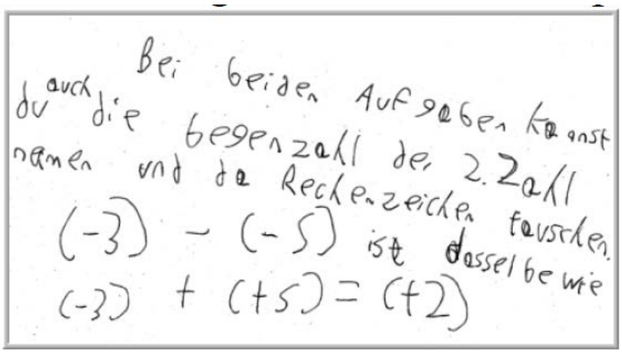
Ja richtig, weil wenn man mehr negative Zahlen dazu rechnet muss es weniger werden

Figure 2: Task for justification from Leo from the experimental class

this exemplary task provides evidence for our success in building up GVs for mathematical concepts. Altogether, the students worked on 20 PALMA-items dealing with negative numbers. The solution rates of the experimental class were at least 10% higher compared to the PALMA-rates for six tasks. One example of these tasks is the following: "The water level of a water reservoir declines to 8cm. On the next day the water level rises about 3cm. How does the water level change in these two days." In contrast to that, the experimental group solution rate was at least 12% lower than for the PALMA-group for four items. These are the following items: (+9) · (-8) = ; (-27) + (+3) = ; (-6) · (-8) = In the fourth task the probands had to determine a temperature on a thermometer and the solution rates were 71% (experimental group) respectively 86% (PALMA). We explain these results by the fact that the plus-minus-game was not used to explain the multiplication of integers. Furthermore the solution rates of the thermometer task are high in comparison to the other solution rates. This gives evidence that this task is one of the easiest tasks and mistakes occur by carelessness. For the ten tasks remaining, the difference of solution rates did not exceed 9%. One example of these tasks is the following: "Mr. Knodel has 450€ on his bank account. He transfers an invoice amount. The actual balance is -300€. What amount did he transfer?" Despite of the small amount of probands in our experimental class, the results show that the average achievement of our experimental group is comparable to the achievement of the PALMA-probands, who came from Bavaria being one of the high-performance regions in Germany.

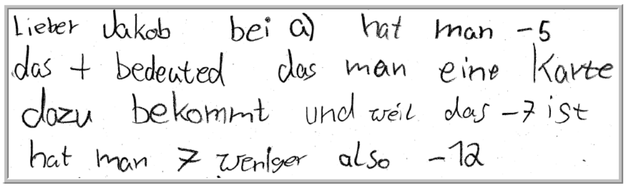
Exemplary results from the evaluation and the interview-study

In the letters that were written to Jacob during the first evaluation, we identified four consistent argumentation schemes that were verified in the following interviews. The first one is called *formal reasoning*: e.g.



Bei beiden Aufgaben kannst du auch die Gegenzahl de 2. Zahl nemen und da Rechezeichen tauschen (-3) − (-5) ist dasselbe wie (-3) + (+5) = (+2)

Figure 3: Example for formal reasoning



Lieber Jakob bei a) hat man -5 das + bedeuted das man eine Karte dazu bekommt und weil das -7 ist hat man 7 weniger also -12

Figure 4: Example for metaphorical reasoning

"In both tasks you can use the additive inverse of the second number and change the arithmetic operator (-3) − (-5) is the same as (-3) + (+5) = (+2)."

The second scheme is called *metaphorical reasoning*. In our case, the students' respective justifications were based on the number line or the Plus-Minus-Game: e.g. "Dear Jacob, in a) you have -5, the + means that you get a card and because this card is -7 you have 7 less so -12. "

The third scheme is called *mixed argumentation.* It contains parts of both *formal* and *metaphorical reasoning*: e.g. "In task b) your score is -9 and then you give your [card] -4 so you get -5 you get the same by calculating -9 + additive inverse (+4)." This example shows both metaphorical reasoning and formal reasoning. Whereas the explanation with the card game is identified as a kind of metaphorical reasoning, the identification of +4 as the additive inverse of -4 hints to a formal understanding.

In the fourth category *no justification* can be identified, e.g.: "Because I learned it this way." In a last step we compared the students' argumentation scheme(s) with their results on 14 computational tasks on negative numbers as for example "(-12) + (-7) =" (Figure 6). We show only the results of these 17 out of 21 students who wrote a letter to Jacob and took part in the evaluation.

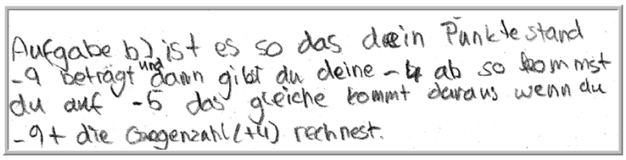

Aufgabe b) ist es so das dein Punktestand -9 beträgt und dann gibt du deine -4 ab so kommst du auf -5 das gleiche kommt daraus wenn du -9 + die Gegenzahl (+4) rechnest.

Figure 5: Example for mixed argumentation

As Figure 6 displays, 13 out of 17 students score at least 11 out of 14 points. Four students score less than 11 points whereas three of these students are not able to justify or explain their strategies in the letter to Jacob or to the interviewer. We find very good results (13 or 14 points) of students who used *mixed argumentation*, *formal reasoning* or *metaphorical reasoning*, whereas the *mixed argumentation* seems to be dominant with high achievers. This result is in line with those of Kilhamn (2008). In contrast to her study, however, we find students with results of 11 and 13 points in spite of their preference for only *metaphorical reasoning*. An explanation for this result is the structure of the Plus-Minus card game, which undoubtedly represents a form of *metaphorical reasoning*, while it also represents the mathematical structure very clearly and omits other contexts, such as those of everyday life. In addition, however, we find three students who used *metaphorical reasoning* only and scored only 9 and 11 points of 14. One reason for this might be the fact that the Plus-Minus-Game represents addition and subtraction tasks within a maximal number range from -55 to +55, whereas the students had to solve tasks such as -530 + (-210) in the computation part as well.

PERSPECTIVES

In the near future, we will revise the unit on negative numbers and teach it again in more classes. We will use several models, such as the balance-debt-model, the Plus-Minus-Game and a model dealing with the number line to identify more argumentation schemes that are specific for these models. Our aim is to gain more insight into individual problems, when dealing with specific models. Furthermore we will work on an effective combination of models and underlying argumentation schemes for addition/subtraction and multiplication of negative numbers.

Student number	Argumentation scheme in the letter and the interview	Result in computational tasks (in points, max. 14)
1	mixed argumentation	14
2	formal reasoning	14
3	mixed argumentation	14
4	mixed argumentation	14
5	mixed argumentation	13
6	formal reasoning	13
7	metaphorical reasoning	13
8	mixed argumentation	13
9	formal reasoning	12
10	formal reasoning	12
11	mixed argumentation	12
12	metaphorical reasoning	11
13	metaphorical reasoning	11
14	metaphorical reasoning	9
15	no justification	7
16	no justification	7
17	no justification	5

Figure 6: Argumentation schemes and achieved results in computational tasks

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Reversible and irreversible desemantization

Milan Hejný, Darina Jirotková and Jana Slezáková

Charles University in Prague, Faculty of Education, Prague, Czech Republic,
milan.hejny@pedf.cuni.cz, darina.jirotkova@pedf.cuni.cz, jana.slezakova@pedf.cuni.cz

A child's first experience with mathematics comes from their everyday life. The child does not know what three is but knows what three fingers, apples or candies are. Later, in consequence to abstraction, this semantic anchoring of mathematical ideas is expanded by ideas and concepts that are not directly dependent on semantics. We call this process "desemantization". If the process of desemantization ousts semantic links from a pupil's mind as a result of too fast a drill of additive and later multiplicative structures, we speak of irreversible desemantization whose consequence is mechanical knowledge of a child. The paper develops this concept.

Keywords: Desemantization, cognitive process, mental schema, generic model.

INTRODUCTION

Czech pupils' and students' negative attitudes to mathematics which can be observed e.g. in international surveys TIMSS and PISA and their low level of understanding of mathematics are a prevailing problem of school mathematics in the Czech Republic and a challenge for mathematics educators. Our experience and research imply that responsible for inauspicious situation is not the content of mathematics but the way it is taught at school. We are convinced that in most cases the teacher presents and explains new subject matter and their pupils only imitate the teacher, they reproduce what has been said and by repetition try to store it in their memory. A pupil is not expected to use their natural will to discover, he/she is reduced to the role of a consumer of knowledge transmitted by the teacher. Knowledge of mathematics enters a pupil's mind from the outside and makes a mosaic of more or less isolated items of knowledge. The chance that this knowledge will not be forgotten and that it will be linked to other knowledge is very low. It has been known for many years that if transmissive model of teaching is replaced by constructivist approach, where new knowledge is born in a pupil's mind as a result of the pupil's intellectual activity, the situation changes (Noddings, 1990; Pehkonen, 1997; Gruszczyk-Kolczyńska, 2012). The here presented study casts the light on the possible causes of why key knowledge of arithmetic is often stored as mechanical knowledge in a pupil's mind and thus becomes unusable in the future when the pupil meets new topics and solves new problems. It also looks for a solution of the first three main questions formulated for TWG02 both on CERME 8 and CERME 9.

The stories used in the paper as illustration of our points come from several different research projects from different periods of time.

THEORETICAL BACKGROUND

We understand the cognitive process in mathematics as a pentad of stages in the sense of the Theory of Generic Models (for details see Hejný, 2012):

1. motivation, it guides the cognitive process and provides energy;
2. creation of items of initially disconnected experience – isolated models;
3. discovery of generalized knowledge – generic model;
4. discovery of abstract knowledge;[1]
5. crystallization when the new item of knowledge becomes organic part of mathematical knowledge of the individual. This stage in fact underlies all the first four stages.

1 The difference between generalized and abstract knowledge is illustrated in Story 4.

This learning process was introduced to CERME audience in (Hejný & Kratochvílová, 2005). Hejný (2012) states that unlike a number of other theoretical studies describing the cognitive process, the Theory of Generic Models (TGM) is well comprehensible both to researchers and teachers. The division of the cognitive process into stages allows a comprehensible application of the theory both into primary and lower secondary education. TGM explains what the sources of development pupils' mechanical knowledge are and how this can be prevented and reeducated. A complete set of textbooks covering all areas of primary mathematics were developed based on this theory. Over the past 8 years the set of textbooks has spread into 20% of all elementary schools in Czech Republic and currently it is being piloted at several schools in Poland and Slovakia.

The Theory of Generic Models has been also used as a tool for analysis of experiments related to cognitive processes as well as when conceiving textbooks for pre-service teacher education.

Sets of generic models in a pupil's mind create mathematical mental *schemas* that are the bearers of an individual's mathematical knowledge (Hejný, 2012).

Attention should be paid to two abstraction transfers:

isolated models
→ generic models
→ abstract knowledge,(*)

in which a pupil's semantic experience changes into abstract cognition.

Let us remark that the process (*) as a tool for discovery of mathematics happens not only in ontogeny, but also in phylogeny. Ernst Haeckel's biogenetic law, which states that ontogeny recapitulates phylogeny, is inspiring also for didactics of mathematics. P. M. Erdnijev (1978, p. 197) formulated the idea as follows: "The growth of the tree of mathematics knowledge in an individual's mind will be successful only if we recapitulate to a certain degree the history of development of mathematics." We also work with this idea. When exploring the issue of desemantization we build on our former studies as well as works (Krpec & Zemanová, 2011) and (Zemanová, 2014).

THE CONCEPT OF DESEMANTIZATION

The term *desemantization* refers to the process (*). It emphasizes the fact that once abstract knowledge is formed, it is no longer dependent on the initial semantic ideas and exists independently.

Story 1

I ask a five-year-old Adam how much two plus three is. The boy looks at apples on the table and asks: "Two apples and three apples?" and when I agree he first takes two apples, then three apples, puts them together, counts them and says: "five apples". Then I point at a bowl with candies and ask: "How much is two candies and three candies?" The boy proceeds analogically and says: "five candies". He does not realize that in both cases it is the same calculation. This is surprising for Adam's father who was observing us. The father was convinced I should have told the boy that he could use the first calculation in the second case. The father was disappointed that his son failed to see the analogy.

Half a year later, when asked how much two and three is, the boy used his fingers to find out the result is "five". Another year later the boy answers "five" was without counting. When he later mastered numbers in the language of higher abstraction, he was able to record this knowledge in the abstract form: $2 + 3 = 5$.

What has taken place in the boy's mind is *desemantization.* If the symbol 2^* describes a semantically anchored number 2 and similarly symbols 3^* and 5^* semantically anchored numbers 3 and 5, then the described desemantization can be described as the transfer $(2^* + 3^* = 5^*) \rightarrow (2 + 3 = 5)$. When practicing addition and subtraction at school, the link gets automated. However, when his younger sister asks him how much two plus three is, his advice is to count it on fingers. This implies that the abstract knowledge $2 + 3 = 5$ is still connected to semantics in Adam's mind. The boy is able to make the abstract item of knowledge $2 + 3 = 5$ comprehensible to his sister using the relation $2^* + 3^* = 5^*$. The desemantization that Adam went through has not broken the individual stages of the process (*). The boy naturally goes back to the stage of isolated models. That is why we speak in this case of *reversible desemantization.*

Story 2

Adam is in the second grade and is one of the fastest arithmeticians in his class. He can add, subtract, multiply and partially also divide very quickly and reliably. He is not so good at word problems. The class is solving the following problem:

Problem 1. Mum paid 163 CZK for her shopping. She has 509 CZK after this purchase. How many crowns did mum have before she went shopping?

Adam sees the signal word "spent" but is not sure whether to subtract, as he finds the text a bit strange. He prefers to ask his teacher: "Miss, is it plus or minus?" The teacher answers plus. Adam then quickly answers six hundred and seventy two. And the teacher commends him for his answer.

The teacher overvalues calculation skills and fails to realize that Adam fails to understand mathematics. She does not realize that calculation in word problems is only secondary, what is of primary importance is understanding the assignment, the pupil's ability to grasp in their mind what the problem asks and requires. When the teacher commends the boy, she deforms his metacognitive belief that mathematics is about being fast in calculations rather than about thinking. The boy's mathematical cognition is no longer supported in his semantic ideas. If we ask him to pose a problem with addition 163 + 509, he will use the standard addition of two sets of data. If we ask him to use the word "spent" in the assignment, he will not manage to do so. Desemantization in the boy's mind is in this case *irreversible.*

Story 3

Adam is in the seventh grade. He can now add fractions. This knowledge entered his mind not by the process (*) but through structural deduction. First the pupils were introduced to reduction and raising of fractions and based on this knowledge the teacher deduced the relation: $a/b + c/d = ad/bd + bc/bd = (ad + bc)/bd$. Adam does not understand the presented deduction and thus has no idea of what is actually happening as there are very few real-life problems based on this concept of fractions. Adam has only learnt a rule and knows that when adding fractions he must follow this specific rule.

When his younger sister asks him how to add one half and one third, he shows her the rule. When she asks him for an explanation of this rule, he says it cannot be explained, it must be learnt.

In this case Adam's knowledge of addition of fractions is not supported by semantic ideas as it entered the boy's mind from the outside with very little semantic support which is, moreover, not further developed in the subsequent lessons. On the contrary, the little semantic support that existed was forced out through the subsequent drill. Desemantization is very weak in this case and the support in semantic ideas ceases to exist. In this case we also speak of *irreversible* desemantization. More precisely we speak of mechanical knowledge that does not enter pupils' minds by (*) but by direct transmission.

ISOLATED MODELS

Adam's story shows how a pupil with clear ideas about mathematics turns into a pupil without any ideas. The story also illustrates the important stages of isolated models. Once we master abstract thinking and cognition, we often believe it is a waste of time to be learning isolated models. That is the case of Adam's father from Story 1 and the teacher who was teaching Adam to add fractions. This common mistake can be avoided if the teacher carefully monitors how their pupils understand mathematics and if they look for ideas in history of mathematics. Let us present here one historical illustration which is underlain by phylogenetic parallel to Adam's ontogenetic activity in Story 1.

Phylogenetic parallel

We can find evidence of the fact that addition can be dependent on the objects we calculate with also in phylogeny. For example the Japanese numeral "two" is "ni" and is recorded by the character (二). However, this concept is not on the level of abstract cognition as in the semantic context the Japanese supplement this numeral by a particle called numerative. This means that in our transcription they do not work with number 2 but with anchored number 2*. Thus two people are futari (二人), two small animals are nihiki (二匹), two large animals are nitó (二頭), two elongated cylindrical objects are nihon (本), two glasses are nihai (二杯) etc.

The number of Japanese numeratives shows that creation of a generic model may require a considerable number of isolated models. It is almost impossible to find out how many different calculations $2^* + 3^* = 5^*$ had been carried out by Adam in Story 1 before he grasped the relation $2 + 3 = 5$.

Both isolated models of the relation 2* + 3* = 5* presented in Story 1 were of the same type:

A) number + number = number.

But the relation 2 + 3 = 5 has more semantic contexts. For example two more types:

B) address + operator of comparison = address (I live on the 2nd floor, Michal lives 3 floors higher. Which floor does Michal live on?)

C) operator of change + operator of change = operator of change (I make 2 steps, then 3 steps. How many steps have I made?)

The spectrum of different contexts of isolated models is crucially important for the quality of future desemantization. First of all it brings a variety of ideas into the nascent schema of "addition and subtraction" and enriches semantic background of arithmetic. Moreover it prepares the grounds for new concepts and relations and also type

A) prepares the grounds for the concept of fraction as we can add 1/2 of an apple + 1/3 of an apple;

B) prepares for grasping the concept of a number line on which there will be all numbers;

C) prepares for work with vanishing models of numbers (e.g. three steps, three winks, they vanish once they have been carried out) and negative numbers.

The presented stories, their analysis and further considerations show that a well-built rich system of isolated models is prerequisite to good and successful desemantization. Although we discuss here only introduction to arithmetic, the conclusion is universally valid. It applies also to fractions, decimal numbers, equations, perimeters and areas of plane figures, combinatorics. Combinatorics will be the setting of the story in the following chapter.

FIRST ABSTRACTION TRANSFER

Based on the analysis of several dozens of video recordings of the cognitive processes, the stage of isolated models was divided into four sub-stages:

1) First experience enters our mind – the seed of future knowledge.

2) Gradual entry of more isolated models that are not interlinked yet. It may happen that we accept quasi-models and refuse surprising models.

3) Some of the models start pointing at each other, we group them together and separate them from other models. We get the feeling that these models are somehow alike.

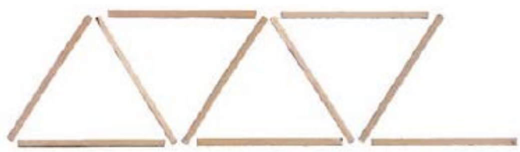
4) Discovery of this alikeness results in creation of a community of at least a subset of all isolated models.

As soon as the pupil has reached stage four, he/she chooses one of the isolated models and says "and this is how it always works". This isolated model becomes a generic model, which shows how to solve problems of this type. In Adam's case from Story 1 this generic model was the addition 2 fingers + 3 fingers = 5 fingers.

Story 4

In the second grade pupils were solving problems with sticks. The following is one of the problems.

Problem 2. Continue making other triangular windows and write down into the table how many sticks you need.



cells	1	2	3	4	5	6	7
sticks							
cells	8	9	10	11	12	13	14
sticks							

Figure 1: How many sticks?

The problem was solved about 10 days ago and the pupils had found out that for creation of 3 (4, 5) windows they needed 7 (9, 11) sticks. None of the pupils had noticed the number of sticks was increasing by two. It means that the pupils reached the 1st and 2nd sub-stage of isolated models.

Later, Ben was solving the problem at home on his own. When he put numbers 3, 5, 7 and 9 in the table, he saw the numbers were increasing by two. (Ben reached the 3rd sub-stage.) Therefore he added number 11 beneath number 5 and checked the result. He was happy to see that his conjecture worked. (Ben reached the 4th sub-stage.) He filled in the whole table and ran to his father to show. The father commended his son and asked how many sticks would be needed for 50 windows. The boy realizes that this will require a lot of writing and calculations. He takes his things and goes to work in his room. He makes Table 1.

While filling in the table he notices that beneath numbers 19 and 29 there are numbers 39 and 59. Then he realizes that beneath numbers 20 and 30 there are numbers 41 and 61. He is convinced that beneath numbers 40 and 50 there will be numbers 81 and 101. He runs to his father to show him this discovery. He commends his son and asks him what number there will be beneath number 57. Ben creates table for numbers 50, 51 to 57 and beneath numbers 101, 103 and 115. The father asks what number there will be beneath number 100. Ben writes 100 and beneath immediately 201. The father asks what there will be beneath number 113. Ben writes 110, 111, 112 and 113. Beneath 221, 223 and 225. The father applauds.

The father resisted the temptation to disclose to his son that the rule works for all numbers, not just tens. Ben discovered this rule later together with his other two classmates and recorded it in a very simple way: $sticks = 2 \cdot windows + 1$. Later when Ben was explaining it to his friend, he wrote briefly $s = 2 \cdot w + 1$.

Ben's solving procedure contains all 4 sub-stages of isolated models. The first two took place in the classroom when pupils were solving the problem for 3, 4 and 5 windows. The third sub-stage was supported by the use of the table thanks to which the first discovery was made: the numbers increase by two. Ben discovered the instruction on how the process continues. This was the 4th sub-stage that immediately transcended into a generic model. Generic models of this type are therefore called *processual*.

The father, by asking about number 50, guided the boy to search for the rule how the number of sticks can be derived from the number of windows immediately, without having to make a long table. The boy first discovered the answer for numbers 10, 20, 30, ... This is a discovery of the model that we will call *partially conceptual*. Finally the third discovery made with classmates is a fully *conceptual model*.

The story shows a complex, several day long process of discovery of a generic model. Each of the three AHA-effects accompanying the process was a very exhilarating experience. And this experience guarantees that the knowledge about the relation of windows and sticks given by the abstract formula $s = 2 \cdot w + 1$ is the consequence of reversible desemantization. Ben is able to recapitulate the whole process even a year later.

ISOMORPHISM OF GENERIC MODELS

In the paragraph on Isolated models we presented several different semantic types of anchoring of the knowledge $2 + 3 = 5$. The rich spectrum of sematic anchoring of most knowledge in mathematics helps reversibility of desemantization. Let us illustrate this on an example of knowledge of combinatorics: () = 10.

Problem 3. Alice has birthday. She invited her friends Betty, Cecil, Dee and Elis to her party. Each who came kissed each other girl who had already come to the party. How many kisses were there?

Problem 4. Find out how many matches will take place in a football tournament if there are 5 teams, each of them playing each other once.

Both problems serve as a semantic illustration of combinatorial number ().

Problem 3 is of processual, problem 4 of conceptual nature.

Story 5

Problem 3 was assigned to 5th graders. Jana was solving it by a simulated dramatization and then she drew 5 points *A*, *B*, *C*, *D* and *E* on a sheet of paper. She joined

15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36
31	33	35	37	39	41	43	45	47	49	51	53	55	57	59	61						

Table 1: How many cells and sticks?

A to points *B*, *C*, *D* and *E* and wrote 4. Then she joined *B* with *C*, *D* and *E* and wrote + 3 next to 4. Thus she continued with *C* and *D* and finally gained the result 4 + 3 + 2 + 1 = 10. Jana finished very fast so the teacher asked her to try to solve the problem for 6 and even 10 girls. Jana started to work. She got engulfed and was working only on this problem until the end of the lesson. Ivan found the problem difficult but together with his friend managed to get the result.

A month later the same class was working on problem 4. Hynek wrote down the names of 5 teams: Sparta, Ostrava, Brno, Hradec, Jihlava. Then he wrote down all the matches: Sparta – Ostrava, Sparta – Brno, ... In the end he stated that there were 11 matches because he had calculated Brno – Hradec and Hradec – Brno.

Jana gave up looking for the solution. She said she did not understand football. The teacher knew Jana liked playing chess and so she advised the girl to solve the same problem for a chess tournament. Jana then started to solve the problem and later, when the problem was discussed by the whole class, she contributed with a very powerful idea.

The first to speak was Hynek (deliberately, because his solution 11 was wrong). When he wrote Sparta – Ostrava, Mirek suggested he should write only S – O to speed up the process. So Hynek changed it and discovered on his own that if there is B – H, he cannot add H – B. There were more pupils who had

	A	**B**	**C**	**D**	**E**
A		X	X	X	X
B			X	X	X
C				X	X
D					X
E					

Table 2: How many tournament matches?

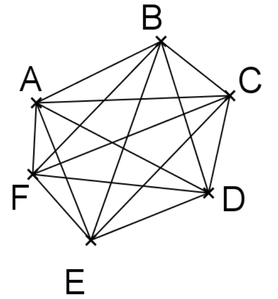


Figure 2: Jana's drawing

been solving the problem analogically to Hynek. Each such solution is an isolated model of the result of problem 4. The most popular solution (especially with boys) was Ivan's solution. He created a table of the tournament and showed there would be 10 matches. This table (Table 2) is closer to a generic model than to an isolated model.

Lada was waving her hand to get the attention of the class to explain that this was the same as kisses on Alice's party. Some pupils agreed but most did not understand. Then Jana suggested it was pretty clear: before each match the captains shake hands, which is the same as when the girls kiss. She ran to the board, drew 5 points and joint each point with all the other points (Figure 2). She said this was how it could be drawn – both Alice and the matches. Everybody could understand this explanation. Jana's drawing became a generic model and Hynek's and Ivan's solution were in this perspective merely isolated models.

When they reached 7th grade, these pupils learnt that the combinatorial number () is

the number of all 2-element subsets of an n-element set. The first to understand this difficult definition was Jana. She pointed out that for $n = 5$ it is the same as with Alice or with the matches. Understanding of the abstract concept () is now for some pupils based on the generic model "Alice's party for n girls", for other pupils on the generic model "tournament for n teams" but for more pupils on isomorphism of both these models represented by an n-gon, its all sides and diagonals. Obviously the understanding of the last group of pupils is the deepest.

CONCLUSION

The above presented stories were used to cast light on the process of desemantization and to point out the phenomenon of reversibility and irreversibility of this process, which is of paramount importance with respect to understanding arithmetical phenomena. We showed that desemantization is impossible where the abstract idea enters a pupil's mind from the outside, without previous semantic preparation. Story 5 illustrated the case where semantic preparation took place three years before the abstract concept was introduced from the outside.

The scope of this paper does not allow us to discuss the issue of desemantization in detail. Some of the phenomena had to be omitted. For example the importance of *counting backwards* for reversible desemantization, *amalgamation process-concept* (Gray & Tall, 1994), or *grasping, coding and transformative power of language* for second abstraction transfer (Kvasz, 2010).

Let us conclude this study by a summary of the main findings of our research:

— Desemantization is a long-term mental process whose mechanism can be described by two abstraction transfers (*).

— Desemantization is reversible if the pupil even after having achieved the abstract level is able to project abstract ideas into generic models.

— Reversibility of desemantization has positive influence on the richness of generic models and links between them.

— Knowledge that enters a pupil's mind from the outside in its final form (i.e. mechanical knowledge) has no semantic anchoring. In this case we cannot speak of desemantization. However, semantic anchoring can be developed later.

ACKNOWLEDGEMENT

The research was supported by the research project No. 15 (PRVOUK) School and the teaching profession in the context of increasing demands on education.

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Sixth grade students' explanations and justifications of distributivity

Kerstin Larsson

Stockholm University, Stockholm, Sweden, kerstin.larsson@mnd.su.se

Equal groups and rectangular arrays are examples of multiplicative situations that have different qualities related to students' understanding of the distributive and the commutative properties. These properties are, inter alia, important for flexible mental calculations. In order to design effective instruction we need to investigate how students construct understanding of these properties. In this study sixth grade students were invited to reason with a peer about calculation strategies for multiplication with the goal of explaining and justifying distributivity. Their discussions demonstrate that the representation of multiplication as equal groups helps them to explain and justify distributivity. At the same time this representation hinders their efficient use of commutativity.

Keywords: Multiplicative reasoning, distributivity, commutativity, equal groups.

INTRODUCTION

Three fundamental properties of arithmetic, the distributive, the commutative and the associative property all apply to multiplication. These properties underpin flexible mental calculations and later algebraic understanding (Carpenter, Levi, Franke, & Koehler, 2005; Ding & Li, 2010; Lampert, 1986; Young-Loveridge, 2005). Although the significance of these properties is well known, researchers have only recently "begun to discuss ways to teach these ideas in the elementary grades" (Ding & Li, 2010, p. 147). To design effective teaching of the arithmetical properties more knowledge about students' understanding of the properties is needed. This study's aim is to investigate how students make sense of the arithmetical properties in multiplicative calculations. The distributive property (DP) is the main focus, but the commutative property (CP) for multiplication is also investigated since students need to manage the CP when they undertake calculations involving the DP. The associative property is not discussed here, which is not a reflection on its importance but this paper's focus. In the next section some general concepts central to this study are presented followed by a review of findings concerning students' understanding of the DP and the CP before, finally, the aims for this study are clarified.

BACKGROUND

The DP, which states that $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$, underpins mental multiplication by splitting one factor to make two multiplications which are then summed. For example one might solve $7\cdot14$ as $7\cdot(10+4) = (7\cdot10)+(7\cdot4) = 70+28$. Students can develop this mental strategy when they view multiplication as repeated addition and focus on the invariance of the total (Schifter, Monk, Russel, & Bastable, 2008). When this implicit use of the DP is transferred to problems where both factors are multi-digit numbers, a common error is to solve multiplications such as $26\cdot19$ by only multiplying the first terms and the second terms with each other; $(20+6)\cdot(10+9) = 20\cdot10 + 6\cdot9$ (Lo, Grant, & Flowers, 2008). Ding and Li (2014) suggest that the difficulties students have learning arithmetical properties, leading to calls for more concretisation, stem from their abstractness and "lack [of] close relevance to learners' lives" (p. 103). Concretisation by contextual and visual representations, in order to build a mental image of the operation and its properties, is argued to help students to structure and "organize their thinking and reasoning" (Yackel, 2001, p. 27). Both contextual and visual representations can reflect different multiplicative situations such as equal groups and rectangular arrays. A simple equal group situation is 4 bags of 8 apples in each bag, while a simple rectangular array can be a chocolate bar with 4 rows of 8 squares. In asymmetrical situations, such as equal groups, one factor is the multiplier (number of bags) and the other the multiplicand (number of apples). In symmetrical situations, such as rectangular arrays, the two factors have the same role.

A contextual representation suitable for illustrating the DP is the total cost for 4 cups of coffee and 4 cakes, where the answer would be the same whether you first multiply the cost for one coffee by 4, then the cost for one cake by 4and then add the products or if you first add the cost for one coffee and one cake and then multiply that sum by 4. A visual representation for equal groups can be used for discussing the ways the total number of objects could be calculated, by means of the DP (Lampert, 1986). A rectangular array of dots or squares can also illustrate how 7 rows, each with 14 objects, can be partitioned flexibly by use of the DP into for example $2 \cdot 14 + 5 \cdot 14$ rectangles (Young-Loveridge, 2005).

The CP, $a \cdot b = b \cdot a$, allows numbers to change places in multiplication and addition. Young students seem to discover and understand the CP for addition without instruction but not for multiplication (Squire, Davies, & Bryant, 2004). The rectangular array model is well suited to concretise the CP, as its rotational quality makes it perceptually self-explanatory, for example a box of 18 eggs in 3 rows of 6, which, when rotated 90° is simply perceived as 6 rows of 3 eggs. With an equal groups situation, it is not perceptually transparent that 7 bags of 14 coins is equivalent to 14 bags of 7 (Lo et al., 2008). Carpenter and colleagues (2003), however, found young students justifying an equal groups approach to the CP by rearranging the objects in the group, as exemplified by a students who said: "I would always get new groups that are the same size as the number of groups I started with and the number of new groups I would get would be the same as the number I had in each of the old groups" (p. 95). Some researchers argue that rectangular arrays should be introduced to enrich students' images for multiplication and to illustrate the DP (Carpenter et al., 2003; Young-Loveridge, 2005). Indeed, the illustration of integer multiplication as a rectangular array can explain standard algorithms and illustrate the extension of multiplication to rational numbers.

Squire and colleagues (2004) investigated 9–10 year old students' ability to use the DP and the CP in varied contextual situations. They gave a multiple-choice test whereby a multiplicative situation was given as a cue, and the problem was to solve a word problem of the same situation by means of either the DP or the CP. The DP problems presented the total number of objects in a groups of b objects (an equal groups problem) and asked for the total number of objects in a+1 groups of b objects. For example the students were given a multiplicative word problem incorporating the assertion $26 \cdot 21 = 546$ and invited to solve a similar problem involving $27 \cdot 21$. The CP problems were constructed analogously; if the cue stated the total number of objects in a groups of b objects the task objective was to find total amount of objects in b groups of a objects. They concluded that 9–10 years old English students could manage the CP in all situations but not the DP. DP problem reflecting equal groups were more often solved correctly than any other situations, leading Squire and colleagues to suggest that the representation of equal groups might be a natural way for young students to imagine what happens when the multiplier is changed by one. They conclude that equal groups should be employed when introducing students to the DP. This is in line with Lampert's (1986) findings that fourth grade students (about 10 years old) made sense of the DP by means of stories in combination with drawings illustrating equal groups. She argues that equal groups is more intuitive for young students than array models as that is how they model multiplication. This is also confirmed by literature in the field of early algebra (Schifter et al., 2008).

In short, we know that younger students (9–10 years old) do not invoke the DP as easily as the CP; that equal groups are more intuitive for the understanding of the DP than rectangular arrays, even though rectangular array is proposed to enrich students' understanding of the DP. But for middle grade students (12–13 years old) there is a lack of research of how they understand the DP and what representations they employ when reasoning about multiplication. Given the importance of arithmetical properties for the understanding of algebra as well as flexibility in calculation, it is appropriate to pose the question: how do sixth grade students understand distributivity?

METHOD

Students from two sixth grade classes already enrolled in a research project were invited to take part in this study. The 19 students who agreed to participate do not form a random or representative sample but a typical mix of Swedish students; some have diagnoses concerning concentration or dyslexia, some struggle with mathematics while others excel. In order to en-

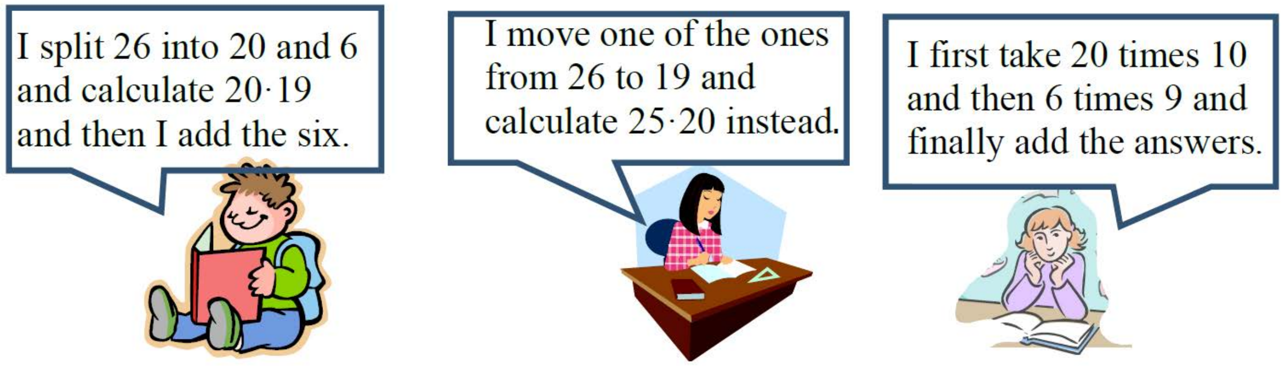

Figure 1: Problem 1 (P1), Problem 2 (P2) and Problem 3 (P3)

hance the possibility of a rich discussion, students were placed in homogeneous pairs (one worked alone) based on the evidence of earlier tests and interviews concerning the forms of multiplicative reasoning they had previously shown. They were presented with three problems written on separate cards, see Figure 1, each comprising a strategy from a fictitious student for the calculation of 26·19. The students' tasks were to discuss each problem with their peer and a) evaluate the validity of the suggested strategy and b) reason why the suggested strategy was valid or not. They were not informed that the suggested strategies were incorrect. A separate card with the multiplication problem (26·19) was visible throughout the discussions.

Problem1, (P1), reflects an incomplete use of the DP, where one factor is partitioned and multiplied by the other factor but the last part of the partitioned factor is added to the product without any multiplication.

Problem 2, (P2), is analogous to a common method for addition, where a suitable part is moved from one term to the other, in order to make an easier calculation.

Problem 3, (P3), reflects a well known error (Lo et al., 2008). However, all three strategies derived from (incorrect) strategies exploited by this group of students the previous year when they were tested on multiplication of two-digit numbers.

Each of the three problems is an example of how students have partitioned the numbers in order to simplify calculation. When using the DP correctly partitioning is the starting point, but to demonstrate understanding of the DP involves explaining what to do with the parts as well as why. By inviting students to evaluate and explain an incorrect use of the DP it was possible to draw conclusions from their reasoning about how they understand the DP.

The students' discussions, which took place in a room adjacent to their normal classroom, were video and audio recorded and all written material collected. The tasks were explained to the students, who were explicitly told that they did not need to do any calculations themselves. Each card was read aloud and left on the table. The discussions for all three problems lasted between 10 and 25 minutes including the oral information about the tasks. The transcribed student discussions were repeatedly read and their answers categorised according to the arguments they employed to explain their decisions about the validity of the strategy. Some students used multiple arguments for each problem and some arguments were used in all problems while other arguments were used for one or two of the problems.

RESULTS

In this section the categories of arguments that emerged from the data are presented and exemplified by excerpts from the students' discussions. These are followed by a discussion about the different types of reasoning in respect to the DP.

General justification reflects the arguments of four students who not only solved P2 by justifying why the suggested strategy was invalid but also investigated the strategy further in order to find out under what conditions it would work and when the answer would be bigger or smaller. Their arguments clearly reflect a discussion *about* multiplication and its properties.

Emil	If you increase the smaller number and decrease the larger number, then it always gets bigger.

Marcus: It might not work with zero point, but with ordinary numbers, whole numbers, then it always...

Here, Emil concluded that the product gets bigger if you move one from the larger factor to the other, and Marcus' statement reflected a discussion-derived argument concerning which numbers Emil's claim is valid for. In this case, "zero point" refers to decimal numbers.

Equal groups is a category where students made contextual references and represented 26·19 as 26 sets of 19 objects or 19 sets of 26. This argument was used in all three problems and reflected an awareness of why the DP is valid in multiplication.

Lucas If you take away one pile, then you take away nineteen. And that should be put in twenty-five piles. That doesn't work.

Here Lucas showed why the strategy was invalid, that one in the number 26 represents a group of 19, which is construed as understanding the DP.

Counterexample was used to evaluate the calculation strategy by use of numerical examples. Hence, it reflected multiplicative reasoning by implicit use of the DP. Students gave examples of moves of one from one factor to the other would not yield the same product, as exemplified by Ida.

Ida It is not the same, if it is eight times nine or seven times ten.

Here, Ida showed an awareness that if she used the same strategy as suggested by the fictitious student in P2, the calculation would yield an incorrect result, since it transformed the problem. She transferred the strategy into easier numbers to make arguments of why the strategy did not work. There were also students who gave counterexamples to the suggested strategy in P1, which reflected knowledge of the DP, stating that 6 needs to be multiplied by 19 and then the two partial products can be added.

Check the answer was used to assess the result rather than the strategy, and drew on a calculation of the answers for both the suggested strategy and for 26·19; if the answers were not the same the strategy must be invalid. This reflected a result-oriented view on multiplication without argumentation as to why the results differ; hence, such an argument does not demonstrate understanding of the DP. It was used for P2 and P3 and is exemplified by Hanna.

Hanna I will calculate that [25·20] and then this [26·19] and see if it is the same.

Experience was when students knew that the strategy was invalid or "knew" that it was valid from their experience of calculations. This argument offered no evidence of understanding of the DP since there was no reasoning as to why the strategy does not work. Alice used it to justify the falseness of P3 and Wilma its truth.

Alice That does not work. [...] I thought it worked before, but it doesn't.
Wilma It works. That is how I calculate.

Additive reasoning reflects the students' incorrect additive reasoning when calculating. This type of reasoning occurred in relation to all three problems.

Matilda If you first take twenty-six, and split it, that's the same. You can take nineteen and then take [one] to first make a twenty, that is, you can take one from the six to the nineteen, so it becomes twenty times twenty, and then just add five. Then you will get the same answer, just that you split it in different [parts].

Here, Matilda described an additive way of handling big numbers to partition the numbers into parts that are easier to handle. She suggested that 26 is split into 20 and 6, and then add 1 from the six (splitting 6 into 5 and 1) to make 20, take 20·20. Then finally add 5, ((19+1)·(26-1-5)+5).

The category *other* consists of vague and unclear arguments as well as no answer. Felicia gives an example of a vague argument and Alva of unclear reasoning to P1.

Felicia Then the six isn't timesed [multiplied].

Alva It is hard to explain, but it is just wrong.

Some of the students who reasoned like Felicia meant to take 20·19 + 6·19, others meant 20·19·6, while others never explained further how the six should be multiplied. Alva's statement is an example from which it is impossible to draw any conclusions about the student's understanding of the DP, which is this study's aim.

When the categories of arguments were analysed further, and in relation to the DP, different types of reasoning were found. Four students investigated the suggested strategy in P2 on a meta-level; for example, under what premises their arguments were valid, which can be described as an *investigative reasoning* on a *meta-level* about multiplication. These students not only considered the structure of multiplication but also the generality of their claims. The arguments, which drew on 'equal groups' and 'counterexamples', were construed as *multiplicative reasoning by the DP* since arguments in both these categories reflected an implicit understanding of the DP as a theorem-in-action (Vergnaud, 2009). The difference between the two categories of argument was that in 'equal groups' all arguments were validated by contextual examples, e.g., Lucas' piles (see above), while the 'counterexamples' were validated by numerical examples without references to objects in groups. In contrast to the category 'check the answer', 'counterexample' was focussed on explaining why the strategy was invalid while 'check the answer' was focussed on calculating answers, and hence labelled as *procedural reasoning*. To use 'experience' reflects a *descriptive reasoning* by stating that the suggested strategy was invalid, but not why. The descriptions were focussed on the calculations as a procedure to get the right answer, implying that the student did not understand the DP. Finally, there were students who gave arguments *not showing multiplicative reasoning*, some by 'additive reasoning' and some categorised as "other". Vague arguments and unclear statements are not necessarily indicative of their not being able to use multiplicative reasoning; they might have had problems verbalising their understanding. All arguments are presented in Table 1.

Since many students gave multiple arguments for the same problem, the sum of arguments for P2 and P3 exceed the number of participating students. The category "other" is only presented when students did not provide any other argument, hence reflecting the number of students unable to give any clear argument to each problem.

The distribution of arguments for each problem varies. For P1 six students employed multiplicative reasoning when they explained why the suggested strategy was invalid and thirteen used arguments showing no multiplicative reasoning. For P2 the distribution of arguments spread over all categories except 'experience', and the 19 students used 33 arguments, demonstrating students' use of multiple arguments. This problem also engaged students in general justifications, which did not occur for the other problems. For P3 six students drew on 'experience', three correct and three incorrect, in the evaluation of the erroneous strategy. The incorrect evaluations drawing on experience are the only arguments, besides additive reasoning, which led students to make incorrect conclusions about the validity of strategies.

In the rich discussions where students reasoned about P2 it was possible to infer understanding of both the DP and the CP. The following utterance from Emil shows his distinguishing the multiplier from the multiplicand. The transcript also demonstrates that he was aware of the CP when speaking about calculating "the other way around", a common way for Swedish students to talk about the CP.

Emil — But if you take less there [pointing at 19] then it has to be fewer times multiplied. Or it depends if you do it the other way around, so if I calculate 19·26, then it is anyway twenty-six times multiplied...

Type of reasoning	Category of arguments	P1	P2	P3
Investigative reasoning on meta-level	General justification	0	4	0
Multiplicative reasoning by the DP	Equal groups	2	6	7
	Counterexample	4	8	0
Procedural reasoning	Check the answer	0	9	8
Descriptive reasoning	Experience	0	0	3+3
Not showing multiplicative reasoning	Additive reasoning	1	4	4
	Other/no answer	12	2	3

Table 1: Number of students using different arguments for each problem

When students reasoned about equal groups they distinguished the multiplier from the multiplicand. Some of them interpreted 26·19 with 19 as the multiplier, others interpreted 26 as the multiplier. In some pairs this different interpretation of a fixed factor as the multiplier caused some confusion. However, all students were aware of the CP being valid for multiplication and overcame confusion by stating, that it did not matter which factor was multiplier, as exemplified by Johanna and Ida.

> Johanna Ida, it is ten times nineteen. Not ten times twenty-six.
>
> Ida What? You did it like this before, and then I thought twenty-six times ... but it doesn't matter what way around you do it.

During the discussions about all three problems some students offered other suggestions as to how to calculate 26·19. These suggestions clarified how they understand the DP, as with Hugo's statement where he gives a suggestion how to proceed with the strategy in P2 to get a correct answer.

> Hugo She has multiplied twenty times, and then she must take away what stands for one time, that is twenty-five. She has to take away twenty-five. [...] Then she gets that one times nineteen, so she has plus nineteen.

Here, Hugo demonstrates his understanding of the DP as he proposes compensating for the erroneous strategy of taking 25·20 instead of 26·19 by subtracting 25 and adding 19 to the product of 25·20; 26·19 = (26 - 1)·(19+1) - (1·25) + (1·19).

In summary, by engaging in the evaluation of incorrect strategies for two-digit multiplication, students with an implicit understanding of the DP demonstrated their understanding, mainly by reasoning about multiplication as equal groups. In equal groups the multiplier is distinguished from the multiplicand and this representation helped students to offer valid justifications about the incorrect strategies and to suggest other valid strategies employing the DP, but it also contributed to miscommunication connected to the CP. The arguments to explain and justify strategies demonstrated different types of reasoning involving both the DP and the CP as building blocks for understanding of multiplication. Students showing additive reasoning did not show knowledge about the arithmetical properties.

DISCUSSION AND CONCLUSIONS

The different arguments that students gave for the invalidity of the three strategies reflected different degrees of understanding the DP. When students cannot explain why or how a multiplication strategy works or not works it might be due to difficulties in expressing what they mean. It might also reflect shallow understanding of multiplication or explanatory difficulties due to perceptions of self-evidence. From my readings of students' explanations to P1, I infer that it was too easy for the majority of students to explain why you need to multiply the six as well. Still, one student, Matilda, who showed additive reasoning to all three problems, was convinced that P1 was a valid strategy, which she had trouble reconciling with the fact that the answer was wrong.

> Matilda It should work, but it doesn't. It might work if you take 20·19 and then multiply by 6. (After checking the calculation.)

Even though students like Matilda can be exposed as additive reasoners, P1 was not very productive since most students' answers and arguments were vague. This is in contrast to the other two problems, especially P2. The suggested strategy in P2, to move one from one factor to the other, caused long and elaborate discussions among most of the pairs. One reason might be that this strategy was new to most students. The novelty, and the analogous strategy for addition, might have evoked students' curiosity to investigate and engage in discussion about the strategy on a more general level than the other problems. In contrast to novelty, six students drew on experience to P3, the common mistake to only multiply tens by tens and ones by ones (Lo et al., 2008). Experience might have decreased the interest to engage in discussions about the strategy; students just knew that it "worked" or did not work.

Students who represented multiplication as equal groups in order to make sense of the DP and calculation strategies were successful, see for example Lucas' explanation why the suggested strategy in P2 did not work. The representation of equal groups as piles of objects served as a thinking tool to sort out the multiplication. To use the representation of equal groups as a thinking tool was demonstrated both for invalid strategies in the problems and for valid strategies employing the DP that the students offered as an alternative calculation. These Swedish middle school students seemed to prefer thinking about multiplication as equal groups just as the younger students from other studies (Lampert, 1986; Squire et al., 2004) and prospective teachers (Lo et al., 2008). However, the successful representation of multiplication as equal groups in respect to the DP proved to have a drawback concerning the CP. Even though students knew the validity of the CP for multiplication, there were instances where their view of one of the factors as the multiplier hindered their understanding of their peer's reasoning. This may be due to the fact that they represented the expression 26·19 differently, either as 26 groups of 19 or as 19 groups of 26. The students' explicit statements about the CP being valid can be construed as if the students did not take the CP as something self-evident when they were engaged in discussions drawing on equal groups. Interestingly, there were no utterances at all where students drew on rectangular array to make sense of calculation strategies or the CP.

The results of this study suggest that if we want students to learn and understand the DP we might better introduce the DP by equal groups and discuss the limits of its validity as well as how it can be used. On the other hand, the rectangular array, also an important representation of multiplication, highlights the CP by making it self-evident that $a \cdot b = b \cdot a$ (Carpenter et al., 2003) and can also be used to illustrate the DP (Carpenter et al., 2003; Young-Loveridge, 2005). If the underlying reason for illustrating the abstract properties of multiplication by contextual and visual representations is to build mental representations that can enhance understanding of the concepts (Yackel, 2001), it would be of interest to introduce multiple representations of multiplication. The findings from this study also suggest that more effort might be needed to incorporate representation of multiplication by rectangular arrays (and areas) in the instruction as complimentary representation to the equal groups, not as a substitute. Further research might give suggestions to how instruction can enhance the possibilities for students to build solid and useful mental representations of multiplication and its properties.

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Investigating fourth graders' conceptual understanding of computational estimation using indirect estimation questions

Sabrina Lübke

TU Dortmund University, Faculty of Mathematics, Dortmund, Germany, sabrina.luebke@math.tu-dortmund.de

This paper presents selected findings from a qualitative study that has the main goal to investigate the procedures and concepts of fourth grade pupils solving computational estimations problems within the context of spending money. It gives theoretical background with respect to understanding computational estimation, and it points out a hitherto unmentioned concept underlying computational estimation – the interrelation of an estimate and the exact calculation – which is closely related to task characteristics. Therefore, the data analysis and the discussion focus on the role of this concept as well as of the role of task characteristics.

Keywords: Computational estimation, elementary arithmetic, conceptual understanding, problem types, approximate numbers.

INTRODUCTION

The necessity of computational estimation in the mathematics curriculum has been discussed for years, especially since the 1980s. Theoretical reflections and prior research have shown that it is an important skill in everyday life as well as in mathematics because it is strongly interwoven with number sense and flexibility in mathematical thinking (e.g. Sowder, 1992; Verschaffel, Greer, & De Corte, 2007). A considerable number of research about computational estimation exists, mainly focussing on estimation strategies used by secondary pupils and adults in America and Asia (e.g., Alajmi, 2009; B. Reys, R. Reys, & Peñafiel, 1991; R. Reys, B. Reys, Nohda et al., 1991; R. Reys, Rybolt, Bestgen, & Wyatt, 1982), and on the adaptivity of strategy choices (e.g., Lemaire & Lecacheur, 2002; Lemaire, Lecacheur, & Farioli, 2000). Depending on the specific problems and its underlying mathematical structures as well as on individual preferences there are different strategies that can be used to make an estimate (see Table 1). At the same time empirical studies gave evidence that rounding is the predominant one being used to solve computational estimation tasks, as soon as it is brought up in the classroom (cf. R. Reys et al., 1991; R. Reys et al., 1982) and whether or not it leads to the correct answer. According to this it is assumed that this strategy is often used without understanding (e.g. Schoen, Blume, & Hoover, 1990). These findings are mainly based on quantitative studies. Thus, it is only little known about pupils' thinking, and about their concepts of computational estimation. Therefore, this paper deals with the concepts underlying computational estimation by means of a qualitative study with fourth grade pupils.

THEORETICAL FRAMEWORK

What is computational estimation?

To clarify the underlying concepts of computational estimation, the term itself will be defined first, since there are several, sometimes inconsistently used terms and definitions. In the English language, probably one of the most widely used expressions is *estimation*. On the one hand, this term refers to activities like *measurement* and *numerosity estimation:* In these cases some information is missing, and therefore an exact calculation is not possible and consequently assumptions about a quantity have to be made. On the other hand, estimation refers to *computational estimation,* which is the focus of this paper. In this case all quantities, respectively numbers, are known, but an exact calculation is not necessary due to the context. Thus, an exact calculation is consciously 'refused' in favour of a *simplification* (cf. Breidenbach 1969). This idea of simplifying a calculation to make it easier is finally one of the main goals of computational estima-

139 + 119 + 124 ≈	
Rounding	100 + 100 + 100 = 300
Translation	e.g. 3 · 130 = 390
Compensation (Adjustment while or after an estimate)	100 + 100 + 100 = 300, it has to be more because of rounding down, ≈ 370

Table 1: Examples of computational estimation strategies (cf. R. Reys et al., 1982)

tion (cf. LeFevre, Greenham, & Waheed, 1993; Star & Rittle-Johnson, 2009).

Concepts of computational estimation

Although most of the previous research deals with strategy execution, there are some studies that are concerned with the concepts of computational estimation as well (e.g., Cochran & Dugger, 2013; LeFevre et al., 1993; Sowder & Wheeler, 1989; Star, Lee, Chang, Glasser, & Rittle-Johnson, 2007). Sowder & Wheeler (1989) differentiated five components involved in computational estimation with *conceptual components* as one of them (p. 132). According to this, there are three concepts pupils have to understand in order to make an appropriate estimate (Cochran & Dugger, 2013; Sowder & Wheeler, 1989):

— **Role of Approximate Numbers**: Recognition that the process of computational estimation requires computing with approximate numbers, and that an estimate is an approximation.

— **Multiple Processes/Multiple Outcomes**: Acceptance of several possible processes for obtaining an estimate, and of more than one possible value as an estimate.

— **Appropriateness:** Recognition that appropriateness depends on the context and the desired accuracy.

An interview study focusing on these concepts (Sowder & Wheeler, 1989) found that pupils (grades 3, 5, 7 and 9) hardly seem to recognize these concepts. For instance, many pupils preferred a compute-then-round method in place of a simplification of the problem, and hence, did not seem to recognize the role of approximate numbers in the sense of Sowder and Wheeler (1989).

In addition to the conceptual components provided by Sowder and Wheeler (1989), Star and Rittle-Johnson (2009) operationalize conceptual knowledge of computational estimation by tasks that focus on the *impact of estimation strategies on distance from the correct answer*. In accordance with this, van den Heuvel-Panhuizen (2001) points out a peculiarity for so-called indirect estimation questions. These are types of estimation problems like "Quickly decide if the total of 186, 495 and 197 points is more or less than 1000" (van den Heuvel-Panhuizen, 2001, 184). For this sort of question the answer is not a number, but a decision. Since an exact calculation is not necessary, a simplification is sufficient. Therefore, an estimate is an adequate, but not a sufficient solution strategy:

> However, in order to answer the estimation question with some degree of certainty, some reasoning is still necessary. With the problem about the total number of points, students must be able to perceive that rounding up three times (besides the fact that the sum of the rounded off numbers is only 900) allows them to be certain that the total must be under 1000 (van den Heuvel-Panhuizen, 2001, 184).

This quotation points out two aspects: It is not only important to take into account the impact of estimations strategies in an operational way, as emphasized by Star and Rittle-Johnson (2009), but it is another relevant concept of computational estimation to perceive that an estimate is directly intertwined with the original problem. In other words, an estimate is a model for an exact calculation. This means, pupils must understand that there is an interrelation of an estimate and the exact calculation and an estimate finally has to be interpreted (adjusted) to lead to a correct answer of an indirect estimation question.

Combining the concepts pointed out by Sowder and Wheeler (1989) on the one hand, and Star and Rittle-Johnson (2009), and van den Heuvel-Panhuizen (2001) on the other hand, the concepts of computational estimation can be summarized as follows:

— Role of Approximate Numbers

— Multiple Processes/Outcomes

- Appropriateness
- Interrelation of an estimate and exact calculation (operational and conceptual)

Not investigated before, the last concept became one of the four foci of a study on the concepts and procedures of grade four pupils solving direct and indirect estimation questions (cf. Hunke, 2012) and leads to the following question:

To what extent do pupils respect the concept of the interrelation of an estimate and exact calculation when they solve indirect estimation questions ("Is it enough money?")?

Since this aspect is closely related to the necessity of interpreting an estimate, the findings in this paper will be discussed along the following questions (Q)

Q1: How do children arrive at the answer to an indirect estimation question when they start with an estimate? Do they interpret their estimates?

Q2: If the situation should arise – How do they carry out such an interpretation?

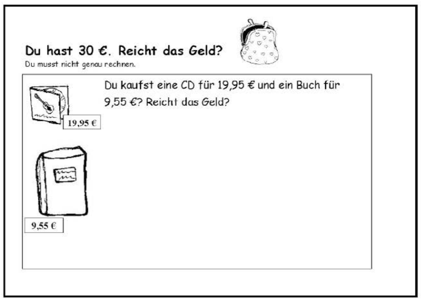
Q3: What are possible reasons for not interpreting estimates?

METHOD

Since the estimation process is quite complex it would be difficult to get insights into pupils' concepts and thinking only by analysing written documents and outcomes of a task (cf. Heirdsfield, 2002; Schoen et al., 1987). Carpenter, Coburn and Reys (1976, 299) pointed out that "to obtain a valid measure of a student's ability to estimate it is probably necessary to observe that student estimating". Thus, all in all 42 fourth grade pupils from four different primary schools in the state of North Rhine-Westphalia (Germany) were interviewed. To gain a broad and balanced overview of strategies and concepts pupils showing low, average and high performance in mathematics were chosen for the interviews. This ranking was made according to the teachers' evaluation of the pupils. However, the pupils' background was not deeper taken into account for analysis.

Two different sets of problems were used in the interviews, following the idea of van den Heuvel-Panhuizen (2001) to differentiate between direct and indirect estimation questions. As pointed out earlier, especially the indirect estimation questions require the concept of *interrelation of an estimate and exact calculation* to be answered successfully. One problem set only consisted of direct estimation questions ("*Approximately how much does it cost?*"), where a number as a result was necessary. The other set only consisted of indirect estimation questions ("*Do you have enough money?*"), where a mathematically based decision had to be made (see Figure 1). With regard to the aforementioned research questions, only data gained by solving the indirect estimation questions are taken into account for this paper.

The settings contained six addition and six multiplication problems. Whole numbers as well as decimals were used. To get an insight in the concepts of computational estimation, some of the problems contained numbers that make an interpretation necessary when a conventional rounding strategy is used first (e.g., the problem in Figure 1). Since there also were also other research foci not discussed in this paper, the numbers in some of the other problems were rather chosen to operationalize other aspects of computational estimation (Hunke, 2012, 117ff.). Nonetheless, all these problems were taken into account for the following analysis.



You have got € 30. Do you have enough money?
An exact calculation is not necessary.

You buy a CD for € 19.95 and a book for € 9.55. Do you have enough money?

Figure 1: Indirect estimation question (addition) (cf. van den Heuvel-Panhuizen, 2001)

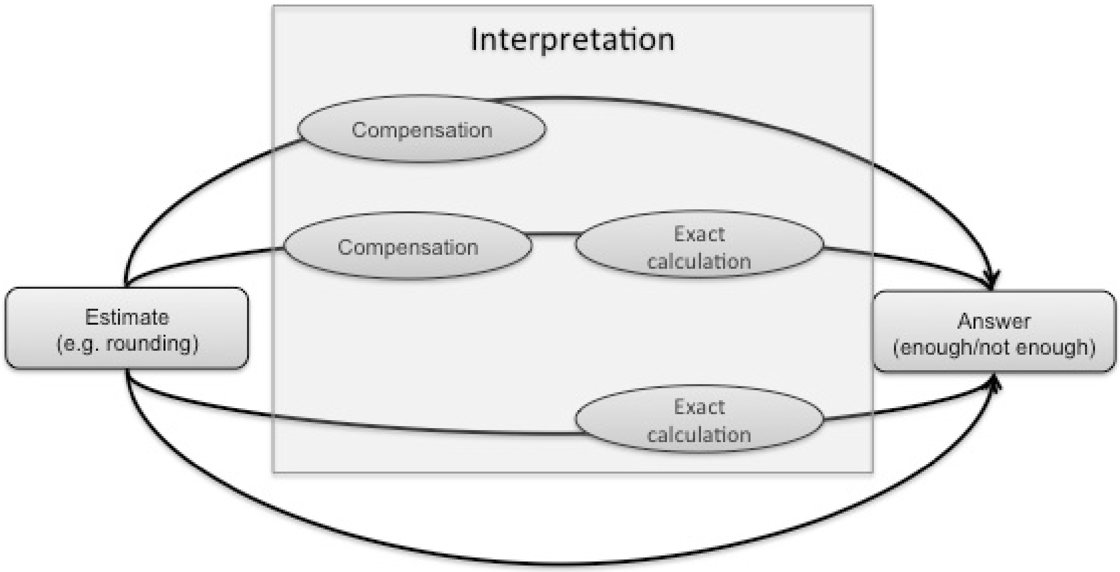


Figure 2: Four approaches of finding an answer to an indirect estimation question

All interviews were video recorded and transcribed for data analysis. Afterwards the data were coded along the four research foci. Therefore, categories were developed according to the theoretical framework, the research questions as well as to the data itself (cf. Schmidt, 1997).

FINDINGS

Data analysis on behalf of question 1 (Q1) led to four considerably different approaches (categories) how children deal with their estimates gained by rounding, to arrive at the answer to the "Do you have enough money?"-problems [1]. Each approach is represented by one arrow in Figure 2. It shows that an interpretation of the estimate, which would indicate a certain amount of conceptual knowledge, is not naturally included in the children's approaches.

Q2: *How do the children carry out such an interpretation?*

The children who use one of the upper three approaches all show a certain amount of conceptual understanding, because, to some extent, they recognize the interrelation of their estimate and exact calculation as they conduct some sort of interpretation. The following excerpt of the interview with Franziska illustrates the second approach 'estimate – compensation – exact calculation – answer' used by her for finding an answer to the addition problems illustrated in Figure 1. Franziska starts with an estimate and then argues as follows:

> *"You have got € 30. You buy a CD for € 19.95 and a book for € 9.55. Do you have enough money?"*

Franziska: That would be 30 Euro. *Since I rounded up here and here, well, I don't believe that it makes 30 all together.* Give me a second, now I calculate it like this (*writes down and executes an exact calculation, see Figure 3*). Yes, because here, I rounded up and rounded up, it's actually more, and thus 30 Euro.

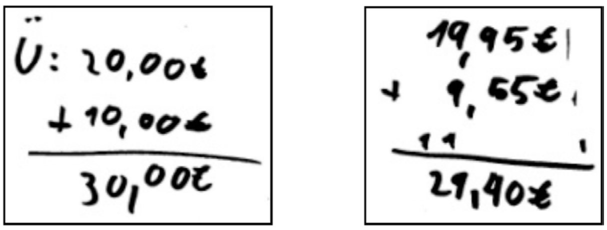


Figure 3: Franziska's estimate and subsequent exact calculation

Franziska is conscious of the fact that the estimate is 'only' a model for an exact calculation, which for this certain problem has to be interpreted. She carries out this interpretation with a compensation strategy first ("Since I rounded up here and here"). But it seems that Franziska does not feel comfortable with this because she checks her arguing by an additional exact calculation. At least, this example shows that Franziska already has an understanding of the concept of the interrelation of an estimate and the corresponding exact calculation from an operational as well as from a conceptual perspective. On the contrary, some children had difficulty in carrying out a compensation strategy although they were conscious about the need of it (cf. Hunke, 2012). Nonetheless, Franziska has to gain more confidence in her number sense, because for this certain problem a further exact calculation was not necessary.

Q3: *What are possible reasons for not interpreting estimates?*

Next to those three approaches, that to some extent reflect conceptual knowledge, it is the fourth of the aforementioned approaches that seems to be especially interesting, since it represents those children who do not take into account the interrelation of an estimate and the exact calculation. In fact, some pupils directly generate their answers from their estimates, which is exemplified by Jessica's and Anna's reasoning:

As well as Franziska, Jessica easily finds an estimate for the problem "*Larissa has got € 25. She wants to buy seven wooden ornamental letters to put on her bedroom door. Each letter costs € 3.55. Has she got enough money?*":

Jessica: [...] I prefer taking € 4 ... multiplied by 7, makes 28! Thus, it's not enough money, she hasn't saved enough. € 3 are missing.

4€ · 7 = 28

Figure 4

She rounds up the price from € 3.55 to € 4 that leads to an estimate above the budget. Jessica then directly generates her answer to the problem on basis of her estimate (approach 4) instead of taking into account her rounding. This is evident by her statement "€ 3 are missing". Thus, Jessica finally gets an incorrect answer to the problem. In fact, due to the process of rounding up, the exact result must be smaller than € 28 and thus some fine-tuning is still necessary to find out if eventually there is enough money. This demonstrates that for Jessica an estimate seems to be an autonomous form of calculation with no relation to the original task. Consequently, it has to be supposed that she does not yet have a comprehensive understanding of the concepts of computational estimation.

More or less the same can be stated for Anna. In addition, her reasoning leads to a further problem which emphasizes the importance of the concept of computational estimation concerning the interrelation of an estimate and the exact calculation. Anna solves all the problems with an exact calculation at first. She only makes estimates when the interviewer explicitly encourages her to do so. For some problems this leads to two competing results as in the following case: *"You have got € 30. You buy a CD for € 15.55, a woolly hat for € 5.99 and a scarf for € 8,98. Do you have enough money?"*

At first, Anna chooses the written algorithm to get to the exact result of € 29.42, and according to that she arrives at the answer that she has got enough money. After that, the interviewer asks her to solve the problem with an estimate as well. Consequently, she gets to the estimate 16 + 9 + 6 = 31, and comes to the conclusion that "it doesn't work with an estimate, actually":

Interviewer: So, what is the answer, finally?

Anna: There is enough money, because 58 Cent are left, but you cannot buy anything expensive in addition to that.

Interviewer: And what about the estimate?

Anna: Yes, well...(speaks quietly) 15 plus 9, 25. It's 31 with estimation. *That doesn't work.*

Here, the assumption comes up that, although Anna is able to execute an estimate, she regards it as an autonomous calculation form which leads to another conclusion to the question than the exact calculation, and therefore, to Anna's mind, estimation is not possible here. Thus, Anna's approach is also categorized as approach number 4 – she tries to generate her answer to the question directly on the basis of her estimate. This can be a hint that Anna does not have a comprehensive understanding of the concepts of computational estimation. In Anna's case this might as well be the reason for preferring exact calculations. For her, estimation does not seem to make sense, and thus she resigns it.

CONCLUSION

This paper supports previous findings that computational estimation is a complex task for which a comprehensive understanding is needed. Theoretical analysis of the concepts of computational estimation led to a hitherto unmentioned and therefore unexamined concept of computational estimation, namely the concept of the interrelation between an estimate and the exact calculation.

The empirical snapshot underlined the theoretical considerations that this concept is a very important aspect of understanding computational estimation. Of course, these results cannot be generalized because of the small scope of the study. Nonetheless, for the case of indirect estimation questions it could be pointed out that although the subjects of this study were generally able to carry out an estimate they did not necessarily understand what an estimate really is. This especially was apparent in those approaches where the children directly generated an answer to the question "Do you have enough money?" from their estimates. While this approach is sufficient for direct estimation questions ("Approximately how much...?"), it is not generalizable for every type of estimation problem.

In consequence, indirect estimation questions turned out to be a useful tool to make this certain concept of computational estimation visible. At the same time the findings give useful indications for teaching computational estimation. If we want to teach computational estimation with understanding, we need to provide tasks that allow pupils to develop the aforementioned concepts. Indirect estimation questions as used in this study seem to be adequate for this because they give the opportunity to discuss the interrelation of the estimate and the exact calculation from an operational as well as from a conceptual perspective.

Although estimation questions like "How much does it cost approximately?" (direct estimation question) and "Do you have enough money?" (indirect estimation question) have been typical estimation tasks in textbooks in the German primary classroom before, their respective potential does not yet seem to be utilized. According to van den Heuvel-Panhuizen (2001) a learning-teaching trajectory is suggested that starts with indirect estimation questions. Not only can these questions evoke other strategies than rounding (Hunke, 2012; van den Heuvel-Panhuizen, 2001), but they can also be used to discuss the concepts of approximation and interrelation, *before* the children learn any estimation strategy that might be executed without understanding. Thus, further research is needed. Design experiments therefore seem to be especially desirable. Pursuing such an experimental approach, further research questions could be "Which design principles are needed for fostering children's understanding of computational estimation? How can direct and indirect estimation questions be used systematically in the primary classroom? Which numbers are especially suitable to make this certain concept obvious? How can children be encouraged to evoke more number sensed based strategies for computational estimation problems?". Finally it is particularly necessary to support teachers to become more sensitive for this very important topic, especially what it means to teach estimation with understanding.

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ENDNOTE

1. More than one half of the indirect estimation questions have been solved with an estimation strategy, mainly rounding (Hunke, 2012).

The impact of a teaching intervention on sixth grade students' fraction understanding and their performance in seven abilities that constitute fraction understanding

Aristoklis A. Nicolaou[1] and Demetra Pitta-Pantazi[2]

1 Ministry of Education of Cyprus, Strovolos and University of Nicosia, Egkomi, Cyprus, educati@cytanet.com.cy

2 University of Cyprus, Strovolos, Cyprus

In a previous study, we found that students' abilities in fraction recognition, definitions and explanations, argumentations and justifications, relative magnitude of fractions, representations, connections and reflection constitute fraction understanding of sixth grade students. In the present study, we examine the impact of an intervention comprising of lessons for developing the seven abilities on students' fraction understanding and their performance in the seven abilities. The sample comprised of 343 sixth grade students. Repeated measures analysis showed that the students of the experimental group outperformed those of the control group in their level of fraction understanding and their ability in fraction recognition, definitions and explanations, argumentations and justifications, connections and reflection.

Keywords: Fraction understanding, repeated measures analysis, sixth grade, students' abilities, teaching intervention.

INTRODUCTION

Fraction complexities and students' difficulties have led a number of researchers in the past to study fraction understanding carrying out research programs and teaching experiments (e.g. Cramer, Post, & delMas, 2002). In the present study, we examine the impact of a teaching intervention which is based upon a different perspective compared to the studies already carried out. More specifically, in a previous study conducted by the authors (Nicolaou & Pitta-Pantazi, 2011), we confirmed a theoretical model with seven abilities that constitute fraction understanding at elementary school. The seven abilities were: (a) fraction recognition, (b) definitions and mathematical explanations for fractions, (c) argumentations and justifications about fractions, (d) relative magnitude of fractions, (e) representations of fractions, (f) connections of fractions with decimals, percentages and division, and (g) reflection during the solution of fraction problems. Five of these abilities, definitions and mathematical explanations, argumentations and justifications, representations, connections and reflection correspond to the mathematical processes suggested by NCTM (2000) that are important for understanding mathematical concepts. We consider that apart from the various issues related to fraction understanding that were already investigated, students' ability to engage in processes such as reflection during the solution of fraction problems, explanations for fractions, argumentations and justifications about fractions, representations of fractions and connections of fractions with decimals, percentages and division are essential for fraction understanding. Additionally, we included fraction recognition and the relative magnitude of fractions in the abilities required for fraction understanding at elementary school (the reasons for this decision are provided in the next section "Theoretical Background").

In the present study, we examine the impact of a teaching intervention comprising of lessons for developing the seven abilities on sixth grade students' fraction understanding and their performance in the seven abilities. The aim was twofold: (a) To examine the impact of the intervention on sixth grade students' fraction understanding, and (b) To examine the impact of the intervention on students' performance in each of the seven abilities.

THEORETICAL BACKGROUND

The seven abilities

According to NCTM (2000), mathematical processes are very important for understanding mathematical concepts. Based on this rationale, we considered that students' abilities in these processes are also important for fraction understanding at elementary school. We also included two more abilities as explained below. In this section we will briefly refer to these seven abilities (for more details see Nicolaou & Pitta-Pantazi, 2011).

Reflection during the solution of fraction problems refers to students' ability to reason their thinking and their answer while solving fraction problems, to support the reasonableness of their answer and verify a given answer. In order to solve fraction problems, students should be able to use the various fraction sub-constructs; they should be able to think of fractions embedded in mathematical problems corresponding to these sub-constructs and utilize them accordingly to solve the problem. Moreover, students should be able to carry out operations with fractions. Argumentations and justifications about fractions refer to a kind of "informal proof" at elementary school. For the purpose of the present study, argumentations and justifications refer to students' ability to judge statements about fractions as true or false, justifying at the same time their choice. Definitions and mathematical explanations for fractions refer to students' ability to define in their own words what a fraction is and also to explain in various ways (verbally, by using drawings, examples etc.) other issues concerning fractions (e.g., fraction equivalence, comparison, density). Connections refer to students' ability to connect fractions to other forms of rational numbers (decimals and percentages) and division. Representations of fractions refer to students' ability to translate to visual, verbal and symbolic representations and their ability to construct drawings for fractions.

In order to propose a model that would provide a sufficient description of fraction understanding, we also included fraction recognition and relative magnitude of fractions in the abilities required for fraction understanding. Fraction recognition refers to students' ability to recognize structural characteristics of fractions and detect similarities and differences between fractions. It also includes students' ability to categorize fractions on the basis of a common characteristic. The relative magnitude of fractions refers to students' ability to compare and order fractions and is crucial for fraction understanding, since in the case students are not able to compare and order fractions, then they probably do not understand the meaning of fractions.

METHODOLOGY

The rationale of the intervention – Its principles

The intervention comprised of activities which aimed at developing the seven abilities. The design of the intervention was based on some principles. According to the first principle, the activities and the problems should be interesting, arise from everyday life and attract students' interest (Elbers, 2003). The second principle referred to the sequence of the activities from the easiest to the more difficult ones. The third principle was about the way students worked; either individually or in small groups favoring discussion and exchange of ideas (Elbers, 2003; Martino & Maher, 1999; Terwel, Van Oers, van Dijk, & van den Eeden, 2009). After working individually or in small groups, students discussed their ideas in the whole classroom and this procedure helped them to share their views and interact (Elbers, 2003; Martino & Maher, 1999; Terwel et al., 2009). Additionally, in every activity students had to explain their thinking and provide adequate reasoning for their decisions. The fourth principle referred to the modification of the lesson plans according to the strengths and weaknesses of the students and their previous knowledge of fractions. The fifth principle referred to the role of the teacher as a facilitator of learning. While students worked individually or in small groups, the teacher moved around the classroom, provided support and feedback. During the whole class discussion and exchange of ideas, the teacher was expected to guide the discussion and pose questions that would stimulate further inquiry (Martino & Maher, 1999).

Participants, instruments and procedure of the study

Participants in the present study, which was quantitative in nature, were 343 sixth grade students (age range 10.8–11.8 years old). A test comprising 37 tasks for measuring the seven abilities was developed. The tasks of the test were content and face validated by four experienced primary school teachers and two

Fraction recognition	One of the following fractions differs from the others. Find that fraction and circle it. $\frac{2}{7}$ $\frac{3}{2}$ $\frac{14}{49}$ $\frac{10}{35}$ $\frac{4}{14}$
Definitions and mathematical explanations for fractions	Imagine that your teacher asked you to explain to one of your classmates what a fraction is. Use as many different ways you can.
Argumentations and justifications about fractions	If I double both the numerator and the denominator of a fraction, then the formed fraction has twice value compared to the initial one. T F Justify your answer:
Relative magnitude of fractions	Order the fractions $\frac{1}{2}, \frac{4}{3}, \frac{2}{3}, \frac{1}{4}$ starting from the smallest one.
Representations of fractions	Marinos ates $\frac{1}{2}$ of a cake and Marinas $\frac{3}{8}$ of the same cake. Construct a drawing to show what part each child ate and what part the two children ate together.
Connections of fractions with decimals, percentages and division	Convert the following fractions to decimals. a) $\frac{1}{4}=$ b) $\frac{2}{5}=$ c) $\frac{3}{10}=$ d) $\frac{1}{20}=$
Reflection during the solution of fraction problems	In order to prepare a cake, I need $\frac{1}{10}$ L milk. If I have $\frac{4}{15}$ L milk, then how many cakes can I prepare? **You should definitely reason about your answer.**

Table 1: Tasks used to measure each ability

university tutors of mathematics education before their administration. Tasks 1–3 were used to measure fraction recognition, tasks 4–7 definitions and mathematical explanations for fractions, tasks 8–12 argumentations and justifications about fractions, tasks 13–15 and 32–37 were about reflection during the solution of fraction problems, tasks 16a-16c and 17 were about the relative magnitude of fractions, tasks 18–27 measured representations, while tasks 28–31 examined students' ability to link fractions to decimals, percentages and division. Examples of tasks used to measure each one of the abilities are presented in Table 1.

The first measurement was conducted at the beginning of the school year in the first two weeks of October. After the first measurement, the participants were split into experimental and control group. The experimental group comprised of eight classes (144 students), while the control group comprised of eleven classes (199 students).

The intervention comprised of nine lessons, four of which had a duration of 80 minutes, two 60 minutes, while the other three 40 minutes. The total time devoted was 14x40 minutes. The time devoted to the development of each of the seven abilities was about the same (2–3 40 minute periods) with the exception of the relative magnitude of fractions, which was devoted less time (about 40 minutes). The reason for this is that similar activities for developing this kind of ability are included in school textbooks. It should also be stressed that some lessons aimed at developing more than one ability and comprised of activities that served towards this goal.

The implementation of the intervention started at the end of October after the first administration of the test for measuring the seven abilities (pre-test). Its duration was about nine weeks until the end of January before the second administration of the test (post-test). The teachers that participated in the study were asked to teach one lesson every week. The students of the control group during this time were taught according to the Cyprus National Mathematics Curriculum which included topics such as recognizing representations of fractions, some activities of explaining what a fraction is, recognizing fractions as the division of the numerator by the denominator, equivalent fractions, fraction comparison and ordering, improper fractions and mixed numbers, decimals and percentages and their conversion to fractions. It must be noted that the total time devoted to fraction teaching was about the same for the two groups.

After the completion of the intervention, all students (including those of the experimental group) were taught according to the Cyprus National Mathematics Curriculum until the end of the sixth grade. They practiced simplifying and comparing fractions, adding and subtracting fractions with the same or different denominator, adding and subtracting mixed numbers, solving fraction problems, fraction multiplication and division, solving problems of mixed numbers and multiplication and division of mixed numbers. About three months after the second measurement, a third measurement took place (retention-test) for examining the duration of the impact of the intervention.

The lessons

The activities designed for implementing the goals of the intervention shared a scenario that was challenging, attractive and pleasant for students. The scenario referred to the trip of the "Mathematician" to the land of fractions. The trip comprised of various activities and the participants were called to answer questions and solve some problems. The scenario was designed in order to please students and motive them to engage in the activities with enthusiasm.

The first lesson with duration 80 minutes aimed at developing students' ability in "defining" what a fraction is, recognizing fractions in various representational systems, placing fractions on number line and linking the concept of fractions to the division numerator ÷ denominator. The goal of the second 80 minutes lesson was the development of students' ability to connect fractions to the other two forms of rational numbers, decimals and percentages and was complementary to the first lesson. During the lesson, the teacher raised questions: "What are decimals?", "What are percentages?" that would lead to the definition of decimals and percentages. Students were also asked to convert fractions to decimals and percentages and the reverse. Emphasis was placed on understanding the conversion and not just applying the rules. Students also worked on textbook activities. The third 40 minutes lesson cultivated students' ability to construct visual representations of fractions and acquire a feeling of the relative magnitude of fractions. Students worked in pairs and were encouraged to construct various representations. Students also worked on textbook activities that referred to fraction comparison and ordering by utilizing visual representations. The fourth lesson with duration 80 minutes referred to the development of students' ability to convert verbal and symbolic representations to visual ones and vice versa. The fourth lesson was complementary to the first and third lessons regarding the development of students' abilities in representations. In some activities, problems of fractions were presented to students and they had to write the equation and construct drawing/drawings in order to solve them (translation from verbal to symbolic and visual representations). Other activities asked students to translate from symbolic to visual and verbal representations. In these activities, the equation was given and students had to write a problem that could be solved by this equation or construct a drawing. Finally, students were called to write problems on the basis of visual representations (from visual to verbal representation). Lessons 5 and 6, with duration 60 minutes each, aimed at developing students' ability in reflection during the solution of fraction problems. Five carefully selected problems were presented to students and they had to solve them and were encouraged to reason their thinking, explain the strategy they used, express their confidence about their solution, examine whether the path they followed was correct or not and what they did correct and what wrong. Furthermore, they were called to think about the reasonableness of their answer and verify their answer. The seventh and eighth lessons with duration 40 minutes each aimed at developing students' abilities in fraction recognition, mathematical explanations for fractions, justifications about fractions and reflection. The two lessons included activities requiring students to detect fraction similarities, fraction differences and write fractions that share a common property. The ninth 80 minutes lesson aimed at developing students' abilities in argumentations and justifications about fractions and reflection.

Statistical analyses

To answer the two research questions about the impact of the teaching intervention on students' fraction understanding and their performance in the seven abilities, the z-scores that emerged from the Confirmatory Factor Analysis utilized in a previous study carried out by the authors (Nicolaou & Pitta-Pantazi, 2011) were used. Descriptive statistics (means and standard errors of estimate) for students' fraction understanding and the seven abilities were first found for the control and the experimental group respectively. Afterwards, Repeated Measures Analysis was used with dependent variable the score for fraction understanding and then for each of the seven abilities separately and independent variables the kind of condition (control or experimental) and the time of the measurement (pre-test, post-test, retention-test). The independent samples t-test was also used to test for the equality of the two groups in the pre-test.

RESULTS

Table 2 shows means and standard errors of estimate of fraction understanding for the experimental and the control group for each of the three measurements.

Table 2 illustrates that before the intervention, the two groups of students were equal with respect to their level of fraction understanding. This was also confirmed by the application of t-test (t=.182, p=.856>0.05). In the period of the implementation of the intervention, both groups improved, but the improvement of the experimental group was greater. In the period three months after the intervention the two groups continued to improve and the difference found in the post-test was maintained in the retention test. Repeated Measures Analysis showed that the means of fraction understanding differed significantly between the three measurements (Pillai's $F_{(2,340)}$=.516, p<0.01), and additionally, there was a statistically significant interaction of fraction understanding with the kind of the condition (Pillai's $F_{(2,340)}$=.029, p<0.01). Therefore, the differences found between the two groups of students could be attributed to the implementation of the intervention which gave the experimental group superiority over the control group.

Descriptive statistics for students' performance in each of the seven abilities are shown in Table 3.

From Table 3, we observe that the experimental and the control group had about equal means for all the seven abilities at the pre-test. This was also confirmed by the application of the t-test, which showed the equality of the two groups of students before the conduction of the intervention (for fraction recognition t=-0.603, p>0.05; for definitions and mathematical explanations t=0.170, p>0.05; for argumentations and justifications t=0.600, p>0.05; for relative magnitude of fractions t=-0.203, p>0.05; for representations t=0.901, p>0.05; for connections t=0.827, p>0.05; for reflection t=-0.641, p>0.05). Regarding the evolution of students' abilities, the experimental group showed continuous improvement during the period of the three measurements in the relative magnitude of fractions, representations, connections and reflection, whereas this was not the case for fraction recognition, definitions and mathematical explanations and argumentations and justifications where there was improvement from the pre-test to the post-test and then a decline in the period three months after the intervention. Concerning the control group, students' ability in the relative magnitude of fractions and reflection showed continuous improvement in the period of the three measurements. For representations, a very small improvement was observed during the period from the pre-test to the post-test, but this improvement was much greater in the period from the post-test to the retention test. Students' ability in fraction recognition and connections had an increase during the period from the pre-test to the post-test and then a decline in the period from the post-test to the retention-test. For definitions and mathematical explanations and argumentations and justifications there was improvement in the period between the pre-test and the post-test and then stabilization. The improvement of the students of the control group in some of the abilities can be attributed to maturity and teaching. Nevertheless, both at the post-test and at the retention-test the experimental group outperformed the control group in all the abilities with the exception of the relative magnitude of fractions where the two groups were about equal. The application of Repeated Measures Analysis revealed that the means for all the seven abilities differed significantly between the three measurements. Additionally, for five abilities: fraction recognition, definitions and mathematical explanations, argumentations and justifications, con-

	Pre-test		Post-test		Retention test	
Group		SE*		SE		SE
Experimental *(n=144)*	11.48	0.49	17.04	0.51	18.56	0.60
Control *(n=199)*	11.37	0.42	15.28	0.44	16.45	0.51

*SE: Standard Error of Estimate

Table 2: Means and standard errors of estimate of fraction understanding for the experimental and the control group for each of the three measurements

	Fraction recognition					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	1.91	0.05	2.43	0.05	2.17	0.05
Control	1.95	0.05	2.22	0.04	2.01	0.04
	Definitions and mathematical explanations for fractions					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	0.50	0.04	0.84	0.04	0.70	0.04
Control	0.49	0.03	0.59	0.04	0.59	0.03
	Argumentations and justifications about fractions					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	0.99	0.07	1.60	0.07	1.42	0.07
Control	0.94	0.06	1.25	0.06	1.26	0.06
	Relative magnitude of fractions					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	1.70	0.08	2.27	0.08	2.49	0.08
Control	1.73	0.06	2.27	0.07	2.41	0.07
	Representations of fractions					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	3.06	0.10	3.29	0.10	3.87	0.12
Control	2.95	0.09	2.98	0.09	3.63	0.10
	Connections with decimals, percentages and division					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	4.67	0.31	7.25	0.32	7.51	0.33
Control	4.59	0.26	6.43	0.27	6.18	0.28
	Reflection during the solution of fraction problems					
	Pre-test		Post-test		Retention test	
Group		SE		SE		SE
Experimental	1.02	0.08	1.85	0.11	2.05	0.13
Control	1.08	0.07	1.52	0.09	1.69	0.11

Table 3: Means and standard errors of estimate of students' performance in each of the seven abilities for the experimental and the control group for each of the three measurements

nections and reflection there was an interaction with the condition (experimental or control) (the results are presented in Table 4). Therefore, the higher means of the experimental group at the post-test and retention-test can be attributed to the implementation of the intervention, which according to the results of the Repeated Measures Analysis was effective in improving students' abilities in fraction recognition, definitions and mathematical explanations, argumentations and justifications, connections and reflection.

DISCUSSION

The results of the present study revealed that the intervention was effective in developing sixth-grade students' fraction understanding and their abilities

	Between the three measurements	Interaction with the condition
Ability	Pillai's $F_{(2,340)}$	Pillai's $F_{(2,340)}$
Fraction recognition	0.248**	0.031**
Definitions and mathematical explanations	0.188**	0.065**
Argumentations and justifications	0.322**	0.046**
Relative magnitude	0.316**	0.002
Representations	0.289**	0.008
Connections	0.239**	0.021*
Reflection	0.342**	0.038**

*p<0.05, **p<0.01

Table 4: Results of Repeated Measures Analysis for each of the seven abilities

in fraction recognition, definitions and mathematical explanations for fractions, argumentations and justifications about fractions, connections with decimals, percentages and division and reflection during the solution of fraction problems. Therefore, the content of the intervention and the principles followed were effective in improving students' fraction understanding and their performance in five out of the seven abilities.

The conduction of the retention test added value to the findings of the present study, as it permitted the extraction of conclusions about the duration of the effects of the intervention. The conduction of three measurements also provided some evidence for the evolution of each of the seven abilities for the students of the control classes that did not receive the intervention.

For two of the abilities, the relative magnitude of fractions and representations of fractions, the intervention did not have significant effects. Concerning the relative magnitude of fractions, the results might be attributed to the fact that the curriculum provided much emphasis in developing this kind of ability. For representations, the results can also be due to the moderate emphasis given by the curriculum towards developing students' ability in representations of fractions. However, in the case of representations it must be stressed that the experimental group was superior to the control group both in the post and the retention test, but this superiority was not statistically significant.

The contribution of the present study in the area is situated in that the intervention focused on the abilities that are essential for fraction understanding at elementary school, while the focus of other studies was different (for example, Lamon (2012) examined fraction understanding addressing fraction sub-constructs). The present study also has implications for teaching fractions with understanding as teachers can utilize the content of the intervention for developing students' fraction understanding and their performance in the five abilities applying at the same time its principles.

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Processes of mathematical reasoning of equations in primary mathematics lessons

Marcus Nührenbörger[1] and Ralph Schwarzkopf[2]

1 Technische Universität Dortmund, Dortmund, Germany

2 Carl von Ossietzky Universität Oldenburg, Oldenburg, Germany, ralph.schwarzkopf@uni-oldenburg.de

Learning of mathematics in primary classes should not be reduced to learning algorithms and routines or procedures. As we see the learning of mathematics as a process of interpreting mathematical structures and generalizations, it is more important to foster children`s thinking and learning of meaningful relations between objects and operations in the context of equations. In this sense children have to learn algebraic relations in primary classes without using algebraic signs. In this paper the results of a design study and a video-based qualitative analysis of teaching/learning situations in the field of reasoning of equations are discussed.

Keywords: Argumentation, equation, algebraic thinking.

INTRODUCTION

This paper is about the well-known problems around the equal-sign in mathematics learning processes: Many pupils understand the equal sign as a clear action-symbol, e.g. the left side of the sign has to be interpreted as a computing-demand, whereas on the right side there can only be found a single number, i.e. the solution of the computing-task (Seo & Ginsburg, 2003; Kieran, 2011; Russel, 2011; Steinweg, 2013). Those pupils don´t accept equations like "8 + 5 = 10 +3" or "7+9 = 2 · 8" because they offend against the clear rule to have the computing term on the left and the result on the right side of the equal sign. For most situations in primary level, this "task-result-interpretation" (Winter, 1982) of equations may be sufficient. But, having in mind the algebra of secondary level curricula, someday the learners will have to overcome this restricted view on equations.

> It's more important, I think, to give them many opportunities to use symbols in many situations than simply to tell them, let´s say: this is the equals sign and what´s one side of the equals sign should be the same as what´s the other side of the equals sign (Seo & Ginsburg, 2003, p. 169).

To realise such *learning opportunities* we need rich learning environments that make algebraic structures accessible to children in primary schools. But, this is not enough. In the following, we give an example to illustrate that the rich learning environment alone does not lead to thinking about equalities in a structural way. Afterwards, we will discuss the necessity of argumentation-processes to motivate *fundamental learning processes*, i.e. to support the children on their way to an algebraic view on equalities. At least, we will give an example from our design study of how to initiate such *collective argumentations*.

An example of a fourth grade class

The example took place in a learning unit that was on the structure of "*Rechendreiecke*" (for an example see Figure 1), a substantial learning environment based on the well-known "arithmogons" (Wittmann, 2001): Every two numbers in the inner fields of the triangle are added, and the sum is noted in the appropriate field outside of the figure.

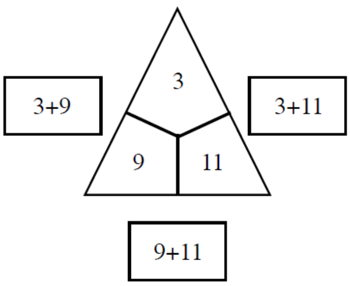

Figure 1

At the beginning of the unit the teacher gave the children some ordinary tasks in computing, i.e. the numbers for the inner fields were given and the children had to calculate the sums for the fields outside of the triangle. After some of these standard-exercises the children made the first discovery on the essential structures of the triangles: The sum of the outer numbers is twice the sum of the numbers in the inner fields, because every inner number appears in two outer fields.

In the present lesson, the teacher´s goal was to discuss the most difficult tasks within this learning environment: She wanted the children to find the inner numbers for a triangle with given numbers in the outer fields. So, she wrote an example for this kind of task on the blackboard (Figure 2), and asked the whole class for first ideas about how to solve this problem. The teacher clearly expected that the children would use the discovery mentioned above to find out the inner numbers: Firstly, they could calculate *the sum* of the inner numbers (by halving the sum of the given outer numbers). Then, they could find a fitting partition of this sum for the inner numbers by trying a systematic way.

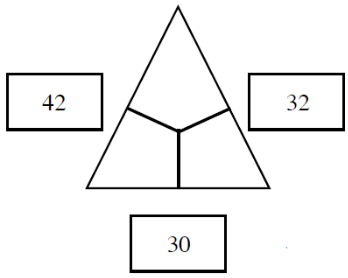

Figure 2

Having in mind this problem-solving approach, the teacher asked Robert for his idea:

Robert	I think that this is the same problem as we had with the number-walls, that 30 plus 32 take away 42 divided by two, the result must be noted in the bottom field (while talking, the pupil arrives at the blackboard).
Children	(make some noise to demonstrate their incomprehension)
Robert	Then we will have the ten there (notes 10 in the bottom field on the left) and twenty there (notes 20 in the bottom field on the right, then notes 12 in the field on the top, Figure 3)

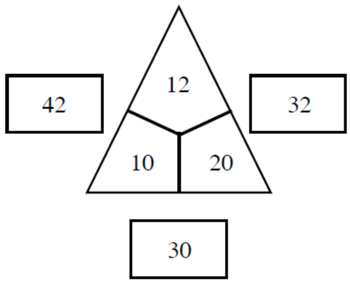

Figure 3

Teacher	Okay, now we have a ten here and a twenty over there (points to the numbers in the figure on the blackboard). How can we go on? The idea of Robert is not bad!
Clarissa	We could change the places of the twenty and the ten. Because, I think it would fit better. Then one would note twenty-two at the top and we had the result. (Clarissa modifies the numbers, Figure 4)

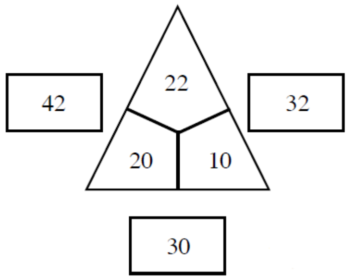

Figure 4

After this episode, the teacher and the pupils seemed to be satisfied, having found the fitting inner numbers. Neither the pupils nor the teacher not even Robert himself had the need to analyse whether it is coincidence or not that Clarissa only had to change the places of Robert's numbers in order to find the right solution of the triangle. Hence, even though the class found a solution for the special task, the underlying mathematical structure of the general problem remained covered. In other words: The *learning opportunity* given by the idea of Robert did not unfold its structural potential, we name it a "missed substantial learning opportunity".

Briefly, two questions remain, resulted from our observation of this episode:

1) Why is Robert's idea so hard to understand?

2) Why is there obviously no need to analyse the proposal of Robert, and no need to question whether the idea may lead to a general way of problem solving?

Analysis: The mathematical core of the missed learning opportunity

Even if Robert´s proposal does not lead to the correct solution, the underlying idea is correct: Given the inner numbers *a, b, c,* Robert firstly builds the sum of two outer numbers (i.e. $a+b+b+c$). Afterwards, he takes away the third outer number ($a+c$), the result is the double of one of the inner numbers (here $2b$).

Although we do not want to shift the mathematics curriculum of the secondary level into the primary school, from our point of view, it should be possible to discuss arithmetical structures like the one above in a fourth grades class. For example, one could organise the learning process within a reflective exercise (s. Figure 5), forcing the children to calculate with the outer numbers in the way Robert does: "Build the sum of two outer numbers and take away the third one – what do you observe?" Having in mind that every outer number is built by the sum of two inner numbers, the children might be able to explain the result of their calculations, and then to use this new knowledge as an effective solving strategy for problems like the one above.

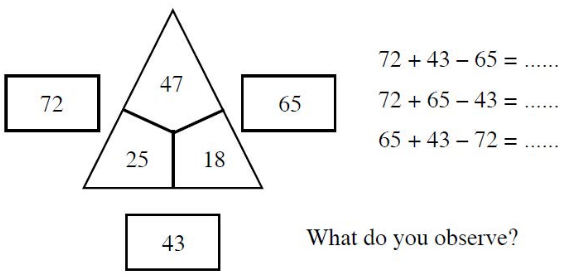

Figure 5

The main difficulty on the way to find an explanation like this is the necessary flexibility in interpreting the inner and outer numbers: Sometimes, they are computing numbers, sometimes they are the result of a computation and the problem-solver has to decide which number would be used to calculate and which one must be understood as a computing-result.

From our point of view, we should help the children to construct a flexible understanding of equations already in primary level. But, again, we do not want to shift the mathematics curriculum of the secondary level into the primary school. Hence, we propose to promote the development of a content-related, flexible *understanding of equality* rather than teaching the children *how to handle equations* in an adequate form to prepare algebraic notations. According to Steinbring (2005), the children in primary school should work on a flexible concept of mathematical equality mainly by constructing several adequate reference contexts rather than learning how to use the standard mathematical signs within the solution of equations.

THE PRESENT PROJECT

Within our project we started a variety of different teaching-learning experiments, like whole class instruction, group working and peer interviews. The experiments were planned on the basis of already existing substantial learning environments, mainly taken from the project "mathe 2000" (Wittmann, 2001). Our goal is to strengthen the *content-related concept* of equality. Hence, the equal *sign* and its algebraic correct and formal rules to use do not play the leading role within the learning environments. In some of them (like arithmogons, number walls, etc.; Wittmann, 2001), this special sign does not even appear in the tasks. But nevertheless, the learning environments focus on equality. For example, the children have to find different number walls with equal numbers in the top stones.

Our main research interest is to understand the micro-processes of teaching and learning mathematics rather than to measure the success of a learning environment. Hence, our analysis of the teaching-learning-experiments follows the interpretative paradigm, mainly using approaches of symbolic interactionism and ethnomethodology (Bauersfeld, Krummheuer, & Voigt, 1998; Voigt, 1994; Yackel & Cobb, 1996), epistemological theories (Steinbring, 2005) and theories of argumentation (Schwarzkopf, 2003). In the following sections we discuss some aspects of the first results of the study.

Theoretical embedding: Different types of learning processes

What does it mean to modify a restricted "task-result" interpretation of the equal-sign to a more sophisticated, flexible and structurally sustainable concept of equality? Following Steinbring (2005), the construction of an adequate equality-concept moves between two epistemological poles:

On the one hand, there is an *empirical, situated description* of mathematical knowledge like finding the result for a computing task in the sense of Clarissa´s offer for the solution of the problem above. This kind of knowledge is easy to handle in communication, one simply has to offer some empirical facts, and everybody will know what they mean. But, in conclusion, the pupils can only learn new facts; the associated learning opportunities will not help them to construct a better structural understanding of the mathematical background.

On the other hand, the understanding of Robert's idea for example, requires a *relational generality* of mathematical knowledge. Learning opportunities that concentrate on this kind of knowledge have to offer the need for the children to create a new interpretation of the thematic mathematics. At the same time one has to consider that the pupils cannot create interpretations that are completely detached from their experienced points of view.

These fundamental learning processes (cf. Miller, 1986, 2002) can only be realised in situations of *collective argumentations*, i.e. the children must be confronted with a problem that makes it somehow impossible for them to go on by routine and they first have to solve that problem in an argumentative way (Miller, 1986; Schwarzkopf, 2003). But, according to Miller, argumentation is a very stressful kind of interaction, and, normally, people try hard to solve their problems in a non-argumentative way.

Even within mathematics classrooms, where argumentation is one of the learning goals, there are many opportunities to avoid a content related argumentation – at least one can always ask the teacher as an expert in mathematics. Remember the example above: There is no need for an argumentation around the idea of Robert because Clarissa found an easier way to solve the problem by changing numbers, and presenting directly the correct solution. In the sense of Steinbring (2005, p. 194–213), the offers of Clarissa and Robert stand for the two poles of communication between which the interactive constructions of knowledge move: Whereas Clarissa´s gives a simple "mediation of facts" (correct calculations), an adequate understanding of Robert´s idea needs a "construction of a new interpretation" that is obviously not accessible to the communication partners.

Hence, collective argumentations do not emerge in mathematics lessons in a somehow natural way. On the contrary, the teacher has to *initiate* the concerning processes in both, a careful and persistent way (Schwarzkopf, 2000). For this, it is important to think about social requirements and mathematical learning goals that are necessary to initiate substantial learning opportunities by cooperative argumentation.

To enforce the emergence of substantial learning opportunities, we develop tasks that initiate mathematical needs for collective argumentation. Our intention is to confront the children with a "productive irritation" (Nührenbörger & Schwarzkopf, 2013), concerning their social experiences in classroom discussions. The tasks or problems, for example, provide phenomena that were not expected by the children so that they have to reflect on the given structures and see the need to re- interpret the experienced mathematics behind the problem.

This approach bases on Piaget´s (1985) work on cognitive conflicts. Roughly speaking, Piaget points out that a child develops new ideas when it is confronted with facts or beliefs that contradict their expectations, depending on their individual experiences. If the concerning cognitive schemas resist the child´s possibilities of assimilation, there is a need for the child to generate a cognitive consensus, i.e. a learning-process emerges.

To initiate productive learning opportunities in this sense, one of the main difficulties is that an observation of a pattern or a surprising discovery is not enough to create the need of argumentation – we gave an example at the beginning of this paper. Moreover, it is necessary that the observation becomes an amazing phenomenon for the pupils. For example, the children discover a pattern in a series of tasks. Then, they are confronted with another task that does not exactly fit to the previously solved series and they are asked to make a prognosis: Will the result of the next task fit to the pattern or not? By this we try to force the children to create an expectation on the result of the next task. The initiation of a "productive irritation" is successful, when this expectation fails while computing the next result – by this, there is an interactive need to find explanations for the failure of the prognosis. In this sense, a productive irritation can widen the implicit working, often sense-restricting "sociomathematical norms" (Yackel & Cobb, 1996) that influence the interpretation of arithmetical terms in routinized (cf. Voigt 1994) classroom discussions.

Initiating productive irritations: Arguing for amazing equalities

In the following part we give an example for a short learning unit with the goal to initiate an argumentation, initiated by a productive irritation within a primary class (3rd or 4th class). The content comes from the well-known substantial learning environment "number-walls" (see Figure 6), where you have to note the sum of the numbers in every stone from the two stones under it. Typically, children discover different types of terms to describe the top stone of the wall (e.g. as a result of a calculation or as a relation between the bottom stones: $a+b + b+c = a+2b+c$).

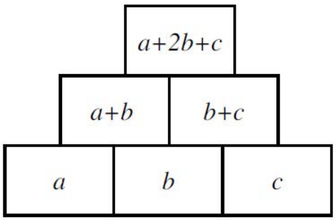

Figure 6

In the first part of the present learning unit, the children calculate a series of number-walls and discover a pattern, that is very familiar to them: Increasing the number in the right bottom-stone and decreasing the number in the left bottom-stone by the same difference will leave the number at the top of the wall constant (keeping the same number in the middle of the bottom, of course) in this example 650 (Figure 7).

The children can calculate the number of the top as well as they can argue with the relations between the stones of a number wall. To point out the equalities between the number walls the children can use so called "term walls" (see Figure 8), noting calculations instead of their results in the stones. These term-walls might build bridges for the children to change their interpretations of the numbers as results (e.g. 650 = 460 + 190 and 650 = 450 + 200) and to see them as computing numbers (650 = 380 + 2 × 80 + 110 = = 370 × 2x80+ 120).

However, having calculated some of these tasks, the children are confronted with another number-wall that does not exactly fit to the previously discovered pattern (e.g., the one in Figure 9), and they are asked about their expectations on the result: Will the number at the top of the wall change or not?

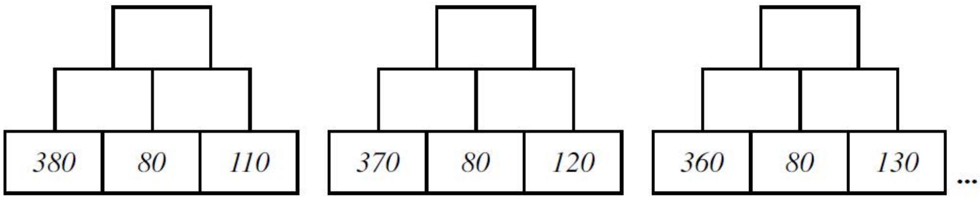

Figure 7

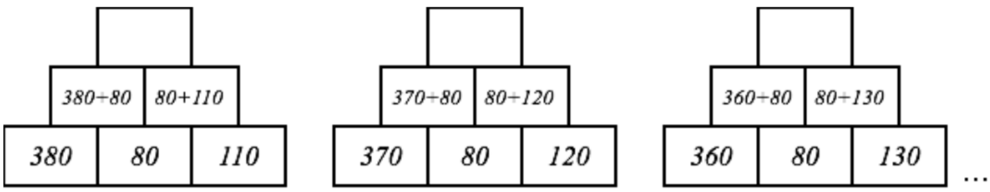

Figure 8a

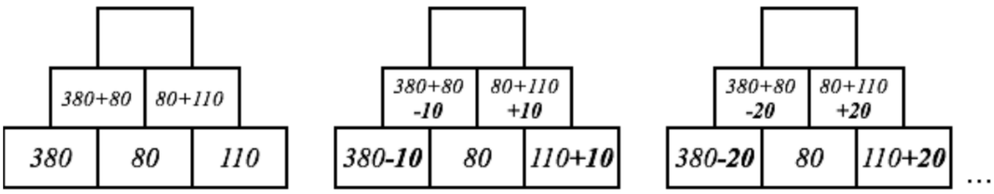

Figure 8b

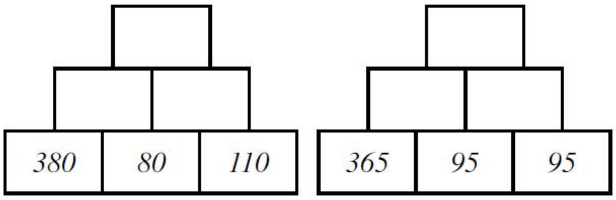

Figure 9

The following episode takes place in an interview-situation, where a teacher-student works together with two children on the above given number walls. The children have already discovered that the top numbers of the number-walls in Figure 7 are all the same, namely 650. This pattern seemed to be clear for the children, having in mind that from wall to wall one of the bottom numbers increases and one of the others decreases by the same difference. Afterwards, the teacher showed the children some number walls like the one in Figure 9, asking for a prognosis about the number of the top-stone. As expected, both of the children gave the prognosis that the top stone must change, because two of the bottom-numbers increase by fifteen and only one of them decreases by fifteen.

Teacher	Do you think that in this top stone (pointing to the right number-wall in Figure 9) there will be the 650 again?
Moritz	no.
Luisa	no.
Teacher	Why do you say no?
Moritz	Well, here are fifteen more than there (points to 380 and 365), and here are fifteen more than there (points to 110 and 95 in the right bottom stone), but here are fifteen more than there (points to 95 in the middle bottom stone and 80), and fifteen plus fifteen are together thirty and not fifteen (.), exactly.

Moritz reasons by comparing the changes of the two increasing bottom numbers with the decreasing number. In this way, he activates his experiences with the constancy-law of additions that was successfully used in the first part of the unit. The teacher moderates the interaction, so that also Luisa has to reason for her prognosis:

Teacher	Luisa, do you have the same opinion?
Luisa	Yes, the result here above is the same.
Teacher	How do you mean it?
Luisa	Well, if you add the bottom stones (points to 365, 80, 110) then they must have the same result as there (points to 380, 95, 95) for getting the same result on the top, and this is another sum.

Luisa points out a hypothesis for a general rule: Two number walls with the same numbers in the top stones must be equal in the sum of the bottom numbers.

Even if the two given arguments are different in detail, from an epistemological point of view, both of them can be characterised as an *empirical, situated description of knowing* (Steinbring, 2005; Schwarzkopf, 2003): The comparisons of the (unknown) top numbers are based on empirical observations of the given examples without leading to a structural deeper understanding of the mathematical structures.

After the children have verified their hypothesis and found out that their observation offended against their expectation, they rethink the arithmetical structures between the stones of the number walls and find new arguments:

Luisa	Mmm, they are the same because this (points to the left and right bottom stone, Figure 10) is coming to the other stones, and this (points two times to the middle bottom stone) meets twice. So we have to calculate them together.

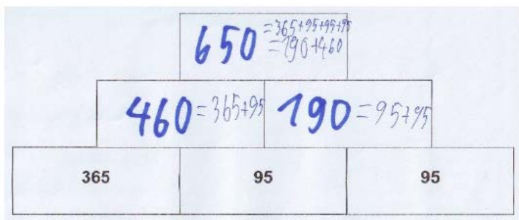

Figure 10

Moritz	Mmm, because in the middle we have 460 is equal 365 plus 95 (.) that is 460 (.), and because there are not four bottom stones you have to calculate once again 95 plus 95 is equal 190 (.). You have to take once again the 95.
Teacher	Ok, and why do we get in both number walls the same number 650 in the top?
Moritz	Because you need the 95 for both sides. However, for example, here you have to

> take 15 more and here exactly the same (points to both number walls).

In their arguments, both children become aware of the special function of the number in the middle bottom stone. On the first view, Luisa argues in a somehow dynamic, still empirical way – as if the numbers would crawl through the stones to the top. But regarding her offer in detail, the argument already shows qualities of a more theoretical approach: The middle stone "meets twice", although the number exists only once. This discovery builds the bridge from the empirical view to the structural understanding of number walls: The meaning of the numbers result of their position in the wall. Moritz modulates this approach (Krummheuer, 1992) of Luisa to a more static view, and by this he strengthens the theoretical character of the argumentation: The number in the middle bottom stone is not a concrete, single object, but it is part of two calculations. His remark "there are not four bottom stones" builds the bridge between the empirical understanding of the numbers as objects (that crawl through the wall) and the numbers as part of calculations that depend only on their position in the wall: An empirical version of the number wall would have four bottom stones (see Figure 11), providing every number as often as it will be needed in the wall.

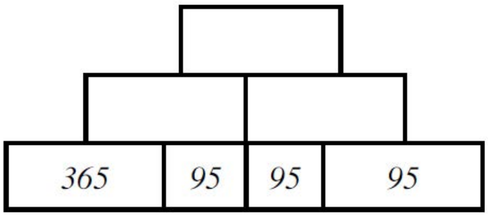

Figure 11

Finally, Moritz is able to interpret the new view on the number walls on his hypothesis of the beginning of the episode: Changing the numbers in the bottom stones means changing the operators in the calculation terms.

The main aspect within this argumentation seems to be that the numbers of the task have changed their computational functions: Previously, the pupils interpreted the number wall in the sense of a task-result-view on equations: At the bottom there are three numbers, and at the top there is one result. Within the argumentation for the unexpected equality, they changed this view to a more flexible, theoretical one (Figure 12): No longer the numbers, but the computing-terms are the main objects of the number-wall (Steinweg, 2013). In this sense, the children have to construct this as a common object a new term concerning twice the decrease of 15 and the increase of 15: $(380 + 80) + (80 + 110) = (380 - 15 + 80 + 15) + (80 + 15 + 110 - 15) =$ $= (365 + 95) + (95 + 95)$.

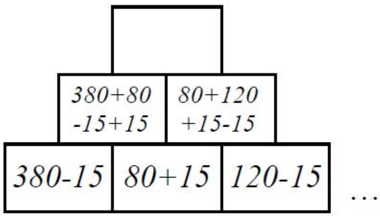

Figure 12

In conclusion, the pupils construct a new interpretation of the arithmetical relations which is related to their old knowledge. According to Steinbring (2005), the children construct a *relational generality* of mathematical knowledge.

FINAL REMARKS

By initiating substantial learning opportunities we try to promote the development of a flexible and structural sustainable concept of mathematical equality. In our work we mean by "understanding equalities in primary classes" that the children operate with the structures of computing-terms rather than only focussing on pure numbers as if they were concrete objects. To understand equality between two terms means to find one theoretical interpretation that fits for two different looking terms (Winter, 1982). In this paper, we discussed a somehow reflective approach to the promotion of accompanying activities, concerning the initiation of collective argumentation through productive irritations.

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The rules for the order of operations: The case of an inservice teacher

Ioannis Papadopoulos

Aristotle University of Thessaloniki, School of Education, Thessaloniki, Greece, ypapadop@eled.auth.gr

In this paper a two-stage project is presented concerning the rules for the order of operations. During the first stage the mal-rules used by an experienced teacher as he evaluated arithmetical expressions were recorded and a session for repairing these misinterpretations followed. During the second stage the influence of the teacher's teaching on his sixth graders was examined. The findings showed that the initial understanding of the teacher was so persistent that almost all his students and in order to evaluate the same arithmetical expressions used exactly the same mal-rules.

Keywords: Mal-rules, order of operations.

INTRODUCTION

The major issue in mathematics teaching and learning is whether students really know mathematics or whether they just memorize rules or conventions. The rules for the order of operations that are introduced during the fifth and sixth grades constitute a representative example. Even though, one can argue that the precedence rules are a matter of procedural knowledge. Lampert (1986) claims that there are contexts that order of operations matters, and according to Merlin (2008) the order of precedence is not simply the sort of convention adopted without careful consideration; rather, it reflects something essential and deep about the operations themselves. Students are taught that symbols of inclusion (i.e., ( ), [ ]) can be used to show which operation is to be performed first in an expression. If there are more than three operations then there would be so many parentheses and brackets that the expression would look extremely complex. Therefore, in order to avoid this, mathematicians have agreed on an order for performing the operations and the parentheses are used only to change this order (Foerster, 1994). The rules for the order of operations are:

i. Brackets first

ii. Evaluate expressions with exponents

iii. Carry out multiplications or divisions from left to right

iv. Carry out additions or subtractions from left to right.

For young students these rules might appear random and therefore meaningless. So, according to Wu (2007) the situation is like this: students are encouraged to memorize things without understanding their use or meaning and teachers are also becoming part of the game as they know that exam points can be surely achieved by this useless memorization. This way of teaching order of operations results in some overgeneralizations that are used equally by primary school students (Linchevski & Herscovics, 1994), middle school students (Blando, Kelly, Schneider, & Sleeman, 1989), university students (Pappanastos, Hall, & Honan, 2002), and prospective elementary teachers (Glidden, 2008). However, there is no research on the teaching practice of in-service teachers in this topic. This leaves out in-service teachers. Therefore, in this study we try to find out how such misconceptions of an experienced in-service primary school teacher may influence the way his students conceive the topic of the order of operations.

BRIEF LITERATURE REVIEW

Perhaps, the whole story starts during the early years of schooling when teachers use problems of the type 5 + 2 x 3 + 10 − 5 = ?. In these problems each operation is followed by "and then" (i.e., 5 plus 2 and then what you found times 3 and then that answer plus 10, etc.). Indeed, very often this kind of problems is considered as a proper one since the students are encouraged to think and carry out mental operations. However, this may – unintentionally- contribute to students' difficulties related to the order of operations convention (Schrock & Morrow, 1993). They disregard the order of operations practice and therefore learn that operations are simply worked from left to right (Kieran, 1989). Linchevski and Livneh (1999), working with 53 sixth graders, found that possibly students have generalized incorrectly the rule for the order of operations and believe that addition takes precedence over subtraction and multiplication over division. This is not unrelated to the fact that in many textbooks the order of operations is given via the known mnemonic BOMDAS (Brackets first then "Of", Multiplication, Division, Addition and Subtraction) (Herscovics & Linchevski, 1994). PEMDAS (Please Excuse My Dear Aunt Sally, which stands for Parentheses, Exponents, Multiplication and Division (from left to right), Addition and Subtraction (from left to right)) is another example of students' reliance on mnemonics. Pappanastos, Hals, and Honan (2002) surveyed over 300 business school students at two universities and the results showed that (a) they recalled the above mentioned acronym, and (b) one third of the respondents applied incorrectly the order of operations. Rambhia (2002) states that as a result of PEMDAS focused teaching many students are convinced that multiplication has to be done before division and that addition is more important than subtraction. Thus, PEMDAS and similar mnemonic devices either hide or assist the learning of operations. But what is especially interesting in the case of these mnemonics is that their use is a symptom of a lack of attention to structure. Finally, Glidden (2008) who investigated how well 381 prospective elementary teachers solved four arithmetic problems that required using the order of operations found that fewer than half of them answered more than two questions correctly.

All these findings are important since (a) teachers must know the correct order of operations to teach the concept correctly, and (b) light might be shed on whether students do in fact understand the order of operations completely or merely interpret the mnemonics literally (Pappanastos et al., 2002). However, despite the importance of these findings there is an aspect that is not represented in the body of the research literature. The above mentioned presented studies concern how very young students overgeneralize the order of operations (Linchevski & Herscovics, 1994), and then how older students tend to perform the operations sequentially from left to right (Kieran, 1979), and then how college students use mnemonics devices to remember order of operations since they do not understand the proper order of precedence that should applied to mathematical operations (Pappanastos et al., 2002), and then how prospective elementary school teachers give incorrect answers attributable to misunderstanding the order of operations (Glidden, 2008). All this incorrectly learned information perhaps continues to handicap learning into their next math courses and beyond for several years. So, what if an in-service teacher did not manage to "repair" this incorrect knowledge acquired years ago? How this influences the way he teaches the specific concept to his young students? How easily these errors are transferred as an incorrect knowledge to them? Actually, these are the research questions of this paper.

THE SETTING OF THE STUDY

This is a two-stage study. During the first stage a collection of tasks was designed to uncover errors with precedence rules. The collection of the tasks was administered to an in-service primary school teacher before teaching the unit of order of operations to his 6th graders. Four of the tasks are presented in Table 1.

Item 1:	$14 : 2 \cdot 14 - 12 =$
Item 2:	$18 + 19 + 14 \cdot (11 + 22) =$
Item 3:	$9 + 23 + (11 - 3) \cdot (4 : 2) =$
Item 4:	$7 \cdot (6 \cdot 6 + 14) \cdot 5 + 6 =$

Table 1: Tasks of the study

The answers of the teacher were examined and the incorrect answers were coded on the basis of the work of Blando and colleagues (1989). In their work, errors are mentioned as *mal-rules* which stand for violations of legal mathematics rules. Their theoretical model relies on *repair theory* which states that errors occur when a student is faced with a difficult or unfamiliar feature of a task. In this case the student may react by modifying a known procedure and applying it (incorrectly) to the task. Blando and his colleagues (1989) described the students' incorrect solutions in term of mal-rules. In relation to Precedence Errors they listed six mal-rules using acronyms (see examples in Table 2).

PAM	Add before multiplying	Example: 4 + 2 x 3→ 6 x 3
PAD	Add before dividing	Example: 10 / 2 + 3→10 / 5
PSM	Subtract before multiplying	Example: 9 – 2 x 3→7 x 3
PSD	Subtract before dividing	Example: 8 – 6 / 2→2 / 2
PAS	Add before subtracting	Example: 6 – 4 + 3→6 – 7
PIP	Ignore Parenthesis	Example: 8 - (2 + 4)→6 + 4

Table 2: List of mal-rules for precedence errors (from Blando et al. (1989))

Thus, in our study, the teacher's errors were coded according to the above mentioned list.

In Item-1 the solution given by the teacher was:

$14 : 2 \cdot 14 - 12 = (14 : 2) \cdot (14 - 12) = 7 \cdot 2 = 14$

The general format of this task is $a : b \cdot c - d$, the correct rule is $[(a : b) \cdot c] - d$ and one of the possible mal-rules is $(a : b) \cdot (c - d)$. This could be regarded as "subtract before multiplying" (PSM) mal-rule. The teacher correctly made the division 14 : 2. But then he ignored the multiplication and chose to subtract 14 - 12 before multiplying.

In Item-2 the solution of the teacher was:

$18 + 19 + 14 \cdot (11 + 22) = (18 + 19 + 14) \cdot (11 + 22) =$
$51 \cdot 33 = 1683$

The general format of the task is $a + b + c \cdot (d + e)$. The correct solution is $a + b + [c \cdot (d + e)]$ and the mal-rule applied by the teacher was $(a + b + c) \cdot (d + e)$, which means "add before multiplying" (i.e., PAM). The teacher correctly evaluated the part inside the parentheses but then he violated the precedence rule and carried out addition prior to multiplication.

In Item-3 the answer given by the teacher was:

$9 + 23 + (11 - 3) \cdot (4 : 2) = [(9 + 23) + (11 - 3)] \cdot (4 : 2) =$
$(32 + 8) \cdot 2 = 40 \cdot 2 = 80$

Even though the general format seems to be more complex than the previous one, the core idea of the teacher's error was the same. He did not ignore the parentheses (11-3=8, 4:2=2), however, he added before multiplying (PAM) instead of following the correct rule (i.e., $9 + 23 + [(11 - 3) \cdot (4 : 2)]$).

Finally, in Item-4, the PAM mal-rule was again prevalent:

$7 \cdot (6 \cdot 6 + 14) \cdot 5 + 6 = [7 \cdot (6 \cdot 6 + 14)] \cdot (5 + 6) =$
$[7 \cdot (36 + 14)] \cdot 11 = 7 \cdot 50 \cdot 11 = = 350 \cdot 11 = 3850$

The correct solution was $[7 \cdot (36 + 14) \cdot 5] + 6 = (7 \cdot 50 \cdot 5) + 6 = 1750 + 6 = 1756$. For one more time what actually the teacher did was to add (5 + 6 = 11) before multiplying (PAM). Once more he respected the parentheses but then he incorrectly evaluated the expression 5+6 before multiplying. These answers signalled the need for an intervention session aiming to let the teacher become aware of his errors. Research has suggested that interventions focused on developing teachers' subject-matter knowledge can positively impact teachers' knowledge (Swafford et al.. 1997) and makes them able to improve their ability to make sense of and evaluate students' thinking strategies in a variety of mathematical context (Tyminski et al., 2014)

The session was decided to take place at school and lasted about 90 minutes. During the session a lot of examples were discussed offering the opportunity for the teacher to internalize (a) the order of operations as an introduction to the necessity for structure and rules, and (b) the need to have a unique answer in tasks and also to have rules required to achieve it. For each example it was clarified that there was a unique answer. The application of various mal-rules result to different outcomes but only one answer is the correct and certain rules are required to achieve it. When multiple operations are included in an expression, different numerical results are obtained according to the precedence given to some operations over others. Visual representations were used to illustrate the notion of operation precedence. More specifically, the *tree diagrams* were used (Kirshner & Awtry, 2004). Their central feature is an iterative procedure for analysing the syntactic structure of a mathematical expression, and representing it as a partially ordered hierarchical structure. In a tree diagram the operation that is more precedent appears lower on the tree than an operation less precedent. In this tree notation it is easy to see the independence of

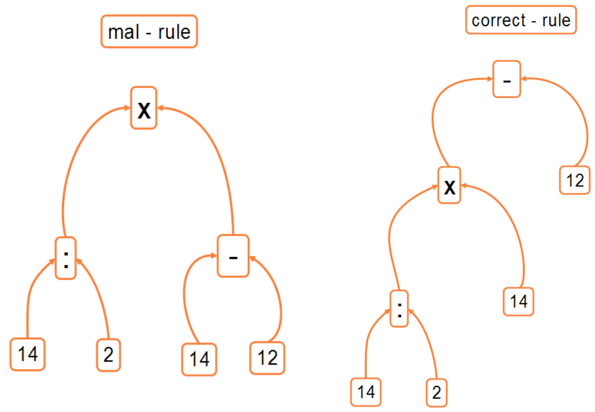

Figure 1: Tree diagrams for Item-1

the operations in the separateness of the tree's main "branches". In Figure 1 two tree diagrams illustrating the precedence decision for Item-1 are presented.

Following Ernest (1987), the teacher's errors might be attributed to a failure to discern the hierarchical syntactical structure or orders of precedence within mathematical expressions. So the tree model was used exactly to exhibit this structure explicitly to the teacher. He was 'trained' to analyse expressions into written tree forms.

It is worth mentioning that the teacher was an experienced one in the sense that he had been teaching for at least 25 years. This means that during this period he was carrying this incorrect perspective about the order of operations which may influence the learning of his students. Consequently, the question is: To what extend an intervention can sufficiently help the teacher to correct his stable errors relevant to order of operations?

This is why the teacher was left to teach the specific unit to his students which lasted for about a week. The second stage of the study took place almost a month after completing the unit and at that moment the teacher was not aware of our intention to come back again and work with his 6th graders. This decision was made on purpose since it was important to let the teacher work on his own terms and not under the feeling that he has to be cautious due to our future return into his classroom. We requested permission to work with the students on this topic. He gladly agreed and confessed that in the meantime he was wondering about the possible impact of this experience on him and/or on his students.

So, the students were invited to cope with the same collection of the tasks that was initially administered to their teacher. Twenty two 6th graders participated. The study took place a month after the beginning of the school year. The students had been taught comparison between natural and decimal numbers as well as the four operations among these numbers. Their worksheets were collected and their incorrect answers were coded according to two criteria: (a) whether their incorrect answers corresponded to the mal-rule list of Blando and colleagues (1989), and (b) whether their answers reflected their teacher's incorrect ones.

RESULTS AND DISCUSSION

The students' answers were classified into three categories:

(i) Incorrect answers that were identical to their teacher's ones (i.e., the ones during the first stage of the study).

(ii) Answers that were correct, and

(iii) Answers that did not fit to any of these two categories (i.e., the students did not answer the item at all or the students answered but there was not a clear way of showing their thinking for the computations).

	Incorrect answers identical to the teacher's ones (students followed mal-rules)	Correct solutions (correct application of the rules for the order of operation)	Other
Item-1 (PSM)	6 (27.27%)	12 (54.54%)	4 (18.19%)
Item-2 (PAM)	**19 (86.36%)**	3 (13.64%)	-
Item-3 (PAM)	**18 (81.81%)**	3 (13.64%)	1 (4.55%)
Item-4 (PAM)	**12 (54.54%)**	4 (18.19%)	6 (27.27%)

Table 3: Summarized view of the student's answers

A summarized view of the students' answers is presented below in Table 3.

It is interesting that many students used symbols of inclusion that facilitated their erroneous way of evaluating the arithmetical expressions. Even though they followed mal-rules for the order of operations they respected the symbols of inclusion they used. Some examples of the students' usage of inclusion symbols that supported their incorrect generalization for the order of operations can be seen in Figure 2.

It is evident that almost the whole class reacted exactly in the same way their teacher did a month before. The percentages for Items 2–4 are extremely high. More than half the students repeated the mal-rule of their teacher for Item-4 (12/22), all but three students for Item-2 (19/22), and all but four students for Item-3 (18/22). It seems that this did not happen for Item-1 since only 6 students out of 22 repeated the mal-rule of "subtract before multiplying" (PSM). A potential explanation for this might be that the students were accustomed to work sequentially from left to right. During their early grades the students were given exercises which disregarded the order of operations and therefore many of them learned incorrectly that operations are simply worked from left to right. The fact is that in Item-1 following the operations from left to right happened to be the same as following the rules for the order of operations and this possibly explains the small percentage of the PSM mal-rule.

The results were also equally surprising for the teacher since he realized how similarly incorrect were his own errors to his students' ones. So, the question is: Given that during the intervening session the rules for the order of operations were examined and became clear, then what would be a possible explanation for having almost all the students repeating their teacher's initial errors?

Before presenting our thesis it has to be acknowledged that teachers' content knowledge in the subject area does not suffice for good learning. However, it is also true that the knowledge of mathematics obviously influences the teachers' teaching of mathematics and subsequently they cannot help children learn things they themselves do not understand. This could explain the impact of the specific teacher in his students' performance for the time period before this study. The difference now was that the teacher was led to face his weak mathematical background concerning order of operations and moreover he participated in a session that made him to see why the rules he applied were mal-rules as well as to get practice on a series of tasks that challenged him to apply now the correct rules for the precedence of operations. He declared that he understood the violation of the rules for the order of operations he used to follow. However, the findings of the study did show that the teaching that took place after the session was dominated by his persistent misinterpretation on the order of operations. This contradiction may be explained by accepting that the session that took place was not sufficient to confront the teacher's erroneous long-time way of teaching. Thus, a stronger intervention might be needed to establish a more compact knowledge on order of operations accompanied with guidance concerning instructional strategies for the unit. Moreover, it can be said that learners generalize in a way that they are initially taught and this can lead to the construction of schemata at an early stage that have a strong inherent robustness (Waren, 2003). Linchevski and Livneh (2002) claim that occasionally these old schemata become tacit models of comprehension and this could mean that –as in our case- despite the intervening session, initial understanding persists.

$[(14 : 2)(14 - 12)] = 7 \cdot 2 = 14$

$[(9 + 23) + (11 - 3)] \cdot (4 : 2)\ (32 + 8) \cdot 2 = 40 \cdot 2 = \underline{80}$

$(18 + 19 + 14) \cdot (11 + 22) = 51 \cdot 33 = 1683$

$[7 \cdot (6 \cdot 6 + 14) \cdot 5] + 6 = [7 \cdot (36 + 14) \cdot 5] + 6 = [(7 \cdot 50) + 5] + 6 = (350 + 5) + 6$
$355 + 6 = \underline{361}$

Figure 2: Students' usage of inclusion symbols

CONCLUSIONS

The deep content knowledge of mathematics is – among others- necessary for teaching successfully mathematics. Until a few years ago, the subject matter knowledge of teachers was largely taken for granted in teacher education. But recent research focused on the ways in which teachers and prospective teachers understand the subjects they teach, reveals that they often have misconceptions or gaps in knowledge (Ball & McDiarmid, 1990). In the same paper Ball and McDiarmid also argue that as teachers are themselves products of elementary and secondary schools in which pupils rarely develop deep understanding of the subject matter they encounter, we should not be surprised by teachers' inadequate subject matter preparation. This was clearly presented in our study. An experienced teacher, teaching more than 25 years, introduced a specific mathematical topic (i.e., order of operations) based on certain mal-rules that probably influenced the quality of learning of his students.

It also seems that these misinterpretations of the correct rules for the order of operations made it difficult for the teacher to have control over his own teaching under the light of the new information. Despite a session that aimed to support the teacher to confront the situation the findings showed that he continued to be based on the same mal-rules. It seems that the session was not really efficient. This is verified by the fact that his students showed similar behaviour to his towards the same collection of tasks. Their responses followed the same incorrect formats that were used by their teacher. This is very helpful to depict clearly the situation in our country. And the situation is like this:

The mathematical content knowledge of primary school teachers takes place only during their university courses. Later on, for pre-service and in-service teachers there are training programs but all of them are focused on the new curriculum, on the new teaching methods, on the usage of technology in mathematics classrooms, etc. Implicitly, all these training programs take for granted that these people know mathematics and this is why they give emphasis to pedagogical aspects of mathematics teaching. So, in this paper we tried to show that this is a rather false image of the real situation. Even though the work of only one teacher was presented and this could be considered as a case study (and therefore we could not generalize) we still believe that this is evidence that cannot be ignored. Part of the training programs must give emphasis on the subject matter knowledge of the persons who are responsible for teaching mathematics in young students and influence by their teaching the mathematical thinking of their students.

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Cracking percent problems in different formats: The role of texts and visual models for students with low and high language proficiency

Birte Pöhler, Susanne Prediger and Henrike Weinert

Technische Universität Dortmund, Dortmund, Germany, birte.poehler@math.uni-dortmund.de

Students with low academic language proficiency are often considered to meet specific comprehension challenges with word problems. But how do conceptual and reading challenges interfere in these situations? We approach this question by investigating how performance depends on the problem format for different problem types. A test with N=250 students investigates cracking percent problems in pure, text, and visual format. The results show that text formats are most difficult for elementary problem types, whereas (con-)text can enhance the accessibility for more complex problem types. Item difficulties differ similarly for students with high and low language proficiency, hence reading challenges seem as crucial as conceptual challenges. A deeper analysis shows specific linguistic challenges with the expressions reduced to and reduced by.

Keywords: Percentages, word problems, visual models, language proficiency.

BACKGROUND

Word problems in mathematics tests – reading challenges or conceptual challenges for students with low language proficiency?

Secondary students' academic language proficiency is a crucial factor for their performance in mathematics tests, as has been found in various empirical studies (Abedi, 2006; OECD, 2007; Prediger et al., 2013). Especially in the context of high stakes testing in the US, language biases in mathematical word problems have been investigated for students with low language proficiency (in brief: low LP) (Abedi, 2006). Many researchers emphasize that students with low LP have specific *reading comprehension difficulties with word problems* (e.g., Duarte et al., 2011, for an overview) whereas test items with less text are often assumed to be "language fairer".

However, at least for the German high stakes test ZP10 NRW, this assumption turned out to apply only partially (Prediger et al., 2014): In nearly all items, language proficient students outperformed their less language proficient peers. But the items in which the former had even more difficulties than expected due to their general performance (the items with significant DIF-values) could *not* be characterized by reading challenges, but by conceptual or process-oriented challenges. We tentatively concluded that reading might not be the main obstacle for students with low LP to crack problems, but their restricted conceptual understanding accumulated in ten years of schooling with language disadvantages. However, the phenomenon requires a deeper investigation.

In many different mathematical topics and for students of all levels of language proficiency, word problems have proved to be more difficult than pure items (e.g., Kouba et al., 1988). For students with low LP, these text formats seem to pose specific challenges (cf. Duarte et al., 2011) that require further exploration. However, some studies (especially in primary schools) have shown that using contexts in problems can also support students' performance since a context can enhance the accessibility of the problem and the underlying mathematical concepts (van den Heuvel-Panhuizen, 2005).

Whereas the role of text and context is discussed incoherently in mathematics education research throughout age levels, there is a consensus on the role of visual models as having the potential to facilitate the accessibility of a test item (shown, for example, in Walkington et al., 2013). One could even assume that students with low LP do equally well as their more language proficient peers in visually presented items if there were no problems in conceptual understanding, only in text comprehension.

These different considerations motivated our research interest on comparing difficulties in *different problem formats*. We treated it for the exemplary mathematical topic of percentages with the following research questions:

— How do students perform in parallel test items on percentages with text format, visual format and pure format?

— How does the role of texts and visual models in test items on percentages differ for students with high versus low language proficiency?

Cracking percent problems as identifying different problem types

The mathematical topic percentages was chosen because it is important in many everyday contexts, and percent problems in assessments bear various difficulties for students (Parker & Leinhardt, 1995; Prediger et al., 2013). Compared to other areas of arithmetic and proportions, relatively few recent studies exist that explore students' competencies and difficulties, (historical exceptions are named in Parker & Leinhardt, 1995; recent exceptions are Dole et al., 1997; Jitendra & Star, 2012 and Walkington et al., 2013).

Typical for percent problems is that students' mathematizing process is shaped by one core step, *identifying the problem type* (Dole et al., 1997). Classically, *three elementary problem types* are distinguished (ibid., with different names): 'Find the amount (if rate and base are given)', 'Find the rate (if amount and base are given)', and 'Find the base (if amount and rate are given)'. Empirical studies show different success rates for different problem types, often 'Find the amount' is easier than the two others (e.g., Kouba et al., 1988, p. 17), and this type being overgeneralized to 'Find the base'.

Beyond these three elementary problem types, *more complex problem types* pose even bigger challenges for students, for example 'percentage growth', 'percentage comparison' or 'Find the base after reduction (if discount and reduced amount are given)' (Parker & Leinhardt, 1995, p. 439). These complex problem types bear reading challenges as well as conceptual challenges and are therefore interesting to compare to elementary types in this study.

Existing empirical studies have compared students' performances on percent problems mainly with respect *to problem types* (Kouba et al., 1988; Dole et al., 1997). In contrast, the comparison of *problem formats* have been less considered (an exception is Walkington et al., 2013). Furthermore, little is known on difficulties with percent problems of students with varying language proficiency. Especially the more complex problem types seem to pose additional comprehension challenges that are worth being considered in more detail. These indications and the limited state of research on percent problems suggest the following additional research question:

— How successful do students of low / high language proficiency identify different percentage problem types? What supports them for more complex problem types?

RESEARCH DESIGN

The study presented in this paper was conducted as a mixed methods study with a paper and pencil test on percent problems for N=250 students (age 13 to 15) in two countries, Germany and German-speaking Switzerland, and complementary interviews. Here, we mainly focus on the tests.

Test design

Language proficiency. Students' language proficiency was assessed by a C-Test in German, an economical and reliable measure of a complex construct of general language proficiency of first- and second-language learners (Grotjahn et al., 2002). By its specific construction in a gap text receptive and productive skills in lexical and grammatical areas are addressed.

Three problem types for percent problems. Students' varying performances for percent problems were measured by a paper-and pencil test with 15 items, systematically constructed in three selected problem types 'Find the amount', 'Find the base' and 'Find the base after reduction' (Table 1 shows the 14 items, some taken from Hafner, 2012, which are relevant for this paper). The problem type 'Find the rate' was omitted as it is the easiest to distinguish from the others by merely considering the involved units.

<table>
<tr><th colspan="2">Item set for problem type “Find amount”</th><th>Frequencies</th></tr>
<tr><td>Pure format</td><td>(Item Find the Amount) What is 5 % of 400 €? Find the amount.</td><td>78 %</td></tr>
<tr><td>Visual format</td><td>(Item Bar Amount) How many GB have already been downloaded? Find the missing value.

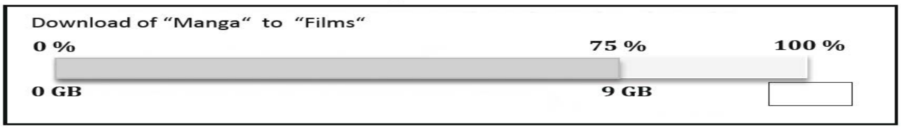
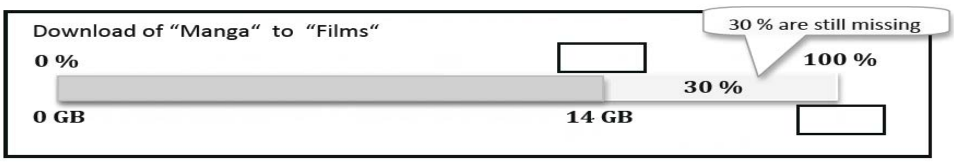
</td><td>67 %</td></tr>
<tr><td>Text formats</td><td>(Item 1 Potatoes) Potatoes consist of 75 % water.
How much water (in g) is contained in 1000 g potatoes?
(Item 2 Foundation) A school transfers 60 % of the revenue earned with a school celebration to the “Aktion Mensch” (a foundation). The revenue was 1400 €. How much money does the school transfer?
(Item 3 Sport Festival) 30 % of the revenue won at a sport festival in the context of a tombola, being at the amount of 700 €, were dedicated to philanthropy. How much was the donation?</td><td>46 %
I1: 33 %
I2: 61 %
I3: 45 %</td></tr>
<tr><th colspan="3">Item set for problem type “Find base”</th></tr>
<tr><td>Pure format</td><td>(Item Find the Base) 30 % are 60 €. Find the base.</td><td>67 %</td></tr>
<tr><td>Visual format</td><td>(Item Bar Base) What is unknown here? Find the missing value.

</td><td>67 %</td></tr>
<tr><td>Text formats</td><td>(Item 4 Jeans) Mr. Koch pays 40 € for jeans in the summer sale.
The jeans were reduced to 80 %. How much did the jeans cost before?
(Item 5 Kitchen) When buying a new kitchen, Family Mays receives a discount of 250 €, that was 5 % of the regular price. What is the normal price of the kitchen?
(Item 6 Holiday Trip) Mrs. Fuchs has prepaid 40 % of the price for her holiday trip. These were 800 €. How expensive is the trip?</td><td>53 %
I4: 34 %
I5: 62 %
I6: 63 %</td></tr>
<tr><th colspan="3">Item set for problem type “Find base after reduction”</th></tr>
<tr><td>Pure format</td><td>(Item Find the Base after reduction) Calculate the former price (base).
New price: 750 € Discount: 25 %</td><td>44 %</td></tr>
<tr><td>Visual format</td><td>(Item Bar Reduction) What is unknown here? Find the missing values.

</td><td>59 %
67 % for percent, 51 % for base</td></tr>
<tr><td>Text formats</td><td>(Item 7 Dress) Mrs. Schmidt pays 30 € for a dress in the summer sale.
The dress was reduced by 40 %. How much did the dress cost before?
(Item 8 Cross trainer) A customer buys a cross trainer in a shop. She pays 450 € for the equipment. As she is a member of a sports club, she receives a discount of 10 %. What is the normal price of the cross trainer?</td><td>41 %
I7: 39 %
I8: 43 %</td></tr>
</table>

Table 1: Item sets in three different formats for three problem types (translated) with frequencies of correct identification and mathematization in the whole sample

Three problem formats for percent problems. Each problem type was presented in three formats: For the *pure format*, exercises were given together with the technical terms (hence the decision of problem type is already explicit, see Table 1). The *visual format* roused the bar model, an established visual model for percentages (van den Heuvel-Panhuizen, 2003), here contextualized in download bars, a familiar everyday context for teenagers (Prediger, 2013). Three or two items for each problem type were constructed in *text formats* with varying language difficulties.

Sampling and subsampling

The sample consisted of 15 classes in 7 schools, in sum N = 250 students (age 13 - 15). In order to investigate robustness of the findings to national curricular specificities, the test was conducted in two countries, Germany (in the metropolitan Ruhr region) and Switzerland (in the German-speaking region of Lake Constance). Due to differences in the school systems (streaming in Germany since Grade 5 versus a more comprehensive Swiss system), the selected Swiss schools have a higher general achievement level. For investigating differences in students' achievements with varied language proficiency (LP), the sample was split into three groups with high, medium and low LP (the medium group is not considered here); the cut-offs being set by the standardized test norms.

Complementary clinical interviews

In order to deepen the insights into students' difficulties and to gain explanations for difficulties, the quantitative data were triangulated by a small interview study in which students solved the problems in a thinking aloud format. So far, we analyzed interviews with forty students, in sum 594 minutes of video, which were completely transcribed. Due to space restrictions, we only briefly refer to the results of these analyses for selectively strengthen possible explanations even if there is only limited space for transcripts.

Data analysis and hypothesis

For the written tests, an evaluation of success in problem type identification and mathematization for each item was binary-coded in order to allow hypothesis-testing on achievements for different formats etc. with t-tests (Davison, 2003). More precisely, the following hypotheses were tested through attempting to falsify the corresponding null hypotheses:

(H1) Problems in text format are more difficult than in pure format due to comprehension difficulties for word problems (Kouba et al., 1988).

(H1*) Problems in text format are easier than in pure format since contexts can enhance students' accessibility of the problem (as shown for elementary arithmetic problems by van den Heuvel-Panhuizen, 2005).

(H2) Problems in visual format are easier than in text and pure format since visual models can enhance the accessibility of the problem (Walkington et al., 2013).

(H3) Students with low language proficiency have difficulties with *other* problem formats than students with high language proficiency; especially they have specific difficulties with problems in text format.

For testing hypothesis (H3), the data were treated in a Rasch-Model for identifying differential item functioning with respect to the students' language proficiency (Fischer & Molenaar, 1995). Additionally, a categorization of students' written solutions allowed deeper insights into students' challenges and resources, shown here for two similar items (in Section Reduce to versus reduce by: An example for linguistic challanges).

Language-driven subsampling **Regional subsampling**	Whole sample	Subsample with low LP	Subsample with high LP
Whole sample	N = 250	n = 60	n = 84
German Subsample	n = 98	n = 25	n = 33
Swiss Subsample	n = 152	n = 35	n = 51

Table 2: Sample and subsamples

Problem type Problem format	Find amount			Find base			Find base after reduction		
	Pure	Visual	Text	Pure	Visual	Text	Pure	Visual	Text
Whole sample	78 %	67 %	46 %	67 %	67 %	53 %	44 %	59 %	41 %
German subsample	70 %	60 %	36 %	51 %	47 %	40 %	15 %	38 %	21 %
Swiss subsample	82 %	72 %	53 %	78 %	80 %	61 %	63 %	73 %	53 %

Table 3: Frequencies of correct identification / mathematization in two regional subsamples

RESULTS

First results and their discussion

For each problem format, Table 3 shows the frequencies of successful problem type identification and mathematizations (here interpreted as finding adequate expressions but not necessarily successful calculations of the result). Frequencies are given for the whole sample and split for the German and Swiss subsample.

These empirical results show that *hypothesis (H1)* must be restricted to well known problem types: Problems in text format are significantly more difficult than in pure format only for the basic problem types "Find the amount" and "Find the base" ($p<0.0009$). In contrast, the ranking of difficulty differs for the more complex and less acquainted problem type "Find the base after reduction" (no significance for higher difficulty with $p=0.175$). In the German subsample, this problem type is marginally easier in text format than in pure format ($p=0.097$), hence hypothesis (H1*) tend to apply. The interviews strengthen our interpretation that the context of shopping discount can enhance the accessibility, and this role of context can compensate potential comprehension problems posed by the text format in this less known problem type (at least for the subsample of high language proficiency, see below). This is illustrated by the written comment of a student (see Figure 1) referring to Item Dress (printed in Table 1).

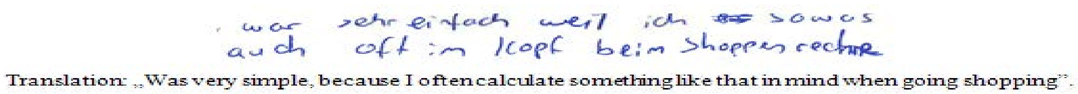


Figure 1: Statement of a student referring to Item Dress

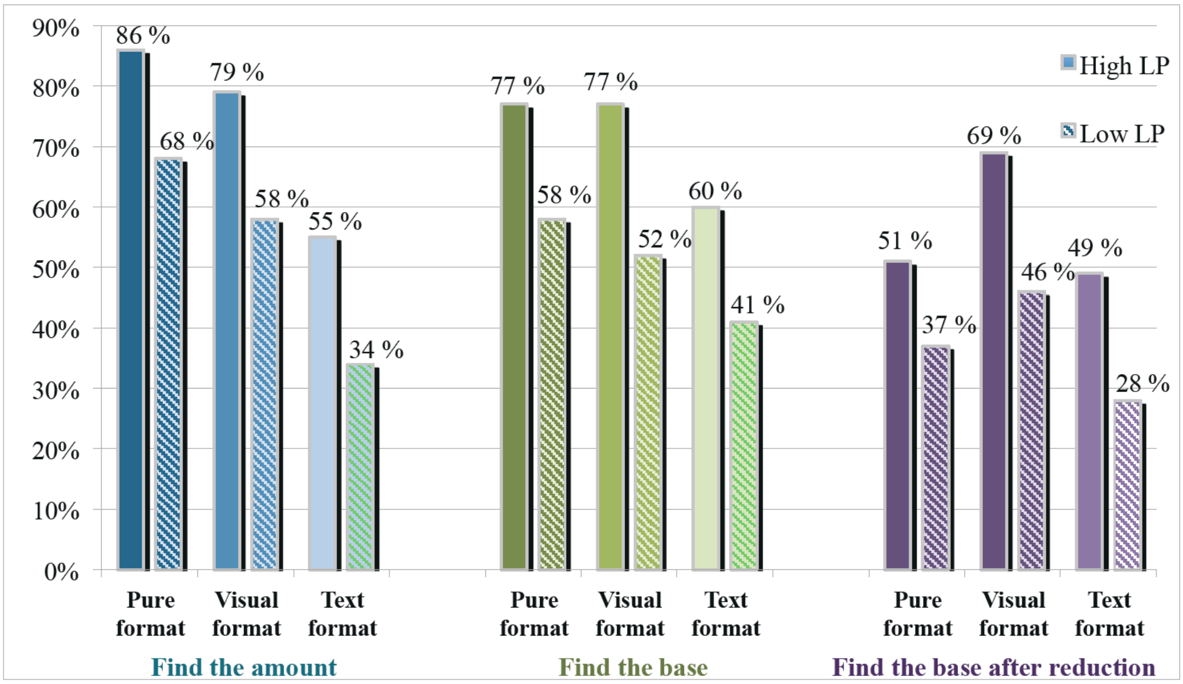


Figure 2: Frequencies of correct mathematization in subsamples with different language proficiency

In contrast, the Swiss subsample, which was more acquainted with the problem type, could solve the pure format more frequently than the text format.

Hypothesis (H2) on the role of the visual format is confirmed for all problem types for the text format: In both subsamples, problems with the visual bar model were solved significantly more frequently than in text format (p <0.0009). In contrast, the visual format is easier than the pure format only for the less known problem type "Find the base after reduction" in both subsamples (p<0.021). Also the interviews show that the visual bar model can enhance students' access to the problem when the key concepts do not suggest routine solutions.

Except for one deviance (text format for less known problem type "Find base after reduction"), the general pattern of difficulties between problem formats seem to be parallel in the German and Swiss subsamples. Unexpectedly, the mentioned differences are even significant in the separate subsamples (except for (H2) in the German subsample).

As Figure 2 shows, the described tendencies seem to apply for students with low as well as with high language proficiency in similar ways: the difference between pure format and text format is very similar for "Find the amount" (31 percent points for high LP and 34 percent points for low LP; of course with a higher rate) and "Find the base" (17 percent points for both samples).

These rather heuristic comparisons are confirmed by statistically more elaborate methods: The estimated item-difficulties in the Rasch-model had no differences for both subsamples, low LP and high LP. Significant differential item functioning was only found with respect to students' mathematics achievement, but *not* between the language proficiency subsamples. Hence, LP does not seem to determine specific difficulties. As a consequence, *hypothesis (H3)* can partly be falsified: Students with low language proficiency perform lower in all items, but their weaker achievement in the text formats is not per se an evidence for reading challenges being most dominant. In contrast, the differences between the formats of the problems proceeds in a related way for students with low LP as for students with high LP. The visual format which might be assumed to be relatively less difficult for students with low LP shows similar differences to the pure format for "Find the amount" (7 percent points for high LP and 10 percent points for low LP) and "Find the base" (0 percent points for high LP and 6 percent points for low LP). These comparisons of formats allow us to conclude that not the language alone, but the conceptual understanding (needed when the item format does not betray the problem type) is the highest difficulty for students with low LP.

This conclusion is strengthened by the analysis of the interviews, from which we show only one singular example, Tom's way of solving the Item Cross trainer (cf. Table 1). The student with low LP makes evident his conceptual understanding when subtracting the number indicating the percentage from the price (#2). He even validates and corrects his solution (#6 / 8), but still with no understanding that Euros and percent cannot be combined directly.

2	Tom:	Well, first, the client buys a crosstrainer in a sport shop. Yes, she pays 450 € for the equipment. That is the price which – how much it costs. She receives a discount of 10 % because she is a member. And now, we should calculate the normal price…. of the cross trainer. And I received: 440. And I have calculated 450 minus 10 %.
…	…	…
6	Tom:	But actually, the normal price should be [reads the text again] should be higher.
…	…	…
8	Tom:	Yes, now I have 460.

Like Tom, many students with low LP succeed in understanding the situation in the text but have too limited conceptual understanding to mathematize correctly. As a consequence, not only problems in text format are difficult for them, but all problems in which the mathematization is not pre-given by technical terms (like in pure format).

Reduce to versus reduce by: An example for linguistic challenges

Even if conceptual challenges are most crucial for students with low LP, there exist also linguistic challenges. We give an example from the deeper analysis of two similarly formulated items, both in text format:

Students' identifications of the problem types "Find the base" (Item Jeans) versus "Find the base after reduction" (Item Dress)	Frequency in *whole sample* (N=250)	Frequency in *subsample low LP* (n=60)	Frequency in *subsample high LP* (n=84)	Significance of differences low-high LP
recognizing a difference	40 %	36 %	49 %	p=0.049
(*both* problem types correctly identified)	14 %	7 %	21 %	p=0.004
(*one* problem type correctly identified)	6 %	7 %	4 %	n.s.
(*no* problem type correctly identified)	20 %	22 %	24 %	n.s.
recognizing no difference	59 %	65 %	52 %	p=0.049
(*one* problem type correctly identified)	39 %	35%	37 %	n.s.
(*no* problem type correctly identified)	6 %	12 %	5%	n.s.
(*both* items not treated)	14 %	18 %	10 %	n.s.

Table 4: Identifying the difference between Item Jeans ("reduced to") and Item Dress ("reduced by")

Item Jeans (Find the base): 40 € for jeans, were *reduced to* 80 %. What did they cost before?

Item Dress (Find base after red.): 30 € for dress, was *reduced by* 40 %. What did it cost before?

The prepositions in the expressions "reduce to" versus "reduce by" determine the problem type, and the typical challenge to recognize this difference (as discussed by Parker & Leinhardt, 1995, p. 439). Table 4 shows students' mathematizations for both items.

Only 40 % of the students have identified different problem types and have hence recognized a difference between the two items, 36 % of the students with high LP and 49 % of those with low LP (the difference being significant with p =0.049). In contrast, 59 % have not recognized the difference while treating the items (65 % with low LP and 52 % with high LP again a significant difference with p =0.049).

However, the fact that only 14 % identified both problem types correctly show that al-though there is a significant linguistic challenge that requires attention in classrooms, the conceptual challenges are still virulent. This can again be illustrated by Tom's way of solving the Item Jeans (cf. Table 1):

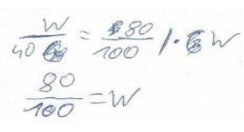


Figure 3

1 Tom: [after reading and calculating]: Yes okay, okay the amount is to be found, short W.

CONCLUSION AND OUTLOOK

The results of the presented test with N=250 students confirm that the same problem formats can provide different challenges: students have most difficulties in cracking percent problems in text format compared to those in pure or visual format. Whereas for elementary problem types, items in pure format are solved better than those in visual format, the visual model seems to enhance the accessibility for more complex, more unknown problem types. A methodological limitation of the study is that we did not account for the specific classroom curricula, which might influence students' varied abilities.

Although the language proficient students outperformed the students with low language proficiency in all items, the general pattern of differences in item difficulties is similar for both groups. This contradicts commonly held assumptions that disparities are bigger for text formats than for pure formats. It suggests that not exclusively the students' restricted reading proficiency is responsible for difficulties in tests, but also their lacking conceptual understanding in percentages (similar in grade 10, cf. Prediger et al. 2014).

In the future research, the findings will be extended by (1) an extended sample, (2) by deeper qualitative insights into students' difficulties by the interview study, and (3) by investigating whether these findings also apply to other problem formats.

Already the current state of the results is taken into account when designing a remediating course for enhancing students' conceptual understanding for percentages and dealing with word problems in this topic.

ACKNOWLEDGEMENT

We thank Esther Brunner for collecting data in Switzerland.

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Cognitive flexibility and reasoning patterns in American and German elementary students when sorting addition and subtraction problems

Elisabeth Rathgeb-Schnierer[1] and Michael Green[2]

1 University of Education, Weingarten, Germany, rathgeb-schnierer@ph-weingarten.de

2 University of North Carolina Charlotte, Charlotte, USA

This study focuses on sorting and reasoning patterns in second and fourth grade elementary students. Based on a theoretical interpretation of flexibility in mental calculation we argue that sorting and reasoning for addition and subtraction problems provide information about students' cognitive flexibility. For the study reported here, a problem-oriented, guideline-based interview incorporating 12 two-digit addition and subtraction problems was conducted with American and German students. Following the theoretical background and design, this article gives an overview of results concerning reasoning patterns.

Keywords: Reasoning patterns for easy and hard problems, flexibility in mental addition and subtraction, cross cultural study.

SYNOPSIS OF THEORETICAL BACKGROUND

In recent decades math educators and researchers have taken an increasing interest in students' techniques for performing mental addition and subtraction. Similarly, math educators have emphasized the importance of developing flexibility in performing mental calculations (e.g., Anghileri, 2001; National Council of Teachers of Mathematics, 2000; Selter, 2000). In this context, mental calculation, in the sense of solving multi-digit arithmetic problems mentally without using paper and pencil procedures, has received increasing attention among researchers (e.g., Blőte, Klein, & Beishuizen, 2000; Heirdsfield & Cooper, 2004; Rathgeb-Schnierer, 2006; Threlfall, 2009).

Recent research on flexibility in mental addition and subtraction has revealed the following general patterns: (1) a negative impact on flexibility when addition and subtraction are learned by examples (Heirdsfield & Cooper, 2004); (2) students' mental strategies influenced by multiple factors (Blöte, Klein, & Beishuizen, 2000; Rathgeb-Schnierer, 2006; Torbeyns, De Smedt, Ghesquière, & Verschaffel, 2009), (3) some approaches more than others support the development of mental math flexibility in elementary students (Heinze, Marschick, & Lipowsky, 2009; Rathgeb-Schnierer, 2006), (4) greater mathematical competence among students who exhibit flexibility in mental calculation (Heirdsfield & Cooper, 2004; Threlfall, 2002), and (5) recognition of number patterns and relationships correlated with computational flexibility (Macintire & Forrester, 2003; Rathgeb-Schnierer, 2010; Threlfall, 2009).

At the same time, inconsistent perspectives appear in the research literature on mental calculation flexibility (Rathgeb-Schnierer & Green, 2013; Star & Newton, 2009). For example, nearly all definitions have the same basic idea of flexibility in mental calculation as an appropriate way of acting when faced with a problem, which is to say that flexible strategies are adapted dynamically to problem situations. Nevertheless, there exist in current research crucial differences concerning the meaning of what constitutes "appropriate" as well as the use of different methods to measure "flexibility" and "appropriate ways of acting" (Rechtsteiner-Merz, 2013). Many researchers define flexibility as the choice of the most appropriate solution to a problem (Star & Newton, 2009; Verschaffel, Luwel, Torbeyns, & Van Dooren, 2009). While Torbeyns and colleagues (2009) had the same notion, they enhanced their definition to incorporate both computational accuracy and timeliness: "strategy flexibility is conceived as selecting the strategy that brings the child most quickly to an accurate answer to the problem" (Torbeyns et al., 2009, p. 583). Even contrasting views of "flexibility" focus on mental outcomes, including elements of speed and accuracy.

The project described here refers to a notion of flexibility based on the model of process of calculation (Rathgeb-Schnierer, 2011; Rathgeb-Schnierer & Green, 2013) and current research results which underscore the crucial role of number patterns and relationships for flexibility in mental math (see above). Therefore, the following definition of flexibility has evolved: cognitive actions that match the combination of *strategic means* [1] (Rathgeb-Schnierer & Green, 2013) to the recognized number patterns and relationships of a given problem in the context of processing a problem solution. In this sense, our definition incorporates the capacity to use several strategies and adaptive thinking and is similar to Threlfall's "interaction between noticing and knowledge" (2002, p. 29):

> *When faced with a fresh problem, the child or adult who follows different solution paths depending on the numbers does not do so by thinking about what the alternatives are and trying to decide which one to do. Rather, he or she thinks about the numbers in the problem, noticing their characteristics and what numbers they are close to, and considering possibilities for partitioning or rounding them. (Threlfall, 2002, p. 41)*

Our work seeks to understand the mental processes that underlie problem solutions, the *cognitive elements* [2] that support the solution process. Therefore, we examine directly whether students recognize problem characteristics, number patterns, and relationships, and whether they use this knowledge for solving a problem (Rathgeb-Schnierer & Green, 2013). Our definition of flexibility differentiates between solution processes based on learned procedures (step-by-step mental calculations) and recognized problem characteristics, number patterns, and relationships. Hence, our project is aimed at identifying degrees of flexibility in students' mental arithmetic by identifying the cognitive elements that sustain the solution process (learned procedures or problem characteristics, number patterns, and relationships).

SYNOPSIS OF THE PROJECT

The project incorporated interview questions directed at students' recognition of problem characteristics, number patterns, and relationships. Students were encouraged to look at the given problems, sort them into either 'easy' or 'hard' categories, and give reasons for the sorting (details in Rathgeb-Schnierer & Green, 2013). In addition, we examined country and classroom-related differences.

Generally, our study was aimed at answering the following empirical questions: Do students recognize problem characteristics, number patterns, and relationships, and what reasoning do they exhibit about these elements? To what extent is cognitive flexibility in mental calculation related to sorting and reasoning patterns? Do students of different grade levels or countries exhibit differences or similarities (or both) in sorting and reasoning? In this paper, we focus predominantly on the first question.

Design

A qualitative study with guideline-based interviews was conducted with 78 second and fourth grade American (Charlotte, North Carolina) and German (Baden-Württemberg) students from ten different classrooms (three from 2nd grade and two from 4th grade in each country). These classrooms were chosen purposely by observing several math lessons and characterizing them by the degree of directness of instruction and openness of tasks. Typically, seven students per classroom were interviewed, and these were teacher selected as being either average or strong in arithmetic.

The qualitative, problem-oriented and guideline-based interview contained twelve two-digit addition and subtraction problems that were displayed on small cards. Each problem was purposely designed with at least one special numerical pattern or relationship feature: 31-29 (renaming required; range of numbers; 29 is close to thirty), 46-19 (renaming required; 19 is close to twenty), 63-25 (renaming required), 66-33 (no renaming required; double and half relation; double digits; inverse problem to 33+33), 88-34 (no renaming required; double and half relation of the ones), 95-15 (no renaming required; fives at the ones place), 33+33 (no regrouping required; double digits; double facts at the ones place; inverse problem to 66-33), 34+36 (regrouping required; double facts at the tens place; ones add up to ten), 47+28 (regrouping required), 56+29 (regrouping required; 29 close to thirty), 65+35 (regrouping required; fives at the ones place add up to ten), 73+26 (no regrouping required).

Students were interviewed one-on-one for 15 to 30 minutes, with video recording. Normally, each interview had two parts: (1) sorting problems into categories "easy" and "hard" and talking about the reasons for sorting, and (2) solving problems (if not already done during the sorting process). If it was indicated by the situation (for instance by utterance or action of the student), a third part was sometimes added to compare selected problems and direct student's attention to the characteristics of the problem 46-19 ("Is there way to make this problem easier?"). The cards were mixed and laid out on the table, and students were asked to sort cards one-by-one under "easy" or "hard" labels placed on opposite sides of the table. After a card was placed, students were asked, "Why is this problem easy/hard for you?"

Interviews were conducted by one researcher and took part in the last two months of the academic year (Germany 2010 and 2012, USA 2011). All interviews were transcribed in their original language for content-analysis.

Based on theory and student data, a coding system was developed. Two types of reasoning were applied to both easy and hard problems: reasoning by problem characteristics for easy (RCE) and hard (RCH) problems and reasoning by solution procedures for easy (RSE) and hard (RSH) problems. Reasoning by problem characteristics was coded when students referred for instance to features of numbers (e.g. similarity of tens and ones), numerical relations (e.g. double and half, analogies), and relations of tasks (e.g. invers problem). Reasoning by solution procedures was coded when students immediately started to describe a technique of mental, step-by-step computation (e.g., Selter, 2000; Threlfall, 2002).

All interviews were coded using event sampling, the individual sort for each of the 12 cards. An event was defined as one complete statement of reasoning that began with a student's first utterance after sorting a card and ended either when the student stopped talking or was interrupted by the interviewer. Multiple codes in the same category could be assigned to an individual event, but the assignment to one of the four core categories was exclusive. For example, a student's reasoning could either be coded as reasoning by characteristics (RCE/RCH) or reasoning by solution procedure (RSE/RSH), never to both; the deciding factor was always the very first statement. For validation, the entire data set was scored by two independent judges, and all disagreements were resolved through discussion.

RESULTS

Sixty-nine students were included in our analysis (eight interviews were dropped [3]): 28 fourth graders, 17 American students and 11 German students, as well as 41 second graders, 19 American students and 22 German students.

Reasoning (31-29)	Coding
This one is easy, cause both of the numbers are very close to each other, so – uhm, you can find out that there're only a couple numbers apart and so the answer would be two.	reasoning by problem characteristics – easy (core category) • relation of numbers (code) • range (sub-code)
This is an easy one, because I do 31 – 20 – uhm, and this is 11, and 11 – 9 equals 2.	reasoning by solution procedures – easy (core category) • compose and decompose (code) • jump method (sub-code)

Figure 1: An example of categorizing

	Sorting	Reasoning by Problem Characteristics	Reasoning by Solution Procedures	Other
Total	902 (100.0%)	492 (54.54%)	284 (31.48%)	127 (14.07%)
Easy	644 (71.39%)	351 (38.91%)	236 (26.16%)	58 (6.43%)
Hard	258 (28.60%)	141 (15.63%)	48 (5.32 %)	69 (7.64%)

Figure 2: Frequency of sorting and reasoning

General patterns in sorting and reasoning

In terms of sorting and reasoning, Figure 2 displays frequencies for easy and hard problems sorted by reasoning category.

In total, 902 reasons appeared in the sample, an average of 13.07 per student (minimum 8, maximum 17). More than two-thirds of the problems were judged as easy. Regarding the two types of reasoning, 54.54% belong to reasoning by problem characteristics, 31.48% to reasoning by solution procedure, and 14.07% to other reasons (which were excluded here from further data analyses). Regarding the category "reasoning by problem characteristic" the ratio for easy to hard is 2.48 to 1; within the category "reasoning by solution procedure" the ratio is 5.24 to 1.

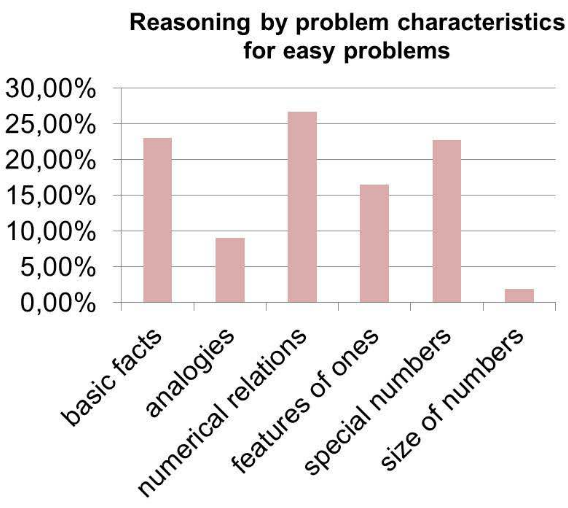
The percentage of each reasoning type related to all given arguments in each grade (separated by problem characteristics and solution procedures) is shown in Figure 3. Differences between grades occurred for both problem characteristics and solution procedures. Regarding the fourth graders, 69.53% of the arguments referred to problem characteristics and 30.47% to solution procedures. In contrast, second graders exhibited 10% fewer arguments belonging to problem characteristics and 10% more to solution procedures. Prior research by Selter (2000) had led us to expect fourth graders to argue more using solution procedures, since they are more familiar with the standard algorithms than second graders (in Germany the standard algorithms are introduced in third grade). Interestingly, our fourth graders exhibited less solution procedure reasoning than second graders, and they used nearly 50% more reasoning by problem characteristics than by solution procedures. This finding might be affiliated with the range of numbers used in the interview tasks and is probably explainable by the greater familiarity of fourth graders with two digit numbers.

In order to examine students' recognition of problem characteristics, number patterns, and relationships, we focused on the core categories "reasoning by problem characteristics." Figure 4 shows the percentage of the entire sample for each code in the two core categories "easy" and "hard." Distinct differences between easy and hard problems are readily apparent. With

	4th (n=28)	2nd (n=41)
Reasons - total	**340 100%**	**436 100%**
Characteristics - total	**234 68.82%**	**258 59.17%**
Characteristics - easy	165 48.52%	186 42.66%
Characteristics - hard	69 20.29%	72 16.51%

	4th (n=28)	2nd (n=41)
Reasons – total	**340 100%**	**436 100%**
Solution - total	**106 31.17%**	**178 40.82%**
Solution - easy	89 26.17%	147 33.71%
Solution - hard	14 5%	31 7.11%

Figure 3: Percentage of reasoning in each grade



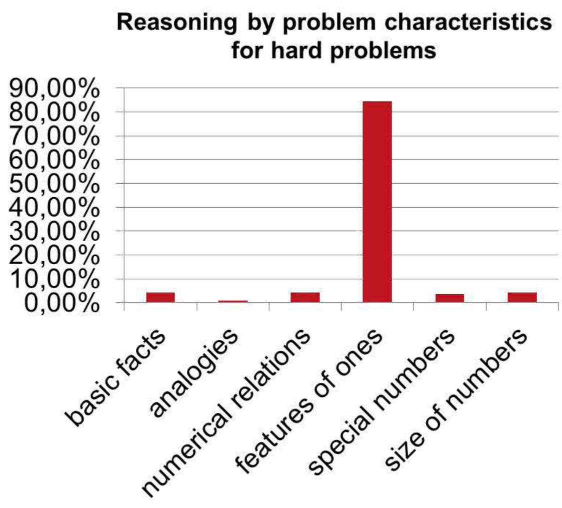


Figure 4: Reasoning by characteristics for easy and hard problems

reasoning for easy problems, students consistently referred to a variety of number patterns and problem characteristics. They pointed, for example, to basic facts [4] (23.36%), analogies between tens and ones (8.83%), numerical relations (27.06%, e.g. range of both numbers, double half, inverse problem), special features of numbers at the ones place (16.23%, e.g. two fives at the ones place or numbers that add up to ten), special numbers (22.50%, e.g. double digits or numbers close to the next ten), and also size of numbers (1.99%).

A completely different pattern was exhibited with reasoning by problem characteristics for hard problems. Students referred predominantly to "features of ones" (84.39%), referring directly to the numbers in the ones place. One typical characteristic for hard problems was the specific constellation of numbers in the ones place that leads to the need for renaming.

Student: So *((selects the card 46-19))* ok *((puts it in the "hard" column))* I'll put that one in the hard. So 46 plus, um, minus 19 *((points at the numbers on the card))* I think it's a little harder because first *((points at the tens on the card))* you can do four minus one which is three, and that one *((points at the 6 and the 9))* is kind of hard, because this one *((points at the 9))* is bigger than that one *((points at the 6))*.

Even if problems that required renaming were often considered as hard, we also found opposite judgments especially with those problems that include one number which is close to the next ten (predominantly numbers with a nine) for instance 31-29 (see Figure 2), 46-19 or 56+29. Whenever students discovered problem characteristics other than the requirement of renaming, they considered those problems to be easy.

Example for 56+29:

Student: This one is easy *((points to the problem 56+29))* because 29 is close to 30, I only have to add one.

Interviewer: That means you add one to 29?

Student: Here is 30 *((points to 29))*, and here 55 *((points to 56))*, um, and I do 30 plus 55.

Example for 46-19:

Student: *((points on 46-19))* One could easily subtract 20 *((points on the minuend))*.

Interviewer: Mhm.

Student: So, for 46 minus 20 you need no special trick, you know the answer immediately.

Interviewer: Yes.

Student: - this is 26, and then you need (..) to (..) – moment (..) to add one.

Interviewer: Why do you need to add one? Is this correct?

Student: Yes, because we have one more subtracted, that means we have subtracted one number more.

Summarizing, our data analyses suggest that addition and subtraction problems that require renaming are not regarded by students to be hard in general. In fact, in our sample, students depicted a more differentiated assessment of problems that required renaming: They considered those problems to be easy when additional problem characteristics (e.g., 9 in ones place) were obvious.

Reasoning in different classrooms

Qualitative analyses suggest classroom-related tendencies concerning reasoning patterns. Figure 5 shows students' reasoning patterns from three different second grade classrooms. Each column represents the arguments of one student, and the size of the dots displays the quantity of arguments in the category. The math lessons in classrooms A (American) and C (German) showed the same high degree of open tasks and self-regulated learning, whereas in classroom B (German) predominantly direct instruction and closed tasks were observed.

Students from classroom A referred almost exclusively to problem characteristics to explain why a problem seems to be easy or hard. Students from classroom B showed a distinct preference for reasoning by solution procedures and exhibited a quite restricted range in their reasoning: Whenever reasoning based on solution procedures appeared, "composing and de-

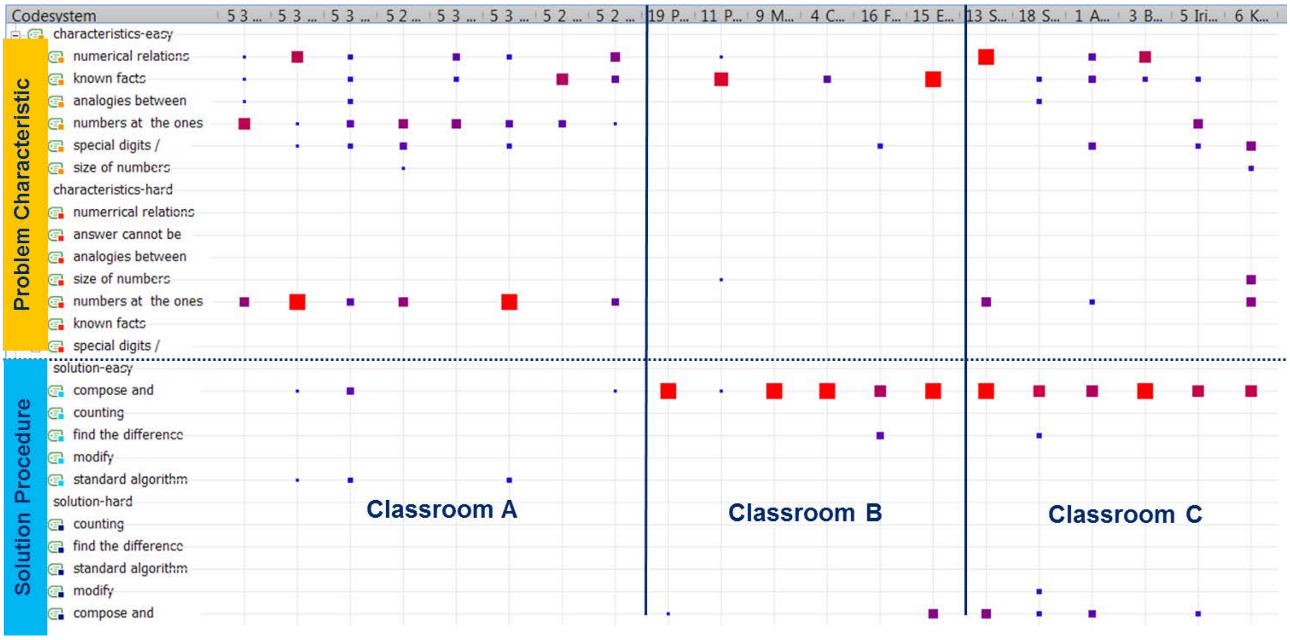


Figure 5: Reasoning patterns in classrooms

composing" strategies were exhibited. The few times reasoning based on problem characteristics emerged, students referred predominantly to "known facts" (basic facts). Classroom C students can be described as more comprehensive, since problem-based and procedure-based reasons coexist. In that sense, this group showed a great variety of problem related arguments and no variety at all in the section of procedure related arguments (note that solution procedures for both hard and easy problems were coded "compose and decompose"). This is an important pattern revealed throughout the whole data: Whenever students' reasoning referred to problem characteristic and number patterns, a great variety was noticeable. Therefore, we presume that those students act dynamically and can be considered as mentally flexible. On the other hand, whenever students depicted reasoning based on solution procedures, a quite restricted range of reasons was empirically observable. Hence, those students acted very statically and can be considered as mentally rigid.

CONCLUSION

In summary, we have reported a variety of patterns in students' reasoning for easy and hard problems that reflected the characteristics we built into the problems. There was obviously a greater variety of arguments for easy problems than for hard ones. Based on individual and classroom related reasoning patterns, we were able to identify three forms of reasoning: flexible (multiple reasons predominantly referring to characteristics), rigid (one reason referring to a solution procedure) and mixed (multiple reasons when referring to characteristics, one reason when referring to a solution procedure). Data analyses suggested that students who exhibited reasoning by characteristics were more cognitively flexible than students who didn't. Our next step will be to identify whether the recognized characteristics (that were visible in students' reasoning) function to sustain the solution process, and therefore can be considered as cognitive elements.

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ENDNOTES

1. Strategic means are neither holistic strategies nor cognitive menus that complete a solution path; they are distinct devices that can be combined in flexible ways to modify complex problems to make them easier. Strategic means include decomposing and composing, modifying a problem, deriving the solution from a known problem, and using decade analogies.

2. Students' solution processes are based on specific experiences that we designate with the term "cognitive elements." Such cognitive elements that sustain a solution process can be learned *procedures* (such as computing algorithms) or *number characteristics* (such as number patterns and relationships).

3. Eight interviews were dropped from the analysis for inaudibility, early termination or irreversible deviation from the interview guideline.

4. The code "basic facts" was purposely assigned to the core category "reasoning by problem characteristics." In our opinion, referring to basic facts is not a sign of procedure, but an indicator of the recognition that parts of the problem belong to basic facts. In this context, we only assigned an argument to the code "basic facts" when students expressed clearly and consciously that part of a problem or a whole problem is known by heart. The naming of a result by itself was not assigned to the code "basic facts."

A network of notions, concepts and processes for fractions and rational numbers as an interpretation of Didactical Phenomenology

Rubí Real and Olimpia Figueras

Centro de Investigación y de Estudios Avanzados del IPN, Departamento de Matemática Educativa, Ciudad de México, México, creal@cinvestav.mx; figuerao@cinvestav.mx

One of the main purposes of the research project that is being carried out in Mexico consists on building, from Filloy's point of view, a Local Theoretical Model (LTM) for fractions and rational numbers. For this purpose, four components of the LTM have to be constructed: models of formal competence, teaching models, models for the cognitive processes and models of communication. In this paper, a network of notions, concepts and processes for fractions and rational numbers that constitutes a fundamental part of the models of formal competence is described. This network was structured taking into account five mathematical processes grounded on Freudenthal's specimen of Didactical Phenomenology of fractions.

Keywords: Fractions, rational numbers, mathematical processes, teaching models.

INTRODUCTION

In the sixties, prestigious university educators discussed two approaches for the teaching of fractions in basic education. One group argued that decimal numbers could replace common fractions while the other one asserted that fractions should be taught, but only those that are more frequently used in everyday life (Rappaport, 1962). Those different positions generated discussions about the teaching and learning of fractions. Rappaport (1962) argued that fractions should be taught, since it is a necessary mathematical concept in order to understand rates and ratios. Usiskin (1979) claimed that a fraction is a representation of many numbers, and identified different uses of fractions involved in situations of splitting up, rates, proportions, mathematical formulae and equations.

Kieren's research (1976 & 1988) showed the complexity of the rational number concept. He stated that in order to construct a permanent knowledge of rational numbers, a person has to understand several constructs: measure, quotient, operator and ratio. For Kieren, part-whole relationship serves as a basis for connecting those constructs, which supports a construction of more formal knowledge that includes equivalence, and additive and multiplicative structures of a quotient field.

Lesh, Landau and Hamilton (1980) pointed out that the concept of rational number is a group of sub-constructs and integrated processes related to a wide range of basic concepts, appearing in a variety of problem situations.

Aware of the constructs identified by Kieren (1976); Behr, Lesh, Post and Silver (1983) proposed a model by grouping them in view of their relationships and links to equivalence, operations and problem solving. They also considered that the part-whole relationship is a fundamental meaning in order to understand the other meanings.

Although the part-whole relationship is an important meaning of fractions, Freudenthal (1983, p. 144) argued that the teaching based only on this relationship is restricted not only phenomenologically, but also mathematically, for it yields only proper fractions.

The research findings influenced worldwide educational reforms in the 90s, in order to improve the teaching and learning of fractions and rational numbers. In Mexico, in 1993, three different fraction meanings (quotient, measure and ratio) were included explicitly in the study programmes for primary and secondary school. However, nowadays secondary school students go on obtaining low scores on questions about fractions in national tests. For instance, a comparative study (data from 2005 to 2008 provided by the *Instituto Nacional para la Evaluación de la Educación* [INEE], 2012) showed that the correct answers obtained by third grade secondary school students in the Excale test, applied in Mexico, were lower than 40%.

For many years – at least since 1962, as mentioned above– it has been thought that knowledge of fraction concept is important in understanding other mathematical concepts. Siegler and colleagues (2012) found that "elementary school students' knowledge of fractions and division uniquely predicts students' knowledge of algebra and overall mathematics achievement in high school" (p. 691). Siegler and collaborators' findings imply that almost 60% of Mexican students would have a poor performance in mathematics in high school. Therefore, the teaching and learning of fractions and rational numbers pose still a problematic area for mathematics education.

As a consequence, it was decided to design a research project so as to characterise the structure and organization of the current teaching model of fractions; to analyse the interpretation made by authors of textbooks of the study programmes, and to characterise the effectiveness of the overall design and the understanding of those authors. This information will provide knowledge for designing an alternative teaching model of fractions for secondary school, and carrying out its experimentation in the classroom.

In this paper the most important result of the building up of the models of formal competence of the LTM for fractions and rational numbers is described. This outcome is a network of notions, concepts and processes for fractions and rational numbers, which is used as a methodological tool for analysing and designing teaching models.

THEORETICAL FRAMEWORK

Filloy's Local Theoretical Models theory (see Filloy, Rojano, & Puig, 2008) is used as a theoretical and methodological framework to organize the research bearing in mind its four components: 1) models of formal competence, 2) teaching models, 3) models for the cognitive processes, and 4) models of communication. Models of formal competence are related to fractions and rational numbers as mathematical objects; the teaching models and models of communication are associated to those mathematical objects as teaching objects, and models for the cognitive processes deal with fractions and rational numbers as learning objects.

METHODOLOGY

To construct the LTM, a specific methodology has been defined for each component. The models of formal competence were built mainly considering Freudenthal's Didactical Phenomenology of fractions (1983). The teaching models comprise analyses of the Mexican current study programme for secondary school and mathematics textbooks used by students. The models for the cognitive processes and communication include: 1) Kieren's explicative model (1988) which provides a global overview of individual construction of rational numbers knowledge; 2) a review of the results found in empirical studies carried out among secondary school students; 3) an analysis of other theoretical perspectives; 4) a characterisation of Mathematical System of Signs (MSS) (Filloy et al., 2008) of fractions and rational numbers, and 5) an analysis of items designed to assess these kinds of knowledge.

MODELS OF FORMAL COMPETENCE

This component of the LTM is related to mathematical knowledge, emphasising the structure and properties of fractions and rational numbers as mathematical objects. It is an important component due to: 1) the observer of educational experimentation should be capable to interpret the messages and texts generated by students and their teacher in the classroom. Therefore, the researcher needs more abstract MSS that encompasses all the MSSs used in the observed processes (Filloy et al., 2008, p. 36); 2) the design of a teaching model requires the knowledge of the characteristics of a competent user of fractions and rational numbers, and 3) the task designer must have examples of diverse situations in which these numbers are used readily available.

For the authors of this paper, the fulfilment of those requirements is possible through analysis of Didactical Phenomenology of fractions done by Freudenthal (1983). In these types of analyses, those phenomena organized by fraction and rational number concepts are described; consequently they provide diverse examples of how those numbers are used in various situations. In the processes of identifying different phenomena, the characteristics of the necessary competencies for using fractions and rational numbers for solving problems emerge.

Freudenthal starts the characterisation of phenomena from the first ideas of fractions embedded in everyday language through to the complex concepts included in the mathematical theory of rational numbers.

The specimen of Didactical Phenomenology of fractions was reinterpreted by the authors, identifying classes of phenomena: description and comparison of quantities, magnitude values or objects, division of substances measured by magnitudes, distribution of quantities, measurement, and numbers as part of a numerical system. For each one of the first five classes, different notions and concepts of fractions and rational numbers were identified, organized, schematized and linked with five mathematical processes: describing, comparing, dividing, distributing and measuring. The sixth class refers to the mathematical constructions of rational numbers.

In the following sections of this paper a brief description of these processes/classes of phenomena associated with fractions and rational numbers is made.

Describing. In everyday language there are expressions in which fractions are used to describe: i) a quantity or magnitude value through another quantity or magnitude value, ii) a measure expressed by numbers, iii) cyclic or periodic processes, and iv) ratios. These expressions are related to the process of 'describing' (see Figure 1) and are linked with four kinds of phenomena in different contexts in which *the fraction is acting as a descriptor*.

Expressions such as: 'half of a way', 'three quarters of an hour' and 'one third of the world's population will suffer water shortages by 2015' are examples in which a description of a quantity or magnitude value through total distance of a way, of a number of minutes of an hour, and of the number of people that live in the world is made. Measures can be described using phrases as '½ litre of milk' or '¾ of a pound'. Different cyclic or periodic processes are described using fractions and mixed numbers, for example: 'a driver only could run ½ times the track due to a mechanical failure' and 'a Chinese competitor presented an almost perfect dive in which he rotated 2 ½ times'. Likewise, fractions are employed to describe ratios relating two quantities or magnitude values. For instance, '3 out of 5 parts' and 'six of every one hundred Mexican people speak an indigenous language'.

Comparing. Three types of phenomena associated to the process of 'comparing' in which *fractions represent comparers* were identified (see Figure 2). Fractions are

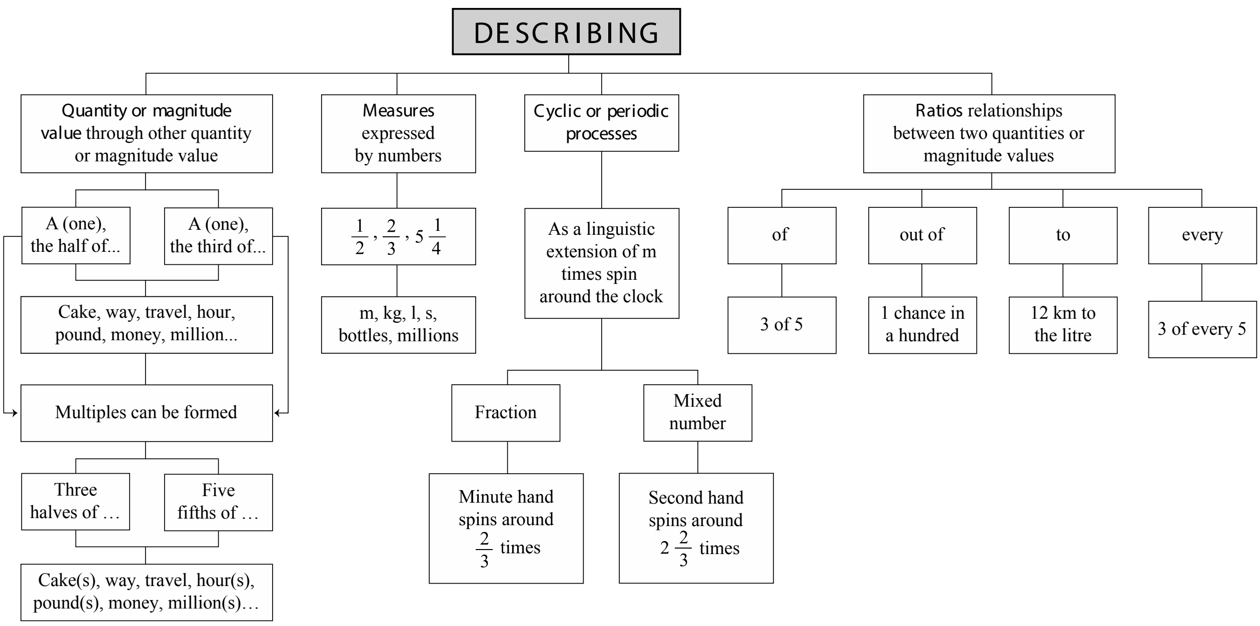

Figure 1: Schematic representation of the 'describing' process as part of the network

also used in everyday language to express the result of a comparison between two quantities or magnitude values; this comparison is related to a first abstraction level. In a comparison of the salary of two persons, one that earns 1 800 € and the other 1 200 € it is possible to say: 'one person earns 3/2 times than the other'.

In a second abstraction level, diverse aspects of the fraction were incorporated: fracturing operator, fracturing relation, ratio relation, ratio operator and transformer.

Two objects that are brought close together, or are in some other way considered, as though the smaller were part of the bigger, can be compared using the fracturing operator or fracturing relation. For example, if we compare the height of two bookcases, one has six shelves and the other has seven shelves, the fractions 6/7 and 7/6 represent these two possible comparisons.

Fraction as a ratio relation is the result of comparing two objects that are separated. This comparison can be made with respect to a number or a magnitude value of each object. To compare the height of two of the tallest towers in the world: the Tokyo Skytree and the Canton Tower, having in mind that one of them measures 634 m and the other 610 m, the fractions 634/610, 317/305, 610/634 and 305/317 are ratio relations that represent the comparisons of the towers' heights.

The aspect of fraction as ratio operator acts on a quantity or magnitude value, transforming it into another quantity or magnitude value. For example, to obtain the amount that each type of heir gets from an inheritance of 60 000 €, knowing that 2/3 are for the sons and 1/3 for the widow, the fractions 2/3 and 1/3 are ratio operators that transform 60 000 € in 40 000 € and 20 000 € respectively.

In its intermediate stage as transformer, the fraction performs on dimensions of objects by deforming or mapping to a scale a/b. To determine a scale of the transformation of an image whose size is 1280 x 960 pixels to another size of 1600 x 1200 pixels, the fraction 5/4 represents both a scale and a transformer.

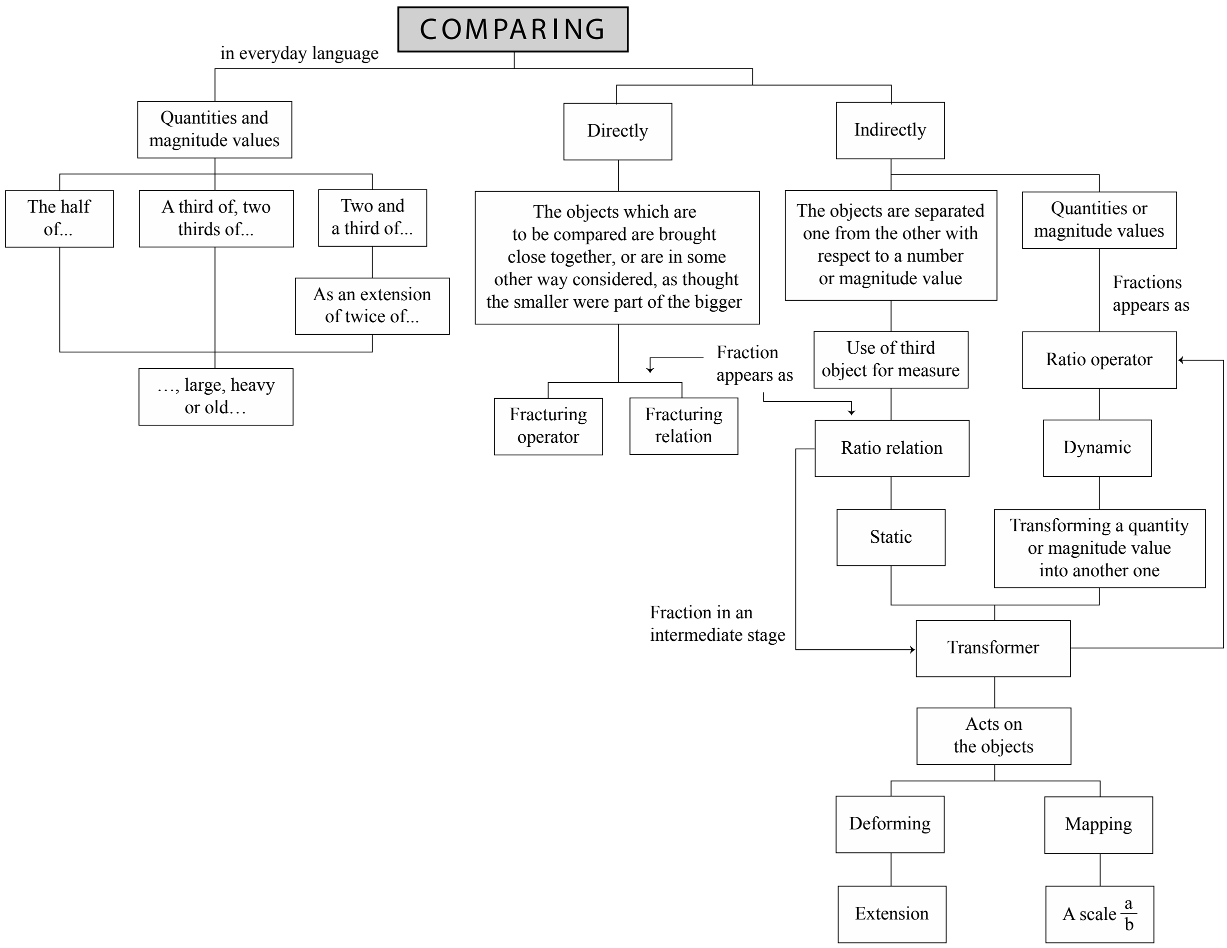

Figure 2: Schematic representation of the 'comparing' process as part of the network

Dividing. To divide substances measured by magnitudes, the whole is partitioned in equal parts, relating it to one or more of those parts (see Figure 3). In that way, *the fractions represent part-whole relationships and are acting as fracturers*. For instance, 18/27 parts of the roulette are black or 11/25 of the numbers of a bingo card are even numbers. The expression numerator/denominator represents the part-whole relationship.

There are different methods of fracturing in equal parts: in an irreversible, reversible or symbolic way. The estimating of equality of the parts can be made at sight, by feel or by others methods like the use of congruences or symmetries or employing different measuring instruments. Regarding the whole it can be discrete, continuous, definite, indefinite, structured or lacking structure. There are transitions among those types of wholes. The parts of the whole can be connected or disconnected.

Distributing. A distribution process of small or large quantities can be related to a finite set model and/or to a magnitude model (see Figure 4); *the fraction represents the result of this process*. On the one hand, when 10 leaf packs are distributed among 5 persons, each person gets 1/5 of the total of leaf packs (10 leaf packs ÷ 5 persons = 2 leaf packs per person = 1/5 of 10 leaf packs for each person). In this case, the process has finished and is associated with the finite set model.

On the other hand, if 3 sacks of rice whose mass is 50 kg each are distributed between 2 persons, each person would get 1 whole sack of rice and it would be necessary to distribute the remaining sack considering its mass, thus each person would get ½ of a sack more (50 kg ÷ 2 persons = 25 kg per person = ½ of a sack per person). In this case, the process is related to a magnitude model because the use of the mass unit has made possible the distribution.

When the magnitude model is introduced to distribute the remainder, it is necessary to constitute a magnitude in a system of quantities that include different requirements ranging from an equivalent relation to the division of an object in an arbitrary number of partial objects. The expression numerator/denominator represents, in this case, the quotient of a division; in other words, it is the result of the distribution process.

Measuring. Fractions also represent measures of magnitudes (see Figure 5). Magnitudes can be measured with unconventional units or using metric and non-metric systems. The metric system is related to decimal fractions. The non-metric systems are the an-

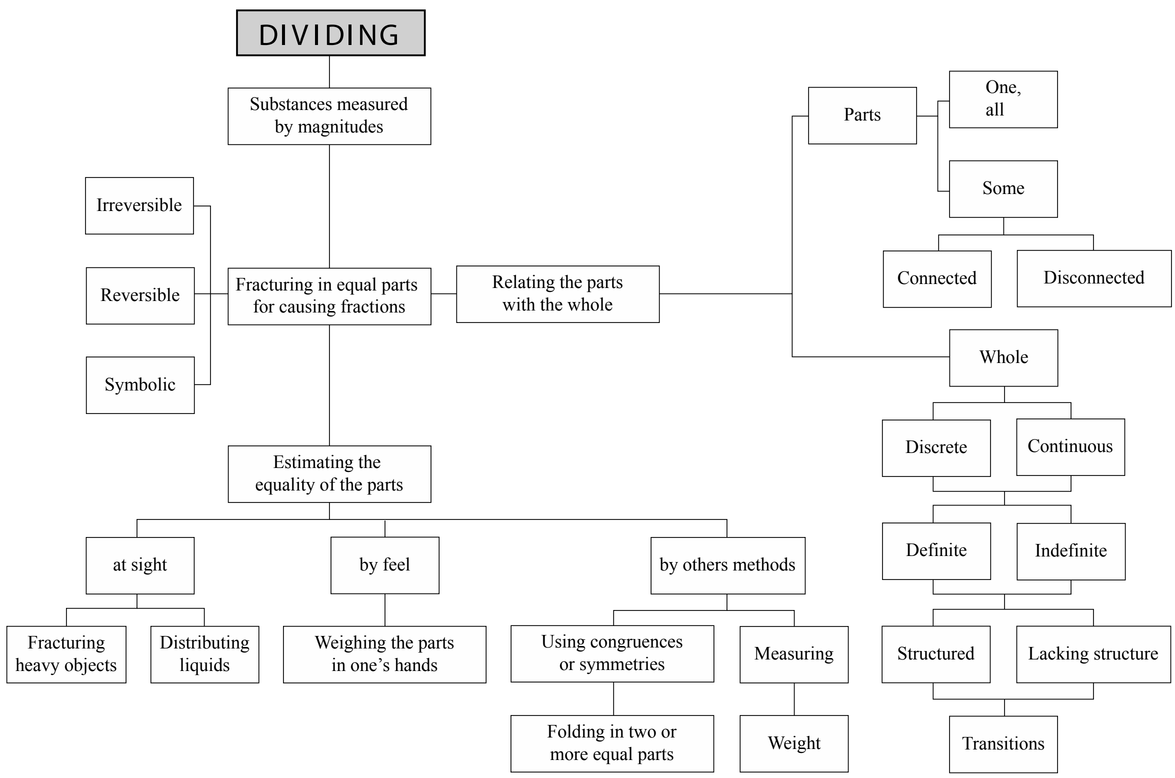

Figure 3: Schematic representation of the 'dividing' process as part of the network

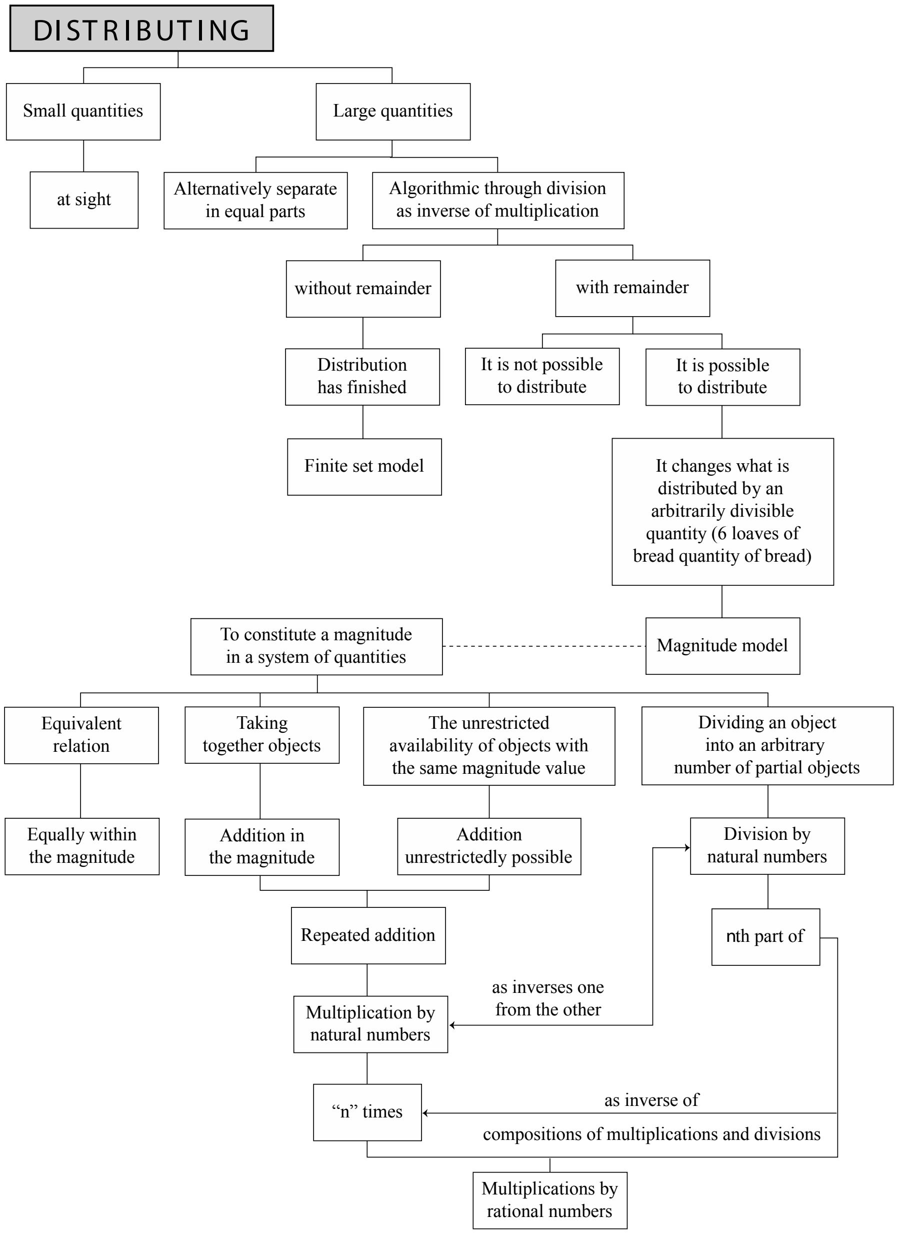

Figure 4: Schematic representation of the 'distributing' process as part of the network

cient systems of measurement and the English system; some of the former have various decimal relations; for example, in the Egyptian system 1 khet = 100 sq cubits.

Decimal fractions can be thought of as elements of ever finer nets. There are connections among these elements in the same net through addition, subtraction and comparison, or between elements from two different nets using multiplication.

To measure magnitudes it is also possible to use the number line, selecting an arbitrary length to represent a unit of measurement to be used for building different scales of length, time, temperature and mass measurements among other magnitudes. *The fraction represents the result of a measurement process and acts as a measurer*.

In the last part of the network of notions, concepts and processes, the phenomena that are associated with the rational numbers as a number system are included. Two ways of constructing this kind of numbers outlined by Freudenthal – an *a posteriori* axiomatic way (as equivalence classes of fractions) connected with the algebraic nature of these numbers, and an *a priori* genetic way (as a commutative semigroup) linked with the ratio operator – were considered as the sixth class of phenomena. These constructions of rational numbers correspond to a more abstract level. In the latter construction, an arbitrary length was selected

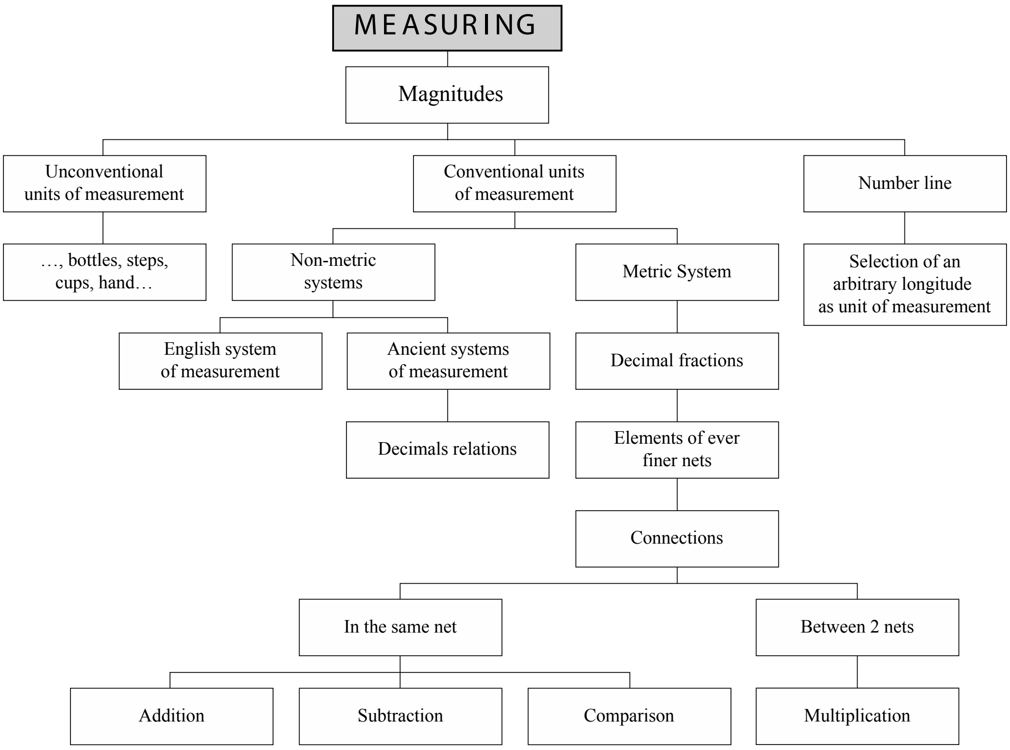

Figure 5: Schematic representation of the 'measuring' process as part of the network

in order to discover rational numbers as measures of a segment or points on the number line.

The network encompasses from the use of first ideas of fractions embedded in everyday language through to the mathematical constructions of rational numbers. Due to the extension of the network, it cannot be included in this paper, but the authors expect that readers could get a general idea from the above paragraphs.

THE USE OF NETWORK AS METHODOLOGICAL TOOL

In order to analyse a teaching model of fractions, the network was used as a methodological tool through examination of the activities included in mathematics textbooks, which were used by secondary school students in Valencia, Spain (Real, Figueras, & Gómez, 2013). The results of that analysis showed that the aspects of fractions in such teaching model are: descriptor, fracturer, ratio operator, and measurer (this last aspect appears in an implicit way). For representing a part-whole relationship the fraction as descriptor is used. The fraction as comparer is utilised for comparing two quantities or magnitude values. For transforming a quantity or magnitude value into another one, the fraction as ratio operator is employed. The fraction as a measurer is a means for solving some activities, but its use to compare different fractions in the number line is not included. In an overall view, some phenomena within the network are included in this teaching model of fractions, even though the authors state that it is based on problem solving.

FINAL REMARKS

All these mathematical processes related to different notions and concepts of fractions and rational numbers, included in the network, describe an overall perspective about teaching and learning of these mathematical objects. The network and its description constitute a potential theoretical element to analyse and to design an activity or a teaching model. Its use as a methodological tool has been proved through the analysis of two different teaching models: one from Spain (Real et al., 2013) mentioned above, and the other from Mexico (Real, 2014). Additionally its validation to design a teaching model for secondary school students in Mexico is being carried out.

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Flexible mental calculation and "Zahlenblickschulung"

Charlotte Rechtsteiner-Merz[1, 2] and Elisabeth Rathgeb-Schnierer[1]

1 Pädagogische Hochschule Weingarten, Weingarten, Germany, rechtsteiner@ph-weingarten.de

2 Pädagogische Hochschule Freiburg, Institut für Mathematische Bildung Freiburg, Freiburg im Breisgau, Germany

The study focuses on the development of mental calculation of elementary students who show difficulties in learning math. In total, 20 children in 8 classes were observed during their first year at school. The math education of five classes was based on a special approach called "Zahlenblickschulung", whereas three classes experienced more regular lessons. The collected data allowed a development of a typology of flexibility in mental calculation. Additionally, it was possible to describe the development of each student. The data analysis shows that instruction with "Zahlenblickschulung" also supports less advanced students in developing flexibility in mental calculation. Another result indicates that the recognition of number patterns and numerical relationships is crucial for learning to calculate (beyond counting).

Keywords: Flexible mental calculation, less advanced students in mathematics, elementary arithmetic.

THEORETICAL FRAMEWORK

For over more than a decade, developing flexible mental calculation has been considered as an important goal in elementary school (Lorenz, 1997; Selter, 2009). Nevertheless, there is still no consensus on instructional approaches and support for less advanced students in learning calculation. The study described below focuses on the development of flexible mental calculation of less advanced students in mathematics. Thereby, we define less advanced children as those who have problems in learning mathematics and need a special support (Schipper, 2005).

Notions and related research results

Current literature offers different definitions of flexible mental calculation (Rathgeb-Schnierer & Green, 2013; Threlfall, 2009; Verschaffel, Luwel, Torbeyns, & van Dooren, 2009). Most of these definitions involve two common aspects: flexibility and adaptivity. Thereby, flexibility is commonly understood as the ability to switch between different tools of solution (Rathgeb-Schnierer & Green, 2013; Verschaffel et al., 2009), whereas adaptivity "puts more emphasis on selecting the most appropriate strategy" (Verschaffel et al., 2009, 337). What is meant by adaptivity is considered differently (Rechtsteiner-Merz, 2013; Verschaffel et al., 2009):

- adaptivity of solution methods and problem characteristics (Blöte, van der Burg, & Klein, 2001),
- adaptivity of solution methods and speed of obtaining a solution (Torbeyns, Verschaffel, & Ghesquière, 2005; Verschaffel et al., 2009),
- adaptivity of cognitive elements that underlie the solution process (Rathgeb-Schnierer & Green, 2013; Threlfall, 2002, 2009).

In this project, flexible mental calculation involves both aspects: flexibility as mentioned above and adaptivity. Referring to Rathgeb-Schnierer and Green (2013, 2015), this project is based on the assumption that the aspect of adaptivity in flexible mental calculation is related to the recognition of problem characteristics, number patterns and numerical relationships.

Number sense – structure sense – "Zahlenblick"

Our basic assumption of flexible mental calculation influences the notion of how to teach towards flexibility. If flexible mental calculation is related to problem characteristics, number patterns and numerical relationships, it is necessary to provide activities that encourage students to focus on these aspects. Therefore, the crucial aim is to develop "Zahlenblick" (Schütte, 2004; Rathgeb-Schnierer, 2006; Rechtsteiner-Merz, 2013). To describe the meaning of "Zahlenblick", it is necessary to regard the constructs number sense und structure sense.

The term number sense is connected with two different notions: as a result of experience based development or as an inherent skill.

> "With respect to its origins, some consider number sense to be part of our genetic endowment, whereas others regard it as an acquired skill set that develops with experience." (Berch, 2005, 333f.)

Regarding the construct structure sense, the notions are quite similar. Lükens' definition (2010) of early structure sense reminds us of an inherent competence, whereas Linchevski and Livneh (1999) point out the necessity of its development. "Zahlenblick" is considered a result of development and means the competence to recognize problem characteristics, number patterns and numerical relationships immediately, and to use them for solving problems (Schütte, 2004). Comparing number sense, structure sense and "Zahlenblick" it is obvious that the meaning of number and structure sense as acquired skills that can be developed by special activities is quite similar to our notion of "Zahlenblick". Since there are still discussions about the different definitions, we use the term "Zahlenblick" in the previously described sense of Schütte (see above). To support the development of "Zahlenblick", it is crucial to provide activities, which highlight problem characteristics, patterns and numerical relationships (Rechtsteiner-Merz, 2013; Schütte, 2004). Generally, these activities target the development of number concepts, understanding of operations and strategic means [1]. They encourage students to recognize number patterns, problem characteristics and relations between numbers and problems, and to sort and arrange problems by using structural relations. These activities include cognitively challenging questions to provoke students' thinking and reflection. By combining mathematical topics with challenging questions, an increase of metacognitive competences is intended (Rechtsteiner-Merz, 2013). This can be illustrated by an activity called "Problem-Family" (Figure 1): The students start with one problem, for instance 5+5=10. Then, they were asked to arrange lots of cards with related problems (e.g. 5+6, 6+6, 4+6 etc.) around the first one with the aim of making the relations visible. Subsequently, the students were encouraged to describe their arrangements, and give reasons for their decisions. This activity does not focus on solving problems, but on recognising problem features and relationships.

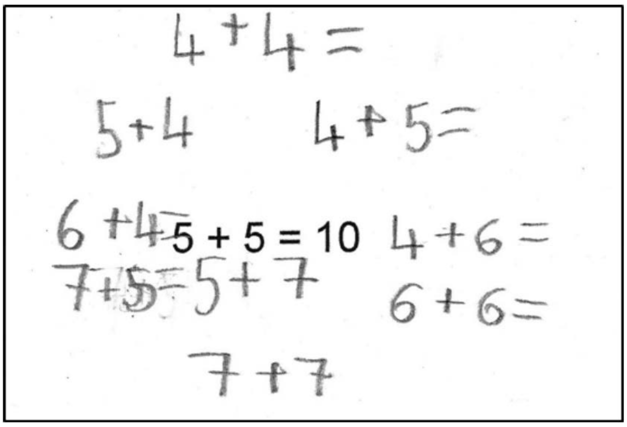

Figure 1: A "Problem-family" (Rechtsteiner-Merz, 2013, 113)

"Zahlenblickschulung" is not considered as an additional program. Rather, it can be understood as an essential principle of teaching arithmetic.

OVERVIEW OF THE PROJECT

Questions

Referring to prior research, we assume that

- "Zahlenblickschulung" is a good vehicle for developing flexible mental calculation (Rathgeb-Schnierer, 2006; Schütte, 2004) and

- not only middle and high achievers, but also less advanced students can develop flexible mental calculation (Torbeyns et al., 2005; Verschaffel et al., 2009).

These assumptions lead to the following research question: Are first graders with difficulties in learning math (numbers and operations) able to develop flexible mental calculation when educated with "Zahlenblickschulung"?

Design

Based on the theoretical notion of flexibility introduced above, a qualitative study that focuses on learning processes has been designed. The study included two parts: the instructional approach and the investigation of learning processes (Figure 1).

The investigation started with an extended period of observation to find students with problems in learn-

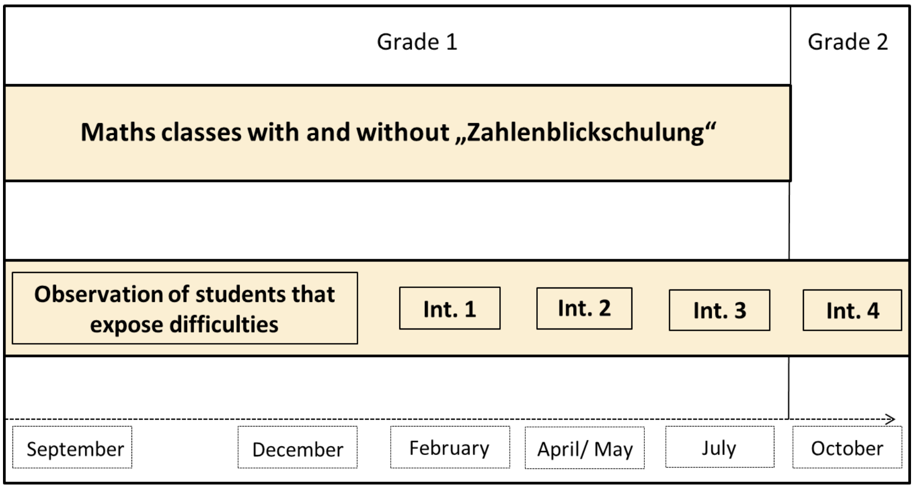

Figure 2: Design

ing numbers and operation. Therefore, two different not standardized tests were conducted with all students. Those who performed poorly were subsequently observed over a period of 6 to 8 weeks. As Schipper (2005) describes, approx. 20% of each class develops problems in learning mathematics. Based on this 20% benchmark, we decided to choose 20 students from eight different classes. Twelve students (five classes) experienced "Zahlenblickschulung" in one of four math lessons per week; eight students (three classes) had regular math classes.

From February until October 2008, each student was interviewed four times. The interviews were guideline-based, problem-orientated, and documented by videotape. Due to different moments in the academic year, each interview included different activities, except the one that was finally analysed. This special activity did not change and contained two parts: First, students were asked to sort addition problems and to talk about their reasons for sorting. Second, students were encouraged to solve the problems and to describe their solution procedures. The first three interviews included addition problems with single-digit numbers. In the last interview, several problems, up to 100, were added additionally.

Data analysis

For data analysis, interviews were transcribed. With the aim to reveal solution procedures and cognitive elements that sustain these procedures, two coding systems were designed based on the "Qualitative Inhaltsanalyse" of Mayring (2008): To classify the solution procedures, an a priori system was used. The analyses of sorting and reasoning were done with an inductively developed coding system (Rechtsteiner-Merz, 2013). Since there was a huge difference in the quality of students' reasoning, it was necessary to judge the value of arguments. Based on a theory called "Argumentationsanalyse" (analyses of arguments) (Fetzer, 2011; Toulmin, 1996) and the theory of proof (Almeida, 2001; Sowder & Harel, 1998), it was possible to judge different arguments according to the theory of flexible mental calculation (see above).

According to Kelle and Kluge (2010), types were constructed by interrelating three dimensions: (1) the amount of correct solutions, (2) the solution procedures, and (3) the reasoning for sorting and solving (Rechtsteiner-Merz, 2013). Therefore, it was necessary to build two feature spaces: First, the dimension "amount of correct solution" and the dimension "solution procedures" were combined. At this level, it was possible to construct pre-types which describe calculation in first grade. In the second step these pre-types were linked to the dimension "reasoning for sorting and solving". On this level, it was possible to develop a typology of flexible mental calculation in first grade (Figure 2).

RESULTS AND OUTLOOK

Finally, nine types could be derived from the data, four main types and five temporary types (Rechtsteiner-Merz, 2013) (Figure 3). The main types focus on a typical phase at the beginning of first grade (*counting strategies*) or on an intended phase at the end of first grade (*consistent use of procedural mastery*, *partly basic facts with relational expertise* or *basic facts extended with relational expertise*). The temporary types represent stages of developments when students learn calculating (beyond counting).

The arrangement of the types in Figure 3 must be understood as the combination of the two dimensions "counting subsumed by calculating" (horizontal dimension) and "reliance on numerical relationship in argumentation" (vertical dimension).

Subsequently, we describe the main types, followed by the temporary types: Students with *counting strategies* [2] solve each problem by counting, usually starting from the large number. Students with *consistent use of procedural mastery* are able to solve most of the problems up to twenty by calculating. Therefore, they always use the same solution procedure without noticing any problem characteristics. They argue for example "I do always like this" or "like always up to ten and then the rest". Students who exhibit *partly basic facts with relational expertise* use different strategic means by relying on problem characteristics. They are able to describe the solution process and give reasons for their strategic means in an elaborate way as the following example shows: "These problems are easy (*points to* 8+5 *and* 4+9), because here it's one less and here it's one more (*points to* 4 *and* 9)". Students who depict *basic facts extended with relational expertise* relay on basic facts with addition problems up to twenty. Additionally, they are able to solve problems with two digit numbers (higher than twenty) based on recognized characteristics and numerical relationships (even if this is not a topic in first grade). This type is special for first grade since all addition problems up to twenty can be memorized by heart.

Students who solve problems *predominantly by counting* are divided in two groups: those who *rely on procedures (temporary type 1)* and those who *rely sometimes on numerical relationships (temporary type 3)*. Students who belong to the *temporary type 1* predominantly practice counting as a rule. Sometimes, they suddenly use strategic means or number facts, although they cannot describe their approach or give a reason for it. Students who belong to the *temporary type 3* also use predominantly counting to solve addition problems, but sometimes they notice numerical relationships, and they are able to describe and reason their approach.

There are also students who solve problems *predominantly by calculating relying usually on procedures (temporary type 2)*. Based on procedures, they can solve many problems up to twenty. Exceptionally, they rely sometimes on numerical relationships when solving a problem or giving reasons for the sorting. On the other hand, there are students from the *temporary type 5* who solve problems *predominantly by*

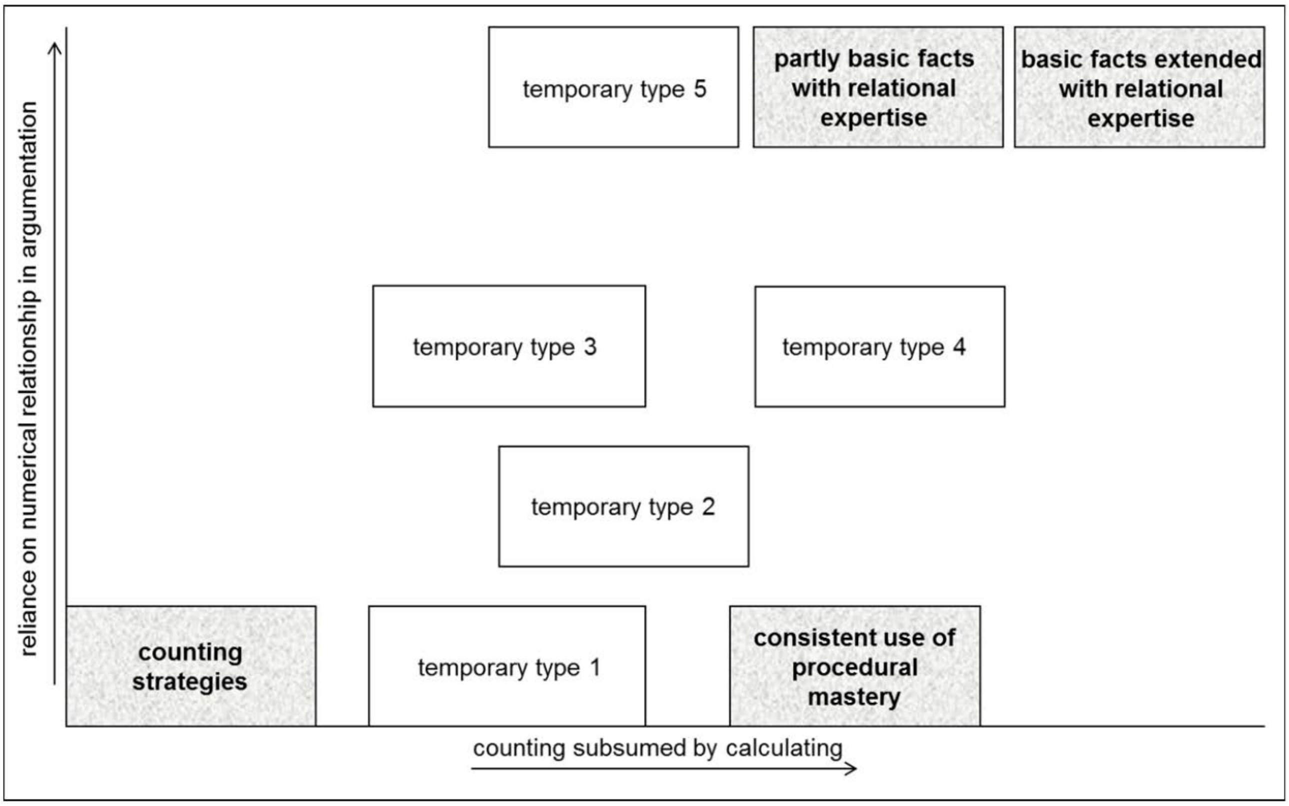

Figure 3: Typology of flexible mental calculation (Rechtsteiner-Merz, 2013)

calculating relying on numerical relationships. They are also able to solve a lot of problems up to twenty, but they recognize and use problem characteristics and numerical relationships.

Students who solve problems by *calculating relying sometimes on numerical relationships (temporary type 4)* are able to solve all the problems up to 20 by going beyond counting. Some solutions rely on procedures, others on numerical relationships. Whenever a solution is based on numerical relationships, it can be described and reasoned.

Finally, we give examples of students' development that were apparent in the timeframe from January in first grade to October in second grade.

Two students who used *counting strategies* at the beginning of second grade exhibited a *predominance of counting relying on procedures (temporary type 1)* in January. Finally, they showed a kind of regression, since their use of basic facts or strategic means were higher in the middle of first grade than at the beginning of second grade.

Four students who used *counting strategies* in January switched to the *temporary type 1* and solved problems by *predominantly counting relying on procedures* between April and July (end of first grade). After this change, no further development was obvious; it seems that they were trapped in counting.

Two students who reached the type *consistent use of procedural mastery* showed different ways of development. However, both exhibited relying on numerical relationships by calculating at least in one interview.

Five students who belong to the type *partly basic facts with relational expertise* at beginning of second grade started in January from *temporary type 1*, and solved problems *predominantly by counting relying on procedures*. They relied obviously on numerical relationships in April; some still by counting, others overcame counting. Lena, for example, solved the same number of problems predominantly by counting in January and April. But, there was a big difference in her reasoning: In January she did not recognize any problem characteristics, in April she used numerical relationships in solving and reasoning at least sometimes.

When comparing students' developments with and without math education based on "Zahlenblickschulung", some crucial differences can be described. Most students (only two exceptions) who did not experience "Zahlenblickschulung" stuck with their counting strategies based on procedures. Actually, those students did not show any progress in the second term of first grade. In contrast, all students who experienced "Zahlenblickschulung" (except Yannik) were able to overcome their counting strategies at least until the end of first grade. Additionally, all these students (except Amelie) used numerical relationships for solving problems, and they were able to reason sorting procedures in very elaborate ways.

Developed hypotheses: Conclusions

Focusing on students who have difficulties in learning addition, data analysis suggests the development of four central hypotheses:

- Relying on numerical relationships is an absolute condition for developing calculation strategies that go beyond counting.
- "Zahlenblickschulung" supports the development of conceptual knowledge.
- "Zahlenblickschulung" supports the development of flexible mental calculation.
- Activities in "Zahlenblickschulung" are a fundamental condition for developing calculation strategies and flexible mental calculation.

Subsequently, two hypotheses will be reported in detail:

Relying on numerical relationships is an absolute condition for developing calculation strategies that go beyond counting.

The knowledge of basic facts and strategic means seems to be insufficient for the development of a deep understanding of calculation that goes beyond counting. Therefore, the focus on numerical relationships and structures is essential. All students who overcame their counting strategies were able to "calculate without counting" at the beginning of second grade, and relied on numerical relationships at least in one stage of development. On the other hand, all students who were predominantly counting relying on procedures remained in this stage and could not progress. This *temporary type 1* seems to be like a dead-end road. Thus, the recognition and use of number patterns and numerical relations seems to be a crucial prerequisite for going beyond counting.

Activities in "Zahlenblickschulung" are a fundamental condition for developing calculation strategies and flexible mental calculation.

In order to develop flexible mental calculation in elementary school, Rathgeb-Schnierer (2006) and Schütte (2004) emphasized the necessity of "Zahlenblickschulung". Focusing on calculation competence of middle and high-achieving first graders, Torbeyns and colleagues (2005) showed that they are much more flexible than students who are considered as low-achieving peers. This observation indicates that middle- and high-achiever students develop a minimum of number patterns and numerical relationships for going beyond counting independently. However, students who have difficulties in learning arithmetic benefit from the "Zahlenblickschulung" approach; first to overcome counting, and second to develop an appropriate degree of flexibility in mental calculation.

The study reveals that less advanced first grade students are also able to develop competences in flexible mental calculation. Thereby, "Zahlenblickschulung" is an important and supportive vehicle. Especially for less advanced students, the recognition of number patterns and numerical relationships is the key for learning to calculate (beyond counting) and developing flexibility.

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ENDNOTES

1. Strategic means are distinct devices to modify problems to them easier. They can be flexibly combined in a solution process, and include for instance composing and decomposing, modifying a problem, deriving the solution from a known fact, and using analogies (i.a., Rathgeb-Schnierer, 2006)

2. For calculating you can use different tools for solution: counting, basic facts, strategic means. Counting can be distinguished if it's with or without models and in there are counting-all or counting-on strategies used (Carpenter & Moser, 1982).

Foundational number sense: Summarising the development of an analytical framework

Judy Sayers and Paul Andrews

Stockholm University, Stockholm, Sweden, judy.sayers@mnd.su.se, paul.andrews@mnd.su.se

What number-related competences do grade one students need to ensure later success and avoid later failure? We address this question by summarising recent work on the development of an eight component framework, which we call foundational number sense (FoNS), in which those necessary learning outcomes are categorised. We then present summaries of three case studies undertaken to evaluate the robustness of the FoNS framework. Each case study, which focused on the teaching of a different mathematical topic, was undertaken in two different European grade one classrooms. Analyses confirm not only the sensitivity of the FoNS framework to both cultural and mathematical contexts but also its power as a tool for both cross cultural research and teacher education practices.

Keywords: Foundational Number Sense, grade one, teaching framework.

INTRODUCTION

The influence of a child's basic number understanding in later mathematical successes (Aubrey & Godfrey, 2003; Aunola et al., 2004) and failures (Geary, 2013; Gersten et al., 2012) is well known internationally. Consequently, it would seem sensible for researchers and teachers to be clear as to what such basic understandings entail. In this paper, focusing on the first year of schooling and from an international perspective, we do two things. Firstly, we summarise the competences research has shown will both avoid later difficulties and ensure later success in an eight component framework. Secondly, we summarise three recent case studies, each of which evaluates the framework's analytical efficacy by comparing specific topic-related practices of teachers from two different European countries.

DEVELOPING THE FRAMEWORK

In relation to the role of children's basic number understanding in their later mathematical development, the expression *number sense* dominates the literature. However, despite its ubiquity its definition has been elusive (Griffin, 2004). Indeed, despite its importance, "no two researchers have defined number sense in precisely the same fashion" (Gersten et al., 2005, p. 296). Our reading indicates that this has, in no small way, been due to psychologists and educators working with different conceptualisations (Berch, 2005), a problem exacerbated by psychologists employing different definitions according to whether they work in general cognition or learning disabilities. That said, our reading reveals three distinct but related perspectives on number sense, which we label *preverbal, applied* and *foundational*.

Preverbal number sense reflects those number insights innate to all humans and comprises an understanding of small quantities that allows for comparison (Ivrendi, 2011; Lipton & Spelke, 2005). For example, young babies can discern 1:2 but not 2:3 ratios (Feigenson et al., 2004). This, numerical discrimination is independent of formal instruction and develops as a consequence of human, and other species' evolution (Dehaene, 2001; Feigenson et al., 2004). *Applied number sense* concerns those number competences related to arithmetical flexibility that prepare learners for an adult world (McIntosh et al., 1992). *Foundational number sense* (FoNS) comprises those understandings that precede *applied*, typically arise during the first year of school and require instruction (Ivrendi, 2011; Jordan & Levine, 2009). Unlike preverbal number sense, it is a "construct that children acquire or attain, rather than simply possess" (Robinson et al. 2002, p. 85). Unlike applied number sense, it does facilitate a world beyond school but later arithmetical competence. FoNS is to the development of mathematical competence what phonic awareness is to reading (Gersten & Chard, 1999).

Below we summarise the key components of FoNS. Our intention was not to offer an extensive list of learning outcomes, as found in Berch (2005) or Howell and Kemp (2006), but a concise conceptualisation that would support a range of activities, including developments in curriculum, teacher education or assessment, as well as cross-cultural classroom analyses. To achieve these objectives we exploited the constant comparison analysis advocated by grounded theorists, a process we describe in full in Andrews and Sayers (2015).

In brief, research papers typically addressing grade one students' acquisition of number-related competence were identified. These were read and FoNS-related categories identified. With each new category, previous articles were re-examined for evidence of the new. This approach, drawing on literature from psychology, mathematics education, learning difficulties and generic education, placed, for example, *rote counting to five* and *rote counting to ten*, two narrow categories discussed by Howell and Kemp (2005), within the same broad category of systematic counting.

Number recognition: Children recognise number symbols and know their vocabulary and meaning. They can identify a particular number symbol from a collection of number symbols and name a number when shown that symbol;

Systematic counting: Children count systematically and understand ordinality and cardinality. They count to twenty and back, or count upwards and backwards from arbitrary starting points, knowing that each number occupies a fixed position in the sequence of all numbers.

Awareness of the relationship between number and quantity: Children understand not only the one-to-one correspondence between a number's name and the quantity it represents but also that the last number in a count represents the total number of objects.

Quantity discrimination: Children understand magnitude and can compare different magnitudes. They use language like bigger than or smaller than. They know that eight represents a quantity that is bigger than six but smaller than ten.

An understanding of different representations of number: Children understand that numbers can be represented differently, including the number line, different partitions, various manipulatives and fingers.

Estimation: Children can estimate, whether it be the size of a set or an object. Estimation involves moving between representations of number; for example, placing a number on an empty number line.

Simple arithmetic competence: Children perform simple arithmetical operations, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction.

Awareness of number patterns: Children extend and are able to identify a missing number in simple number sequences.

In sum, our systematic review identified eight distinct and not unrelated FoNS components. The fact that they are not unrelated is important as number sense "relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures" (Gersten et al., 2005, p. 297). In other words, without the encouragement of such links children may be able to count but not understand that four is less than six.

EVALUATING THE FRAMEWORK'S EFFICACY

Having derived an eight component FoNS-related entitlement for grade one students, our purpose was to evaluate the framework's efficacy for identifying FoNS-related opportunities in different cultural contexts. Such a process facilitates both instrument refinement and an evaluation of its sensitivity to cultural nuances. In the following we summarise three recently reported case studies in which we evaluated the efficacy of the FoNS framework. The first examined the teaching of sequences in England and Hungary (Back et al., 2014), the second the development of students' conceptual subitising in Hungary and Sweden (Sayers et al., 2014) and the third focused on teachers' use of the number line in Poland and Russia (Andrews et al., 2015).

Each examined lesson, typically drawn from video-based teacher professional development programmes independent of not only the research presented here but also each other, involved a teacher construed against local criteria as effective. Thus, no lesson was captured with a FoNS-related analysis in mind. Two lessons were identified from those available for topic-based case studies. Such an approach, drawing on data intended for purposes other than a FoNS analysis, made these lessons ideal for evaluating the framework's capacity for identifying topic-related, but essentially incidental, FoNS-related occurrences. In all cases teachers had been video-recorded in ways that would optimise the capture of their actions and utterances. Each video, with transcripts, was repeatedly scrutinised for evidence of FoNS components by two researchers independently. These analyses were then compared and agreements reached with respect to which FoNS-related components were being encouraged at different times. Significantly, such an approach allowed lesson episodes to be multiply-coded according to which components were observed.

As data derived from different projects in five different countries, ethical procedures were managed according to local norms. In all countries permission from school principals and participating teachers was obtained by means of letters confirming the right of teachers to withdraw without notice or reason and anonymity. With respect to the Hungarian, Polish and Russian students, all parents, at the point at which their children entered their school, had signed to agree their child's participation in ethically conducted classroom based research. In England and Sweden, parental permission letters explained the projects and, alongside the promise of minimal classroom disruption, guaranteed the same protective principles as above.

RESULTS

In the following we do two things. Firstly, we summarise qualitatively the pilot studies introduced above. Space prevents detailed summaries but we believe that sufficient has been included to demonstrate the FoNS framework's sensitivity to both cultural and mathematical context. Secondly, acknowledging the limitations of case study, we present a simple, frequency-based, quantitative analysis to highlight not only how FoNS-related learning was managed but also interesting similarities and differences in the ways the various codes interact in the case study episodes. In so doing we show how the FoNS framework can facilitate the sorts of complex analyses discussed above in relation to our earlier European study of mathematics teaching.

The qualitative analyses

In the first study (Back et al., 2014) episodes focused on number sequences were analysed. In addition to examining the functionality of the FoNS framework an aim was to examine how teaching, focused explicitly on one FoNS component, would yield other components. The analyses, based on three episodes from each lesson sequence, indicated that Klara in Hungary addressed six of the eight FoNS components while Sarah in England addressed four. Both encouraged, throughout their respective episodes, students' recognition of number symbols, vocabulary and meaning. Both encouraged the awareness of number patterns and the identification of missing numbers and both exploited simple arithmetical operations, typically to facilitate finding the next values in a sequence. In respect of differences Klara addressed three categories, the relationship between numbers and quantities, comparisons of magnitude and different representations of number that Sarah did not, while Sarah was seen to address systematic counting when Klara did not.

However, while both teachers encouraged various FoNS components, Klara's teaching was more didactically complex, with an average of four components per episode, than Sarah's, with an average of barely two. Moreover, Klara's practice resonated with earlier research highlighting the cognitively demanding but coherent learning outcomes of Hungarian classrooms, while Sarah's reflected the relatively unsophisticated promotion of modest and less coherent goals of her English colleagues.

In the second study (Sayers et al., 2014), analyses focused on conceptual subitising in grade one lesson sequences taught by Klara, again, in Hungary, and Kerstin, in Sweden. Conceptual subitising, the ways in which individuals identify large quantities through identifying smaller quantities that comprise the whole, has been promoted as a key component of early number learning. In both cases, an average of five FoNS components were identified in each of the teacher's three analysed episodes, indicating that claims for the efficacy of teaching focused on conceptual subitising are not without warrant.

It was interesting that in neither case was conceptual subitising an explicit intention, nor were teachers expecting to address FoNS categories of learning. It is also interesting to note that despite substantial differences in the management of their lessons - Klara spent all her lesson orchestrating whole class activity with only occasional expectations of students working individually, while Kerstin spent the great majority of her time managing and supporting students working in pairs - the FoNS components addressed in their respective excerpts were remarkably similar.

Finally, the third pilot study (Andrews et al., 2015) examined episodes drawn from lesson sequences focused on the introduction and exploitation of the number line taught by Olga, in Russia, and Maria, in Poland. Here the analyses, as in the first case study, showed that such a didactical emphasis on one FoNS component does not necessarily restrict opportunities for other FoNS outcomes. For example, Olga's episodes addressed an average of almost five components, while Maria's almost four. Not surprisingly, bearing in mind the number line emphasis, all analysed episodes addressed number recognition and systematic counting, while all bar one showed evidence of children being asked to work with a different representation of number.

With respect to differences, whenever Olga asked her students to represent a number on the number line, she insisted on their pointing simultaneously to zero with their left hand and the desired number with their right. In this manner her teaching focused on the relationship between number and quantity. By way of contrast, Maria presented simultaneously three distinct number lines, each showing zero to eight but with different sized intervals. In so doing she highlighted the arbitrary size of the interval alongside the need for a consistent interval size. Both teachers also used the number line to facilitate simple arithmetical operations, including tasks involving several operations simultaneously. Finally, Maria used the number line in relation to number patterns, particularly even numbers and the identification of missing numbers. Interestingly, key differences were also found in Maria's frequent use of number line representations drawn from the real world, thermometers, measuring tapes, measuring jugs and so on, something that Olga did not do.

The quantitative analysis

It is important to remember that when teaching their respective classes, none of the case study teachers were focusing explicitly on FoNS-related learning opportunities. Moreover, despite the quality of instruction focused on it, neither Kerstin nor Klara were explicitly aware of conceptual subitising as a learning goal. In other words, all project teachers, in varying degrees and in varying ways, addressed a range of FoNS-related learning outcomes in incidental rather than planned ways. The extent to which these varying ways played out can be seen in Table 1. This shows a summary of the codes applied to each analysed episode. In addition, the mean number of codes calculated for each teacher's three episodes is included alongside the teacher's name. Finally, the table shows the total number of occurrences for each FoNS component.

At a very crude level, one could argue that the mean number of categories applied to a teacher's episodes could be construed as a measure of didactical complexity. For example, Sarah's number patterns-related practice, as reflected in a mean of 2.3 categories per episode, seemed considerably lower than that of her colleagues. In this respect, the next lowest mean, Maria's 3.7, was almost one and a half categories per episode more. Thus, Sara's practice seemed to lack the didactical complexity typically found in her colleagues' episodes. However, the more interesting differences, it could be argued, emerged at the level of the topic. Notwithstanding Sarah's low didactical complexity in relation to Klara when teaching number patterns, Olga's episodes, with respect to the number line, appeared more didactically complex than Maria's, particularly in the former's repeated opportunities for her students to explore the relationship between number and quantity. With respect to conceptual subitising such differences were minor, although it could be argued that Klara paid much more attention to systematic counting than did Kerstin. However, such conclusions remain tentative, although they allude to the sensitivity of the FoNS framework to culturally-located differences.

It is also interesting to note that the topics themselves seemed to invoke different levels of complexity. Admittedly, such distinctions are crude, but are supported by the fact that Klara was involved in both the highest and lowest topic means. As can be seen from the topic means in Table 1, episodes focused on number patterns invoked, in general terms, relatively few FoNS categories, while those focused on conceptual subitising the most. Indeed, the data suggest that some topics have a greater propensity for teachers to address a range of FoNS-related learning possibilities. However, such conclusions, while tentative, tend to confirm the sensitivity of the FoNS framework to topical differences.

Finally, with respect to this particular analysis, Table 1 highlights the relative paucity of opportunities for students to engage in quantity discrimination and the complete absence of encouragement to estimate. While it could be argued that such FoNS categories may not be suited to the three topics examined here, the fact that Klara managed to invoke quantity discrimination in two of her three number sequence-related episodes may suggest otherwise. Also, the complete lack of invitation to estimate may say something different about how teachers construe mathematics as a precise rather than imprecise discipline, not least because it is not difficult to imagine a teacher asking students to estimate, say, the twentieth member of a sequence or where a given number would be placed on an empty number line.

DISCUSSION

Our aim was to introduce and summarise recent work on the development of foundational number sense (FoNS). Our uncovering of three forms of number sense has gone a long way with respect to the problem of definition. Our atypical use of constant comparison has facilitated the development of an eight component FoNS framework that we have shown to be functional in different cultural contexts and with

			FoNS categories							
	Topic and mean	Episode	Number symbols, vocabulary and meaning	Systematic counting	Relationships between numbers and quantities	Quantity discrimination	Different representations of number	Estimation	Simple arithmetical operations	Number patterns and missing numbers
Sarah (2.3)	Number patterns (3.2)	1	X							X
		2	X						X	X
		3	X	X						
Klara (4.0)		1	X						X	X
		2	X		X	X	X		X	X
		3	X			X	X			
Olga (4.7)	Number line (4.2)	1	X	X	X		X			
		2	X	X	X		X		X	
		3	X	X	X		X		X	
Maria (3.7)		1	X	X			X			
		2	X	X			X			X
		3	X	X			X		X	
Kerstin (4.7)	Subitising (4.8)	1			X		X		X	X
		2	X	X	X		X		X	
		3	X		X		X		X	X
Klara (5.0)		1		X	X		X		X	X
		2		X	X		X		X	X
		3	X	X	X		X		X	
Totals			15	11	10	2	14	0	12	9

Table 1: Codes applied to each episode with summary statistics

different mathematical topics. Importantly, the case studies show that different teachers, while attending to similar core topic-related outcomes, also privilege different things. For example, Olga and Maria, in their number line-related teaching, encouraged several similar outcomes, although Olga focused attention on the relationship between number and quantity while Maria emphasised real world representations of the number line. In similar vein, when working on number patterns, Klara addressed almost twice as many FoNS-related outcomes as Sarah, confirming earlier research about the relative didactical complexity of Hungarian and English mathematics teaching. The analyses also show that the three topics failed to yield any episodes in which teachers encouraged their students to estimate, a key predictor of later arithmetical competence (Booth & Siegler, 2006), while only two, identified during Klara's teaching of number patterns, alluded to quantity discrimination, also a key predictor of later competence (Titeca et al., 2014).

In closing we speculate a little and suggest that the FoNS framework has the potential to inform the practices of teacher education for elementary teachers; its simple structure makes it a suitable starting point for students' professional learning, particularly from the perspective of practicum-related planning and teaching. It can also be used as a simple assessment tool for provoking post lesson discussion.

ACKNOWLEDGEMENT

We are grateful to our colleagues Roger Fermsjö; Jenni Back and Lisa Björklund Boistrup for sharing their Russian, Hungarian and Swedish data respectively.

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Additive adaptive thinking in 1st and 2nd grades pupils

Lurdes Serrazina and Margarida Rodrigues

Instituto Politécnico de Lisboa, UIDEF, Escola Superior de Educação, Lisboa, Portugal, lurdess@eselx.ipl.pt

This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues". It discusses what is meant by adaptive thinking and presents the results of individual interviews with four pupils. The main goal of the study is to understand pupils' reasoning when solving numerical tasks involving additive situations, and identify features associated with adaptive thinking. The results show that, in the case of first grade pupils, the semantic aspects of the problem are involved in its resolution and the pupils' performance appears to be related to the development of number sense. The 2nd grade pupils seem to see the quantitative difference as an invariant numerical relationship.

Keywords: Adaptive thinking, numerical relationships, quantitative difference.

INTRODUCTION

This paper is part of the Project "Adaptive thinking and flexible computation: Critical issues" being developed by the Schools of Education of Lisbon, Setúbal and Portalegre that has two main focus: (a) to characterize the development of pupils' numerical thinking and flexibility in mental calculation from 5 to 12 years, and (b) to describe teachers' practices that facilitate that development.

In this paper we will discuss different perspectives on adaptive thinking and flexible calculation with regard to addition and subtraction in the literature, and present preliminary findings obtained by conducting individual clinical interviews with four pupils (two from first grade and two from second grade). We intend to understand pupils' reasoning when solving numerical tasks involving additive situations, and identify aspects related to adaptive thinking. Besides this goal, the interviews were also conducted to test tasks to later implement a teaching experiment.

THEORETHICAL FRAMEWORK

In the last decade, flexible calculation has been considered an ability that all pupils should develop in elementary school (Anghileri, 2001). Being proficient in mathematics implies the ability to use the knowledge in a flexible way and apply in an appropriate way what is learned in a situation to another (NCTM, 2000).

The flexible idea appears associated to mental calculation and arithmetic problem solving. There are different ways of solving an arithmetic problem mentally, usually mentioned as strategies. Strategic flexibility in mental calculation refers to the way as the problem is affected by circumstances to be solved (Threlfall, 2009). These circumstances may be related with specific features of the tasks or with individual characteristics or contextual variables. Threlfall (2009) refers to the mechanism behind calculation-strategy-flexibility as *zeroing-in*, still referring that it is not a fully conscious and rational process, involving partial exploratory calculations arising from noticing specific features of the numbers involved and their respective relationships. "The calculation-strategy is not selected and applied, it is arrived to" (Threlfall, 2009, p. 548). In a different perspective, Star and Newton (2009) define flexibility as knowing multiple solutions as well as the capacity and tendency to choose the most appropriated for a given problem and a particular objective of problem solving. These authors also stated that flexibility exists at a continuum; when the pupils gain flexibility they may first show a greater knowledge of multiple strategies, then particular preferences, and finally, the appropriate use of the preferred strategy. The 'appropriate' term refers to the more effective strategy: one which requires the least number of intermediate calculation steps to arrive at the result. Other authors (Baroody & Rosu, 2006; Rathgeb-Schnierer & Green, 2013) reported that flexibility in calculation is related to the fact that children discover patterns and relations, as they develop number sense, thus building a network of relationships. For example, pupils who recognize the commutative property of addition, given the need to calculate 3 + 9, know they can do 9 + 3. The way this property is mobilized, revealing or not the contextual aspects of the tasks, can vary depending on the age of the children. In this regard, several authors (De Corte & Verschaffel, 1987; Greer, 2012) report that the semantic aspects of the tasks influence how young children solve them. Pupils who understand the various compositions of a number in different parts (for example, 1 + 7, 2 + 6, 3 + 5, and 4 + 4 = 8…) and decompositions (e.g., 8 = 1 + 7, 2 + 6, 3 + 5, 4 + 4) are more likely to develop ways of thinking as "doubles +1" (e.g., 7 + 8 = 7 + 7 + 1 = 14 + 1) or making a "ten" (9 + 7 = 9 + 1 + 6 = 10+6). As the network of relationships is being built, children acquire the flexibility to use these relationships in concrete situations of calculation, which depends on their knowledge of numbers and operations (Rathgeb-Schnierer & Green, 2013).

In our perspective, adaptive thinking refers to a thinking that can be flexibly adapted to new as well as familiar tasks. Its focus is not on calculation-strategy, but on quantitative reasoning. Children can mechanically use learned strategies without considering the context or the numbers involved in the task (Brocardo, 2014) and in this sense, they can compute accurately without flexibility. The flexible calculation and the additive quantitative reasoning are two dimensions that are interrelated to each other. Because the quantitative reasoning focuses on the description and modeling of situations and comparative relationships involved (Thompson, 1993), it ultimately underlies the development of flexible calculation as a calculation that mobilizes numerical relationships, in an intelligent and adaptive way to situations and numbers themselves. So the adaptive thinking involves the development of a flexible and relational understanding enabling the pupils to produce new known facts from old ones.

The quantitative reasoning involves reasoning about relationships between quantities. It "is the analysis of a situation into a quantitative structure — a network of quantities and quantitative relationships" (Thompson, 1993, p. 165). What matters are the relationships between quantities and not the numbers and number relations. In this regard, this kind of reasoning approaches the algebraic reasoning. To clarify the distinction between quantity and number, Thompson (1993) connects the idea of measure to the notion of quantity, although this is not only applicable to continuous measurable quantities, and the reasoning does not depend on their measures. It is important to develop research in children's abilities to deal with complexity in situations. A relationally complex situation involves at least six quantities and three quantitative operations. Comparing two quantities to find the excess of one relative to the other is a quantitative operation. The result of the quantitative operation of comparing two quantities additively is the excess found, that is to say, the *quantitative difference*. The author stresses also the distinction between the concepts of numerical difference, as the result of subtracting, and quantitative difference. On the one hand, a quantitative difference is not always evaluated by subtraction and on the other hand, subtraction can be used to compute quantities that are not quantitative differences. The results of the teaching experiment held with 5th-grade children referred in Thompson (1993) show that these children (i) did not distinguish the quantitative and the arithmetical operations, and (ii) had trouble with two aspects of the concept of quantitative difference, namely the difference as an additive comparison of quantities and when they conceived the quantitative difference as an invariant numerical relationship as they assumed the relative change as an absolute amount and needed to know absolute values before they could make comparisons.

The additive comparison is closely linked to inverse reasoning, involving the mobilization of reversible thought. According to Greer (2012), the inversion is of central importance to the arithmetic of natural numbers and the four basic operations involving these numbers, with important implications in relation to flexible computation. Regarding comparison problems, the author draws attention to the fact that the inverse relationship relates the difference between A and B to the complementary difference between B and A, which is "quite a different conception" (p. 434). So, although this author refers to the inverse relationship between addition and subtraction and the quantitative reasoning involves additive quantitative operations that are distinct of these arithmetic operations, we can consider that the inversion is an intrinsically topic underlying the quantitative reasoning.

METHODOLOGY

This study follows a qualitative approach within an interpretive paradigm. It aims describing and interpreting an educational phenomenon (Erickson, 1986). Data collection for this paper was done through clinical interviews (Hunting, 1997). It is a technique that is directed by the researcher and seeks a description of the ways of thinking of respondents.

Individual interviews were conducted in January 2014 by the authors of this paper, both members of the research team of the Project. The four pupils were attending for the first time the respective grades and were selected by their teachers. The selection criterions were: (i) pupils that usually express what they think, and (ii) pupils with reasonable performance in Mathematics. For the ethical principle of confidentiality, we use fictitious names for the children interviewed. The interviews were audiotaped and occurred in a room outside pupils' classrooms and had lasting less than 30 minutes. We also used the observation technique in the course of interviews, recording after its end the children's performance observed in field notes.

Each pupil solved three/four tasks but not all solved the same tasks. The task *boxes with balls*, inspired by Cobb, Boufi, McClain and Whitenack (1997) (Figure 1), was proposed to two first graders (Ana and Rui) and two second graders (João and Diogo).

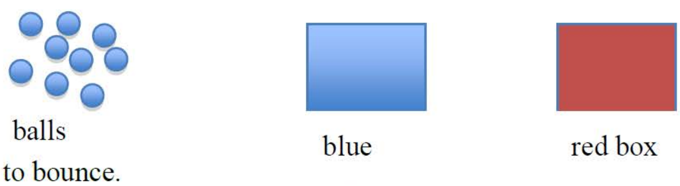
The task *game of marbles,* adapted from Thompson (1993) (Figure 2), was only solved by João of 2nd grade.

Because the limited size of the paper we do not present all five tasks proposed in the interview. We chose these two tasks for the paper because they have relevant features to reveal adaptive thinking. The task *boxes with balls* was chosen because it reveals the children's thinking about flexible partitioning (e.g., a collection of 9 items conceptualized in imagination as: five and four, three and six, etc.). It is embedded in a fairytale context with the purpose of captivating young children and appealing to their imaginary world in which inanimate objects (such as balls) come to life and jump from one box to another, without human interference. The idea of movement (change of state) was central for the task design. So, consideration of the dynamic part of that movement will induce the pupils to explore different possibilities of decomposition of 9, since the balls are not distributed statically into two boxes but continue to jump from one box to the other, varying in number at each moment. The existence of two and no more boxes relates to the fact that it is desired to induce the representation of 9 into two groups, facilitating the development of the additive structure of **N** and the obtaining of certainty of the totality of solutions by the use of some organization in the disposition of them. For instance, in this scheme, we can see a structure of increasing and decreasing sequences and a central symmetry that support the exhaustion of splittings (Freudhental, 1983):

$$9 = + \frac{9}{0}\ \frac{8}{1}\ \frac{7}{2}\ \frac{6}{3}\ \frac{5}{4}\ \frac{4}{5}\ \frac{3}{6}\ \frac{2}{7}\ \frac{1}{8}\ \frac{0}{9}$$

The other task was chosen because it aims to emphasize the notion of quantitative difference as signifi-

Once upon a time two boxes, one blue and one red, and 9 magical bouncy balls that spent their lives jumping from one box to another.



The balls will continue to bounce.
Imagine the different ways that we can see all nine balls in the two boxes.

Figure 1: Task *boxes with balls*

Ana, Luís and André played two games of marbles together.
Ana won 3 marbles from Luís and 7 marbles from André.
Luís won 4 marbles from André and 9 marbles from Ana.
André did not win marbles.
a) What is the least number of marbles that André had at the beginning of the games?
b) Compare the number of Luís's marbles before and after these two games.

Figure 2: Task *game of marbles*

cantly independent of the knowledge of the values of the additive and subtractive. It reveals the children's ability to reason inversely and to distinguish between quantitative difference and the absolute value. This is a fundamental aspect in flexible calculation.

Based on our adopted approach to flexible calculation and adaptive thinking in which we integrate the theoretical framework of Threlfall (2009) and Thompson (1993) that is presented in the previous section, the data were analyzed trying to understand how the children were able to establish a network of connections through their reasoning about different representations of the numbers, and about relationships between numbers and quantities. The analytical categories built from the theoretical framework and inductively emerged from the data focused on pupils' process of solving the tasks: applied relationships (relating the numbers, relating the operations, using the inverse relationship, comparing quantities); applied properties of addition (exhausting all possibilities).

SOME EMPIRICAL RESULTS

Addition and subtraction: From concrete to abstract thinking

In the task *boxes with balls* the collection of balls represented in the sheet of paper remained visible throughout the interview so that each child might be able to propose various ways in which the balls could be in the boxes. A sheet of paper with two drawn boxes was available for the pupils.

Once the first grade pupils had trouble with reading, the researcher read the task to be sure that there were not problems of understanding. Ana answered immediately:

Ana: 5 + 4

Researcher: And other ways?

Ana wrote immediately on the paper: 2+7, 7+2, 8+1, 1+8, 4+5, 3+6, 6+3.

After these records, Ana added there could still be 9 + 0. When questioned if it would be needed to write 2 + 7 and 7 + 2, Ana replied: "It gives the same, but it is not the same. Here [*pointing to the firs box*] there are 2 and there 7 [*pointing to the second box*] and here [7 + 2] is the opposite."

It seems that Ana thought about the concrete balls, and looked at the sums as ordered pairs of numbers.

In the case of Rui, after registering 5 + 4, the sequence of registration was: 4+5, 6+3, 3+6, 8+1, 1+8, 9+0.

After a moment, Rui wrote on the paper: 7+2 and 2+7.

Rui also responded to the question whether or not 6 + 3 is the same as 3 + 6: "It is. Only that is unlike."

Both children, Ana and Rui, represented first the situation of 5+4. Both pupils were able to visualize all decompositions of 9 thinking about the real situation — boxes and balls, although they did not need to draw them in the boxes. So, it seems that both thought about ordered pairs of numbers, trying to write all the pairs, using their different images of 9. Both seem to identify the commutative property because they wrote commutative pairs of numbers in a consistent way. Rui, after having written 5+4 and 4+5, he wrote 6+3, 3+6, using the increase/decrease 1 property. The same seems to happen with Ana, when she wrote 8+1, 1+8 following 2+7, 7+2.

In case of the two second grade pupils, João wrote "4+5, 3+6, 2+7, 1+8" and then stopped. Asked if there were more chances, João replied: "No, because I could change the order of numbers, but it would be the same thing, the sum is the same." Diogo wrote on the paper: 5 + 4, 6 + 3, 8 + 1, 7 + 2.

When the researcher asked if he already had written all the possibilities, he said: "Yes. If I change, its sum is the same, 9".

Both second graders made all possible non-empty decompositions of 9, not having considered the possibility of 9 + 0 (or 0 + 9). It should be noted that João used the increase/decrease 1 property to write all the decompositions, while Diogo appeared to start in that way, but changed his strategy when wrote 8+1 after 6+3 (increase/decrease 2?) and then seems to come back to the first one. But they did not express their ways of thinking.

These pupils seem to be able to think about the numbers abstracting from real situations. More, it appears that they have already understood the commutative property of addition.

Thus, we see that the consideration of the contextual situation was taken over by first graders but not by second graders who ignored the fact that the parcels play different roles in the proposed situation. The first grade pupils understand the concrete situation, and their thoughts are close to real situations since they considered ordered pairs of numbers. Instead, the second graders overcame the concrete situation.

Quantitative difference

In the task *game of marbles,* João used a tabulated registration. He began to register the total wins of marbles for each player: "+10"; "+13"; "+0". After, he put the total losses for each player by reading the sentences allusive to the wins: "-9"; "-3"; "-11". For that, he mobilized an inverse reasoning, understanding that the number of marbles won from someone is the number of marbles that someone had lost. His resolution can be seen in Figure 3.

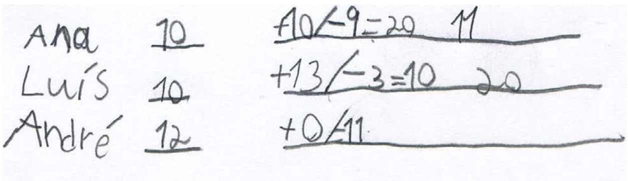


Figure 3: João's resolution of the task *game of marbles*

Then he focused on the item a) of the task, considering that André would have the minimum number of 12 marbles at the beginning of the games to have lost 11. He raised other hypotheses for this initial number like 20 or 30.

Researcher: And less than 12, no?

João: No, he had to have marbles.

This question focuses the absolute quantity of marbles. João held various hypotheses for the number of marbles of André before the games, all above 11, but assumed that at least André would have had 12 marbles. Probably, he did not equate the hypothesis of 11 marbles for having discarded the possibility of André having no marbles at the end of the games.

After, João did the balance of wins and losses of Luís's marbles, concluding that Luís would have 10 marbles more at the end of the games and recorded "=10" ("+13/-3=10"). In the trace corresponding to the beginning of the games, João wrote "10".

João: [At the end] Luís got ten more (...).

Researcher: At the beginning, did Luís have 10 or 10 more?

João: He had ten marbles.

João wrote "20" in the final of Luís's line corresponding to the total number of Luís's marbles at the end of the games. Next, the researcher guided João to the case of Ana:

Researcher: And Ana? She won ten and lost nine. After all, did she have more or less marbles at the end of the game?

João: Less. Before, she had ten marbles.

Researcher: If she started the games with ten marbles, won 10, with how many would she have at the end?

João: Twenty.

Researcher: And then she lost nine...

João: She had eleven.

João failed to make the balance between wins and losses of Ana's marbles, and concluded that Ana would have less marbles without mentioning how many. He wrote "10" in the trace corresponding to the beginning of the games. After, he registered "=20" ("+10/-9=20") as the number of Ana's marbles after 10 marbles won in the two games, and finally recorded "11" after losses, corresponding to the total number of Ana's marbles at the end of the games. João did not confront this final record with his previous statement ("Less") or verbalized that Ana had one more marble at the end of the games.

The critical issue inherent to this task is the distinction between quantitative difference and the absolute value. João did not make confusion between one thing and another, distinguishing the relative change (plus 10) from the absolute amount (20) in the second trace of the Luís's line, allusive to the end of the games. However he was not able to express the quantitative difference for the start of the two games (minus 10), needing to put absolute numbers for each player. João used the same equal symbol in the lines of Ana and Luís ("=20"; "=10"), but he attributed different meanings to the numbers: in the case of Ana, 20 is the absolute amount; in the case of Luís, 10 is the quantitative difference, that is to say, it is the result of relative change — the amount by which one quantity fell short or exceeded of another. The need to refer to the absolute amount — the concrete number of marbles — is also evident in the way João determines the absolute value of the number of marbles of Ana and Luís at the end of the games ("11"; "20").

FINAL REMARKS

In the task *boxes with balls* pupils established different decompositions of the number 9 using their network of connections to have 9 based on properties of addition or making a direct subtraction and realizing that, for instance, by taking 4 from 9 they get 5. In both situations, pupils dealt with the operations addition and subtraction as being intrinsically inverse to one another (Greer, 2012). The quantity of nine balls can be symbolized by the number nine (Thompson, 1993) expressed by different sums representing the partition of a collection of objects or the decomposition of the number nine. The results show that all pupils dealt with the number, and not properly with the quantity. They did not divide the set of balls. Although the collection of balls represented in the sheet of paper remained visible, the pupils ignored them, thinking in a higher level about the number 9 conceptualized in imagination as sums representing its decompositions. For that, they seem to understand the relationships between those sums.

Although all the pupils started with the numbers 4 and 5 (or 5 and 4) adopting the approach of double through the decomposition of 9 into almost equal groups, the first and second graders solved the first task in different ways. Even though, first grade pupils did not need to materialize the situation with manipulatives or drawings, they solved the task very close to its context. We suspect that they were sure that they had generated all the possibilities because they applied some organization in the generation of the pairs of numbers writing consistently commutative pairs. So they looked for ordered pairs of numbers that together make 9. Instead, the second graders disengaged from the concrete situation, and only considered the fact that they had to obtain 9 from a sum. João did it systematically (increase/decrease 1) while Diogo did not do it for all decompositions. As they already know the commutative property (though not in a formal way) they applied it to justify that they found all the cases. So, they solved the problem in mathematics terms but not the actual proposed problem, where it should be considered the two different boxes and, in this perspective, it is not the same to have the balls in the red box or in the blue box. All the pupils already seem to understand the commutative property, but use it in a different way: the first graders use it to exhaust all possibilities through the symmetry of ordered commutative pairs of numbers (Ana: "It gives the same but it is not the same"), and the second graders use it not to present the commutative pairs, whereas the symmetrical parts would be the same (Diogo: "If I change, its sum is the same 9"). De Corte and Verschaffel (1987) stress that younger children are more influenced by the semantic aspects of the tasks. So it seems that the differences in the answers of first and second graders may be related with their age and their level of mathematical thinking.

In the task *game of marbles*, João shows an additive adaptive thinking as he seems to apply the inverse relationship between the wins and losses of marbles. Unlike the pupils of 5th grade reported by Thompson (1993), João did not confuse the notions of quantitative difference and absolute value of marbles nor needed to know the initial number of marbles to be able to think about wins and losses. However, he needed to be anchored in concrete numbers of marbles as happened with the pupils studied by Thompson (1993). Because João felt the need to attribute absolute values to the initial numbers of marbles, he showed to reason in terms of difference as quantitative operation without separating it from the involved particular arithmetical calculations. It is to say, João could not speak of relative changes associated to additive comparisons without referring to absolute amounts, showing conceiving the quantitative difference as an invariant numerical relationship. Thompson (1993) argues that it is important to conceive a quantitative difference independently of numerical information about quantities and relationships. However, we suspect that young pupils of 2nd grade are not able to conceive the independence of the values of the additive and subtractive when they reason quantitatively. For that reason, we consider that this task is not suitable for pupils of this grade, and we will not implement it in the teaching experiment.

The results reported here support the idea that flexible calculation is related to the knowledge and use of numerical relationships, being richer as pupils are developing their number sense, and are able to use the network of relationships that are building (Baroody & Rosu, 2006; Rathgeb-Schnierer & Green, 2013).

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Improving classroom assessment in primary mathematics education in the Netherlands

Michiel Veldhuis[1] and Marja van den Heuvel-Panhuizen[1,2]

1 Utrecht University, Freudenthal Institute for Science and Mathematics Education, Utrecht, Netherlands, m.veldhuis1@uu.nl

2 Faculty of Social and Behavioural Sciences, Utrecht University, Utrecht, Netherlands

We report on the results of the Improving Classroom Assessment (ICA) project in the Netherlands that was aimed at improving primary students' mathematics achievement through improving their teachers' classroom assessment. Towards this end we first investigated primary teachers' assessment practice in a large-scale survey study. After having described and profiled teachers' current assessment practice, we designed a number of classroom assessment techniques, which were tested for feasibility in two small-scale sub-studies. Finally, we evaluated the effectiveness of the use of these classroom assessment techniques in a large-scale evaluation. Results indicate that students generally benefit from their teachers' improved use of classroom assessment techniques in mathematics.

Keywords: Classroom assessment; primary school; assessment technique; achievement.

BACKGROUND AND FOCUS OF THE PROJECT

Developing and keeping track of students' mathematical abilities are important parts of every primary mathematics teacher's daily practice. In order for teachers to gauge their students' learning, classroom assessment plays a pivotal role (Cizek, 2010). By using classroom assessment teachers can gather information about their students' mathematical skills and level of understanding. Collecting information on students' learning is primordial for at least two reasons: to find out whether the instruction has had its desired effect and to generate ideas for how to proceed in the subsequent lessons. Based on assessment information teachers can align their teaching to their students' needs, which in turn can result into adapting their teaching, but can of course also mean not changing anything and continuing with what was planned before.

Many of the characteristics of classroom assessment appear to be part of merely good teaching practice, as Ginsburg (2009) wrote in the context of mathematics education:

> Good teaching […] sometimes involves the same activities as those comprising formative assessment: understanding the mathematics, the trajectories, the child's mind, the obstacles, and using general principles of instruction to inform the teaching of a child or a group of children (p. 126).

Classroom assessment is broader: it comprises all activities that permit teachers to find out where their students are at a particular moment in terms of comprehension of the subject and to give information on what is going right and wrong. Policymakers as well as influential researchers have urged the educational community, and in particular teachers, to embrace (formative) classroom assessment in their practice. For instance, the U.S. National Council of Teachers of Mathematics (NCTM, 2013) recently took the following position on formative assessment in mathematics education:

> The use of formative assessment has been shown to result in higher achievement. The National Council of Teachers of Mathematics strongly endorses the integration of formative assessment strategies into daily instruction (p. 1).

Teachers are the only ones that can actively integrate these formative assessment strategies into their practice. Advocating positions such as these were mainly inspired by the influential review study by Black and Wiliam (1998) that reported the different practical expressions of classroom assessment to be the most effective interventions for teachers to improve student learning. Recently, several researchers have questioned the size of the effectiveness of (formative) assessment on student learning through reviews or meta-analyses of existing studies (e.g., McMillan, Venable, & Varier, 2013). Common to these critical examinations, although their specificities differ, is that they do not contest the positive effect formative assessment is purported to have on student achievement, but only the size of this effect.

Why then is classroom assessment by teachers supposed to lead to improved student learning in mathematics? In order to answer this question researchers have drawn parallels between the concepts and practices of formative assessment, self-regulated learning, feedback, and scaffolding (see for an overview, among many others, Clark, 2012). An intuitive way of saying it would be: if teachers are better aware of their students' mathematical abilities and understanding, then they can undoubtedly better adapt their teaching to the needs of the students. In doing this and providing explicit and implicit feedback students also become more aware of their own functioning, and the circle is complete: students and teacher simultaneously advance. This does have its limits, because "the teacher *must actually use the assessment data* to inform some change in the conduct of instruction" (Erickson, 2007, p. 189, original emphasis). In order for teachers to be willing to use classroom assessment techniques, these have to provide them with valuable and easily usable information about students' understanding of mathematics in a timely manner, otherwise it would not contribute to better teaching and, in the end, to better student achievement.

In the ICA-project we strived to improve primary students' mathematics performance through improving their teachers' use of classroom assessment in Grade 3. As a start we investigated the current classroom assessment practice of primary mathematics teachers in the Netherlands (Study 1). Secondly we identified a meaningful profile characterization of these teachers' mathematics assessment practice (Study 2). Now that the current practice was known we could test the feasibility of classroom assessment techniques that were designed to match the mathematics curriculum of the second half of Grade 3 in the Netherlands and provide valuable information to the teachers (Study 3). Finally we evaluate the effectiveness of the use of these classroom assessment techniques in a large-scale experimental study (Study 4).

STUDY 1: PRIMARY TEACHERS' USE OF CLASSROOM ASSESSMENT IN MATHEMATICS

We conducted a survey of the classroom assessment practices of Dutch primary school teachers in mathematics education (Veldhuis, van den Heuvel-Panhuizen, Vermeulen, & Eggen, 2013). International studies have shown that teachers use a wide range of methods to collect information about their students' learning (e.g., Suurtamm, Koch, & Arden, 2010). To find out students' skills and comprehension level, teachers can use methods ranging from standardized tests and tests that come with a textbook, to asking questions and observing students while they are working. The assessment methods teachers choose to reveal their students' learning processes depend on several factors. A first factor that has been found to affect this choice is teachers' beliefs concerning classroom assessment (Dixon, Hawe, & Parr, 2011). A second factor in choosing a particular assessment method, beside beliefs, concerns the assessment purpose teachers have in mind (Suurtamm et al., 2010), for instance a formative or summative purpose. A further determining factor of using particular assessment methods is the view on education in which the assessment takes place. The methods used for assessment often correspond to the approach to education as reflected in the adhered learning theory and the curriculum that is taught (Shepard, 2000).

Method

We investigated, using an online questionnaire, how primary teachers in the Netherlands collect information on their students' progress in mathematics and how teachers' assessment methods, purposes, and beliefs about the usefulness of assessment are related. This questionnaire contained 40 items, pertaining to the teachers' (i) background characteristics, (ii) mathematics teaching practice, (iii) assessment practice, and (iv) perceived usefulness of assessment. Questions with different formats were included: fixed-response and items with a rating scale, but also some open-ended items. The sample of participating teachers was obtained through an open invitation by e-mail, which was sent successfully to 5094 primary schools for regular education in the Netherlands. Teachers who were willing to respond to the online questionnaire were promised a set of digital mathematical exercise material as a reward.

Results and discussion

In total 960 teachers at 557 Dutch primary schools responded to the questionnaire. Observation-based assessment methods of questioning, observing, and correcting written work, were the most frequently – that is weekly – applied methods, whereas instrument-based methods, particularly using textbook tests and student monitoring tests were employed several times a year (see Figure 1).

Teachers used assessment mainly for formative purposes and they considered the assessment methods they used themselves as most relevant. We found that teachers in primary mathematics education in the Netherlands use a variety of assessment methods, use instrument-based and observation-based assessment methods on average just as frequently and find assessment generally useful. This perceived usefulness is shown by the overall very positive reactions teachers gave on the different uses of assessment. The two main instrument-based assessment methods, textbook tests and tests from a student monitoring system, are reported as the most relevant, with asking questions and observing students the most relevant of the observation-based assessment methods. Furthermore, the teachers' responses to the questionnaire revealed that they used assessment both for formative and summative purposes. The results of our survey indicate that teachers do use assessment information for various purposes, from giving feedback via adapting instruction to stimulating thinking.

STUDY 2: PRIMARY TEACHER PROFILES IN MATHEMATICS ASSESSMENT

After this general overview of the current assessment practice we were interested in finding out more about individual teachers. The second study was aimed at gaining knowledge of how the assessment practices of individual teachers could be characterized within the universe of assessment skills and activities. In fact, we wanted to understand assessment from the conglomerate of choices a single teacher is making when collecting information about his or her students' learning process. To achieve this we performed a secondary analysis of the earlier gathered questionnaire data to identify a profile characterization of every teacher's assessment practice (Veldhuis & van den Heuvel-Panhuizen, 2014). The rationale for distinguishing assessment profiles of teachers is that these can contribute to our theoretical understanding of assessment as it teachers carry it out. In addition, knowledge about these assessment profiles can help us in a practical sense with designing tailor-made courses for professional development that fit the teachers' needs.

Analyses

We analyzed the survey data in two steps. To identify the latent structure of what was measured by the questionnaire and be able to construct assessment profiles of teachers we used a combination of latent variable modeling techniques. To explore the underlying structure of the items measuring teachers' math-

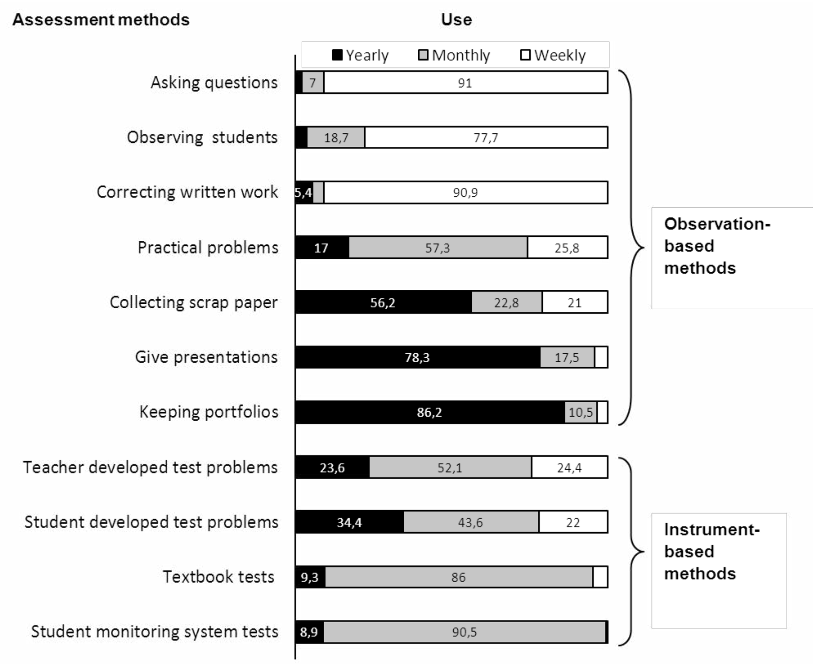

Figure 1: Percentages of frequencies of assessment methods' use (*ns* > 940)

ematics assessment practice, we performed exploratory factor analyses. To investigate whether these latent factors could also be used to interpret classes of teachers, we performed a latent class analysis. This is a statistical technique permitting the identification of underlying classes of individuals based on differences in their responses on items in a questionnaire or test. The teachers in our sample were assigned to the different latent classes–that we will call assessment profiles–through modal assignment, i.e. they were assigned to the latent class to which they had the highest probability of belonging.

Results and discussion

After comparing one- to seven-factor solutions and eliminating items with cross loadings over |0.4|, an exploratory factor analysis delivered a five-factor solution that had a good enough fit ($\chi^2(1076, N = 960) = 5494.1$, $p < .001$, RMSEA = .064, CFI = .961). We named the five factors based on the items they contained: *Goal centeredness of assessment* (items on teachers' purposes of assessment), *Authentic nature of assessment* (items on authentic assessment methods), *Perceived usefulness of assessment* (statements on usefulness), *Diversity of assessment problem format* (items on problem formats), and *Allocated importance of assessing skills and knowledge* (items on the importance of assessing particular skills and knowledge). To be able to characterize teachers' assessment practice and assign them to different assessment profiles we performed a latent class analysis using all variable scores as input. As such we were able to check whether we would be able to show differences between the latent classes of teachers on the five factors we found in the separately performed factor analysis. Four latent classes provided the best fitting solution. To find out whether teachers thus assigned to the four latent classes differed on the five factors of assessment practice identified before, we performed several analyses of variance. The results showed that teachers from one latent class to another differed significantly from each other (see Figure 2 for the size and the direction of these differences). These differences suggest that teachers with particular assessment profiles have qualitatively different assessment practices.

The assessment profile to which most teachers (35.5%) in our sample belonged was the *mainstream assessors* profile. In this profile most teachers regularly used different types of assessment, test-based and observation-based, for both summative and formative purposes. On all factors teachers with this profile scored around the mean.

The next biggest group (28.5%) were the *enthusiastic assessors*. Teachers with this profile were very aware of the different possibilities assessment offers them and used them likewise. On all components these teachers scored above the mean, with a peak on *Goal centeredness of assessment*. An almost equally large

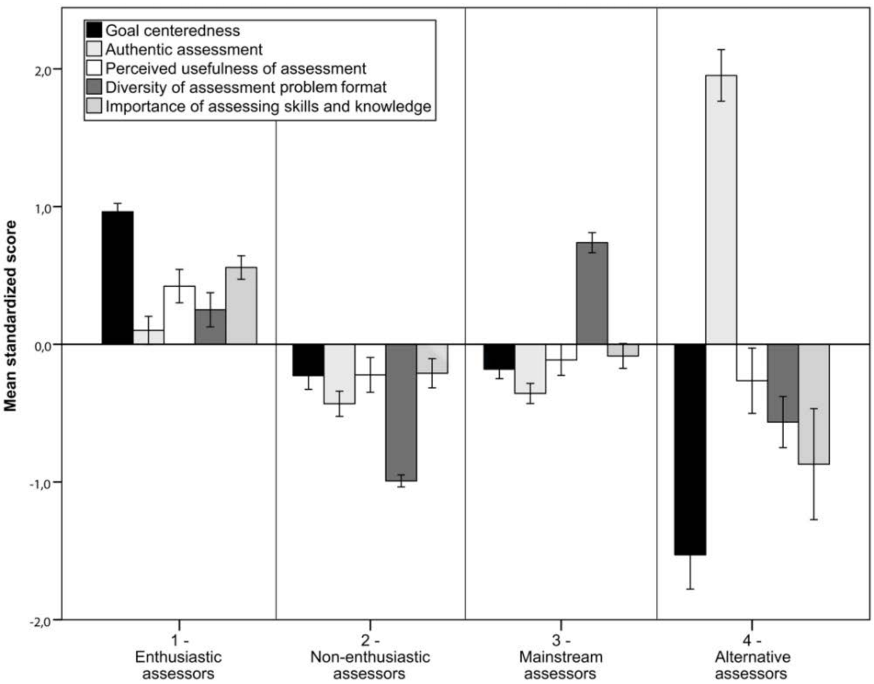

Figure 2: Mean standardized scores on factors for teachers in the four latent classes

group of teachers (25.8%) were the *non-enthusiastic assessors*. These teachers viewed assessment more often in a negative way and used it accordingly less. On all factors, teachers with this profile scored below average. Finally, there were the *alternative assessors* (10.3%). Teachers with this profile had an ambiguous view of assessment. Although they reported a lot of own input in assessment and devised their own tasks and tests, they did not find assessment important or necessary.

Through this profile characterization of teachers' assessment practice we were able to select some of the skills and activities from the universe of assessment skills and activities of teachers. In this way we brought structure to the many possible characterizations of assessment practice and gained a more clear idea of what to expect from teachers prior to the introduction of the assessment techniques in our consecutive studies.

STUDY 3: EXPLORING THE FEASIBILITY AND EFFECTIVENESS OF CLASSROOM ASSESSMENT TECHNIQUES IN MATHEMATICS

In this small-scale study the focus switched from the current practice of teachers to how this practice could be improved. Many types of formative or classroom assessment techniques have been proposed and used in international research (e.g., Black, Harrison, Lee, Marshall, & Wiliam, 2004) or in more practice-oriented work (e.g., Keeley & Tobey, 2011). In mathematics education there exist many different forms of these classroom assessment techniques, through our survey of current practice we could determine that these techniques were not very often used by primary teachers in the Netherlands. The Dutch Inspectorate (Inspectie van het onderwijs, 2013) also pointed out that many primary (40%) and secondary schools (33%) do not systematically use assessments to monitor their students' progress. In any case, from available findings it becomes clear that there is a need for investigation of the use of classroom assessment techniques in mathematics education.

The purpose of this study was to investigate the feasibility and effectiveness of classroom assessment techniques for mathematics in primary school. We wanted to find out whether teachers and students were prone to use assessment techniques and whether the use of an ensemble of these techniques would be related to an increase in achievement.

Method

Ten teachers (with 214 students; 14 to 29 students per class) participated in monthly workshops in the second semester of Grade 3 in two consecutive sub-studies (four teachers in the first; six teachers in the second). In the workshops, consisting of three or four teachers and the first author, classroom assessment techniques were presented, discussed, and evaluated. The teachers were approached by e-mail and volunteered to participate. The schools were all situated in urbanized areas with highly mixed student populations, and the teachers used four different textbooks that were all based on realistic mathematics education principles as is common in the Netherlands.

The feasibility of the classroom assessment techniques was investigated by conducting regular classroom observations of every teacher in between workshops. These observations were intertwined with short informal interviews. To investigate the effectiveness of the use of classroom assessment techniques we used a pre-/post-test evaluation of students' mathematics achievement. The pre-test data consisted of the results from the midyear student-monitoring test for Grade 3 (Cito LOVS M5) and the results from the end of year student-monitoring test for Grade 3 (Cito LOVS E5) served as post-test data (Janssen, Verhelst, Engelen, & Scheltens, 2010). These biannual student-monitoring tests are used in virtually all primary schools in the Netherlands to monitor students' development in mathematical ability over the years. The teachers administered the tests in their own classes as is common in educational practice in the Netherlands. The scores on these tests are mathematical ability scores calculated through item response theory models.

We proposed a collection of classroom assessment techniques consisting of short activities of less than 10 minutes to the teachers. The techniques were supposed to help teachers to quickly find out something about their students' mathematical skills and understanding, provide teachers with indications for further instruction, and focus on some of the mathematics content of the second semester of Grade 3. Most techniques were centred on the assessment of number knowledge, mainly in the context of addition and subtraction, but they could also be used to assess multiplication and division tables. In Figure 3 we provide an example of a technique called Red/Green cards.

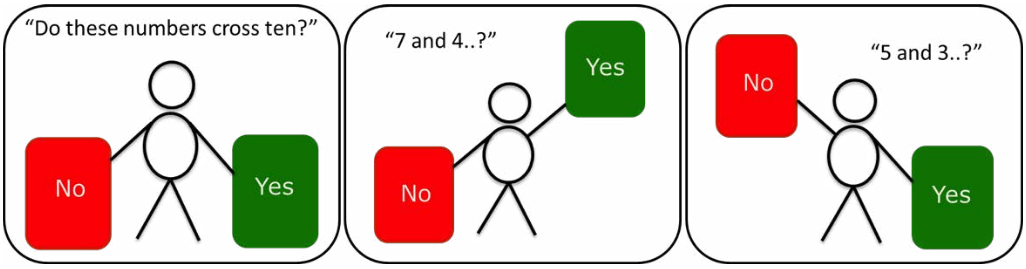

Figure 3: An example of the Red/Green cards. The focus is on number sense: the comprehension that two numbers together can be more or less than 10, 100, or 1000

The teacher asks all students a series of questions that can be answered quickly with Yes (green) or No (red). By inspecting the waving red and green cards the teacher gets an immediate overview of all students' responses. Especially when students have to carry out mental additions and subtractions with two-digit numbers it is crucial that they can instantaneously identify whether two numbers cross ten or not, because this has consequences for the strategy to be applied. This technique provides the teacher quickly with information on particular number sense knowledge of the students.

Results and discussion

Teachers and students reported enjoying the techniques and finding them useful in the sense that they provided them with valuable information that supported their teaching and learning. Teachers also mentioned that the techniques were easy to apply in their classrooms. In terms of mathematics achievement, results indicate students improving considerably ($M_{\text{gain substudy 1}}$ = +9.7; $M_{\text{gain substudy 2}}$ = +7.6). It could of course be expected that students advance in their mathematical ability, whether teachers perform specific assessment activities or not; the scores of the national norm sample also showed this direction ($M_{\text{gain norm}}$ = +5.1).

Even though the treatment group was relatively small and there was no control group in this study, these results do provide an indication for the feasibility and effectiveness of the use of the classroom assessment techniques in mathematics: teachers use the techniques and students appear to advance more from the midyear to the end of the year testing than expected.

STUDY 4: TEACHERS' USE OF CLASSROOM ASSESSMENT IMPROVING STUDENTS' MATHEMATICS ACHIEVEMENT

To verify whether the students' achievement improvement we found in Study 3 was really due to the teachers' use of the classroom assessment techniques in mathematics and not just to an attention (also Hawthorne) effect, we replicated this investigation in a large-scale experiment.

Method

The same pre-/post-test design was used as in Study 3, but now with a control group and an extra manipulation. Thirty teachers (and their 616 students) participated and were randomly distributed over the three experimental conditions and one control condition. In the experimental conditions teachers participated in the same type of workshops as in Study 3. These experimental conditions differed on the intensity of the professional development teachers received: in Experimental I there was one workshop, in Experimental II, two workshops, and in Experimental III, three workshops, and in the control condition teachers did not have any workshops. In these workshops the same classroom assessment techniques were discussed; there was more time for every technique if teachers had more workshops.

Results and discussion

Teachers' use of classroom assessment techniques in mathematics was associated with students' improved mathematics achievement. More specifically, when teachers participated in three workshops and, as such, developed more ownership of the techniques, their students showed more improvement in terms of mathematics achievement than in the other conditions ($M_{\text{gain Experimental III}}$ = +8.1; $M_{\text{gain other}}$ = <+5.9; cf. Figure 4 with the results of an ANCOVA, correcting posttest scores for pretest differences). Supporting teachers in the use of classroom assessment techniques for math-

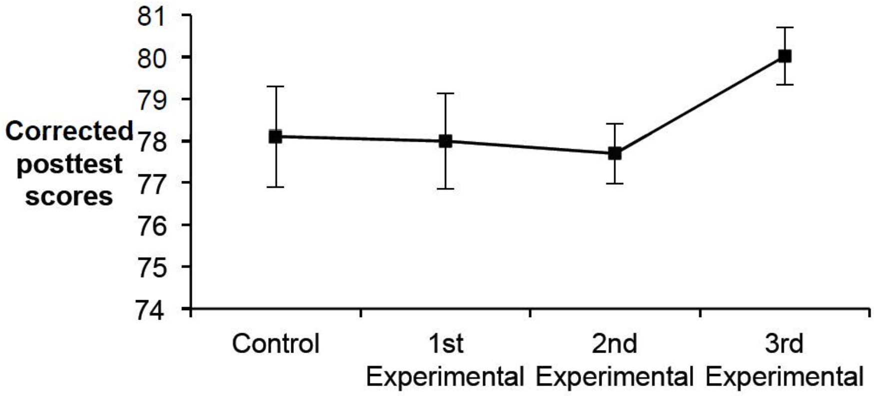

Figure 4: Posttest scores corrected for pretest differences for the four conditions (with error bars)

ematics in three workshops clearly benefits students' mathematics achievement.

ACKNOWLEDGEMENT

The ICA (Improving Classroom Assessment) project was supported by a grant from the Netherlands Organization for Scientific Research (NWO MaGW/PROO: Project 411-10-750).

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TWG02

Poster

A socially built understanding of rational numbers

Helena Gil Guerreiro[1] and Lurdes Serrazina[2]

1 Universidade de Lisboa, Instituto de Educação, Lisboa, Portugal, hg@campus.ul.pt

2 Universidade de Lisboa, UIDEF, Escola Superior de Educação Lisboa, Lisboa, Portugal, lurdess@eselx.ipl.pt

This poster focus on an exploratory study which is part of a research study which aims to understand how to develop students' emerging understanding of rational numbers, at elementary school, looking into how number sense development is promoted, through the use of different representations - percents, decimals and fractions - being the percentage the introductory one. Through a design research, based on a teaching experiment, guided by a conjecture, we intend to analyse the interactions, the strategies and the students' productions when solving tasks. In this poster some data analysis from the exploratory study, concerning the use of percents, will be presented. This data show that percentage seems to be a good starting to introduce rational numbers.

Keywords: Learning, rational numbers, number sense, taken-as-shared meaning.

Rational numbers is a complex content for the development of a sustainable construction of mathematical knowledge (Fosnot & Dolk, 2002). Moss and Case (1999), developed an experimental curriculum to introduce rational numbers where the first topic was percentage, following by two-place decimal numbers and fractional notation as introduced last. Based on this idea, the first author of this poster is developing a design research study, based on a teaching experiment, guided by a conjecture. This conjecture sustains that a detailed and sequenced work with percentage, and the subsequent interrelation with the other representations (decimal and fraction), can be a powerful learning pathway in emerging student's understandings of rational numbers, as they participate in social activity in the classroom and build taken-as-shared meanings, in a number sense development perspective. This conjecture has two dimensions: a mathematical content dimension and a pedagogical dimension. The first one focuses on a comprehensive concepts' learning construction related with rational numbers, in a perspective of number sense development. Teaching for development of number sense, as Fosnot and Dolk (2002) say, can be seen as the development of powerful strategies, models and big ideas that provide practical, flexible, and efficient computation, to handle numerical problems. The pedagogical dimension is based on a sociocultural perspective, where the construction of knowledge happens in the classroom, through active engagement in communication and interaction. These two dimensions complement each other and allow the study of pupils' learning and its evolution, by articulating a social perspective with a psychological perspective of the learning process (Cobb et al., 2011).

An exploratory study was developed at the beginning of third grade, in a class of 20 students and aimed to identify students' perception on percentage and their intuition regarding proportions. The results show that all the involved students recognize the expression 100% and associate it, in the context of a mobile phone battery, to a "full" battery, that is completely charged. The majority of those students are able to correspond 50% to half. Justifications like "Codfish with 50% discount costs half the price" or " It's 20€ because the jumper was 40€ and since it has a 50% discount it is half-price" reveal that this intuition exists and seems to come about, in some way, associated to the student´s experiences, mainly outside school.

Also, in the exploratory study, students were asked to take a stand concerning the statement "More than half of the document is saved!" in a situation where it was possible to see a computer status bar, when 80% of a document would already be processed. Agreement statements like "saving document ends at 100% and half has passed, which is 50%, because it is already in

82%" or "Because the saving document ends in 100% and that has already passed half 100, which is 50 and it is already in 82%" show that most of the students in the classroom reveal some knowledge related to this domain. Other situations presented involved mainly reference numbers and were associated to strong visual models, as percentage bars. It is worth noting that percentage had not been a topic worked in the context of the classroom. Using communication technologies in life creates a set of opportunities that can enhance school learning contexts. The data analyses of this exploratory study pointed towards integrating in school the knowledge built from the use of technologies in life. Percentage makes sense in school because it is part of life. Percentage bars provide a powerful visual concrete representation and can further contribute to build a connection between children's intuitions about proportions and rational numbers. The purpose is to grasp those experiences and trigger the construction of a knowledge network, towards rational numbers.

The ongoing study will seek to reconstruct episodes of student's work, in a detailed analysis of the events that took place, guided by the chosen interpretative theoretical framework. This analysis should allow the presentation of student's understandings embedded in the emergence of taken-as-shared meaning, in the learning of rational numbers, which may provide some clues for research in this area.

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TWG03

Algebraic thinking

Introduction to the papers of TWG03: Algebraic thinking

Jeremy Hodgen[1], Reinhard Oldenburg[2], Valentina Postelnicu[3] and Heidi Strømskag[4]

1 University of Nottingham, Nottingham, UK, Jeremy.Hodgen@nottingham.ac.uk
2 University of Augsburg, Augsburg, Germany
3 Texas A&M University-Corpus Christi, Corpus Christi, USA
4 Norwegian University of Science and Technology, Trondheim, Norway

In CERME9, as a long-standing group, TWG03 "Algebraic thinking" continued the work carried out in previous CERME conferences (e.g., Cañadas, Dooley, Hodgen, & Oldenburg, 2013).

There were a total of 15 papers and 7 posters with a total of 24 group participants representing 18 countries: Argentina, Canada, Cyprus, Czech Republic, Germany, Hungary, Ireland, Italy, Mexico, Norway, Portugal, Romania, Spain, Sweden, Turkey, Ukraine, UK, the USA.

THE RANGE AND DIVERSITY OF RESEARCH FRAMEWORKS

As we have observed in previous CERME working group reports, algebraic thinking is a "mature" sub-domain within mathematics education research (e.g., Cañadas et al., 2011). As a result, our group discussions touched on many familiar themes. Of particular interest to the group was the range of research frameworks, models and theories that participants drew on. In order to understand this diversity, we developed a categorisation of the frameworks used in TWG03 papers, where authors of TWG03 papers are given in square brackets.[1]

A. Models for conceptualising algebra, algebraic activity and algebraic thinking

— Kaput's (2008) model for conceptualisation of algebraic thinking [Chimoni & Pitta-Pantazi] [Glassmeyer & Edwards] [Twohill]

— Arzarello, Bazzini and Chiappini's (2001) model for analysis of algebraic thinking [Cusi & Malara]

— Drijvers, Goddijn and Kindt's (2011) model for categorisation of algebra [Pittalis, Pitta-Pantazi, & Christou]

— Kieran's (2004) model for conceptualisation of algebraic activity [Strømskag]

— Driscoll's (1999) framework for algebraic habits of mind [Eroglu & Tanisili]

— Pittalis and colleagues' (2013, 2014) model of early number sense [Pittalis, Pitta-Pantazi, & Christou]

B. Frameworks of variables and equation solving

— Bloedy-Vinner's (1994) dichotomy of "algebraic-analgebraic" to analyse students' difficulties with parameters [Postelnicu & Postelnicu]

— Hadjidemetriou and Williams' (2010) concept of linearity prototype for graphs [Pilous & Janda]

— Lima and Healy's (2010) notion of didactic cut in equation solving [Block]

— Star and Rittle-Johnson's (2008) strategies for solving linear equations [Block]

C. Frameworks of functions and functional thinking

— Vinner and Dreyfus' (1989) distinction between concept definition and concept image for the concept of function [Panaoura, Michael-Chrysanthou, & Philippou]

— Isoda's (1996) application of van Hiele levels as a model for development of the function concept [Szanyi]

— McEldoon and Rittle-Johnson's (2010) framework for functional thinking assessment [Xolocotzin & Rojano]

— Rivera and Becker's (2011) framework for pattern generalisation [Twohill]

D. General theories about teaching and learning mathematics

— Duval's (2006) theory of semiotic registers [Cusi & Malara]

— Bikner-Ahsbahs and Halverscheid's (2014) theory of interest-dense situations [Janssen & Radford]

— Radford's (2007) theory of objectification [Janssen & Radford]

— Godino, Batanero and Font's (2007) onto-semiotic approach to research in mathematics education [Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa]

— Dekker and Elshout-Mohr's (1998) model for interaction and mathematical level raising [Simensen, Fuglestad, & Vos]

— Matute, Roselli and Ardila's (2007) framework for neuropsychological children assessment [Xolocotzin & Rojano]

— van der Niet, Hartmann, Smidt and Visscher's (2014) framework for modelling relationships between bodily movement and academic achievement [Henz, Oldenburg, & Schöllhorn]

— Wertsch's (1991) concept of mediating tools [Wathne]

E. Holistic theories encompassing instructional design

— Brousseau's (1997) theory of didactical situations, TDS [Norquist] [Strømskag]

— Chevallard's (2003) anthropological theory of the didactic, ATD [Mavongou & González-Martin]

— Marton, Runesson and Tsui's (2004) variation theory [O'Neil & Doerr]

Interestingly, the research frameworks are at different levels. The most common type of research framework (as outlined above in A, B and C) can be considered as what Eisenhart (1991) refers to as conceptual frameworks. They are skeletal structures of justification, rather than structures of explanation based on a formal theory (which would be the case with a theoretical framework). The frameworks described in D are conceptual frameworks that are "general" in their roots, where algebra is the focal topic "imported" into the framework by the authors.

The frameworks in F are holistic theories that encompass a methodology of instructional design. The methodological principle of TDS is that a piece of mathematical knowledge is represented by an epistemological model – a situation – that involves problems that can be solved in an optimal manner, using the targeted knowledge. The general epistemological model provided by the ATD proposes a description of mathematical knowledge in terms of mathematical praxeologies whose main components are types of tasks, techniques, technologies, and theories. In this way, TDS and ATD provide tools for both designing and analysing mathematical activities. The concepts and models of these theories provide guidance for task design, so that the mathematical tasks – as research instruments – will be an integrated part of the whole research enterprise.

FUNCTIONAL THINKING

We were struck by the large number of papers at this conference that addressed the nature and role of functional thinking in the development of algebraic thinking and focused on students' and teachers' difficulties with functions and functional thinking (Cusi & Malara; Eroglu & Tanisli; Glassmeyer & Edward; Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa; O'Neil & Doerr; Panaoura, Michael-Chrysanthou, & Philippou; Pilous & Janda; Postelnicu & Postelnicu; Prendergast & Treac; Szanyi; Xolocotzin, & Rojano).

The following considerations were prompted by the mathematical content of the tasks of the research studies presented in the papers and posters of the Algebraic Thinking group. Euler's, Dirichlet's, and Bourbaki's definitions of function were used, paralleling the historical development of the concept of function and matching the students' developmental stage. With few exceptions, like O'Neil and Doerr's paper on logarithmic functions, or Pilous and Janda's poster with examples of rational functions, linear and quadratic functions and equations were predominant. There seemed to be a consensus on the importance of students' and teachers' fluency within and between various perspectives of functions, and connecting between multiple representations of functions. Some papers and posters presented tasks specific to Early Algebra approaches such as pattern-based or quantitative reasoning approaches (Mavoungou & González-Martín; Strømskag; Twohill; Ugalde & Zazueta). Some tasks reflected our traditional algebra curriculum, influenced by the historical quest for solving equations, by focusing on the equation approach to algebra (Block).

In several papers, the contexts of tasks and the focus of research departed somewhat from the functional thinking, like in papers that dealt with factors that may influence algebraic thinking. Among those factors were other ways of thinking (Chimoni & Pitta-Pantazi; Norqvist), the interaction of the task with the teacher's actions, type of learner, and the learning environment and its affordances (Henz, Oldenburg, & Schöllhorn; Janssen & Radford; Pittalis; Pitta-Pantazi & Christou; Simensen, Fuglestad, & Vos). Theses shifts in the focus of research, away from the nature of algebraic thinking and thought, prompted a discussion about the new borderlines of the research on algebraic thinking.

BORDERLINES: LOOKING FORWARD TO CERME10

In the working group, we noticed various borderlines that define but also limit the scope of the algebraic thinking TWG. It is an interesting strategic question how the group, particular as a mature group, should react to these borderlines.

As in past conferences, the TWG has concentrated much on the core of algebra and algebraic thinking that includes approaches to algebra, early algebra, functional thinking, algebraic reasoning, development issues related to all that, misconceptions, epistemic actions in algebra, learner generated examples and teachers' goals. In recent conferences (but not 2015), we have discussed papers on the history and philosophy of algebra and the use of technology to promote algebraic thinking. Yet, although revolutionary new insights have become rare, but still the situation of algebra in schools is not satisfactory. What hinders progress is, among other things, the existence of subtle differences in understanding notions (e.g. what is a functional approach) that result in borders within the subject. Moreover, differences between teaching cultures in different countries (and within countries) are enormous and restrict generality of results very much.

Furthermore, we observe a number of emerging borderlines which have been studied to some extent, but in our view not yet sufficiently. For example, how much algebraic thinking is needed, how is it applied and how can algebraic 'defects' be hindered in various other aspects of education in schools? For example:

- Probability, e.g. what algebraic competence is needed to work with expressions like $P(|X - np| < k) > 95\,\%$.
- Geometry: $g \perp h \wedge h \perp l \Longrightarrow g \parallel l$
- Computer science: f vs. $f(x)$
- Physics: $U = RI$

The crucial question that should be cleared in further discussion is if these points are within or beyond the border of algebraic thinking.

Another related question is how do other school subjects act back on algebra? How do students cope with the fact that in other areas different rules apply? For example:

- A random variable is not a variable
- A physical quantity may not be a variable (but a function of time)
- Letters in geometry are labels or names of objects, not variables

Some of the paper in CERME9 hinted at emerging opportunities to cross boundaries to other subjects of research:

— Inclusion of lower achieving students raises questions usually studied in social sciences

— Gestures, bodily movement and brain research are traditionally more central in cognitive science

These new 'borderline' areas certainly open up the opportunity to understand algebraic thinking from new perspectives. The group acknowledges this but has a strong view about maintaining a strong mathematical focus to the algebraic thinking group.

Finally, a feature of the group has been continuity and we hope very much to be joined by many of the CERME9 TWG03 participants at CERME10.

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ENDNOTE

1. The reader is referred to the TWG papers for references to the research frameworks.

TWG03

Research papers

Flexible algebraic action on quadratic equations

Jan Block

Technische Universität Braunschweig, Braunschweig, Germany, jan.block@tu-braunschweig.de

This paper describes a study that explores the competencies of flexible algebraic action of German students in grade nine and ten when dealing with quadratic equations. A theoretical framework for the concept of flexibility in algebraic action in the context of quadratic equations is provided. Further on, data analysis and some important early results of the study are discussed. The study examines which features of quadratic equations the students perceive, what meanings they infer from these features and to what extent this is conducive to or obstructive for flexible algebraic action. Two types of meta-tasks were used in the study and analyzed with qualitative data analysis methods.

Keywords: Quadratic equations, flexibility, algebra.

INTRODUCTION

When professional mathematicians solve quadratic equations like (1) $x^2 + x - 6 = 0$, (2) $x^2 + 2x = 0$, (3) $(x - 3)(x + 5) = 0$ or (4) $(x - 3)(x + 5) = 7$ they will use different methods to find the solutions in effective and less error-prone ways. They do so, because they recognize different features of the equations and they are able to draw appropriate conclusions for solving the equations. E.g., equation (1) and (2) look very similar with a sum as the term on the left-hand side and zero on the right-hand side. The difference is the missing constant in (2), which indicates that this equation can be easily solved by factoring. In contrast, using the pq-formula[1] is a suitable procedure for solving equation (1). The equations (3) and (4) have the same structure on the left-hand side. The only, but important difference is the number on the righthand side. The solutions of (3) can be immediately determined without any calculation, whereas for solving (4), it is necessary to expand the brackets and use the pq-formula. Although all quadratic equations can be solved by using the pqformula, for (2) und (3) it is not an effective way because the calculations performed before or while using the pq-formula are not necessary. Furthermore these calculations are error-prone, especially for students with problems in algebraic conversions. When they use the pq-formula for (2) a common mistake is using a wrong value for q in the formula. The expanding of brackets in (3) is also a wellknown field of mistakes. Using different solution methods depending on the characteristics of the equation can be called flexibility in contrast to the use of only one standard routine like the pq-formula for each type of quadratic equations.

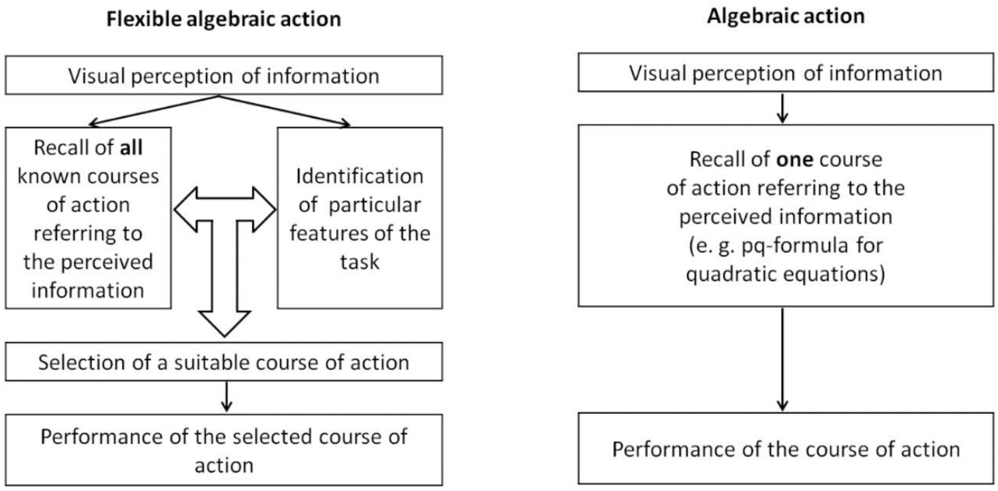

Figure 1: Comparison of flexible algebraic action and algebraic action

QUADRATIC EQUATIONS AND FLEXIBLE ALGEBRAIC ACTION

Flexible algebraic action is defined as the ability to choose an adequate processing method depending on the specific features of the task and the abilities of the individual. This definition refers to the concept of flexibility in mental calculation (e.g., Rathgeb-Schnierer, 2006; Threlfall, 2002) and a general discussion about what flexibility can mean (e.g., Star & Newton, 2009). Figure 1 shows the comparison of flexible algebraic action and algebraic action just with one standard routine.

For quadratic equations a didactical map can show the complexity of the situation students have to cope with, when they learn to solve this type of equation. A didactical map is a graphic depiction on an issue which contains important information for didactical considerations under special questioning. To clarify the difference between linear and quadratic equations in situations of learning and regarding the necessity of flexibility, a didactical map of linear equations (Figure 2) will be contrasted to a didactical map of quadratic equations (Figure 3). The construction refers to the "Didactical cut" which was first named by Filloy and Rojano (1984, 1989) and later on discussed by several researchers (e.g., Herscovics & Linchevski, 1994; Lima & Healy, 2010; Vlassis, 2002).

The map shows, that linear equations can be divided into two main groups: In the first, the unknown is only appearing once on one side of the equation. These equations can be solved by arithmetical procedures. It is not necessary to act on or with the unknown because they can be solved by using the reverse operations, e.g. $3x + 7 = 19$ can be solved by calculating $(19 - 7) \div 3$. To solve the second group of equations, in which the unknown occurs on both sides or more than once on one side, it is necessary to use algebraic procedures to act on or with the unknown. According to this classification, Lima and Healy (2010) call these two groups 'evaluation' and 'manipulation' equations which resembles the classification by Filloy and Rojano (1984, 1989) for linear equations, but which is farther-reaching also for classifying quadratic equations. Lima and Healy focus on the activities which are necessary to solve an equation and not on the question, where or how often the variable occurs. In contrast to the evaluation equations, for the manipulation equations it is necessary to manipulate algebraic symbols. The group of manipulation equations can be divided into two subgroups. For the first, where the variable is only appearing on one side but more than once, algebraic procedures are only necessary for the terms on one side. For the second, where the variable appears on both sides, equivalent transformations on both sides of the equation are necessary. To describe the important differences between the three groups regarding the different requirements of arithmetical and algebraic skills, the author suggests using the term "cognitive step" which is also suitable for the quadratic equations.

The aim of each algorithmic solving process for linear equations is to transform the equation into the form of the first group. The possible transformations are indicated by the arrows on the map. One facet of flexibility on acting with linear equations should be e.g. the ability to recognize that $3x + 4 = 3x + 5$ has no solution without starting algebraic procedures on this equation. The map shows that the field of linear equations has got a manageable number of cases. Nevertheless, there are also flexible and intelligent strategies to solve linear equations by simplifying the given equation without strictly using algebraic

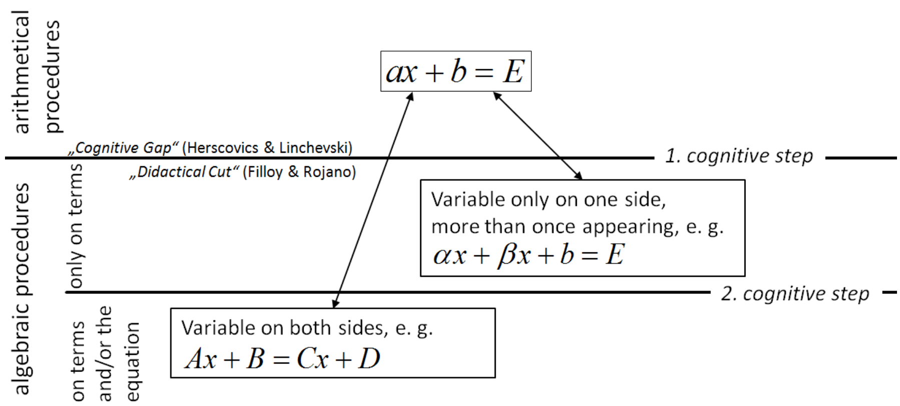

Figure 2: Didactical map of linear equations

algorithms (e.g., Star & Rittle-Johnson, 2008). For example for solving the equation $5(x + 2) = 20$ it is not necessary to expand the brackets, when recognizing that the term in brackets has to be 4 and then solving the equation $x + 2 = 4$ instead of the given equation. But this type of flexibility is depending only on special numbers. The idea behind this is to get an equation which is solvable with arithmetic procedures but the way to achieve this type depends on the numbers and the structure and it is not strictly algorithmically performing.

In contrast to this fairly simple model, the didactical map of quadratic equations (Figure 3) shows the wide variety of types of quadratic equations under the view of different effective solution methods. The study is only focussing on basic types of quadratic equations and not on non-standard examples like $\sin^2(x) + 2\sin(x) + 1 = 0$ or $x^4 - 6x^2 + 9 = 0$, for which identifying and interpreting of features as a basis for acting flexible is also very important. There are two main groups of equations: The first is solvable with quasi-arithmetical procedures such as inverse operations, extracting radicals or using the fact, that a product equals zero if one of the factors equals zero. The last case is indicated as a special case by the vertical spotted line, because a special knowledge is needed and no arithmetic operations are necessary. To solve the second group, algebraic procedures are necessary. This group can be divided into two subgroups: The first is characterized by the fact that the algebraic procedures are explicitly done when factoring the equation with the missing constant term. After this, the solutions are obvious using the knowledge about cases when a product equals zero. Using the pq-formula for all other types, the algebraic procedure of solving is not completely visible, so it is called implicit. This classification of the two main groups is according to the terms 'evaluation' and 'manipulation' equations Lima (2007) used. In contrast to the linear equations, the cognitive steps do not depend on the fact where and how often the variable appears.

The dashed arrows indicate that some equations can be interpreted as special cases of other types of equations. The types of equations differ in the types of the terms appearing. The structure of the terms is indicated by the form of the frames. A product is indicated by a frame shaped like an ellipse and a sum by a hexagon. The rectangle indicates the special case, where the unknown appears linear in a product but the term is a sum. To indicate the different suitable methods of solving, the frames have different kinds of lines. The dashed lines indicate the *pq*-formula, the dot-dash-lines extracting radicals as an appropriate method and the dot-dot-dash-lines stand for the possibility to get the solutions by factoring the term.

It is obvious that the types of quasi-arithmetical and algebraic solvable equations correspond in a complex way. For choosing an effective solution method it is not sufficient to look just at the type or structure of the term on one side of the equation. It is also necessary to look at the structure of the equation as a whole and the appearing numbers at special places in the equation (e.g. zero on one side of the equation). The main difference between the linear and the quadratic equations in the process of flexible solving is, that for quadratic equations it is not the aim to transform all types just into one, which can be solved by a stand-

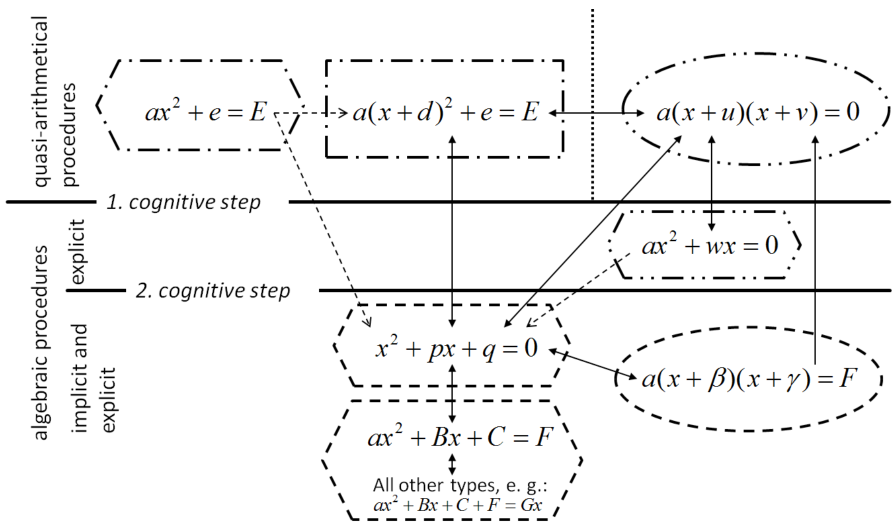

Figure 3: Didactical map of quadratic equations

ard method, as it is with the linear equations where it is the aim to produce a type of equation which can be solved by arithmetical procedures. Flexibility in solving quadratic equations means choosing different algorithmic solving methods, depending on special features of the equations. To master this, it is required to consider the relationships shown in the didactical map and to know, which features of an equation are important to indicate a suitable more or less algorithmic solving method.

METHOD OF THE STUDY

The three main questions in this study are: 1. Which features of quadratic equations do students perceive? 2. What meanings do they infer from these features? 3. To what extent is this conducive to or obstructive for flexible algebraic action?

To answer these questions two types of studies were made: 1. Laboratory study in a one-to-one situation (researcher and participant) with eleven students from grade nine from four different classes from two different German high-schools (Gymnasium). 2. Classroom study with 26 students from grade nine and 20 students from grade 10 from yet another Gymnasium. In the classroom study the tasks were integrated in a lesson by a teacher who was exactly instructed how to moderate the lesson. In the German high-school curriculum normally quadratic equations are a topic at the end of grade eight, so that all participants had taken part in lessons about quadratic equations and regarding this the groups are comparable. From another point of view potential varieties of the two groups in the classroom study can be recognized during the process of data-analysis. The frame-data from the participants in the laboratory study, marks for mathematics and results of the DEMAT 9-Test (Schmidt, Ennemoser, & Krajewski, 2013) reveal that this is a mixed group regarding the level of general mathematical skills. For the classroom study frame-data couldn't be collected. The teacher reported that both classes show no abnormality regarding the level of general mathematical skills.

In the laboratory study, the participants had to process three tasks. In the first task, a quadratic equation was given and the students were asked to create new equations by varying the given one. In the second task, they had to solve five quadratic equations of different types to check which strategies the students use to solve the equations.

The third task, which is at the centre of the study, is a meta-task like the first one. 20 quadratic equations were given on cards of carton and the participants had to assort them. These equations represent the different categories shown in the didactical map of quadratic equations, e.g. the equations discussed in the introduction are part of the selection. The number 20 was chosen based on the time needed to get an overview of the equations and the capacity of the visual field. There were no rules given and the participants had all freedom to assort the equations as they like. It was remarked that there was not only one possibility to assort them. While working on the tasks, the participants were video recorded and asked to think aloud. The students were asked to explain the meanings of the features of the groups.

The participants of the classroom study had to process two tasks. The variation-task was left out because the laboratory study showed that the most interesting information of this task was given by the thinking aloud of the participants which was not recordable in the classroom study. Deviating from the laboratory study, the students had to work in pairs on the sorting-task to encourage multiple solutions for the assorting. This seemed to be necessary because there was no researcher beneath the participants during process on the task like in the laboratory study to initiate more than one solution. The participants had to write down their assorting with an explanation of the meanings of the features on a special documentation sheet which referred to a tool used in sorting-tasks in a study with teachers by Zaslavsky and Leikin (2004). Selected pairs of students presented their arguments for assorting and their reasoning in the class which was recorded by video.

The transcriptions of the videos and the documentation sheets for the sorting-task were analyzed with qualitative data analysis methods, like an open coding, with the aim to develop categories (cf. Corbin & Strauss, 2008) of reasoning for assorting and the meanings of the recognized features of the equations. The sorting-task as an analyzingmeta-task is particularly suitable to examine the questions of the study. To process on this task it is insufficient to know a routine to handle quadratic equations. It is necessary to have an explicit look on the features of the equations and to detect the syntactic or semantic differences which are preconditions for flexible algebraic action. By reasoning for assorting, the meanings of the features can be explained. Evaluating the identified categories of assorting and meanings can show how far the mental structures of the students are conducive to or obstructive for flexible action.

DATA ANALYSIS: SELECTED RESULTS OF THE SORTING-TASK

Analyses of the data in the classroom studies reveal that there were six main categories for assorting the equations, which were also found in the laboratory study. These categories and some main sub-categories are shown in Figure 4.

The meanings inferred from the features for assorting the equations were first and mainly evaluated by analyzing the interviews from the laboratory study because the videos contain much more information than the documentation sheets from the classroom study. The meanings can be divided into helpful and conducive to or obstructive for flexible algebraic action. There are also features and meanings which are obscure for flexible action but they show insight into the students' mental concept concerning the dealing with equations. In the following selected examples, features and reasons for assorting are discussed.

One dominating reason for the assorting in the category "Term" was the appearing of brackets in the equations. A lot of participants argued, that terms with brackets had to be expanded to simplify them. This result is in accordance with a result Lima and Tall (2006) found in a study, where the quadratic equation $(x - 3) \cdot (x - 2) = 0$ was (not successfully) solved by the most participants by expanding the brackets. Similar difficulties with equations in this structure were also noticed by Vaiyavutjamai, Ellerton and Clements (2005). The brackets operate as a signal to expand the term regardless whether it is necessary or useful or not. In other studies (e.g., Dreyfus & Hoch, 2004; Wenger, 1987), other signals like radicals or fractions, which evoke routines regardless of the context or the questions that should be answered, were identified. Focussing on such signals could be an indication, that the students do not plan their approach (cf. Wenger, 1987) which is a necessary step in flexible action. Expanding the brackets can prevent an efficient solving process and can lead to mistakes during expansion or the subsequent use of the pq-formula. One reason for this habit can be the way transforming terms is taught. When transforming is called "simplifying" and most of the tasks require expanding products then it is obvious that for most students, brackets have to be expanded in any case. This hypothesis is supported by the explanation that terms with brackets are more complicated, which was remarked by a lot of participants of the laboratory study.

The remarks about brackets often occur together with remarks about solving methods. The participants describe that it is more difficult to isolate the variable when it occurs in brackets. This argument mirrors the strategy for linear equations, i.e. isolating the variable on one side of the equation, which was used in the solvingtask for 26.8% of the equations (from all samples) but successful only for 45.1% of these equations. The correct solutions with this strategy were all produced for the equation $(x - 8)^2 = 0$ by using the inverse operations or arguing with the semantics of

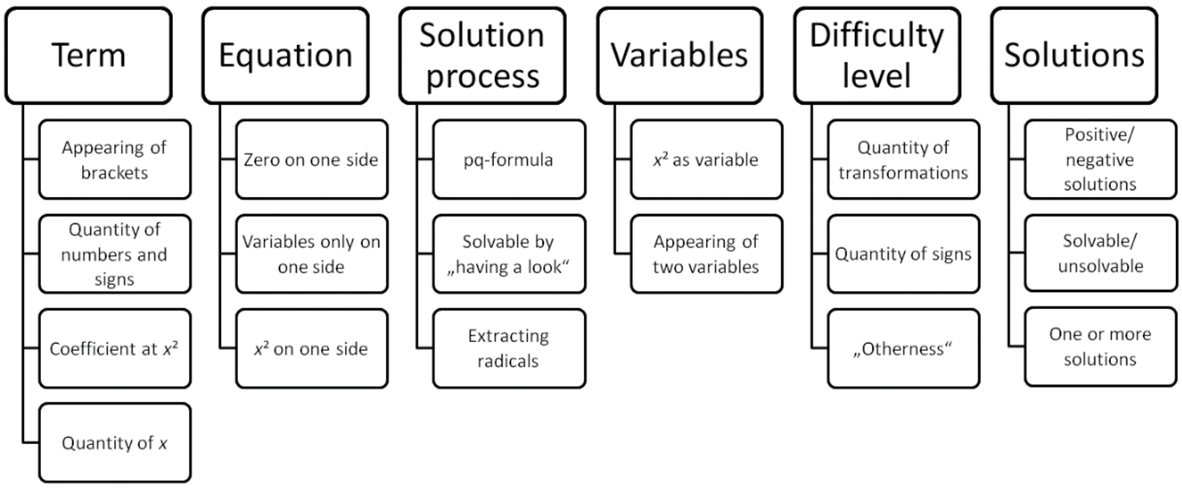

Figure 4: Categories and main sub-categories of assorting quadratic equations

this equation. The other strategies used for isolating the variable, misapplied on equations for which the pq-formula or factorizing is a suitable solving method, were division by the variable x or eliminating the exponent of x^2 by a division by 2, by extracting radicals just from this monomial or by summarizing monomials with different exponents. These faulty strategies of isolating the variable are all related to the idea of solving the equation by extracting the linear variable on one side of the equation to find the solution then on the other side. This is the idea of solving linear equations which works for quadratic equations only in special cases. Obviously, a generalization of this strategy leads to the effect that it is used in situations where it is unsuitable. The argumentation of the participants in the sortingtask points to a lower level of understanding of the concept of equations in general and especially of different types of equations like linear and quadratic equations. This hypothesis is promoted by results of studies about the understanding of equations (e.g., Lima & Tall, 2008). The isolation-strategy, which evokes the expanding of brackets, is opposed to strategies for solving quadratic equations where brackets are produced when factoring a term or where the brackets indicate a simple way to find the solutions because they show that the term is a product.

A more general problem is the mentioned difficulty to identify the equations as quadratic when brackets are appearing. This topic responds to the aspect of symbol sense in algebra in the context of understanding the meaning of variables and parameters in equations (cf. Postelnicu & Postelnicu, 2015) and to the ability to anticipate the effect of transforming terms.

Another feature that appeared in the category "Equation" for nearly every participant as a category was the presence of zero on one side of the equation. In contrast to the reasons of the importance for the occurrence of brackets, no blocking points for flexible action were found in the argumentation for this feature. However, only one pair of students in the classroom study explained, that the occurrence of zero on one side of the equation is the necessary precondition to use the pq-formula or to find the solutions when the term on the other side is a product. A lot of participants explained that zero on one side is necessary to use the pq-formula. This explanation is not wrong, but it was referred to all equations with zero on one side, regardless of the type and structure of the term on the other side of the equation, e.g. $(x-3)(x+5)=0$ or $4x^2-10=0$ which can be easily solved by the fact that a product should be zero or by extracting a radical. The students focussed only on one single feature not following the need to analyse potential sub-features of this group of equations. If this feature works as a signal to use the pq-formula, it can be obstructive for flexible algebraic action. This is as much more remarkable when looking on the results of the solving-task. From all samples 34.3% of the equations were solved by using the pq-formula, 70.3% of these correct. This indicates, that the pq-formula as a standard-method is not executed as well as you can expect for a standard algorithm. These results are compatible to a study by Lima and Tall (2006) where most participants solved quadratic equations with trial-and-error or with the pq-formula, but mostly unsuccessfully. Similar to this focussing on only one single feature, for the feature x^2 on one side of the equation some participants argued that if x^2 is on one side, extracting radicals is a suitable solving method without regarding what is on the other side of the equation, e.g. $x^2=x$ or $x^2=-16x-64$. Following the faulty meaning of this feature and the revealing solving method can produce individual faulty strategies to handle the other side and wrong solutions.

Another explanation for assorting by zero on one side was that there are special rules to be regarded when a zero appears. It is true that the special rules (like division by zero is not possible) are valid for handling equations. This argumentation, at a first glance, seems not to be connected to flexibility therefore it is neither conducive to nor obstructive for flexible action. But if this is the only and dominating importance of this feature in the awareness of the students, it can overlay meanings which are important for flexible action and in this way it can be obstructive.

CONCLUSION

The first data analyses of the sorting-task show some important findings regarding the competencies of flexible algebraic action in the context of quadratic equations. A lot of explanations in the sorting-task which can be obstructive for flexible action were identified and only a few participants show that their understanding of quadratic equations and solving methods is based on a concept of flexibility. The qualitative analyses used in this study are appropriate to identify the reasons for the established deficits. This knowledge can be used to develop improvements for teaching. The results show that some problems the students have with quadratic equations are founded in the teaching of previous topics (like transforming terms and the dominance of expanding brackets). Other problems like focussing just on one feature (e.g. zero on one side or x^2 on one side) should be addressed in the lessons by using suitable types of tasks which focus not on finding solutions of equations but on classifying different types of equations. It would seem that the meta-tasks, used in this study, have the potential to be a useful tool for the design of mathematic lessons which aim to enable the learners to act flexibly.

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ENDNOTE

1. The pq-formula for solving quadratic equations:

$$x^2 + px + q = 0 \Leftrightarrow x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$

Connections between algebraic thinking and reasoning processes

Maria Chimoni and Demetra Pitta-Pantazi

University of Cyprus, Department of Education, Nicosia, Cyprus, chimoni.maria@ucy.ac.cy

The aim of the present study is to investigate the relationship of algebraic thinking with different types of reasoning processes. Using regression analyses techniques to analyze data of 348 students between the ages of 10 to 13 years old, this study examined the associations between algebraic thinking and achievement in two tests, the Naglieri Non-Verbal Ability Test and a deductive reasoning test. The data provide support to the hypothesis that a corpus of reasoning processes, such as reasoning by analogy, serial reasoning, and deductive reasoning, significantly predict students' algebraic thinking.

Keywords: Algebraic thinking, cognitive mechanisms, reasoning processes.

INTRODUCTION

In recent years, researchers, policy makers and curriculum designers have recommended that algebraic thinking should become central to all students' mathematical experiences across K-12 grades (e.g., NCTM, 2000; RAND Mathematics Study Panel, 2003). The realization of this need stems primarily from the fact that algebra and algebraic thinking are closely linked to the development, establishment and communication of knowledge in all areas of mathematics, including arithmetic, geometry and, statistics (NCTM, 2000). Secondly, students' abrupt and isolated introduction to algebra in the middle school has led them to experience difficulties in understanding core algebraic concepts (Cai & Knuth, 2005).Thirdly, it has been argued that the mere focus of elementary mathematics on arithmetic and computational fluency deprives the conceptual development of mathematical ideas (Blanton & Kaput, 2005). Fourthly, the call for reconceptualising the nature of school algebra across all grades is underlined by the belief that algebraic thinking is within the conceptual reach of all students. This belief is supported by several research findings which offer evidences that as early as the elementary grades students are able to develop algebraic thinking in supportive classroom environments (e.g., Radford, 2008). Moreover, available research provide insights into appropriate pedagogical factors, such as curriculum materials, technological tools, and instructional strategies that facilitate this development (e.g., Blanton & Kaput, 2005).

Despite the considerable advances in the field, still realizing and achieving the goal for developing algebraic thinking as early as the elementary grades is challenging. The NCTM's research agenda (Arbaugh et al., 2010) highlighted that a main topic of focus is the identification of mathematical concepts and reasoning processes which facilitate the learning of algebra. English (2010) also stressed that one of the main priorities in the field of mathematics education is to define the key mathematical understandings, skills, and reasoning processes that students need, in order to succeed in mathematics. In this context, the present study aims to unfold the relationship between algebraic thinking and specific reasoning processes. This analysis might provide useful insights onto the skills which enable younger students to think algebraically. Students of the 4th, 5th, 6th and 7th grades were selected, in order to illuminate the ways by which abilities involved in reasoning processes might facilitate or restrict algebraic thinking within this age range.

THEORETICAL FRAMEWORK

The notion of algebraic thinking

Several researchers made efforts to analyze the nature and content of algebraic thinking, focusing on what individuals do and the way in which their abilities for generalizing and using symbols develop. One of the most influential developments of the past decades in respect to conceptualizing the notion of algebra as a multidimensional activity is Kaput's theoretical model. Kaput (2008) specified that there are two core aspects of algebraic thinking: (i) making generalizations and expressing those generalizations in increasingly, conventional symbol systems, and (ii) reasoning with symbolic forms, including the syntactically guided manipulations of those symbolic forms. In the case of the first aspect, generalizations are produced, justified and expressed in various ways. The second aspect refers to the association of meanings to symbols and to the treatment of symbols independently of their meaning. Kaput (2008) asserted that these two aspects of algebraic thinking denote reasoning processes that are considered to flow through varying degrees throughout three strands of algebraic activity: (i) generalized arithmetic, (ii) functional thinking, and (iii) the application of modeling languages for describing generalizations.

This conceptualization breaks down the wide field of algebraic thinking into major components of mathematical activity. Furthermore, Kaput's (2008) ideas articulate ways in which algebraic activities might be applied both in early and secondary school algebra contexts.

Algebraic thinking and reasoning processes

The view of algebraic thinking reported above focuses on the establishment of generalizations, taken to mean the detection and expression of structure and a growing understanding of symbolization. This approach raises the question: "which are the cognitive mechanisms that regulate this process?". English and Sharry (1996) show that analogical reasoning constitutes an essential mechanism when students resolve algebraic tasks. Specifically, they describe analogical reasoning as the mental source for extracting commonalities between relations and constructing mental representations for expressing generalizations. For example, the action of noticing differences and commonalities among different expressions of equations is considered as cognitive in nature and includes the formulation of a generalized concept that does not completely coincide with any of its particular cases.

Likewise, Radford (2008) developed a definition of the process of generalizing a pattern which unfolds the involvement of various forms of reasoning:

> Generalizing a pattern algebraically rests on the capability of grasping a commonality noticed on some particulars (say $p_1, p_2, p_3,..., p_k$); extending or generalizing this commonality to all subsequent terms ($p_{k+1}, p_{k+2}, p_{k+3}, ...$), and being able to use the commonality to provide a direct expression of any term of the sequence. (p. 84)

As the quotation suggests, this process first involves the identification of differences and similarities between the parts of the sequence – described as analogical reasoning by English and Sharry (1996). Then the commonality founded is generalized through predicting a plausible generalization. This stage is considered by Rivera and Becker (2007) as abductive in nature since it is abductive reasoning that boosts conjecturing and adopting a hypothesis that is considered testable. Finally, the tested commonality becomes the basis for inducing the generalized concept of the sequence. Here, the role of inductive reasoning is considered as pivotal (Ellis, 2007).

The role of processes of induction and deduction has also been highlighted by recent literature. Ayalon and Even (2013) emphasized the role of inductive reasoning when students investigate algebraic expressions. Martinez and Pedemonte (2014) have shown that a prerequisite for linking inductive argumentation in arithmetic and deductive proof in algebra is the co-existence of arithmetic and algebra for supporting the arguments developed within an argumentation.

METHODOLOGY

Research Question

The purpose of the present study is to investigate the way specific reasoning processes influence achievement in tasks that examine their algebraic thinking. Specifically, the present study addresses the following question: *Is there a relation between specific reasoning processes and individuals' algebraic thinking abilities?*

Participants

The participants were 348 students that were selected by convenience from four different schools. The students were divided to four age groups: 55 were students of Grade 4 (10 years old), 89 were students of Grade 5 (11 years old), 101 were students of Grade 6 (12 years old) and, 120 were students of Grade 7 (13 years old). Taking into consideration the fact that the data collection instruments would be the same for all of the participants of the study, no younger or older groups of students were selected. On the one hand, third grade students would not be able to manipulate the tasks, probably due to developmental reasons and absence of experience. On the other, eighth grade students were considered as more skillful in solving algebraic tasks due to their intensive involvement in algebra courses.

The tests

The participants were tested with three tests. Forty minutes were allowed to complete each of the three tests.

Algebraic thinking test

The test consisted of 25 tasks that were adapted from previous research studies related to the notions of algebra and algebraic thinking or algebraic proof (e.g., Blanton & Kaput, 2005; Mason et al., 2005) and past studies that evaluated students' mathematical achievement in international or national level (e.g., TIMSS, 2011; NAEP, 2011; MCAS, 2012). These were accordingly categorized into four groups:

(a) The use of arithmetic as a domain for expressing and formalizing generalizations (generalized arithmetic). These tasks involved solving equations and inequalities. The participants had to treat equations as objects that expressed quantitative relationships, without any reference to the meaning of the symbols.

(b) Generalizing numerical patterns to describe functional relationships (functional thinking). These tasks required finding the n^{th} term in patterns and functional relationships and expressing them in a verbal, symbolic or any other form.

(c) Modeling as a domain for expressing and formalizing generalizations: These tasks required the expression and formalization of generalizations by analysing information that are presented verbally, symbolically or in a table.

(d) Algebraic proof: These tasks reflected different activities and associated abilities of algebraic proof. For example, one of these tasks required the use of a generalization that was previously established (what is the sum of two odd numbers) for building a new generalization (what is the sum of three odd numbers).

The first three groups reflected the three strands of algebra as these were described by Kaput's (2008) theoretical framework. The fourth group was added to the test addressing the strand of algebraic proof. The examination of the construct validity of the items in the test to measure the factors of algebraic thinking was assessed through Confirmatory Factor Analysis using the MPLUS statistical package. The results indicated that the data fit the model well (CFI=0.95, x2=103.345 df=131, x2/df=1.19, RMSEA=0.03), verifying the structure of the proposed model. Table 1 presents examples of tasks for each of the four categories.

The Naglieri Non-Verbal Ability Test (NNAT)

The NNAT measures cognitive ability independently of linguistic and cultural background (Naglieri, 1997). There are seven different levels of the test corresponding to different age-groups of students. The test is a matrix reasoning type of exam that contains patterns formed by shapes that are organized into designs. All the tasks are multiple choice and students are asked to choose the answer that best completes the pattern.

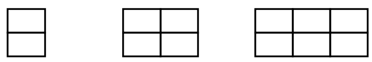
Generalized Arithmetic	The sum ***245676 + 535731*** is odd or even number? Explain your answer.
Functional thinking	Bill is arranging squares in the following way. How many squares there will be in the 16th figure? Figure 1 Figure 2 Figure 3 
Modeling as a domain of expressing and formalizing generalizations	Joanna will take computers lesson twice a week. Which is the best offer? **OFFER A:** €8 for each lesson **OFFER B:** €50 for the first 5 lessons of the month and then €4 for every additional lesson
Algebraic Proof	What is the sum of three odd numbers?

Table 1: Examples of tasks in the algebraic thinking test

The NNAT was selected among other tests that are used extensively for assessing students' cognitive ability due to the fact that the NNAT includes different categories of questions which reflect different types of reasoning skills. Specifically, it contains four different groups of questions: pattern completion, reasoning by analogy, serial reasoning and spatial visualization.

Based on available mathematics education literature, the reasoning processes that seem to be related to algebraic thinking and at the same time are measured by the NNAT are:

(a) Reasoning by analogy: In this category of items, students have to recognize commonalities between several geometric shapes and determine which answer is correct by focusing on how the objects change as one moves across the rows and down the columns in the design. Correspondingly, English and Sharry (1996) have described as analogical reasoning the process where students map similarities between algebraic expressions.

(b) Serial reasoning: The items in serial reasoning matrices are constructed using a series of shapes that change across the row horizontally and the columns vertically throughout the design. As the design moves down the matrix it also moves one position to the right, creating a series of designs that changes over the matrix. Students have to identify where the sequence finishes and starts again from a different starting point. The strategy that students follow in this kind of items shares common features with inductive reasoning. According to mathematics education literature, inductive reasoning is pivotal when students explore pattern sequences (e.g., Rivera & Becker, 2007). In these tasks students have to make generalizations, based on recognizing that a series of numbers or figures constitute a sequence that follows a specific rule.

The NNAT test' reliability was tested with norms based on a sample of more than 100,000 students (Naglieri, 1997). In this study, the internal consistency of scores measured by Cronbach's alpha was satisfactory for the NNAT test (a=0.84).

Deductive Reasoning Test

A test on deductive reasoning was constructed, guided by existing theory and research on deductive reasoning. In particular, items in this test were adapted from a test that was used by Watters and English (1995). This test was considered as appropriate due to the fact that it was used and validated for measuring deductive reasoning among students that were approximately of the same age as the participants in the current study. In Watters and English's study, students' performance in the deductive reasoning test was related to their performance in scientific problem solving. In our case, students' performance in the deductive reasoning test will be related to their performance in algebraic thinking. The items in this test represented 10 syllogisms which requested the students to reason deductively. This process included the analysis of premises that describe formal truth relationships, without reference to their empirical or practical truth value and the extraction of a logical fact, result or consequence. The internal consistency of scores measured by Cronbach's alpha was satisfactory for this test (a=0.79).

Analysis

The quantitative analysis of the data was carried out using the SPSS statistical package. Pearson correlation analysis and Regression analyses were performed. This study assumes that reasoning processes (as these are indicated by available literature) might predict algebraic thinking abilities. To test this assumption, Regression analysis was selected since this kind of analysis informs on the way one or more independent variables predicts the variance of a dependent variable.

The assumptions of multilinear regression are met since the Tolerance and VIF values were for all of the independent variables close to 1 (.972, .863, .876 and 1.03, 1.16, 1.14). This fact indicates that the multicollinearity and singularity assumptions are met. Moreover, standardized predicted X standardized residuals plot showed that the residuals did not violate the homoscedasticity of residuals and linearity assumptions.

RESULTS

The question of the present study addressed the relationship between algebraic thinking abilities and specific reasoning processes. Therefore, a correlation analysis was conducted in order to find out whether algebraic thinking and abilities involved in reasoning processes are significantly correlated. According to Pearson indicator, there is a statistically significant correlation between the individuals' achievement in the algebraic thinking test and the NNAT test (R=0.510, p=0.000<0.05). Moreover, the results show that there is a statistically significant correlation between the achievement in the algebraic thinking test and the deductive reasoning test (R=0.278, p=0.000<0.05). These results support previous reports which denoted that successful engagement with algebraic tasks involves several types of reasoning processes.

The nature of the relationship between algebraic thinking and specific reasoning processes was further explained by conducting regression analyses. Specifically, the analysis examined the way in which the achievement in the NNAT test and the deductive reasoning test (control variables) predict the achievement in the algebraic thinking test (depended variable).

Table 2 presents the results of the regression analysis. The B is the regression coefficient and represents the change in the outcome resulting from a unit change in the predictor, whereas, the beta coefficient (β) is the standardized version of the B coefficient where all variables have been adjusted to standard score form (Field, 2005). As the R-square shows, a percentage of 54.2% of the variance can be explained by the independent variables NNAT and deductive reasoning. This result indicates that as achievement in the two tests increases, the total achievement in the algebraic thinking test also increases. NNAT categories and deductive reasoning are indicated as predictors of algebraic thinking abilities. In order to further examine this relationship, multiple regression analysis was conducted with criterion (depended variables) the total achievement in the algebraic thinking test and predictors (independent variables) the abilities in three reasoning processes: reasoning by analogy, serial reasoning, and deductive reasoning. The results of the multiple regressions are presented in Table 3. According to these, all of the three reasoning processes exert a significant influence on the prediction of individuals' achievement in algebraic thinking.

Algebraic thinking	B(SE)	B
NNAT categories	.391 (.040)	.473*
Deductive reasoning	.471 (.122)	.188*
R^2=.542 , *p=.000		

Table 2: Regression Analysis of the achievement in NNAT test and the deductive reasoning test with dependent variable the achievement in the algebraic thinking test

The data show that the factor with the greatest effect on the prediction of achievement in algebraic thinking tasks is reasoning by analogy (β=.308). Serial reasoning also seems to be a significant predictor of individuals' total achievement in the algebraic thinking test (β=.238). Serial reasoning addresses the recognition of sequences and finding changes in the sequence. The abilities involved in the serial reasoning tasks share common features with the abilities involved in activities with pattern. According to the model, deductive reasoning (β=.180) explains a respectable proportion of variance in individuals' total achievement in the algebraic thinking test. It is anticipated that the effect of deductive reasoning could be higher if deductive reasoning was measured through non-verbal methods, as in the NNAT. The deductive reasoning test was not a language-free test of ability. Students from different linguistic groups where tested through a test that involved logical premises written in Greek. In contrast, the NNAT is not dependent on verbal abilities.

Algebraic thinking	B(SE)	B
Reasoning by analogy	.883 (.161)	.308*
Serial reasoning	.962 (.229)	.238*
Deductive reasoning	.452 (.123)	.180*
R^2=.544 , *p=.000		

Table 3: Regression Analysis of the achievement in each of the three reasoning processes with dependent variable the achievement in algebraic thinking test

DISCUSSION

As English (2010) suggested, a priority for mathematics education research is the definition of fundamental skills and reasoning processes which enhance students' efforts for achieving understanding in mathematical learning. NCTM (Arbaugh et al., 2010) also emphasized the need for coherently defining mathematical concepts and reasoning processes that enable individuals to develop algebraic thinking. Motivated by this argument, the present study aimed at examining students' algebraic thinking abilities with different reasoning processes. The importance of this study also lies in the fact that aims to capture a more holistic view of the algebraic thinking concept, by using Kaput's theoretical model as a referent point.

The findings obtained from the quantitative data indicate that students' achievement in algebraic thinking tasks is influenced by reasoning by analogy, serial reasoning, and deductive reasoning. Reasoning by analogy appears to be the factor with the most significant effect on algebraic thinking abilities. This result lends support to the findings of previous studies which indicated a relationship between algebraic thinking and analogical reasoning. According to English and Sharry (1996), analogical reasoning provides the basis for algebraic abstraction in tasks where students have to identify similarities and differences between a group of algebraic equations. Therefore, it seems reasonable to argue that analogical reasoning constitutes a basic process for succeeding in tasks of identifying structure and relationships.

Our findings also suggest that serial reasoning has a significant role in algebraic thinking. This result might be attributed to the fact that serial reasoning shares common features with inductive reasoning. This ability has been reported by related literature as crucial for the engagement in activities of determining pattern rules, recognizing the part that is repeated, and finding not observable terms (e.g., Rivera & Becker, 2008). Deductive reasoning also seemed to account for some variance in the algebraic thinking test. One plausible explanation for this result might be the fact that deductive reasoning is associated to the notion of proof. According to Blanton and Kaput (2005) activities such as using generalizations to build other generalizations, generalizing mathematical processes, and testing conjectures, and justifying reflect categories of algebraic thinking that are interwoven with proof.

As Kaput (2008) recommended, several reasoning processes run through algebraic activities. The analysis of the data provides empirical validation of these ideas and sheds some light on the crucial issue of which might these processes be and what is their nature. Furthermore, these findings can be used for informing educators about the sources that act as means for mastering different forms of algebraic thinking. Future teaching interventions might support the development of different types of reasoning, in order to test advances in developing algebraic thinking. According to English (2011), the key to reach advanced forms of reasoning is the creation of cognitively demanding learning activities in appropriate contexts.

The present study seemed to provide some evidence regarding the associations between students' algebraic thinking and fundamental reasoning processes. Yet, one limitation of the study is the context in which it was conducted. The relationships found in the present study need to be further examined, in other educational systems in which algebraic thinking might be approached through the mathematics curriculum in a different way. In respect to the methodology, a limitation of the study seems to be the fact that analogical reasoning and inductive reasoning were examined with a non-verbal test and deductive reasoning was tested with a verbal test. Future research could examine these associations within tests that follow similar design and features. Also, students of lower primary grades or higher secondary grades could be added to the sample. Finally, future research could identify the associations of algebraic thinking with other core processes and mental operations, in order to approach a wider picture of the algebraic thinking concept.

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Which algebraic learning can a teacher promote when her teaching does not focus on interpretative processes?

Annalisa Cusi[1] and Nicolina A. Malara[2]

1 University of Torino, Torino, Italy, annalo@tin.it

2 University of Modena and Reggio Emilia, Modena, Italy, malara@unimore.it

In this paper, we will focus on the effects of teachers' lack of the algebraic knowledge and sensitiveness that are necessary to effectively control the plurality of levels of interpretation involved within activities aimed at making students aware of the meanings associated to their use of algebraic language. Our analysis of class discussions conducted by a group of middle-school teachers involved in university training courses enabled us to highlight profound gaps, between modeling and interpretation, that prevent them from becoming aware of what it is important to stress when facing this kind of activities with students.

Keywords: Interpretative processes, algebraic thinking, algebraic teacher knowledge, teacher education.

FOCUS ON INTERPRETATIVE ASPECTS IN THE TEACHING OF ALGEBRA: KEY ROLE OF THE TEACHER

Since the development of the first studies on the problems related to the learning and teaching of algebra, many researchers reject the previous widespread idea that students' difficulties are mainly related to the complexity of algebraic syntax (see, for instance, Ursini, 1990; Kieran, 1992). The focus has therefore shifted on students' control of the meanings associated to their use of algebraic language. Different studies stress the importance of stimulating them with the aim of fostering their aware use of symbols as tools to represent, communicate, generalize, solve problems, develop reasoning (Arcavi, 1994; Arzarello, Bazzini, & Chiappini, 2001; Kieran, 2004). Bell (1996) states, in particular, that if students are given the possibility to have experience of the use of algebraic symbolism as a tool to express regularities and represent relationships, they could be guided through what he calls the "essential algebraic cycle", characterised by three main typical algebraic activities: to represent, to manipulate and to interpret.

In this work we focus on the third component of the essential algebraic cycle: the interpretative processes that are typical of algebraic activities. When we use the term '*interpretative processes*' we refer both to Duval's (2006) idea of conversions between different representation registers (in the case of algebra, from symbolic to verbal register and vice versa, but also from graphic, to symbolic, to verbal) and to Arzarello, Bazzini and Chiappini (2001)'s model for teaching algebra as a *game of interpretations*. These last researchers highlight, in particular, how the activation of conceptual frames (defined as an "organized set of notions, which suggests how to reason, manipulate formulas, anticipate results while coping with a problem") and the changes from a frame to another and from a knowledge domain to another could represent fundamental steps in the activation of interpretative processes.

The problem of algebraic modeling and of the coordination between verbal and algebraic language have been a central focus of research since the eighties. A paradigmatic study is the one developed by Clement (1982), who documented erroneous approaches developed by science-oriented college students in facing simple kind of algebra word problems. As regard to this problem, Smith and Thompson (2007) suggest that students' difficulties with algebra result not only from algebra curricula that lack meaning and coherence, but also from elementary curricula that fail to develop students' abilities to reason about relationships. This observation has been shared by different research studies developed in the realm of Early Algebra. Blanton and Kaput (2005, 2011), for example, highlight the crucial role played, in elementary school mathematics, by the algebraic reasoning embedded in finding, describing, justifying and symbolizing mathematical relationships between quantities, advocating a functional approach to algebra in the elementary grades. This is in tune with Kieran (2004), who refers to the shift of focus from the calculation of numerical answers to the relationships between quantities as one of the fundamental adjustments that students should make in their transition from arithmetic to algebra. This vision is in tune with the approach that we have developed within the ArAl Project (see, for instance, Cusi, Malara, & Navarra, 2011), aimed at proposing a relational and linguistic approach to Early Algebra and also meant to constitute an integrated teacher education program.

A recent study by Magiera, van den Kieboom and Moyer (2013) has highlighted that pre-service middle school teachers demonstrated a rather limited ability to recognize the full potential of algebra-based tasks to elicit algebraic thinking in students. They suggest, therefore, that there is a need of making teachers understand the contexts in which the various features of algebraic thinking might arise in order to enable them to effectively engage students. In tune with these results, Strand and Mills (2014), in their survey of research literature on prospective elementary school teachers' knowledge of algebra, state that it is well documented that these teachers tend to struggle to effectively interpret and use algebraic symbols (even those that they have produced themselves), to interpret graphical representations and to make connections between representations.

Our experience as teacher educators enabled us to observe similar problems also in the Italian context. Although research results about the importance of guiding students through the whole essential algebraic cycle have been widespread, Italian teachers are frequently not used to focusing on interpretative aspects. As a result, formalism and application dominate. This requires us to shift the focus on the teacher knowledge of the interpretative aspects connected to algebraic thinking and on the effects of this knowledge on students' learning. A framework to identify the knowledge for teaching school algebra has been developed by McCrory, Floden, Ferrini-Mundy, Reckase and Senk (2012), who stress the role played by teachers' capability of selecting, applying, and translating among mathematical representations and of making connections between manipulatives and mathematical ideas explicitly. But this way of behaving in the class must be educated. In order to foster teachers' overcoming of these difficulties and their effective approach in guiding students in the transition from arithmetic to algebra, in our in-service training courses for middle-school teachers we propose specific laboratorial activities focused on this theme.

In this paper we will present the main results from the analysis of the first of these activities, which was aimed at making trainee-teachers (in the following TT) directly experience how (and if) they are able to foster students' activation of interpretative processes during whole class discussions about a task involving proportional relations. In particular, we will highlight how this analysis enabled us to document the effects, in the way TT conducted class processes, of their lack of the specific knowledge that could favour students' development of interpretative attitudes.

METHODOLOGY OF WORK WITH TRAINEE-TEACHERS

The laboratorial activity we present was the first one proposed to a group of 58 middle-school (grades 6–7–8) TT involved in a training course aimed at making them achieve a teaching qualification. This course, which lasted six months, was specifically devoted to temporary teachers that have been working in school for at least three years. In Italy, in fact, also people who do not have a teaching qualification could work as teachers in school. As regards, in particular, the middle-school context, mathematics is mainly taught by teachers who do not have a mathematics background (their degrees could be in biology, chemistry, natural science, geology...).

During the whole training period, the TT attended to different courses in Mathematics Education, carried out by the authors themselves and by another colleague involved in the ArAl project. Many lessons were devoted to the problem of the teaching of algebra, with a specific focus on the main new trends in Early Algebra and on the role played by the teacher in the class. The first laboratorial activity in which the TT were involved was focused on a problem that we are going to analyse in detail in the following section. The

TT were asked to propose the problem to their students and to carry out its resolution during a whole class discussion. This is the text of the problem:

"A florist sends to a flower grower an email, asking to send him plants of sage and rosemary. However he does not indicate the exact number of plants, but specifies that for every 4 plants of sage he wants 6 plants of rosemary. Let r represents the quantity of plants of rosemary and s the quantity of plants of sage. Represent: (a) the relation between these two quantities; (b) the number of plants of sage through the number of plants of rosemary; (c) the number of plants of rosemary through the number of plants of sage.

Draw, in the Cartesian plane $O(r, s)$, the graph of the relation that expresses the quantity of plants of sage through the quantity of plants of rosemary. Then draw, in the plane $O(s, r)$, the graph of the relation that expresses the quantity of plants of rosemary through the quantity of plants of sage.

The flower grower delivers 66 plants of sage and tells him that he will send later the plants of rosemary. How many plants of rosemary does he have to deliver?"

This problem can be located among the activities, usually proposed in grade 7, which are aimed at linking the discrete arithmetic of natural numbers to the arithmetic of rational and real numbers.

We have chosen this specific problem because it could be both faced: (1) Adopting an "arithmetical approach" – typical of the Italian school tradition – focused on the application of properties of proportions; (2) Referring to an idea of proportionality as a functional relation, idea that could open the way to the development of algebraic reasoning. The approach that we have suggested TT to adopt in their classes was the second. Our aim was, in fact, to make them become aware that classical problems that, according to the Italian school tradition, are usually faced through the application of "rules" could instead be solved through the study of relations and, therefore, through the activation of an interesting game of interpretations. This approach to the resolution of the problem involves, in fact, the intertwining of different interpretative levels (as we will highlight in the next section) and the activation of different representations of the proportional relation involved.

The objectives of the problem and the main processes to be activated during a class discussion on its resolution were therefore shared with TT with the aim of making them recognise the potential of the task to elicit algebraic reasoning in students.

RESEARCH AIMS AND RESEARCH METHODOLOGY

In this paper we are going to focus on our analysis of the transcripts of the class discussions conducted by TT. Our main aim is to highlight what kind of difficulties they faced when they implemented the problem in their classes, trying to follow our suggestion of adopting an 'algebraic approach' to its resolution (namely an approach focused on the algebraic representation of the relation involved and aimed at the activation of different interpretative processes).

In the following we will analyse the problem introduced in the previous paragraph. This a-priori analysis of the problem has two main aims:

(1) The first aim is to highlight the multifaceted interpretative processes that could be activated in its resolution. Since this analysis was shared with TT, this "hypothetical" resolution represents also the path that they were asked to follow when facing the problem during the whole class discussions with their classes.

(2) The second aim is to identify specific indicators for our analysis of the transcripts of the class discussions conducted by TT. These indicators refer to the games of interpretations and the meta-level reflections that TT should have tried to develop in their interaction with students.

Analysis of the problem and identification of the indicators for the analysis of the way TT guided their classes in its resolution

As we stated above, if we focus on an "algebraic approach" to the resolution of the problem, it could be characterised by a deep twine between aspects related to the use of different representation registers and aspects related to the activation of interpretative processes at different levels.

To carry out the task in the class the teacher should initially guide students in the analysis of the text of the problem and in the identification of the key verbal relation that has to be translated into a mathematical sentence. This requires considering proportions between couples of numbers that exemplify the relation itself and their classical representation (i.e., 8:12=4:6, 12:18=4:6...), then translating them into fractional terms (i.e., 8/12=4/6, 12/18=4/6...).

A discussion aimed at a real sharing of the meaning of the letters *r* and *s*, introduced in the text of the problem, should precede the formalization of the relation between the number of plants of sage and the number of plants of rosemary. The possible formalizations of this relation should be compared and interpreted through their verbalisation. For instance: if the relation is represented through the proportion , it can be interpreted as "the quantity of plants of sage is to the quantity of plants of rosemary as 4 is to 6"; but it could also be represented through , that can be interpreted as "the fourth part of the quantity of plants of sage is equal to the sixth part of the quantity of plants of rosemary"; etc. This moment, devoted to the interpretation of the formalizations proposed by students, prevents from the uncritical acceptance of erroneous symbolic formalizations such as , where letters play the role of simple labels.

During this discussion, the teacher should also foster the representation and subsequent verbal interpretation of the previous proportions in terms of equivalence between fractions. This leads to the required representation of the number of plants of sage through the number of plants of rosemary and vice versa. For example, starting from the equality and the simplified , the class could interpret the latest as "the ratio between *s* and *r* is 2/3" and then as "*s* is 2/3 of *r*", that can quickly lead to , which is the required symbolic representation of the number of plants of sage through the number of plants of rosemary.

After the identification of the two representations – and – of the relations between *s* and *r*, another interpretative process should be activated: the particularization of the two formulas through the analysis of specific numerical cases, which could foster students' acquisition of the concept of variable. The couples of values determined through this particularization could be inserted into two tables that better enable to highlight the interrelations between one couple of values and the one that is obtained inverting the values.

After the construction of the two required graphs through the identification of the points corresponding to the couples of values collected in the two tables, the teacher should guide students in noticing that, although the alignment of the points could induce the idea of drawing a continuous line, not all the couples of numbers that are solutions of the two equations are also representatives of the phenomenon that has been modelled. Another aspect to be discussed in this phase is the pertinence of the solution $r = 99$, $s = 66$ according to the florist's request. These observations could be followed by a geometrical examination of the model, discussing the meaning of the two ratios 2/3 and 3/2 and introducing the neutral reference system (x,y) to compare the graphs of the two equations and empirically identify their geometrical relation.

This analysis enabled us to identify four main key-phases that characterise the resolution of the problem and four main groups of indicators, corresponding to the different phases, which are summarised in the Table 1.

As we stated above, this a-priori analysis was shared with TT before they proposed the problem to their classes. TT were asked to audio-record the discussions they conducted with their students and to reflect on these discussions referring to three different perspectives: the mathematical content at play, the role played by the teacher, the students' approaches to the problem and reactions to the teacher's interventions. After they performed this task, they sent us both the transcripts of their class discussions and the corresponding reflections.

We therefore analysed both the transcripts themselves and the reflections of the TT and the results of our analysis were discussed with them during a further lesson. In this paper, because of space limitations, we are focusing only on our analysis of the transcripts of the discussions of the TT, aimed at highlighting the difficulties they faced when they implemented the problem in their classes trying to follow our suggestions. The transcripts' analysis, whose main results will be presented in the next section, was performed referring to the four key-phases and the corresponding indicators that we have identified thanks to the a-priori analysis of the problem.

Key-phases in the resolution of the problem	The teacher guides the students in the:
Phase 1: *The transition from proportions to the formalization of the proportionality law*	Identification of the couples of numbers that satisfy the condition required in the problem; Generalization and corresponding construction, through the introduction of letters, of the proportions representing the relation between the two quantities; Verbalization of the constructed proportions; Interpretation of the proportions in fractional terms.
Phase 2: *The twine between syntactical and semantic aspects in the transition from the implicit forms of the relation to its two explicit forms*	Control of the syntactical transformations that lead to the two explicit formulas; Identification of the calculation process subtended to each formula; Verbalization of the meaning expressed by each formula; Conceptualization of the letters as variables and identification of their different roles (independent vs dependent variable); Discovery of the predicting power of each formula; Conceptualization of each formula as an equation and of the couples of numerical values that verify it as solutions of this equation; Discovery of the direct connection between the two formulas and exploration of the interrelation between their solutions.
Phase 3: *The representation of the two relations on the Cartesian plane.*	Coordination between symbolic and graphic registers to represent the graphs of the two formulas in the Cartesian planes (r,s) and (s,r); Re-nominalization of the variables to represent both the formulas in the Cartesian plane (x,y); Interpretation of the graphs in relation to the problem and discovery of their predicting power.
Phase 4: *Control of the adherence of the mathematical model to the specific problem-situation.*	Comparison between the domain and codomain of each relation and the domain and codomain of the corresponding restrictions that model the problem-situation; Reflection on the acceptability of certain couples of values (e.g., $s = 66$ and $r = 99$) as solutions of the problem.

Table 1: Indicators for the analysis of the transcripts of the discussions conducted by TT in their classes

DATA ANALYSIS: IDENTIFICATION OF SOME PROBLEMATICAL ASPECTS IN THE DISCUSSIONS CONDUCTED BY TT

Through our analysis of discussions of TT we have identified specific problematic aspects, connected to their incapability of guiding students within the games of interpretation and reflection necessary to make them develop those competencies and awareness that are objectives of an "algebraic approach" to this kind of activities.

As regards the first phase in the resolution of the problem (*the transition from proportions to the formalization of the proportionality law*), some TT were not able to support students in abandoning the quantitative/ numerical level. They adopted a procedural approach, making students only formulate the numerical proportions necessary to determine the couples of values that satisfy the relation and directly construct the graphs from the table of values. The modeling process was therefore inhibited.

In some classes students were able to formulate different proportional laws starting from the numerical examples they constructed, but the TT accepted all these laws without asking students to interpret them through their verbalization. Many TT declared, during the class discussions, that "all the constructed proportions are the same proportion", instead of highlighting that, although their equivalence, the relations expressed by these proportions are different. This erroneous conception, probably induced by some textbooks, could be overcome only through the activation of interpretative processes.

As regards the second phase in the resolution of the problem (*the twine between syntactical and semantic aspects in the transition from the implicit forms of the relation to its two explicit forms*), we have highlighted a really alarming phenomenon. Most of the TT did not support students in controlling and discussing the meaning of the two formulas derived from the proportional laws. In fact, very few of them made students analyse the formulas through their particularization to introduce the role played by the letters as variables;

almost none of them made students verbalize the formulas they had determined.

Another problematical aspect connected to this phase is that different TT often used terms such as "find *s/r*; search for *s/r*; calculate *s/r*" – instead of "make *s/r* explicit" or "express *s/r* through *r/s*" – to invite students to construct the two explicit formulas. Also the use of this procedural language inhibits the conceptualization of variable. Finally, most of the TT guided students in the determination of the two explicit formulas starting from the constructed proportions, but they did not highlight how they could be obtained from each other, nor foster a reflection on their relationships.

As regards the *third phase* in the resolution process (*the representation of the two relations on the Cartesian plane*), an aspect that we have highlighted is that some TT let that students construct the graphs determining the couples of values that are solutions of the two explicit equations through proportions, without stressing on the predicting power of the two formulas. In our view, this behaviour testifies that they conceive the modeling phase as something unnecessary and that they do not interpret the two formulas as the representation of all the possible pairs of numbers having the same ratio. Moreover, the fact that students do not refer to the formulas in constructing the graph is an evidence of a lack in their control of the meaning that the formulas convey.

Another widespread problem related to the capability of TT of coordinating the verbal, the algebraic and the graphic registers is that, although many TT examined with students the geometrical properties of the graphs, almost none of them made students interpret the graphs, highlighting their predicting power. Moreover they accepted continuous lines or semi-lines as graphs of the relations, without developing a reflection on the fact that only some points of these lines are representatives of the phenomenon that has been modelled.

This last aspect is also connected to the problematic ones that we have noticed in analysing the discussions of the TT according to the indicators concerning the *fourth phase* of the resolution process (*control of the adherence of the mathematical model to the specific problem-situation*). Few TT, in fact, made students reflect on the domain and codomain of the relations that model the problem-situation and on the acceptability of certain couples of values as solutions of the problem (during most of the discussions the value 66 as a possible number of plants of sage was uncritically accepted).

CONCLUSIONS

Through the analysis of the transcripts of discussions of TT, we highlighted that, although they had shared with us the aims of the activity and the *a priori* analysis of the problem, many of them were unable to activate the necessary interpretative processes and meta-level reflections and therefore to exploit the potential of this task to elicit algebraic reasoning in students.

The evident blocks in the games of interpretation that should have been activated during the class discussions testify corresponding profound gaps in the knowledge of the TT. These gaps, in fact, prevent them from becoming aware of what it is important to stress when facing this kind of activities with students. If teachers do not overcome these difficulties, they will not be effective models for their students, making them develop erroneous attitudes. This is a crucial problem in the didactic of algebra, because of the plurality of levels of interpretation that a teacher is required to effectively control.

These results suggest that research should better scrutinize teachers and students' difficulties in coordinating the different levels of interpretation often involved in the algebraic activity, in order to identify possible strategies to overcome them. At the same time, as it was also stressed in the discussion within the TWG03, a reflection should be developed on how teacher education programs must be engineered according to these results with the aim of enabling teachers to become effective activators of interpretative processes.

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Does bodily movement enhance mathematical problem solving? Behavioral and neurophysiological evidence

Diana Henz[1], *Reinhard Oldenburg*[2] *and Wolfgang I. Schöllhorn*[1]

1 University of Mainz, Mainz, Germany, henz@uni-mainz.de

2 University of Augsburg, Augsburg, Germany

The present pilot study investigates how bodily movement and mental motion interact during solving of different types of mathematical tasks. In an experimental study, subjects performed algebraic, geometric, and numerical reasoning tasks at three complexity levels under static and dynamic postural control affordances during sitting. Electroencephalographic brain activity was recorded at resting baseline and during all experimental conditions. Results support the hypothesis that bodily movement has a positive effect on cognitive processing of demanding cognitive tasks. Moreover, our results indicate that mental symbolic transformations are processed within a visuo-motor representation that is aligned with the mental representation of physical space.

Keywords: Mental motion, bodily movement, electroencephalography.

INTRODUCTION

Current research in cognitive science demonstrates close interrelations between the cognitive and the motor system. One interesting research question for teaching and learning mathematics is how bodily movement interacts with cognitive processing of mathematical problems, both in learning of concepts and in the application of learned procedures. Recent research shows that spatial abilities predict mathematical performance (Tosto et al., 2014), but it is unclear how this correlation depends on the specific type of the task. Moreover, this influence may result from prior learning that has been influenced by bodily experiences or by bodily movement during cognitive task processing. In the present study, we focus on mathematical thinking that might have been developed using metaphors that ultimately build on experiences of moving in space. We assess students' behavior in terms of mathematical performance, and measure the corresponding electrical brain activity during solving mathematical tasks under different manipulations of movement behavior.

Bodily movement and cognitive performance

Ongoing research has demonstrated effects of bodily movement on cognitive and psychological functioning (for a meta-analysis see Etnier, Nowell, Landers, & Sibley, 2006). Positive effects of moderate aerobic exercise have been shown for attentional, executive, and sensorimotor task performance (van der Niet, Hartmann, Smith, & Visscher, 2014). A large body of research shows close interrelations between the cognitive and the postural control system (e.g., Hwang et al., 2013). To test interactions between the cognitive and motor or postural control system, usually a dual task paradigm is administered in an experimental setting in which participants are asked to perform two tasks, one task taxing motor, and the other task cognitive demands, at the same time. Most studies show that performance is affected in either one or both tasks depending on task difficulty and type of tasks administered compared to conditions where only one task has to be performed. One explanation is that cognitive processing is impaired when two tasks are performed taxing the same informational processing subsystem and therefore occupy its capacity, completely. For instance, the famous working memory model proposed by Baddeley and Hitch (1974) postulates specific subsystems, one for verbal, and the other for visuo-spatial information processing where cognitive resources are allocated depending on their coding format. Several studies have shown that movement information and visuo-spatial information are processed within the same subsystem. This was shown for spatially directed movements (e.g., Logie & Della Sala, 2005), and gestures (Rumiati & Tessari, 2002).

Bodily movement and mathematical problem solving

The famous book "Where mathematics comes from" by Lakoff and Núñez (2000) puts out the strong thesis that "all abstract ideas are built by metaphors that are based on experience made possible by our body interacting with the physical world" (p. 496) and thus has triggered research in mathematics education that investigates the role of bodily movement in learning and explaining mathematics. A similarly strong conception is found in Wittmann , Flood and Black (2012):

> All human concepts, including mathematical concepts, are based in the perceptual motor system experiences we have while interacting with the world around us.

With this background they investigate gestures of mathematics students while solving differential equations and formulate and give support for the hypothesis that algebraic symbols are moved during computation in a way similar to physical objects. When students multiply an equation by a denominator d their behavior is consistent with the view that d moves along some path from one side of the equation to the other.

In judging the strong claim that all mathematical concepts are formed by metaphors that are based on bodily experiences it might be useful to introduce the notion of 'metaphorical distance'. Concepts that are directly linked to bodily experiences have a short metaphorical distance to bodily experiences, while others may have a long distance that spans a chain of metaphors. It seems likely that concepts with smaller metaphorical distance to bodily experience might be stronger related to other concepts with short distance to similar bodily experiences. We suppose, although we have no a-posteriori empirical evidence for this, that the metaphorical distance increases, e.g., in the following sequence: spatial rotation, moving symbols in equations, applying inverse operations, performing algorithms of numeric calculations. This line of argumentation suggests that arithmetical tasks are affected less by bodily movement than spatial geometry tasks or algebra tasks that afford moving symbols mentally.

Besides the metaphorical distance another dimension concerns the question whether bodily experience was essential in forming the concepts but is no longer relevant when these concepts are applied or if actual bodily experience (such as movements or gestures) becomes relevant during applications.

Further, we elaborate the above mentioned point that algebraic sub-expressions may be moved in a way similar to physical objects. It is interesting to note that researchers from different teaching traditions have noted that students tend to use the language of moving objects (e.g., Tall, 2013, p. 12). From the point of view of diagrammatic thinking signs do not refer to mathematical objects, but they are the mathematical objects. If this is combined together with the above theory of moving symbolic objects, the distinction between physical objects and symbolic mathematical objects is completely blurred.

From this discussion, we extract the following *hypothesis of the existence of an algebraic symbol space*: algebraic manipulations are carried out in a visuo-motor representation, either physically or mentally. Mental processes within this visuo-motor representation are supposed to be metaphorically close to experiences of bodily movements.

Moving in physical space may involve similar brain regions as moving in algebraic symbol space. Thus, it may be, that prior exercising one of them may have positive effects on the other, and it may be that simultaneous exercising may decrease (due to capacity limits) or increase (due to psychophysiological activation) performance. Neurophysiological evidence for a pre-motor implementation of metaphorical motion could be demonstrated by Fields (2013). In the present study, we investigate effects of unspecific bodily motion (i.e. motion that is not directly related to the structure of the mathematical task) on mathematical performance by variation of postural control affordances during sitting while participants are working on the mathematical tasks. The corresponding brain activation is assessed as a neural substrate for the postulated common visuo-motor representation that underlies the processing of algebra and geometry tasks. Increases in electroencephalographic (EEG) theta (4.0–7.5 Hz), and alpha (8.0–13.0 Hz) activity in central, and posterior (parietooccipital) brain areas should reflect visuo-motor information processing in algebra and geometry, whereas increases in EEG beta (13.0–30.0 Hz), and gamma (30.0–70.0 Hz) activity should indicate concentrative, attentionally mediated information processing.

In summary, the present paper advances two research questions: The first is whether bodily movement has an effect on mathematical performance measured in terms of behavioral data and corresponding brain activity. The second section tests the hypothesis that transformational algebraic manipulations are processed within a visuo-motor representation by investigating correlations between tasks that differ in the metaphorical distance to bodily movement and measure corresponding brain activation in areas related to processing of visuo-motor information in a frequency range (theta and alpha activity) that indicates working memory processes. We therefore expect increases in EEG theta and alpha activity in central, parietal, and occipital brain areas indicating visuo-motor working memory processes in algebra, similar to brain activation patterns in geometry.

MATERIALS AND METHODS

Participants

In the present study, $n = 15$ university students (mean age = 22.1 years, age range = 19–25 years) were tested. For a sub-sample of students ($n = 6$), electroencephalographic (EEG) activity was recorded for the entire duration of the test. All subjects were right handed, had normal or corrected to normal vision, and no history of neurological impairments. All participants gave informed consent and were naïve as to the purpose of the study. Due to small sample size and its selection, the present pilot study has to be considered as an exploratory study.

Study design and tasks

The laboratory study was carried out in a 3 (mathematical task: algebra, geometry, numerical calculation) x 3 (task difficulty: low, intermediate, high) x 2 (postural control: static, dynamic) within-subject design. We presented three different types of mathematical tasks within each type three levels of difficulty have been distinguished. The last factor was that of control of bodily movement. In the static condition, students were instructed not to move, while in the dynamic sitting condition participants sat on a stool that allows to move in all directions, and therefore promotes a dynamic control of bodily posture. For a sub-sample of students, EEG was recorded for the entire duration of the test.

The three types of mathematical tasks were arithmetics or numerical calculations (Num), algebra (Alg) and spatial geometry (Geo). All tasks were presented in a multiple choice format and processed mentally by the students, i.e. they were not allowed to write down any calculations or notes. The 3 x 3 x 2 item sets were presented in a randomized order. Within each cell students worked on the items for five minutes. To avoid exhaustion the whole test was split up into three sessions.

The arithmetical items were constructed ad hoc but informed by established theories of task difficulty in calculations (e.g. number of digits and carries). Example items are shown in Table 1.

Algebraic items tested the ability to determine the solution of linear equations in one unknown. On the basic level 1 the unknown was located on the left-hand side of the equation and could be determined by arithmetical calculation as read of directly from the equation. At level 2, difficulty is increased by larger number involved and by flipping right and left side of the equation. Both of these levels concern equations that are classified by Filloy (2008) as arithmetical equations, as they do not require to operate on the unknown itself, but only on numbers around it. At level 3, equations are of Filloy's algebraic type, i.e. they require true operating with unknown. All test items require to move the unknown across the equal sign as in $20x + 4x = 50 - x$. Example items for algebra are shown in Table 2.

Performance in spatial geometry tasks was measured by the "Bausteine-Test" (Birkel, Schein, & Schumann, 2002). The test is not designed to have three levels of difficulty but the solution probabilities of all items in a large sample are published in the test manual and we used this to group the items into three level sets.

A constraint of our setup is that all items are presented in a multiple choice form so that taking possible solutions as distractors might lead participants to find the right answer not by transformational algebra but by checking which of the numbers matches the equation. However, it is well known that students usually apply learned transformational methods even when inserting is more effective (e.g., Kouki &

Level	Task	Distractors
1	279 – 69 =	191, 190, 210, 220, 230
2	283 – 144 =	125, 139, 129, 149, 141
3	1980 / 44 =	47, 46, 45, 44, 43

Table 1: Examples for arithmetic items

Level	Task	Distractors
1	$8x + 7 = 47$	1, 2, 3, 4, 5
2	$79 = 11x + 2$	2, 7, 3, 5, 6
3	$x + 15 = x + 10 + x$	5, 10, 15, 20, 25

Table 2: Examples for algebra items

Chellougui, 2013, for a recent confirmation) so that we expect that most students choose transformational strategies. Violations of this assumption would tend to decrease sensitivity of our tests, so that results that we can show would remain valid.

Analysis of behavioral data

As measure of the students' performance in each experimental condition we take the number of correct answers achieved in the fixed time frame of five minutes. These numbers are denoted by a type signifier (Num, Alg, Geo), and the level, e.g., Geo1, Alg3, Num2. Lower levels are easier so that there are more correct answers, e.g., Alg1 > Alg2 > Alg3. When forming sum scores for the types of tasks, we calculated weighted sums to achieve approximately equal weight of all levels, e.g., Alg = Alg1 + 2 * Alg2 + 3 * Alg3. Classical test theory was applied to determine effects of the factors on achievement.

EEG recordings and data analysis

Electrical brain activity was recorded at resting baseline with eyes open before and after experimental tasks, and during each experimental condition. EEG was recorded (Micromed Brainquick, Micromed Systems Evolution) from 19 electrodes positioned according to the international 10–20 system. Vertical and horizontal electrooculogram was recorded from two electrodes. Impedances were kept below 4.0 kΩ. The EEG signal was digitized at 256 samples/s. After removal of oculomotor and electromyographic artifacts EEG data were subjected to Fast-Fourier-Transformations. Power density spectra were calculated for the theta (4.0–7.5 Hz), alpha (8.0–13.0 Hz), beta (13.0–30.0 Hz), and gamma band (30.0–70.0 Hz) for each subject. Data of power density spectra were averaged over all participants and were subjected to a 2 (postural control: static, dynamic) x 3 (mathematical task: algebra, geometry, numerical calculation) x 3 (level: low, intermediate, high) analysis of variance for repeated measurements with Bonferroni-corrected post-hoc t-tests.

RESULTS

Behavioral data

For all conditions we found that – according to what one would expect – the higher the level, the smaller was the number of correct answers. Cronbach's alpha for the scales formed by the three levels for each type were 0.79, 0.95, 0.74 for Num, Alg, Geo respectively. However, the Shapiro test for normality of this scales had p-values of 4.9e-05, 0.028 and 0.21 due to the skewness introduced by one exceptional well performing student. In the smaller sample without this student, normal distribution can be assumed. We checked that the results presented below vary only very little when run with the full or the smaller sample. We decided to report results including this student because n is already rather small.

The unspecific effect of bodily motion was positive in all cases, i.e. working in the dynamic condition yield-

Task type	Num	Alg	Geo
Cohen's d	0.41	0.39	0.12

Table 3: Cohen's d effect sizes

ed higher scores. The Cohen's d effect sizes for paired samples are shown in Table 3.

However, all of these differences fail slightly to be significant (significance level $p = 0.05$) as measured by the Wilcoxon test due to the rather small sample size.

Considering the first research question we performed a linear regression (R 2013, method lm). Results are shown in Table 4. In this case, it is important to note that with the smaller sample that satisfies normality assumption almost the same results appear. The most interesting β-weight of Geo in Alg3 is 0.064 in this case and significant as well. To complete data analysis, we calculated correlations of the relevant variables (see Table 5). The high correlation between Alg1 and Num is to be expected as reading these equations backwards yields a calculation task. Basically, to deal with Alg1 items one needs to determine by calculation a numerical unknown. This last aspect explains the rather high correlation between Alg1 and Alg3.

In all 3 x 3 combinations of levels and task types students performed better under the dynamic sitting condition. However, due to the small number of participants the effects could not be shown to be significant although effect sizes (Cohen's d for paired samples) indicated at least medium effects going up to $d = 0.49$ for Alg3.

EEG data

Results for EEG brain activity are depicted in Figure 1 (the nose of the head models is directed towards the top of the page). Significant main effects were obtained for posture control, $F(1, 5) = 7.02$, $p < .05$, task, $F(2, 10) = 5.46$, $p < .05$, and level, $F(2, 10) = 5.72$, $p < .05$, with a significant posture control x task x level interaction, $F(4, 20) = 4.13$, $p < .05$. EEG data show increased theta and alpha power in central and posterior regions during algebraic and geometric tasks at high complexity level in the dynamic postural control condition ($p < .05$) indicating an increase of activity in brain regions related to processing of visuo-motor information. Gamma power was increased over all brain regions during numerical reasoning at high complexity level under dynamic postural control ($p < .05$) which is a correlate for an internalized attentional processing mode that is not dependent on sensory modality of information.

DISCUSSION AND CONCLUSION

To our knowledge, this is the first study examining effects of postural control manipulation on mathematical performance. Behavioral and neurophysiological data show positive effects of dynamic postural control on mathematical reasoning performance. Different patterns of brain activation could be observed depending on postural control affordances, mathematical task, and task difficulty. Task-dependent EEG activation patterns indicate that mathematical reasoning is affected differently by manipulation of postural control affordances. We suppose that stimulation of the postural control system activates a visuo-motor representational mode during solving of algebraic and geometric tasks which is indicated by an increase in EEG theta and alpha activity in central and posterior brain areas, whereas attentional information processing is enhanced in numerical reasoning tasks indicated by increases in gamma activity in all brain regions. Therefore, our results confirm the hypothesis that algebraic and geometric tasks are processed in a different mode than arithmetic tasks.

	Alg1 ~ Geo + Num		Alg3 ~ Geo + Num	
	β-weight	p-value	β-weight	p-value
Geo	0.004	0.89	0.067	0.039 *
Num	0.246	0.3×10^{-9} ***	0.135	0.45×10^{-6} ***

Table 4: Linear models for Alg1 and Alg3

	Num	Alg1	Alg3	Geo
Num	1	0.90	0.76	0.47
Alg1		1	0.87	0.43
Alg3			1	0.58
Geo				1

Table 5: Correlations of task types

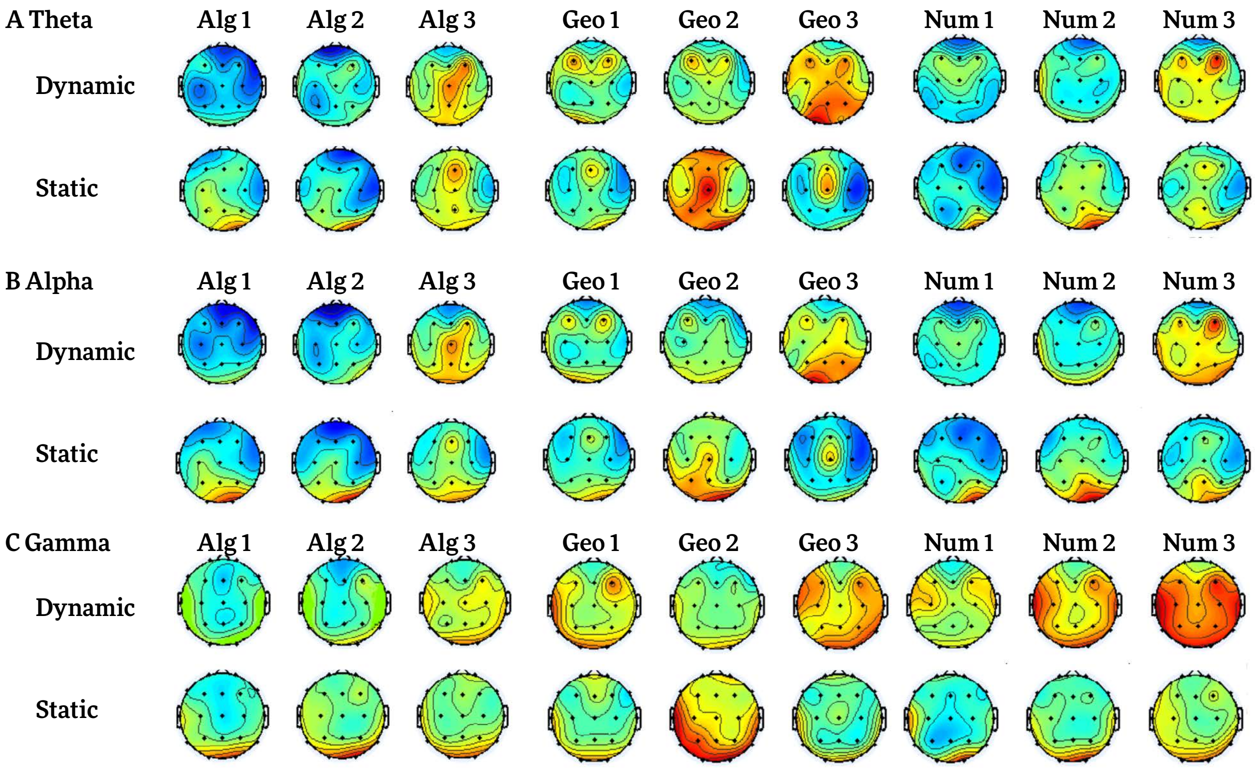

Figure 1: EEG activity during mathematical problem solving in static and dynamic postural conditions. The nose of the head models is directed towards the top of the page. EEG frequency bands: A Theta (4.0–7.5 Hz). B Alpha (8.0–13.0 Hz). C Gamma (30.0–70.0 Hz)

Further, our results support the hypothesis of an algebraic symbol space. The significant non-zero correlation found in behavioral data supports the hypothesis of the existence of an algebraic symbol space. Increased EEG theta and alpha in algebraic and geometric reasoning tasks indicate that both tasks are processed within a common working memory representation when the cognitive system is stimulated by bodily movement. However, further research is needed to clarify this hypothesis.

Finally, we present two theoretical explanations for the found patterns of results: (1) Increased mathematical performance is found in conditions of dynamic postural control affordances due to an increase of level of psychophysiological arousal, and therefore enhanced wakefulness. In a previous study (Maus, Henz, & Schöllhorn, 2013) increased attentional performance during dynamic sitting was demonstrated. EEG brain activation mirrored the found patterns of results as shown by an increase in beta activity in brain areas related to visual processing. EEG brain activation at high task difficulty level under postural control supports the hypothesis of increased psychophysiological arousal. Our results are in line with a study conducted by Vourkas and colleagues (2014) who observed differences in EEG brain activity in arithmetic tasks depending on task difficulty in children. (2) The occurrence of different EEG brain activation patterns in algebraic and geometric tasks in contrast to arithmetic tasks under dynamic postural control confirms the hypothesis that the presented tasks are processed within different cognitive subsystems. Increased central and posterior EEG alpha and theta activity in algebraic and geometric tasks at high task difficulty levels under dynamic postural control indicates that visuo-spatial working memory processes are stimulated by additional bodily movement, and therefore are responsible for the observed enhanced mathematical performance.

With the design of the current pilot study, we present a new methodological approach to investigate the underlying cognitive and neurophysiological processes in mathematical problem solving and their interaction with bodily movement. The found results contribute to a better understanding of cognitive processes that occur during solving of different types of mathematical problems, and encourage to design movement interventions which alleviate mathematical processing in learners. Our results have important implications for designing environments that promote bodily movements in learners of mathematics to increase their academic performance as could be shown in current research (e.g., van der Niet et al., 2014). Further, our results encourage to apply visuo-motor learning and teaching strategies in algebra, such as gestures (for a discussion see Janßen & Radford, 2015), due to the shown physiological preference of the brain for visuo-motor processing of algebra.

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Solving equations: Gestures, (un)allowable hints, and the unsayable matter

Thomas Janßen[1] and Luis Radford[2]

1 University of Bremen, Bremen, Germany, janssent@uni-bremen.de

2 Université Laurentienne, Sudbury, Canada, lradford@laurentian.ca

The goal of this paper is to contribute to the research on the introduction of solving linear equations. Subsumed in the "Comparing and Contrasting" category introduced in Prediger, Bikner-Ahsbahs and Arzarello's (2008) networking strategies, we contrast two episodes informed by two distinct theories and offer an insight into the teacher's role in introducing new knowledge in the classroom and the meaning-making narratives of hands-on didactic approaches to algebra. We examine the teachers' gestures and hints and what appears to be unsayable in the teacher-students' interaction.

Keywords: Linear equations, networking theories, teaching-learning, gestures.

INTRODUCTION

In the discussion of a particular classroom episode we found that our distinct research projects resorted to a very similar approach to introducing the process of solving linear equations. However, the equation's contexts, as well as the student-teacher interaction – in particular what the teachers were willing to say – were substantially different. The analysis of the episodes through the lenses of the theories that informed the design and implementation of the tasks as well as the interpretation of the data – the Gathering-Connecting-Structure-seeing (GCSt) model (Bikner-Ahsbahs & Halverscheid, 2014) and the Theory of Objectification (Radford, 2007) – led us to achieve a deeper understanding of the teaching-learning that is usually involved in a hands-on introduction to solving linear equations. In particular, resorting to a comparative analysis, which corresponds to what Prediger, Bikner-Ahsbahs, and Arzarello (2008) identify as "Contrasting and Comparing" theories, gave us new insights into the constraints and affordances with which teachers are endowed in their interaction with the students. Our comparative analysis also makes visible the limits of what is considered to be unsayable (i.e., that which would be *improper* to mention by the teacher) and how this unsayable shapes the contour of the space and kind of gestures teachers deploy in the interaction with the students. Last but not least, we reached a new awareness about the learning impact that the didactic context has as a potential horizon of narrative-based meaning production in the introductory steps in learning to solve linear equations.

RESEARCH BACKGROUNDS AND THE CONTEXT OF THIS RESEARCH

The authors of this paper both study the development of algebraic thinking. The first author is interested in algebraic structure sense (Hoch & Dreyfus, 2010) as a dynamic entity. The second author is interested in the social co-transformative sense-making processes through which the students gradually become critically acquainted with historically constituted cultural meanings and forms of reasoning and action.

In the course of a discussion about two classroom episodes dealing with students solving linear equations, one informed by research following the GCSt model and the other following the Theory of Objectification, an important distinction became apparent between the social-constructivist first theory that leaves lots of freedoms to the actors in the classroom on the one hand, and on the other hand the second theory that stresses the importance of the cultural nature and basis of the mathematical content – the idea that the algebra we teach in school is not a natural developmental outcome, but the outcome of a historical-cultural evolution. This important distinction turned out to set limits to what teachers can say in the classroom and thus defines what they cannot say – the unsayable. It also has an impact on the teacher's hints, gestures, and their meaning. This paper is an attempt to describe what we learned from comparing the same phenomenon – teaching-learning linear equations – and to formulate it in terms of the specificities of the target algebraic knowledge and the teacher's role in the introduction of a new concept.

THEORETICAL BACKGROUND

The Theory of Objectification (TO) considers knowledge as a historically developed cultural synthesis of actions and reflections (e.g., how to solve linear equations), which is concretized or realized in certain activities. In most cases, and especially in school, students do not enter this process on their own. Teachers and students engage in joint activity in order to make the cultural synthesis of actions and reflection noticeable to the students. In doing so, knowledge becomes an object of consciousness and thought. In the TO, the teacher's and students' joint activity or joint labour is referred to as "teaching-learning activity." The joint nature of teaching-learning does not mean that teachers and students play the same role. There is an asymmetrical division of labour that makes teaching-learning a tense process (Radford & Roth, 2011) filled with emotionality and fragility. Under this premise, the TO can be used as an instrument to thoroughly plan teaching-learning, however always with an awareness for the ever-developing relation between the actors.

The Gathering-Connecting-Structure-seeing (GCSt) model (Bikner-Ahsbahs & Halverscheid, 2014) aims at describing the epistemic actions that are carried out in so called interest-dense situations. In these situations, the class or parts of it collectively participate in the name-giving epistemic actions: *Gathering* refers to the collection of bits of mathematical meaning in the given situation, e.g. empirical values. These are then *connected* with limited scope. In the example that may be a table or a graph. Based on the connections, the students may come to *see structures*, an event which is understood as constituting the construction of new knowledge. In the example, the students may see linearity as a feature of the examined function. Therefore, what qualifies as knowledge is not so much defined a priori, but rather by the observed behaviour of the students. This also implies a rather open task design and requires the teacher to be open towards the learning routes taken by the students.

METHODOLOGICAL CONSIDERATIONS

In the TO, learning is mediated by teaching-learning activities underpinned by a range of semiotic resources, such as signs (e.g., spoken and written language, diagrams), embodied actions (gestures, tactility, perception), and rhythm. Furthermore, the relationship between the involved individuals is seen as an important factor. In the GCSt model, the epistemic actions form the centre of the researchers' attention. As discussed above, no presuppositions are made about the nature of the actions, thus, depending on the context, they may cover the same semiotic resources that are of interest in the TO. As a result, to investigate learning, both the TO and the GCSt model privilege video analyses.

The analysis of the classroom episodes below is an instance of the "Comparing and Contrasting" category introduced in Prediger, Bikner-Ahsbahs, and Arzarello's (2008) networking strategies, seeking to conceptualize the role of the individuals, social interaction, and the specificities of the target knowledge.

DATA OVERVIEW

The data to be discussed here was originally collected in two projects with different foci and scopes. In the German project a grade 8 class (13–14-year-old students) in an integrated school in Bremen was filmed for about seven months in those lessons where algebraic structures were the target topic. The episode discussed here is from the very first of these lessons. In the Canadian project a Grade 2 class (7–8-year-old students) in Sudbury was followed for 5 years when the students were learning algebra. The episode discussed here is from the second day of the algebra lessons.

In both cases the solving of linear equations was introduced in a non-mathematical context that emulated the mathematical rules of linear equations. In the Canadian project, the students were presented a task that went as follows (the equation was illustrated by envelopes and cards on the blackboard see Figure 1):

Sylvain and Chantal have some hockey cards. Chantal has 3 cards and Sylvain has 2 cards. Her mother puts some cards in three envelopes making sure to put the same number of hockey cards in each envelope. She gives 1 envelope to Chantal and 2 to Sylvain. Now, both

Figure 1: The envelope equation can be seen on the blackboard in the background

children have the same amount of hockey cards. How many hockey cards are in an envelope?

In the German project, the students had matchbox equations on their tables that were introduced as puzzles. The students were told that on both sides of the equal sign there was the same total number of matches, some of them hidden in matchboxes. All of the matchboxes would contain the same amount of matches.

Abstracting from the two scenarios in both cases there were a) representations of unknown quantities with each of them representing the same quantity, and b) two sets composed of known and unknown quantities of objects with the same total quantity of objects in each set. Based on these two rules, linear equations may be presented in many more imagined contexts.

As implied by the GCSt model, the teacher of the class in Germany had instructions to give as little help as possible to allow the students develop their own ways of finding the correct solution. In contrast, the teacher in Canada had talked about the method of isolation on the previous day: the method that consists of removing same quantities from both sides of an equation in order to isolate the unknown.

For the purpose of this analysis, both transcripts were translated into English. Where the transcripts indicated important actions by the students or the teacher, stills were created from the video to accompany the transcript.

ANALYSIS OF THE EPISODES

The episode from the Canadian study is framed as a classroom discussion, while the episode from the German study shows a discussion solely between the teacher and two students who work on the task together. In both cases the students first followed an arithmetic trial-and-error approach and had already found and tested the correct solution to the equation. However, each teacher still wanted the students to get to the target algebraic approach.

As mentioned above, the teachers' instructions were very different in the two cases. The teacher in Canada engaged the class in a discussion about various methods to solve equations and was comfortable asking questions, submitting ideas and a new vocabulary. Thus, in the discussion below, which happened after the students suggested a trial-and-error method (see Radford, 2014), she suggests to use what the class has come to term the previous day the "isolating strategy," that is, the strategy based on removing equal terms from both sides of the equation. As we shall see, the teacher follows the still not fully linguistically articulated actions of Cheb and Cheb's pointing gestures, by moving the concrete envelopes and cards on the blackboard, making thereby apparent to the class:

94 Teacher: I'll go with the isolating strategy, Ok? Cheb? (*see Figure 1*)

95 Cheb: Umm... you remove one of Sylvain's envelopes and one of (*the teacher has already put the hand on the envelope, yet she stops to wait for the next part of C's utterance, turning her head towards C)* Chantal's envelopes

96 T: Is it important to remove the same thing from each side of the equal [sign]? (*she makes a two-hand gesture around the equal sign moving the hands to the bottom of the blackboard, where envelopes and cards have been moved, to indicate that removing action is happening in both sides of the equality*)

97 C: Yes. And you can remove the other envelope... Oh non! One of Sylvain's cards and one card from Chantal's (*the teacher removes one card from Chantal's, see Figure 2, left image*).

98 T: Aw! Again, one envelope, we remove one envelope (*see Figure 2, centre image, where the teacher points to the removed envelopes*), one card , [and] one card (*see Figure 2, right image, where the teacher touches the two removed cards*) ...

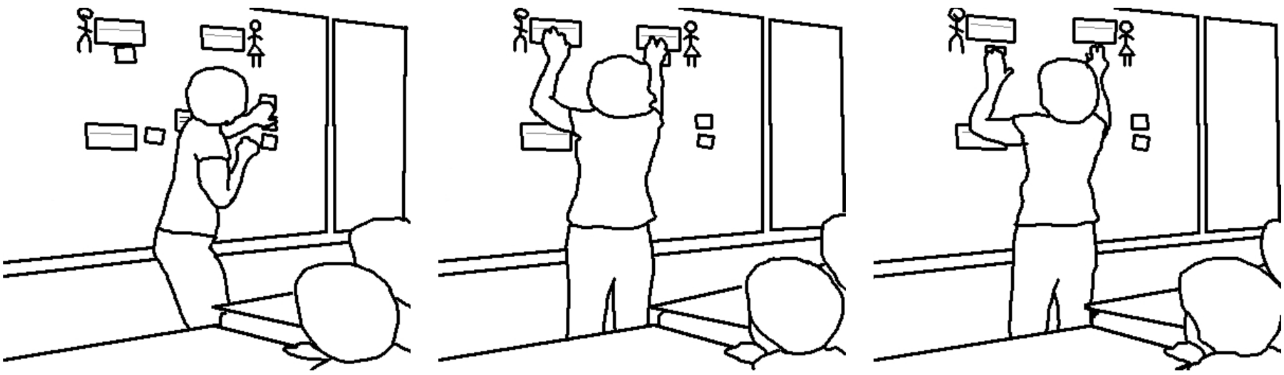

Figure 2: The teacher takes a card from the right side and moves it to the top of the blackboard (left image); she indicates the two envelopes (centre image) and then the two cards (right image) that have been removed

99 C: You remove one of Sylvain's cards and you remove one of Chantal's cards (*the teacher moves the cards towards the top of the blackboard*)
100 T: We remove another card of Chantal's cards. Then, that gives us...
101 C: The answer!

As we can see, the teacher moves the objects and performs the proposed actions. Where appropriate, she interrupts the flow of the discussion to emphasize for the whole class the algebraic conceptual meaning of the actions.

In contrast, the teacher in Germany has trouble doing so due to her professional self-concept (that was, at least in part, a result of the layout of the study she and her class were involved in). This leads to an interaction that is much longer than the one laid out above, and can thus only be presented in a condensed form. We particularly investigate how the teacher's struggle becomes apparent through her gesturing.

In the beginning of the interaction analysed here (line 10), the teacher gestures towards the two sides of the equation:

10. Teacher: (*briefly lays her hand on the right side, where Herbert is just finishing his counting – see Figure 3, left image*) Well what can one change here for example. so that it stays (*briefly holds both her hands above the two tables*) the same. (*multiply taps the tables with all of her fingers – see Figure 3, right image*) that must always stay the same that is very important.

A closer analysis of this first scene reveals that there are two gestures (see Figure 3). First, the teacher briefly and unspecifically lays her hand on the right side of the table and refers to this gesture by the word "here". Her idea is to focus on one side of the equation. However, at the same time the equation must stay an equation, which the teacher tries to stress by saying "so that it stays the same". The pronoun is concretised by the gestures shown in the right image: She means that the number of matches on both sides stays the same. This is too complex for the students, as neither the matches nor the sides of the equation are explicitly named as the relevant objects. Of course, this problem also applies to the aforementioned gesture. As a result, the students can make no sense of the teachers hint.

Figure 3: The two gestures in the first scene: Pointing at the right side of the equation (left) and highlighting the equality of the two sides (right)

After this first occurrence of gestures the teacher refrains from using any for almost two minutes. This is even more striking as she does in two instances use unspecific pronouns that would require clarification about what they refer to. The teacher's behaviour is probably due to the agreement that she should refrain from direct hints to the solution of the problem – an approach founded in the idea that the students should come to see structures on their own. However, during this time, she does talk about the two sides of the equation as the relevant objects. One could argue that from this explicit talk about the two sides it should indeed be clear for the students where they are expected to act, in terms of the GCSt model, the teacher helps with the gathering to make connecting and structure-seeing happen. But the structure is a new one, and it is hard to see without a break with the existing view.

In the scene that ends the comparatively long absence of gestures, the teacher uses gestures that point more directly at the two sides of the equation:

64 Sabine: (*moves her torso backwards, then to the front again, gestures with both arms*) The simplest would be if you simply tell us the solution. (*runs her fingers through her hair, Herbert laughs*)
65 T: Well it must (*points with her right hand first at the left and then at the right side, then turns it with the palm up*) always be the same but maybe- a bit (*makes a sudden upward movement with her right hand*) clearer. (*stands up straight again, crosses her arms*)
66 S: (*looks up at the teacher*) What does clearer mean? (*Sabine and the teacher look at each other*) (2sec)

However, she still uses the unspecific singular pronoun "it", again referring to the number of matches on the two sides. At the same time the word "it" stands for the whole situation that should become "a bit clearer". The students thus focus on what she means by "clearer". They demand a plan for action – this becomes visible already in the line before. In the last two utterances to be discussed here, the teacher reproduces the two gestures from the beginning, as can be seen in Figure 4. The teacher multiply taps on the right side (see Figure 4, left image). Here, for the first time, she adds a hint that taking away something might help, by claiming that "that are so many". However, the students still keep on aimlessly guessing what to do (88–91), meaning that they have no goal for their actions. Finally, the teacher additionally makes clear where the equality is to be preserved (see Figure 4, right image).

But only when the teacher gives her very concrete advice ("take away something"), Sabine is very quick at finding out what to take away. Here it proves that the groundwork laid before may not have been in vain, but can now be activated:

98 T: Okay what can one do on both sides so that it gets clearer you must always do the same. (*Herbert removes some dust from the table, the teacher looks at Sabine for some time, Sabine looks at the table, still holding the three boxes in her hand*) (3sec) (*gets up and walks away, whispering in Sabine's ear*) ‚take away something.

Figure 4: The two gestures re-enacted: Indicating the relevant side of the equation (left) and making clear that equality must be preserved (right). See Figure 3 for comparison

DISCUSSION

What are the students to learn and how can they learn it?

The isolation method is not the first method to which children resort when they are asked to solve an equation. Indeed, it is far from trivial to "isolate" the unknown to solve the equation. This is the method of analysis that the ancient Greeks devised. It is a deductive method, where relationships are deduced through a long chain of deductions, the last one being one in which you have the unknown equal to something. For students, it is at first much more reasonable to assume numbers and try and see if the assumption confirms the story (trial-and-error method), which is indeed what can be seen in both episodes.

In the examples presented here, the (linear) equation is supposed to emerge from an original context from where things and actions acquire an initial meaning (cards and envelopes in one case, matches and boxes in the other). This context is set in terms of a narrative that establishes an equality involving known and unknown numbers. Formally speaking, the two contexts explored here are similar. We can say that, in principle, the context offers the same potential in terms of algebraic meaning-making. Our analysis suggests, however, that the narrative (i.e., the linguistically connected account of events) is much more emphasized in the Canadian study. Cheb and the teacher talk much more in terms of cards and envelopes than the students and the teacher talk about matches and boxes in the German study. The potential significance of the context appears hence not to be equally exploited.

However, the exploitation of the significance of a meaningful context is not enough for the students to envision the algebraic isolation method. Indeed, to proceed to the simplification of the equation, the original narrative has to be disrupted by a (mathematical, in this case algebraic) sense that is already a real-life counter-sense. It is hardly natural to think about *removing* cards from the individuals in the story, while in fact the question is about the number of cards in an envelope. There is a shift from quantities as such to relations between quantities. The teacher and the students have to expand the narrative so that the removing actions and their results may acquire a new meaning. Hence there is a need for the teacher to interrupt the flow of actions and to make sure that the class finds a new mathematical meaning in what has been done to the equations, after the removal of same quantities. In the Canadian episode, the teacher shows a developed sense of the importance of this interruption and the special value of the algebraic solution so that she can refer to it at the appropriate moment. This developed sense is not natural. It was nurtured during the design of the classroom activity. The German episode ends with the teacher telling one of the students what to do on the two sides of the equation, which she has just identified as the place of action. The course of action appears improvised under the impression of the difficulties that arise rather than didactically planned.

Teacher intervention by gestures and its limitations

Until she decides to help the students in a more direct manner, the German teacher's attempts to guide her students consist mainly of gestures, as if the contextual actions required to simplify the equations had an ostensive meaning. However, gestures always work within the parameters of how teachers conceive of themselves in their teaching. They are directed to oneself and to others. It supposes that they work within the parameters of what we take a good teacher-students interaction to be, i.e., the meaning ascribed to interaction in the classroom. We see this tension in the manner in which the teacher in Germany refrains from telling the students. She tries to point to a solution solely by gestures.

However, to be understandable, to be meaningful as a hint to make connections (in the GCSt model) or as a guide into a cultural activity (in the TO), gestures need an explanation about what they refer to, and what it is that can be done. As has been pointed out, this is far from trivial, especially when we consider the framings of the problem: In both cases it is embedded in a narrative that makes the required actions meaningful in an abstract sense – an abstract sense that is opposed to the more quotidian sense where one may try to use trial-and-error methods.

The central hypothesis about the teacher's gestures in this episode is that they are too abstract. They require a deeper students' understanding than the one they have at this point. In particular, the teacher presumes that the students already see the same objects and relations between these objects as she does. It seems that the gestures are made from the standpoint of someone for whom the equation is already of a symbolic nature.

Maybe an understanding of the sides of the equation as the useful unit of analysis would suffice for the gestures to be fruitful, but the teacher does not even try to induce that explicitly. The teacher's gestures cannot find a kind of contextual narrative support to provide a rationale for the algebraic actions to find a meaning that may be accessible to the students.

CONCLUSIONS

The introduction of algebraic methods and ways of thinking is a crucial point in students' individual paths through mathematics – a point where many lose touch with the subject. The two episodes make clear that a teacher with an appropriate understanding of his or her role can help students substantially. Awareness for the novelty of algebraic methods is the essence of this understanding. It can help realize the decisive steps that the students have to take and to position other forms of help, such as gestures, in the teaching-learning situation. Without the consideration of context, otherwise helpful gestures are at risk to stay opaque to the students.

The result is surprising from the point of view where the learning should come from the students and is seen as an autonomous act of construction, as it is in the GCSt model. The episodes and their analysis presented here raises the question how one could even expect students to develop the complex deductive method of solving an equation without getting an introduction by an experienced person. In both episodes, the learning that happens in the end (or more precisely: that begins in the end, as the new knowledge will need to be consolidated) is based on an input from the teacher-students' interaction. To inform better teaching, it should be a goal of mathematics education researchers to better understand what this input is in different content fields, as we have tried here regarding the solving of linear equations.

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Algebraic reasoning levels in primary and secondary education

Juan D. Godino[1], Teresa Neto[2], Miguel R. Wilhelmi[3], Lilia Aké[4], Silvia Etchegaray[5] and Aitzol Lasa[3]

1 Universidad de Granada, Granada, Spain, jgodino@ugr.es
2 University of Aveiro, Research Centre „Didactics and Technology in Education of Trainers", CIDTFF, Aveiro, Portugal
3 Universidad Pública de Navarra, Pamplona, Spain
4 Universidad de Colima, Colima, México
5 Universidad Nacional de Río Cuarto, Río Cuarto, Argentina

In a previous paper, we proposed a characterization of algebraic reasoning in primary education, based on the onto-semiotic approach to mathematical knowledge and instruction, where we distinguish three levels of algebraization. In defining these levels we took into account the types of representations used, the generalization processes involved and the analytical calculation performed in mathematical activity. In this paper, we extend this previous model by including three more advanced levels of algebraic reasoning in order to analyze the mathematical activity carried out in secondary education. These new levels are based on the following considerations: 1) using and processing parameters to represent families of equations and functions; 2) the study of algebraic structures themselves, their definitions and properties.

Key words: Algebraic reasoning, primary education, secondary education, onto-semiotic approach, teachers' education.

INTRODUCTION

Recognizing the characteristic features of algebraic thinking is an issue that has attracted many mathematics education researchers, because it is necessary to promote such reasoning at different levels of elementary and secondary education (Kieran, 2007; Filloy, Rojano, & Puig, 2008; Kaput, 2008). Depending on how the school algebra is conceived, decisions are taken concerning whether to introduce such algebra since early levels, or to delay its teaching until secondary education as well as to change the corresponding instructional strategies. In fact, the "early algebra" research and development program (Carraher & Schliemann, 2007; Cai & Knuth, 2011) is supported on a conception of algebra that recognizes signs of algebraic thinking in mathematical activities of initial educational levels, as shown in NCTM (2000). While there has been progress in the characterization of school algebra, the interconnection between primary and secondary education algebra is not completely solved.

In previous publications (Aké, Godino, Gonzato, & Wilhelmi, 2013; Godino, Aké, Gonzato, & Wilhelmi, 2014) we proposed a model of algebraic thinking for primary education, with three different levels of algebraic thinking. Furthermore, we established criteria to delimit these algebraic levels from 0 (arithmetic nature of mathematical activity) to 3 (clear algebraic activity), with two intermediate levels of proto-algebraic activity. The criteria to define these levels were based on the type of mathematical objects and processes involved in mathematical activity, according to the onto-semiotic approach (OSA) to mathematical knowledge (Godino, Batanero, & Font, 2007; Godino, Font, Wilhelmi, & Lurduy, 2011)[1]. These algebraization levels are assigned to the operative and discursive practices performed by a mathematical subject that solves a mathematical task, rather than to the task itself, which can be solved in different ways, and may bring into play different algebraic activity.

1 The Onto-Semiotic Approach of mathematical knowledge and instruction (OSA) is a theoretical framework that adopts semiotic and anthropological assumptions about mathematics, and socio-constructivist and interactionist principles for the study of teaching and learning processes. Due to space limitation, it is not possible to include a synthesis of the main theoretical tools that compose this framework; the readers might consult Godino and colleagues (2007) and Godino and colleagues (2011).

In this paper, we extend that model of algebraization levels to secondary school, mathematical activity. This extension is also supported by the onto-semiotic distinctions considered in the OSA; particularly by the presence, use and processing of functions and equations parameters. The work is organized in four sections. In the following section we summarize the features of algebraic reasoning levels in elementary education; next, we define three new levels of algebraization, include some illustrative examples and connect the new levels to the presence of discontinuities in the onto-semiotic configurations involved in mathematical practices.

LEVELS OF ALGEBRAIC REASONING IN PRIMARY EDUCATION

In Table 1 we summarize the essential features of the three preliminary algebraization levels described by Godino, Aké, Gonzato and Wilhelmi (2014), completed by level 0 (absence of algebraic characteristics). An example is also included to help understanding the distinction among levels. In summary, the definition of levels is based on the following onto-semiotic distinctions:

- Presence of *intensive* algebraic objects (i.e., entities of general or indeterminate character).
- Transformations (operations) based on structural properties applied to these objects.
- Type of used language (natural, iconic, gestural, symbolic).

Obviously, these levels do not exhaust the algebraization processes of school mathematical activity. Instead, they describe the gradual enrichment of solving problems tools with an increasing degree of symbolization in other contexts of use. These processes, in the end of primary school and junior secondary school, may evolve to higher algebraization levels. The criteria used to distinguish the different algebraic levels have been gradually refined through its application to the analysis of responses from different samples of student teachers (Aké et al., 2013; Godino et al., 2014).

LEVELS OF ALGEBRAIC REASONING IN SECONDARY EDUCATION

In this section, we extend the model of algebraization levels to secondary and high school mathematical activity, in describing three additional algebraization levels for this educational stage.

The use and treatment of parameters is a criterion for defining higher levels of algebraization, as it is linked to the presence of equations and functions families, and, therefore it implies new "layers" or levels of generality (Radford, 2011). The intervention of parameters will be linked to the fourth and fifth algebraization levels, while the study of specific algebraic structures will mark a sixth algebraization level of mathematical activity.

Fourth algebraization level: using parameters

The use of parameter for expressing equations and function families is indicative of a higher level of algebraic reasoning, as compared to the third algebraization level considered by Aké and colleagues (2013), which is linked to operations with unknowns or variables. This "first encounter" with parameters and variable coefficients involves the discrimination of the domain and range of parametric functions, i.e. functions that assigns a specific function or equation to each value of the parameter. As suggested by Ely and Adams (2012, p. 22) "A significant conceptual shift must occur in order for students to be comfortable using placeholders in algebraic expressions rather than just numbers".

Example 1: The linear function

In the algebraic expression, $y = 2x$, the literal symbols x and y represent variables, symbols that can take any value from a previously established number set, usually R. The numerical values x and y co-vary in terms of each other, according to the rule laid down in the corresponding expression; in this case, y is twice the value assigned to x. The factor multiplying x can be generalized to any value in a certain domain; as we see in the expression $y = ax$. Here the letter a intervenes as a parameter: it can take different values within a certain domain, so that for each possible value a, we obtain a particular function. For example, for $a = 2$, we have $y = 2x$.

Consequently, a parameter is a literal symbol involved in an expression with other variables, such that for each particular value assigned to it, a function is obtained. We express such families of functions as $F = \{f(x) = ax/aR\}$, or more precisely, a family of functions that depend on the domain D of definition of the functions f: $F_D = \{f(x) = ax \mid aR; xD\}$.

<table>
<tr><td colspan="4">Task: Students either go by car or they walk to a certain school. There are 3 students walking for every 3 student going by car. If the school has 212 students, how many of them use each means of transportation?</td></tr>
<tr><td>LEVELS</td><td>OBJECTS</td><td>TRANSFORMATIONS</td><td>LANGUAGES</td></tr>
<tr><td rowspan="2">0</td><td>No intensive objects are involved.
In structural tasks unknown data can be used.</td><td>Operations are carried out with extensive objects.</td><td>Natural, numerical, iconic, gestural; symbols referring extensive objects or unknown data can take part.</td></tr>
<tr><td colspan="3">Example of resolution:
For every 3 students who walk, there is 1 going by car. Hence, in every group of 4 students (3 + 1) there is 1 going by car (a fourth of students). Thus, 50 out of 200 students go by car and 3 out of 12 students use the car. Therefore, 53 students use the car and three times that amount, that is, 159, walk to the school.</td></tr>
<tr><td rowspan="2">1</td><td>In structural tasks unknown data can be used.
In functional tasks intensive objects are recognized.</td><td>In structural tasks relations and properties of operations are applied.
In functional tasks calculation involve extensive objects.</td><td>Natural, numerical, iconic, gestural; symbols referring to intensive recognized can be used.</td></tr>
<tr><td colspan="3">Example of resolution:
For every 4 students there are 3 which walk. We write out the following proportion:
4 (children) ------> 3 walk
212 (children) ------> x walk
$\frac{4}{3} = \frac{212}{x}$; $x = 3 \times \frac{212}{4}$; $x = 159$
Once we obtain the number of children who walk to the school, the number of students going by car is easily obtained, 212 – 159 = 53.</td></tr>
<tr><td rowspan="2">2</td><td>Indeterminate or variables are involved.</td><td>In structural tasks equations are of the form $Ax \pm B = C$.
In functional tasks generality is recognized but operations with variables are not carried out to get canonical forms of expressions.</td><td>Symbolic – literal, used to refer the intensive recognized, although linked to the spatial, temporal and contextual information.</td></tr>
<tr><td colspan="3">Example of resolution:
$212 = x + 3x$
$212 = 4x$; $x = 212/4$; $x = 53$
53 children go by car and 212 – 53 = 159 walk.</td></tr>
<tr><td rowspan="2">3</td><td>Indeterminate or variables intervene.</td><td>In structural tasks equations are of the form $Ax \pm B = Cx \pm D$.
Operations with indeterminate or variables are carried out.</td><td>Symbolic – literal; symbols are used analytically, without referring to contextual information.</td></tr>
<tr><td colspan="3">Resolution example:
x = Children going by car
y = Children walking
$x + y = 212$ $x + 3x = 212$;
$y = 3x$ $4x = 212$; $x = 212/4 = 53$</td></tr>
</table>

Table 1: Characteristic features of elementary algebraic reasoning levels

The symbols x and y ($f(x)$) are variables indicative of a first level of generality; their definition domains and range are the numeric sets in which they are defined. The symbol a is also a variable; however a second level of generality is involved, since its definition domain could either be (D) as before or just another number set, and the range of values is the family of functions F_D.

Example 2: Quadratic equation
Parameters are used not only to express and operate with function families, but also with equation families (Ely & Adams, 2012). For example, $ax^2 + bx + c = 0$ ($a \neq 0$) is the general expression for the quadratic equations family. There is only one unknown, x. The letters a, b, c, usually considered as variable coefficients, take

specific values within a set of possible values (real numbers and a ≠ 0) to produce a particular equation.

Therefore, a parameter is a variable that is used with two or more other variables to specify a family of functions or equations. For families of equations the parameter is commonly named coefficient. In some way, the parameter plays the role of independent variable in a function whose domain is the set in which the parameter takes its values and whose rank is a set of functions. For each value assigned to the parameter a function image is obtained. Therefore, the expression $y = ax^2+bx+c$, is not a function but a family of functions, though it is usually referred to as "the quadratic function." It is an expression in which three parameters indicated by the letters *a, b, c* are involved. Giving a particular value to each of the parameters a specific quadratic function is obtained.

Fifth level of algebraization: treatment of parameters

We can assign a higher level of algebraization to mathematical activity displayed, when analytical (syntactic) calculations are carried out involving one or more parameters. Operations with parameter involve a higher semiotic complexity level, since the objects emerging from these systems of practices are built on algebraic objects of the previous level (equations or functions families).

Example 3: Obtaining the general formula for quadratic equations

To obtain the general formula for quadratic equations we perform symbolic manipulation and use successive equivalences. Assuming the director coefficient a is not 0 (a ≠ 0) – otherwise the equation would not be quadratic – we have:

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{b}{a} = 0 \Leftrightarrow x2+ \frac{b}{a}x = -\frac{c}{a} \Leftrightarrow$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \Leftrightarrow$$

$$(x + \frac{b}{2a})^2 = \frac{b^2-4ac}{4a^2} \Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2-4ac}{4a^2}} \Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a} \Leftrightarrow$$

$$x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Thus, in this case the solution is written in terms of the parameters linked by rational operations (addition, subtraction, multiplication, division) and square roots.

Example 4: Geometric progressions

We define the general term of a geometric progression (Figure 1) by discursive practices in which two parameters, a_1 (first term of the sequence) and *r* (progression ratio) are involved. The sequence is a function with domain *N* and range *R*; therefore the parameters a_1 and *r* define a family of functions (sequences), and consequently this discursive practice uses an algebraization level 4. The description and proof of the sum of the first *n* terms of a geometric progression (r ≠ 0) involves a computation with parameters, as shown in Figure 1; therefore it implies the algebraization level 5.

Sixth level of algebraization

The introduction of certain algebraic structures (such as vector spaces, or groups) and the study of functional algebra (addition, subtraction, division, multiplication, and composition) start at high school. These practices bring into play higher level algebraic objects and processes according to its onto-semiotic complexity than those considered at level five. It may be useful, therefore, to characterize a sixth algebraization level to focus our attention on the specific nature of the mathematical activity involved. High school

36.8 Finite Geometric Series

When we sum a known number of terms in a geometric sequence, we get a finite geometric series. We know that we can write out each term of a geometric sequence in the general form:

an = $a_1 \cdot r^{n-1}$

where

- n is the index of the sequence;
- a_n is the nth-term of the sequence;
- a_1 is the first term;
- r is the common ratio (the ratio of any term to the previous term).

By simply adding together the first n terms, we are actually writing out the series

Sn = $a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$

We may multiply the above equation by r on both sides, giving us

$rS_n = a_1r + a_1r^2 + a_1r^3 + \ldots + a_1r^{n-1} + a_1r^n$

....

Dividing by (r – 1) on both sides, we arrive at the general form of a geometric series:

$$\boxed{S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1\,(r^n - 1)}{r - 1}}$$

Figure 1: Finite geometric series (Free High School Science Texts, Mathematics Grades 10 – 12, p. 469, 2008)

books include texts and activities corresponding to this sixth algebraization level:

Example 5: Vector space
Figure 2 shows a general formulation for the vector space algebraic structure. In this first encounter with this algebraic structure a set of mathematical objects (vectors) are defined on which operations satisfying a set of specific properties are carried out. An initial "structural study" of vectors is required, since in this type of (axiomatic) presentation, the properties of the vector addition and multiplication by numbers have to be established.

Example 6: Composition of functions
In Figure 3 the notion of function is used in all its generality, in replacing a particular family of functions by any function. Operations are carried out over functions to produce new functions, whose properties will be studied in general. For example, properties such as "the composition of functions is not commutative" would arise. In fact, a set of functions (polynomials, for example) satisfying certain operations (addition, multiplication, etc.) is an "algebra".

Algebraization levels and strands of algebraic reasoning

In various studies Kaput has proposed a model of algebraic reasoning as a complex composite organized around five interrelated forms, or strands of reasoning listed below (Kaput & Blanton, 2001; Kaput, 2008):

1) Algebra as Generalizing and Formalizing Patterns & Constraints,

2) Algebra as Syntactically-Guided Manipulation of Formalisms.

3) Algebra as the Study of Structures and Systems Abstracted from Computations and Relations.

4) Algebra as the Study of Functions, Relations, and Joint Variation

5) Algebra as a Cluster of Modeling and Phenomena-Controlling Languages

The algebraization levels we propose are related to strands 1 and 2. Strand 1 is specified in our model by levels 1 and 2 of proto-algebraic reasoning, while strand 2 is associated with level 3, where algebra is already consolidated. Strands 3, 4 and 5 basically correspond to fields or areas of school algebra (generalized arithmetic, study of abstract structures, functions, modeling).

Our algebrization levels of primary and secondary school mathematical activity can be identified in each mathematical content strands, and involve a progressive epistemic and cognitive complexity degree due to the level of generality of mathematical objects, ostensive representations and syntactic calculation used. The presence and manipulation of parameters associated with levels 4 and 5 take place within the strands "Algebra as the study of functions", and "Algebra as a cluster of modeling of phenomena". Kaput's (2008) algebraic reasoning model is oriented mainly to characterize algebra as institutionalized mathematical content, while our model attempts to

Imagining a vector idea as an arrow help conceive the vector space: sets of vectors among which some operations satisfying certain properties are defined. But there are other mathematic entities with the same operations and properties. So, the definition of vector space is much broader and open than collections of "arrows". We have a set, V; among their elements (called vectors) two operations are defined:

SUM OF TWO ELEMENTS OF V: if $\vec{u}, \vec{v} \in V$, then $\vec{u}, \vec{v} \in V$

PRODUCT BY A REAL NUMBER: if $a \in R$ and $\vec{u} \in V$, then $a \cdot \vec{u} \in V$

If $(V, +, \cdot)$ satisfies the following properties then is a vector space on R.

	SUM OF VECTORS
ASSOCIATIVE	$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
COMMUTATIVE	$\vec{u}+\vec{v}=\vec{v}+\vec{u}$
NULL VECTOR	It is a vector called $\vec{0}$ such that if $\vec{v} \in V$ fulfils: $\vec{v}+0=\vec{v}$
OPPOSITE VECTOR	All v has its opposite $-\vec{v}$: $\vec{v}+(-\vec{v})=0$
MULTIPLYING A VECTOR BY A NUMBER	
ASSOCIATIVE	$(a \cdot b) \cdot \vec{v} = a \cdot (b \cdot \vec{v})$
DISTRIBUTIVE I	$(a+b) \cdot \vec{v} = a \cdot \vec{v} + b \cdot \vec{v}$
DISTRIBUTIVE II	$a \cdot (\vec{u}+\vec{v}) = a \cdot \vec{u} + a \cdot \vec{v}$
PRODUCT BY 1	If $\vec{v} \in V$ then $1 \cdot \vec{v} = \vec{v}$

Figure 2: Vector space (Colera & Oliveira, 2009, p. 62)

It is possible to combine two functions by adding, subtracting, multiplying or dividing two given functions.
There is another way to combine two functions to create a new function. It is called composition of two functions. It is a process through which we will substitute an entire function into another function.
First let's get acquainted with the notation that is used for composition of functions. When we want to find the composition of two functions we use the notation $(f \circ g)(x)$.
Another way to write this is $(f(g(x))$. This is probably the more practical notation although the first notation is what appears most often in books.

Figure 3: Composition of functions (AlgebraLAB. Project Manager. Mainland High School)

characterize the algebraic activity performed by the individuals solving mathematical tasks. Therefore both theoretical school algebra models are compatible and complementary.

ALGEBRAIZATION LEVELS AND ONTO-SEMIOTIC DISCONTINUITIES

Algebraization levels are basically generality levels, combining various registers of semiotic representation (RSR), theirs transformations and conversions (Duval, 1995). Under the OSA these levels can be characterized by the presence of different types of onto-semiotic configurations (Godino, Font, Wilhelmi, & Lurduy, 2011) which involve practices, objects and processes implying new levels of generality or syntactic calculus, supported by symbolic representations of the corresponding objects. Furthermore, they imply unitization, materialization and reification processes involved in generalization and representation (Godino et al., 2014).

Considering algebraization levels of mathematical activity can help raise awareness of gaps or discontinuities in didactical trajectories. These gaps involve the use of different registers of semiotic representation, their treatment and conversion, as well as the establishment of relations between conceptual, propositional, procedural and argumentative objects of higher generality. In other words, these gaps can be explained by analyzing how the numerical-iconic and analytical – algebraic onto-semiotic configurations involved are articulated, and not only by the treatment or conversion of RSR. Strømskag (2015) referring several studies emphasizes that it is not generalization tasks that are difficult for students but they are related to the way tasks are designed. We think that taking into account the levels of algebraization could help in selecting and design tasks that increase students' opportunities to learn algebra.

SYNTHESIS AND IMPLICATIONS FOR TEACHER EDUCATION

In this work we complemented the work by Ake and colleagues (2013) and Godino and colleagues (2014) on the identification of algebraization levels of mathematical activity in primary education, including three new levels that characterize secondary mathematics. As a summary we propose the following six levels of algebraic thinking in primary and secondary education (along with level 0, indicating absence of algebraization):

Level 0: Operations with particular objects using natural, numerical, iconic, gestural languages are carried out.

Level 1: Use of intensive objects (generic entities), the algebraic structure properties of N and the algebraic equality (equivalence).

Level 2: Use of symbolic – alphanumeric representations to refer the intensive recognized, although linked to the spatial, temporal and contextual information; solving equations of the form $Ax \pm B = C$.

Level 3: Symbols are used analytically, without referring to contextual information. Operations with indeterminate quantities or variables are carried out.

Level 4: Studying families of equations and functions using parameters and coefficients.

Level 5: Analytical (syntactic) calculations are carried out involving one or more parameters.

Level 6: Study of algebraic structures themselves, their definitions and structural properties.

These algebraic reasoning levels have implications for teacher training, both in primary and secondary education. In addition to develop curricular proposals (NCTM, 2000) including algebra from the earliest levels of education, the teacher need to act as the main agent of change in the introduction and development of algebraic reasoning in elementary classrooms, and its progression in secondary education. Reflecting on the recognition of algebraic thinking objects and processes can help identify the features of mathematical practices on which the teachers can intervene to gradually increase the algebraization levels of students' mathematical activity.

Consequently, recognizing the algebraization levels 4, 5 and 6 by secondary school teachers, along with its articulation with the previous levels, can help raise their awareness of the gaps or onto-semiotic discontinuities which may appear when carrying out tasks proposed to their students.

ACKNOWLEDGEMENT

The research reported in this article was carried out as part of the following projects: EDU2012-31869 and EDU2013-41141-P (Ministry of Economy and Competitiveness, Spain)

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Using variation theory to design tasks to support students' understanding of logarithms

AnnMarie H. O'Neil and Helen M. Doerr

Syracuse University, Syracuse, USA, ahoneil@syr.edu

In this paper, we discuss three implementations of a task in which students were asked to generate examples of logarithm expressions equal to a given value. We situate the design of the task in variation theory and in research on learner generated examples, which describe learning as developing students' ways of seeing, particularly in regards to the dimensions of variation and the range of permissible change. The analysis of the three implementations reveals students' understanding of logarithms, as well as what is possible to learn given the task-as-implemented, or the enacted object of learning. We claim that using variation theory in task design can support students in developing important capabilities for reasoning about logarithms in powerful ways.

Keywords: Logarithms, variation theory, learner generated examples.

While exponential functions have been emphasized as a key mathematical understanding in secondary school (Confrey & Smith, 1995), their inverses, logarithmic functions, have received very little attention in the research literature. Besides the significance of logarithms for their relationship to exponential functions, the applications of logarithms to various phenomena, such as sound, earthquakes, and human growth (Wood, 2005), are important in their own right.

According to the Common Core State Standards (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010), recently adopted in the United States, the emphasis on the algebra of logarithms in a typical Algebra II course, as well as subsequent courses in mathematics (e.g., precalculus), means that upwards of 2.5 million American students will be expected to engage with and develop an understanding of these ideas each year (National Center for Education Statistics, 2014; National Science Board, 2010). Despite this, there is a dearth of research literature involving student learning of logarithmic functions, in general, and the algebra of logarithms, in particular. Of the literature available, much of the focus is on suggested mathematical and pedagogical approaches to logarithms with no empirical data related to students' learning using these approaches. Weber (2002) concluded that many students do not have a process understanding of exponentiation and logarithms and suggested numerical approaches to encourage the development of these understandings, but not for logarithms directly. Wood (2005) suggested verbal explanations from students about the meaning of logarithmic expressions such as in order to build an understanding of this expression as a numerical value. Confrey and Smith (1995) and Panagiotou (2011) both suggest drawing on the historical development of logarithms as a basis for teaching about logarithms. Confrey and Smith further claim that building the isomorphism between the *counting*, or additive, world and the *splitting*, or multiplicative, world is building the rules of logarithms. The purpose of this study is to address this research gap by exploring a task design that develops students' understanding of logarithms. In the recent ICMI 22 Study, Margolinas (2013) indicated the importance of tasks for generating mathematical activity that afford students the opportunity to encounter concepts and strategies. This study focuses on three iterations of a task designed to elicit students' current understanding of logarithms, as well as lead them to generalizations related to the properties of logarithms.

The task design and analysis is situated within both variation theory, developed by Marton and colleagues (Marton, Runesson, & Tsui, 2004), and research related to learner generated examples, LGEs, (e.g. Watson & Mason, 2005). Variation theory is most concerned with the object of learning, comprised of three aspects: (1) the intended, (2) the enacted, and (3) the lived (Marton et al., 2004). The intended object of learning is what the teacher intends the students to learn at the outset or in the planning of a lesson. The enacted object of learning is what was actually made possible for students to learn in the implementation of a lesson. The lived object of learning is what the students actually did learn at the completion of the lesson, and beyond. Marton and colleagues (2004) defines learning as the development of capabilities, where a capability is described as seeing, experiencing, or understanding something in a certain way. In order to develop a particular capability (way of seeing, experiencing, or understanding), one must simultaneously focus on the critical features of the particular object of learning. A variation theory perspective claims that we can only focus on that which we discern; we can only discern what we experience to vary; we can only experience variation if we have experienced different instances previously and can juxtapose our previous experiences with our current experience simultaneously.

Marton and colleagues (2004) contend that we can only learn that which we experience to vary. In order to ascertain the enacted object of learning, it is necessary to be concerned with what varies and what remains invariant in a learning situation. Marton et al. describe what varies and what remains invariant as a pattern of variation and identify four of these: (1) contrast, (2) generalization, (3) separation, and (4) fusion. The first pattern of variation, contrast, refers to the comparison between what something is and what it is not. For example, in the context of logarithms, $\log 100 = 2$ can be contrasted with $\log_2 8 = 3$ in order to discern which aspects of a logarithm statement can vary. In order to understand what it means for a logarithm to have base ten, students need to experience logarithms that are not base ten. The second pattern of variation is generalization, which refers to experiencing the varying appearances of an object of learning in order to separate it from irrelevant features. For instance, seeing $\log_2 8$, $\log 1000$, $\log_3 27$, and $\ln e^3$ as equivalent log expressions that all equal three, can help students to generalize what it means for a logarithm to be equal to three. The base of the logarithm is an irrelevant aspect here; but the power of the input in terms of the base is significant. The third pattern of variation, separation, involves varying a particular aspect of an object of learning while holding the other aspects invariant. This draws attention to the particular aspect that is allowed to change. The example above held the value of the logarithm invariant while changing the base, which then determined the input. Marton et al. contend that systematically varying certain aspects, while keeping other aspects invariant, can prepare students for various other situations related to the capability in question. Fusion, the last pattern of variation, is the experiencing of all of the critical aspects simultaneously. Through fusion, learners develop the ability to make generalizations that link the critical aspects of an object of learning (Holmqvist, 2011). For instance, discerning the relationship between the base of a logarithm and the input, in order to hold the value of the logarithm invariant, is a result of fusion and experiencing simultaneous changes in both the base and the input of the logarithm.

Drawing on Marton's work, Watson and Mason (2005) suggest that LGEs are an appropriate way to introduce new concepts in mathematics. The use of LGEs, however, may appear to be in conflict with variation theory, as the task designer/instructor concedes control of the presentation of specific examples to the students. Variation theory, however, does not suggest particular ways of arranging for learning, only that variation must be present for discernment. Rather, Marton and colleagues (2004) claim that the particular way of arranging for learning is dependent upon the thing to be learned, and research can be undertaken to determine the most conducive arrangement for student learning of that particular thing. In this sense, then, there is no tension between variation theory and the use of LGEs, as LGEs, when used in conjunction with collaboration, can create the variation in features necessary for discernment.

Watson and Shipman (2008) found that the use of LGEs, in a supportive classroom atmosphere, can successfully introduce new concepts for both advanced and low-achieving learners. Watson and Shipman suggest that while the discernment of critical features of a concept (Marton's *dimensions of variation*) through a set of examples may reveal the structure of the concept, learning through exemplification occurs through discerning the generalization of relationships across the dimensions of variation. Drawing on this work, the task used in this study was developed with the intended object of learning as generalizations related to the properties of logarithms, such as $\log_b(xy) = \log_b x + \log_b y$, $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, and $\log_b x^n = n \cdot \log_b x$, as well $\log_b b^x = x$, which proceeds from the equivalence relationship between logarithmic and exponential statements: $\log_b a = x \leftrightarrow b^x = a$. As will be discussed in the analysis below, over the course of the three implementations of this task, the intended object of learning was shifted based on the insights gained in the first two implementations of the task.

METHODOLOGY

A task that involved learner-generation of examples of logarithm statements was enacted by the first author as the teacher/researcher with three groups of students: two sections of pre-freshman engineering students enrolled in a summer mathematics course, one of which was composed of students who had previously studied calculus (n=13) and the other of which was composed of students who had previously studied precalculus (n=12), and a small problem-solving session comprised of two pre-freshman students who were recruited from a summer precalculus course. The task was enacted with groups of students of varying achievement levels in order to expand on the teacher/researcher's understanding of the variation in students' reasoning about logarithms as revealed by the task. The first enactment was with the group of engineering students who had previously studied calculus; the second enactment was in a problem-solving session with two pre-calculus students; the third enactment was with the group of engineering students who had previously studied precalculus. Each enactment of the task was carried out by the teacher/researcher, and each of the enactments was video-recorded. Students' written work was also collected.

By the time students encounter logarithms in a typical precalculus course in the United States, students have already been introduced to logarithmic functions, the properties of logarithms, and have used logarithms to solve exponential equations (in a typical Algebra II course). Hence, logarithms were not a new concept for these students, but rather a concept that many of the students still struggled with in terms of recalling, applying, and reasoning with and about the properties of logarithms. Prior research also suggests that students tend to struggle with the relationship between logarithms and exponents, as well as the properties of logarithms (Weber, 2002; Wood, 2005). This study, then, expands on previous work related to the use of LGEs to introduce new concepts (Watson & Shipman, 2008) by using LGEs to deepen students' understanding of and reasoning about a previously introduced concept. This research study addressed the following questions:

1) What does the task, as implemented, reveal about students' understanding of logarithms?

2) What is the *enacted object of learning*, or what is possible to learn, given the task-as-implemented?

Task design and data analysis

As the task itself cannot be separated from its enactment, the way in which the task was implemented in each iteration varied. The variation in the task-as-implemented was influenced by the teacher/researcher's insights garnered from the previous implementation(s), as well as the particular *space of variation* opened in that implementation. Despite the differences in the implementation of the task among the three groups of students, commonalities between the *spaces of variation* opened in each of the implementations revealed much about students' understanding of logarithms and served to focus the teacher/researcher's *intended object of learning* for later implementations, as well as to refine the task.

The basic structure of the task in all three implementations involved (1) individual student generated examples, (2) group assessment and group generation of examples, (3) collective class organization/categorization of the student generated examples, and (4) generalization. In each iteration of the task, the teacher asked the students to write a log expression that was equal to three. Then the students were asked to write another. The student generation of examples was meant to draw on students' prior knowledge of logarithms and elicit students' understanding of logarithms, in general, and the value of a logarithm, in particular. The students were then arranged into groups of two to four students to share what they wrote with each other. The teacher distributed a set of index cards to each group, asking them to write a different log expression that equalled three on each card. This potentially required that the group generate additional examples as their original examples may have been duplicative. As groups of students finished writing their examples on the cards, the teacher asked them to tape their cards on the board. The teacher then gathered the students around the cards on the board and asked them to collectively categorize the cards. The collective student sorting of the log statements allowed the teacher to gain insight into what critical features of logarithms the students were attending to, as well as the structure of logarithms, as perceived collectively by the students. A discussion of the students' categories followed, along with additional student generation of examples of logarithm expressions equal to two and five, and finally, student generalizations.

Each of the three implementations were analysed using the framework of variation theory in an attempt to understand *the intended object of learning* and *the enacted object of learning* (or *the space of variation*). While this framework served to answer the second research question most directly, the analysis also addressed what was revealed about students' understanding of logarithms. The variation in the student generated examples, as well as what was revealed about students' prior understanding of logarithms, served to answer the first research question.

RESULTS

This section describes the insights garnered by the teacher/researcher primarily during the first two implementations of the task and the refinement of the task through the third implementation. The intended object of learning shifted from the properties of logarithms to the equivalence relationship between logarithms and exponents and the generalization $\log_b b^x = x$ to generate additional examples, equal to any given number, x. In all three implementations of the task, the categorization of the LGEs, largely by the base, showed that the base of the logarithm and the input of the logarithm, were brought to the fore as aspects of a logarithm that were possible to change, and the range of permissible change for each of these aspects was explored, to some extent. While all students were able to generalize the process of writing a log statement equal to a given number, higher achieving students were more readily able to recognize a generalization comprised of a single statement, such as $\log_b b^x = x$. Extending the task to include combinations of logarithm expressions may create an opportunity for students to verify the properties of logarithms and explore the relationship between the properties of logarithms and the properties of exponents.

The intended object of learning

The intended object of learning in the first implementation of this task was the generalizations related to the properties of logarithms, such as $\log_b(xy) = \log_b x + \log_b y$, $\log_b(\frac{x}{y}) = \log_b x - \log_b y$, and $\log_b x^n = n \cdot \log_b x$, as well as the equivalence relation: $\log_b a = x \leftrightarrow b^x = a$. We had anticipated LGEs of the form $\log_2 8 = 3$, but had also anticipated that some students would extend their thinking to include an example such as $\log_{10} 10 + \log_{10} 100 = 3$. Goldenberg and Mason (2008) discuss how example generation is not just a memory lookup, but rather found that students often start with some known example(s) and through combinatorial approaches can construct new examples. As students began generating examples in the first part of the task, the teacher/researcher quickly realized that while the combinatorial approach was not appearing, other examples that had not been anticipated were being generated by the students, such as examples with fractional bases.

In the second implementation the generalization $\log_b b^x = x$ was the intended object of learning, with b and x as the *dimensions of variation*. We were also interested in how to extend the task in such a way that allowed students to gain access to examples that included a combinatorial approach, as they did not appear in either the first implementation, or in the first portion of this problem solving session. The students were asked to use the logarithm statements that they had already generated and combine them in some ways (using addition, subtraction, multiplication, or division) to generate other logarithm statements equal to three. While this allowed students to recall and verify the properties of logarithms that they had previously learned, students could not discern why these properties were valid and they were unclear about the relationship between the properties of logarithms and the properties of exponents.

The third implementation of the task was a refined version, based upon our insights from the first two implementations of the task. In this implementation, the intended object of learning focused on the equivalence relationship between logarithmic and exponential statements: $\log_b a = x \leftrightarrow b^x = a$, as well as the general logarithm statement $\log_b b^x = x$ for generating logarithm statements equal to a given value, x. Students were not asked to combine logarithm expressions to generate new statements equal to a given value in this task. Rather, the combination of logarithms was separated into a distinct, but linked task, for the sake of time and depth.

Categorization of LGEs

After students shared their generated examples in a small group and on the board, the whole class categorized the examples they had generated. The first class separated the examples into four categories, determined largely by the base of the logarithm: (1) whole number, (2) base 10, (3) fractional bases, (4) special, and (5) wrong. The students in the third implementation were not able to use graphing calculators (that can calculate logarithm expressions with various bases), but rather used simple four-function calculators to calculate a larger power of a number. As suspected, perhaps due to the change in the type of calculator available for a tool, fractional bases for the logarithm expressions did not appear in the third implementation of the task. In the first portion of the second implementation, the students still organized their examples according to the base, first separating them into even and odd categories, then deciding to list them out from statements with base two up to statements with base ten. In the second part of the session, the students were asked to generate combinations of logarithm expressions equal to three and reorganize the examples, in consideration of the additional examples. The students chose, then, to categorize the LGEs by operation. The students in the third implementation also categorized the LGEs according to the base of the statement, however, they included four subdivisions or "branches": (1) Odd, (2) Even, (3) Base 10, and (4) Fractional, under the "big tree" of the equivalence relationship $\log_b a = x \leftrightarrow b^x = a$. This is similar to the students in the first implementation who also explained how all of their correct examples "followed the same rule," after they had determined a generalization for logarithms with whole number bases. The students in the third implementation also included a "wrong" category, a "natural log" category, which were actually exponential statements that included natural log in the exponent (e.g. $e^{\ln 3} = 3$), and a "unique" category, which included only the statement $\log_3 (\frac{729}{27}) = 3$.

In all three implementations of the task, the students' categorization of the LGEs showed that the base of the logarithm, as well as the input of the logarithm, was brought to the fore as aspects of a logarithm that were possible to change. The combinatorial approach did not spontaneously appear. In terms of revealing student understanding, however, Watson and Goldenberg (2008), point out that, "the fact that [students] don't display an example does not imply that it is not within their accessible [example] space, just that they have not perceived a reason to express it" (p. 189). As such, it appears that the task as enacted did not cue or trigger students to think of examples using the combinatorial approach. The *enacted object of learning*, in the first and third implementation, as well as the first portion of the second implementation, was restricted to the generalization $\log_b a = x \leftrightarrow b^x = a$, the equivalence relationship between logarithm and exponential statements, and the generalization $\log_b b^x = x$, which expresses the structure of the correct logarithm statements.

Range of permissible change

The LGEs served to reveal students' understanding of the *range of permissible change*, particularly in the base of the logarithm statement. As mentioned above, the combination of logarithm statements did not arise spontaneously, indicating that students did not, in that instance, recognize combinations of statements as within the *range of permissible change* for the structure of a logarithm expression equal to three. Without this range of LGEs, students tended to categorize the examples of logarithm statements according to the value of the base, despite the common structural form. Their choice to separate logarithm statements with a whole number base from those with base ten was perhaps indicative of their greater familiarity with base ten, or their understanding of the common log as somehow more important than logs in other bases. We did not anticipate the use of fractional bases students' generated examples; this may have been related to students' graphing calculator usage while generating their examples, particularly since this did not occur in the subsequent implementations when students were restricted to four function calculators. Students' choice of separating base e and base π logarithm statements as "special" could perhaps be indicative of students' sense of e and π as symbols that represent something other than a specific number. Students' recognition, however, of $\log_1 1 = 3$, $\log_0 0 = 3$, and $\log_{-1} -1 = 3$ as "wrong" logarithm statements served to restrict the *range of permissible change*. Student attempts at both justifying these statements and explaining why these logarithm statements were incorrect created an opportunity to deepen students' understanding of both exponents and logarithms.

Only a single example began to directly confront the *range of permissible change* for the input of the logarithm. The students seemed to recognize that the input of the logarithm would change, dependent on the base of the logarithm, but failed to see the range of possibilities in writing the input. In the third implementation, the example $\log_3\left(\frac{729}{27}\right) = 3$ was placed in the "unique" category. The statement $\log_3\left(\frac{729}{27}\right) = 3$ is equivalent to the statement $\log_3 27 = 3$. Therefore, it has the same structure as the other LGEs, but the students did not recognize it as such. This indicates that students could use more exposure to variation in the input of the logarithm. This was also the only instance when combinations of logarithm statements arose spontaneously. Two students in the third implementation insisted that the way that you would deal with this statement is to rewrite it as $\log_3 729 - \log_3 27 = 6 - 3 = 3$. Thus, students recalled the logarithm property $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$, and applied it, but this appeared to overshadow a more flexible and efficient means of simplifying the statement to show its equivalence to three.

Generalizations

In the last part of the task, the teacher/researcher asked the students to write a logarithm statement that was equal to any number. Some students generated two separate statements that was often a restatement of the equivalence relation: $\log_b a = x \leftrightarrow b^x = a$. Others were able to succinctly describe the generalization as $\log_b b^x = x$. Being able to symbolically write down a generalization did not indicate that students would be successful at verbalizing the generalization. For instance, Evette, in the third class, wrote the symbolic statement $\log_b a = x \leftrightarrow b^x = a$, and the verbal statement: "That 'any number' that you want it to equal, must be raised to the base value." While Evette correctly wrote the symbolic statement, she incorrectly stated that the exponent would be the base value. It is also ambiguous what she is referring to with the use of "it". This is perhaps related to a lack of opportunities to explain and communicate verbally in the mathematics classroom.

Lower achieving students had more difficulty than higher achieving students in discerning a generalization comprised of a single statement, such as $\log_b b^x = x$. This is perhaps related to an underdeveloped sense of variable and equality, and the failure to recognize the substitution of equivalent expressions. This could also be related to a preference of seeing each "part" of the logarithm statement (the base, the input, and the output) as distinct and a failure to fuse these critical aspects together.

Combinations of logarithm expressions

The expansion of the example space to include combinations of logarithm expressions in the second implementation served to broaden the *space of variation* when compared to the first and third implementations. Through opening up the variation of the statement to include combinations of logarithm expressions, it becomes possible to discern both *how* the logarithm expressions can be combined and the ways in which the *dimensions of variation* are related within a given statement. Watson (2000) described these two ways of seeing pattern as 'going with the grain,' indicating a recursive continuation of pattern to generate more instances that may not indicate structure (in this case, *how* the logarithm can be combined), and 'going across the grain', a metaphor that indicates the revelation of the internal structure itself (here, the relationship between the *dimensions of variation*). Thus, this opening of the *space of variation* has the potential to provide students with the opportunity to "see" the relationship between logarithms and exponents in ways that they had, perhaps, not experienced before. One of the students wrote about what he had learned during the problem solving session:

> The exponents in exponential equations are used as the values for logarithmic functions. For example:
>
> $$5^{2} + 5^{1} = 5^{3}$$
>
> $$\log_5(25) + \log_5(5) = 3$$
>
> $$2 + 1 = 3$$

While the exponential statement is erroneous, this student is beginning to discern variation 'across the grain', and we would argue, is on his way to developing a certain way of seeing logarithms, and hence developing important capabilities for reasoning about logarithms in powerful ways.

CONCLUSION

Based on the three iterations of this task, using LGEs with students who have had previous exposure to a concept can serve to reveal students' understanding of the *dimensions of variation* of a concept, as well as the *range of permissible change* in those aspects. The use of LGEs, in this particular task, revealed students' understanding of the *range of permissible change* in the base of the logarithm and led to opportunities to connect exponential and logarithmic functions. Holding the value of the logarithm statement invariant (for a time) created an opportunity for students to explore what happens when a single aspect, namely – the base of the logarithm, is varied.

While this task draws students' attention to the relationship among the *dimensions of variation* in a logarithm statement, as well as the *range of permissible change* of those dimensions, further consideration of how tasks can be designed to generate and develop an understanding of the relationship between combinatorial properties of exponents and logarithms is needed to more fully develop students' facility with logarithms.

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Teaching the concept of function: Definition and problem solving

Areti Panaoura[1], Paraskevi Michael-Chrysanthou[2] and Andreas Philippou[2]

1 Frederick University, Department of Education, Nicosia, Cyprus, pre.pm@fit.ac.cy

2 University of Cyprus, Department of Education, Nicosia, Cyprus, pmicha@ucy.ac.cy, andreas.philippou@cytanet.com.cy

The present study investigates students' abilities to understand the concept of function. Secondary education students were asked to (i) define the concept of function and present examples of functions, (ii) translate between different representations of a function and (iii) solve function problems. Findings revealed students' great difficulties in proposing a definition of function, in solving tasks of conversions between different modes of representation, and in solving function problems. Based on the students' abilities and misconceptions about functions, teaching practices for improving the students' understanding of functions are discussed.

Keywords: Function, definition, use of representations, teaching methodology.

INTRODUCTION

For more than twenty years, the concept of function has been internationally considered as a unifying theme in mathematics curricula (Steele, Hillen, & Smith, 2013). Students face many instructional obstacles when developing an understanding of functions (Sajka, 2003; Sierpinska, 1992). Kieran (1992) questions whether students' inability of conceptually understanding functions is related to its teaching or is due to students' inappropriate way of approaching function tasks. Sajka (2003) indicates that students' abilities in solving tasks involving functions are influenced by the typical nature of school tasks, leading to the use of standard procedures. According to a standard didactic sequence, students are asked to infer the properties of a function using the given graph, by following a specific procedure.

In relation to the above, our study examines students' conceptions of function, as it is one of the most important topics of the curriculum and it is related to other subjects, such as physics (Sanchez & Llinares, 2003). In fact, the results we present in this paper are a part of a large scale cross-sectional study examining the use of different modes of representations in functions at the secondary school level. Adopting a developmental perspective at different grades of secondary education, we aim to trace students' abilities in defining functions, recognizing and manipulating them across representations and in problem solving, emphasizing the approach used (algebraic or geometric). Thus, our main questions are: (i) What abilities do students have to define and flexibly manipulate functions and solve function problems?, and (ii) What are the differences in students' performance at the 3rd, 4th and 5th grades of secondary education? Based on our results, we provide suggestions for teaching practices that can facilitate students' understanding of functions.

THEORETICAL FRAMEWORK

The role of multiple representations in the understanding of functions

There is strong support in the mathematics education community that students can grasp the meaning of a mathematical concept by experiencing multiple mathematical representations of that concept (Sierpinska, 1992). One of the main characteristics of the concept of functions is that they can be represented in a variety of ways (tables, graphs, symbolic equations, verbally) and an important aspect of its understanding is the ability to use those multiple representations and translate the necessary features from one form to another (Lin & Cooney, 2011). In order to be able to use the different forms of representations as tools in order to construct a proof, students have to understand the basic features of each representation and the limitations of using each form of representation.

According to Steele and colleagues (2013), it is typical in the U.S. for the definition of a function and a connection to the graphic mode of the function concept to occur in the late middle grades, while the formal study of function with an emphasis on symbolic and graphical forms occurs in high school. According to Bardini, Pierce and Stacey (2004), in their brief overview of Australian school mathematics textbooks, symbolic equation solving follows graphical work. A proper understanding of algebra, however, requires that students be comfortable with both of these aspects of functions (Schwartz & Yerushalmy, 1992). It is thus evident that the influence of teaching is extremely strong, and the promotion of specific tools or processes enforces the development of specific cognitive processes and structures. For example, it has been suggested that one way to improve learners' understanding of some mathematical concepts might be the use of graphic and symbolic technologies (Yerushalmy, 1991).

The role of definition in the understanding of functions

In mathematics definitions have a predominant role in the construction of mathematical thinking and conceptions. Vinner and Dreyfus (1989) focus on the influence of concept images over concept definitions. Over time, the images coordinate even more with an accepted concept definition, which in turn, enhances intuition to strengthen reasoning. A balance between definitions and images, however, is not achieved by all students (Thompson, 1994). Actually, pupils' definitions of function can be seen as an indication of their understanding of the notion and as valuable evidence of their mistakes and misconceptions. Elia, Panaoura, Eracleous and Gagatsis (2007) examined secondary pupils' conceptions of function based on three indicators: (1) pupils' ideas of what function is, (2) their ability to recognize functions in different forms of representations, and (3) problem solving that involved the conversion of a function from one representation to another. Findings revealed pupils' difficulties in giving a proper definition for the concept of function. Even those pupils who could give a correct definition of function were not necessarily able to successfully solve function problems.

METHODOLOGY

Participants

The participants were 756 secondary school students from eight schools in Cyprus. There were 315 students at 3rd grade of gymnasium (15 years old), 258 students at 4th grade of lyceum (16 years old) and 183 students at 5th grade of lyceum (17 years old). The students at 5th grade follow a scientific orientation and attend more advanced levels of mathematics courses. Students' participation was due to the voluntarily participation of their teachers, thus our sampling procedure was not randomized.

Procedure

Two tests including different types of tasks were developed. The tests were developed by the researchers and secondary mathematics teachers. The tasks were mainly aligned to the level of the 3rd graders and the tests were piloted to almost 100 students at each grade. At the middle of the school year, each test was administered to students by a researcher or by a teacher who had been instructed on how to correctly administer the test. The testing period was 40 minutes. The tests were scored by the researchers, using 1 and 0 for correct and wrong answers, respectively.

The tasks focused on (a) defining and explaining the concept of function, (b) recognizing, manipulating and translating functions from one representation to another (algebraic, verbal and graphical), (c) solving problems.

Costas has €20 and spends €1 per day. His sister has €15 and spends €0,5 per day.

i. Find the function expressing the amount of money (y) each person will have in relation to the number of days (x).
ii. Design the graph showing this function for each person.
iii. In how many days will the two brothers have the same amount of money? What will this amount of money be?

Figure 1: Example of a problem solving task

The first test asked students to (i) present a definition of function and an example, (ii) explain their procedure for recognizing that a graph does not represent a function and present a non-example of function, (iii) write the symbolic representation of six verbal expressions (e.g. "the area E of a square in relation to its side"), (iv) draw a graph to solve a problem, (v) recognize graphs of functions, (vi) explain a graph in terms of the context, and (vii) examine whether symbolic expressions and graphs represent functions. In the second test students had to (i) present their procedure for examining whether a graph represents a function, (ii) draw graphs of given functions, (iii) recognize graphs of given verbal or symbolic expressions, and (iv) write the symbolic equations of given graphs. The reliability for the total of the items on both tests was high (Cronbach's alpha=0.868).

RESULTS

According to our research questions, we first present the results of students' performance on defining the concept of function and their ability to present an example in order to explain the definition. Then, we concentrate our attention on the procedure students follow in order to justify whether a verbal expression or an equation is a function. A crosstabs analysis gives further insight into the performance of students in defining and recognizing functions. Finally, differences between students in the three grade levels are presented.

Based on the results of our descriptive analysis, 159 students were able to present a correct definition, 242 presented a wrong definition, and 358 students did not give any answer. Most of the students who presented a correct definition (Table 1) were at the 4th and 5th grade. The results were similar to the task that asked students to present an example for explaining the definition of function. Only 401 of the students presented an example, and of these, 323 were correct. The highest percentages of correct examples (Table 1) were at the 4th and 5th grade. This result is the first indication that students are perhaps more capable of providing an example in order to explain a mathematical concept than of defining it verbally by using formal language and/or symbolization.

In the first test students were asked to explain their procedure for identifying a graph that does not represent a function and provide a relation that does not represent function. In the second test, students were asked to explain their procedure for identifying a graph that represents a function and to provide an example of a function. Table 2 indicates the percentages of correct answers for the specific tasks. In all cases, the results were higher in the 5th grade, as was expected, but there were especially negative results for students in the 3rd grade. The majority of the 3rd graders did not give an answer and many of those who answered presented a wrong procedure.

	Correct definition	Correct example
Grade 3 (N=315)	8.2%	23.0%
Grade 4 (N=258)	58.5%	57.0%
Grade 5 (N=183)	32.7%	55.7%

Table 1: Students' correct definitions and examples for each grade

Tasks	3rd (%)	4th (%)	5th (%)	Total (%)
Procedure indicating that a verbal expression or an equation is not function	2.2	38.4	42.0	24.3
Procedure indicating that an expression is function	2.5	32.6	47.5	23.7
An example of a function	19.4	50.4	55.7	38.6
A non-example of a function	3.8	38.4	38.3	23.9

Table 2: Percentages of students' correct answers at specific tasks

We analysed, by using crosstabs analysis (Table 3), the performance of students who correctly described a procedure for determining whether a graph represents a function in relation to their ability to correctly identify functions presented graphically. Results indicated that less than half of the students, who correctly described a procedure, correctly recognized the graph that represented a function (44.1%).

We further analysed, by using crosstabs analysis (Table 3), the characteristics of students who were able to correctly define function. Consequently, these students seemed to have a more conceptual understanding of function. Results indicated that 89.4% of the students presented both a correct definition of function and a correct example, showing that they likely have a strong theoretical understanding of the concept. At the same time, of the students that provided a correct definition of function, 74.2% also succeeded in describing a procedure for determining whether a graph did not represent a function, and 62.7% also succeeded in describing the procedure for recognizing whether a graph did represent a function. These initial results permit us to assume that the students who are able to define function and explain by providing an example are the students with the highest conceptual understanding.

In the second test, students who were able to correctly describe a procedure for determining whether a graph represented a function were generally able to present an example of function (90.4%). Of the students who were able to correctly describe a procedure for determining whether a graph did not represent a function, 83.7% were also able to present a symbolic non-example of a function. It is important to note, however, that of the total sample, only 153 students correctly presented a procedure for recognizing a graph that represented a function, while only 135 students correctly presented a procedure for recognizing a graph that did not represent a function. Similarly, only 241 students correctly presented an example of a function, and only 165 students correctly presented a non-example.

	Example for definition	Describing a procedure for determining whether a graph represents a function	Describing a procedure for determining whether a graph does not represent a function
Recognition of function	–	44.1%	–
Correct definition	89.4%	62.7%	74.2%
Function Example	–	90.4%	–
Function Non-example	–	–	83.7%

Table 3: Crosstabs analysis for students' (total sample) understanding of functions

The second objective of the study was to identify statistically significant differences ($p<0.05$) concerning the concept of function in respect to students' grade level in order to investigate the developmental aspect of a conceptual understanding of function. ANOVA analysis was used for comparing students' means in presenting the definition of the concept and giving an example of a function in respect to the categorical variable of their grade. In the first case, Scheffé analysis indicated that there was a statistically significant difference ($F_{2,397}=33.396$, $p<0.01$) between the students at the 3rd grade with the students at the 4th and 5th grade ($\bar{x}_3=0.11$, $\bar{x}_4=0.53$, $\bar{x}_5=0.48$). Although the highest mean was at the 4th grade, there was not any statistically significant difference between the two grades at the lyceum. Concerning the presentation of an example in order to explain the definition of the concept, the difference was statistically significant ($F_{2,398}=5.896$, $p<0.01$) only between the 3rd and the 4th grade ($\bar{x}_3=0.72$, $\bar{x}_4=0.88$, $\bar{x}_5=0.78$). The same analysis was conducted concerning students' ability to describe a procedure for recognizing an equation which did not represent a function, in respect to grade level. There were statistically significant differences between the 3rd grade and the two other grades ($F_{2,292}=33.510$, $p<0.01$) with a large difference between the means ($\bar{x}_3=0.12$, $\bar{x}_4=0.47$, $\bar{x}_5=0.55$).

Items from both tests were grouped in order to be able to further analyse students' performance concerning the specific aspects which were the main interest of this study: (i) Propose definition, (ii) present examples to explain a concept or a procedure, (iii) recognition of the concept, (iv) translation of the concept from one representation to another, (v) construction of graphs which represent functions, as an indication of manipulating the concept and (vi) problem solving tasks. Table 4 presents the means and standard deviations of students' performance on these specific dimensions.

	$\bar{X}$	SD
Definition	0.718	0.261
Examples	0.837	0.231
Recognition	0.449	0.152
Translation	0.619	0.212
Construction	0.648	0.354
Problem solving	0.393	0.169

Table 4: Means and standard deviations of students' performance

ANOVA analysis was conducted for examining the statistically significant differences for each of the above aspects of a conceptual understanding of function in respect to the three grade levels. Statistically significant differences ($p<0.05$) were found only concerning three of the dimensions. There was a statistically significant difference ($F_{2,182}=7.678$, $p<0.01$) concerning the proposed definition of the concept. The Scheffé analysis indicated that the difference was between the 3rd grade with the 4th and the 5th grade ($\bar{x}_3=0.33$, $\bar{x}_4=0.75$, $\bar{x}_5=0.71$). The second statistically significant difference was for the students' ability to recognize functions presented in different forms of representations ($F_{2,570}=18.926$, $p<0.01$). The difference was between the students at the 3rd grade in comparison to the students at the 4th and 5th grade ($\bar{x}_3=0.48$, $\bar{x}_4=0.41$, $\bar{x}_5=0.40$). It was unexpected that in this case the performance at the 3rd grade was higher than the two other grades. The third statistically significant difference was for the students' ability to translate the functions from one type of representation to another ($F_{2,165}=11.077, p<0.01$). The students at the 3rd grade had a lower performance than the older students ($\bar{x}_3=0.343$, $\bar{x}_4=0.547$, $\bar{x}_5=0.702$).

DISCUSSION

We have concentrated on students' ability to define the concept of function and on their ability to handle flexibly the different modes of representation of function. The results of the present study confirm previous findings that students face many difficulties in understanding function at different ages of secondary education (Sajka, 2003; Tall, 1991). Findings revealed serious student difficulties in proposing a proper definition for function or a tendency to avoid proposing a definition due to their possible belief that the intuitive and informal presentations of their conceptions cannot be part of the mathematical learning. Those difficulties are consistent with Elia and colleagues' (2007) findings. Although formal definitions of mathematical concepts are included in the mathematics textbooks for secondary education, mathematics teachers do not focus on definitions. Instead they promote the use of algorithmic procedures for solving tasks, as actions and processes (Cottrill et al., 1996), and underestimate the word and meaning (Morgan, 2013) as interrelated dimensions of the concept.

Secondly, it seems easier for students to present an example for explaining a mathematical concept, rather than using a non-example in order to explain the respective negative statement. At the same time, it is easier for them to use an example for explaining the concept of function rather than explaining the procedure they follow for determining whether a graph represents a function. This is probably a consequence of the teachers' method of using examples in order to explain an abstract mathematical concept. Thirdly, students had a higher performance on manipulating the concept in graphical form and translating from one type of representation to another, than on recognizing functions in algebraic and graphical forms. Symbolic equation solving follows the graphical work (Llinares, 2000), and probably for this reason, the performance on graphical functions was higher. Finally, the results of the students at the 3rd grade of secondary education were especially negative, thus we have to further examine whether their processing efficiency and cognitive maturity prevent us from teaching the specific concept at that age. Despite the tendency to use the spiral development of concept in the teaching process and the curriculum (Ministry of Education and Culture, 2010), we have to rethink the teaching methods we use at the different ages and the cognitive demands of tasks at each age.

We believe that it is not adequate just to describe the students' knowledge of a concept, but it is interesting to design and implement didactic activities and examine their effectiveness. Brown (2009) suggests that the construction of concept maps will enable teachers to have in mind all the necessary dimensions of the understanding of the concept. The distinction of the procedural understanding and the conceptual understanding and the lack of interrelations between aspects such as the definition of the concept, the manipulation of the concept and problem solving related to a concept have as a possible result the phenomenon of compartmentalization (Elia, Gagatsis, & Gras, 2005). Thus, the use of multiple representations, the connection, coordination and comparison with each other and the relation with the definition of the concept should not be left to chance, but should be taught and learned systematically.

The verification of the interrelations between the different aspects of understanding functions are within our next steps, which include the confirmatory factor analysis (CFA), for confirming the theory about the structural organization of the conceptual understanding of functions. We are going to examine whether the students' abilities in defining the concept, in recognizing and manipulating the concept and in translating the concept from one representation to another are important dimensions of the conceptual understanding of functions. Specifically we will focus on the interrelationship between these dimensions. Special emphasis will be given to the role and influence of the definition on the remaining dimensions of the conceptual understanding of functions, as our results revealed students' great difficulties in this particular aspect of the concept.

Concluding, the present study enables us to know and understand how students conceptualize the notion of function and to realize the students' obstacles and misunderstandings. Teaching processes and teaching materials need to be enriched with problem solving situations. The given examples presented by students were mathematically structured by using a formalistic way and there was not any reference to a daily experience. In the context of the interdisciplinary social reality, the concept of function has to be related to other relevant domains such as physics, engineering, and technology.

Limitations

A limitation of our study concerns the administration of the tests. In cases where the administration was not performed by the researchers, there is no certainty that the proper amount of time was given to the students. Thus we cannot be sure that the same conditions held in every classroom during the administration of the tests, and this may have affected the reliability of the tests. The teachers who administered the tests, however, were provided with all the necessary instructions. A further limitation of the study was our inability to control the teaching method which was used for the specific concept. In Cyprus, however, teachers receive the same instructions by the Ministry of Education for the teaching methods they have to use in their classes and the same in-service training. There is also a common curriculum and a common textbook for students.

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The development of student's early number sense

Marios Pittalis, Demetra Pitta-Pantazi and Constantinos Christou

University of Cyprus, Nicosia, Cyprus

The present study adopted a theoretical model, suggesting that number sense consists of elementary number sense, conventional arithmetic and algebraic arithmetic, to trace the development of students' early number sense. Two hundred and four 1st grade students were individually tested on five time-points. Data analysis suggested that number sense follows a linear growth rate and six groups of students were identified that follow different growth patterns. Two of the groups had a low initial value, two had a moderate initial value and two had high initial value. Systematically, one of the two groups with the same initial value exhibited a greater growth rate. The gap of the groups with same initial value in algebraic arithmetic progressively increased across the five time-points of the study.

Keywords: Early number sense, arithmetic, algebraic arithmetic.

INTRODUCTION

The development of students' number sense is considered as an important outcome and key ingredient of school curricula and a foundation for developing formal mathematical concepts and skills in elementary school (Yang, 2005). Research findings support that number sense is a powerful predictor of mathematics outcomes and a vital prerequisite to success in mathematics (Malofeeva, Day, Saco, Young, & Ciancio, 2004). Recent longitudinal studies documented that number sense skills developed in pre-K and kindergarten are not only foundational, but also correlate with first grade mathematics achievement (Jordan et al., 2006). It was found that the mastery of counting principles correlates significantly with arithmetic abilities. Moreover, Jordan, Glutting, Ramineni and Watkins (2010) showed that core number competences in kindergarten have strong predictive validity on mathematics computation and problem solving even at third grade.

The present study capitalizes on previous studies asserting that early number sense consists of three distinct, but interrelated components (Pittalis, Pitta-Pantazi, & Christou, 2013, 2014). Our adopted model hypothesizes that student's early number sense consists of (a) elementary number sense, (b) conventional arithmetic and (c) algebraic arithmetic. The innovative aspect of this model lies on the inclusion of algebraic arithmetic as a component of number sense and builds on the theoretical premise that the artificial separation of arithmetic and algebra deprives students of powerful ways of thinking about mathematics in early grades and makes it more difficult for them to learn algebra in later grades. Algebraic arithmetic adopts Drijvers and his colleagues (2011) description of algebra as a synthesis of (a) patterns and formulas, (b) restrictions and (c) functions. This conceptualization of algebraic arithmetic leads to a broader, more dynamic and flexible construct that underlies the importance of grasping the relations among numbers and manipulating numbers and number relations with flexibility.

In this study, we extend the work presented in PME 38 and present the results of a longitudinal study that traced student's number sense in grade 1 and at the beginning of grade 2. In particular the aims of the study were to: (a) to trace the development of six year old students' number sense and (b) to identify different growth patterns.

LITERATURE REVIEW

Number sense is one of the most important concepts to be developed in early mathematics (Baroody et al., 2009; McGuire & Wiggins, 2009). Children's number sense is a key predictor of later mathematical success, both in the short (Aunio & Niemivirta, 2010) and the longer term. For instance, research findings suggest that early number sense development contributes in learning more complex mathematics concepts, it promotes numerical fluency and it is foundational to all aspects of early mathematical skills (Baroody et al., 2009; Jordan et al., 2010). Inadequate development of number sense in early grades may be related to mathematics learning difficulties. In addition, it is widely supported that students who enter school with strong number sense are more likely to benefit from mathematics teaching in the elementary grades and that the effect of weak number sense may be cumulative (Jordan et al., 2010).

Number sense develops gradually and matures with experience and knowledge (Reys & Yang, 1998). It develops as a result of exploring numbers in a variety of contexts and relating them in ways not limited to traditional algorithms (Sood & Jitendtra, 2007). Moreover, number sense is highly personal and is a by-product of teaching for understanding. Longitudinal studies provided evidence regarding the growth of students' early number sense in kindergarten and in elementary early grades. For instance, Jordan and her colleagues (2010) traced empirically three growth trajectories: (a) children who started with low number competence and stayed low, (b) children who started with high number competence and stayed high, and (c) those who started with low number sense but made relatively good growth. A growing body of research with converging data (Gersten et al., 2005; Jordan et al., 2010) assert that the identification of growth groups is extremely important to trace students that need early intervention.

A number of researchers proposed that number sense is difficult to define, while different research studies identified different conceptual understandings and components of number sense (Malofeeva, Day, Saco, Young, & Ciancio, 2004).A well-accepted and broad definition of number sense refers to a coherent understanding of what numbers mean, numerical relationships and the ability to handle daily life situations which involve numbers (Yang, 2005). In the present study we define early number sense as a synthesis of elementary number sense, conventional arithmetic and algebraic arithmetic (see Pittalis, Pitta-Pantazi, & Christou, 2013, 2014). Elementary number sense refers to key elements of numbers sense. It consists of counting, number identification, number knowledge, quantity discrimination, enumeration and non-verbal calculations (Jordan, et al., 2006). Counting relates to grasping one to one correspondence, knowing the count sequence and mastering counting principles, such as stable order and cardinality (Jordan, et al., 2006). Number knowledge refers to discriminating and coordinating quantities and making numerical magnitude comparisons. Enumeration relates to the ability of enumerating sets, irrespectively of the orientation of the objects. Number identification refers to the ability of naming written symbols. Non-verbal calculations refer to the ability of manipulating non-verbal addition-subtraction situations with objects, while quantity discrimination relates to the ability of recognizing the relative magnitude size (Reys, & Yang, 1998). Conventional arithmetic includes students' ability to calculate number combinations, conceptualize the effect of operations (Yang, 2005) and solve story problems. Number combinations relates to students' ability of executing calculations with no object referents (Jordan et al., 2007). Understanding the effect of operations involves students' ability to conceptualize, judge and interpret the meaning of operations. Story problems conceptualize the ability to solve simple word problems, without using manipulative.

Algebraic arithmetic refers to the dimension of early number sense that moves beyond conventional arithmetic. It encompasses the development of a more sustainable and abstract understanding of the relations among numbers. Thus, the inclusion of algebraic arithmetic as a component of number sense could be considered as a move from particular numbers and measures towards relations among sets of numbers and measures. It can be conceived as a generalized arithmetic of numbers and quantities in which the concept of function assumes a major role. The inclusion of algebraic arithmetic as a factor of number sense results in reconceptualising the conception of number sense. The proposed nature of number sense defines a more dynamic and flexible construct that could facilitate students' advancements and transition to a more abstract and relational system of thinking. In particular, algebraic arithmetic encompasses the general cognitive ability of relational thinking, since it focuses on relations than on calculating answers (Carpenter, Levi, Franke, & Zeringue, 2005). In particular, algebraic arithmetic refers to number patterns, functions and restrictions (equations and balance scale restrictions), as proposed by Drijvers and his colleagues (2011). The parameter of number patterns involves searching for regularity and patterns to recognize a common algebraic structure. The dimension of restrictions describes students' ability in finding what value(s) of the unknown satisfies the required conditions in various situations; balance scale tasks or in more formal setting, such as equations. The function component involves students' ability to investigate arithmetic relations between quantities/variables.

METHODOLOGY

Subjects

The subjects of the study were 204 first grade students from 7 urban primary schools in Cyprus. The school sample is representative of a broad spectrum of socioeconomic background. Students were assessed on the number sense measures four times during the period October to June (approximately one administration per two months). A fifth measure took place at the beginning of the following year, when students attended grade 2. During each measurement students were interviewed in two sessions of approximately 30 minutes each. Students had a time restriction for each type of task (one minute for the majority of tasks). Students were individually tested in all five occasions. The order of the parts was rotated in the five time series.

Measures

The majority of the test items were adopted from the Curriculum Based Measurement (Fernstrom & Powell, 2007) and the rest ones were developed based on the theoretical considerations of the study. Six types of tasks were used to measure elementary number sense: (a) counting tasks, (b) number recognition, (c) quantity discrimination, (d) number knowledge, (e) enumeration, and (f) non-verbal calculation. In the counting tasks, students were asked to enumerate objects, in the number recognition tasks, students had to read numbers, in the quantity discrimination tasks students were asked to decide which was the largest number, in the number knowledge tasks, students were asked to find smaller and bigger numbers of a given one and in the non-verbal calculation tasks students had to add or delete objects in a given set so the number of objects corresponds to a given number. Conventional arithmetic was measured by three types of tasks: (a) story problems, (b) understanding of operations and (b) number combinations. In story problems tasks, students had to select the appropriate number sentence for a list of story problems. In the understanding of operations tasks, students were presented with simple word problems and were asked to select out of 4 mathematical sentences, the one that fitted the problem. In number combinations tasks students had to find mentally the result of addition, subtraction, multiplication and division combinations. Finally, algebraic arithmetic was measured by four types of tasks: (a) number patterns, (b) restrictions-equations, (c) restrictions-balance scale and (d) functions. In number patterns tasks, students had to extend or complete number patterns, such as 5, 8, 11, ... The ability to solve number patterns implies that a student can conceptualize the relations among numbers to fill in or extend a number pattern. For the assessment of students' abilities in number restrictions two types of tasks were used, number equations and balance scales. In the number equations, students were asked to complete the missing terms of equations, such as 3+5=4+□. Solving number equations encompasses conceptualizing the equal sign as equity of two quantities and the relations among all its components. In addition, estimating the missing term by analysing the relations among the components of the equation, instead of executing calculations, indicates students' flexibility and it is less time consuming. The other restriction task appeared in the form of a balance scale. Students had to identify the value of two or three shapes which balanced in the balance scale with a given number. Finally, regarding student's abilities with functions, students were presented with function machines and a table which showed the input and output values. Students were requested to provide the input or output numbers which were missing. The task was an adaptation of a task presented by Drijvers and his colleagues (2011). Students had to identify the relation between the input and the output value, identify the effect of change and examine whether their hypothesized rule was valid in all cases.

Data Analysis

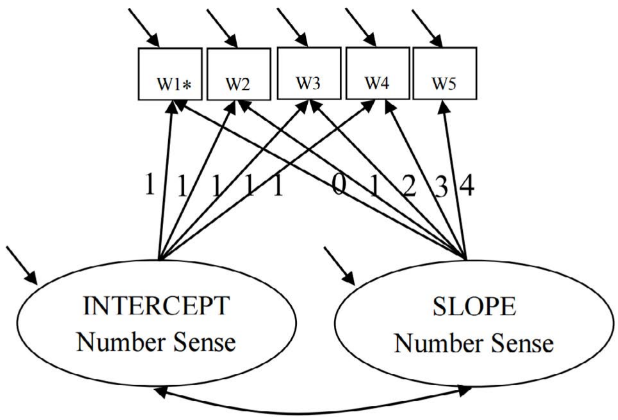
To trace the development of number sense we used latent growth models. Growth models examine the development of individuals on one or more outcome variables over time. In order to evaluate model fit, three widely accepted fit indices were computed (Muthén & Muthén, 2007): The chi-square to its degree of freedom ratio (x^2/df should be <2); the comparative fit index (CFI should be >.9); and the root mean-square error of approximation (RMSEA should be close to or lower than .08). Latent class analysis was used to trace categories of students reflecting different developmental patterns in number sense; this is a statistical method for finding sub-types of related cases from multivariate data. To make the extraction of an average score for each dimension of number sense feasible, it was thought appropriate to set the highest score in each test across the five time-points as 1 and then make the necessary adaptations. That was necessary because all the tasks were equivalent and the time restriction did not allow the setting of an ultimate maximum score. The same procedure was followed for the extraction of an average number sense score.

RESULTS

Student's early number sense development

To evaluate change in students' number sense components, the repeated measures longitudinal date were analysed using latent growth curve analysis. The latent intercept of number sense was defined by fixing all loadings at 1, whereas the loadings for the latent slope of number sense were set to the values of each time-point (i.e, t_1=beginning of Year 1, t_2=January, t_3=March, t_4=June, t_5=beginning of Year 2). It was assumed that t_1=0, indicating that the intercept of each component can be interpreted as the individual initial value and the slope as the individual rate of change over time. The hypothesized latent growth model assumed that student's early number sense followed a linear growth rate across the five time-points. Alternative models hypothesizing (a) a quadratic trend in the slope and (b) a logarithmic trend were examined. Figure 1 presents the way in which the factors of the model relate to each other.

The results of the study showed that the hypothesized linear growth model had an excellent fit to the data (χ^2/df=1.94, CFI=.99, RMSEA=.07). The fit of the two alternative models was extremely poor (χ^2/df>10, CFI<.70, RMSEA>1). Thus, it was concluded that student's early number sense followed a linear growth rate. Two latent factors were estimated for number sense, the intercept (or initial value) of the latent number sense and the slope (or growth rate). Each contains a latent growth factor mean and variance, with the covariance being allowed between them. Both the mean intercept and mean slope of number sense were significantly greater than zero. The results of the model fit and significant slope mean and variance supported the proposed growth model regarding the linear development of the parameters of number sense. The analysis showed that the mean value of student's number sense intercept was 0.20 and the mean value of the slope across the five time-points was 0.07. The $M_{N.S.}$ value of 0.07 can be interpreted as the increase in performance per unit of time (i.e., 0.07 × 4 = 0.28), which corresponds to the actual increase of the mean value of number sense from time-point 1 to time-point 5. The amounts of variation of the intercept factor and slope factor were significant larger than zero. This finding is extremely important because we could conclude that there is variability. The correlation between the intercept and the slope was positively strong ($\underline{r}$=.55, $\underline{z}$>1.96), indicating that students entering first grade with a strong number sense may exhibit greater growth rate.



* w corresponds to time wave

Figure 1: The hypothesized growth model

Tracing growth patterns

To trace categories of students that follow different growth behaviour, latent class analysis was used. To do so, we first examined whether students varied at entering grade 1. Thus, we applied a stepwise method to validate the model under the assumption that there were one, two, three, four or five categories of students at time 1 (first measurement). The best fitting model with the smallest AIC and BIC indices and the largest entropy was the one involving three categories of students. The mean value of the three categories were 0.13 (n_1=62), 0.20 (n_2=99) and 0.30 (n_3=36), respectively. Then, we repeated the same procedure for each category of students, using the data of the four remaining measurements. The results of the three latent class analysis showed that each of the three initial categories of students could be more accurately modelled by two sub-categories of students. Thus, in total we traced six groups of students that varied across the five time-points. Figure 2 presents the development rate of the six groups.

	Group 1	Group 2	Group 3	Group 4	Group 5	Group 6	Total
N.S.T_1	0.12	0.13	0.20	0.22	0.29	0.31	0.20
N.S.T_2	0.18	0.21	0.27	0.30	0.36	0.40	0.27
N.S.T_3	0.22	0.27	0.31	0.38	0.41	0.49	0.32
N.S.T_4	0.28	0.35	0.40	0.52	0.48	0.59	0.40
N.S.T_5	0.35	0.41	0.46	0.62	0.56	0.68	0.47

Table 1: Means of the six groups

Groups 1 ($n_{Gr.1}$=32) and 2 ($n_{Gr.2}$=30) represent the two subgroups of the first initial category. Groups 1 and 2 had almost the same initial value, but progressively Group 2 adopted a greater growth rate. Groups 3 ($n_{Gr.3}$=70) and 4 ($n_{Gr.4}$=21) represent students entering grade 1with moderate initial value. However, Group 4 exhibited a remarkable growth rate, overtaking after the third measurement a group that had a greater initial value (Group 5). Groups 5 ($n_{Gr.5}$=19) and 6 ($n_{Gr.6}$=17) exhibited the greatest initial value. Overall, the mean value of Group 5 increased from 0.29 to 0.56 (see Table 1), while the mean value of Group 6 increased from 0.31 to 0.68, indicating a significantly greater growth rate.

Summing up, we traced six groups of students that followed different growth patterns. Two groups with low initial value, with the one following a moderate growth rate and the other one adopting a remarkable growth rate, two groups with moderate initial value, with the first adopting a moderate growth rate and the second exhibiting an amazing progress, and two groups with high initial value, with the one adopting a constant greater increasing rate. To get a better insight of the reasons that may explain the different developmental trends of the six groups of students, we examined their growth rate in the three dimensions of number sense.

Figure 3 presents the development of the six groups of students in the three dimensions of number sense. Comparing groups 1 and 2 (with low intercept), it could be concluded that in the first four time-points the gap between the two groups in elementary arithmetic got constantly bigger and it was reduced in the last measurement (beginning of grade 2). The gap between the two groups in conventional arithmetic increased until the third measurement and then it stabilized. However, the gap between the two groups in algebraic arithmetic increased systematically, especially after the third measurement. It should be noted that the gap between the two groups with moderate initial value (Groups 3 and 4) got impressively larger every successive time-point in all number sense components. On the contrary, the gaps of the three components between the two groups with high initial number sense (Groups 5 and 6) followed three different patterns. Their gap in elementary number sense increased until the third time point and then

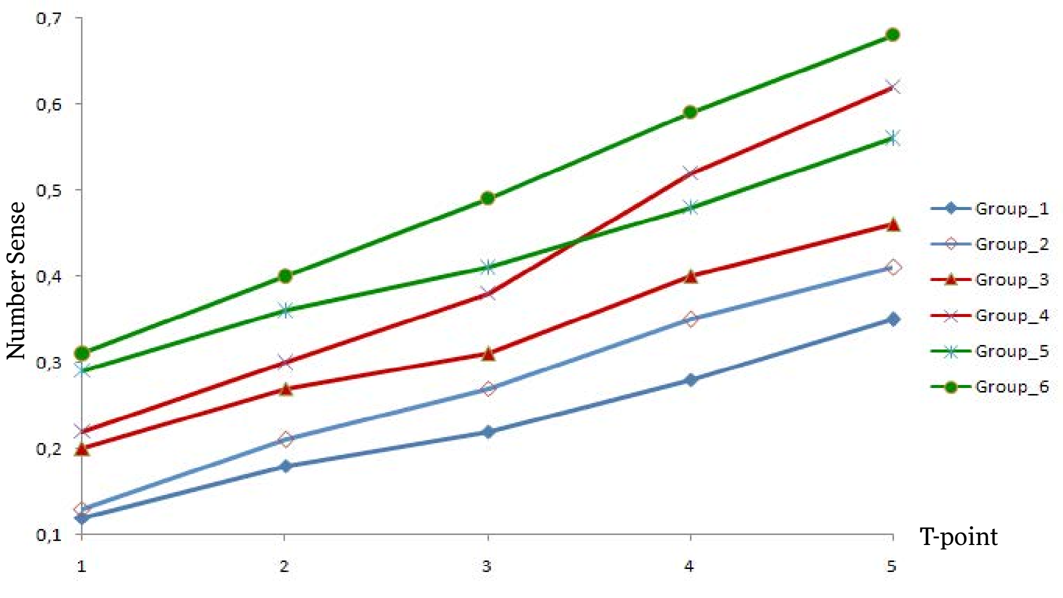


Figure 2: The development rate of the six groups

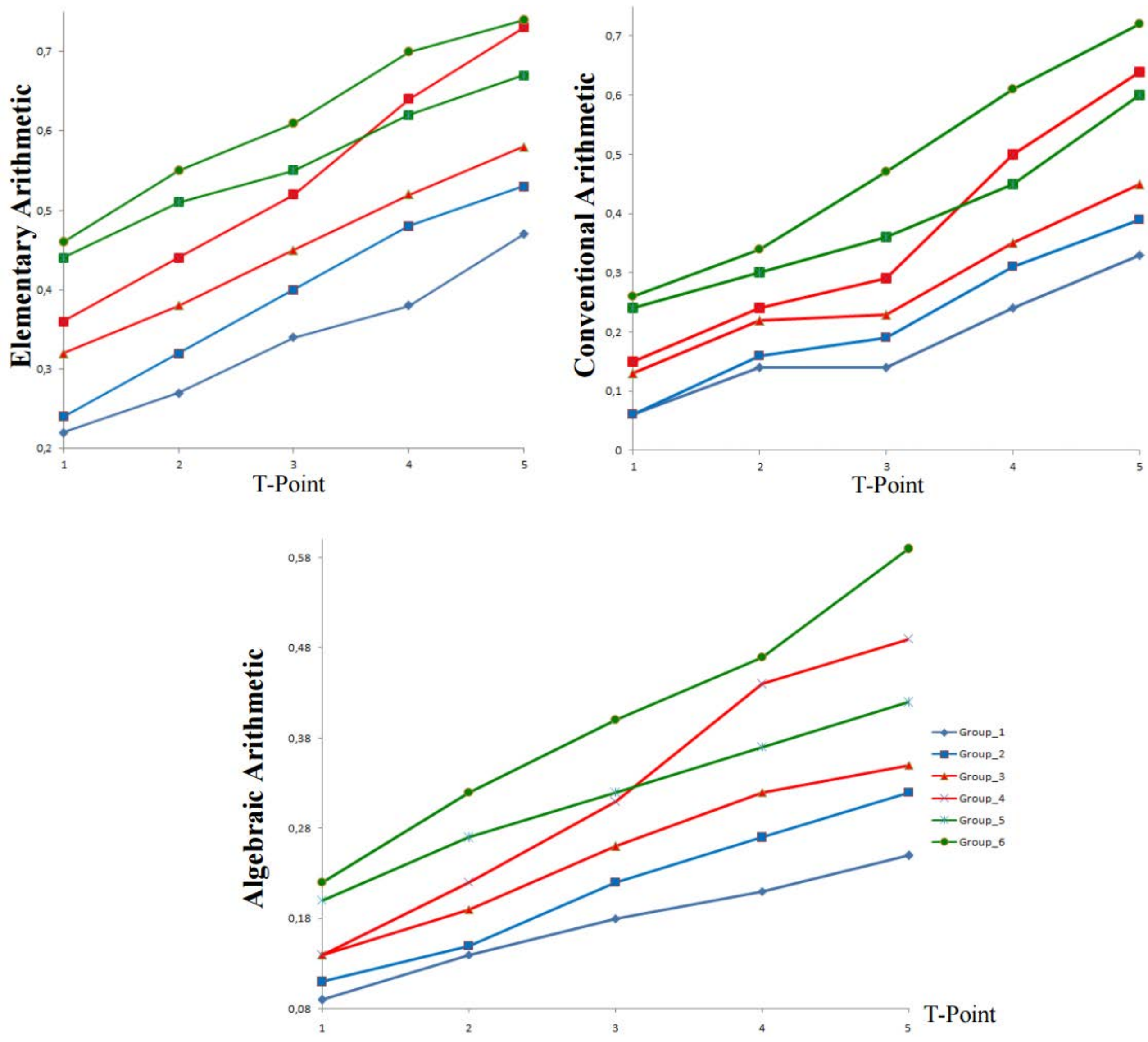


Figure 3: Development of the six groups in the dimensions of number sense

stabilized. The gap in conventional arithmetic got systematically bigger until the fourth time point and then reduced, while their gap in algebraic arithmetic progressively increased.

DISCUSSION

The present study utilized the model proposed by Pittalis, Pitta-Pantazi and Christou (2013, 2014), suggesting that students' early number sense consists of elementary, conventional and algebraic arithmetic, to trace six year old student's development in number sense. The results of the study showed that student's early number sense followed a linear growth rate from the beginning of grade 1 to the beginning of grade 2, suggesting a constant increase across the five time-points. However, the variation in the intercept and slope factors suggest that there are individual trajectories in the development of number sense. In addition, mixture growth analysis showed that six groups of students could model more accurately student's variances in the five time-points. We identified two groups of students with low initial number sense, two groups with moderate initial value and two groups with high initial value. Systematically, the one of the two groups with the same initial value adopted a greater growth rate in all the dimensions of number sense.

We might suggest that the gap between the groups with same initial value could be explained by their progress in algebraic arithmetic. This could be concluded by the fact that the gap between the groups with the same initial value progressively increased across all the five time-points only for algebraic arithmetic. Thus, we might suggest that algebraic arithmetic might be the source of large individual differences in number sense development. In addition, it might be assumed that the development of students' algebraic arithmetic might positively affect the development of the other number sense components. For instance, Group 4 every major improvement in algebraic arithmetic was followed by a corresponding huge improvement in conventional arithmetic in the following time point. These findings highlight the dynamic nature of algebraic arithmetic and underlie the significance of studying (a) more extensively the factors that may enhance or prohibit the development of algebraic arithmetic and consequently the development of student's early number sense, (b) the interaction among the growth of the three components of number sense and (c) the effect of instruction and teaching material on their growth rate.

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College students' understanding of parameters in algebra

Valentina Postelnicu[1] *and Florin Postelnicu*[2]

1 Texas A&M University-Corpus Christi, Corpus Christi, USA, Valentina.Postelnicu@tamucc.edu

2 Alhambra High School, Phoenix, USA

A study was conducted with 26 college students with the purpose of gaining insight into students' conceptual understanding of parameters in algebra. Participants contributed to a whole-class discussion, solved problems with parameters, and identified the parameters in each problem. About one third of the students had difficulty identifying parameters. Even when successful at identifying parameters, students had great difficulty solving the problems with parameters. The difficulty was even greater when the mathematical object was a family of quadratic equations. This suggests that the true difficulty lies not with identifying parameters, but with parameters in action, that is to say when solving problems with parameters.

Keywords: Parameter, algebra, college students.

INTRODUCTION

At the time Šedivý (1976) wrote "A note on the role of parameters in mathematics teaching," the New Math era in the United States was replaced by the Back-to-Basic movement. The New Math secondary algebra curriculum, characterized by a structural approach and deductive reasoning, was replaced by a collection of "basic" algorithms to solve simple equations (Kilpatrick & Izsák, 2008). Solving equations with parameters like, $x + \sqrt{x^2 - 2ax} = b$ (Šedivý, 1976), using symbolic manipulation, disappeared from the secondary algebra curriculum. During the Back-to-Basic movement, the research on parameters was very scarce (Bloedy-Vinner, 1994). According to Furinghetti and Paola (1994), in the journals *For the Learning of Mathematics*, and *Educational Studies in Mathematics* only one article with the word "parameter" in the title was published, the one written by Šedivý (1976). The secondary algebra curriculum of the standards-based era that followed the Back-to-Basic movement departed from the static equation solving and gradually introduced the dynamic functional approach and the use of graphing technologies (Kilpatrick, Mesa, & Sloane, 2007). The research on parameters remained scarce (Ursini & Trigueros, 2004) and only in the last decade gained some momentum by focusing on ways in which graphing technologies may contribute to student understanding of parameters (Abramovich & Norton, 2006; Green, 2008). The purpose of the study reported here is to revisit the students' conceptual understanding of parameters in algebra, given the curriculum shifts from the past five decades.

THE CONCEPT OF PARAMETER

There is a consensus that the concept of variable is multi-facetted and context-dependent (Kuchemann, 1978; Philipp, 1992; Schoenfeld & Arcavi, 1988; Usiskin, 1988). Variables as parameters have the role to stand for values or numbers "on which other numbers depend" (Usiskin, 1988). Parameters are "general constants" (Philipp, 1992) or "general numbers, but of second order, that is, required when generalizing first order general statements." (Ursini & Trigueros, 2004) Discriminating between parameters and other variables implies both a variable hierarchy (Bloedy-Vinner, 1994; Philipp, 1992; Šedivý, 1976), and a reification of the mathematical objects defined with the help of parameters (Sfard & Linchevski, 1994). With respect to the mathematical objects with parameters, in the secondary algebra curriculum in the United States, the students encounter mostly families of linear and quadratic equations and functions in which one particular value of the parameter generates one specific equation or function.

One example given by Philipp (1992) is conceiving of the parameter k in the family of linear functions

$C = kg$, where C is the cost per gas (in \$), k is the price of gas per gallon (in \$ per gallon), and g is the quantity of gas (in gallons). First, one has to imagine selecting the price of gas, i.e., instantiating k with posible values (e.g., \$2.49, \$2.69, \$2.99), and only then can one construct the specific mathematical object, i.e., the specific linear function that describes the quantitity of gas, g, and the cost, C, varying together with a constant rate of change, k. Thus, discriminating between the parameter k, the dependent variable C, and the independent variable g, implies both a variable hierarchy and a reification of the family of linear functions. With respect to the family of functions $C = kg$ or $C(k, g) = kg$, from the student point of view, it may be difficult to conceive of k as the literal coefficient of the variable g and instantiate it with one value at a time to obtain particular linear functions $C(g) = kg$. If we use the subscript notation for the parameter k in the family of functions $C_k(g) = kg$ as suggested by Šedivý (1976), we fulfill the need to specify explicitly which variable is considered a parameter, and which ones are considered the independent and dependent variables. As well, it may be even more difficult for students to conceive of k as a variable with a specific domain, attend to all its values simultaneously, and conceive of the whole family of linear functions $C(k, g) = kg$ as a mathematical object.

Another example used in this study (see Problem 1, in the Method chapter) is conceiving of the parameter m in the family of quadratic functions $E_m(x) = mx^2 + 2x + 3$. Within this context, we can pose the problem:

> Find the values of m, where $m \in R - \{0\}$, such that the equation $E_m(x) = 0$ has only real solutions.

Unfortunately, in our secondary algebra curriculum we rarely use the subscript notation. Therefore we formulate Problem 1 this way:

> Find the values of m, where $m \in R - \{0\}$, such that the equation $mx^2 + 2x + 3$ has only real solutions.

Assuming that a student with "symbol sense" – a "feel for symbols," "at the heart of competency in algebra", as described by Arcavi (1994), identifies the parameter m, and conceives of the quadratic equation $mx^2 + 2x + 3$, there remains the hardest part yet, requiring both "symbol sense" and knowledge of quadratic equations, linear inequalites, and intersection of sets. The student should interpret the paramenter m as a literal constant, impose the condition of positivity for the discriminant of the quadratic equation, accept the role change for the symbol m (now a variable for the linear inequality), solve the linear inequality $4 - 12m \geq 0$, and intersect the solution set with the domain of the parameter m. The final solution, $A = (-\infty, \frac{1}{3}) - \{0\}$, should have meaning for the student – every value of the parameter m from A instantiates a quadratic equation $E_m(x) = 0$ that has two real solutions. In short, the student should conceive of the family of quadratic equations $E_m(x) = 0$ ($m \in R - \{0\}$) as a mathematical object (Sfard, 1991) at the beginning of the problem, unpack the mathematical processes the mathematical object entails, and reify the new mathematical object $E_m(x) = 0$ ($m \in (-\infty, \frac{1}{3}) - \{0\}$) that satifies the condition imposed by the problem.

The literature on student understanding of parameters points to student difficulty to discriminate between parameters and other variables, reify mathematical objects (Sfard, 1991), and succefully solve problems with parameters especially when the context is unfamiliar (Bloedy-Vinner, 1994; Furinghetti & Paola, 1994; Šedivý, 1976; Ursini & Trigueros, 2004).

The theoretical framework we propose for analyzing student understanding of the concept of parameter has two levels: I) *identifying parameters*; and II) *parameters in action*.

Each level has three categories, corresponding to student actions as observed. When asked to *identify parameters*, students: i) correctly identify parameters; ii) identify "actual" parameters; or iii) identify other variables as parameters, like the independent and dependent variable in a function.

The term "actual" is borrowed from computer science and represents constants or expressions used in places where parameters might have been used in another context. For example, a student may reason this way: a, b, c are parameters in $ax^2 + ax + c = 0$, therefore "m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$."

Across the level *parameters in action*, we have three categories for student actions. When asked to solve the problem with parameters, students: i) solve the problem and check/discuss the solution given the constraints of the problem; ii) apply the right algorithm to solve the problem without considering the constraints of the problem; or iii) cannot solve the problem.

This theoretical framework is a departure from the dichotomy "algebraic-analgebraic" used by Bloedy-Vinner (1994) to analyze students' difficulties with parameters. By treating all the incorrect answers as "analgebraic" we lose access to valuable information with respect to students' difficulties and concequently to ways of overcoming those difficulties. In our framework, the two levels capture those two reifications needed to solve problems with mathematical objects defined with the help of parameters, the second level posing more difficulties to students than the first one. Morevoer, the intermediate categories capture the students' over-reliance on pseudo-empirical abstractions (Piaget, 2001). Piaget (2001) discriminates between pseudo-empirical knowledge abstracted from individual actions on objects, and reflective knowledge abstracted from coordinated actions on objects. For example, in Problem 1, when the student identifies "m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$," we may infer that the parameters are recognized as the coefficients of x^2, x, and x^0 in the mathematical object $ax^2 + ax + c = 0$. As such, the student performs individual actions on the quadratic equation $mx^2 + 2x + 3 = 0$, and therefore relies on pseudo-empirical abstractions. Reflective knowledge may be inferred if the student identifies m as the parameter after conceiving of the family of quadratic equations $mx^2 + 2x + 3 = 0$, where $m \in \mathrm{R} - \{0\}$, poses the condition of positivity of the discriminant to ensure real solutions for quadratic equation, solves the inequality that results from posing the condition of positivity, and intersects its solution with the domain of the parameter. Thus, the student performs coordinated actions on the family of quadratic equations. Moreover, after solving Problem 1, the student conceives of a new mathematical object-the family of quadratic equations with real solutions only, $mx^2 + 2x + 3 = 0$, where $m \in (-\infty, \frac{1}{3}) - \{0\}$. The coordination of actions can be interrupted at any time, for example if a student stops solving Problem 1 after substituting m, 2 and 3 the quadratic formula for solving $mx^2 + 2x + 3 = 0$. We may consider this latter case as another example of generalization via pseudo-empirical abstractions, as the student fails to link it to a new action, at a higher level (Piaget, 2001), in this case the reification of the family of quadratic equations with real solutions only.

METHOD

This study is part of an ongoing study on student understanding of parameters. We report here only the exploratory phase, used to inform our teaching intervention. Participants in this study were 26 college students, enrolled in an Introduction to Proofs class at a university in the United States. All students completed at least the Calculus I course, and at the time of the study they have just began an introduction to logical quantifiers and elementary methods of proof. There was no lesson taught on the topic of parameters, therefore students' knowledge on parameters was acquired prior to this study. To answer the question "What is a parameter?" all students were asked to come prepared to class with written examples of problems with parameters. The task was to identify the parameters, and justify the choice. A whole-class discussion on the concept of parameter was conducted, and every student attempted to answer the question "What is a parameter?" and commented on the other students' previous answers. The instructor (the first author) wrote the students' examples and comments on the whiteboard, took pictures for analysis (see Figure 1, in the Analysis chapter), and collected the students' written answers for analysis. The discussion lasted the whole class period, 50 minutes. The consensus was reached by the students, without the instructor's validation. The next day, a 30-minute questionnaire was administered to all students. Students were asked to solve four problems, and identify the parameters in those problems. We report here on only two of those problems. Problem 1 was inspired from the research literature on parameters (Šedivý, 1976), and Problem 2 from the current mathematics curriculum at the secondary level. We wanted to minimize the role of context in students' difficulties with parameters, therefore we proposed only problems with familiar contexts (e.g., linear and quadratic equations and functions) and familiar tasks (e.g., solving a quadratic equation, graphing a linear function). At the same time, we wanted to compare our students' difficulties with those reported in literature (Problem 1), and to gain insight into our students' understanding of parameters, given their exposure to graphing technologies (Problem 2):

> Problem 1. Find the values of m, where $m \in \mathrm{R} - \{0\}$, such that the equation $mx^2 + 2x + 3 = 0$ has only real solutions.

> Problem 2. Graph the function $f\colon \mathrm{R} \to \mathrm{R}$, $f(x) = kx$, where $k \in \mathrm{R}$.

Students' answers were scored, first using two rubrics ("0" for correct, "1" for incorrect, and re-scored using three rubrics ("0" for correct, "1" for partial correct, "-1" for "incorrect") by two raters, with high inter-rater agreement, measured using Cohen's k statistic (Cohen, 1960), $k = .90$ ($p < .05$). Open coding techniques and procedures described by Strauss and Corbin (1998) were used to develop the theoretical framework used for analysis. A conceptual analysis (Postelnicu & Postelnicu, 2013; Steffe & Thompson, 2000) of the data was performed, with the goal of answering the following research question: What is the students' conceptual understanding of parameters? We tried to infer students' conceptual understanding of parameters by analysing their constructions and actions. The data collected was subjected to repeated linkage processes, and systematic inferences, both inductive and deductive. We adjusted our working hypothesis. i.e., our proposed models to account for students' conceptual understanding of parameters, until the data no longer contradicted our hypothesis. The last viable hypothesis was reported as the students' conceptual understanding of parameters.

ANALYSIS AND RESULTS

Identifying parameters

During the whole-class discussion, the students seemed to agree that parameters are some sort of

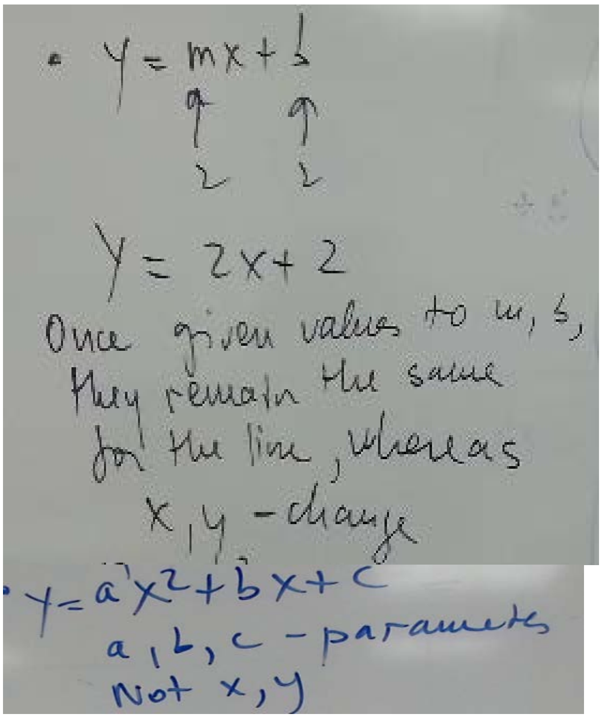

Figure 1: Identifying parameters in families of linear and quadratic functions

"general constants" (Philipp, 1992) or variables that appear in the definition of a mathematical object, but do not affect the structure of the mathematical object. When parameters change their values, the specific mathematical object changes, but keeps its structure (see Figure 1).

As can be seen from Figure 1, during the whole-class discussion, the students could discriminate between parameters and other variables, in the context of linear and quadratic functions.

There was some debate if parameters are variables or constants, illustrated by the following comments:

> "Parameter is the quantity that influences the output of a function and is usually constant. Example 1: m and b are parameters in $y = mx + b$ because they're constants in this case that affect what the function looks like. This is a linear function so it'll always look like a line."

> "[A parameter is] a constant or variable that determines the specific form of the function but not the general nature. E.g., $y = ax^2 + ax + c$ where a, b, and c are all parameters."

> "Parameters are variables that have fixed number. An example is the equation of a line $y = mx + b$, [where] m is a parameter that is equal to the slope of the line, and b is a parameter that is equal to the y-intercept of the line."

The idea of variable hierarchy (i.e., first parameters are instantiated, and only then the mathematical object becomes specific) did not appear explicitly in students' discussions.

After the whole-class discussion, our working hypothesis was that our students conceived of parameters as "general numbers of the second order" (Ursini & Trigueros, 2004), and they could identify parameters in the context of linear and quadratic functions and equations. It was not clear if our students considered the parameters variables or constants. Students' answers on the questionnaires showed that the issue of the nature of parameter – variable or constant that appeared during the whole-class discussion was more problematic (see Table 1).

	Identifies parameters	Identifies "actual" parameters	Identifies other variables as parameters
Problem 1	(N=18) m is the parameter in $mx^2 + 2x + 3 = 0$	(N=3) m, 2, and 3 are all parameters in $mx^2 + 2x + 3 = 0$	(N=5) x is the parameter in $mx^2 + 2x + 3 = 0$
Problem 2	(N=21) k is the parameter in $f(x) = kx$	(N=2) the parameter depends on what value takes k in $f(x) = kx$	(N=3) x is the parameter in $f(x) = kx$

Table 1: Identifying parameters

In Table 1, we present the frequencies of student answers per problem and category across the level *identifying parameters,* accompanied by corresponding examples of student answers. We have already commented on students' answers to Problem 1 from the mid-column, when the students rely on pseudo-empirical abstractions, and identify the parameters based on their places in the symbolic representation of the quadratic equation. With respect to students' answers to Problem 2 from the mid-column, we can infer pseudo-empirical knowledge, too. We interpret the observation that the parameter depends of the value taken by k as the students' need for instantiation. We infer that the students operate with instances of the mathematical object defined with the help of parameters, i.e., one specific linear function instantiated by a specific value of k, and not with the whole family of linear functions. Looking at the students' answers from the last column, when the other variables, x or y were identified as parameters that can affect the mathematical object, equation or function, we infer that the parameters might have been considered variables by students if they attended to the domain of the variable x (at least in the linear function in Problem 2), or might have been considered constants by students if they attended to particular instances or values of x, one at a time. In both situations, a failure to identify the parameters implies that the student operates with a particular instance of the mathematical object defined with the help of parameters. This raises the question whether, in the absence of the reification of the mathematical object, one can perform any operations on the object. The obvious answer is no, and the analysis across the second level, *parameters in action,* supports this answer.

Parameters in action

The students who could not identify the parameters in Problem 1 could not solve the problem. Most students (N=21) tried unsuccessfully to solve it by manipulating it symbolically in various ways (e.g., solving the quadratic equation by factoring, completing the square, using the quadratic formula). All students exhibited a weak competency in algebra (Arcavi, 1994), when they tried to solve the equation and write the solutions explicitly, instead of just posing the condition of positivity of the discriminant and solving the linear inequality obtained. From those 18 students who identified the parameter m in Problem 1, none solved the problem completely. Five students almost solved the problem, without completing the last necessary step – checking that the solution they found ($m < \frac{1}{3}$, sometimes $m < \frac{1}{3}$) fulfils the constraints of the problem ($m \neq 0$). In this case, when $m = 0$, we obtain a degenerated equation. Considering the students' schemes of operations (Piaget, 2001), we may infer that their reifications of the family of quadratic equations from Problem 1 are structurally weak (Sfard, 1991). Indeed, when solving Problem 1, about one third of the students failed the first reification the family of equations $mx^2 + 2x + 3 = 0$, where $m \in \mathrm{R} - \{0\}$ and there is no evidence of a successful second reification of the family of equations $mx^2 + 2x + 3 = 0$, where $m \in (-\infty, \frac{1}{3}) - \{0\}$.

Under the assumption that our students have been exposed to graphing technologies, we did not anticipate difficulties with Problem 2, which required graphing a family of linear functions. In Problem 2, only one student provided the correct graphic representation and highlighted the fact that the line $x = 0$ is not part of the solution, fifteen students graphed several lines, while all the other students graphed only one line. We can infer that the students who graphed only one or several lines conceived only of an instance or several instances of the family of linear functions, respectively. With the exception of one student, all the students failed to represent graphically the family of linear functions as the geometric locus of all the lines passing through the origin, except the line with the equation $x = 0$.

DISCUSSION

In retrospect, during the past five decades, the concept of parameter has remained an elusive one, even for college students. Our study supports previous findings with respect to student difficulty identifying parameters. We believe that this difficulty can be addressed by using subscript notation or logical quantifiers when using mathematical objects defined with the help of parameters. This means that the curriculum ought to be augmented by special topics, like connecting symbolic representations and logical quantifiers with ways of defining mathematical objects with the help of parameters. Identifying parameters is only the tip of the iceberg, the true difficulty lies with *parameters in action*. Of note, Godino, Neto, Wilhelmi, Aké, Etchegaray, & Lasa (2015) also proposed two levels of algebraic thinking involving parameters, the superior level referring to the "treatment of parameters" in problems requiring higher algebraic competency. Indeed, the real issue seems to be the weak algebraic competency (Arcavi, 1994; Sfard, 1991) inferred from the students' pseudo-empirical knowledge that hinders the reification of mathematical objects, or from the lack of active knowledge, like the knowledge about the role the discriminant of a quadratic equation in Problem 1. Our findings suggest that in spite of the use of graphing technologies, students continue to have difficulty connecting symbolic and graphic representations of mathematical objects defined with the help of parameters, and thus they have difficulty conceiving of those mathematical objects as geometric loci, like in the case of the family of linear functions (family of lines passing through the origin, except the line $x = 0$) in Problem 2. To conclude, the students' difficulty when solving problems with parameters – is the reification of the mathematical objects, reification that is dependent on the students' fluency in unpacking and packing the mathematical processes behind the mathematical objects.

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Analysing Ireland’s Algebra Problem

Mark Prendergast[1] and Paraic Treacy[2]

1 Trinity College Dublin, Dublin, Ireland, Mark.Prendergast@tcd.ie

2 University of Derby, Derby, United Kingdom

In recent years, there has been concern regarding the performance of Irish second level students in mathematics and in particular algebra. In response, the Irish Government have introduced ‘Project Maths’ which is a major reform of second level mathematics. It was introduced on a phased basis in 2008 and involves changes to what students learn, how they learn it and how it is assessed. One of the main changes is the promotion of a functions based approach to teaching algebra. This is much different from the transformational rule based approach which had dominated Irish classrooms. The new algebra strand and functions based approach was introduced in second level schools in September 2011. This paper aims to investigate the immediate effect, if any, of such an approach on student’s transformational algebraic ability. This will be done by analysing the mathematics diagnostic scores of incoming third level students at an Irish University.

Keywords: Algebra, curriculum change, teaching approaches.

THE STUDY

This paper aims to investigate the immediate effect (if any) of the functions based approach to teaching algebra that has been implemented in Irish schools instead of the transformational (rule and procedure) based approach which was previously used. The paper will do this through analysing the results of a diagnostic test taken by incoming first year students at an Irish University. There are eight algebra based questions on the diagnostic test and all of them are transformational based in nature. Hence this paper will compare student’s algebraic ability from a technical perspective in aspects such as manipulating terms and solving equations.

BACKGROUND TO THE STUDY

Irelands ‘Algebra Problem’

Algebra has long been identified as an area of difficulty in the teaching and learning of mathematics. In the introductory note to his monumental *Arithmetica*, written ca. 250 AC, Diophantus of Alexandria mentions the discouragement that students usually feel when learning what we now term ‘algebraic techniques’ to solve word-problems. Fast forwarding to 1982, Cockcroft identified algebra as a source of considerable confusion and negative attitudes among students. This was followed by Herscovics and Linchevski (1994, p. 62) who reported that many students consider algebra an unpleasant, even alienating experience and find it difficult to understand.*”How can we multiply by x when we don’t know what x is?”* (12 year old student). Furthermore Artigue and Assude (2000) posit that many students see algebra as the area where mathematics abruptly becomes a non-understandable world.

Evidence of this confusion is common in Irish classrooms. Chief Examiners’ Reports have identified algebra as an area of weakness over the past number of years. According to these reports, Irish student performance in algebra has shown little or no progress in the last fifteen years. In the 1999 Junior Certificate (JC) Higher Level paper, there were two questions based primarily on algebra, while other parts of questions also involved algebra. The long questions on algebra were both low scoring and unpopular choices. Question 3, yielding an average mark of 24.3 out of 50, was the lowest scoring on the paper thus reflecting the extent of candidates’ difficulties with algebra. Furthermore, candidates often ignored parts of other questions which involved the topic. In the 2003 JC Higher Level paper, the Chief Examiner Report (2003) concluded that while there was some improvement in relation to algebraic skills, further improvement was still needed. Questions 3 and 4 relating to algebra demonstrated that the algebraic skills of candidates need to be enhanced so that they can handle with ease topics such as manipulation of formulae, quadratic equations, solving inequalities and setting up equations. Again the lowest scoring question on the paper (Question 4, yielding an average mark of 29.1 out of 50), was based on algebra (Chief Examiners Report, 2003). This was also the case in the most recent Chief Examiners Report (2006). On this occasion Question 4 yielded an average mark of 27.1 out of 50. This report noted that improvements were required in areas such as simplifying and removing brackets from algebraic expressions, particularly expressions containing minuses and also simplifying algebraic fractions. Hence on the evidence of these reports it is clear that although algebra has long enjoyed a place of distinction in the mathematics curriculum, many students have difficulty in understanding and applying even its most basic concepts. *"Algebra means hours of instruction that you don't even come close to understanding"* (seventh-grade student as cited in House, 1988, p. 1).

Reasons for such poor performance in algebra

In Ireland, research has suggested that there has been an over reliance on traditional methods when teaching algebra. Transformational based activities have dominated lessons. Algebra was a paper and pencil activity involving the following of rules and procedures. A minimalist approach to algebraic sense making took place. Each day of instruction was textbook led and focused on a particular type of manipulation. For example, the textbook started by introducing the concept of a variable, followed by the notion of algebraic expressions and then equations (Kieran, 1992). This structure ensured that algebra was considered as a series of skills to be mastered (Chazan, 1996). Success in the subject was determined by the ability to memorise procedures by rote, nothing else (Bracey, 1992).

Research has found that many mathematics teachers felt there was no alternative to teaching mathematics through the traditional chalk and talk or the common method of following sections through a textbook (Lyons et al., 2003). Using such a method with algebra forced students to memorise procedures and solve artificial problems that had no meaning to their lives. They were drilled on the possession of mathematical rules and manipulations and they were graded not on their understanding of the mathematical concepts, but on producing the right symbol series. As a result what students learned was a collection of procedures and skills to be performed, having no logical coherence, very little connection with previously learned arithmetic, and no applications in other school subjects or in the outside world (MacGregor, 2004). Although such procedures and skills are important outcomes of learning algebra, what students need even more is a sound understanding of algebraic concepts and the ability to use knowledge in new and often unexpected ways. Students need to be given the opportunity to construct their own mathematical knowledge along with understanding its importance and usefulness in every day applications.

The 'Solution'

As a result of Irish students' poor performance in mathematics and indeed algebra, the Irish Government have introduced 'Project Maths' [1]. This is an ambitious reform of Irish second level education [2] which has an overall aim to teach mathematics in a way which leads to real understanding (Department of Education and Skills (DES), 2010). It involves changes to what students learn in mathematics, how they learn it and how they will be assessed. The initiative is designed to ensure an appropriate balance between understanding mathematical theory and concepts and developing practical applications skills. There is a much greater emphasis placed on student understanding of mathematical concepts, with increased use of contexts and applications. The focus is on students understanding the concepts involved, building from the concrete to the abstract, from the informal to the formal and learning to apply their knowledge in familiar and unfamiliar contexts (DES, 2010). Changes have being phased in over a number of years covering 5 strands of mathematics (Number, Algebra, Statistics and Probability, Geometry and Trigonometry, Functions), with assessment in the examinations being adapted as each strand of mathematics comes on stream. The assessment reflects the different emphasis on problem solving and applications in the teaching and learning of the subject.

Changes to the teaching, learning and assessment of algebra

Although the initiative first began in 24 pilot schools in 2008, it was not rolled out nationally until 2010 and the new algebra strand was not introduced to all schools until September 2011. In a major shift from the transformational based approach which had dominated Irish classrooms, the new strand advocates a functions based approach to teaching algebra. The functions based approached envisages that students may be able to work with variables and the rules of arithmetic and learn to use algebraic notation and techniques themselves. There is an opportunity for students to see algebraic notation arising as a natural and useful consequence of expressing generality (Pegg & Redden, 1990).

Through Project Maths, the new approach reflects inquiry methods through which students take responsibility when dealing with new problems rather than rehearsing known procedures. Students examine functions derived from some kind of context, e.g., familiar everyday situations, imaginary contexts or arrangements of tiles or blocks. They express generalisations mathematically using algebraic symbolism, interpret expressions as rules for functions and use the Cartesian plane as a space to display and consider a variety of meanings of the results (Chazan & Yerushalmy, 2003). Therefore students represent the problem using words, numbers, symbols, tables and graphs. This approach builds on the learner's prior knowledge and allows them to see different representations which should enable a deeper understanding of the topic.

The change in teaching approach to algebra has also led to a change in the assessment approach for the strand. The Project Maths assessment reflects the changes in emphasis of a functions based approach in which students are required to take an everyday problem, solve it mathematically using tables, functions and graphs and then interpret their results in the context of the problem.

Success of Project Maths and the Functions Based Approach?

The phased implementation of Project Maths means that 2014 was the first year in which all strands of the revised syllabi were examined. Indeed it will be 2017 before a first cohort of students who have experienced all 5 strands of Project Maths from 1st to 6th year [3] will be examined. Thus it is very early to make any conclusions regarding the successes / failures of the initiative. However, an interim report commissioned by the NCCA and conducted by the National Foundation for Educational Research (NFER, UK) has been published and includes findings on students' attitudes and achievements. Overall this report found that there is emerging evidence of the positive impacts of Project Maths on students' experiences of, and attitudes towards, mathematics (Jeffes et al., 2013). Furthermore, students' are achieving more at individual strand level, and in some instances students appear to be successfully drawing together their knowledge across different mathematics topics (Jeffes et al., 2013). This suggests that students are beginning to acquire a deeper understanding of mathematics and how it can be applied (Jeffes et al., 2013).

With a specific reference to algebra, the revised strand was first examined at LC level in June 2012. The NFER report found that out of the five strands, algebra is in the lowest two strands, both in terms of student confidence and student achievement (Jeffes et al., 2013). Furthermore although students' performance in other strands is similar to international students who participated in TIMSS 2007, students appear to find algebra especially difficult when compared to international standards (Jeffes et al., 2013). However on a positive note for the functions based approach, the report found that students feel more motivated to learn algebra when it is taught in a way that makes it seem more relevant to everyday life and when they can see that it interlinks with other mathematics topics.

METHODOLOGY

The methodology of this study involves comparing student's results on a number of transformational based algebra questions from a university diagnostic test. The results will be compared between two cohorts, 2011 in which students were taught using traditional methods, and 2013 in which students were taught using a functions based approach.

The instrument

The diagnostic test from which the data is compared was designed and implemented in the University of Limerick (UL) in 1997. It was designed to help identify students who may be at risk of failing service mathematics examinations (O'Donoghue, 1999; Gill, 2006). Within the design process, a team of experienced service mathematics lecturers analysed and adjusted an initial list of 70 questions, reducing this to the final 40 question version. Thirty four of these questions are set at a LC Ordinary Level standard or below, with the other six questions set at a LC Higher Level standard. To ensure the validity of this test, it was then piloted in Irish second level schools and compared with the SEFI Core Level Zero syllabus for engineers, the Irish

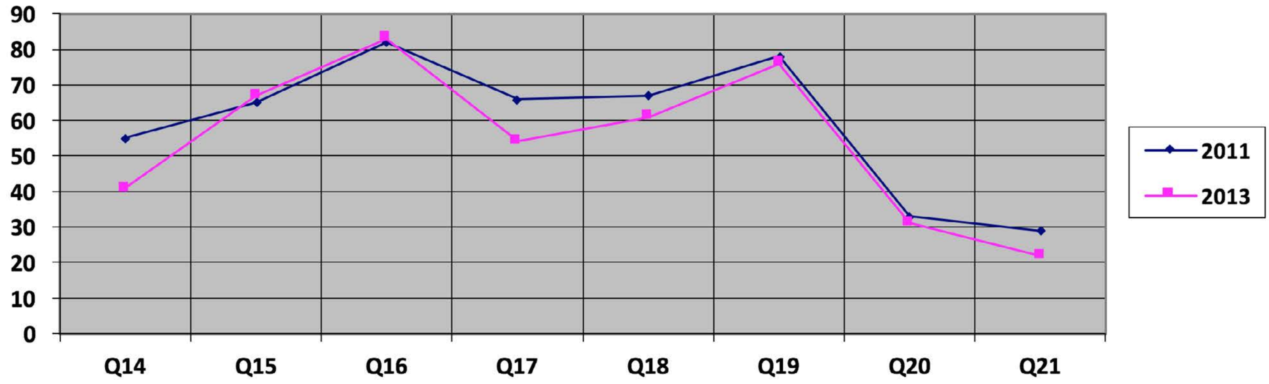

Figure 1: Student results in eight algebra questions

JC mathematics syllabus, the Irish LC mathematics syllabus and further diagnostic tests (Gill et al., 2010).

The test was intended specifically to suit the mathematical level of the students in each of the targeted service mathematics modules in UL, namely Science Mathematics 1 and Technological Mathematics 1. Since 1998 students from these cohorts have been assigned the 40 question diagnostic test which assesses basic skills and procedures in various mathematical topics. These topics include arithmetic (13 questions), algebra (8 questions), geometry (4 questions), trigonometry (3 questions), co-ordinate geometry (4 questions), complex numbers (2 questions), differentiation (3 questions), integration (2 questions), and modelling (1 question). The test is presented to students in their first mathematics lecture of each year since and they are not informed beforehand that they are required to take it.

The layout of the test and each of the 40 questions have remained unchanged over the years to ensure reliability. The UL database which contains data from these diagnostic tests dating from 1998 to 2013 currently holds information on over 10,100 students.

THE STUDY

This paper will focus on the results of students from the 8 algebra questions (see appendix 1) in the year 2011 in comparison to the results from 2013. It is important to note that the algebra questions included in the test are transformational based in nature and hence do not reflect a student's overall algebraic ability. The years 2011 and 2013 were selected as the algebra strand of Project Maths was first phased into Irish schools in September 2011. Hence the students who took the 2011 UL diagnostic test would have been taught algebra using the transformational based approach. The students who took the 2013 UL diagnostics test would have been taught algebra using the functions based approach. 685 students took the diagnostic test in 2011 and 645 took it in 2013. Analysis of this data will aid the authors is answering the following research question:

Is there a difference in the transformational algebraic ability of incoming third level students who have been taught using different approaches?

RESULTS

Descriptive analysis of the data found that there was a statistically significant difference ($t=4.463$, $p=.000$) between the mean algebra scores of the students in 2011 (M: 59.01; SD: 24.07) compared to the mean scores in 2013 (M: 53.16; SD: 23.74). In order to get a more in-depth analysis of this finding, the results of each of the eight algebra questions from both years were compared (see Figure 1).

Students who had been taught using the transformational based approach scored higher in six out of the eight algebra questions. There was a statistically significant difference in the mean scores of students in four of the eight algebra questions (Q14, 17, 18, 21) in 2011 when compared to 2013 (see Table 1).

A closer inspection of the four questions (See Appendix 1 Q14, 17, 18, 21) in which there were statistically significant differences between the cohorts does not reveal any correlation between the questions. Question 14 involves the rearrangement of formula, Question 17 and 18 involve solving equations (quadratic and simultaneous respectively) and Question 21 concerns the subtraction of algebraic fractions. Similarly, no correlation appears to exist between

Question	Mean and SD 2011	Mean and SD 2013	Independent T-test
14	.54 (.499)	.40 (.490)	t=5.390, p =.000
15	.65 (.478)	.66 (.475)	t=-3.52, p =.725
16	.82 (.383)	.82 (.386)	t=0.156, p =.876
17	.66 (.476)	.52 (.500)	t=5.087, p =.000
18	.66 (.474)	.59 (.492)	t=2.551, p =.011
19	.77 (.420)	.75 (.436)	t=1.131, p =.258
20	.33 (.471)	.30 (.460)	t=1.138, p =.255
21	.29 (.453)	.22 (.414)	t=2.895, p =.004

Table 1: Mean Score, Standard Deviation and Independent T-tests

the questions in which there were no statistically significant differences (See Appendix 1 – Q15, 16, 19, 20). Question 15 involves substitution, Question 16 involves solving a linear equation, Question 19 is the expansion of brackets while Question 20 concerns solving inequalities. Each of the eight questions are JC Higher Level / LC Ordinary Level standard and would have been on the traditional syllabus, in addition to being on the current Project maths syllabus.

DISCUSSION AND CONLCUSION

Despite the perceived success of Project Maths and the functions based approach to teaching algebra (Jeffes et al., 2013), the analysis of the UL diagnostic test results shows that there is statistically significant differences in the 2013 mean algebra scores of students when compared to 2011. Analysis of the LC grades shows that both cohorts had very similar grades in mathematics upon completing second level. The mean number of LC points achieved by these students through mathematics in 2011 was 54.27 while, in 2013 this cohort produced a mean number of 53.64 points. However the performance of the 2013 cohort in the algebra section of the diagnostic test is significantly lower than the 2011 cohort. This would suggest that the introduction of the new algebra strand in Project Maths and the change in teaching approach has had a detrimental effect on student's transformational algebraic ability.

The findings of the study are hardly surprising given that diagnostic test used to gather the data focuses solely on transformational activities and the emphasis on such activities has been diminished under the new curriculum. However the eight algebra questions contained in the test (See Appendix 1) are of a very basic standard and all Irish students would have encountered such concepts at Junior Cycle / Senior Cycle level in both syllabuses. Thus the results of this study, in this regard are surprising. Whatever the approach to teaching algebra at second level, it is expected that students entering third level on degree programmes with a mathematics component, should be competent in rearranging basic formula, expanding brackets, substitution and solving basic equations. The authors argue that such skills should be acquired by second level students regardless of whether they are being taught algebra using a transformational, a functions based or indeed any other approach.

Thus the challenge for Irish mathematics educators on the basis of this study is to ensure that the transition to the functions based approach does not result in a neglect of teaching the technique and rule-bound aspects of the algebraic language (Prendergast & O'Donoghue, 2014). While the transition to the functions based activities of Project Maths is a welcome move, it is also important that the algebraic purpose behind such activities is not lost. This results with students not knowing how to move from their informal constructions to a formal and algebraic relationship (Stacey & MacGregor, 1997).This was evidenced in the UK where the search for meaning and the consequent suppression of symbolism led to a situation in the early 1990s where students were doing hardly any symbol manipulation (Sutherland, 1997). Problem solving by whatever means had all but replaced algebra (Kieran, 2004). The hope was that, in focusing on algebraic understanding, the techniques would take care of themselves. However, a study carried out by Artigue in France in the mid 1990's on the use of DERIVE in French classrooms found that the techniques did not take care of themselves (Kieran, 2004). As anticipated, the researchers found that the teachers were emphasising the conceptual elements while neglecting the role of the procedural work in algebra learning. However, this emphasis on conceptual work was producing neither a clear understanding of the procedural aspects, nor a definite enhancement of students' conceptual understanding, "easier calculation did not automatically enhance students reflections and understanding" (Lagrange, 2003, as cited in Kieran, 2004, p. 28). Thus traditional exposition and practice must be retained alongside more opportunities for practical work, problem solving, investigations and discussion and providing purpose to the activities (Sutherland, 1997). The work of 'Project Maths' in Ireland must facilitate the curriculum and teachers in making such an evolution. While the new curriculum is having many positive benefits it is important that techniques and conceptual understanding are taught together rather than in opposition.

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ENDNOTES

1. 'Project Maths' is the name under which the reform of the Irish mathematics curriculum has been implemented.

2. There are three levels to the Irish education system – primary level, second level and third level. Second level students are typically aged 12–18 years.

3. 1st to 6th year – Second level education is typically of 6 years duration.

APPENDIX 1

Algebra Q.14 - Q.21

14. Solve for $h : V = \pi r^2 h$

Ans ____________________ Don't know ☐

15. Evaluate $ab + 2bc - 3ac$ when $a = 3$, $b = -2$ and $c = 4$

Ans ____________________ Don't know ☐

16. Solve the equation: $3(x + 2) - 24 = 0$

Ans ____________________ Don't know ☐

17. Solve for x: $x^2 + x - 6 = 0$

Ans ____________________ Don't know ☐

18. Solve the set of equations:

$$\begin{aligned} 2x + y &= 7 \\ x + 2y &= 5 \end{aligned}$$

Ans Don't know ☐

19. Write out $(x + 3y)(a - 2b)$ in an equivalent form without brackets

Ans ____________________ Don't know ☐

20. Solve for x: $3 - 6x < 21$

Ans ____________________ Don't know ☐

21. Simplify $\dfrac{1}{x-1} - \dfrac{2}{x+1}$

Ans ____________________ Don't know ☐

How much space for communication is there for a low achieving student in a heterogeneous group?

Anita Movik Simensen, Anne Berit Fuglestad and Pauline Vos

University of Agder, Kristiansand, Norway, anita.m.simensen@uia.no

This paper reports on a case study aiming to deepen our understanding of low achieving students' learning of algebra, in particular when they work with pattern problems. We observed one low achieving student, May, who participated and worked in three different heterogeneous settings. Data were analysed from a multimodal perspective on key and regulating activities in the groups. The analysis revealed that May's contribution varied, depending on the composition of the groups, and that her contributions were influenced by regulating activities by peers and access to physical artefacts. The findings show that a low achieving student is able to generalise beyond her arithmetic knowledge, but the environment in a heterogeneous group did not offer her the space to do so.

Keywords: Regulating activities, low achieving, pattern problems, heterogeneous groups.

INTRODUCTION

Pattern problems have been studied extensively in classroom situations, particularly in group work (Radford, 2009; Ryve, Larsson, & Nilsson, 2013). Pattern problems can be tackled using different strategies, including arithmetic strategies (counting) and grouping strategies, before advancing to generalised explicit formulae (Lannin, Barker, & Townsend, 2006). Radford (2012) has explained that generalising is central in algebraic thinking, not symbolising. With pattern problems, students can reason informally, yet algebraically, they can even generalise their answers, without immediate need to symbolise.

To supplement the studies of students working on pattern problems, we are interested in how to facilitate low achieving students' learning of algebra within heterogeneous groups. We define low achieving students as students following regular mathematics lessons, but achieving low on tests compared to their peers. Many studies have demonstrated that these students are able to reason algebraically, but their potential is often not captured by standard tests (Karsenty, Arcavi, & Hadas, 2007; Watson, 2002).

The rationale for our study of low achieving students within heterogeneous groups is that in Norway, the official policy is an inclusive school system: it is not allowed to teach in fixed ability groups over time (The Educational Act, 1998). Research on learning in group settings has identified that heterogeneous groups are beneficial to students' learning and invite all students to become active participants (Dekker & Elshout-Mohr, 1998; Webb, 1991).

In the current paper we present a case study of May, who is considered as a low achieving student by her teacher and based on test results. We observed her work with pattern problems in three different groups.

THEORETICAL FRAMEWORK

Algebra learning is defined to involve both symbolic expressions and reasoning about generalisation (Caspi & Sfard, 2011; Måsøval, 2011). In our study, we focus on the algebraic reasoning and generalising, and to a lesser extent on symbolising. Strømskag (2015) emphasises the importance of natural language as a basis for symbolic expressions which might be even more crucial for low achieving students. According to Karsenty and colleagues (2007), low achieving students do have a potential to think algebraically, yet they are not always capable of working with symbols. Therefore, low achieving students' algebraic potential might be more visible in natural language than in symbolic expressions.

A relation between students' communication and algebraic thinking is emphasised by Caspi and Sfard (2011) when they define algebra as a discourse. This definition is promising for researching low achieving students' algebra learning because it "transfers algebra from the category of passive tools to that of human activities" (p. 470). In their study, students' communication is analysed according to verbal interaction and written products, not focusing on gestures. However, research on low achieving students has identified gestures to be crucial for students' active participation in meaning making processes (Simensen, Fuglestad, & Vos, 2014).

Radford (2009) demonstrates how students' written products from generalisation processes can be traced back to their use of speech, gestures and actions (such as using physical artefacts) during the solution process. He also argues that gestures in isolation do not tell much; it is when all three actions work together that observing communication can give insight into students' meaning making. Radford refers to activities where students combine speech, gestures, and actions as *multimodal activities* (2009, p. 120). Multimodal activities are not only local products of the situation in which they take place; they have roots in previous situations and contribute to future communication (Civil & Planas, 2004). Therefore, facilitating all students to become active participants is not only about an individual's empowerment to become an active contributor by offering appropriate tasks and physical artefacts; it is also about previous communication, and mutual encouragement of communication.

Dekker and Elshout-Mohr (1998) have defined two categories that can be used to analyse multimodal activities: *regulating activities* and *key activities*. Regulating activities are activities that encourage students to communicate about their work, namely by questioning or critisising it, for instance: *"What are you doing? Why are you doing that?"* (Pijls, Dekker, & van Hout-Wolters, 2007, p. 312). Regulating activities can be practiced by other students, the teacher, or by the student herself.

Regulating activities do not only encourage communication, they can also be used to define students' *key activities*. Four key activities have been identified as indicators for level raising in mathematics (Dekker & Elshout-Mohr, 1998) and they can be observed in students' communication. The four key activities are: showing one's work; explaining one's work; justifying one's work; and, reconstructing one's work.

Pijls and colleagues (2007) have carefully demonstrated how to classify utterances according to regulating and key activities. They emphasise particularly the key activities' dependence on regulating activities, because the regulating activities are crucial to categorise key activities. For instance, Pijls and colleagues (2007) claim that it can be difficult to decide whether a student is showing or justifying her work. The decision should then be based on the regulating activity that initiated the actual key activity. For example, justifying one's work is always a response to critique, while showing is not (p. 313). For further explanations of regulating and key activities, we refer to work by Dekker and Elshout-Mohr (1998) and Pijls and colleagues (2007). Heterogeneous groups in which all four key activities were identified showed to be the most beneficial for attaining mathematical level raising (Pijls et al., 2007). Nevertheless, students who demonstrate fewer key activities, like showing and justifying, can attain a certain degree of level raising. We build on these studies, but extend the analysis beyond utterances, and also analyse other multimodal activities.

Despite the large amount of research concerning algebraic thinking and communication, not much of it relates to low achieving students' communication and their algebraic learning potential. For this reason, we want to investigate: *How do regulating activities encourage or hinder a low achieving student's communication about pattern problems in heterogeneous groups?*

METHODS

To answer our research question, we designed tasks on hexagon patterns in such a way that the questions should be easy to understand and such that they could be solved by multiple methods. Motivated by Radford's (2009) claim that students' use of gestures and artefacts is crucial for communication on pattern problems, we made hexagon tiles and sheets with hexagon patterns available for the students. We assumed that these factors (questions that were easy to understand, multiple methods for solving, availability of physical artefacts) would invite all students to become

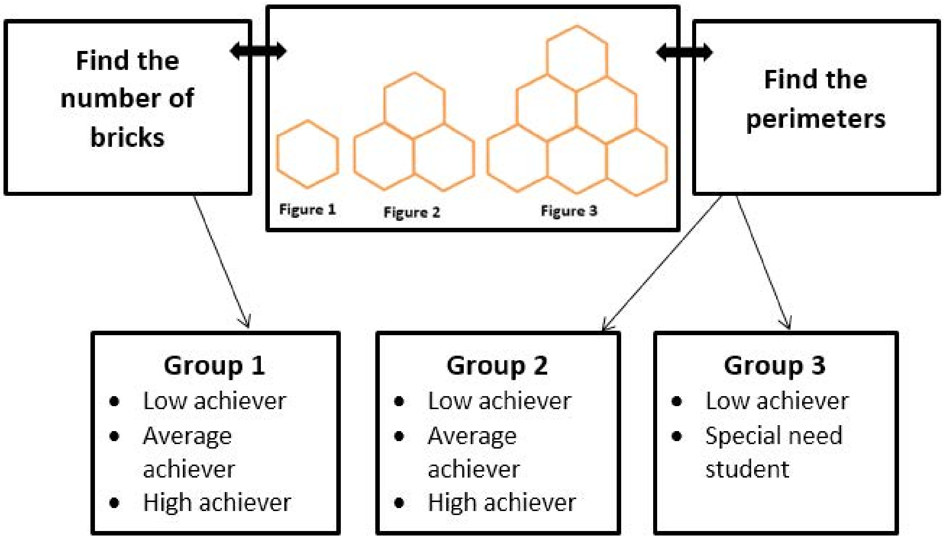

Figure 1: Tasks and organisation of heterogeneous groups

active participants. Based on these problems, we carried out a task-based intervention on three different days during one week in a Grade 8 class.

For this case study, we will focus on one student, May, who performed low on a National Test for Grade 8. Before the intervention, we had communicated with the teacher, and created a group in which we assumed that May would become an active participant: May (low achieving), Tom (average achieving) and Bo (high achieving). But Bo was sick when our intervention started and we had to reorganise the groups, replacing Bo with Eva (high achieving). After that first session, Bo was back and May could work with Bo and Tom in the ensuing session. This accidental change in our study enabled us to observe May in two heterogeneous settings, which turned out to be very different. Because of the different outcomes, we became interested in learning more about what encouraged or hindered May's communication in heterogeneous settings. Therefore, we carried out an additional, third session in which we asked May to work together with Siv. Siv is not only May's friend, but she has special educational needs as well. We assumed that these two factors, working with a friend and being the most knowledgeable, might empower May to take an active role in the work. A third choice we made to strengthen May's role as the most knowledgeable peer, was to give May and Siv a task with which May was familiar from the work in Group 2 (see Figure 1).

All the sessions were video recorded and transcribed. We then counted the number of utterances made by May in each group, and this gave us a quantification of her participation in the communication between the group members. However, counting the number of utterances does not give insight into the nature of utterances nor the chain of activities that the utterances are part of. We therefore analysed the communication according to key and regulating activities, as developed by Dekker and Elshout-Mohr (1998). In this analysis, the four key activities can be identified independently, one activity is not dependent on others, and they are incremental in the sense that we can see attempts to carry out key activities, partly or strongly. Because of this, we consider key and regulating activities as suitable to identify both students' actual level raising and their potential level raising. In previous studies (Dekker & Elshout-Mohr, 1998; Pijls et al., 2007), key activities and regulating activities are only observed in students' speech, and not in their gestures and actions. However, we decided to analyse communication according to key and regulating activities from a multimodal perspective, following Radford (2009).

RESULTS

We carried out a frequency count of May's utterances in the three groups. In the first group she hardly said a word (3 % of the group's utterances are from May), in the second group her contribution raised to 16% of the groups' utterances, while in the third group she contributed with 53 % of the utterances. Below we will report exemplar episodes from the three groups.

Group 1

On the first day of the intervention, May sat with Tom and Eva. They discussed pattern problems on finding

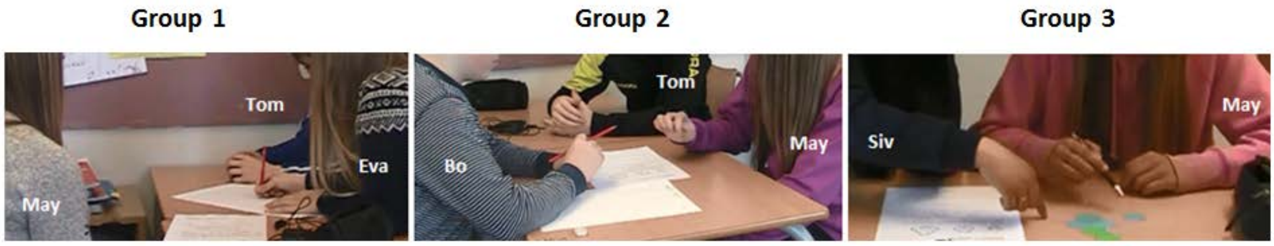

Figure 2: Organisation of the three groups

the number of hexagon tiles required to build various figures. In their discussions, Tom and Eva agreed about how to find this number, reasoning from a recursive approach, namely by adding the next figure number to the number of tiles in the present figure. May said little and her utterances were in general short, consisting of one or two words.

Tom asked May several times if she understood how he and Eva had found the number of hexagon tiles in the figures without counting. Every time May nodded, but except for the nodding, she did not move her arms or body. As can be observed in Figure 2 she was sitting further away from the other group participants, as if she was not part of the group (see Figure 2). There were a few moments, when May's nodding was followed by Tom asking if she really understood. This could indicate that he did not accept May's nodding as sufficient confirmation that she understood.

Eventually, Tom asked May to explain it to him:

122 Tom: Ok, explain it for me. I need an explanation. [Looks at May and moves the hexagons in her direction]
123 Eva: You know the explanation. [Looks at Tom and gathers the hexagons]
124 Tom: Yes, but I want her to understand. [Points to May]
125 Eva: To get the next figure number, you have to add the next figure number. [Looks at Tom]

In utterance122 Tom extended his attempt to invite May to contribute. Here he asked her explicitly to explain her understanding and he offered her artefacts (hexagon tiles) that could have supported her explanation. However, the invitation was not answered by May but by Eva, both verbally and physically (123). May did not give any explanation.

We interpret Tom's question about May's understanding to be a regulating activity, aiming at inviting May to contribute. However, we do not interpret the nodding to be a key activity, because we cannot learn about May's ideas from the nodding. In utterance 122, Tom makes a verbal regulating activity, and he strengthens it by physically offering the hexagon tiles to May. Eva's subsequent contribution (123) can be interpreted as a barrier to May's acceptance of the invitation. Eva is hindering at two levels: by taking the hexagons physically and by verbally answering. We therefore consider both contributions from Tom as examples of regulating activities, asking May to explain. However, they do not result in May using key activities.

Group 2

On the second day of the intervention the groups were reorganised and May's group was joined by Bo (high achieving) instead of Eva. Therefore, Group 2 consisted of May, Tom, and Bo. Their work was still related to the hexagon patterns, but now the question was to find the perimeter of the figures.

The discussion started similarly to what happened on the previous day, with Bo asking May if she understood. She responded in exactly the same way as she did in Group 1, by nodding. However, Bo did not repeat the question like Tom did in Group 1. Bo rephrased it into "May, do you want to count how many sides the second figure has?", at the same time as he pointed at the illustrations on the worksheet. Later on Bo initiated an invitation to May to give an explanation:

691 Bo: Why is it multiplied by 6?
692 May: Because it is 6 more each time.
...
722 May: You take the number of sides on one brick.
[Teacher nods]
723 May: And multiply by the figure number.

...

738 Bo: Instead of writing it in words, can you write it like you would have written it in your notebook? What do you write if you want to do a calculation?

739 May: N multiplied by 6.

The excerpt shows how Bo acted like a teacher, guiding May to make explanations without using artefacts. In utterance 691 he asked May to justify *why* the perimeters can be found by multiplying by six. In utterance 692 May indicates that this is about the multiplication table of six continuing by adding 6. Further, in utterances 722 and 723 May explained in words how to find the number of tiles in a random figure, and in utterance 739 she showed how it can be generalised with symbols.

We interpret Bo's explicit questions at the beginning of the discussion in Group 2 as regulating activities, inviting May to contribute with speech, gestures, and use of artefacts. May shows her ideas by pointing at the physical artefacts and we interpret it as key activities demonstrated with multimodal expressions. In the episodes given above, we identified key activities such as: justify (692), explain (722 and 723), and show (739) one's ideas.

To sum up, in the heterogeneous setting of Group 2 we first observed how Bo asked May to explain her understanding, which did not evoke May's use of key activities. When he asked her explicitly to show her ideas ("do you want to count?"), this was followed by May demonstrating the key activities of showing her ideas. Finally, when Bo asked May again to explain her ideas (691), she demonstrated the key activity explanation by offering the generalised formula "N multiplied by 6".

Group 3

The third setting in which May participated was different from the previous ones in several ways: (1) only two students were involved, Siv and May; (2) Siv is identified to be in need of special education and, therefore, May was the relatively high achieving student in this setting; (3) the students were given the task from Group 2, so May was knowledgeable about the task, while Siv was not.

First May demonstrated that the perimeter could be found by counting the physical sides one by one. We interpret this as the key activity 'showing one's idea'. May then carried out a regulating activity by asking Siv to count the sides. Thereafter May initiated other strategies to find the perimeters:

51 May: Do you know what we can do instead of counting one by one?

52 Siv: Hm?

53 May: We can, or it is fine to just count. We only have to add these. 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37. [Points on the "new" sides in Figure 5] Then we know that it is 37 sides around. Do you see a pattern?

54 Siv: Where?

55 May: Here. [Points on the diagram in the task]

56 Siv: No.

57 May: It is the times six table. 6, 12, 18, 23, 37. For example: 6 plus 6 equals 12. It is 6 twice. If we have 6 three times, then we have 18. 12, 13, 14, 15, 16, 17, 18. [Counts on her fingers]

In the above episode (53) May suggested that the perimeter can be found by counting on from the previous figure's perimeter, which is a recursive strategy. Thereafter, she claimed that the perimeters are equal to numbers in the "times six table" (57), which is an explicit formula for the perimeter of the triangular hexagon pattern.

We interpret the utterance "Do you know what we can do instead of counting..." (51) as a regulating activity. This regulating activity is followed by May showing her idea (53). Within that very same utterance (53), we also observe a regulating activity: asking for a pattern. Finally, when May showed her idea (57), she demonstrated another key activity. In utterances 53 and 57 she scrutinises the counting process, which we interpret as searching for generalisation. She did not detect that the numbers she found were incorrect, which illustrates how her willingness and potential to generalise goes beyond her skills in mathematics.

May's contributions in Group 3 differed from the other groups, because her contributions did not depend much on others' regulating activities directed at her. May's contributions in Group 3 demonstrated both regulating activities and key activities. In Group 3 we observe that May's contributions relate little to the other participant (Siv gives little feedback), and more to the nature of the tasks, use of gestures, and access to physical artefacts.

DISCUSSION AND CONCLUSION

The empirical material presented in this paper shows how differently a low achieving student can contribute in various heterogeneous settings. When working with higher achieving students, May's communication was strongly affected by the regulating activities from the other students. She was mostly awaiting and only answering when others invited her to contribute. In contrast, when working with a special needs student, May was the leader; she was the one initiating and regulating the communication. Our findings are consistent with work by Civil and Planas (2004), and it indicates that being categorised as a low achieving student might position a student into a similar position as students who are marginalised because of social inequality. Therefore, we assume that regulating activities are crucial for the lowest achieving student's contribution in heterogeneous groups.

Although May's contributions in both Group 1 and Group 2 were strongly related to the regulating activities by the other students, her contributions in these two groups were different. In the first group we observed how regulating activities can fail to encourage low achieving students' contribution. Tom asked May several times about her ideas but May just nodded and did not answer in words. Pijls and colleagues (2007) reported similar observations and claim that two main factors are crucial for students' use of key activities. First, the use of key activities depends on whether someone asks you about your work. Second, students should think their contribution makes sense to other students. In our study we observed that there was no communication space for May's response because the regulating activities by Tom were blocked by Eva.

Further, in Group 2 we observed how Bo's attempts to encourage May to carry out key activities, did not become successful until May had access to artefacts. This suggests that access to artefacts might be another important factor for low achieving students to be able to carry out key activities. Therefore, our findings indicate that regulating activities can encourage low achieving students' communication about pattern problems, but it is crucial that they have access to combine speech, gestures, and use of artefacts in order to communicate their ideas. A new finding is that regulating activities can also hinder low achieving students' active participation in communication. We observed how artefacts were moved away, and speech blocked the regulating activities that were originally aiming to invite for contributions. This is in line with Radford (2009) who advocates that thinking happens not only as a mental process, but in and through the use of speech, gestures, and actions (like use of artefacts). In our study we observed that these multimodal expressions were important for May in showing her ideas, specifically when she used counting (relying on arithmetic) to communicate about the pattern problem. Later on, when she demonstrated more abstract strategies like recursive reasoning and how to express the problem explicitly in a formula, in Group 2, she did not use gestures and artefacts. This may indicate a development: in the first session gestures and artefacts were important for the key activities, and May needed them to express ideas that were not yet completely her own. But at the end of the session in Group 2 she expressed her ideas verbally without use of gestures and artefacts, which was possible because Bo encouraged her. In Group 3 she could confidently claim that the pattern of the perimeters is equal to the multiplication table of six. This indication of development needs more research addressing low achieving students' use of multimodal expressions when generalising beyond their arithmetic knowledge.

Our results show the complexity of how to empower low achieving students' in order to become active contributors in heterogeneous groups. We observed in Group 2 that a regulating activity for a low achieving student can be an explicit invitation to combine speech, gestures, and artefacts for showing one's ideas. The key activity of 'showing one's ideas' is a prerequisite which enables low achieving students to demonstrate other key activities. Therefore, teachers can empower the students to use key activities by giving access to artefacts and by designing tasks inviting all students to show their ideas. While further work is required to explore low achieving students' learning and their ability to generalise, our findings also indicate that further research is needed on low achieving students in heterogeneous settings where the low achieving student is relatively high achieving, for instance when they work with younger students.

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A pattern-based approach to elementary algebra

Heidi Strømskag[1]

Sør-Trøndelag University College, Faculty of Teacher and Interpreter Education, Trondheim, Norway, heidi.stromskag@hist.no

With a focus on epistemology, this paper discusses what pattern generalisation as an algebraic activity involves. Further, it presents a review of empirical studies where a pattern-based approach is used to teach algebra. This shows that students' problems with establishing algebraic rules from patterns and tables can be explained by: 1) difficulties caused by students' use of invalid methods to identify explicit formulae; 2) difficulties caused by students' tendency to focus on recurrence relations; and 3) institutional constraints. As an alternative to a traditional task on a shape pattern, the paper presents an epistemological model designed to implement the equivalence statement: $1 + 3 + 5 + L + 2n - 1 = n^2$.

Keywords: Algebraic activity, pattern generalisation, epistemological model, milieu.

INTRODUCTION

According to Reed (1972), humans have a natural inclination to observe patterns, and to impose patterns on different experiences. Inspired by Steen (1988), Devlin (1994), and others, I consider mathematics as the science of patterns. Mathematicians seek patterns in different areas, including numbers (arithmetic and number theory), form (geometry), motion (calculus), reasoning (logic), possibilities (probability theory), and position (topology). In the development of mathematical knowledge, generalisation is an essential process. This is asserted by for instance Krutetskii (1976), who classifies generalisation as one of the higher cognitive abilities demonstrated by mathematics learners.

Generalisation of shape patterns and numerical sequences is part of the elementary and secondary curriculum in many countries, for example England (Department for Education, 2014); the United States (National Council of Teachers of Mathematics, 2000); Canada (Ontario Ministry of Education and Training, 2005); and, Norway (Directorate for Education and Training, 2013). A purpose of students' engagement with patterns is to provide a reference context (physical, iconic or numerical) for generalisation and algebraic thinking.

A shape pattern is usually instantiated by some consecutive geometrical configurations in an alignment imagined as continuing until infinity. In this paper, geometrical configurations will be referred to as *elements*, and the constituents of an element will be referred to as *components*. Generalising a pattern algebraically rests on noticing a commonality (a structure) of the components of some elements of the pattern, and using it to provide an expression of an arbitrary member of the number sequence mapped from the pattern. This will be explained in more detail below.

PATTERN GENERALISATION AS AN ALGEBRAIC ACTIVITY

A model for conceptualising algebraic activity is proposed by Kieran (2004), where she introduces three interrelated principal activities of school algebra: generational activity, transformational activity, and global/meta-level activity. The *generational activities* involve the creation of algebraic expressions and equations like (i) equations that represent quantitive problem situations; (ii) expressions of generality arising from shape patterns or numerical sequences; and (iii) expressions of the rules that determine numerical relationships (Kieran, 2004). I interpret the letters used in the three examples as having the role as unknowns, variables and parameters, respectively. The *transformational activities* involve syntactically-guided manipulation of formalisms including: collecting like terms; factoring; expanding brackets; simplifying expressions; exponentiation with polynomials; and, solving equations (Kieran, 2004). These are the activities with which school algebra has traditionally been associated. The *global/meta-level activities* involve activities for which algebra is used as a tool, and include:

1 The author has earlier published under the name

problem solving; modelling and predicting; studying structure and change; analysing relationships; and, generalising and proving (Kieran, 2004).

The elements of a shape pattern are carriers of multiple structures which have to be interpreted by the students. The process of interpreting and representing these structures algebraically, involves generalisation of arithmetical relationships in members of the sequence mapped from the shape pattern. A shape pattern can be generalised either through an indirect approach, where the result is a recursive formula (a relationship between consecutive elements), or through a direct approach, where the result is an explicit formula (a functional relationship between position and numerical value of an element).

Måsøval (2011) distinguishes between two types of shape patterns: *arbitrary patterns* (Figure 1), and *conjectural patterns* (Figure 2).

These patterns correspond respectively to two different mathematical objects aimed at in the process of generalising: *formula* (for the general member of the sequence mapped from the shape pattern; e.g., $a_n = 3n + 1$ in Figure 1), and *theorem* (in terms of a general numerical statement; e.g., $1 + 3 + 5 + ... + 2n - 1 = n^2$ in Figure 2). Institutionalisation of the knowledge in the case when algebraic generalisation aims at a formula (for an *arbitrary* pattern) is not institutionalisation of the formula *per se*. It is institutionalisation of how the formula can be derived through identification of an invariant structure in the elements of the pattern. Further, it is institutionalisation of how the invariant structure is interpreted into arithmetical relationships and how these in turn are generalised algebraically in terms of a formula. The cultural, reusable knowledge in this case is the nature of the relationship between the algebraic expression and its referent (a generic element of the pattern). On the other hand, institutionalisation of the knowledge in the case when algebraic generalisation aims at a theorem (illustrated by a *conjectural* pattern) involves decontextualisation of the general numerical statement from the shape pattern on the basis of which it is developed. The cultural, reusable knowledge in this case is a general relationship between sequences of numbers (in Figure 2, between odd and square numbers).

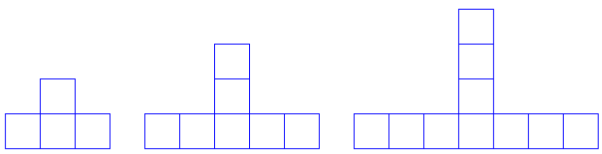


Figure 1: Example of the first three elements of an arbitrary pattern

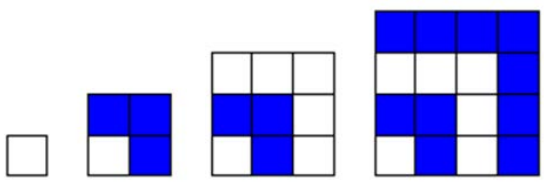


Figure 2: Example of the first four elements of a conjectural pattern

In the following, I illustrate briefly a strategy for generalisation of an *arbitrary pattern*, where I focus on the connection between the iconic, the arithmetical, and the algebraic representation of the pattern. For a detailed epistemological analysis of shape pattern generalisation, see (Måsøval, 2011, Chapter 5). The target knowledge is the nature of the relationship between the sought generalisation (an algebraic expression) and its referent (a generic element of the pattern). A direct approach to generality is employed. The invariant structure of a shape pattern provides the possibility to decompose the elements into different repetitive parts. Decomposition refers to diagrammatic isolation (encircling, painting with different colours, or other techniques) of various parts of the elements in order to visualise the invariant structure of the pattern. The point is to express the number of components of each partition of an element as a *function* of the element's position in the shape pattern. These arithmetical expressions are used to express the total number of components of the element. Generalisation of the sequence of arithmetical expressions will lead to a formula where letters are placeholders for positions. An example of the first three elements of a quadratic pattern is presented in Figure 3.

Figure 4 presents a possible decomposition of this pattern, with corresponding arithmetical expressions

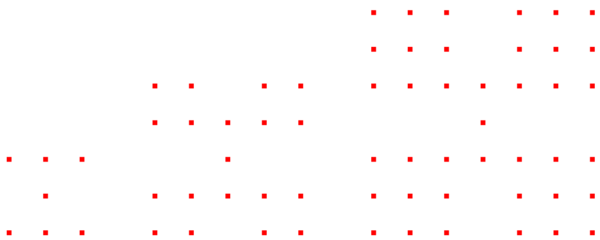


Figure 3: The first three elements of a shape pattern

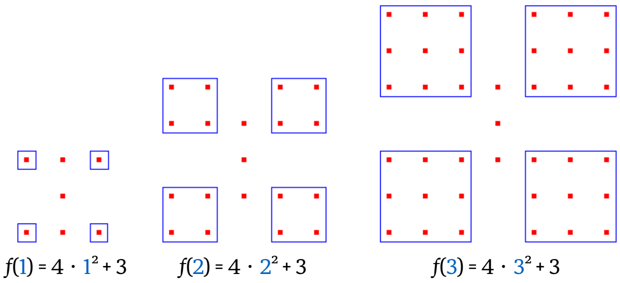


Figure 4: A possible decomposition of elements of the pattern in Figure 3

representing the number of components in the partitions of the first three elements.

The decomposition shown in Figure 4 corresponds to an interpretation which means that each element consists of three components plus four times the square of the position of the element. The corresponding arithmetical expressions suggest a generalisation in terms of the formula, $f(n) = 4n^2 + 3$. An alternative decomposition (discussed without illustration) involves filling in components at "empty places" to make each element into a square; this is compensated by subtraction in the arithmetical expression. In order to build a square, the n-th element would need to get an extra of $4n - 2$ components. The resulting formula would then be given by $g(n) = (2n + 1)^2 - (4n - 2)$.

The example in Figure 3 conceptualised through Kieran's (2004) framework

The process of interpreting and decomposing the pattern is a *global/meta-level activity*. It is a modelling process that involves studying and representing the quadratic relationship between the position and the corresponding member of the number sequence mapped from the pattern. Establishment of a formula (here, a function) is a *generational activity*, where the variable n is a placeholder for an element in the domain (natural numbers). The two different decompositions presented above result in different formulae, where *transformational activity* can be used to justify that they are equivalent.

Justification of the new knowledge (the formula), which is a global/meta-level activity, can be done through decomposition of a *generic example*. This is done to illustrate references between the partitions of the generic element, on the one hand, and the mathematical symbols of the formula, on the other.

EMPIRICAL STUDIES OF STUDENTS' PATTERN GENERALISATION

Students' difficulties in establishing algebraic rules from patterns and tables

Several studies have documented students' difficulties in establishing algebraic rules from patterns and tables. Stacey (1989) reports responses to linear generalising problems of 140 students aged between 9 and 13. Generalisation of the given problems was of the type $f(x) = ax + b$ with $b \neq 0$. It turned out that mainly two ideas were used. Stacey refers to these as the *difference method* and the *whole-object method*. The difference method involves multiplying the common difference between members of a sequence by the rank of a member to calculate its numerical value. The whole-object method involves taking a multiple of the numerical value of a member of a sequence to calculate the numerical value of a member with a higher rank; that is, implicitly assuming that $f(mn) = mf(n)$. The two methods will be applicable only when the linear problems are direct proportionalities. Because the problems used in Stacey's study were not of this type, the difference method and the whole-object method were invalid. The erroneous generalisations were not discovered by the students because they failed to check the validity of the rules they produced.

Another finding from Stacey's (1989) study was that students showed a tendency to focus on recurrence relations in one variable rather than on functional relationships between two variables. The same conclusion about students' tendency to focus on recurrence relations was reached by MacGregor and Stacey (1995). They tested approximately 1200 students in Years 7 to 10 in ten schools on recognising, using, and describing rules relating two variables; 14 students were interviewed. The results showed that the students had difficulties in perceiving functional relationships and expressing them in words and as equations. The students' tendency to find recurrence relations in patterns and tables were in most cases counter-productive to identification of a relationship between two variables. Hence, MacGregor and Stacey recommend teachers to use examples where it is not possible to find differences between consecutive members of a sequence.

Orton and Orton (1996) conducted a study in which 1040 students from Years 6, 7 and 8 (ages 10 to 13) completed a written test on different pattern questions;

30 of the students were interviewed about their responses. Results from this study are consistent with findings from Stacey (1989) and MacGregor and Stacey (1995): students have a clear tendency to use *differencing methods* and identify a recursive pattern.

Lannin, Barker and Townsend (2006) explored students' use of recursive and explicit relationships by examining the reasoning of 25 sixth-grade students, including a focus on four target students, as they approached three generalisation tasks while using computer spreadsheets as an instructional tool. Their results demonstrate students' difficulty in moving from successful recursive formulae towards explicit formulae. One obstacle to students' ability to connect recursive and explicit formulae was their limited understanding of connections between mathematical operations, such as addition and multiplication.

Måsøval (2011) reports from a case study of six (two groups of three) student teachers' collaborative engagement with four tasks on shape pattern generalisation (with some teacher involvement). Three categories of constraints to students' generalisation processes emerged from a process of open coding of transcripts of video recordings, conceptualised through the theory of didactical situations (Brousseau, 1997). The constraints are explained in terms of: 1) an inadequate *adidactical milieu*, in particular caused by unfavourable design of tasks (where the focus is on number of components rather than on the multiple structures inherent in the pattern); 2) complexity of transforming observations and conjectures represented in informal language into algebraic symbolism (from *action* to *formulation*); and 3) complexity of justifying proposed generalisations (*validation*), in particular caused by students' use of empirical reasoning instead of rigorous mathematical reasoning (Måsøval, 2011).

The foregoing discussion of students' difficulties in establishing algebraic rules from patterns and tables can be summarised in three points. First, there are difficulties caused by students' use of invalid or unsuccessful methods to identify explicit formulae (the *difference method* and *differencing*, and the *whole-object method*). Second, there are difficulties caused by students' tendency to focus on recurrence relations which are not easily transformed into explicit formulae. Third, there is an institutional constraint caused by the use of stereotype tasks (focusing on "How many?") and further, by the way pattern generalisation is taught.

Components of a successful pattern-based approach to elementary algebra

Results from Redden's (1996) study demonstrate a significant correlation between natural language descriptions and symbolic notation used by students. On the basis of investigation of how 1435 children aged 10 to 13 responded on requests to generalise shape patterns, he found that natural language descriptions exclusively in terms of functional relationships appear to lead to students' successful use of algebraic notation. This finding points at the importance of relating the independent variable (the position of a member) to the dependent variable (the member itself).

Warren (2000) demonstrates significant correlation between students' ability to reason visually (identify, analyse, and describe patterns) and successful algebraic generalisations from shape patterns and tables of values. Warren's finding is based on responses on two written tests administered to 379 students (aged between 12 and 15 years); 16 of the students were interviewed in groups of four. Warren, Cooper and Lamb (2006) examined the development of students' functional thinking during a teaching experiment that was conducted in two classrooms with a total of 45 Year 4 students (average age nine and a half years). They found that tables with input values not increasing in equal steps assisted students to search for a relationship between two data sets instead of focusing on variation within one. Randomness of the input values encouraged students to think relationally instead of sequentially, a finding consistent with MacGregor and Stacey's (1995) recommendation referred to above.

The results referred to above can be combined to provide a recommendation for students' engagement with shape patterns: Students' should be encouraged to *express functional relationships in natural language*, because this is important for the ability to use symbolic notation. It is relevant here that students remain connected to the iconic representation and *reason visually*, which is a condition for successful algebraic generalisations. Further, visual reasoning can potentially prevent students from senseless pattern spotting in numerical sequences without connection to the original mathematical situation. The strategy presented above (exemplified by the pattern in Figure 3) for generalising a pattern algebraically is favourable in that it has the recommended features: first, it involves analysis of geometrical configurations (decomposition); and second, it involves identification of functional relationships between two sets (position and numerical value of element, respectively).

Several studies suggest that it is not generalisation tasks in themselves that are difficult; the problems that students encounter are rather due to the way tasks are designed and limitations of the teaching approaches employed (Moss & Beatty, 2006; Måsøval, 2011, 2013; Noss, Healy, & Hoyles, 1997). Motivated by this, I designed an epistemological model – a *situation* (Brousseau, 1997) – that involves a problem that can be solved in an optimal manner by using the knowledge aimed at. In the concluding section of the paper, I explain this epistemological model and its devolution to a class of twenty student teachers enrolled on a master programme for primary and lower secondary education. The data from the experiment are students' notes and solutions in addition to my field notes.

AN EPISTEMOLOGICAL MODEL OF A PIECE OF KNOWLEDGE

Inspired by TDS, the theory of didactical situations (Brousseau, 1997), I created an epistemological model of the general numerical statement that "the sum of the first n odd numbers is equal to the n-th square number". The choice of this piece of knowledge was motivated by Måsøval (2011) who reports from students' (less successful) engagement with this equivalence statement through a traditional task, based on a shape pattern similar to the one in Figure 2.

Devolution

I had cut out sets of twelve paper forms, consisting of from 1 to 25 unit squares (a selection of which is shown below). In the classroom I presented the context (below), and gave each pair of students an envelope with a set of cut-outs.

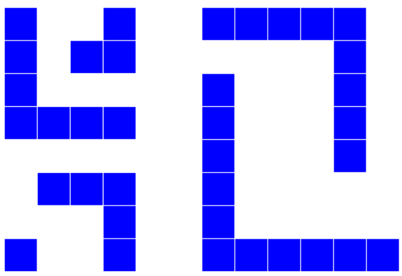
"The company TILEL (in class, represented by the envelopes) sells a special kind of tile formations that can be used to cover *squares*. The tile formations have shapes as Ls, and consist of an *odd number* of unit squares. There is also a degenerated L-form which consists of only one unit square. You and your partner are supposed to construct a quadratic area of tiles, using L-forms from TILEL. You decide on the *size of a square*, and the task is for your partner to go and get a selection of L-forms which *precisely covers* the chosen square. There is a restriction on the L-forms: they shall all be of *different size*.

When your partner returns with the L-forms he/she has bought, the two of you shall arrange them into a square. If you lack some L-forms, or have anyone left over, your partner will have to go back to TILEL for supplements or returns. Each time this is necessary, a *charge* must be paid, so buying the right L-forms at once is important."

Some of the features of the milieu were, due to time constraints, hypothetical (e.g., purchase of tile formations and the fee charged for supplements and returns). The adidactical situation might have been designed as a game, where the winner would be the group that solved the task with least costs, and/or had the best recipe, etc. Figure 5 shows the task given to the students for work in pairs, after the context was presented.

Features of the milieu derived from the knowledge at stake

The target knowledge in this case is the equivalence statement: "the sum of the first n odd numbers is equal to the n-th square number", potentially represented by $1 + 3 + 5 + L + 2n - 1 = n^2$. A model of the target knowledge is created using a dissection of a square into L-forms consisting of consecutive odd numbers



1. One of you chooses the size of a square; the other one gets L-forms to cover it. Collaborate to arrange the L-forms into the chosen square. (ACTION)
2. On the basis of the work you have done, make a recipe for how to cover a square of random size with L-forms of different sizes, without having to go back for supplements or returns. Let another group try your recipe and see if it works. (FORMULATION)
3. Explain why your recipe will always work (for a random square). (VALIDATION)

Figure 5: The task given to the students for work in pairs

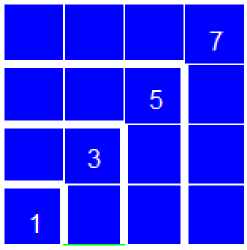


Figure 6: Dissection of the fourth square into the first four odd numbers

of unit squares (1 is represented by one unit square, hence a degenerated L). A generic example is given in Figure 6, illustrating that $1 + 3 + 5 + 7 = 4^2$.

It is important that the L-forms given to the students are of different size; this is what guarantees that the square is made of a sum of *consecutive* odd numbers. The fee charged for not getting the correct L-forms is intended to motivate students' (in the situation of action) to develop a model that relates odd numbers and square numbers. It is expected that the students (in the situation of formulation) express in natural language that it is necessary to add as many odd numbers (from 1 and upward) as the rank of the chosen square number. The situation of validation is intended to motivate reasoning based on the nature of the knowledge at stake, even if algebraic notation might not be used. For this to happen, it is necessary that the students have a technique (prior knowledge) for representing odd numbers in terms of $2 \cdot 1 - 1$, $2 \cdot 2 - 1$, $2 \cdot 3 - 1$, and so on.

Due to limited space, I comment only on the validation phase (in whole class). Two approaches were used to justify the conjecture: One was a visual proof (cf. Figure 6), where students argued by a generic example that the next square is reached by adding the next odd number $(2(n+1)-1)$ to the current square. The other approach started with the statement in algebraic notation, $1 + 3 + 5 + L + 2n - 1 = n^2$; students showed that the sum on the left hand side is equal to n^2 by using the Gaussian method (adding the first and last terms, then the second and last but one term, etc.). There was a discussion of implementation of the model in different grades in school.

The adequacy of an epistemological model is based on the quality of the underlying epistemological analysis (EA). An EA should provide a rationale that would make the students' engagement in the problem situation sensible. An EA is however "work in progress"; experiments feed back to, and might strengthen, the EA and hence the model based on it. A relevant direction for future research on pattern generalisation is therefore the design and study of implementation of epistemological models of pieces of algebraic knowledge – in order to improve the models. Due to its epistemological focus, TDS would be a favourable framework to use in this kind of research.

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A study of the preparation of the function concept

Gyöngyi Szanyi

University of Debrecen, Institute of Mathematics, Debrecen, Hungary, szanyi.gyongyi@science.unideb.hu

Function is a basic concept of mathematics, in particular, mathematical analysis. With an appropriate development of a function approach, it becomes possible for students to use function models to describe mathematical and non-mathematical problems. After an analysis of the function concept development process, I propose a model of rule following and rule recognition skills development that combines features of the van Hiele levels and the levels of language about function (Isoda, 1996). Using this model I investigate students' rule following and rule recognition skills from the viewpoint of the preparation for the function concept of sixth grade students (11–12 years old) in the Ukrainian education system.

Keywords: Function concept, Ukrainian secondary education, features of van Hiele levels, rule following and rule recognition skills.

INTRODUCTION

The function concept interweaves the whole teaching of mathematics. Functions are incorporated in the concepts of numbers, equations, inequalities, ratio, proportionality, geometrical transformations, etc. Through the teaching of functions, it is also possible for students to develop creativity, functional thinking, and other cognitive strategies (Czeglédy, Orosz, Szalontai, & Szilák, 1994).

In her study, Sierpinska (1992) sets out the conditions of understanding the notion of function. These conditions illustrate that it takes time to reach a thorough understanding of the function concept. There is a long journey between beginning to develop an understanding of the links between the elements of sets to the robust function concept. In this study I examined the portion of this journey that happens during the fifth – sixth grade, which is the period before learning the definition of the function (preparation period). The study was based on the analyses of the Ukrainian curriculum framework. The study revealed that the development of rule following and rule recognition (hereafter referred to as RF and RR) skills are missing from the curriculum. Dreyfus and Vinner (1989), however, point out that a function can also be defined as a rule, and the rule is an element of the function concept (Kwari, 2007). The present study examines RF and RR skills that are necessary in the formation of the function concept and in the construction of function tables, which help children to figure out the relationship between quantities (Blanton & Kaput, 2011). The participants were a class of sixth grade (11–12 years old) students, studying in the Ukrainian education system.

THEORETICAL BACKGROUND

Definition plays an important role in mathematics. According to Skemp (1987), definitions have their specific places in mathematical concept development. Concepts of a higher order than those which people already have an understanding of, cannot be communicated to them by a definition, but rather by presenting to them a suitable connection of examples. Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner (Skemp, 1987).

The modern definition of function that frames this study is the Dirichlet-Bourbaki definition, which is "a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)" (Vinner & Dreyfus, 1989, p. 357). So in order to develop a concept of function, the knowledge of both simple and higher level concepts is necessary, and this formation is a long process.

Vinner and Dreyfus (1989) discuss the notions of function put forth by secondary school students after being given the definition of function. The authors, drawing on Vinner (1983), categorized students' definitions of function into six categories: (A) correspondence (the Dirichlet-Bourbaki definition); (B) dependence relation (dependence between two variables); (C) rule (a function is a rule; a rule is expected to have some regularity, whereas a correspondence may be "arbitrary"); (D) operation (a function is an operation or manipulation); (E) formula (a function is a formula, an algebric expression, or an equation); and, (F) representation (graphical or symbolic representation) (Vinner & Dreyfus, 1989, p. 360).

Taking into account these categories and the above mentioned studies, it can be highlighted that the function concept has many elements. Sierpinska (1992) described the "worlds" that the study of functions should focus on: the world of changes or changing objects; the world of relationships or processes; and, the world of rules, patterns, and laws. Likewise, Kwari (2007) showed the constitutive elements or aspects of the function concept that should be developed: change and what changes; relationships (attribute — builds rules to determine a unique y-value from any given x-value); rules (symbolically given by, e.g., $f(x) = ax + b$); representation; and, language/notation. Of the skills that could be linked to the above listed aspects, possession of the rule recognition and rule following skills are significant for the students in this study in order to recognise and express function-like relations. So, this study concentrates on the investigation of these skills and only touches upon the question of symbolisation (the articulation of rule by arithmetic operations), but does not intend to go further (to write function-like links with symbols, e.g., $f(x) = ax + b$).

As a *skill* is considered to be the psychic feature of an individual, that evolves by the practice of some kind of activity, and is manifested in the doing of that activity, then the mentioned skills can also be developed by cognitive operations. The recognition of a rule (regularity), the following of the rule, and in some cases, the appropriate application of the rule, presumes the execution of a series of cognitive operations (categorisation, selection, and link-recognition).

The information acquisition process is strongly influenced by the development of students' cognitive operations. Two Dutch didacticians, Pierre van Hiele and Dina van Hiele-Geldof developed a pedagogical theory in 1957 for the understanding of the process of geometric thinking, which differentiates between five levels of geometric thinking: visualization; analysis; informal deduction; deduction; and, rigor (as cited in Herendiné Kónya, 2003, p. 51).

Freudenthal (1973) and Isoda (1996) extend the van Hiele levels from geometry to other areas. Freudenthal viewed progressive mathematization as the main goal of school mathematics. For this ongoing task, he provided a framework by recursively defined levels: the activity of the lower level, that is the organizing activity by the means of this level, becomes an object of analysis on the higher level. Freudenthal's theoretical approach rests on the Van Hiele levels. Van Hiele, himself, has written about levels in arithmetic and algebra (van Hiele, 2002). He observed 'a change in level' from the act of counting to the concept of number. Isoda's paper (1996) points out features of van Hiele levels and shows that they are also characteristics of the proposed levels of language about function. These features include: (1) Language hierarchy. Each level has its own language and the levels are hierarchical; (2) The existence of untranslatable concepts. The corresponding contents of different levels sometimes conflict; (3) Duality of object and method. The thinking of each level has its own inquiring object (subject matter) and inquiring method (the way of learning); (4) Mathematical language and student thinking in context. While the levels are distinguished as sets of mathematical language, the actual thinking of each student varies depending on the teaching and learning context.

Isoda (1996) first discusses the levels of function from the point of view of language, and shows the duality between object and method in van Hiele's levels (the levels of geometry) and in the levels of function. These levels of language are: Level 1. Level of everyday language (students describe relation in phenomena using everyday language obscurely: students explore phenomena (object) using obscure relations or variation (method)); Level 2. Level of arithmetic (students describe the rules of relations using tables. They make and explore tables with arithmetic: students explore the relations using rules); Level 3. Level of algebra and geometry (students describe function using equations and graphs: students explore the rules using notations of function); Level 4. Level of calculus (students describe function using calculus); Level 5. Level of analysis (an example of language for description is functional analysis which is a metatheory of calculus).

The present study examines the preparatory part of the notion of function in the sixth grade in light of the state framework curriculum, using features (1) and (3) of van Hiele levels and the first three of the five levels of function described by Isoda (1996). I set out the levels of the cognitive operations that are crucial for the possession of RF and RR skills and the criteria for categorising activity forms into levels. Noticing an analogy between these levels and the van Hiele levels, I used the names of the van Hiele levels for the marking of the discussed levels. The levels which I created by joining the features of van Hiele levels and Isoda's levels and using them to develop a deeper understanding in (sixth grade) students' development of the function concept, are the following:

> Level 1 (visualization): Students recognise some kind of rule (functional relation) (method) between the element pairs (object) and follow the recognised rule *(level of everyday language).*
>
> Level 2 (analysis): Students are able to phrase the recognised rule (they can argue in favor of the recognised links between the cohesive element pairs) and follow the rule which is given by words or by simple formulas *(level of everyday language and level of arithmetic).*
>
> Level 3 (informal deduction): At this level the harmony of the simple (2–3 steps) rule-making and its description with formula develops *(level of arithmetic).*

METHODOLOGY

Sample

Participants were 26 sixth grade students (11–12 years old), with moderate abilities, in a school with Hungarian as the language of instruction in Ukraine. The students had four classes of mathematics a week, according to the state curriculum framework. The Hungarian version of the mathematics textbook is used at this level and is approved by the Ukrainian Ministry of Education and Science. As the research was carried out in March, during the second semester of the sixth grade, students were already familiar with the natural numbers, fractions (common fractions and decimals), and had learned arithmetic operations with rational numbers. The introduction of proportional amounts and linear relationships occurred during this period, with the practical application in the initial phase.

Background

In the Ukrainian education system, function as a mathematical concept is defined at the seventh grade of the secondary school. In the lower classes, students are prepared with the use of different materials for introduction of the function concept. Analysing the curriculum and the textbooks for the fifth and sixth grade from the point of view of topics and their content that are supposed to support the development of the function concept, major deficiencies come to the surface in the requirements for developing RF and RR skills (in the lower classes it does not exist at all). In the development requirements of the themes of the curriculum, rule recognition and rule following skills are not mentioned. Prior research (cf., studies cited above), however, suggest that they are neces-

Class	Themes	Development requirements
5.	Number line	The representation of natural numbers on the number line.
	Letter expressions	The recognition of number- and letter expressions and the illustration with examples.
6.	Linear relationship	Illustration of proportional amounts with examples, the definition of the concept of linear relationship, finding the unknown element of the proportion, defining the proportion between amounts.
	Diagrams	Editing column- and circle diagrams.
	Cartesian coordinate system	Finding the coordinate of the point in the coordinate plane and representing the given coordinate point.
	Graphs	Representing correlations between quantities by graphs and analysing these graphs. The student is able to read the data from the graphs.

Table 1: Themes preparing the function concept in the Ukrainian textbooks[1] and curricula[2]

sary for the development of the function concept. In Table 1, I summarized the textbook themes that could support the preparation of the function concept. It can clearly be seen in the table that RF and RR skills are not amongst the development requirements of the materials.

Based on these aspects, in this study I am looking for the answers to the following questions: *At the end of the 6th grade, what level do students reach in their RF and RR skills? What are the typical mistakes students make when carrying out activities at each level and what might explain these errors?*

The questionnaire

A written test was used in order to investigate the RF and RR skills of students. Students worked independently and had 30 minutes to complete the test. The test contained five tasks that were based on the recognition and application of the relationship between the cohesive elements (assignment rules), as well as on the expression of the recognised rule, including as a formula. I was interested in students' possession of the necessary skills for the preparation of the function concept. In some exercises, the cohesive element pairs did not clearly make a function, so more rules might be possible. In the direction to the test, however, I tried to make it clear that I wanted students to find only one adequate rule. When constructing the test, I included tasks for levels 1, 2 and 3. When choosing the tasks, I predominately relied on the literature and used some of them without any alterations.

1) *Find a rule between the first and second row of the table. Fill in the table according to the rule* (Level 1)*! Write down the recognised rule in words* (Level 2).

pék	*tér*	*ló*	*bál*	*görög*		
kép	*rét*	*ól*			*derék*	*savas*

Figure 1

2) *Find a rule for the numbers in the columns and fill in the blank places of the table according to that rule* (Level 1). *Write down the recognised rule in words* (Level 2).

40	80	12	60	44	100	160				
10	20	3					1	13	31	95

Figure 2

I indicate the level of the task, parenthetically, within the instructions (see figures below). The first two tasks (Figure 1 and Figure 2) targeted the recognition, application and verbal expression of the relationship between cohesive elements (words and numbers). The filling in of the blank places of the tables assessed the application of the rule. Although the correct solution of both tasks assumes the same level of cognitive operations and activity forms (level 1 and level 2), the difference can be found in the context of the tasks: While in the first task the cohesive element pairs are words, in the second they are numbers. Because function relationships do not only occur between numbers, it is crucial that students recognise this relationship, as well.

The aim of the third task (Figure 3) was to make students recognise the relationship between the elements, apply it, and to express it with both words and symbols. In order to reach the first level, it is necessary to recognise some kind of relationship between the cohesive elements (x and y), but unlike in the first two tasks, the table is extended by an extra (first) column which serves as a hint to record the recognised relationship in the language of arithmetic (2nd level). When asking students to express the relationship with a formula, I touch upon the question of symbolisation (3rd level), but I do not intend to examine it deeper in this study. The "end product" (y value) should be found with the help of the given "raw material" (x value) according to the recognised rule, while in the previous two tasks knowing the "end product" and using the recognised rule, the raw material should be found.

The fourth task (Figure 4) was aimed at the interpretation and following of a predefined rule. In order to solve the task, the student needed to possess the activity forms of the two levels in order to interpret (analyse) the given formula. A correct completion

3) *Find a relationship between the x and y values of the columns and based on it, complete the table with the missing elements* (Level 1)*! Write down the relationship with words* (Level 2) *and as an expression* (Level 3)*!*

x	1	10	7	0	9	20	38
y	5	23	17				

y=__________

Figure 3

x	-3	4	44	48	-20	0
y						

4) Fill in the table according to the following rule: y=2x+3. Write down the rule in words (Level 2).

Figure 4

5) 2 litres/second of water flows from a tap to a tank. How much water is in the tank at:

a) 1 s, c) 5 s, e) 16 s, (Level 2)
b) 2 s, d) 10 s, f) x s (Level 3)

later if the tank was empty at the beginning? Illustrate the relationship between the amounts in a table.

Figure 5

of the table indicated a correct interpretation of the symbolic rule.

In the fifth task (Figure 5) I examined rule recognition and its mode of illustration during the solution of a task given in context. In this case, the rule is given verbally, in context. I take students' correct responses for parts (a) through (e) (2nd level) as an indication that the student had correctly interpreted the rule. A correct response to part (f) indicated that students' understanding of function had reached the third level, since the student was able to generalise the task, i.e. write down the relationship using a formula.

RESULTS

Analysis of students' answers

All of the students filled in the table in Task 1 correctly. This indicated that the students could recognise some kind of regularity between the first and the second row of the table, and they could apply the recognised rule. This means that when the cohesive element pairs are words, students can recognise the relationship between them. Writing down the recognised rule in words, however, was difficult for eight students. Some students skipped this part of the task or gave a rule that was not supported by the completed table. Some examples of correct responses for recognised rules: "Words should be read backwards"; "If we change the first and the last letters we get another meaningful word".

In the second task, where the cohesive element pairs were numbers, only 18 students' gave a correct solution, while 14 students were able to give the recognised rule in words. The other students made one of the following mistakes: (1) they filled in the blank squares in the second row of the table according to a recognised rule, but in the first row they filled in the blank squares using another rule; that is, they did not apply the inverse of the recognised rule and interpreted this part of the table separately. From the point of view of the function concept, these mistakes indicated issues in recognising and differentiating between the basic set and the image set; (2) students tried to find different rules for each column and filled in the squares according to it. This could be the consequence of being unfamiliar with the table illustration of cohesive amounts. Here are some examples of correct responses for recognised rules:

> "If the square in the second row is empty the number above it has to be divided by four, and where the first row square is empty the second row number has to be multiplied by four" ; "Numbers of the first row are the fourfold of the lower row".

Only two students completed the third task, while other students did not give any indication of their thinking. This let me conclude that those students who possess the skill of one step rule recognition and rule creation may have difficulty with two step rule recognition.

In the fourth task, the rule was given symbolically. Students had to understand the symbolic rule and fill in the table accordingly. The given rule could have been familiar to the students, as letter expressions were from the fifth grade mathematics material, when they had to define the value of the letter expression along the certain values of the variable, but the values were not given in table form. Presumably, this new situation confused many students. Only twelve students could solve the task with only minor calculation mistakes.

Only ten students could answer all of the sub questions of the fifth task, including the last (f), so they could generalise the rule of calculating the amount of water in the tank if the elapsed time was unknown, and they could illustrate the relationship between the results with a table. Eight students could calculate with concrete numbers (parts (a) through (e)), but failed to complete the (f) question.

By analysing the responses of the students in the tasks according to the criteria of the set out levels it can be said that a student reached level 1 if he/she could complete at least one of Tasks 1, 2, or 3. I considered that a student had reached Level 2 when he/she correctly provided the rule in at least three tasks out of the five. The student reached Level 3 if he/she gave the correct answer to all of the questions of the third task. In some tasks students made calculation mistakes (such as in Tasks 4 and 5), but I did not take these into consideration if the student demonstrated the correct reasoning.

The students' answers were analysed based on the levels at which the various parts of the tasks were categorised. The results are summarised in Table 2.

Based on the analysis, the students were most successful at demonstrating a Level 1 understanding in the first task since every student correctly completed it. However, the part of the same task, which was categorised as Level 2, was completed by fewer students (18). It can be concluded, however, that in the case of each task, the highest results were reached on Level 2, as compared to the other levels. The third task was the most difficult. Only two students gave a complete solution.

Based on these aspects, out of 26 students, 14 are on the first level, ten are on the second level, and only two students are on the third level. So, most of the students can recognise some kind of rule between the element pairs and can follow it, but to write these rules down with words cause them difficulties. In addition, interpreting the rules given by symbols and making multistep rules also seems to be problematic. This study also confirmed the hierarchy of the levels. There was no student who could meet the requirements of Level 3, but not Level 1 or Level 2.

CONCLUSIONS

The goal of this paper was to investigate the RF and RR skills of sixth grade students studying in the Ukrainian education system, from the point of view of the development of the function concept. The results showed that students certainly reached level 1, indicating that they can recognise the relationship between simple elements. In many cases, however, I could see that some students fulfilled the requirements of level 1, but could not get to level 2 due to possible deficiencies in the area of communication in the language of mathematics. Many could also not successfully use the table as a tool for displaying cohesive elements. I suspect that students' deficiencies are not only age-specific, but are also related to the absence of tables from the curriculum requirements and from the textbook tasks. Students' lack of success in correctly completing Tasks 4 and 5, which entailed the use of already known concepts (letter expressions and linear relationship) in new situations (problem solving), indicated that this was also a problematic area for the students.

Finally, I would highlight a component of van Hiele's theory: that reaching a level does not only depend on the age of the student, but also on the teaching methods used and the quality of student learning. Based on this suggestion, with appropriate teaching methods and practical exercises integrated into the teaching/learning process, we can ensure that students possess the adequate skills for at least the first two levels in order to make the introduction of the abstract concepts easier in the seventh grade.

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Task / Levels number	Level 1		Level 2		Level 3	
	Number of students	%	Number of students	%	Number of students	%
1.	26	100	18	69		
2.	18	69	14	54		
3.	2	8	2	8	2	8
4.			12	46		
5.			17	65	10	38

Table 2: Counts and percentages of correct solutions to the tasks according to levels

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The development and arithmetic foundations of early functional thinking

Ulises Xolocotzin and Teresa Rojano

Research and Advanced Studies Centre, Ciudad de México, Mexico, uxolocotzine@cinvestav.mx

Functional reasoning is a key strand of early algebraization. This paper presents a cross-sectional study that analysed functional thinking in a sample of 94 elementary school students. Aspects such as following and identifying covariation rules showed dramatic differences between Grade 2, Grade 4, and Grade 6, whereas increases in the abilities to command verbal and symbolic representations were much smaller. After controlling for the influence of nonverbal reasoning, overall functional reasoning was found to be strongly associated with calculation skills, but not with skills such as counting, the understanding of numerals, and arithmetic problem solving. These results are discussed in terms of the nature of functional reasoning and its relationships to the arithmetical skills learnt during elementary education.

Keywords: Functional thinking, early algebra, arithmetic, quantitative methods.

INTRODUCTION

There is an increasing interest for understanding how algebraic ideas might be introduced during elementary education. Studies of early algebraization suggest that young children are capable of acquiring algebraic competences such as understanding the relationship between two quantities x and y, figuring out a co-variation rule (Carraher, Martinez, & Schliemann, 2008), and explaining and symbolically representing such rule (Cooper & Warren, 2007). However, the extent to what children's cognitive development permits their learning of algebraic notions remains to be fully understood (Carraher, Schliemann, Brizuela, & Earnest, 2006), and the interest for linking algebraic ideas with children's existing arithmetic knowledge are fairly recent (Russell, Schifter, & Bastable, 2011). This paper adds to the current literature by studying the acquisition of algebraic ideas such as that of function, by exploring the ways in which this might be linked to the arithmetical skills that children acquire during elementary education. This paper reports a cross-sectional study designed with two aims. First, the study aimed to identify the progression of a range of functional thinking aspects comparing children in Grade 2, Grade 4 and Grade 6. Second, the study analysed the extent to what overall functional thinking might be related to non-verbal reasoning and arithmetical skills, including counting, numerical understanding, calculation and problem solving.

Functional thinking in the early grades

The *functional approach* to teaching algebra is based on situations involving the simultaneous variation of two quantities and a rule governing such variation. Functional reasoning used to be absent in the elementary mathematics curriculum. However, during the last 15 years the development of the function concept gained recognition as a strand of early algebraization and, consequently, education programs addressing the mathematics of variation are increasingly common. For example, the Principles and Standards for School Mathematics introduce the subject 'Analyse change in various contexts' from Pre-K2 to Grade 12 (http://www.nctm.org/standards/content.aspx?id=26853). This is in part due to the recognition amongst researchers of the importance of introducing young pupils to mathematical representations of everyday situations (Blanton & Kaput, 2011), and responds to the evidence that technological environments effectively support the manipulation of dynamic mathematical representations of variation without algebraic representations (Rojano, 2008). The current study addressess two research questions.

How do functional thinking aspects progress across elementary grades?

Function tables are powerful tools for scaffolding and studying functional reasoning in the early grades (Martinez & Brizuela, 2006), since they help children to figure out relationships between quantities (Blanton & Kaput, 2011) and to elaborate algebraic conceptions such as co-variation and generalized correspondence (Tanışlı, 2011). Function tables are used in the current study for assessing a set of aspects of children's early understanding of functions, namely following explicit covariation rules, identifying and using such rules without an explicit definition, and understanding and generating verbal and symbolic representations of such rules (McEldoon & Rittle-Johnson, 2010).

What arithmetic skills are related to functional thinking?

It is well documented that children rely on their arithmetic knowledge to face their first encounters with algebraic problems (e.g., Van Amerom, 2003). Disregarding these intuitive arithmetical attempts and urging children to use formal algebraic methods might be misleading (Smith & Thompson, 2008). Therefore, there has been advocacy for grounding early algebraization on arithmetical knowledge (e.g., Russell, Schifter, & Bastable, 2011). This study adds to this research line by describing the relationship between functional thinking and a set of arithmetical skills that children usually learn during primary school.

METHOD

Design

This descriptive study relied on a cross-sectional design to compare aspects of functional reasoning across elementary school grades.

Participants and context of the study

The final sample for this study consisted of 94 students (58 girls) attending Grade 2 (n = 29, *Mean age* = 8.0 years, *SD* = 0.7), Grade 4 (n = 33, *Mean age* = 10.0 years, *SD* =0.5), and Grade 6 (n = 32, *Mean age* = 11.9 years, *SD* =0.5). The sample was randomly taken from the morning and evening shifts of a school located in a middle-size city in central Mexico. There are two considerations to make about the context of the study. First, at the moment of the study the mathematics performances of the two shifts in the participant school were fairly close to the national average in the nation-wide assessment ENLACE (SEP, 2014), which is similar to national standardized tests such as UK's SATs. This might be a helpful reference to understand how the participants in this study might compare to the rest of the Mexican population. Second, it is known that Mexico tends to rank at bottom positions in international mathematics assessments such as PISA (OECD, 2013). This is helpful to understand the relevance of this study for an international audience.

Measures

Functional Thinking Assessment

We wanted to link the progression of functional reasoning across elementary school grades with children's arithmetical knowledge. Therefore, we employed one instrument designed to do that, namely McEldoon & Rittle-Johnson's (2010) *Functional Thinking Assessment* (FTA). The items in the FTA are based on function tables that, as mentioned above, are adequate for approaching functional thinking. The FTA covers four aspects of functional reasoning:

— Apply rule: The student can use a given rule describing the relationship between the numbers in two table columns in order to determine new values of a table.

— Recognize rule: The student recognizes an $x - y$ correspondence rule in a table, and uses it to determine the next y value.

— Generate and use a verbal rule: The student writes a correspondence rule verbally, e.g., "*you add 4 to the number in the x column to get the number in the y column*".

— Generate an explicit symbolic rule: The student writes a correspondence rule using algebraic symbols, i.e., letters.

These aspects and their order were defined on the basis of both a review of existing educational materials involving function tables, and an empirical analysis of FTA validity and reliability. Item Response Theory measures showed that for all items, the probability of correct response implicated greater ability. Classical measures indicated high internal consistency (McEldoon & Rittle-Johnson, 2010). These properties suggest that the FTA possess adequate construct validity since it reliably identifies progression across ages, and the grouping of its items is consistent with the aspects that it is intended to assess.

A pilot study (Xolocotzin & Rojano, 2014) confirmed the appropriateness of the FTA, but it was also discovered that it might be beneficial to increase the number of items in the FTA, in order to give children a better opportunity to demonstrate their skills. Therefore, we added 4 items that required the generation of a verbal and a symbolic rule, without having to use it. These items added to the existing ones assessing the identification, generation, and usage of a covariation rule, which combination indexed the ability to command the verbal and symbolic representation of relationships between numbers. Other 4 items were added to balance the operations, for instance, to equalize the number of items involving additions and subtractions. Exploratory analyses (not reported) showed that these added items had similar difficulty and discrimination capacity as the original ones.

Non-verbal intelligence

Participants completed the *Matrix Reasoning* subtest of the Weschler Intelligence Scale for Children IV (Wechsler, 2007) according to the standard procedure. In order to have an age-standardized measure of non-verbal intelligence, the raw scores were converted to scaled scores for Mexican population.

Arithmetic ability

General arithmetic abilities were indexed with the raw scores obtained in the Arithmetic scale of the *Evaluación Neuropsicológica Infantil* (Neuropsychological Children Assessment, ENI) by Matute, Rosselli, & Ardila (2007). This is a battery for children aged 5 to 15. This instrument was selected because it assesses a comprehensive set of arithmetic abilities:

— Counting. This includes items that require numbering objects with and without interference, e.g., *"How many stars and bells are in this card?"*

— Numerical understanding. This includes items indexing abilities for commanding numerals, such as reading numbers, writing numbers, comparing numbers, and ordering numbers.

— Calculation. This subtest requires the making of numerical series, both forwards and backwards, as well as the verbal and written solution of arithmetic operations (e.g., 23 + 14).

— Arithmetical problems. This subtest requires the verbal solution of word-problems (e.g., "A second-hand motorcycle was sold in $ 8, 700, which is three-fourths of its original price. What is its original price?").

Procedure

After selecting the school, the corresponding authorities were contacted in order to explain the project intentions and requirements and obtain official permission for accessing the students. Parents of the selected groups were required to give informed consent for their children's participations. After this, complete groups of Grade 2, Grade 4 and Grade 6 were tested collectively with the FTA. Grade 4 and Grade 6 groups answered the FTA in one session lasting up to one hour, whereas children in Grade 2 groups were tested in two sessions of up to 30 minutes each. In total, 196 children were tested with the FTA. A sub-sample of 100 students was randomly selected for this study. Except for few children who required two sessions due to interruptions, the testing of arithmetic ability and non-verbal intelligence was made individually in one session lasting 45 minutes to one hour. Children were made to perform other tasks related to their cognitive development, the results of which are not reported here. The testing took place in a quiet room within the school, with presence of the researcher and the child only. Before each session, children were requested to verbally consent with the activities, after being clarified that their participation was anonymous, voluntary, and that they could stop at any moment. Three children did not complete the testing sessions because they decided to stop, and other three did not complete a second session, leaving a final sample of 94 children.

RESULTS

Comparisons by functional thinking aspect, grade and gender

The percentages of correct responses in each of the FTA aspects were analysed with a mixed ANOVA including the within-participants factor Aspect (Apply rule/Recognize rule/Verbal representation/ Symbolic representation) and the between-participants factors Grade (Grade 2/Grade 4/Grade 6) and Gender (male/female). None of the scores was weighted. The assumption of sphericity was violated, and although the results of the analysis with and without corrections were virtually equal, here we report the Greenhouse-Geisser corrected results. There was a significant main effect of Aspect [F (2.46, 217.037) = 118.323, p <.001, h^2 = .573], suggesting that children were more able to apply a rule than to recognize a rule, representing it

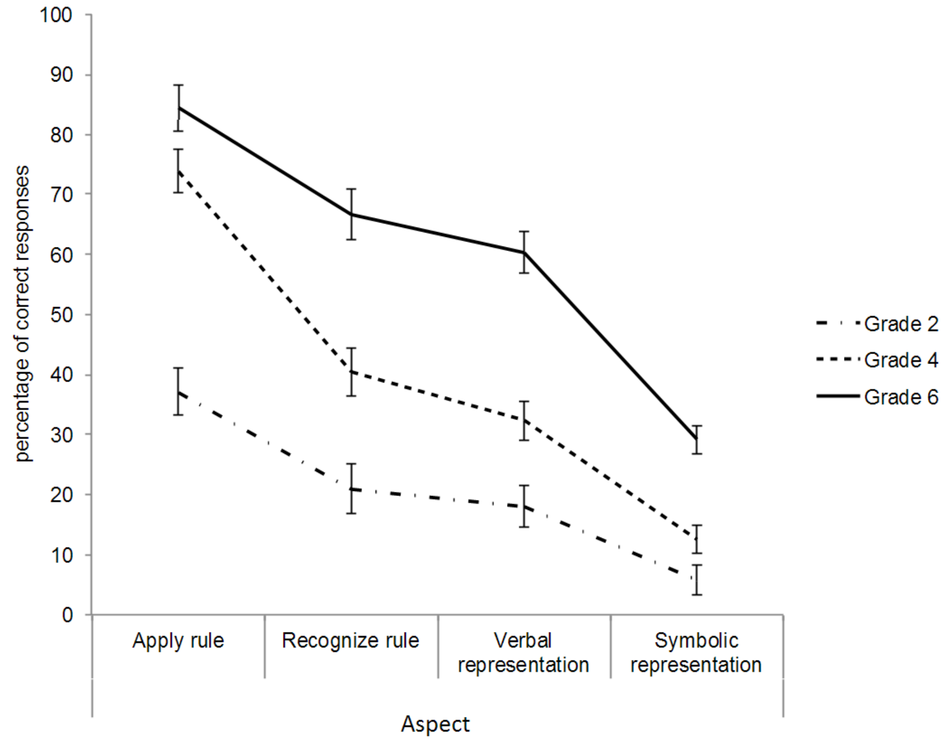

Figure 1: Percentage of correct responses by aspect and grade

verbally, or representing it symbolically, in that order. The significant main effect of Grade [F (1, 88) = 42.789, p <.001, h^2 = .493] indicated that the functional thinking aspects changed from one grade to another. There was also a significant aspect x grade interaction [F (6, 217.037) = 42.789, p < .001, h^2 = .493].

Post-hoc pairwise tests with Bonferroni adjustments revealed that the aspect apply rule increased significantly from Grade 2 to Grade 4, but not from Grade 4 to Grade 6. Also, the aspects verbal representation and symbolic representation increased only from Grade 4 to Grade 6, but not from Grade 2 to Grade 4. Neither the main effect of Gender [F (1, 88) = 0.57, ns] or the interactions aspect x gender [F (3, 217.037) = 0.93, ns], and aspect x grade x gender [F (6, 217.037) = 0.11, ns] resulted significant. Figure 1 illustrates these results.

Relationships between arithmetic and functional thinking

A multivariate regression analysis identified the arithmetic strands that might have been related to children's performance in the FTA. The dependent variable was a composite score indexing overall functional thinking, defined as the summed raw scores of the FTA indexes, namely apply rule, recognize rule, verbal representation and symbolic representation. Positive and significant correlations (not reported) were found between these scores (all rs > .39, all ps < .001), justifying their aggregation.

The independent variables in the regression analyses included two dummy variables that identified Grade 2 and Grade 4 individuals, the scaled scores of the matrix reasoning test, and each of Arithmetic scale

Independent variables	B	SE	β	p
Grade 2	-14.480	1.231	-.777	<.001
Grade 4	-7.614	1.179	-.422	<.001
Matrix reasoning	.776	.233	.231	.001
Calculation	.285	.117	.184	.017
Notes: all *ps* two-sided, only significant results are presented				

Table 1: Summary of the multivariate regression analysis including overall FTA score as dependent variable and Grade, matrix reasoning and the indexes of the ENI Arithmetic scale as independent variables

scores of the ENI, mean-centred by Grade. The results are shown in Table 1.

Children in Grade 2 and Grade 4 scored lower in the FT test than children in Grade 6, which is consistent with the results of the ANOVA presented above. The coefficient of the matrix reasoning score indicated that those who had a more non-verbal reasoning also scored higher in the FT test. As for the arithmetic scores, counting, numerical understanding, and arithmetical problems, were not significantly related to the FT. In contrast, the positive and significant effect of calculation indicated that those who were more able to make numerical series and resolve arithmetical operations were also better at responding the FTA.

DISCUSSION

How do functional thinking aspects progress across elementary grades?

The comparisons by aspect, grade, and gender, are consistent with the results reported with other applications of the FTA (McEldoon & Rittle-Johnson, 2010), suggesting its adequateness for assessing functional thinking. The absence of gender effects suggests that functional reasoning is accessible for both boys and girls. Performance in all of the aspects increased across grades. However, there were differences in their rates of increase from one grade to another. The ability to apply a rule develops steadily, and approaches top performance by Grade 4. This seems only natural considering that applying rules is a form of following instructions, which is something that children get familiar with from early stages of education. Younger children might be unable to apply a given covariation rule because they do not understand it, as is indicated by their lower performance recognizing and representing such rules. Performance recognizing a rule seems to progress regularly across elementary grades, although is not fully commanded by older students. The difficulty to command verbal and symbolic representations is noticeable. The differences across grades are apparently not regular and rather modest, especially for symbolic representation. Recall that children in Grade 4 perform at the same level of children in Grade 2, and children in Grade 6 failed to reach above 30% of correct responses.

The low performance on symbolic representation might be the result of either low students' abilities or hard items. This is difficult to disentangle with the collected data, so the results of this aspect should be interpreted cautiously. Analyses of the items' discrimination and difficulty for the current sample showed that 5 out of the 9 items involved were very difficult, even for top performers. The rest were also difficult but at least 30% of top performers were able to solve them. The items also involved different mathematical relationships, including addition, subtraction, multiplication, and combinations of these. The effects of potential differences in the capacity to handle these operations might be confounded with the capacity to symbolize. Also, the assessment of symbolization as a unitary aspect might not be optimal. This might be addressed by decomposing it in sub-aspects of different difficulty. For instance, symbolizing a covariation rule that has been presented in natural language might be one easier sub-aspect than symbolizing directly from data.

Functional reasoning aspects change differently across grades, suggesting that the progression from one aspect to the other is not necessarily uniform. Further studies should be designed to replicate these results, especially those regarding symbolic representation aspect, considering the apparent difficulty for assessing this aspect.

Non-verbal intelligence and functional reasoning

The significant relationship between matrix reasoning and the FT scores reflects that perceptual reasoning, indexed by the understanding visual patterns, is closely linked with the development of mathematic abilities in general (Parkin & Beaujean, 2012). Nevertheless, future studies should investigate if undersanding numerical patterns could be specifically related to the understanding of functions.

What arithmetic skills are related to functional thinking?

The results of the multivariate regression analysis outline the arithmetic skills that might be associated with functional thinking. The effects of the Grade 2 and Grade 4 dummies are not discussed since they replicate the grade effects discussed above. The significant matrix reasoning effect was expected since performance in this test tends to be correlated with general mathematics ability (Jordan, Glutting, & Ramineni, 2010) and, therefore, it was important to account for the influence of this variable when assessing the relationships between arithmetic and functional thinking.

The lack of significant counting effects might reflect the abstract nature of functional thinking. Counting was assessed with tasks in which children numbered objects accessible to visual perception. In contrast, functional thinking tasks required working with relationships, either explicit or not, between abstract quantities, i.e., represented by numerals. The lack of numerical understanding effects might be seen as counterintuitive. One would expect that skills for mastering the symbolic number system might be also used for functional thinking. Both of these involve generalizations. However, this seems to be not the case, at least for this sample. Children might command the structure of ten base-10 system, generalizing the relationship between the location of a numeral in a number and its value. For instance, knowing what the numeral 3 represents depending of whether is located first (32) or second (132). However, this sort of generalization seems unrelated to generalizations about relationships between different sets of quantities.

The significant effects of calculation suggest that those more able for making sequences and resolving arithmetical operations also resolved more FT items. Probably they attempted the function tables with iterative calculations, or by trial and error, and then they figured out the covariation rules. Another possibility is that they figured out the covariation rule first and then proceeded to do the calculations. The later seems less likely considering that items involving representation were more difficult than items involving the imputation of missing data, such as following a rule or employing a non-explicit rule. Further studies might shed light on this issue. It is important to mention that this result seems consistent with suggestions made about the important role that calculation might have for grounding algebraic reasoning. Russell and colleagues (2011) identified activities that sustain both arithmetic and algebra, namely understanding operations, generalizing and justifying, extending the number system, and using notation with meaning. Whether children might engage with these activities whilst doing function tables deserves attention.

The lack of significant effects of arithmetic problems does not rule out that problem solving is associated with functional reasoning. There was a sizable coefficient and a large standard error (B= .64, SE= .47). This might be related to the instrument. This score was obtained from a single test with eight items, which is small compared to the calculation score, made with 42 items from 4 tests. Thus, a small increase in the arithmetic problems score could be associated with large increases in the FT score, producing the observed large B. However, with standardized scores, calculation's β remained sizeable, whereas the arithmetic problems one decreased notoriously (.18 and .09 respectively). This might suggest that calculation is more related to functional thinking than arithmetic problem solving. However, this is uncertain considering the differences in their measurement. Future studies should include convergent measures of arithmetic problem solving in order to make a fair measurement of this aspect and assess its relationship with functional thinking more accurately.

Limitations and further studies

The sample for this study was selected purposfully, and comes from a very particular population. It would be useful to conduct other studies with samples from different populations to observe similarities or differences in functional reasoning. Also, measurement limitations for symbolic representation and problem solving indicate cautios interpretations of the results related to these aspects. The cross-sectional design does not allow for developmental inferences. It is highly desirable to collect longitudinal data for studying the development of functional reasoning. This study is part of a larger project aimed to identify the cognitive underpinnings of functional reasoning. In future, studies will be carried out to confirm the presented findings, and to investigate a wide set of cognitive capacities in addition to arithmetic.

ACKNOWLEDGMENT

This research was generously funded by the Mexican Council of Science and Technology (CONACYT), Grant No. 168620.

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TWG03

Posters

Windows on students' algebra: Describing their habits of mind

Deniz Eroğlu and Dilek Tanişli

Anadolu University in Eskişehir, Faculty of Education, Eskişehir, Turkey, denizeroglu@anadolu.edu.tr

This paper illustrates students' algebraic habits of minds through interviews and demonstrates how students have difficulties in algebra. In this study we present data from clinical interviews with sixth grade students, which were analyse using Driscolls' (1999) framework of algebraic habits of mind. All students show different algebraic habits of mind and their difficulties in algebraic concepts constrain their ways of thinking.

Keywords: Algebra, habits of mind.

INTRODUCTION

One major learning goal of mathematics education is to improve students' different ways of thinking. When students spontaneously and are likely to use mathematical ways of thinking, it is reasonable to call them mathematical habits. Several researchers have studied mathematical habits of mind (Lim & Selden, 2009). Mathematical habits of mind are viewed as students' mathematical thinking in the same way mathematicians do (Cuoco, Goldenberg, & Mark; 1996). Mathematical habits of mind are general approaches and there are also content-specific habits such as algebraic habits of mind. Driscoll (1999) provides a theoretical framework for algebraic habits of mind. He considers algebraic habits of mind habituated ways of thinking that help students succeed in learning algebra. Further, Driscoll (1999) concentrates on three algebraic habits of mind: (1) doing-undoing, (2) building rules to represent functions, and (3) abstracting from computation. These critical habits of mind help students to use and apply what they know and have learned when they encounter any problem they haven not specifically learned. Students who have not developed different habituated ways of thinking struggle particularly with algebra and mathematics in general. This research, being in the initial stage of an ongoing longitudinal study is designed to bring students in algebraic habits of mind in over two years. The purpose of the study is to investigate sixth grade ele-

Categories	Indicators	NUR	ALİ	GÜL	CAN
Doing-Undoing	Working the steps of a procedure backward	✓	✓	✗	✓
	Finding initial conditions from a solution	✓	✗	✗	✓
Building Rules to Represent Functions	Organizing information in useful ways	✓	✗	✓	✓
	Describing change in a process	✓	✓	✓	✓
	Noticing a rule at work and trying it how it works	✗	✗	✓	✓
	Describing the steps of a rule without using specific inputs	✗	✗	✗	✗
	Justifying why a rule works for any number	✗	✗	✗	✗
	Seeking and using connections across different contexts or different representations	✗	✗	✗	✓
Abstracting From Computation	Looking for shortcut in computation	✗	✓	✗	✓
	Thinking about calculations independently of the particular number used	✗	✓	✗	✓
	Recognizing equivalence between expressions	✓	✓	✓	✓
	Expressing generalizations about operations symbolically	✗	✗	✗	✓

Table 1: Each student's algebraic habits of mind

mentary students' acquired algebraic habits of mind after completing their first year of algebra domain. Initial interview results will help us to plan the next steps of this research to support students' algebraic habits of mind.

METHODOLOGY

Participants of this study were four sixth grade middle school students. Data were collected through clinical interviews at the end of the sixth grade. Students were given problems to solve and, after solving each problem, they were asked for detailed descriptions, and to apply different ways to solve problem. The four interviews were fully transcribed and students' responses were put into one of the Driscoll's algebraic habits of mind indicators and categories.

RESULTS AND DISCUSSION

The results indicate that all students exhibited some algebraic habits of minds at the end of the sixth grade (Table 1). However, students' difficulties in conceptualizing algebraic concepts appeared to constrain their ways of thinking.

As shown in Table 1, the indicators of algebraic habits of mind vary across individual students. In the interviews, all students described change in a process and recognized equivalence between expressions. In addition, none of the students described the steps of a rule without using inputs, and justified why a rule works for any number. Thus, students did not seem to exhibit various and different thinking ways. This has implications for teachers and for us. Teachers should benefit from students' thinking in designing their mathematics courses, and use interviews as a tool to further investigate their students' thinking. As a next step, we will further design our mathematics course primarily to gain students how to describe the steps of the rules without using specific input and how to justify why a rule works for any number. Additionally, we will overcome students' difficulties about algebraic concepts and support all indicators of students' algebraic habits of mind.

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Middle grade teachers' thinking of algebraic reasoning in relation to their classrooms

David Glassmeyer and Belinda Edwards

Kennesaw State University, Kennesaw, USA, dglassme@kennesaw.edu

The purpose of this study was to investigate United States mathematics teachers' thinking about algebraic reasoning within the context of their middle school classrooms. After collecting document and observation data from 19 teachers in a two-week summer professional development workshop, we analysed how teachers defined algebraic reasoning in their classroom and their description of algebraic reasoning tasks for their students. Our findings detail the ways teachers initially described algebraic reasoning in the context of their classroom and the changes in thinking teachers reported during and after the workshop.

Keywords: Algebraic reasoning, professional development, teacher thinking.

AIMS AND RESEARCH QUESTION

Algebraic reasoning is an essential habit of mind for building conceptual knowledge in K-12 mathematics (Kaput, 2008), yet little is known about how K-12 mathematics teachers think about algebraic reasoning in the context of their classroom (Blanton & Kaput, 2005; Ellis, 2011). In this project, we aimed to address this research need by examining how algebraic reasoning was considered by middle school in-service mathematics teachers who taught grades 6, 7, or 8 in the United States. Our research question was: how do teachers develop their understanding of algebraic reasoning in the context of their classroom through a two-week professional development session? This question focused our efforts on characterizing how teachers communicated their understanding of algebraic reasoning throughout the professional development and during the following months, after teachers returned to their classrooms. We use Blanton and Kaput's (2005) definition of algebraic reasoning as "a process in which students generalize mathematical ideas from a set of particular instances, establish those generalizations through the discourse of argumentation, and express them in increasingly formal and age-appropriate ways" (p. 413). We use these authors' depiction of algebraic reasoning as a framework for this study.

METHODS

The participants of this study were 19 middle school teachers from the Southern United States engaged in

Code	Reflection 1	Reflections 2–5	Reflection 6
Single Solution	11 (58%)	0 (0%)	0 (0%)
Single Solution Strategy	6 (32%)	0 (0%)	0 (0%)
Single Representation	2 (11%)	0 (0%)	0 (0%)
Multiple Solutions	1 (5%)	4 (21%)	1 (5%)
Multiple Solution Strategies	5 (26%)	17 (89%)	6 (32%)
Multiple Representations	3 (16%)	8 (42%)	3 (16%)
Procedural knowledge	12 (63%)	3 (16%)	0 (0%)
Conceptual knowledge	10 (53%)	14 (74%)	11 (58%)
Expressing Generalization	2 (11%)	13 (68%)	5 (26%)
Functional Thinking	1 (5%)	7 (37%)	0 (0%)

Table 1: Counts of how many teachers (n=19) made statements coded using our coding dictionary before (reflection 1), during (reflections 2–5) and after (reflection 6) the professional development

a two-week professional development session and a follow-up meeting two months later. We conducted observations and collected documents from teachers during and after a two-week professional development session. Data consisted of teachers' daily reflections prompting teachers to reveal their thinking about algebraic reasoning and researcher notes about teacher work on various activities and conversations during the professional development sessions. We used content analysis to analyze teachers' reflections and our observation notes by creating and refining our codes based on existing literature.

FINDINGS

As summarized in Table 1, we found that teachers' initial thinking about algebraic reasoning included procedural tasks with a single solution, solution strategy, or representation. At the end of the two-week professional development, teachers described algebraic reasoning as requiring conceptual knowledge and multiple solutions, solution strategies, or representations. Some teachers also associated aspects of generalization and functional thinking as part of algebraic reasoning. Two months after the professional development, teachers still described algebraic reasoning as tasks requiring conceptual knowledge rather than procedural skills, and included multiple solutions or solutions strategies. Teachers did not continue to associate generalization or functional thinking as part of algebraic reasoning.

These findings may help other teacher educators anticipate teacher thinking when working to develop algebraic reasoning in professional development settings and identifies more work with teachers' algebraic reasoning is needed to support teachers' use of generalization and functional thinking in their classrooms.

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Developing the notions of variation and covariation through patterns: An institutional analysis of primary school textbooks

Eudes Libert Mamas Mavoungou and Alejandro S. González-Martín

Université de Montréal, Département de Didactique, Montréal, Canada, eulimas19@yahoo.fr

We examine how pattern-based activities presented in textbooks for primary education allow developing the notions of variation and covariation, using the conceptual tool of institutional relationship of Chevallard (2003). Our sample is formed by the textbooks currently approved by the Quebec Ministry of Education. Ours results reveal a didactic void in the textbooks.

Keywords: Variation and covariation, textbook analysis, institutional relationship.

INTRODUCTION AND RESEARCH PROBLEM

Researchers have identified numerous difficulties faced by students in learning functions. Because elementary school students are able to grasp pre-algebraic concepts, researchers recommend introducing pre-algebra at the primary level to facilitate the subsequent learning of algebra and functions at higher levels. The notions of variation and covariation have been highlighted for their role in fostering a better understanding of functions at the secondary level (Hitt & González-Martín, 2015). In primary education, the notions of variation and covariation can be taught using pictorial growth patterns, which foster two modes of reasoning: figural and numerical. Becker and Rivera (2005) found that prospective elementary school teachers predominantly use numerical reasoning to analyse patterns with their pupils. Because both modes of reasoning should be present in the classroom, and considering the tendency of teachers and students to favour numerical reasoning, we are interested in examining how pattern-based activities are presented in elementary school textbooks and how this reveals the institutional relationship with these notions. Specifically, our research questions are: 1) how are notions related to pictorial growth patterns organised in elementary school textbooks and what type of reasoning do they tend to favour, and, 2) do the textbooks' activities effectively allow students to develop the notions of variation and covariation?

THEORETICAL FRAMEWORK

Our research leans heavily on Chevallard's anthropological theory of the didactic (ATD, 2003). In particular, we employ the notion of *institutional relationship* to study institutional choices concerning the use of the mathematical objects of variation and covariation in textbooks. We also use the notion of praxeology to characterise the institutional relationship with the notions of variation and covariation.

METHODOLOGY

The content analysis applied to textbooks in this study was performed with tools provided by ATD. We analysed the three current elementary school textbooks approved by the Quebec Ministry of Education for the final years of primary school (10- and 11-year-old students), focusing on the following elements: the nature of symbols used (numerical or pictorial), the types of tasks and the praxeologies they define, the goal of these tasks, the independent and dependent variables at play in the task, the regularity from one position to subsequent positions, the mode of reasoning favoured, and the relationships between variables. We paid particular attention to how the textbooks lead students to use the information presented to accomplish tasks and develop an awareness of relationships between two given quantities.

DATA ANALYSIS AND DISCUSSION

We found only 18 cases, in two of the three textbooks, in which a task makes use of patterns: 11 tasks using numerical patterns, six tasks using patterns presented both numerically and pictorially, and one task using pictorial patterns. We identified three different types of tasks: 1) introducing the definition of a pattern, 2) predicting subsequent terms, and, 3) generalising through an expression. Each of these types of task defines a different mathematical organisation: the first attempts to familiarise students with the idea of change or variation, the second aims to teach students the idea of covariation, and the third focuses on dependence relations through a rule. The relationship between variables is not specified in either of the two textbooks and both use the symbol n in formulae without explicitly giving any meaning to it. All 18 tasks can be solved using just two standard techniques, which favour the use of numerical reasoning. Moreover, the technology that could reinforce and give meaning to the notions of variation and covariation is absent, and therefore students are not provided with optimal conditions for developing these notions. The institutional relationship with these notions is characterised by a didactic void, especially with regard to the theoretical block. We are aware that this research has been conducted on a small scale and that further study is necessary to investigate whether children are given better opportunities to develop the notions of variation and covariation in primary school. For this reason, we are planning to examine potential difficulties students may encounter when presented with patterns and other generalisation activities. We also wish to study teachers' practices and professional teacher education in the effective use of patterns.

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Creative reasoning more beneficial for cognitively weaker students

Mathias Norqvist[1], Johan Lithner[1], Bert Jonsson[1] and Yvonne Liljekvist[2]

1 Umeå University, Umeå, Sweden, mathias.norqvist@umu.se
2 Karlstad University, Karlstad, Sweden

In a study with 91 upper-secondary students the efficiency of two different types of mathematical practice tasks, procedural based algorithmic tasks and creative reasoning tasks, were studied. It was found that although the algorithmic group outperformed the creative group during practice the latter performed significantly better on a follow-up test. Closer inspection revealed that the difference in test performance was, contrary to common beliefs, driven by the cognitively weaker students.

Keywords: Mathematical reasoning, creative reasoning, cognitive proficiency.

LEARNING BY IMITATIVE AND CREATIVE REASONING

Starting off from research that points to the inefficiency of rote learning, the LICR design research project is studying the efficiency of different kinds of practice tasks. One sort of task where the student is presented with an already complete solution or formula and has to practice this, much like the layout in most textbooks (Algorithmic Reasoning, AR). This is contrasted by a task type that gives no indication on how to solve the specific task, constructed in such a way that it gives the student a chance to, with small creative steps, construct a general solution (Creative Mathematically founded Reasoning, CMR). The tasks are designed based on the mathematical reasoning framework (Lithner, 2008) utilizing a specific adidactical situation (Brousseau, 1997).

METHOD

Our sample consisted of 131 students at the Natural Science program from four Swedish upper secondary schools (16–17 year olds) that after attrition and screening for ceiling effects was reduced to 91 (48 AR and 43 CMR). We used two cognitive tests, Raven's

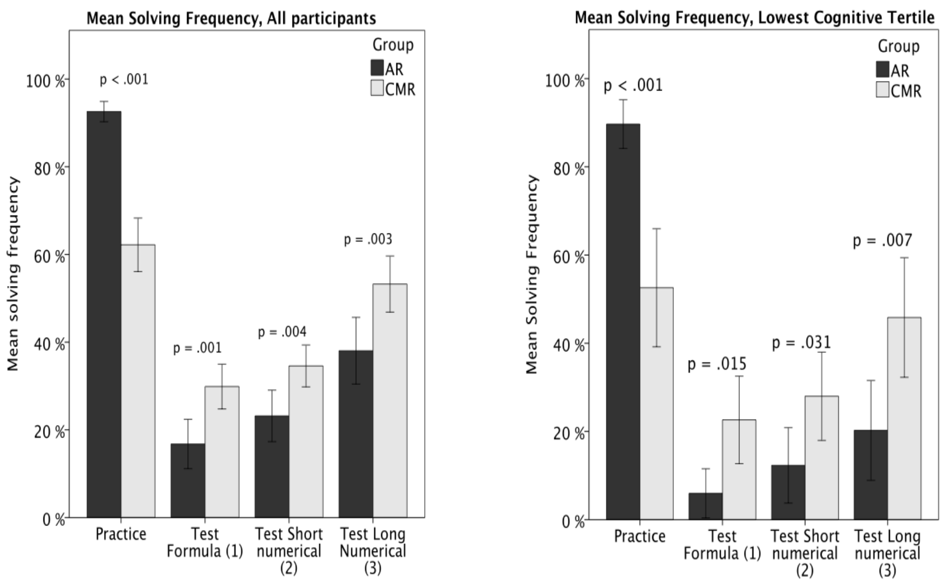

Figure 1: The left diagram shows the mean result for practice and test for all participants while the right shows the mean result for the lowest cognitive tertile (error bars represent two standard error of means)

Progressive Matrices (non-verbal problem solving ability) and Operation Span (working memory capacity), to construct a cognitive composite score. Both of these tests have been proven to be of importance for mathematics achievement (Primi, Ferrão, & Almedia, 2010; Alloway, Gathercole, Kirkwood, & Elliot, 2009). The cognitive composite score, mathematics grade and gender were used to match the participants into two comparable practice groups, AR and CMR.

The two groups got to practice on 14 task sets, each with a specific target knowledge. The CMR-group solved three tasks per task set while the AR-group solved five tasks to compensate for the quicker AR-tasks. The allotted practice time was the same for both groups but there was a slight difference in used practice time, 29 min (SD, 10) for CMR and 21 min (SD, 6) for AR. One week later the two groups took the same test. The test consisted of three tasks per task set. They evaluated: 1) memorized knowledge of a specific formula, 2) if a mathematical principle was memorized, and 3) if the formula or principle could be reconstructed if forgotten.

RESULT

Looking at the test result (Figure 1) the CMR-group significantly outperformed the AR-group, $t(89) = 3.54$, $p = .001$, $d = 0.73$, on all three tasks even though the AR-group performed much better during practice. The analysis also showed that the test performance of the AR-group was highly predicted by their cognitive score. The test performance of the CMR-group was predicted by their practice result and here a teacher can have huge impact. The most interesting result however is that the differences in test results were mainly driven by the cognitively weaker students rather than by the stronger students as might be expected according to common beliefs. Put together, these results would imply that CMR practice is more efficient with regard to remembered knowledge and also more neutral in terms of cognitive prerequisites (Jonsson, Norqvist, Liljekvist, & Lithner, 2014).

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Prototypes in secondary and university mathematical education

Derek Pilous and David Janda

Charles University, Faculty of Education, Department of Mathematics and Mathematical Education, Prague, derek.pilous@seznam.cz

The aim of presentation was to demonstrate our extension of previous theoretical framework of prototypes – a term of cognitive psychology – in mathematics education, illustrated by examples of students' outcomes and sets out a research project based on this framework. The study was supported by Charles University in Prague, project GA UK No 227-364.

Keywords: Prototype, function, example, mathematical concept, classification.

THE THEORY OF PROTOTYPES

The theoretical framework of the concept of prototypes lying in cognitive psychology is presented in two ways. First, traditionally, by a description of historical evolution from first research studies carried out by Eleanor Rosch in the 70'th and introduction of prototypes to mathematics education by Rina Hershkowitz and others in the 80'th of the twentieth century. Second, we will demonstrate inherent usage of prototypes in mental representation of mathematical, well-defined concepts through introspection of poster readers.

Schwarz and Hershkowitz (1999) describe prototypes as construct of probabilistic approach to concept learning –"special examples, that are more central to learning than others". Hadjidemetriou and Williams (2010) offer strong example, linearity, ascribing it to schooling. Our goal is to define and identify prototypes of mathematical concepts and also observe long-term development of prototypes. In this poster, we concentrate on two questions:

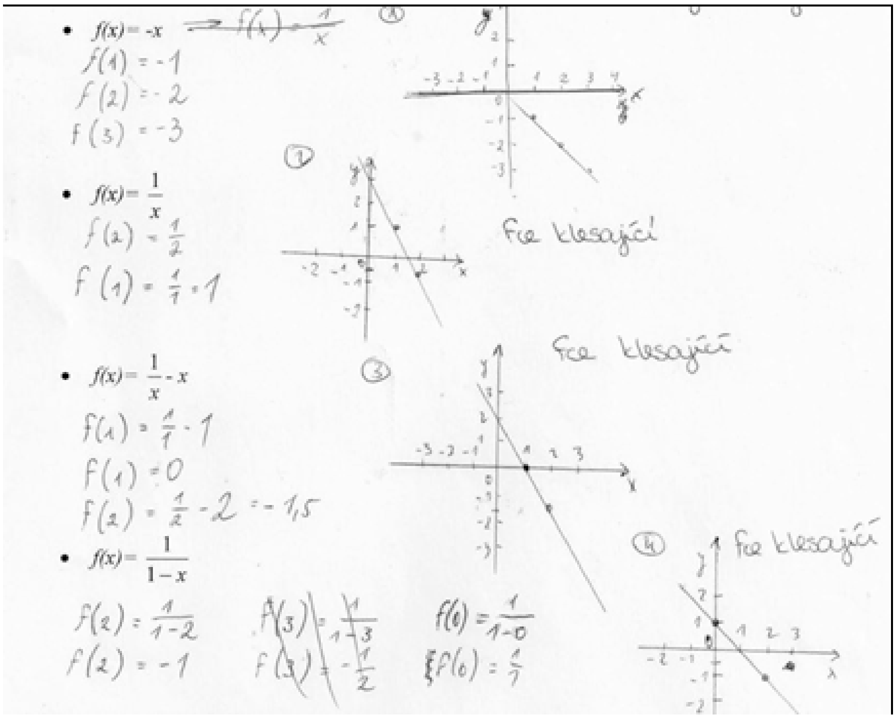

Figure 1: A student's solution

1) How to characterize prototypes of mathematical concepts in a way which will provide criteria for their detection?

2) Which prototypes of common important mathematical concepts like functions and real numbers do students use?

In the theoretical part of poster we pose two conditions for useful definition of prototypes in context of mathematical education and present our working definition: *A prototype is any instance of concept, a set of instances or any property functionally connected to concept, which is statistically preferred by an individual in usage of that concept, if this preference cannot be explained by a mathematical essence of that concept and informational complexity of the would-be prototype.*

Theoretical framework is followed by examples of prototypes from our observations of students' prototypes in ordinary lessons of mathematics and from our research not specifically focused on prototypes (Janda, 2013). One such example is given in Figure 1. A group of 40 secondary school students solved the problem "*How does f(x) change if we replace x by various integers? Describe or represent it.*" for four functions.

As we can see, substituting two or three numbers for x satisfied the student enough to draw supposed graphs of functions. Numbers 1 and 2 were the most often used (hence, we consider small positive integers to be prototypes of real numbers), number 0 was used only once, negative numbers were not used at all (only 14% of the students inserted at least one negative number into any of the rules).

Figure 1 is an example of the tendency to "linearize" graphs of functions which was generally very strong among students; in Figure 1, we can even see that the student omitted the result of the evaluation for $x = 3$ (by crossing out the evaluation itself and the corresponding point in the Cartesian plane) because it did not conform with the linear function passing through points which resulted from previous evaluations.

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Generalising from visual spatial patterns

Aisling Twohill

Dublin City University, St. Patrick's College, Dublin, Ireland, aisling.twohill@spd.dcu.ie

In the research project presented, I categorise the strategies employed by four Fourth class girls and four Sixth class girls when asked to construct a general term for a visual spatial pattern. The approach adopted by each girl is categorised as either explicit or recursive, and the mode of generalisation is categorised as either linear or numeric.

Keywords: Visual spatial pattern, generalisation strategies, reasoning.

THEORETICAL BACKGROUND

Generalisation is considered by many to be a highly significant component of algebraic reasoning (Kaput, 2008). Patterning plays a key role in supporting children's developing skills in generalisation, and internationally visual spatial patterning tasks have been utilised in many research projects to investigate children's success in generalising (e.g., Rivera & Becker, 2011).

Lannin (2004) presents two approaches to the construction of generalisations from patterns. An 'explicit' approach involves identifying a rule for the relationship between a term and its position in the pattern, whereas a 'recursive' approach focuses on a relationship between successive terms. Rivera and Becker (2011) discuss the tendency of some children in their research to adopt a 'numerical' rather than 'figural' mode of generalising, which in some cases caused difficulty in the children's own reasoning about the generalisations they had constructed.

Within the algebra strand of the Irish Primary School Mathematics Curriculum (PSMC) it is proposed that children be facilitated in establishing rules for number sequences, but explicit approaches are not mentioned, and examples given all indicate constant differences between terms (Government of Ireland, 1999). Generalisation is not mentioned and visual spatial patterning is not suggested for consideration, beyond repeating patterns of shapes aimed at the youngest children in primary school. In light of this, the research discussed in this paper aims to explore the responses of primary school children in Ireland when asked to construct generalisations from a visual spatial pattern.

METHOD

Individual clinical interviews were used to gather data on children's constructions of generalisations.

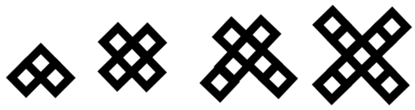

Figure 1: Diamonds pattern

Four girls were randomly selected from each of Fourth class and Sixth class in an Irish primary school. The cohorts from the two classes had mean ages of 10.08 years and 12.19 years respectively. Pseudonyms are used. The girls were asked to describe, extend and construct near and far generalisations from the pattern terms shown in Figure 1. Suggestive questions were asked to support children in making progress in the solution of tasks.

FINDINGS AND DISCUSSION

During the clinical interview, each girl was asked to identify the 100th term in the pattern, as a far generalisation. Each girl's generalisation, or work towards a generalisation, was deemed to be figural or numeric. A response was deemed to be figural if the child referred to the position of the diamonds, by referring to the 'top' or 'bottom' or through gesture and use of the deictic 'there'. Table 1 summarises the mode of generalisation adopted by each girl and whether the approach adopted in working with the pattern was recursive or explicit. An incomplete generalisation

Name	Class Level	Numeric/figural mode of generalisation	Recursive or explicit approach	Validity of generalisation
Natasha	4th	Figural	Explicit	Complete, but flawed numerically
Bella	4th	Figural	Explicit	Complete and valid
Tara	4th	Did not respond	Did not respond	No response
Nikki	4th	Numeric	Recursive and later explicit	Complete and valid
Lisa	6th	Numeric	Recursive and later explicit	Complete and valid
Sarah	6th	Figural	Did not explain her thinking	Complete and valid
Aoife	6th	Numeric	Recursive	Incomplete
Rachel	6th	Numeric	Recursive	Incomplete

Table 1: Classifications of girls' constructions of a far generalisation

refers to an instance when a participant did not construct the 100th term.

Within the PSMC the 6th Class girls may have encountered many number sequences which favour recursive thinking. Lannin (2004) suggests that such immersion may cause difficulty in considering an explicit approach. Additionally, the numeric approach of Aoife and Rachel seemed to inhibit their thinking about the general term, in a manner consistent with the findings of Rivera and Becker (2011).

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Teachers' procedures when introducing algebraic expression in two Norwegian grade 8 classrooms

Unni Wathne

University of Agder, Kristiansand, Norway, unni.wathne@uia.no

We investigate similarities and differences in two teachers' way of introducing algebraic expressions by designed examples. One teacher moves from the specific to the general, and the other moves from the general to the specific. They both mediate the passage from the students' real world and the school mathematics they know, to algebra.

Keywords: The introduction of algebra, designed examples, mediating tools.

Two teachers, Kari and Ola, are introducing algebra in two Norwegian grade eight classrooms. Kari holds up a set of large playing cards and writes on the blackboard what is written in the corner of the cards. Ola starts walking in one direction in the classroom, asking the students to describe what he is doing. These are the starting points of two examples that each teacher has designed as a tool for communicating and explaining new algebraic ideas in her/his classroom. Kari makes the students familiar with using letters for numbers with the help of playing cards. Ola makes the students familiar with algebraic expressions and variables with the help of body movement.

The aim of this study is to investigate the introductory lesson when two teachers mediate algebraic concepts through their examples. The purpose of the analysis is to capture how the teachers approach the complexity students meet in such learning situations. More specifically: Which procedures do the teachers' use to introduce the concept of algebraic expressions?

THE THEORETICAL FRAMEWORK

The shift in school mathematics from arithmetic to algebra is known to be challenging for students. The learning of algebra includes new symbols, new concepts and also new ways of thinking. Examples play a central role in the teaching and learning of mathematics as described among others by Rowland (2008).

The concepts of mediation and mediating tool (Wertsch, 1991) have emerged from our empirical material. The theoretical term of mediating tools facilitates our analysis in making a distinction between the goals of the lessons (including the mathematical objects of variables and algebraic expressions) and the tools (designed examples, concretes and semiotic items) that the teachers employ in their interaction with the students.

THE METHOD

In order to accomplish the aims of the study we use a qualitative approach to collect and analyze the empirical data grounded in a sociocultural theoretical perspective of learning. The data has been collected after the VIDEOMAT design (Kilhamn & Röj-Lindberg, 2013); we observed the first five algebra lessons in each classroom (videotaping), interviewed the teachers after the fifth lesson (audiotaping) and collected written material used in the classrooms (teacher and student material). As a first approach to the collected data, lesson graphs for each lesson were elaborated, and the first lesson in all classrooms was transcribed.

The two examples were chosen through a process involving the lesson graphs and several viewings of the video material. They comprise the introductory part of the first lesson in each classroom. The designed examples stood out as unique in the international video material. The examples are also referred to in later lessons by the teachers and therefore play an important role in the introduction of algebra in these two classrooms.

RESEARCH RESULTS

There are similarities and differences in these teachers' way of introducing algebra. The two teachers both design introductory examples which are used as their central means for explaining the same concepts (variable and algebraic expression). The examples are easily distinguishable in their use of concrete materials, cards versus the body. However, there are more fundamental differences in the examples' structures.

Kari starts with numbers, number operations, and arithmetic expressions, and she makes generalizations introducing algebraic expressions. She continually connects the arithmetic and the algebraic elements, and sees variables as numbers. Kari moves from the specific to the general in her approach to introduce algebraic expressions.

Ola, on the other hand, establishes an algebraic expression directly from the imaginary number line with given direction and units (first step, then foot) without using numbers. He builds the algebraic expression through a transformation chain following this path: bodily movement – words – abbreviations – variables, and sees variables as quantities, numbers included. Ola moves from the general to the specific in his approach to introduce algebraic expressions.

The aim of this study has not been to propose how algebra should be introduced in the classroom. The analysis illuminates the complexity students meet when facing introductory algebra in school, and the challenge it is for teachers to make algebra accessible for all students. The main procedure of the teachers in this study has been to use examples designed by themselves, mediating the passage from the students' real world and the school mathematics they know to algebra.

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TWG04
Geometrical thinking

Introduction to the papers of TWG04: Geometrical thinking

Sona Ceretkova[1], Aurélie Chesnais[2], Michela Maschietto[3], Joris Mithalal-Le Doze[4], Philippe R.Richard[5] and Ewa Swoboda[6]

1 Constantine the Philosopher University, Nitra, Slovak Republic
2 University Montpellier 2 – University Montpellier 3, Montpellier, France
3 University of Modena e Reggio Emilia, Modena, Italy, michela.maschietto@unimore.it
4 ESPE Paris, University Paris 4, Paris, France, joris.mithalal@espe-paris.fr
5 University of Montréal, Montréal, Canada
6 University of Rzeszów, Rzeszów, Poland

INTRODUCTION

TWG04 on geometrical thinking had more than 22 participants from 10 countries. During the sessions, the participants discussed thirteen papers and three posters (one paper and two posters were submitted but not discussed in the working group).

TWG04 about geometrical thinking has been working on this topic for many CERME sessions, and mostly studied what geometrical thinking would mean. It is concerned with research and development of geometrical thinking from pre-school up to University teaching and learning, including any type of geometry. In line with the previous sessions, CERME8 tried to identify four competencies that helped describe geometrical thinking: reasoning, figural, operational and visual (Maschietto et al., 2013). The discussion paper focused on educational aspects related to the development of these geometrical competencies for students, the difficulties of learning and the design of curricula and their implementation. These four poles were very helpful to use a common language and shared reference concepts. They also allowed to show how the many faces of geometrical activity are intertwined, which made necessary a clarification of what was meant by these poles.

In this working group at CERME 9, there were more specific contributions about the way geometry is, or should be, taught: for instance, in-class observations, pre-service teachers education, going from everyday concepts to geometrical knowledge, specific issues in geometry. As a consequence, the four competencies were used as a general way of describing the geometrical activity and for creating links between different points of view. For instance, line symmetry was studied either in a didactical engineering perspective or to characterize ordinary classrooms practices. Then, the papers and posters contributed to study what is at stake in the teaching and learning processes.

During the discussions, the work was structured by four specific themes, specifically studied in the contributions:

- Initial geometrical knowledge
- Visualization
- Transformations and 3D
- Actions on objects (material and mental)

We will emphasize here how the papers about these four themes deal with the following questions, and more specifically:

- Who is the research about?
  - Pupils
  - Teachers
  - Teachers educators
- What are the aims of the research?

  - In the description of existing teaching and learning phenomena
  - In the design of new tasks
- How do we study this?
  - Theoretical tools
  - Methodology
- What are the interactions with the other themes? This question also concerned the role of language and social interactions that appeared as a common issue to several papers.

THE ROLE OF INITIAL GEOMETRICAL KNOWLEDGE

We chose to use this notion, instead of pre-concepts, to emphasize a general issue in geometry: going to everyday concepts to geometrical ones does not only concern early ages geometry. The participants showed that "intuitive notions" are very important for the construction of geometrical knowledge. This is true for many topics as rotation (Swoboda), line symmetry (Chesnais & Mathé), polygons (Ulusoy), polyhedra (Mithalal-Le Doze & Papadaki) and at every age from the beginning of the primary school (Rodrigues & Serrazina) to prospective teachers' education (Brunheria & da Ponte, Kuzniak & Nechache).

This idea of initial geometrical knowledge is linked to the idea of "geometrization", seen as a dynamical and continuous process that turns it into an institutionalized geometrical knowledge, with the development of geometrical competencies. This process is a fundamental part of geometrical thinking, like the more scientific knowledge it helps building. We could see that this process involves at the same time pupils, teachers, teacher educators, and researchers.

It is necessary for teachers to be aware of how pupils' personal previous experience may influence their learning geometry, in order to design efficient teaching sequences (Loureiro & Serrazina, Herendiné-Kónya). Therefore, they need theoretical tools, especially pedagogical and didactical knowledge, to adapt the in-class experience to the pupils and the topics.

Many frameworks were evoked - and had been mentioned in the previous CERME works. Some of those frameworks are very specific to geometry, as Van Hiele levels (Papadaki), Geometrical Working Spaces (Kuzniak & Nechache), concept image and concept definition (Rodrigues & Serrazina, Ulusoy), figural concept (Ulusoy), visualization (Papadaki, Mithalal-Le Doze). Other frameworks are more general in mathematics education, as Theory of Didactical Situation (Douaire & Emprin) or from psychologists (mainly Battista or Gagatsis' works exploited by Loureiro and Serrazina). The papers have shown that in this case, it is possible to design tasks involving outdoor activities, real-life experiences, with high didactical potential (Douaire & Emprin).

ACTION ON OBJECTS

The first point concerns the role of action on objects, both mental ones and material ones. A paradoxical situation was raised, as it is at the same time very natural to pupils, and quite difficult for the teachers to develop their geometrical competencies from it and not to use it only with young pupils to increase their engagement.

"Action on objects" is quite confusing, and we decided to reduce its meaning to action performed by hand on material or tangible objects -- which includes the use of instruments (like transparent paper by Chesnais and Mathé, or Uygan et al.) or Dynamic Geometry Software (Mithalal-Le Doze). We studied the many functions of action (Pytlak): it helps the pupils to develop intuition, concept image and definition, geometrical imagination, and at the same time it makes children's knowledge more visible to the teachers.

This function strongly depends on specific conditions, and for many of us it was essential that manipulation came first, to create a need for anticipation, validation or control that may justify geometrical knowledge. This articulation is organized by the tasks themselves, the backing of the teacher and the social interactions.

TRANSFORMATION AND 3D GEOMETRY

These two topics were developed in many papers, which gave good examples of how complex the relations between action, visualization and geometrical knowledge, are.

The learning of geometrical transformations (rotation by Swoboda, line symmetry by Chesnais & Mathé, or isometrics by Thaqi et al. and Uygan et al.) depends on linking multiple contexts and representations. It also requires articulating a global and a punctual point of view and going from perception to geometrical properties.

The same questions were discussed for 3D geometry, and we showed that the greatest issues were not only "sight" issues. Indeed, a psychological point of view shows the role of getting better "images" to act on (physically or mentally), but many of the previously mentioned mathematical aspects are part of the visualization process. For instance, Dynamic Geometry Software, and more generally geometrical tools, were seen as ways of making geometrical knowledge useful for a better control of the actions and visualization. This knowledge was at the heart of the visualization process when using only the sense of touch: linking the subparts that were perceived by touch is a mathematical process linked to what Duval calls deconstructions (see Mithalal-Le Doze, Papadaki). At the end, we showed that visualization depends on perceptual, psychological, but also – and this is fundamental – on mathematical aspects.

VISUALIZATION

Eventually, we had to clarify what we called visualization. It was involved in studies of mental abilities (especially for initial geometrical knowledge and action on objects), classification (how do we discriminate the information while seeing), and analysis of drawings in a deductive geometry context (Brunheria & da Ponte, Herendiné-Kónya). We underlined that, in this case, visualizing aims at being able to solve geometry problems, so that it both depends on very particular cognitive processes and mathematical knowledge. Therefore, a great difficulty is the gap between visualization by teachers, based on categories and geometrical properties, and by pupils, often based on prototypes. The main theoretical tools used were Laborde's work about drawings and figures (Mithalal-Le Doze), and Duval's distinction between iconic and non-iconic visualization (see Mithalal-Le Doze).

CONCLUSION

The discussions in TWG04 confronted very different points of view on geometry teaching and learning, with complementary issues (e.g., teaching practices studies vs. task design), very different cultural and educational contexts that change the role of geometry in the curricula and the way it is presented. It also appeared that these contexts had an influence on our researcher positioning, which not only concerns geometry: Does it mean our research practice or the teachers' education practice that most of us share? What can we learn from research on everyday practices? How is it possible to better combine our teacher, teacher educators, and researcher positions?

Eventually, let us mention the new issue of the role of language and social interactions in the teaching and learning geometry processes. This has been little discussed during the previous sessions, but it appeared that this could play a great role in each of the topics mentioned above and that some of the phenomena are very specific to geometry.

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TWG04

Research papers

Prospective teachers' development of geometric reasoning through an exploratory approach

Lina Brunheira[1] and João Pedro da Ponte[2]

1 Escola Superior de Educação do Instituto Politécnico de Lisboa, Lisboa, Portugal, lbrunheira@eselx.ipl.pt

2 Universidade de Lisboa, Instituto de Educação, Lisboa, Portugal

We aim to characterize how prospective teachers perform in defining and classifying quadrilaterals through working on exploratory tasks. Data was gathered from the participants' reports and portfolios. Results show most understood the meaning of defining and presented correct definitions, using properties they previously ignored, and showing comprehension of the underlying concepts. They produced economical definitions in few cases, and performed better in inductive than deductive reasoning. The classifications showed conflicts between prior classifications and structural criteria that rules a geometrical classification. The exploratory work allowed participants to construct their knowledge in a meaningful way and reflection played an important role in becoming aware of personal preconceptions and knowledge.

Keywords: Teacher education, geometry, geometric reasoning, exploratory approach.

INTRODUCTION

This paper addresses prospective elementary teachers' preparation in geometry. Recent studies in Portugal show less than satisfactory results concerning the geometric knowledge they present before and after attending their teacher education programs (Menezes, Serrazina, & Fonseca, 2014; Tempera, 2010). A similar conclusion is also found in studies from other countries, concerning teachers and prospective teachers, indicating that geometry is an area in which they perform poorly, have little self-confidence, and show weak geometric vocabulary (Clements & Sarama, 2011; Fujita & Jones, 2006; Jones, Mooney, & Harries, 2002). In addition, there are very different views about what geometry can or should be taught in teacher preparation courses, which is problematic as the success of the teachers' work depends, to a great extent, on their deep understanding of geometry. And, we must also remember that knowing geometry does not ensure effectiveness, how teachers come to know it matters as well (Jones, Mooney, & Harries, 2002).

This scenario challenges us to improve teacher's curriculum and preparation in this area and investigate its outcomes. The work we report in this paper fits into a wider study with that goal. We developed a design research experiment in the context of a curricular unit of geometry based on exploratory work, linking geometry and didactics and valuing prospective teachers' reflection on their learning. We seek to characterize how prospective teachers perform in processes which are components of geometric reasoning, focusing on defining, but also looking at classifying.

CONCEPTUAL FRAMEWORK

Prospective elementary teacher education in geometry

For the National Council of Teachers of Mathematics [NCTM] the knowledge necessary for teaching includes

> the content and discourse of mathematics, including mathematical concepts and procedures and the connections among them; multiple representations of mathematical concepts and procedures; ways to reason mathematically, solve problems, and communicate mathematics effectively at different levels of formality. (1991, p. 132)

This perspective is coherent with the idea advocated by Ma (1999) that teachers need a profound understanding of fundamental mathematics. But what does this mean in geometry? The NCTM (1991) states that all teachers should understand how geometry is used to describe the world we leave in and how it is used to solve concrete problems; analyze a diverse set of two and three dimensional figures; use synthetic geometry, coordinates and transformations; improve their skills in producing arguments, justifications and privilege spatial visualization. In 2000, the Conference Board for the Mathematical Sciences (CBMS) proposed that prospective K-5 teachers must develop competence in the following areas: Visualization skills (projections, cross-sections, decompositions; representing 3D objects in 2D and constructing 3D objects from 2D representations); basic shapes, their properties, and relationships among them (angles, transformations, congruence and similarity); and communicating geometric ideas (learning technical vocabulary and understanding the role of mathematical definition). The recent report of CBMS (2012) updates the main ideas for teaching preparation in geometry, presenting less topics and less complex competencies:

— Understanding geometric concepts of angle, parallel, and perpendicular, and using them in describing and defining shapes; describing and reasoning about spatial locations (including the coordinate plane).

— Classifying shapes into categories and reasoning to explain relationships among the categories.

— Reason about proportional relationships in scaling shapes up and down. (p. 30)

This shift confirms the lack of agreement about the geometric knowledge teachers should hold. In addition, the education of teachers concern also the ways they are taught. Regarding the results of several studies about prospective teachers' knowledge of mathematics, Watson and Mason (2007) propose that courses should prompt participants to engage in mathematical thinking through working on suitable mathematics tasks, develop their understanding about the features and power of those tasks, reflect on the experience of doing mathematics tasks individually or with others, challenge approaches dominated by procedures which depend on rote memorization and observe and listen to learners. These orientations are also consistent with ideas underlined by other investigators: in teacher education, the prospective teachers should learn using the same methods that are recommended they should use in the future (Ponte & Chapman, 2008); connecting subject matter knowledge and pedagogy is a promising strategy to develop both kinds of knowledge and their integration, which is critical to teach well (Ball, 2000). The work we conducted follows these proposals, as we focus on prospective teachers' learning as they work on exploratory tasks and reflect on their own learning. Exploratory tasks demand students to engage actively in the construction of their knowledge by solving situations where there is no clear solution or method. Sometimes, they are also challenged to ask questions or extend the purpose of the task. Students need to interpret the given information, develop strategies, represent and communicate their solutions. This promotes the understanding of representations, concepts, and procedures, and also develops the ability to argue about ideas, as they communicate such ideas to others. Work on exploratory tasks develops usually in three phases (Ponte, 2005): (i) presenting and interpreting the task; (ii) carrying out the task individually, in pairs, or in small groups; and (iii) presenting and discussing results and final synthesis.

Geometric reasoning

The study of geometry is the natural context to develop and use visualization, special reasoning and geometric modeling to solve problems (NCTM, 2000). Despite the growing focus on geometric reasoning and visualization in research, clarification of their meanings is still missing (Gutiérrez, 1996). This is even more complicated by the many expressions used with similar meanings (geometric reasoning, visual reasoning, visualization, spatial thinking...). For example, for Battista (2007) "geometric reasoning consists, first and foremost, of the invention and use of formal conceptual systems to investigate shape and space" (p. 843), a definition we may find too broad. Also, the van Hiele's model describes how students' geometric reasoning develops and includes five levels: 1) visual-holistic reasoning; 2) descriptive-analytic reasoning; 3) relational-inferential reasoning; 4) formal deductive proof; and 5) rigor (Battista, 2009). These levels cover different forms of reasoning. So the need to investigate the development of geometric reasoning drove us to ask what is specific of this kind of reasoning and what main features does it have. A possible approach to study geometric reasoning consists in analyzing it from its processes, which are present in other areas but have some specificity in geometry. In this paper we will focus on the processes of defining and classifying.

Defining is a crucial activity in mathematics. For de Villiers, Govender and Patterson (2009), it is so import as solving problems, conjecturing or proving and, despite that, is much neglected in mathematics teaching. Their work with students in grades 9 to 12 suggests that producing definitions improves students' understanding of geometric definitions and concepts. Zazkis and Leikin (2008) emphasize that, for teachers to be able to support students in this process, they need to be competent in performing it. In a study involving prospective teachers, the construction and analysis of definitions for square showed their ability to distinguish necessary and sufficient conditions, use adequate language and show conceptions about defining.

The process of defining implies also classifying because of their mutual relationship:

> The classifications of any set of concepts implicitly or explicitly involves defining the concepts involved, whereas defining concepts in a certain way automatically evolves their classification. (de Villiers et al., 2009, p. 191)

In the perspective of Mariotti and Fischbein (1997), the process of defining must also be considered as a component of geometric reasoning. For those investigators,

> a classification task consists of stating an equivalence among similar but figurally different objects, towards a generalization. That means overcoming the particular case and consider this particular case as an instance of a general class. In other terms, the process of classification consists of identifying pertinent common properties, which determine a category. (pp. 243–244)

In a study with grade 6 students, those investigators found that classifications often resort to structural criteria which are not immediately clear and very often conflict with perceptual criteria we are used to refer in spontaneous classifications. Hence, achieving correct definitions makes students to question their prototypes which frequently introduces properties perceptually relevant that do not conform to the general requirements of the definition.

METHODOLOGY

This paper addresses an investigation with an intervention, in order to change practices and enhance teachers' preparation in geometry. The research focus is on learning in context, starting from the conception of strategies and teaching tools, following a design-based research as methodology, in the form of a prospective teacher experiment (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) in which the teacher also plays the role of researcher. We expect to run trough cycles of creation and revision, trying to deal with the problems that we will find along the way. At the present time, a first cycle was conducted involving 60 prospective teachers. The participants are in the second year of their teacher education program and attend a curricular unit of geometry taught by the first author of this paper. The study of quadrilaterals was developed in five lessons, following three sequential steps suggested by de Villiers and colleagues (2009): (i) to investigate the properties of quadrilaterals using the dynamic geometry environment (DGE) Geogebra; (ii) to classify them; and (iii) to define quadrilaterals. In the first step, they solved the clown task (Battista, 2007) adapted for Geogebra, where one has to manipulate quadrilaterals to overlap others, forcing them to use the relations among them (e.g., a rectangle may overlap a square but not the opposite). Afterwards, the participants registered all the properties that they found in each quadrilateral. In the second step, they classified the figures using a flowchart and a Venn diagram with the purpose of realizing that different criteria lead to different organizations. In addition, the participants also worked on a definition task.

Data gathered includes the participants' records of tasks solved in the classroom, an assessment task and reflections concerning quadrilaterals collected from portfolios. We also present the results of two multiple choice questions about quadrilaterals addressed in an initial individual diagnostic test. In the first question the participants identified relations among quadrilaterals, and in the second one they had to decide on possible definitions for square. The data was analyzed through several processes. Regarding the process of defining we adopted the categorization of de Villiers' and colleagues (2009): economical definitions, correct definitions and incorrect definitions. In this last case we considered definitions containing necessary properties but insufficient to define the intended set; in this category are also the definitions presenting properties that do not apply to some or all objects. Correct definitions present properties necessary and sufficient; if those properties are minimal, the definition is economical. In respect to the processes of classifying, the categorization emerged from the data, and we refer the comprehension of inclusive classification of quadrilaterals and the use of logical reasoning and communication skills.

RESULTS

In the first lesson, 57 prospective teachers solved the diagnostic test. The results show that only 25% considered that all squares are rectangles (but not the opposite) and 39% considered incorrectly that all quadrilaterals with two pairs of congruent sides are rectangles. Confronted with four possible definitions for square, 86% chose correctly "Polygon with four congruent sides and four congruent angles", but 75% also pointed "Polygon with four congruent sides". Only 23% considered valid "Quadrilateral with congruent and perpendicular diagonals". These results are not significantly different for participants that studied and did not studied mathematics in high school. They show that most of the participants ignored the relations between quadrilaterals and did not notice properties related to diagonals or lines of symmetry. In addition they were very connected to rigid prototypes and reasoned about figures by comparison to those prototypes, which is associated to van Hiele's level 1. Also they seemed to accept that a correct description of the quadrilateral may function as a definition.

The first two tasks of the sequence confronted the participants with their previous conceptions and made them realize there are relations among figures they did not know or expect and helped them to understand these relations:

> When I began to solve this task, I thought I would only recall some ideas about quadrilaterals. However, trough out the activity not only I recall them but also I was able to fit in to my head the hierarchy between some quadrilaterals. (Reflection written in the participant's portfolio about the classification task)

In the definition task, the participants worked in small groups, registered their answers which were discussed collectively at the end of the lesson. They were asked to: 1) Identify all the rectangles' properties; 2) Propose two different definitions for rectangle; 3) Propose two different definitions for parallelogram.

In respect to 1), most of the groups identified correctly all the main properties of rectangles (using sides, angles, diagonals and symmetry). Questions 2) and 3) show that they understood that there is no need to present all properties of an object to define it and most produced correct definitions, which is associated to van Hiele's level 2 (Battista, 2009). The next response is an example of a correct definition for rectangle, in which one of the properties is valid but unnecessary:

> Group A: Rectangles' properties: 4 right angles; 2 by 2 parallel sides; 2 lines of symmetry; bisected diagonals; congruent diagonals.
>
> Definition: quadrilateral with 4 right angles and 2 lines of symmetry.

Although less frequent, some definitions were incorrect:

> Group A: Parallelogram: quadrilateral without lines of symmetry.
>
> Group B: A parallelogram is a figure composed by 2 paires of congruent and parallel sides, forming 2 acute angles (opposite) and another 2 obtuse (opposite).
>
> Group E: Rectangle: The diagonals intercept in the center but are not perpendicular;
>
> 2 symmetry lines (1 horizontal, 1 vertical) passing in the center of the figure.
>
> Group F: Rectangle: Geometric figure with 4 sides where the length should be bigger than the height.
>
> Parallelogram: Geometric figure similar to rectangle, where the shorter lines are oblique.

The definitions for parallelogram proposed by groups A and B exclude all rhombuses in the first case and all the rectangles in the second, so their definitions are not inclusive. Similarly, the first definition presented by group E excludes squares. These examples show some difficulty to abandon previous conceptions and recognize the hierarchical organization of quadrilaterals. Still in group E, the second definition is incorrect because it does not exclude some rhombuses. Yet, the more striking feature of this definition is that it is dependent of the position that rectangles are usually presented. Group F's response is the only one that considers as properties the relations between the dimension of the sides and its position. Although incorrect, these definitions were presented collectively, which led into an important discussion. Some students argued about their validity giving counter-examples or correcting the statements and others noticed and reflected on their own misunderstandings.

Finally, some examples of economical definitions demonstrate an interesting analysis, where students used less usual properties they discovered with Geogebra:

Group C: Rectangle: Quadrilateral with two congruent and bisecting diagonals.

Figure 1

Parallelogram: Each diagonal divides it into congruent triangles.

Group D: Rectangle: Quadrilateral with 2 lines of symmetry passing through the middle points of opposite sides.

In the first definition of group C, the prospective teachers draw a quadrilateral where the diagonals do not bisect so they justify the need to include this property. The second, although roughly written, is very interesting because the word "each" makes a difference (one diagonal would not be enough because of kites). Group D presents a definition focused on the lines of symmetry, but stating their position which is necessary (all rhombuses have also two lines of symmetry in a different location).

Overall, we found four types of problems. Producing economical definitions was the most common difficulty and the hardest to overcome, especially because the participants did not know how to be sure that the properties were sufficient to identify each quadrilateral. A second problem that came up some times was the production of non-inclusive definitions. Even for participants that seemed to understand previously the hierarchical relation between quadrilaterals, sometimes they stopped to consider it, showing difficulties to let go previous conceptions. All these cases correspond to van Hiele's level 2, according do Battista (2009). The third problem, happened in very few cases and corresponds to definitions linked to certain positions or relations between parts of the quadrilaterals, clearly associated to frequent prototypes (corresponding to van Hiele's level 1). Despite their low frequency, these cases must keep us aware of how striking the systematic exposure to rigid prototypes may be (Yu, Barret, & Presmeg, 2009). Finally, there was only one definition containing an insufficient property to define the quadrilateral.

The previous examples demonstrate some difficulties, but also some interesting successes if we remember that it was the first time that these participants defined something. To formulate definitions implies to investigate invariants. We must identify the common properties to all the elements we include in that class, mobilizing inductive reasoning and visual abilities, in particular visual discrimination and perceptual constancy (Gutierrez, 1996). So, given the fact that most of the definitions were correct, we consider that as a positive indicator regarding those abilities and inductive reasoning. The few participants that produced economical definitions moved to van Hiele's level 3 (Battista, 2009) and showed a significant improvement. Given the fact that formulating economical definitions involves also deductive reasoning, it appears the participants showed more difficulty in it.

The production of definitions was a good opportunity for the participants to learn about the quadrilaterals and to revise their conceptions about the process of defining, as this reflection shows:

> This task raised some doubts because, before we done it, I thought I knew the definitions of each figure, I thought there existed only one for each figure . . . I came across basic definitions about square or rectangle completely different from what I learned until then. To define figures I never had use angles, diagonals or even lines of symmetry; indeed, I was unaware of their major role. (Reflection written in the participant's portfolio)

Regarding the process of classifying, the work we have developed prompted most of the participants to consider quadrilaterals as classes of figures. However, this evolution does not happen all at once. It is possible that an individual recognize some relations and others do not. The following response regards a question in the final test, where the participants were asked to comment on two sentences: *All kites are squares*; *there are trapeziums with perpendicular diagonals*.

> The first sentence is wrong. Kites are not squares. The squares can be kites because they have two equal consecutive sides.
>
> The second sentence is wrong. A quadrilateral with perpendicular diagonals is a rhombus, which doesn't belong to trapezium's family.

These answers show several difficulties we also identified in other cases. In the first place, this participant recognized that a square is a kite, but did not recognize that a rhombus is a trapezium, supporting the conclusion that learning to classify is progressive and is not independent of the objects it regards. Second, it shows a logical problem: for the second sentence to be true, it is not necessary that all the trapeziums have perpendicular diagonals, so one counterexample does not deny that statement. Third, a problem of rigor in communicating: instead of "Kites are not squares", one should say "Some kites are not squares" and also the word "pairs" is missing from the kite's description. Communicating using the precise words has a fundamental role in the processes we are dealing with. A prospective teacher asked once during a lesson: "If a parallelogram is a trapezium, why do they have different names?", a question that shows difficulties in interpretation.

CONCLUSION

In the beginning of the experience, the prospective teachers showed weak knowledge about quadrilaterals and their relations. However, the work on the sequence of tasks (investigating quadrilaterals' properties, classifying and defining) seems to have promoted their reasoning and the reconstruction of their knowledge. In the definition task, most of the participants understood the meaning of the process itself and presented mostly correct definitions using properties that they previously ignored, showing the comprehension of the underlying concepts, which supports de Villiers and colleagues's conjecture (2009). However, the participants produced economical definitions in few cases, suggesting that they perform better in inductive rather than deductive reasoning. Classifications (associated or not with definitions) showed, in some cases, a conflict between prior classifications, based on perception, and structural criteria that rules geometrical classifications, which is fundamental to the learning process (also a result indicated by Mariotti and Fischbein, 1997). The process of classifying also mobilized logical reasoning and communication, which presented difficulties for some participants. However, the nature of the work developed in classes favored discussion and negotiation of meanings, which is essential to overcome those difficulties (de Villiers, 1994). This idea lead us to conclude that the exploratory work in which the participants engaged, using a DGE, allowed them to investigate and discuss their findings and construct their knowledge in a meaningful way. As the testimony of a prospective teacher shows, reflection may play an important role in becoming aware of personal preconceptions and knowledge, which is an essential part of teacher education (Ponte & Chapman, 2008).

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Articulation between students' and teacher's activity during sessions about line symmetry

Aurélie Chesnais[1] and Anne-Cécile Mathé[2]

1 Université de Montpellier, LIRDEF (EA 3749), Montpellier, France, aurelie.chesnais@fde.univ-montp2.fr

2 Université d'Artois / ACTé (EA 4281), LML, Arras, Université Blaise Pascal, Aubière, France, a-cecile.mathe@univ-bpclermont.fr

The research presented here builds on an experimental work ran at the end of primary school (9–10 y. o. children) about line symmetry. We intend on questioning the factors that drive the evolution of geometrical activity of students and analyze in that purpose the articulation between student's and teacher's activity. We try to highlight the fact that learning in geometry relies on both an adaptation process when confronted to a task (individual and adaptationist dimension) and a collective and social construction mediated by interactions between teacher and students and between students.

Keywords: Line symmetry, adaptation, social construction, language.

This paper focuses on the teaching and learning of line symmetry at the end of primary school (9–10 year old children). The corpus that we study was collected during 5th grade classroom sessions where situations were used that had been created by a group mixing teachers from primary and secondary school and a researcher (one of the authors of this paper). The group had elaborated situations trying to take into account difficulties related to the transition from primary to secondary school and then implemented them in the classes of the teachers belonging to the group (Chesnais & Munier, 2013). Some videos of these classroom sessions are used here as a corpus for a research which intends on questioning the articulation between students' and teacher's geometrical activities. This research coordinates two different ways to understand the learning and teaching of geometry. The first one tries to describe the geometrical activity of students interacting with a given task, in all its complexity. It studies the factors that drive the evolutions of this activity in a movement for learning (Mathé, 2012; Bulf, Mathé, & Mithalal, 2011; Barrier, Hache, & Mathé, 2013). The other one, following up with previous work about everyday teaching practices and line symmetry, (Chesnais, 2009; Chesnais & Mathé, 2013; Chesnais & Munier, 2013), tries to investigate more precisely how teaching practices influence students' activity and hence students' learning, and also to get a better understanding of what drives teaching practices.

We take as a premise that learning in geometry relies on both an adaptation process when confronted to a task (individual and adaptationist dimension) and a collective and social construction mediated by interactions between teacher and students and between students. Our goal is here to highlight the complementarity of two approaches to get a better understanding of how the two processes articulate. After clarifying the elements of knowledge at stake in the learning of the concept of line symmetry, we will present the task and the methodology we used to analyze the productions of a pair of students and the teacher's activity during two sessions. We finally present on the results of this analysis and conclude.

ABOUT LINE SYMMETRY

What is mainly aimed at in 5th grade in France about line symmetry is that pupils understand an "instrumental definition" of symmetrical figures: two figures are symmetrical to each other with respect to a line if they are superimposable by folding along this line. They are also supposed to be able to find lines of symmetry on simple figures and to know some properties such as the fact that the mirror image is flipped over compared to the initial figure, the two figures are equidistant (in a global way) from the line and have same shape and dimensions.

At this stage, symmetry is mainly handled as a transformation acting on surfaces (2D-elements), and considered as restriction to the plane of a rotation of 180° around an axis included in the plane. Properties are then considered in a global manner and closely related to perception. However, working on symmetry might imply back-and-forth movements from relations between surfaces (and a line) and relations between 2D, 1D or 0D elements of the figures. For example, the dimension conservation property can either be perceived globally or focusing on the length of segment lines. In a similar way, equidistance to the line may be understood in terms of surfaces or elements of surfaces or event points. In fact, the idea of distance from a figure to a line is difficult at this level. It may refer either to the distance between the figure and the line perceived globally (Grenier, 1988), the distance between elements of the figure and the line perceived globally, the fact that the midpoint of a segment joining a point to its image belongs to the line, the distance from points to the line, seen as the length of the segment joining the point and its orthogonal projection on the line or its projection in a given direction.

About instruments, students at this level essentially use folding or tracing paper to control the symmetry of figures or to construct mirror images, mainly working on surfaces. Students might sometimes also use a ruler – as in the corpus presented below – which implies another way of considering figures (which is also the one that is at stake when working on grid paper): working with 1D-objects (segment lines, sides of surfaces). Switching from one way to the other makes it necessary to coordinate a view of figures as surfaces on one hand and as a network of segment lines on the other hand (what Duval (2005) calls "dimensional deconstruction"). The relation between symmetry and a movement in 3D-space is also less obvious.

Moreover, one of the goals of the work about symmetry in primary school is to make pupils overcome some wrong conceptions they might have about line symmetry, which particularly appear when working on surfaces (Grenier, 1988). Particularly those related to vertical lines, making them act as if the mirror image of a horizontal (resp. vertical) segment line is also horizontal (resp. vertical); the conception of alignment (resulting from the conjunction of vertical or horizontal lines and vertical and horizontal figures: the image of a segment line perpendicular to the axis is then aligned with its image); confusion with translation (which is related to the flipping over property); conception of symmetry as a transformation moving figures from one half-plane onto the other one (essentially related to folding) (Chesnais, 2009).

THE IMPLEMENTED TASK

The task is the second one in a sequence about line symmetry in a 5th grade class. In the first task, taking place during a previous session, students were supposed to predict what would paint stains become when folding a paper along various axes and then to identify some properties of line symmetry (equidistance of the two figures from the axis, shape and size conservation, flipping over property). In the present task, pupils were asked to draw the mirror image of a geometric figure (in the shape of an "L") with respect to a given line in various configurations (cf. Figure 1). Didactic variables (orientation of the figure and orientation of the axis) were used to make students

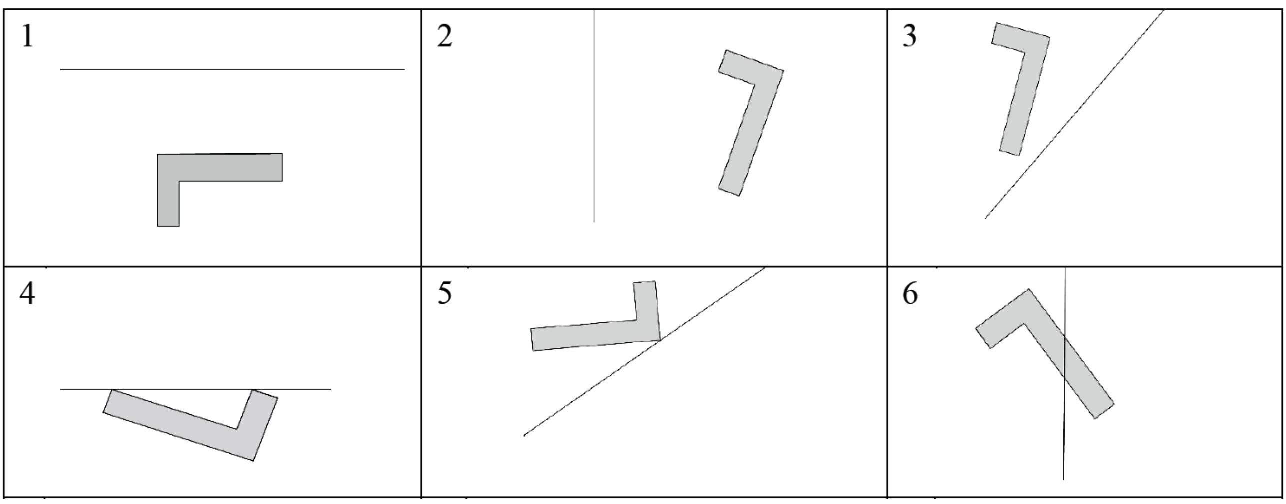

Figure 1: Configurations for the task

encounter one or the other of the properties and/or wrong conceptions.

Work was organized as follows: pairs of students were successively given six sheets – corresponding to each configuration. They were asked to draw the images approximately without folding they were allowed to use a pen and a ruler. A tracing paper on which the initial figure and the line were drawn for the first two cases and which was blank for the other four was provided to control their answer. The teacher went from one pair to another one to help them, and provided tracing paper when they were done drawing. A collective discussion with the whole class, based on some productions selected by the teacher, took place between the work in pairs on each case. The first three configurations were handled in a first session, which ended up by writing a couple of rules (mentioning the flipping over property, the equidistance to the line, conservation of shape and dimensions and the relation with folding), on a paper then posted on the wall. The work on the three other cases took place in a second session.

In this paper, we will particularly focus on one of the major properties of reflection: the "flipping over property" even if some other ones were also mobilized in the task. Mathematically speaking, it corresponds to the fact that line symmetry is an inversion. It is also related to the idea of "mirror image" or "reflection in a mirror". Materially speaking, it implies that a figure and its image are superimposable when the figure is flipped over, which means that a rotation of 180° degrees in 3D-space around a line included in the plane is applied to it. In practical terms, it can be obtained by folding the paper along the line or by flipping over tracing paper (and positioning it such that the line stays invariant). This property is difficult to express with common words because "flipping over" could also refer to a half-turn and then symmetry around a point whereas turning a page would end up flipping it over for example. Eventually, verbal language is not sufficient to handle this property and coordinating it with movements and material actions is necessary to make students identify and understand it (Chesnais & Mathé, 2013).

This property is part of what is at stake in the task. In the first case, flipping doesn't change the orientation of the sides of the figure (vertical stays vertical and horizontal stays horizontal). It does change it in all the other cases. Hence, it will be contradictory with what would other conceptions imply (especially the conceptions related to vertical and horizontal axis, and the one about alignment). In case 6, flipping can be hard to anticipate if the students have a conception of symmetry as transformation acting from one half-plane onto the other one (related to folding).

METHODOLOGY

We try to characterize students' activity, teacher's activity and the way they both articulate. Our indicators are mostly the material actions (folding, use of tracing paper and hands movements) and the interactions between students and students and teacher. We describe difficulties experienced by one pair of students throughout the work on the task, especially when the flipping over property is at stake. Based on the observation of this evolution, we then investigate how the task and social interactions participate in it towards an understanding of the flipping over property and symmetry. We also aim at understanding how the teacher identifies and uses students' activities and how she helps them completing the task.

GEOMETRICAL ACTIVITY OF A PAIR OF STUDENTS AND ITS EVOLUTION, BETWEEN ADAPTATIONISM AND SOCIAL PROCESS

In this part, we present an overview of what happened for each configuration during the sessions, focusing our attention on what the pair of students produced, the way they validated their answer and the content of the interactions with the teacher (when there are some) and of collective discussions. We aim at describing the evolution of geometrical activity of this pair of students throughout their work on the six configurations and try to identify what drives this evolution.

Configuration 1. They used a ruler to draw the image, taking into account the conservation of shape and dimensions, but without flipping the figure over (Figure 2). They probably didn't imagine the movement in 3D-space and then didn't anticipate the image. They used tracing paper for validation, but they couldn't figure out how to proceed differently when they found out that they did not get the right answer.

During the discussion with the class, the teacher chose to comment on their production. The class pointed out the absence of inversion. The teacher accompanied

Figure 3: Construction of configuration 2

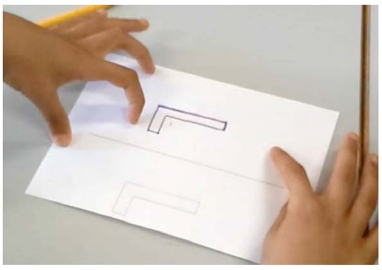

Figure 2: Production of configuration 1

the students trying to put this in words. Students used various wordings ("the other side", "in the other way around", "transferred", "turned over"). Their difficulties ended up in making the use of gestures necessary. The words "turned over" and gestures were also used by the teacher.

Configuration 2. One of the two students of the pair started by drawing globally the image with her finger: the position and orientation seemed correct and she flipped the figure over. Afterwards, she folded (not completely) the sheet of paper in order to find the precise position of one of the vertices of the image and then used a ruler to measure a first side of the figure to construct its image, transferring its length.

They finally gave then a third try, switching the orientation of the biggest part of the "L" but not the one of the smallest one (Figure 4).

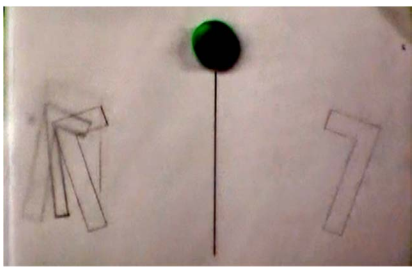

Figure 4: Final

The teacher chose again their production to comment during the final discussion. She makes the class explicit a mistake related to the conservation of shape (one of the students says "we had a 7, we get a 1"), but the link between the change of shape and flipping over the figure or elements of the figure is not pointed out. Flipping is not mentioned.

Configuration 3. They drew an image on the other side of the line, with the same size and same shape but without flipping it over (Figure 5). The students didn't seem to take into account the line and translated the figure, without imagining the rotation in 3D-space. Afterwards, they validated their answer using tracing paper (on which they draw both the figure and the line) but they slided the tracing paper, instead of flipping it over (Figure 6).

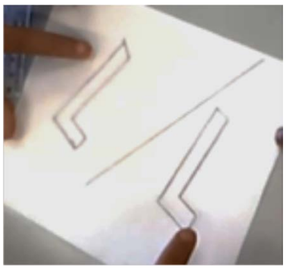

Figure 5: Production

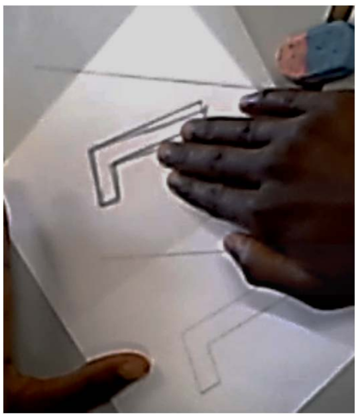

Figure 6: Validation

The teacher asked them how they controlled their answer and she pointed out the fact that the way they tried to control their answer was not correct, without mentioning a correct way to do it.

Part of the collective discussion was devoted to discussing this production. The class immediately pointed out the flipping over "problem". A work on putting it into words was conducted once again by the teacher and expressions like "in the same direction", "a little bit turned around", "completely turned around" were said.

"Turned over" was finally mentioned, repeated by the teacher and accompanied with gestures. At the end of this first session, the teacher elaborated a paper trail. She mentioned the "mistake often made" by this pair of students, pointing out that they often drew the figure "in the wrong direction". The teacher then wrote down that "the image has to be flipped over compared to the initial figure".

Configuration 4 (session 2). At first, the two students translated the figure along a direction given by the small part of the L (Figure 7); one of the two students, taking a wider look at it, realized the mistake. But constructing the image line by line made her do the same mistake again for her second try. Realizing it, they did a third try. The image was flipped over, but they did not switch the orientation, extending the sides of the small part of the L (Figure 7) and acting as if the line of symmetry was perpendicular to these sides. The other student tried to modify it so that the image touches the line at the same place as the initial figure (Figure 7). Taking into account the conservation of the shape led them to end up at the final production (Figure 7).

Trying to validate their answer, one of them drew the initial L and the line on tracing paper and then tried to turn the paper around (keeping it in the horizontal plane). The other one took it, flipped it over and replaced it so that the two lines matched.

The teacher then started a discussion with the pair. She placed the tracing paper sheet in its original position, then flipped it over and placed it such that the two lines matched. She tried here to link explicitly the flipping over movement of the tracing paper sheet and the flipping over property characterizing the relation between the initial L and its image (considered as relation between two figures in the plane). She then accompanied the two students to interpret feedback from superimposition of their production with the flipped over tracing paper. In particular, she showed them that the orientation of the bigger part of the L was correctly switched whereas the orientation of the smaller part didn't match. "Your line is straight, prolonging the other one, while it should have been tilted like that (she showed the side of the flipped over L)". She helped them taking into account the relation between global flipping of the figure and change in the orientation of 2D elements of the figure (the smaller part of the L and the bigger part), relation that was difficult for them to identify in the previous tasks.

The flipping over property was not mentioned during the collective discussion.

Observing the resolution of this four cases, we see the difficulties students encountered to perceive and take into account the fact that the initial figure and its image are flipped over, one compared to the other. They often anticipate approximately the correct position and orientation of the image in a global way, as well as they realize their mistakes when they look globally at their production after drawing. But drawing the image with a ruler makes them consider the figure in a very different way. It is not seen as a surface (2D) anymore, but as a grid of segment lines (1D) which images have to be constructed separately. Students seem then to forget about the symmetry as rotation of 180° degrees around the line in 3D-space and they keep working in the plane. Yet they don't have any other definition of symmetry than this instrumental definition (see above), which refers to folding (real or simulated). The different attempts for case 2 show that it is difficult for them to articulate the switch of orientation for each of the segments (or 2D elements of the figure – smaller or bigger part of the L) and conservation of shape. Students also have difficulties using tracing paper to control their productions,

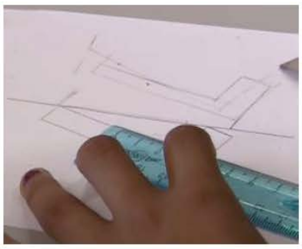
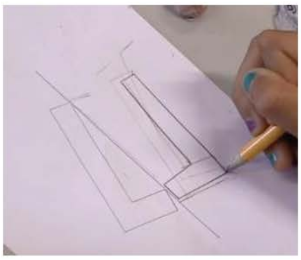
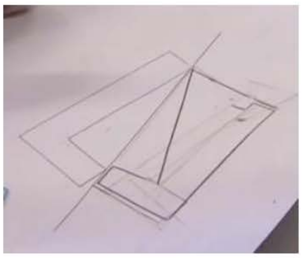
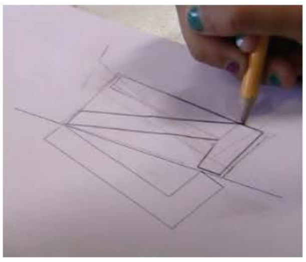

Figure 7: Try 1 Try 3 Try 4 Final production

which makes it hard to use pragmatic feedback as we could observe about cases 2 and 3.

Configuration 5. At first, one of the two students drew an image, flipped over compared to the initial figure, but with an approximate orientation. The other one changed the drawing to get a better orientation and a better corresponding size. (Figure 8)

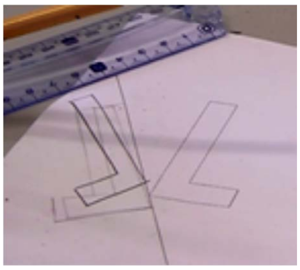

Figure 8

The collective discussion dealt with the question of the difference between "turned around" and "flipped over", about the productions of students who applied a symmetry around a point. Flipping over (rotation in 3D-space of 180° around the line) was again pointed out by the teacher, as opposed to rotation in the plane. Words were accompanied by a lot of gestures.

The teacher then pointed out the link between the need for flipping over and folding, which had been used to construct mirror images during the previous session. She folded a sheet of tracing paper so that folding makes the initial L flip over.

Configuration 6 The pair of students produced a flipped over figure, with same size and same shape as the initial L (Figure 9). The orientation of each part of the figure was correctly reversed. The invariance of points of the line was only partially completed.

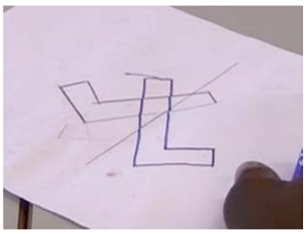

Figure 9

The teacher pointed it out during the collective discussion that followed.

The analysis of students' activity on the first four configurations showed that the interactions between the students and the task didn't seem sufficient to cause an evolution in their activity towards complete conceptualization of the flipping over property. However, an evolution has occurred in their ability to construct and validate, towards a better consideration of this property: their first idea was to translate the figure without flipping it over for the first four configurations, but they flipped it over directly for configurations 5 and 6. How can we understand what drove this evolution?

What we chose to observe leads us especially to pay attention to oral interactions during task completing. Consistently with our theoretical frame, this leads us to identify that, in this example, the evolution of pupils' activity is caused by a double process, adaptationist (students interacting with a task) and social, (essentially here through interactions between students and teacher). How do both interactions between the students and the task and interactions between students and teacher articulate to contribute to the evolution of the ability of students to acknowledge the flipping over property?

We distinguish two types of interactions: the interactions between the pair and the teacher, when she comes to talk to them during their validation; the interactions between the students and the teacher during collective discussions. Each of these types is a place where teacher and students' activities articulate, completing interactions between students' and the task, in order to make their understanding of the task and of the 'milieu' change. During the interactions between the students and the teacher, we identify three objectives of the teacher: helping them to identify mistakes (about the alignment of segments, in configuration 4), asking them to use tracing paper to control their answer to make them go back to material actions (configuration 3), and helping them using tracing paper correctly to flip it over and replace it so that the two lines match: what is at stake here is the link between the flipping over of the figure and the switch in the orientation of 1D elements. The teacher accompanies them in what interactions with the task was not sufficient to ensure. We suggest that, among these interventions, some have a productive function (Robert, 2008) (help students to complete the task) and some have a more constructive function (Ibid.) (help students transform activity into knowledge).

During collective discussions, the teacher based the debate on students' productions and mistakes. She made students explicit the flipping over property, then put it into words, and decontextualized it. However, she didn't emphasize the link between this property and the material action of folding or flipping tracing paper as much as she does when interacting with the two students (except for configuration 6).

RESULTS AND CONCLUSION

Our analysis of the evolution of productions of a pair of students on a task about symmetry allowed us to identify factors of this evolution: the feedbacks provided by the task but also interactions between students and between students and teacher. It points out how an adaptationist and a social process intertwine to ensure this evolution.

Finally, we claim that, in the observed sessions, students' progression towards a better consideration of the flipping over property results from articulation of four types of interactions between students and milieu:

- interactions between the students and the task, their use of instruments and the way they adapt to pragmatic feedback coming from the control with tracing paper;
- the verbal interactions between the two students in the pair;
- the interactions between the pair and the teacher, when she comes to talk to them during their validation;
- the interactions between the students and the teacher during collective discussions.

This study also informs on the way the teacher's activity may articulate with students' one: choice of the task, interactions with them during research, use of productions during collective discussions. Finally, it points out that linking various dimensions of activity (material, verbal and conceptual) is a condition to ensure learning.

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Teaching geometry to students (from five to eight years old)

Jacques Douaire[1] and Fabien Emprin[2]

1 LDAR, Ermel Team-IFé, Paris, France, jacques.douaire@wanadoo.fr

2 University of Reims Champagne-Ardenne, CEREP, Reims, France, fabien.emprin@univ-reims.fr

For students from 5 to 8 years old, the spatial and geometrical learning concern mainly the control of its relations to real space (tracking, travelling), the recognition of objects and shapes, and their representation by drawings using straight lines. The ERMEL team, in France, is experimenting teaching situations on space and geometric learning in the 2nd cycle of primary education (5–8 years old student) and is building complete engineering to teach math in elementary school. Our methodology involves an analysis of the student's way to solve problems and thus their abilities. Experiments conducted in a great number of classrooms allow us to better understand the components of this learning in terms of knowledge and abilities. Thanks to these results, our hypothesis and choices have evolved.

Keywords: Geometry, teaching, learning, primary education, spatial problems solving.

RESEARCH'S PRESENTATION

Issue of this paper

Our research takes place in the French context of geometry teaching in primary school. Our goal is to build proven, complete and reliable teaching engineering and thus to improve geometry teaching.

By submitting our results and remaining questions we want to contribute to the working group's thought. In a European context we also want to compare our approach to others and seek our results significance.

Why working on geometry teaching?

For a long time, Berthelot and Salin (1992, 1993), highlighted the difficulties of geometry teaching in France:

> No one disputes, for example the usefulness of knowing how to do a multiplication, so learning it in school, has been normal for a century or so. [...] For the geometrical knowledge, beliefs are less assertive, [...]. So they feel "authorized" to take leeway with the school curricula, that is to say, to ignore this part (as well as high school teachers ignore 3D geometry) (translated from (Berthelot & Salin, 1993, p. 39))

An analysis of teachers' representation about geometry teaching (in process) confirms that in 2014, things have not really changed. This work, based on interviews, also shows that both geometrical and didactics knowledge of teachers (even experienced) is weak.

As far as geometry is concerned, textbooks mainly focus on students' work, on drawings and being able to write the proper name on the proper drawing.

Geometry learning is not seen as a social necessity by the families and sometimes by teachers. Moreover, its contribution to subsequent learning is often reduced in the classrooms at an early learning, even ineffective, of geometrical vocabulary.

This research is also based on the idea that students' abilities are insufficiently taken into account in geometry teaching in primary school. Thus we have to identify the knowledge at stake in this learning and take former students' knowledge into account.

By starting our research from the analysis of students' abilities and geometric concepts we develop an uncommon methodology (described below).

About us

ERMEL is a research team on mathematics education in primary school (in French "Équipe de Recherche en Mathématiques à l'École Élémentaire"), which belongs to the French institute of education (IFé). This team is made up of primary school teachers, teachers' trainers and researchers working in different cities of France (Châlons-en-Champagne, Grenoble, Lyon, Paris...).

The ERMEL team conducted studies on teaching and learning of number system and arithmetic's form 1985 to 1998 and since 1998 on geometry teaching and learning. Results of theses researches lead to comprehensive book publications on teaching engineering (Équipe de Recherche en Mathématiques à l'École Élémentaire [ERMEL], 1998, 1999 & 2006).

The aim of the current research is to analyse spatial and geometric skills that 5 to 8 years old students can build.

Methodology

Our research has been and is sometimes called "action research" in the French tradition. We do not deny that term in its meaning and use Hugon and Seibel's (1988): "Research in which there is a deliberate transformation of reality; research that has a dual purpose: to transform reality and produce knowledge about these changes." Our research actually intended to produce resources that are analysing issues of education in relation with school curricula and their changes.

On the other hand, our methodology is quite different form the English tradition of "action research in education" as defined by Sagor (2000, p. 3):

> It is a disciplined process of inquiry conducted by and for those taking the action. The primary reason for engaging in action research is to assist the "actor" in improving and/or refining his or her actions. Practitioners who engage in action research inevitably find it to be an empowering experience.

Even if teachers and teachers training have an important place in our research, we define our methodology as a didactic engineering. Our approach has in common with the didactic engineering concept the general questions it allows to address. In fact, the term "didactic engineering" appears in the mathematics teaching in France in the early 80s as a way to answer two fundamental questions translated from Chevallard (1982 as cited in Artigue, 2002, p.59):

> How to take into account the complexity of the classroom in research methodology?
>
> How to think about the relationship between research and action on the education system? [...] As a research methodology, didactic engineering is different from the usual experimental methods by its validation mode. This internal validation method is based on the confrontation between an a priori analysis in which are engaged a number of assumptions and a post hoc analysis that relies on data from the actual implementation.

The ERMEL research team develops an analysis of students' knowledge, issues of teaching, and offers a didactic engineering based on an experiment conducted in many classrooms for several years.

This research is derived from the analysis of the challenges of teaching mathematics in the field. Once the needs are identified, such research involves several steps:

1) An analysis of the mathematical knowledge (problems, properties...) at stake, as well as students' knowledge and abilities;

2) An explanation of educational issues and the organisation of the study of the different notions throughout the years;

3) Development of teaching situations and tests in several classrooms.

These last three components interact: the identification of students' abilities is the outcome of experiments conducted.

4) Writing a book for teachers and trainers with an explanation of the issues of learning and teaching, a description of the identified learning situations, a reasoned choice of learning roadmap and syllabus planning. These books are often references in training in France.

The diagram below (Diagram 1) illustrates our methodology.

Analysing teaching and learning process require theoretical frameworks.

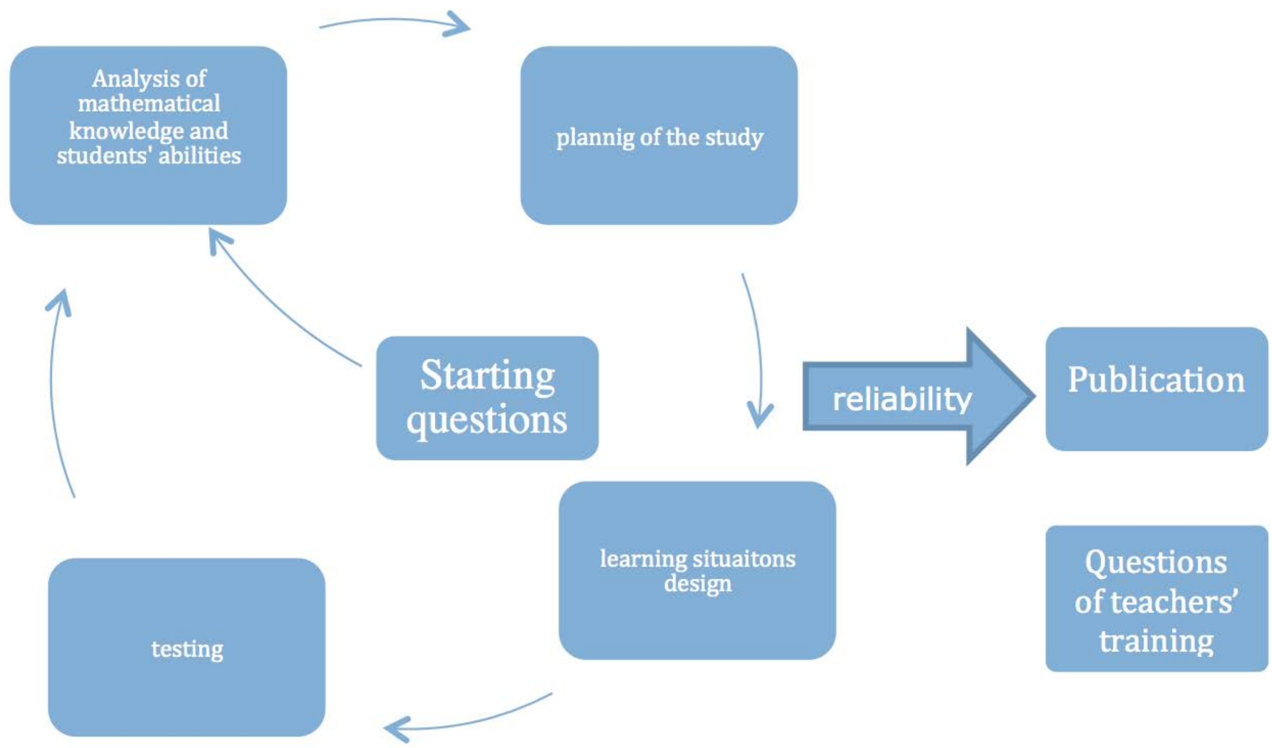

Diagram 1: Steps of ERMEL's methodology

Theoretical frameworks

We mainly focus on a framework that is preeminent in the French context, the Theory of Didactical Situations based on the ideas below:

> Mathematicians don't communicate their results in the form in which they discover them; they re-organize them, they give them as general a form as possible. Mathematicians perform a "didactical practice" which consists of putting knowledge into a communicable, decontextualized, depersonalized, detemporalized form.
>
> The teacher first undertakes the opposite action; a recontextualization and a repersonalization of knowledge. She looks for situations which can give meaning to the knowledge to be taught. But when the student has answered to the proposed situation (...) she will have to redepersonalize and redecontextualize, with the assistance of the teacher, the knowledge she has produced so that she can see that it has a universal character, and that it is re-usable cultural knowledge. (Brousseau, 1997, p. 227)

For teachers, building learning-situations is building a problem where the knowledge is recontextualized. By solving this problem, the student will acquire the knowledge at stake. There are different kinds of problems corresponding to different ways to reach the knowledge (more or less efficiently).

As far as geometry is concerned, we also use concepts highlighted by former researches. We distinguish between spatial knowledge and geometrical knowledge:

> In general we can distinguish between two kinds of problems:
>
> — spatial problems thus characterized: their purpose concerns the sensitive world; they can focus on implementing actions: making, shifting, moving, drawing, etc. communication about actions or findings [...] The success or failure is determined by the subject itself by comparing the expected result and the result;
>
> — geometric problems, like that word is used in mathematics: Solving a geometrical problem is an activity involving the necessary and non-adversarial nature of certain properties of the geometrical objects" (translated from Berthelot & Salin, 1993, p. 41).

We also use the work of these authors, which defines different kinds of space in which spatial problems can be placed: micro space (very close to the subject, object can be moved, touched, turned) meso-space (surrounding space, between the arms of the subject, he can have a comprehensive view, he can move in the space) and macro space (far space, the view is more local, subject has to conceptualize). The space defined by the sheet of paper can be called graphical space and has also special features. Computer screen also form a new type of space depending on software uses (for example dynamical geometry software). Theses spaces are so many choices and so many parameters of the learning situation.

We focus on the alignment and straightness concept to make our approach explicit.

ALIGNMENT AND STRAIGHTNESS IN THE 2ND CYCLE OF EDUCATION

(2nd cycle of French school is 1st and 2nd grade in the USA)

This research was attributable mainly to the fact that the teaching of geometry in the primary grades does not adequately take into account the knowledge children can develop when solving problems. This research therefore requires identifying the skills involved in these learnings and take into account the initial knowledge in these fields.

As often with geometric concepts introduced in elementary school, the concept of straight line has a double aspect:

- it allows to represent real-world objects or actions

- it is a component of a geometric knowledge constitution that has properties that students gradually discover: in this case it is the properties of the straight line and the constraints of its drawing.

This learning raises several questions: in the 2nd cycle, what are the possible links between these two aspects (alignment of points and straightness of lines)? In particular can the procedures developed in the meso-space be reused on the paper? Is it better to start with experiences in the meso-space to show a straight line as a solution of an alignment problem?

Meanings of the straight line

The notion of straight line can be understood by the 2nd cycle's student through different meanings induced by perception or experience, which can be used in problems for:

1) The graphic representation of physical objects soliciting properties of the straight line, including:

— a material object: a stretched wire, a straight edge object, a light ray, the fold of a paper ...

— a border between material objects :

  — two planar regions of space, as the edges of a polyhedron;

  — two regions of the map, as the sides of a polygon, of half-planes ;

— a subject of the graphics world : a straight line (which may be extended beyond the ends);

— a trace :

  — a print trace produced by a path with the ruler ;

  — a print screen trace caused by the "straight" tool in a dynamic geometry software (like Cabri Geometry)...;

— the path of a rectilinear object (ball ...).

For all these problems in connection with these first meanings, the points do not play an important role.

2) a set of aligned points : the locus of points aligned with two points, for instance for sight problems in the meso-space.

Properties

Our previous work research (ERMEL, 2006) on geometric learning has confirmed that the properties attributed to the straight line by the students aged 8 to 9 (CE2), are limited to those related to the perception of lines drawn. From a theoretical point of view, if a straight line can be characterized in various ways, as the effect (invariance) of a transformation, as the intersection of planes, for the student of six or seven years old the concept of the direction (extension) is the first they meet. In fact for students of this age a straight line is simply the straight line drawn on a sheet. Several properties of the line, as a mathematical object (in particular the fact that the line is formed by an infinite set of points) are not accessible to the primary school.

Questions

One of the challenges of our research on 2nd cycle was for us to determine:

— What is the initial knowledge of the students? What perceptions, what experience do they have of straight lines?

— Among the meanings of straight line which must be preferred?

Initial hypothesis

We therefore sought to clarify the meanings of straight lines that may be encountered or that are accessible to 2nd cycle students and to create problems, allowing a passage to geometric knowledge.

We thought that the fact that a line can be extended for solving an alignment problem should be learned.

At the beginning of this research, we considered teaching a grasp of the concept of alignment through the experiment centred on the idea of hiding an object (using the sight) in the meso-space, then again on paper in order to highlight the utility of using the straight line.

We proposed a problem ("Plots") that takes place in the schoolyard (Figure 1): students have to find locations (using the sight) where an object hides another one (I see the red plot hiding the green one and the yellow plot hiding the orange one). The problem was after proposed on the paper sheet (representing a top view of the yard). In solving the problem on paper the students were not using straight lines to find the locations; broken lines have been uses to depict alignments.

These experiments revealed activity transition-related difficulties conducted in the meso-space activities on paper.

First experimental results for CP (1st grade US – Year 2 GB) and CE1 (2nd grade US – Year 3 GB)

Students had developed spatial resolution procedures based on the sight and other gestures in the meso-space. But the modelling by a straight line was not effective in solving a similar situation on paper. In addition there were difficulties for drawing straight lines with a ruler, many productions included broken lines and not straight lines to represent the target.

Analysis of the difficulties in the transition from meso-space to micro-space

They seem related to:

1) An understanding of modelling on a sheet of a situation experienced in the schoolyard:

— Meso 3D / micro 2D (top view ...) and the disappearance of the subject (student) in the micro-space device.

— A representation of physical objects by schematizations (circles, dots).

2) What the straight line was supposed to represent.

3) For the drawing of plots straight lines.

The use of dynamic geometry software (DGS) seemed to enable this switch in CE1 (7-years-old): students using the straight lines to produce with DGS a solution to the alignment problem. However, we have not been

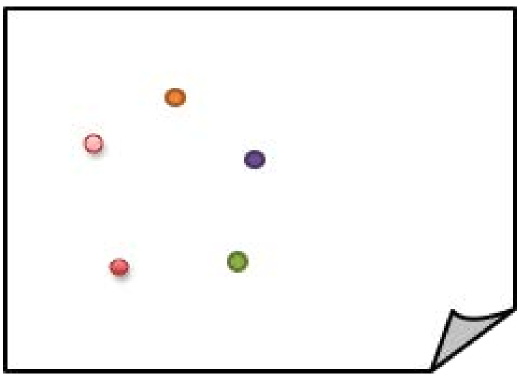

Figure 1: "Plot" situation in the schoolyard (left) and in sheet of paper (right)

able to check the reinvestment in the longer term on the sheet of paper.

The analysis of these problems (items 2 and 3 above) also highlighted the need for a specific work to allow the apprehension of the meanings and some properties right from the 2nd cycle (the term « straightness ») relatively independent of the alignment point problems. This corresponds to the first set of meanings of the straight line as shown on page 5.

Rolet (2003, p. 7) who also analysed 8-year-old students' work in different spaces noticed difficulties: "[students] they took the piece of rope not like a part of a line but like a segment. [...] Cabri-geometry was also a good instrumented space to give a real status to concepts like points: 'visible' vs defined, 'draggable' (or independent) vs fixed (or dependent). But understanding of these concepts still presents many difficulties for all students."

SPECIFIC LEARNING OF STRAIGHTNESS

Meanings of the concept of straight line

The meanings of the straight line can be associated with problems concerning either: the production of straight lines, the identification of straight lines (judgment), the choice and use of instruments that could be used to identify or produce an alignment or a straight line.

Among the meanings of a straight line, we chose to emphasize, for 6-year-old students, those of edge parallel strips, contour of a geometric figure. Validation criteria are either simply perceptual (regularity of the trace or resemblance to a model) or related to a more or less explicit reference to parallelism (direction ...) or consist of a practical validation (coincidence ends, overlay actions objects ...).

Presentation of a situation and common Goals

Students are working on "superpositioned form". They have to do a stack of shapes (rectangles, triangles) and draw their superposition.

These meanings can be associated to problems concerning either:

— the production of straight lines;

— the identification of straight lines (judgment);

— the recognition and use of instruments that could be used to identify where to produce an alignment or a straight line.

This work on the notion of " straightness", although it can be started in kindergarten (Year 1 in GB, called "grande section" in France) with the production of regular lines made in drawing activities, really takes its geometric dimension at 1st grade (6-years-old, CP in France) where the properties of straight line can therefore be apprehended. These properties are not yet objects of study for themselves, but are first experiences:

— a line can represent something that does not leave a physical form (e.g. sight);

— a drawn line may be associated with instruments;

— a line may be extended, for example to represent a hidden object.

Different levels of understanding control may be associated with these meanings:

— understand the need to draw a straight line to solve the problem; this is the role of formulations (validation through language, rather consensual), role of gestures (hands);

— learn how to place the ruler to draw a longer line; it is the knowledge of the method and the technological aspect: name tools, describe the action...

— mastery of drawing (validation by production).

ASSESSMENT FINDINGS AND QUESTIONS

Robustness/reliability of situations

— Devices (progressions, situations) that we have developed favour a knowledge construction based on problem solving. They have a certain "robustness" due in particular to the fact that the results and procedures that will be produced by students in a class are described in the description of situations, allowing the teacher, in general non-specialist in mathematics, to anticipate their decisions based on its own class productions. This reliability seems partly due to the coherence between the concepts of learning and the proposed situations and, secondly, to their experimentation in many classes for many years.

One of the challenges of our methodology is to allow ownership by teachers of educational created devices.

The conditions that we consider necessary for the appropriation of our devices by those teachers are:

— a better perception of the relationship between spatial knowledge and geometrical knowledge that students can develop;

— an awareness of all the meanings of a concept and situations associated;

— the identification of the essential characteristics of teaching situations;

For this we highlight these elements in our publications.

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The level of understanding geometric measurement

Eszter Herendiné-Kónya

University of Debrecen, Institute of Mathematics, Debrecen, Hungary, eszter.konya@science.unideb.hu

The perimeter and area are two important geometric concepts, which are taught through many years in schools. Although the curriculum and the textbooks in Hungary consist of both qualitative and quantitative approaches by teaching area and perimeter, the students' performance is low. The main goal of this research is to investigate students' ideas of the concepts area and perimeter from 5th to 8th grade. We identify typical solving strategies in order to understand students' imagination connected to these mathematical objects.

Keywords: Perimeter, area, geometric measurement, concept formation.

INTRODUCTION

The perimeter and area are two important geometric concepts, which are taught through many years in schools. We know from the works of many researchers and from our experience too that the teaching-learning process on this topic poses many problems. Although the curriculum and the textbooks in Hungary consist of both qualitative and quantitative approaches by teaching area and perimeter, the students' performance is low. In Hungary quite the same misconceptions and troubles are observed as in other countries.

The main goal of this research is to investigate students' ideas of the concepts area and perimeter from 5th to 8th Grade. We find the students' long term memory interesting, because it forms a correct view about the level of understanding concepts examined in the study. It happens in schools usually that teachers examine students' new knowledge in a few days after teaching a certain topic and they find that the students acquired the new concepts and procedures. The good result of such kind of tests can be misleading: some month later students don't even remember the basic ideas.

THEORETICAL BACKGROUND

One of the goals of education is to help students store information in long-term memory and to use it in order to solve problems. There are three different parts of long-term memory. Episodic memory refers to our ability to recall personal experiences from our past. Semantic memory stores concepts, rules, principles, and problem-solving skills. Information is more easily stored in semantic memory when it is easily related to existing, well-established schemata. Procedural memory refers to the ability to remember the steps of performing a task or employing a strategy (Baddeley, Eysenck, & Anderson, 2010; Skemp, 1975).

The topic of measurement is very useful to develop students' skills in problem solving, spatial sense, estimation and concept of numbers (Korenova, 2014).

In primary school by teaching measurement, general measurement principles are used: quantity conservation; direct comparison of quantities without measuring; the need for repeated (standard or non-standard) units; estimation before measuring; exploration of the inverse relationship between the size of the unit and the number required to measure; choosing an appropriate standard unit for a concrete quantity (Curry, Mitchelmore, & Outhred, 2006; Herendiné-Kónya, 2013). In Hungary in the teaching of length, mass or capacity these principles are accepted, but they are not followed in the teaching of area and perimeter. The steps listed are mainly left out; the emphasis is usually placed on the measurement of length and mode of calculation.

The perimeter and area are at the same time geometric concepts and measurable quantities too. This is the reason we use two approaches in the teaching of this topic, a qualitative and a quantitative one.

> In the teaching of area two approaches are generally used. One that can be considered as formal which refers to the calculation of areas with formulae and another, informal, that emphasizes the conservation of area in figures of a different shape. (Acuna & Santos, 2012, p. 1)

We agree with Acuna and Santos, that there is a gap between these aspects of the area, furthermore there is also a gap between the qualitative and quantitative aspects of the perimeter. We know very well, that students used to have deficiency of area and perimeter measurement. Researchers describe that these problems arise among others from the early teaching of formulas (Baturo & Nason, 1996; Vighi & Marchini, 2011; Zacharos, 2006).

Usiskin (2012) speaks about a multidimensional approach of understanding, which helps to clarify the meaning of a certain concept and elaborate teaching materials to develop it. The dimensions of understanding of area and perimeter according to Usiskin are the following:

1) Skill-algorithm understanding: Knowing how to get an answer, i.e. choosing an appropriate algorithm to calculate the area and the perimeter of a given figure.

2) Property-proof understanding: Knowing why the way of obtaining the answer worked, i.e. knowing the derivation of the basic formulas and the relation between area and perimeter of the same figure.

3) Use-application understanding (modelling): Knowing when doing something, i.e. recognizing area and perimeter measurement in everyday life problems.

4) Representation-metaphor understanding: Knowing represent the concept in some way, i.e. area with congruent tiles, perimeter with the length of a fence.

According to Usiskin, the dimensions are relatively independent, and there is no precedence among them in terms of difficulty. In the present study the focus is on the 1st and 4th dimension. We compare the level of the skill-algorithm and the representation-metaphorical understanding in different grades.

RESEARCH QUESTION

The present research is looking for the answer to the following questions:

What do the concepts of area and perimeter mean for students at different ages? What are the students' typical strategies and misconceptions by solving area and perimeter tasks?

Our hypothesis is that the older students perform better with respect to the two investigated dimension of understanding (skill-algorithm and representation-metaphor), due to the expanding knowledge. We assume furthermore that the most frequent mistakes arise from identifying the formulae with the concept itself.

METHODOLGY

The research sample comprises 84 students from the same school in Hungary, from the following classes: 26 students from the class 5/A, 29 from the class 6/A, 19 from the class 7/A, 21 students from the class 8/A. The age of the students was from 11 to 14. The four classes are special language classes, they have more language lessons per week as usual and only the minimum mathematics lessons; 3 lessons per week. Students involved in this study didn't show particular talent and interest in mathematics.

We made interviews with the mathematics class-teachers in order to know the exact teaching-learning situation connected to our topic. By studying the curriculum and relevant textbook we can outline the teaching-learning process in the previous school years.

4th class, 8–10 lessons per year: Recognizing area and perimeter as an attribute of plane figures. Measuring perimeter of polygons by adding the side lengths. Measuring area by counting congruent tiles. Finding the perimeter of a rectangle, applying the formulae $(a+b)\cdot 2$, or $a\cdot 2+b\cdot 2$. Measuring areas by counting unit squares, finding the area of a rectangle with whole-number side lengths by multiplying the side lengths and applying the formulae $a\cdot b$.

5th class, 10–12 lessons per year: Finding areas of rectilinear figures by decomposing them into rectangles and adding the areas of the parts. Finding the area and perimeter of rectangles/rectilinear figures in the context of solving real world and mathematical problems. Recognising rectangles with the same perimeter and different areas or with the same area and different perimeters.

6th class, 4–6 lessons per year: Finding the area of right triangles, other triangles, and parallelograms by cutting and rearranging into rectangles.

7th class, 10–12 lessons per year: Knowing the formulas for the area and perimeter of triangles, special quadrilaterals, circles and use them to solve problems.

To investigate and compare the actual level of the understanding of area and perimeter, we designed a 30-minute written test. In one part of the tasks, the area and the perimeter of concrete shapes had to be calculated with the use of known formulae, where we either gave the required lengths or they had to be measured. In case of solving other part of tasks the development or representation of qualitative images were required. The first three tasks were the same in every class. As these tasks are considered very easy routine tasks for those in the 7–8 grades, we assumed they would need shorter time to solve them, so we assigned two further tasks for them. Students were familiar with the type of the tasks, because they solved such kind of problems earlier. It was considered to be also important to have a real-size picture to every task.

While we are interested in students' long term memory, we did the test on the first week of the new school year, on the 3rd of September 2014. This date provides that the students haven't dealt with mathematics especially with area and perimeter for at least 3 months, so we can consider our test as a delayed test.

TASK ANALYSIS AND RESEARCH FINDINGS

The structure of the test

Task 1: Students had to calculate the perimeter and area of three rectangles with the sides 3 cm x 7 cm, 5 cm x 5 cm, and 2 cm x 8 cm. The rectangle's position on the worksheet was usual; the sides were parallel or perpendicular to the paper edges. We wanted to know if students use the adequate calculating method correctly. The perimeter of the rectangles was the same, the area was not. We asked the following question: "What do you observe?" We wondered whether students notice that the same perimeter not necessary results the same area.

Task 2: *Calculate the perimeter and area of the plane figure.* (Figure 1)

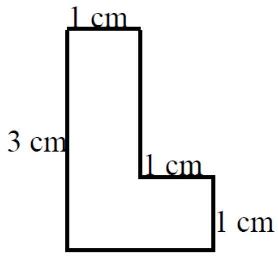

Figure 1

In this case there are no concrete formulae for perimeter and area. Students have to know that the perimeter of a polygon means the sum of the lengths of every side.

They also learned about the additivity of the area and finding area of rectilinear figures by decomposing it. Following Vighi and Marchini (2011), only the necessary numbers are given. We were looking for the level of understanding of these two concepts and typical solving strategies.

Task 3: "*Find the area of the triangle if the distance between two adjacent grid points is 1 cm. Draw two other plane figures with the same area as the triangle has.*"(-Figure 2)

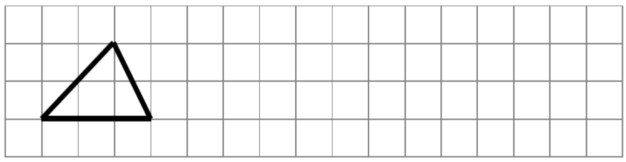

Figure 2

The area of a plane figure means the number of unit squares which cover the figure without gaps and overlapping. Determining the area often requires appropriate cutting and rearranging. There is also a possibility to apply the area formulae for triangles: $b \cdot h/2$. Drawing figure with a given area means more than formal understanding of the procedure for calculating the area.

Task 4 (for Grade 7–8): "*Calculate the area of the triangle.*" (Figure 3)

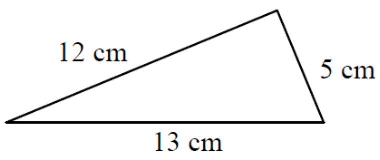

Figure 3

We wonder whether students know the concept of altitude, and apply the formulae learned last school year. The question was the following: is there any student who recognizes that the triangle has a right angle?

Task 5 (for Grade 7–8): "*Calculate the area of the parallelograms. Measure the necessary data by ruler.*" (Figure 4)

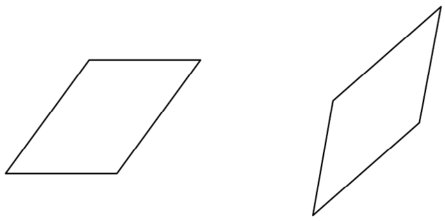

Figure 4

The parallelograms have the same sides but different angles. The first was in "usual" position and the second was rotated. We assume that the unusual position causes problems by determining the altitude of the parallelogram. While the parallelograms have the same perimeter, it's easy to recognize that the areas are different.

The understanding of the idea of perimeter

Every student from the Classes 5 and 8 determined the perimeter of the rectangles in Task 1; except 3 students from Class 6 and 2 from Class 7 gave correct answers too, which means that they are familiar with the calculating method of rectangle perimeter.

The solution of the Task 2 indicates whether the student understands the perimeter concept well or not. In this case there isn't any formula; students have to add all the sides of the rectilinear figure. The diagram below (Figure 5) shows the result of students in different classes. We found that the percentage of the correct answer (10 cm) is not more than 40% in the classes, and what is more, the 5th Graders performed the best. We detected four mistakes as typical. (1) Students often left out one or maximum two sides from the sum. This side is obviously the horizontal side of 2 cm. (2) They added only the given numbers, which indicates the lack of the perimeter meaning. (3) Relatively lots of students applied wrong formulae blindly which is analogue to the rectangular formula. For example: (3 cm + 1 cm + 1 cm +1 cm) · 2. (4) The "additivity of perimeter" also appears in a following way: some students divide the figure in two parts, calculate the perimeters of these parts than add them. It means that the procedure used when calculating the area doesn't work when calculating the perimeter. Vighi and Marchini (2011) call this symptom as "area-perimeter conflict", i.e.: "the use of a procedure for area to compute perimeter". The only distinction is that we experienced this problem not in the Grade 4, but Grade 8.

Reviewing the results of the classes we can establish that the level of understanding perimeter doesn't increase and from the detected solving strategies the wrong rectangle analogy disappeared in Class 8, but we came up with another: using perimeter as an additive quantity.

We noticed that there are 4; 1; 2 and 1 student in Classes 5; 6; 7 and 8, who mixed the words "perimeter" and "area" consequently through all the tasks. The present test was not able to say whether they interchanged only the words, i.e. the label of the concepts or the

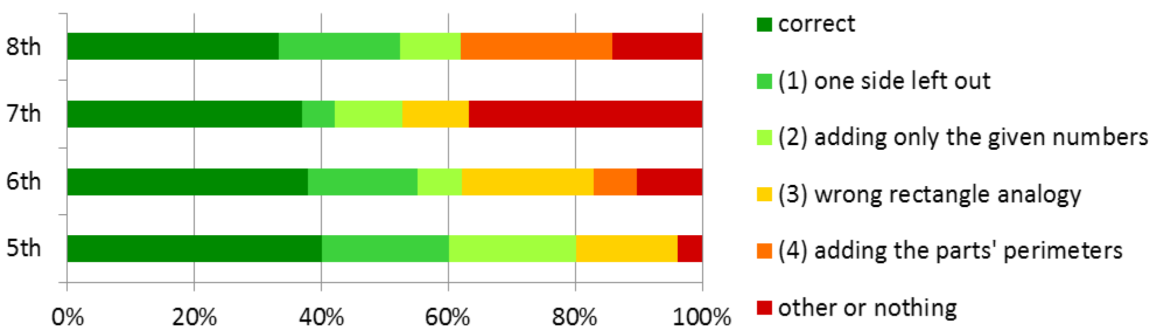

Figure 5: Calculating the perimeter of a rectilinear figure

concepts itself. This finding requires further research on real life word problems.

The understanding of the idea of area

The solution of the Task 3 gives information about the representation-metaphorical understanding (Usiskin, 2012) of the concept of area. The grid suggests the unit squares (*1 cm*^{2}) for determining the area of the triangle. If a student is able to draw another figure with the same area it means that he/she has a correct mental image of the concept even if he/or she can't make the connection between this image and the calculation (Figure 6). Studying the result on the Figure 7 we can see, that the percentage of the correct answer is significantly lower than in case of perimeter. Furthermore students from the Class 8 achieved the best result (close to 50%), it's more than they achieved in the perimeter task.

The most frequent misconception was measuring the sides of the triangle. Most of the students who measured the length of the sides multiplied the three lengths in order to achieve the area. Some of them added the sides or completed the triangle to a minimal rectangle which includes it. The percentage of incorrect answers in Class 6 and 7 are remarkable. 6th Graders forgot about the square grid which is a tool for area measurement, so their solutions were unsuccessful. 7th Graders learnt about decomposing figures and about the formulae of the triangle last year, but the grid-context was "new" for them, so they tried to apply something similar to the well-known rectangular formulae.

Task 2 is based on the knowledge that the area is an additive quantity (area conservation). The diagram on Figure 8 shows that the rate of correct answers in Classes 5; 6 and 7 are very low. Two typically wrong solving strategies can be detected: multiplying sides (some or all) and applying wrong rectangle analogy. In this last case we recognised not only using a procedure for rectangle to compute the area of the rec-

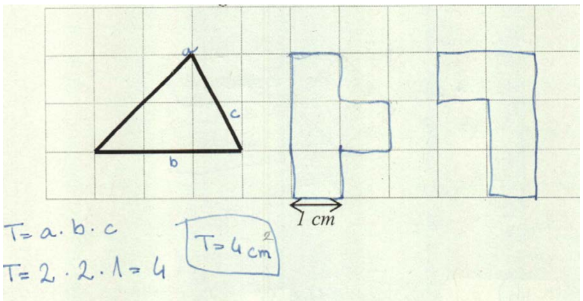

Figure 6: The solution of ***Dorina*** from Class 7

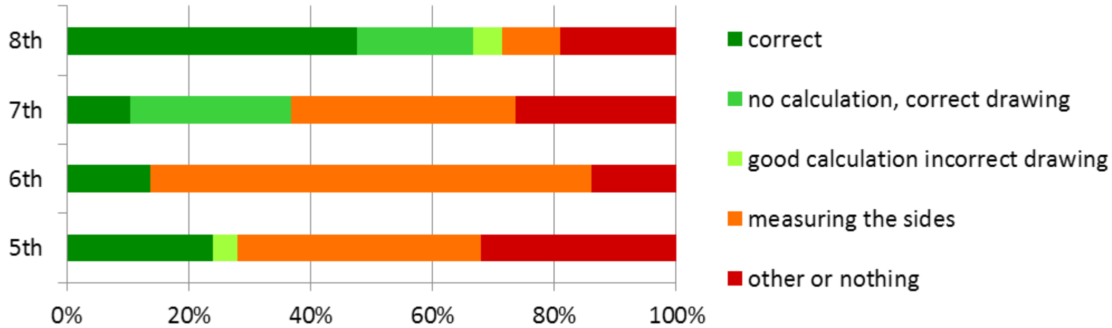

Figure 7: Calculating the area of a triangle drawn on a grid

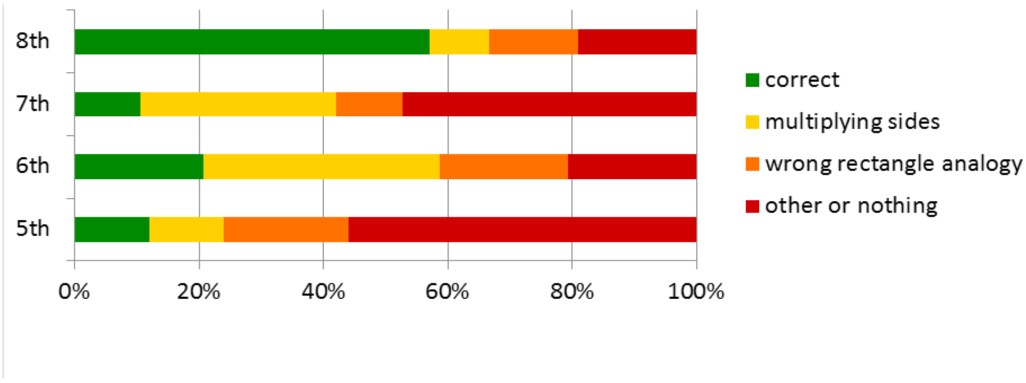

Figure 8: Calculating the area of a rectilinear figure

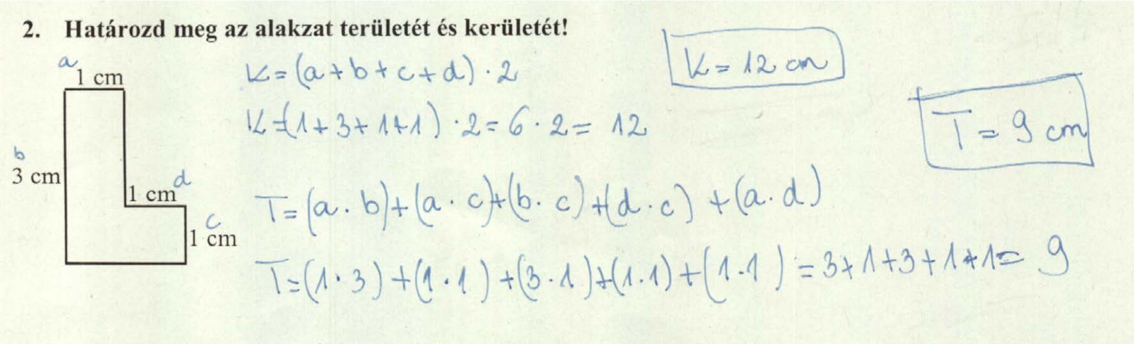

Figure 9: Dorina's (Class 7) solution

tilinear figure, but using a procedure similar to the surface area of a rectangular solid.

For example in the work of *Dorina* (Class 7) (Figure 9), we can notice the use of a procedure for computing the perimeter (K) of the rectilinear figure, analogue to the perimeter of a rectangle. She determines the area (T) with special formulae: she produces each different product of two factors from given sides then ads these factors.

Students who gave the correct solution for Task 2 or 3 were able to solve Task 1 too.

Relation between the area and the perimeter of rectangles

In Task 1, after determining the perimeters and areas of the three rectangles we asked "What do you observe?" There were only a few students (3; 3; 2 and 4 in Classes 5; 6; 7 and 8) who described explicitly that the same perimeter doesn't imply the same area in case of rectangles: "The perimeter of all of the rectangles are the same while the areas are not." One student in Class 8 gave some more explanation:

Lili: The perimeters of the three rectangles are the same, but the areas are different. It depends on the ratio between the sides of the figures.

Determining the area of triangles and parallelograms

While students learnt about the area of triangles and parallelograms in Grade 6, we wonder whether students in Class 7–8 are able to solve such kind of simple tasks.

We constructed the Task 4 and 5 to observe the use of appropriate formulae, the understanding of the concept of altitude and the influence of the position of the figures. In Class 7 there was only one student who calculated the area of the triangle correctly ($12 \cdot 5/2$) and nobody was able to calculate the area of the parallelograms. In Class 8 the result was better: 11 students had success with the triangle and 2 students with the parallelograms. 10 students (Class 7) and 2 students (Class 8) calculated the area of the triangle as the product of the sides. Nearly the same was the situation in Task 5: 10 (Class 7) and 5 (Class 8) students thought that the area of the parallelogram is the product of the two different sides. So the area of the two parallelograms became the same and the students didn't use visual control: it was easy to see that the areas are different. A new example confirmed our earlier opinion that the formulae without any meaning substitute the understanding of the concept itself. 5 students from Class 8 who tried to use the diagonals of the parallelograms ($e \cdot f/2$), did not take into consideration that this parallelogram isn't a rhombus, and the diagonals aren't perpendicular. Of course the position of the figures leaded to misconception too.

CONCLUSION

The findings of our study show the lack of understanding the two geometric concepts we investigated. We thought that the older students perform better due to the expanding knowledge. In contrast we can see for example from the analysis of Task 3, that the mental image of the area concept didn't necessarily develop, students even forgot the ideas established earlier. The formulas cover the meaning of the concepts and cause many misconceptions. One of the main findings of our research is that students think of the concept of area as the product of the sides. The new knowledge also brings mistakes if this knowledge hasn't a strong basis. The additive feature of the area implies the additive property of the perimeter. The introduction of a new formula (e.g. $e \cdot f/2$ for the area of the rhombus) causes trouble in finding the adequate calculating method. We observed and confirmed many misconceptions which were mentioned in the literature before.

Our experiences related to this research highlight the fact that students can easily forget concepts and procedures if they do not have the possibility to establish and practice it. The efficiency requires meaningful and continuous practice. The present study is the part of a wide research which aims at developing a complex teaching experiment in classes 3–8, on the topic of geometrical measurement.

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Using the geometric working spaces to plan a coherent teaching of geometry

Alain Kuzniak and Assia Nechache

University Paris Diderot, Laboratoire de Didactique André Revuz, Paris, France,
kuzniak@math.univ-paris-diderot.fr, assia.nechache@hotmail.fr

A frequently-asked question by in-service teachers during training sessions relates to the overall organisation of the geometric work at primary school level. Indeed, teachers have difficulties to integrate the various resources and activities discovered during their training in a general progression for their teaching of geometry. This issue is difficult even for teacher trainers who have to meet students' expectations. In this paper, we consider some elements that could help teacher trainers devise an overall coherent teaching of geometry. We address this issue from the analysis of a teaching sequence in geometry proposed for Grade 4 - 6 students. This analysis is mainly based on the Geometric Working Spaces model used as a tool to clarify and structure a coherent teaching of geometry.

Keywords: Geometry, geometric working space, pedagogical progression.

INTRODUCTION AND OBJECTIVES

As a teacher trainer, one of the authors (Assia Nechache) is in charge of several training sessions, especially in geometry, for in-service primary school teachers. Various activities are used during these training sessions and are selected from available resources written in French and dedicated to teacher training, such as the book *Concertum* (2002) edited by the French Commission for primary school teachers' training or journals like *Grand N*. The trainees expressed their interest and pleasure in discovering and doing these activities, but at the same time they were very anxious about devising and organising an overall progression of the teaching of geometry using these new, specific activities in their classrooms.

Indeed, it is usually possible to find ideas and examples of activities to implement in a classroom situation: resources such as those we mentioned above are relatively numerous. By contrast, very few are concerned with providing tools to help teachers reflect on and develop their own overall progression in geometry.

In these conditions, we may wonder how a teacher trainer can help teachers integrate new and interesting tasks in their classroom progression. This is by no means an easy question to answer, even for teacher trainers. We therefore focused our study on possible tools to help teacher trainers deal with this specific issue.

In another context, that of the research in geometry, the model of the Geometric Working Spaces (GWS) is used to understand and structure the various ways of thinking about the teaching of geometry throughout compulsory schooling. We wondered whether this model originally designed for education research could also be a tool for the training of primary school teachers, as it could help them conceive and implement a progression in geometry. Thus, as a first step, we decided to explore the use of the GWS model with teacher trainers to highlight the main elements that organise a long teaching sequence on a specific topic and see how it can help structure an overall progression.

In this paper, we first present the model of the Geometric Working Spaces and then report an analysis of a complex and long sequence of activities on the concept of circle. The analysis was conducted with a group of 27 teacher trainers during a working group devoted to the teaching of geometry. The analysis is mainly based on the GWS model and it allows us to highlight some key points to think about the teaching of geometry in elementary school. The reported experiment is not a product of pure mathematics education research according to the standards of didactic engineering, but can be considered as an action research process, and thus as an initial attempt to adapt and transpose a theoretical tool from research to teacher training education.

THE MODEL OF THE GEOMETRIC WORKING SPACES

The Geometric Working Spaces (GWS) and geometrical paradigms have already been presented, including during former CERME sessions. For a general presentation of paradigms, we refer to Houdement and Kuzniak (2003) where the three paradigms GI, GII and GIII are identified. An introduction to GWS was presented in a plenary lecture during the last CERME session in Turkey (Kuzniak, 2013). The GWS are now part of the more general framework of the Mathematical Working Spaces (MWS) described by Kuzniak and Richard (2014)[1]. We will summarise here some key points which we think are useful to understand the analysis that we will report in the paper.

The Geometric Working Spaces

The description of the geometric work done by students in school is the main purpose of the Geometric Working Spaces (GWS). As its name suggests, the geometric work is at the centre of the model and motivates the reflection on the teaching and learning of geometry (Kuzniak, 2014). In this approach, the crucial function of educational institutions and teachers is to develop a rich environment which will enable students to solve geometric problems in an appropriate way.

To describe the specific activity of students solving problems in geometry, the GWS is organised into two planes or levels. The first, "epistemological" plane defines *a priori* expectations about the activity according to the requirements of the mathematical domain, in this instance geometry. As regards geometry, three interacting components are characteristic of geometric activity in its purely mathematical dimension:

- A real and local space as material support, with one set of concrete and tangible objects such as figures or drawings;
- A set of artefacts such as drawing instruments or software;
- A theoretical reference system based on definitions and properties.

The geometry that is taught and learnt at school is not a disembodied set of properties and objects reduced to signifiers which can be manipulated by formal systems – it is first and foremost a human activity. Therefore, it is essential to understand how communities of individuals, but also specific individuals, use and internalise their knowledge of geometry in their practice of the discipline. This implies a second, "cog-

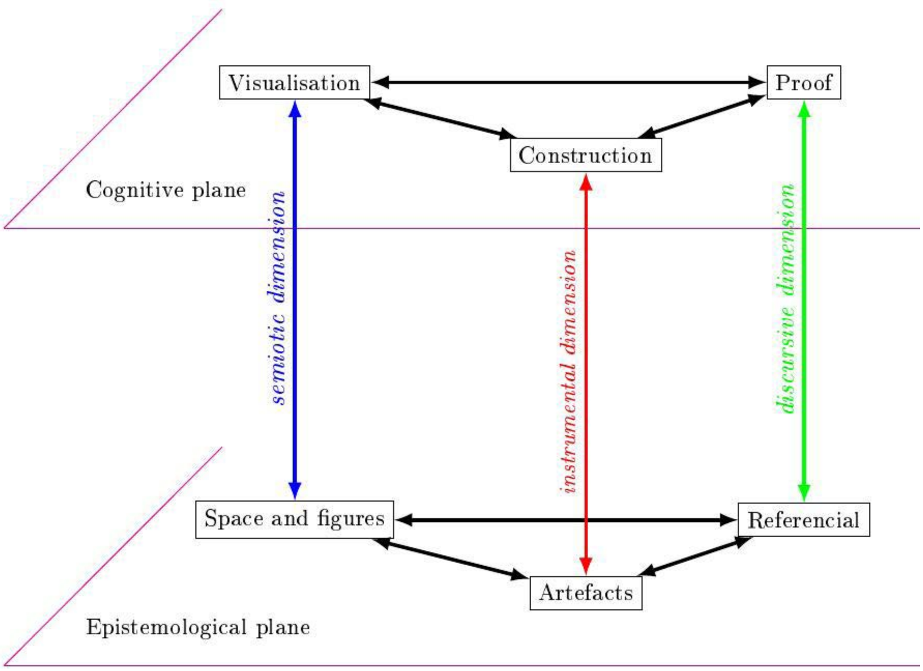


Figure 1: General diagram of the Geometric Working Spaces

nitive" level centred on the subject viewed as a cognitive subject solving problems. Considering geometric activity, these processes are as follows:

— A process of visualisation related to the representation of both space and the material support;

— A process of construction and function of the used instruments (e.g. rulers, compass) and the respective geometrical configurations;

— A discursive process producing arguments and proofs.

This set of relationships could be described proceeding from the elements of the following diagram which, in addition, shows the relationships between the two levels with three different dimensions or geneses: semiotic, instrumental, and discursive.

Various types of input in the geometric work
The above diagram shows three specific work dimensions between epistemological and cognitive planes which will require three specific genetic developments named geneses.

1) A figural and semiotic genesis that gives the tangible objects their status of operating mathematical objects;

2) An instrumental genesis that transforms artefacts into tools within the construction process, which is crucial in the case of geometry;

3) A discursive genesis of proof that gives a meaning to the properties used within mathematical reasoning.

The diagram in Figure 1 shows three vertical planes that match the connections between these dimensions and that will help us later to specify the precise geometric work existing in the GWS when students solve tasks given by their teachers. The three levels can be identified by the genesis they implement [Sem-Ins] (blue), [Ins-Dis] (red) and [Sem-Dis] (green). The objective of the analysis we present in this present was to understand precisely the nature and dynamics of these planes during the resolution of a series of geometric problems.

DIDACTIC ANALYSIS OF A TEACHING SEQUENCE THROUGH THE GWS VIEWPOINT

According to the theoretical framework, we assume that a geometric work can be considered "complete" when a geometric entity is built throughout the three semiotic, instrumental, and discursive dimensions of GWS. For this reason, a geometric entity, here the "circle", will be considered a triplet [sign, artefact (material or symbolic), property]. As a result, building a progression in geometry first requires the identification of key geometric entities in the curriculum and then the analysis of the work associated with these entities in terms of GWS dimensions. We focus here only on the second point related to a long teaching sequence on the circle entity.

The selected sequence "Le cercle sans tourner en rond" was conceived and implemented by two well-known and experienced French scholars in the domain, Fénichel & Taveau (2009). It has been tested in various classrooms under different conditions and is relatively known by teacher trainers. Dedicated to Grade 4 – 6 students, the whole program is very ambitious and includes eight sessions – from half an hour to one hour – which can be administered over three months according to a spiral program. The main objectives of the sequence are to introduce the global notion of circle as the set of all points equidistant from a given point, the centre; to use this property to solve distance problems; to relate it to construction with compass used also to transfer distances. According to our analysis, the "circle" entity targeted by the sequence can be described with the triplet [circle as drawing, compass, equal distance].

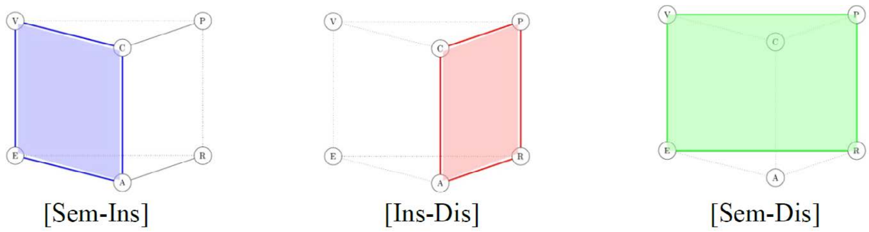


Figure 2: GWS vertical planes

Some activities of the sequence have been chosen for a common analysis with the 27 teacher educators in a working group setting. We here only report on the activities corresponding to classroom sessions 1, 2, 3, 4 and 7.[2] Each of the five selected sessions has been analysed using the GWS model. In particular, participants in the working group were asked to identify, if possible, the different input dimensions (semiotic, instrumental, discursive) in the geometric work and the favoured planes (plane [Sem-Ins], plane [Sem-Dis] and/or plane [Ins-Dis]). This identification highlights the dynamics of the geometric work during the different studied sessions and allows for the characterization of the overall implemented GWS.

We first detail the analysis of session 1.

The objective of the session is to highlight the fact that the circle is the set of all points equidistant from a given point, the centre. A varied amount of material and artefacts is made available: blank and tracing paper, twine, square set, compass…

Students are asked to draw 15 points at a given distance from a point A. During this phase of action, they need to place a point A on the white sheet and then a point B. After that, they need to place 15 points "situated at a distance from A which is the same as the distance of B from A". The geometric work starts in the plane [Sem-Ins].

Then, during a phase of formulation, some students' productions are displayed on the blackboard and discussed. The strategies they used to carry out the task are clarified and formulated. The objective is first to validate the notion of circle as discussed by the pupils based on the pupils' constructions. The notion of equidistance to a given point is expected to emerge. Some geometric terms are institutionalized and the characteristic property of the circle is given by the teacher and enriches the theoretical referential in the GWS.

The analysis made by the teacher educators is summarised in the following table (Table 1).

We have carried out the same work with other sessions. The different forms of the geometric work identified in the five sessions are presented in the following table and linked to a GWS diagram (Table 2).

To summarise, the geometric work is centred on the development of the notion of "circle" viewed as the set of all points equidistant from a given point and it is closely linked to the use of the compass. The epistemological plane can be defined by the triplet [circle as drawing, compass, and equal distance].

The geometric work conceived by the designers of the sequence is based, firstly, on the material artefact to bring out a property and enrich the set of theoretical tools. Then, the material artefact is set aside to promote a discursive reasoning using the theoretical notion of circle associated with the notion of equal distance.

At the end of the sequence, a return to the material artefact is operated to introduce a new use of the compass, which triggers a new circulation of the geometric work trough the different vertical planes. All the aspects of the work pertain to Geometry I but the sequence clearly paves the way for a prospective work in Geometry II at secondary school.

Teacher trainer viewpoints

As mentioned above, the global analysis of the sequence has first been conducted by the two authors and then some of the activities have been chosen for a common work with teacher trainers during a working group: that ensures a stronger relevance to the final

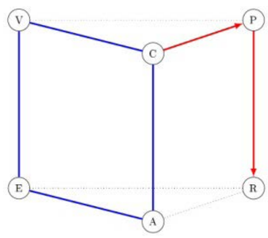
Sessions	Input in geometric work	GWS Diagrams
Session 1 The circle defined as the set of all points equidistant from a given point, the centre	The geometric work starts in the plane [Sem-Ins] and is concluded by the enunciation of the characteristic property of the circle which enriches the theoretical referential in the GWS. Properties and definitions of various figures used in this Grade are included in the referential. At this level, figures are generally introduced by "ostension"; it is worth noticing that such is not the case here: the proposed session clearly contrasts with traditional classroom sessions.	 [Sem-Ins] → Dis

Table 1: The analysis of session 1

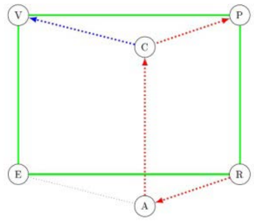
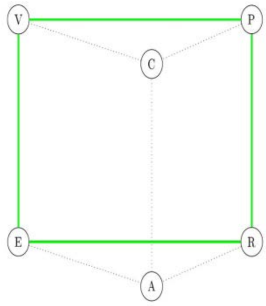
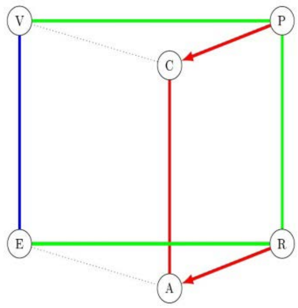
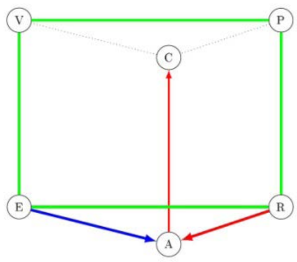
Sessions	Input in geometric work	GWS Diagrams
Session 2 Geometric problem solving with use of the stated property during session 1	The new theoretical property needs to be used to solve a problem. The input into the work is first rather semiotic but the use of a discursive proof is required to validate the solution. The use of drawing instruments strongly depends on the previous identification of the characteristic property of the circle. Artefacts are proposed, in option, as verification tools after the task is solved in the plane [Sem-Dis].	 [Sem-Dis] (→ Ins)
Session 3 Giving sense to Session 1 property by using it on freehand drawings.	The geometric work is this time in the plane [Sem-Dis] but with a clear discursive input because freehand drawings are used and can be considered as symbolic signs. Drawing instruments are set aside. As a consequence, the discursive dimension is now essential to the validation of the solution. The idea is to show that the “circle” is a theoretical object grounded in a characteristic property and not only an empirical object perceptively and instrumentally linked to a drawing. Validation is based on a discursive proof within Geometry I and appropriate to the primary school level, but it is paving the way for Geometry II, which is the challenge of geometry teaching in French secondary education. As was the case previously, the authors still give students the opportunity to return to the experimental validation if it is necessary for their understanding.	 [Sem-Dis]
Session 4 Discovering circle and disk uses to solve equal distances problems.	This time, the activity supposes a construction after modelling the problem. Once the task is interpreted, the geometric work is mainly located in the plane [Dis-Ins] using the theoretical referential: the characteristic property of the circle appears as a theoretical tool to build the solution. The data are provided in the semiotic register and the circle property ensures the validity of the solution.	 [Sem-Dis]→[Dis-Ins]
Session 7 Triangle construction using the circle property and introduction of the compass as length-transferring tool	The compass will acquire a new function. Initially considered as a tool dedicated to the construction of circles, the compass is used to transfer lengths and construct other geometric figures such as triangles. An enlargement of the use of the artefact is intended and it is related to the theoretical referential and the “circle” figure. The geometric work starts in the plane [Sem-Dis] and then enriches the instrumental dimension.	 [Sem-Dis] → Ins

Table 2: The analysis of the other sessions

analysis which is based on the convergence of the different contributions made by participants.

In relation to in-service training and our initial question, the identification of the types of input in geometry seems successful in helping teacher trainers consider an overall progression in the teaching of geometry based on the organisation of a set of geometric tasks to promote a complete geometric work along the three GWS dimensions. Teacher trainers agree with this idea and underline that it is possible to have a global vision of geometry thanks to the GWS model. Moreover, teacher trainers have been aware of the importance of linking the different dimensions (semiotic, instrumental and discursive) within the geometric work.

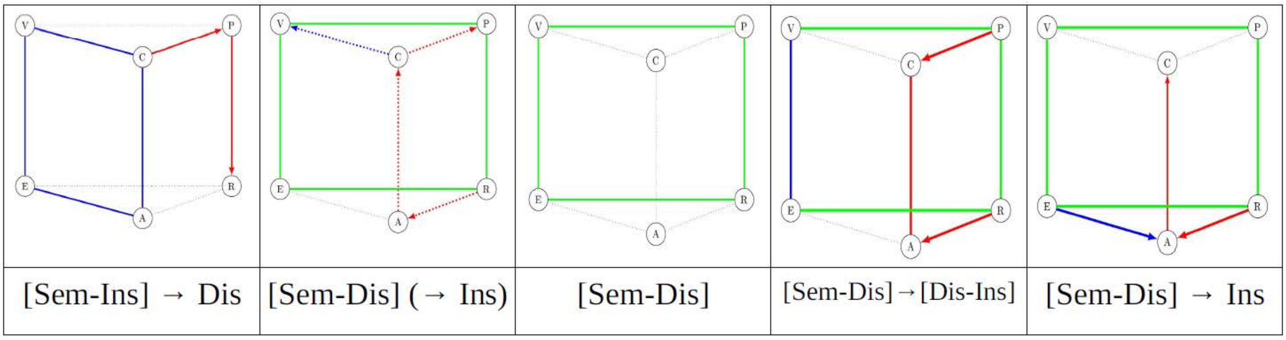


Figure 3: The dynamic evolution of the mathematical work during the different session

Teacher trainers insist on the fact that an analysis using the GWS model is relevant to highlight the dynamic evolution of the mathematical work during the different sessions. The evolution of the work is illustrated in the following global diagram (Figure 3).

CONCLUSION

Our initial question was how to devise an overall progression in the teaching of geometry at primary school. As the issue is very complex even for teacher trainers, we started our study by working with them during a workshop. To deal with this issue, we introduced a twofold approach: first, the foremost mathematical entities targeted by the curriculum need to be identified; then, activities and tasks related to a specific entity can be analysed according GWS model to ensure a global consistency. In the paper, we focused on the analysis of a sequence of activities on the "circle" entity to identify the different favoured GWS dimensions. The GWS which appear in the selected sequence are structured around a set of tasks related to the triplet [circle-drawing, compass, equal distance]. This set of tasks mobilises different articulations[3] between the three vertical planes of the GWS diagram and gives birth to a real and complete dynamic cycle in the geometric work.

According to the teacher trainers involved in the study, this approach gives a global vision of the geometric work and highlights the choices of the designers. It allows teachers to assess the consistency of a sequence and permits them to discuss the choices made. For instance, some teacher trainers did not agree with the idea of focusing on the characteristic property of the circle at this level of schooling. We may then wonder what are the consequences of other inputs giving priority to the use of software or spatial activity on the geometric work. With the GWS tool, it seems possible to discuss the « best dynamics » to favour the geometric work among teachers and pupils.

The question remains of the use of the model in teacher training and teachers' practice. We hypothesize that it is more adapted to teacher trainers than to teachers. Indeed, the priority in teacher education is to explain the mathematical content involved in teaching sequences but our reflection on the didactic transposition of the GWS theoretical model should be furthered and is one of our prospective research topics. What is clear from our experience is that it is not necessarily required to present the GWS model in depth to teacher trainers: a short introduction based on the GWS diagram has enabled us to conduct our analysis with convincing results.

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ENDNOTES

1. For readers interested in the development of the model see the web-page http://turing.scedu.umontreal.ca/etm/documents/Actes-ETM3.pdf where the proceedings of ETM3 can be found.

2. The whole sequence is online on Alain Kuzniak's web-page: http://www.irem.univ-paris-diderot.fr/~kuzniak/publi/Publications. Even though the text is in French, we hope that the reader can understand the main phases of the different geometrical activities.

3. These smooth and graduate transitions between planes or dimensions are called "fibrations" in the GWS model.

Spatial and geometric structuring – contributions for a collective construction

Cristina Loureiro[1] and Lurdes Serrazina[2]

1 Escola Superior de Educação de Lisboa, CIED, Lisboa, Portugal, cristina@eselx.ipl.pt

2 Universidade de Lisboa, UIDEF, Lisboa, Portugal, lurdess@eselx.ipl.pt

This study aims to understand the contributions of collective moments of whole-group discussions for the construction of didactical pathways, based on geometric tasks experimented with children in the 2nd and 3rd years of primary school (6–10 years old). The experiment contributes to the understanding of Battista's model in its two dimensions of spatial and geometric structuring as well as the relationship between them (Battista, 2008). The data we gathered, although its analysis is still in progress, help us to build a framework from which it seems possible to identify a connection between the individual forms of mathematical knowledge for each student, the taken-as-shared mathematical practices of the classroom community and the taken-as-shared mathematical practices of wider society (Cobb, Yackel, & Wood, 1992).

Keywords: Spatial structuring, geometric structuring, whole-group discussions.

INTRODUCTION

In the Portuguese educational context the teaching of elementary geometry is still very poor and has adverse consequences for future teachers (Tempera, 2010). As pre-service and in-service teacher educators we recognize the need to enhance this area, seeking to produce useful materials for teacher education. These requirements are consistent with research interests in this area (Battista, 2007).

This communication is part of a PhD research project of the first author whose purpose is to study the teaching and learning of geometry in the early years. Its main objective was to design, test and evaluate teaching didactical pathways in Geometry and Geometric Measure for the first cycle of basic education (6 to 10 years old children). Our investigation, a design research (Van den Akker, Gravemeijer, McKenney, & Nieveen, 2006), was guided by a framework in three phases (Jackson, Garrison, Wilson, Gibbons, & Shahan, 2013). This framework, which arises from Stein, Smith, Henningsen and Silver (2000), includes the installation of a task, *phase 1*, and its implementation, split in two distinct components, *phase 2*, the moment when students work on the task, and *phase 3*, the moment of whole group discussion. Progressively, implementing tasks in different classes *phase 3* has gained greater importance increasing the role of the researcher as a teacher, but always acting together with the class teachers.

The objective of this study is to identify and understand the contributions of collective moments of whole-group discussions for the construction of didactical pathways, based on tasks focused on the spatial and geometric structuring (Battista, 2008). We try to highlight the importance of these moments and some critical aspects that help to decide the orientation of the pathways and to establish the links between the tasks that integrate them.

THEORETICAL CONSIDERATIONS

The research is based on three fundamental axes: the structuring of geometric reasoning (Battista, 2008; Freudenthal, 1991); didactical pathways based on hypothetical learning trajectories (Confrey & Kazak, 2006; Gravemeijer, 1998; Simon, 1995); implementation of mathematical tasks (Jackson et al., 2013; Stein et al., 2000; Yackel & Cobb, 1996).

We adopt the structuring perspective of Battista (2008) involving three items: (a) Spatial structuring; (b) Geometric structuring; and (c) Logical/Axiomatic structuring. At this level of instruction we only face the first two, although we have in mind that the development of good formal logical structuring depends on a good geometrical structuring, as the development of the later depends on the quality of spatial structuring. According to Battista:

> Spatial structuring determines a person's perception/conception of an object's natures or shape by identifying its spatial components, combining components into spatial composites, and establishing interrelationships between and among components and composites. (p. 138)

For instance, a geoboard is an instrument of spatial structuring through an orthogonal normal structure. We use it to spatially structure rectangles. If the sides of the rectangle coincide with the lines, structuring is immediate. If not, it is necessary some kind of visualization feature to spatial structuring the rectangle. On the other hand, analyzing a number of different rectangles and identifying the existence of four right angles as invariant we are geometric structuring rectangles. This way we are constructing a rectangle model. This conceptual scheme allows the square to be recognized as a rectangle. This example illustrates, as Battista says, how "for a geometric structuring to make sense to a person, it must evoke an appropriate spatial structuring for the person" (p. 138).

In our investigation, we follow the visualization perspective of Presmeg (1997):

> Visualization is taken to include processes of constructing and transforming both visual mental imagery and all of the inscriptions of a spatial nature that may be implicated in doing mathematics. (p. 304)

In the context of geometry, the need for research on the nature of tasks that develop spatial visualization skills and visualization is recognized for the development of spatial and geometric structuring. We connect this with learning trajectories. We use the name of didactical pathways to identify a set of tasks suitably structured, based on hypothetical learning trajectories, after experimental use in a classroom which allows its refinement from experimentation. In a way, a didactical pathway can be considered as a photograph of a journey of learning and teaching, already performed, including their reflective analysis.

We attend to students different roles and responsibilities, distinguishing thinking responsibility and participation responsibility (Wood, 1999; Wood & Turner-Vorbek, 1999), the sense of orchestrating to promote productive classroom discussions (Smith & Stein, 2011; Stein & Smith, 1998; Stein et al., 2008), the role and kind of questions (clarification, argumentation, confirmation) and their importance to obtain productive exchanges (Boaler & Brodie, 2004; Wood & Turner-Vorbek, 1999).

METHOD

Concerning the method, this research is an *educational design research* according to Van den Akker and colleagues (2006). At the end of the fieldwork conducted by the first author, we managed to get four didactical pathways. Many of the tasks have been tried more than once, in different classes and years of schooling, with particular focus in the 2nd and 3rd years. The goal was to keep improving their implementation, as well as sequencing. Each path operates as a learning cycle (Simon, 1995) and the development of research constitutes itself as a cumulative cyclic process where interpretation of a trajectory provides added value for the planning, experience, reflection and interpretation of the following cycles. So intervention dimensions, iteration and orientation processes were valued. The introduction of new tasks was one of the important aspects in the iteration of cycles of experiences.

The tasks are of an open nature and provide collective discussions based on students' productions. They provide an easy adhesion because their understanding is very simple and students can reason and act in a personally meaningful manner (Gravemeijer & Cobb, 2006). Students worked first individually and the tasks were performed with geoboard or dotted paper as a visual thinking support.

The fieldwork was divided into two periods. The 2nd one is marked by a dual role of the researcher who assumed the leadership of phase 3. This change occurred because during the 1st period collective discussions were very poor or nonexistent. This change was possible due to the established relationship with the four teachers involved as well as with their students. In the second period, which occurred in two consecutive school years, all tasks have been tried with the same students from one of the classes in their

2nd (episode 1) and 3rd years (episode 2) of schooling. These lessons were recorded on video. Subsequently, we identified several relevant episodes of collective discussion as the unit of analysis.

EXAMPLES, COUNTEREXAMPLES, SPATIAL AND GEOMETRIC STRUCTURING

The following two episodes pretend to illustrate the role of examples constructed by students, as well as of counterexamples. They also illustrate various aspects of spatial and geometric structuring present in the tasks and the importance of giving them prominence in geometric figures as the work was going on.

Episode 1 – "Almost equal"

The proposed task consisted of identifying pairs of congruent figures. Two squares raised some controversy. The researcher asked the students if the two last copies were or not congruent (Figure 1). Almost all students said they were equal, although one student said they were not. The researcher asked one student,

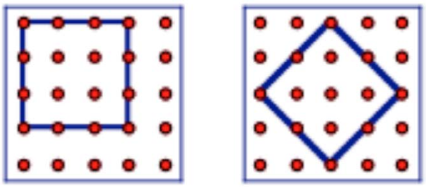

Figure 1: Squares that generated the controversy surrounding the congruence

Beatriz, to show her colleagues why they were equal.

As we expected this could happen, we prepared two acetates with figures and we gave them to Beatriz to experiment, corresponding to the practice of anticipating (Smith & Stein, 2011). She tried to overlap them, at the overhead projector, so they matched. Beatriz said nothing, but went around the acetates to get the match. During this process the other students observed. The researcher asked "who does still think that the two squares are equal?". Several students were saying "they are not equal". However, Beatriz remained stubbornly trying to match them. The researcher asked students to argue:

Ana: One of the squares is larger than the other.

Leonor: We can't put one above the other.

Joel: The number of dots inside the square is different. One square has 4 points and the other 5.

We have three different validations and the last one is more elaborate, denoting a more refined visualization. Leonor and Ana still seem to think based on the figure as a whole, while Joel shows the ability to distinguish the underlying structure to the figure to conclude that the two squares can not be congruent. Should we follow the reasoning of this student and share it? What is the contribution of this reasoning to the geometric structuring? This tension between taking advantage of good individual contributions and deciding the orientation of the collective movement was present in several episodes. It is the basis of significant dilemmas that teachers face during practice classroom (Carter & Richards, 1999; Wood & Turner-Vorbek, 1999) and acting as a filter of students' contributions (Sherin, 2002).

Episode 2 – "The rectangle that is not rectangle"

This second episode occurred later. It takes place in another pathway in which quadrilaterals have been worked as composite figures, highlighting its components and seeking to establish simple relationships between these elements, namely the angles. The pathway consisted of six tasks that allowed four moments of collective discussion. This episode occurred at the end of the second task and the discussion was about the students' work conducted in tasks 1 and 2. The task was to find out as much of squares and rectangles on geoboard. Students' works were exhibited with the solutions they found for squares and rectangles.

In the first part of the discussion the researcher and the teacher tried to interest students on the discussion they intended to create, as the appropriation by the students of social rules of whole-group discussion is a slow process and built over several experiments (Wood & Turner-Vorbeck, 1999). The discussion began with an appeal to respect different opinions, highlighting the value of students' opinions, with attention and care to not repeat what others have said which is very common among small children.

The researcher asked the students if they had any different figure, squares or rectangles. Two students, Inês and Beatriz, raised their fingers. They went to the black board to expose their magnified figures. This preparation of figures occurred during *phase 2*, corresponding to the practice of selecting, according to

Smith and Stein (2011), as we already knew that these figures would be important for the discussion. Those students thought that they had made a rectangle, however, when they showed it to their colleagues these at once identified it as a parallelogram.

Two students in chorus:
It is a parallelogram.

The researcher asked students to remain calm and asked Inês to justify why she thought her figure was a rectangle. The episode evolved with a long dialogue with many voices involving several students trying to show why Inês' parallelogram was not a rectangle.

Hugo: It is so because it has two beaks on one side.

Hugo shows with his fingers and surrounds the parallelogram. He points to two opposite sides and says they were inclined compared to the rectangle he considers not to be inclined. Hugo must reinforce the comparison with another figure, a rectangle in prototypical position "on high", identifying important elements for this decision. Given his difficulty in justifying his reasoning, the researcher decided to ask another student.

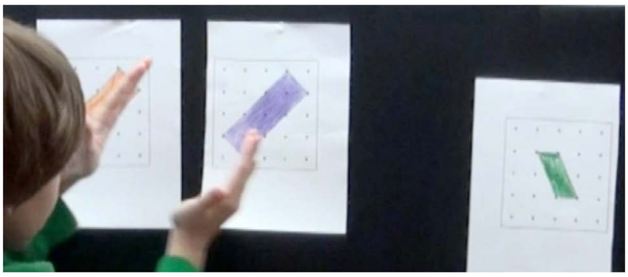

Figure 2: The controversial parallelogram and the rectangle Duarte used to compare

Duarte: It is not a rectangle because it is wry. Instead of being like this, is like this.

Duarte accompanied what he was saying with gestures with his hands (Figure 2). First he made with the hands two parallel segments, ||, then he made two inclined parallel threads, //, and surrounded the parallelogram with his fingers. Moreover, he was able to do with his hands the necessary modification to transform the parallelogram into a rectangle, the angle of two consecutive sides should be a right angle. He compares with another rectangle exposed in a non-prototypical position, that is, "tilted". Although Duarte showed that he had very clear ideas about geometric justifications, he was not able to fully verbalize them.

All this discussion focused on the observation and analysis of quadrilaterals, showed the need to work more aspects of spatial structuring of quadrilaterals and the need to give more attention to spatial structuring, required for geometric structuring of quadrilaterals (Battista, 2008). It also emphasizes the requirement to pay attention to the elements that compose a figure, in this case the angles and sides. It also identified difficulties in verbalizing the reasoning, revealing that in most cases the mental images of students are correct and appropriate to their arguments.

This episode revealed the need to enhance students' language and to provide them with a simple tool of visualization to highlight and compare angles. This object, which we call "right angles' detector", is not more than a corner of an A4 sheet cut so as to be easily manipulated and had already been widely used in previous experiments of these tasks. This episode marked guidance route for the sequence of this didactical pathway, which focused mainly on the spatial structuring. The following tasks were directed to discovering and comparing angles in quadrilaterals. Thus highlights the focus on the right angle as a reference for spatial and geometric structuring, without even resorting to any measurement system to identify acute, obtuse and right angles in plane geometric figures.

Both described episodes, as well as many others experienced, illustrate how students gradually assumed responsibility for discovering examples and counterexamples, incorrect or not complying figures with the established conditions, families of quadrilaterals in hierarchical classification. This was found successful to give students leadership in validation of mathematical knowledge in spatial and geometric structuring tasks.

DISCUSSION

We follow two lines of discussion, the first about geometry and the other on social constructivist aspects of learning.

We consider that the data collected value the potential of the structuring model of Battista (2007, 2008), and help us to better understand spatial and geometric structuring, as well as the relation among them. We emphasize the importance of visualization in spatial structuring of quadrilaterals and the need to find strategies and objects to support visualization, for example, dot papers, colouring the angles, the “right angles detector”. We saw the importance of structuring quadrilaterals as composed by four angles figures, especially the rectangle as a four right angles composite figure, (Battista, 2008), aimed to structure each of the various types of quadrilaterals in hierarchical classification (de Villiers, 1994). We emphasize the need to incorporate several spatial structuring models to obtain personal’s mental models that determines her or his own way of thinking (Battista, 2008).

With regard to the geometrical structure, it seems to give new contributions for the classification of angles embedded in composite figures (quadrilaterals) and for an understanding of the geometric structure of quadrilateral on several levels (isolated figure, as part of a class, class relation). The examples we worked on showed us it is possible to implement meaningful tasks for young children, demanding in terms of geometric structuring (congruence of figures, understanding of a figure as part of a class, the class understanding, organization of quadrilaterals in hierarchical classes) from didactical pathways. Applying these pathways, with special attention to the collective moments of whole-group discussions proved to be a key factor in design tasks and for taking decisions for tasks sequence.

Second part of discussion is about social constructivist aspects of learning. The data collected makes us aware of the challenges associated with a practice focused on student work when you want them to become providers of mathematical knowledge (Wood & Turner-Vorbeck, 1999), and seeks a balance between the mathematical value of the work of each student and ensuring that this work is in the mathematical point of view recognized and accepted (Cobb et al., 1992; Yackel & Cobb, 1996). We are in the process of establishing indicators and frameworks based on videotaped episodes. We are trying to construct a framework on learning environment, roles and responsibilities for learning in whole-group discussions from inquiry tasks (Figure 3).

Comparing columns A and B (Figure 3), we face the relation between teacher and students. Teacher action is more like a double mirror reflecting but also allowing seeing through it. As Wood and Turner-Vorbeck (1999) says this is a matter of ways of structuring social interaction and discourse to create contexts for learners’ personal construction of meaning.

The parallel that may exist between both columns and the approximation of the identified practices, highlight the student’s responsibility in the construction of mathematical knowledge. It seems therefore that this work getting closer forms of mathematical knowledge for each individual student, shared mathematical practices of the community classroom and shared mathematical practices of wider society (Cobb et al., 1992).

The lived episodes help us to understand the requirement of the teacher role at moments of whole group discussions. This requirement is expressed in simultaneous attention needed to give individual student’s contributions and the collective movement that teacher has to provide so the whole class advances, ensuring that the ideas and processes at play are widely accepted as having mathematical value and are necessary for future school mathematics learn-

<table>
<tr><th></th><th>Teacher responsibility (A)</th><th>Students responsibility (B)</th><th></th></tr>
<tr><td rowspan="2">Taken-as-shared Mathematical practices of the community classroom (Cobb et al., 1992)</td><td>Discursive practices
Ask question</td><td>Discursive practices
Validation of own or colleagues knowledge</td><td rowspan="2">Responsibility for thinking (Wood & Turner-Vorbeck, 1999) (Smith & Stein, 2011)</td></tr>
<tr><td>Reflective practices
Decisions on the progress of the discussion</td><td>Reflective practices
(Emerging)</td></tr>
<tr><td rowspan="2">Collective discussion management (Oliveira, Menezes, & Canavarro, 2013)</td><td>Interactive practices
Student participation management</td><td>Interactive practices
Engagement in discussion</td><td rowspan="2">Responsibility for participation (Wood & Turner-Vorbeck, 1999)</td></tr>
<tr><td>Reflective practices
(Underlie teacher decisions on students’ participation)</td><td>Reflective practices
(Underlie initiatives and actions of students)</td></tr>
</table>

Figure 3: Framework for whole-group discussions from inquiry tasks (in construction)

ing. It also illustrates the critical importance of collective moments to construct a challenging learning environment for students, giving them autonomy and authority in the validation of ideas and mathematical knowledge, their own and those of colleagues. Regarding the teacher's role the indicators of orchestration practices are relevant (Smith & Stein, 2011), questions with potential (Boaler & Brodie, 2004) and dilemmas (Carter & Richards, 1999; Wood & Turner-Vorbeck, 1999).

As a final idea for this discussion, we return to the initial purpose of this research to recall the importance this experience has taken to help design and evaluate teaching pathways in geometry, showing as one of the critical aspects of these pathways the links that can be established between the tasks that integrate them.

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Combining epistemological and cognitive approaches of geometry with cK¢

Joris Mithalal-Le Doze

Université Paris 4, ESPE de Paris, Laboratoire de Didactique André Revuz, Paris, France, joris.mithalal@espe-paris.fr

In this article, we study the issues in making students use deductive geometry, both from epistemological and cognitive points of view. We show that working on geometrical objects needs at the same time working on diagrams in a specific way (instrumental and dimensional deconstruction), and that this can be trained by specific tasks. In this process, instrumental work is a crucial point of the work and, using the cK¢ model, we characterize two instrumental deconstruction that are involved in this evolution of the geometrical work.

Keywords: Dynamic 3D geometry, visualization, geometrical paradigm, dimensional deconstruction, cK¢ model.

INTRODUCTION

The issue of pupils using diagrams in order to solve deductive geometry problems is well known and has been studied a lot in this working group: they can *read* results so that mathematical proof is useless to them. We presented (Mithalal, 2009) preliminary results about didactical situations, based on Cabri 3D (a 3D Dynamic Geometry Environment), that aimed at making students use natural axiomatic geometry (Houdement & Kuzkiak, 2006). We shown (Mithalal, 2009, 2010) that such conditions leaded them to work on geometrical objects, and not only on diagrams. At first sight, making them use deductive reasoning could be considered as the major part of this evolution. In this article we will show that, in fact, instrumental activity should be considered as a pivot point that makes the whole process possible. We use Duval's (2005) framework, and show that instrumental deconstruction is the link between iconic and non-iconic visualization. We use the cK¢ model (Balacheff & Margolinas, 2003) to combine an epistemological and a cognitive understanding of geometrical activity into an operational tool for its precise analysis. We show that two instrumental deconstructions are to be distinguished. On the one hand, there are only small differences between them, but on the other hand moving from the first to the second one is fundamental in this process. At the end, deductive geometry turns to be meaningful because of new conceptions about construction tasks.

USING DIAGRAMS FOR DEDUCTIVE GEOMETRY

Epistemological and cognitive points of view

Using diagrams is very ambiguous for pupils when learning deductive geometry: they are used to reading results on it, but suddenly this turns to be forbidden. Now, they have to use diagrams for heuristic work but not for proving, which is very confusing as they *see* what will take much time to be proved. Nevertheless, many authors, such as Parzysz (1988), Chaachoua (1997) or Jahnke (2007), have shown that that geometry involves a mix of practical activities and axiomatic reasoning, and that "inventing hypothesis and testing their consequences is more productive for the understanding of the epistemological proof than forming elaborate chain of deduction" (Jahnke, 2007, p. 79). Working with diagrams is, then, fundamental.

Laborde and Capponi (1994) made a distinction between drawings and figures: a drawing is a graphical object (hand-drawn shape, diagram, digital representation, manipulative....), while a figure is the matching between an ideal object and a set of drawings that represent it correctly. This emphasizes that a drawing can be considered for itself or as a representation of something, and extends Fischbein's (1993) theory of figural concepts. Considering that our question is now "*how could we make pupils work on figures instead of drawings?*", two kind of issues must be take into account: epistemological and cognitive ones. We already gave a few words about epistemological issues, and we will use the well known framework of geometrical paradigms (Houdement & Kuzniak, 2003, 2006) to express it: pupils have to move from natural geometry (GI) to natural axiomatic geometry (GII) which is about figures.

Cognitive point of view

A cognitive point of view is essential, as it explains why using drawings is unavoidable, and why this is so hard to perform. Chaachoua (1997, pp. 32–42) showed that drawings play fundamental roles in the geometrical activity – which not only includes the resolution process –, both for teachers and learners. In order for the resolution to be correctly performed, drawings must satisfy three conditions: (i) they have to display geometrical properties the text doesn't necessarily mention (ii) they may carry out an illustration function, to illustrate either the problem's statements, the resolution steps or the final solution (iii) they may carry out an experimentation function, so that the "geometer" can work on the drawing, which leads to perceive new sub-figures or relations, to make or evaluate conjectures, etc. Therefore, to be able to work with drawings is fundamental, and teaching geometry without using drawings would be a nonsense.

However, Duval (2005) showed that this operations are very hard to perform, because of visualization issues. He explains that there are two ways in which one sees a drawing: the iconic and the non-iconic visualization. **Iconic Visualization** (I.V.) means that the recognition of an object is due to the similarity of its shape with an already known object: if something looks like a square, it is one of those. A well-known consequence is the obstacle of typical configurations, which leads some pupils to identify a square as a rhombus, but not a square, because its position is the typical position of a rhombus (Figure 1). Another consequence is that you cannot modify or analyse the drawings: adding lines changes the shape, and then the nature of the objects decomposing the object into lines is impossible as nothing but the global shape is perceived. This second consequence, more subtle, is fundamental for deductive geometry which involves experimental work and analytic study of the drawings.

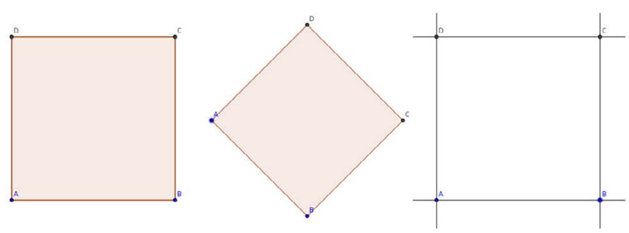


Figure 1: A square, a rhombus and "something from a square" (Iconic visualization)

With **Non-Iconic Visualization** (NI.V.), one considers that a drawing is nothing but one of the representations of a geometrical object, and there is no contradiction between modifying the drawing and considering that it remains a representation of the same object. Moreover, the general shape is no longer a fundamental characteristic of the drawing, which results of assembling lower dimension components, the *figural units* (such as points, lines, segments, circles, etc.) Then, visualization is based on three operations: mereological deconstruction, instrumental deconstruction (one answer to the following question "*How is it possible to construct this drawing with a given set of tools?*") and dimensional deconstruction (a discursive process in which a figure is seen as a set of figural units linked by geometrical properties, which leads to the geometrical object). Mathematical proof becomes meaningful to the pupils when they use non-iconic visualization and dimensional deconstruction. Nevertheless, Duval (2005, pp. 45–48) claimed that going from iconic to non-iconic visualization is neither easy nor natural. We showed (Mithalal, 2009, 2010) with space geometry, iconic visualization is no longer efficient. Construction tasks with 3D DGEs make 10th grade students use dimensional deconstruction. In this article, we assume this evolution is not a *revolution*, and we will study how continuous this process is. More precisely, we will show that instrumental deconstruction can be related both to iconic and non-iconic visualization, depending of tiny differences, and helps the pupils' activity evolution.

GOING FROM ICONIC VISUALIZATION TO DIMENSIONAL DECONSTRUCTION: AN EXPERIMENTAL STUDY

The following case study aims at describing such a continuum and the role of instrumental deconstruction. We will use the results of an experiment more precisely exposed in Mithalal (2010), that consists of a knowledge diagnosis simple construction task.

Let $ABCDEHGF_1F_2F_3$ be a truncated cube (Figure 2), the pupils had to find as many ways as possible to construct the missing vertex, and to verify that the cube was still a cube when dragging point A to modify its size: the construction procedure had to be based,

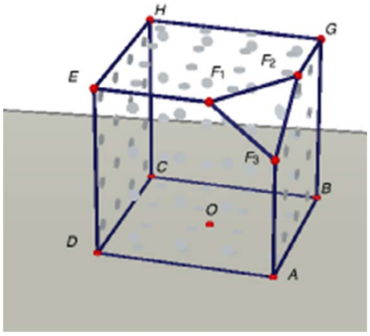


Figure 2: The truncated cube pupils had to complete

explicitly or implicitly, on geometrical properties so that the result was robust (Healy, 1994). They were asked to describe each method in a few words.

Interpreting the question with the various points of view we mentioned led to distinct aims and methods. For instance, with an "iconic" interpretation, the aim was to construct the missing part of the shape without considering geometrical properties, whereas a "non-iconic" interpretation should lead to use the cube properties (such as symmetry) to construct nothing but F. We identified (Mithalal, 2010) four distinct interpretations corresponding to specific resolution strategies: (1) to draw the missing part so that the result looks like a cube (iconic) (2) to draw the missing part so that the result looks like a cube and with the same length as the original (iconic) (3) to draw a point using a construction procedure that fits with a cube (instrumental deconstruction) (4) to construct a point linked to the truncated cube with its geometrical properties (dimensional deconstruction)

Instrumental deconstruction: Connected both to I.V. and to NI.V.

We propose now to illustrate the evolutions of two groups' geometrical work, representative of usual evolutions: on the one hand Pierre and Ludovic, moving from iconic visualization to instrumental deconstruction, on the other hand Paul, Julie and Marie, whose interpretation of the task went from instrumental (3rd strategy) to dimensional deconstruction (4th strategy). Pierre and Ludovic's aim was to construct the shape: "Why do you absolutely want to use 'triangle'?" – "Because it *is* a triangle!" (5'43) They spent about 25 minutes trying to put a point at the right place, but this is almost impossible. They concluded so after multiple trials and viewpoint changes, and decided that they needed new methods:

16'06 Ludovic: Anyway, we can't see anything.

21'57 Pierre: It looked good, but when we move it, it looks like nothing...

26'22 Pierre: We have to find another solution.

Their aim still was that "it looks good", but they realised that the means for it had to be improved. Then, they tried various methods, including intersecting the three truncated edges (two would have been enough), constructing parallels, or trying to use vectors (they failed): they started using instrumental deconstruction. Very often, pupils interpreted robustness properties as mechanical ones, but did not connect it to geometrical properties. This is why Duval (2005) considered there was no link between the two deconstructions. However Cabri 3D's mechanical tools are based on geometry, so even "mechanical" problems make dimensional deconstruction very useful for anticipating the strategies and their validities, and for convincing each other. This happened with Marie, Paul and Julie. They first decided that "reconstructing the cube" was not sufficient and that they had to fill it in: it was a shape problem, and they used similar methods to the former group.

13'19 Paul: Maybe we should try with a point, just a point here, at the right place.

13'24 Marie: But in this case you leave it to chance!

13'27 Paul: No! I mean, yes! Well, you make it...

Their problem was to be sure that the construction was good, and they eventually decided that it was "approximatively good", which was enough for them. This is why they decided to use more sophisticated methods based on dimensional deconstructions of a cube, such as edges intersections or symmetry, but during half an hour the correctness was determined by the shape (iconic) and the robustness (instrumental). But the fact that a new method was *really new* turned to be a dimensional deconstruction problem. They previously constructed F by intersecting 3 edges (Figure 3, left). Then, they tried another solution, constructing the symetrical of the truncated cube, and using it to draw lines (Figure 3, 2nd & 3rd pictures).

25'46 Julie: Look, you construct the same next to it. You'll see, the shape will do schlack, schlack, you see it will be a rhombus. Then, you use from this one to that one. To its symmetrical. Then, you only have to do this... Then you create the line, and you're done.

31'50 Marie: Wait, wait, because I just thought about it: the [second] cube is absolutely useless...

32'14 Paul: Yes, in fact you just did the same!

32'15 Marie: Exactly. That's it, you just added a cube on the two sides.

The result was visually very different, and so was the construction process, but they used the same lines: this statement made them decide this was not a new method, no matter they constructed it differently, because they used *the same dimensional deconstruction of the cube*. This shows how the two deconstructions can be strongly linked: the dimensional one is a good way for designing instrumental deconstruction and for controlling its validity, and reciprocally the instrumental problems make dimensional deconstruction (and deductive geometry) play a greater and greater role.

The fundamental role of Instrumental Deconstruction

The example we mentioned here show the two main reasons for the pupils to move from iconic to non-iconic visualization. First, they needed to act more easily and to make vision more efficient. Then, they needed to better control and anticipate their actions, so that dimensional deconstruction was required to control instrumental processes. This points out that, in fact there are two kinds of instrumental deconstruction. The first one (we call it I.D.$_{iv}$) is linked to iconic visualization problems, and a second one (I.D.$_{niv}$) is a way for dimensional deconstruction to be operative. More than these two processes, moving from I.D.$_{iv}$ to I.D.$_{niv}$ is fundamental because it strongly modifies the way problems are interpreted, becoming more theoretical, which makes deductive geometry meaningful. Our aim is now to confirm this statement. External signs of these two instrumental deconstructions are usually very similar, which make it hard to analyse. Same processes might be used with different purposes, and a similar aim may generate different strategies, depending on the pupil's knowledge, cognitive abilities, and interpretation. To get a more precise description of the deconstructions, we used the cK¢ model that perfectly fits the duality between acting and controlling the actions.

ANALYSING INSTRUMENTAL DECONSTRUCTION WITH CK¢

The cK¢ model

cK¢ is a knowledge model (Balacheff, 2011), linked to the Theory of Conceptual Fields (Vergnaud, 1990) and to the Theory of Didactical Situations (Brousseau, 1997). Mathematical knowledge is characterized by the problem it solves. It is both determined by a subject and the milieu that generates the problem, so this is a [subject<>milieu] system balance, with an action/feedback loop: the subject acts, and the feedback from the milieu has to be good. This is very local, deeply linked to a specific context, and Balacheff (2011) calls it *conception*. Then, *knowledge* is a set of one subject's conceptions, and concept is a more general set of "knowledges". The social, more general, textual construct is called *savoir*.

Conceptions are what pupils work with during problem solving, and the cK¢ model describes it as a collection of 4 sets. **P** is a set of problems (p_i). A problem is basically a disturbance of the system balance, and the conception may solve these problems. **R** is a set of operators (r_i), that turn a problem into another problem belonging to P. It causes action, and this is the most directly visible part of the conception. **L** is a representation system for P and R expression. Σ is a control structure that ensures the conception is coherent and judges whether an operator has to be used

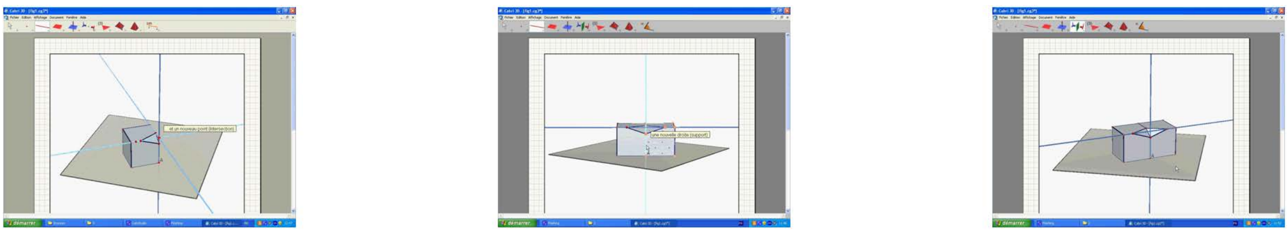


Figure 3: Construction with three lines or with the symmetrical of the cube

or not, whether a problem is solved or not... Σ is a set of controls (σ_i). As we can see, this is a very formal description, and it is essential to be conscious that it does not give any theoretical explanation for what happens, but it methodologically helps describing observations in a way that is supposed to be precise enough. In our case, L is always the same – drawings, oral language – and the possibilities of initial problems of a conception are well identified. The main evolution during the experiment is about R, Σ, and the relation between it.

Some nuances in very similar examples

Let us go back to the former examples. Pierre and Ludovic, Marie, Paul and Julie, used exactly the same construction process: construct the lines from the truncated edges, make the intersection, fill in the cube so that "it looks like a cube". From a mathematical point of view, the third line was useless, so that we consider that all of them tried to draw the original shape. Pierre and Ludovic "[had] to find another solution" (26'22) and reproduced what another group was doing (*We'll do the same as they do!*, 27'25), but the shape was so important that they also constructed the three "missing" triangles. Marie, Paul and Julie tried this strategy at the beginning:

3'11 Paul: (Marie created a line on a truncated edge.) Tell me what you're doing!

3'20 Marie: You perfectly see what I'm doing.

3'22 Paul: Yes, you put a line, but explain us what you want to do.

3'28 Paul: (Marie: created the two other lines.) Ok, and then?

3'30 Marie: And then, you think I'm stupid! Don't you see the cube is done?

Marie knew what she was doing and that the solution was right, because she used the properties of the cube to control, a priori, the construction process. This helps understand why she didn't need visual control to validate the construction, and at the end little differences are very meaningful.

In other words, the two procedures and the results were very similar, but our interpretations of it are different: Marie, Paul & Julie considered that all was about the shape, but they could use geometrical knowledge to control their actions. Pierre & Ludovic failed when geometrical controls were unavoidable. The great difference was about controlling, which leads to the following conclusion: the first group used a I.D._{iv} interpretation, whereas the second one interpreted the task with I.D._{niv}, so that geometrical knowledge and mechanical properties were connected.

Operators and controls

Analysing these examples, we didn't fully describe the students' conceptions. Would it be possible, this paper is too small. We used cK¢ to point out the main aspects of their conceptions, but this is more accurate when studying operators and controls.

An operator is an association between a purpose (I want to do this) and a mean for it ([aim ⇒ action]). For instance, an operator could be "I want to construct a parallel to (d), so I have to use the "parallel tool" of Cabri 3D to select (d) and a point". Nor the aim, neither the action, reflect the kind of geometry used, the interpretation has to be based on their association (⇒). Let us consider another frequently used operator, "I want to construct a parallel to (d), so I have to use the line tool, select two points to create a line (d'), and move one of the points so that (d) and (d') are parallel". The purpose is the same, but the action shows that neither robustness nor geometrical properties are taken into account, and that "are parallel" mean something very perceptual, so that this operator is associated more to iconic than to non-iconic visualization. A control isn't directly linked to an action, it is a judgment on it, and then it is a [Statement1 ⇒ Statement2] system. The first one is a fact, the second one is a conclusion, and here again the association (⇒) mean something about the kind of geometry used.

	Pierre and Ludovic	Marie, Paul and Julie
Interpretation, final control	Shape (iconical)	
Operators	Three lines, 3 triangles	Three lines
Control of actions	Visual-instrumental	geometrical

Table 1: Two interpretations of the problems

One of the greatest issues with defining operators and controls is the not so clear frontier between them. Balacheff (2003) mention that, for instance, "the symmetrical of a segment is another segment" could be either an operator — that indicates how to draw it — or a control — to judge the correctness of a construction. Our formal description helps understanding this: a single property may be linked to an operator or a control, because it mainly justifies the (⇒) link in both cases, but it is a tool in one case — linked to an action — and a fact, an object, in the other one. Operators are easier to catch, and also give information about their visualization. For instance, constructing three lines was useless, so this combination of operators (we call it procedure) indicated an iconic interpretation of the problem.

The two main ways to catch controls are dialog analysis and inferences from the observed procedures: an observation is associated to a procedure, linked to a set of controls and a type of conception. It is all the more important that the differences we shown between the two former instrumental deconstructions were mainly based on some specific controls, and on their role, but not on operators. Indeed, controlling the final validity with some dimensional deconstruction-based control mean something very different than using this deconstruction only as a tool to control the action. This is why we need to take into account the moment when the controls are used.

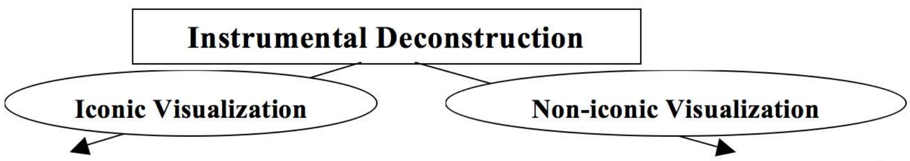
1. Before acting, controls determine the interpretation of the problems, a set of possibly adequate operators, i.e. a selection function (Vadcard, 2000), and a priori reasons for the validity of a strategy.
2. During the action, controls ensure that operators are adequate and help interpreting the feedback.
3. After acting, controls provide a posteriori judgment about the validity of a solution.

(2) is close to a "tool" role for using operators, and then for the kind of geometry they are based on. (1) and (3) are more about how geometry problems are interpreted and about the pupils' knowledge. Of course, using deductive geometry requires that (1) and (3) mainly involve controls based on dimensional deconstruction, but identifying (2) controls is very important. It gives information about the coherence of the conception, about knowledge that are used only for practical reasons, and most of all about the evolution mechanism of the conception.

Characterization of two instrumental deconstructions

With this description, it is now possible to make the difference more clear. The two instrumental deconstructions both consist in considering the objects as the result of a construction process that involves figural units and some specifications about "how it works". But the kind of objects involved, their properties, the tools pupils use and the controls that are needed, can be different. Of course, these are prototypes.

Pierre and Ludovic gave us a very good example of I.D.$_{iv}$ we already analysed. Leelah and Catherine's deconstruction was very different. At the very beginning they tried approximate shape constructions, but they gave up immediately and repeated many times that the shape didn't matter (6'06, Catherine:



	Instrumental deconstruction.$_{iv}$	Instrumental deconstruction.$_{ni.v}$
Properties	Shape, mechanical	Geometrical
control (time 1 and 3)	about shape or mechanical properties	about mechanical or geometrical properties
control (time 2)	visual; knowledge about the tools of Cabri 3D; prototypes of procedures	geometrical properties, knowledge about the tools of Cabri 3D
operators	geometrical properties, knowledge about the tools	no need for visual control

Figure 4: The two instrumental deconstructions

Yes, but no, the point can be anywhere.; 17'44, C.: But you don't need to create the tetrahedrons to get the points, the intersection of the [planes] already made it.; 32'44, C.: No, but the aim is not the numerical stuff, for us...). Contrary to the other groups we mentioned, they anticipated and validated every strategy before the action:

6'06 Leelah: I'd like to put a plane like this, a plane like this, a plane like this. [...] By extending the truncated sides...

22'51 Catherine: Later I thought of a much easier solution! Symmetry with respect to this point: hop, done!

Then, a priori controls based on dimensional deconstruction and deductive geometry allowed these pupils to validate a strategy before action, and to design precisely their procedure by transforming the geometrical information into an operational process. One great consequence is that visual control is no longer required, as Catherine expressed it very clearly: "Middle of this segment... We can't see it, but it's done! " (21'59) Eventually, we can describe the differences this way:

CONCLUSION

Many studies about teaching geometry underline how difficult it is to make pupils use axiomatic geometry and proof, and consider that there is no continuity of the cognitive process (Duval, 2005). We showed here that in particular cases, such as construction tasks with 3D DGEs, a continuous evolution is possible. This is linked to a more general statement: there are in fact two instrumental deconstructions, which are at the same time very different cognitive process, but very close ways of acting. Eventually, characterising the differences needs a very precise tool combining epistemological and cognitive points of view, which was provided by cK¢. We only could underline the similarities between I.D.$_{iv}$ and I.D.$_{niv}$, and the evolution process between it remains to be studied.

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Working with visually impaired students: Strategies developed in the transition from 2D geometrical objects to 3D geometrical objects

Chrysi Papadaki

University of Bremen, Bremen, Germany, chrysi@uni-bremen.de

In this paper, I will present some of the results of research that was carried out, during my master studies, that aimed to examine the strategies that visually impaired students develop while coping with the transition from 2-dimensional (2D) to 3-dimensional (3D) geometrical objects, and also their correlation with the concepts of visualization, haptic perception, gestures and language. A teaching experiment took place in a support unit for visually impaired students. The results showed that the students develop geometrical thinking procedures that are influenced by the concepts of visualization, haptic perception, gestures and language.

Keywords: Geometry, visualization, haptic perception, visual impairment.

INTRODUCTION

The subject of geometry has always been and continues to be a "headache" for most students. For many years researchers in didactics of mathematics, have studied the levels of students' geometrical thinking (Van Hiele, 1986; Gutierrez, Jaime, & Fortuny, 1991), as well as the processes that students follow in geometry (Battista, 2007; Duval, 2011). But if geometry is one of the most demanding subjects for the "mainstream" students, what happens in the case of students with visual impairments? How does a student who doesn't have the luxury of vision, cope with the demands of geometry, which besides mathematical abilities also includes visualization and spatial abilities?

Such questions and concerns led us to address these issues and conduct research in order to observe how visually impaired students interact with the objects of geometry and what strategies they develop while solving geometrical problems that connect 2D with 3D geometry. The basic concepts I will use to examine the strategies developed by students are: visualization, gestures and language that students use in order to describe geometrical objects and also haptic perception. Through these concepts, I will observe the processes of reasoning developed by students during their transition from 2D to 3D geometry, in the context of two tasks.

THEORETICAL FRAMEWORK

Important theories on the evolution of geometrical thinking

Many researchers have studied students' geometrical thinking and here I will present some significant theories of the evolution of geometrical reasoning, as well as some other concepts that relate to it, such as similes. In the field of didactics of mathematics and especially as far as geometry is concerned, much research has been done regarding the levels of students' geometrical thinking. The Dutch mathematician Van Hiele proposed a widely accepted description of the development of students' geometrical thinking (Van Hiele, 1986) through five levels, which show the way students think about figures and other geometrical concepts. Gutierrez, Jaime & Fortuny (1991), extended the Van Hiele levels for the case of three-dimensional shapes. They describe four stages concerning students' thinking levels regarding the geometry of solid objects and their properties. Gutierrez (1992) extended these levels, examining them also from the perspective of visualization.

An important contribution to research concerning students' geometrical thinking is the research work of Duval. Duval (2011) argues that when observing a figure from a mathematical perspective, we should determine its figural units, depending on its dimen-

		Figural Units (Components)	
Shape	Cube (3D shape)	0D/3D	Vertices (zero-dimensional components)
		1D/3D	Edges (one-dimensional components)
		2D/3D	Faces (two-dimensional components)
		3D/3D	Cube (three-dimensional object)
	Square (2D shape)	0D/2D	Vertices (zero-dimensional components)
		1D/2D	Edges (one-dimensional components)
		2D/2D	Square (two-dimensional object)

Table 1: Figular units of cube and square

sions, i.e. its different dimensional components (see examples in Table 1). This procedure can help us see the same shape as a composition of different figural units. In this way we choose the composition that is useful for solving a specific problem.

Similes, haptic perception, gestures and geometry

Visually impaired students often use, phrases like: "I see it" or "I do not see that yet". These phrases are usually verbal expressions of their haptic experience. Another phenomenon in language is the great influence of everyday life on visually impaired students' vocabulary. It is many times that influence that makes it difficult for these students to describe their thoughts in the formal terms of geometry. The results of Potari, Diakogiorgi & Zanni (2003), showed that similes are an important tool for students' access to geometrical concepts, which were inaccessible only with the use of formal terminology.

According to Figueiras and Arcavi (2012), haptic perception provides us with access to spatial details that would not be easy to observe in another way. This is due to the fact that haptic exploration combines the process of visualizing an object and "action", since it actually is the translation of a haptic stimulus to a mental representation, without the existence of a visual stimulus. According to Triantafillidis (1995), when exploring an object by touch, a series of steps is required in order to identify its shape. As a first step, the key characteristics of the shape are identified, though without following any particular strategy. Then a more detailed exploration of the shape begins, which is this time based on some strategies that will reveal in greater detail its characteristics and properties. For students with visual impairments, haptic exploration of shapes or figures plays for them the role of vision. It is the way through which they can "see" the shapes. According to Williams (1983) and Millar (1981), blind students explore an object in such a way that provides them a comprehensive view of it, thus leading them to a more accurate mental representation of its figure. According to Triantafillidis (1995), there is a correlation between the strategies a student chooses to use and the level of his/her geometrical thinking (Van Hiele levels). As a consequence, vision is not the only source of mental images in mathematics. To "feel" some physical objects with our hands, without seeing them with our own eyes, is also a source of a rich production of mental images.

Our hands though, are not a tool useful only for our ability to perceive haptic stimuli. They can also be used as mediations of our thoughts, through the gestures they perform while we argue or think. McNeill (1992) categorizes gestures in Iconic, Metaphoric, Deictic and Beat gestures, depending on the role they had in the talk of the speaker. Radford (2009) argues that thinking is not something that occurs only in the mind, but also through language, body and the tools one has at his disposal. He argues that gestures are operative components of thought. This means that gestures do not only make apparent to us what someone thinks, but they are in fact the ones to generate ideas, the ones that incite thinking.

Visualization and geometry

Gutierrez (1996) describes visualization in mathematics, as the kind of reasoning activity based on the use of visual or spatial elements, mental or physical, in order to solve a problem or prove properties. But what happens in the case of students with visual impairments? If someone cannot see, how can (s)he visualize? Just because someone is blind, it does not mean that (s)he cannot "see". Miller (1987) argues that mathematical visualization also involves intuition through pictures formed in the eyes of the mind. As described by Jackson (2002), visualization may go far beyond the sense of vision, because it implies the kind of understanding that comes from the intuition described by Miller. However, in order to achieve this kind of understanding, we should not isolate visualization from a general mathematical context, but instead connect it with other kinds of reasoning.

Lohman (1988) correlates directly the concept of visualization to that of spatial ability, defining the concept of spatial visualization, which refers to the ability to comprehend imaginary movements in a three dimensional space or the ability to manipulate objects in imagination. According to Kospentaris and colleagues (2011), a student's visualization process can influence the strategies (s)he chooses in order to cope with a geometrical task, as also the way in which (s)he implements these strategies. It is therefore apparent that there is a strong correlation between visualization and the formation of geometrical thinking.

RESEARCH QUESTIONS – METHODOLOGY

My research questions in the study were the following:

1) What kind of strategies and procedures do the students follow during the transition from the 2D to 3D geometry?

2) What is the contribution of gestures, the use of similes, and visualization in the selection and implementation of these procedures?

Participants. The research took place in a support unit for students with visual impairments. Five students participated in the study. The students were chosen according to their age, visual ability and their mental ability (visually impaired students with mental disabilities did not take part in this research as its focus was on students with only visual impairments). The reason why we wanted to have students of different ages and grades is that at different ages, students have gathered different experiences from their everyday lives and different levels of geometry knowledge from school. The research was conducted within the frame of interviews. There was no classroom situation. During the interviews the researcher was alone with one or two of the participants at a time. Students M and A worked together (group 1) as also students D and T (group 2), while student B worked alone.

Tasks. For the purpose of the study, four tasks were given to the students. Here I will present and analyze the results of only two of these tasks.

Procedure – Aim of Task 1 (Plane figure Rotation). The students were asked to think about what kind of solids will be created by the rotation of a plane figure (Rectangle, Right triangle, Disk), around a vertical axis of their choice. They used 2D objects (plastic plane figures) which they could rotate and manipulate freely. The objects were neither attached nor fixed on the desk or anywhere else. The students did not use any 3D objects. The procedure of the task was that the students touch the plane figures, create a rotation, imagine the solid being "created" and finally identify it. Students' prior knowledge regarding rotation was related mainly to what they knew about the earth's rotation around its axis.

Procedure – Aim of Task 2 (Nets of Solids). The students were asked to identify a solids' nets that were given to them. They started with the net of a cube and I asked of them to construct as many different nets as they could, which when folded would create a cube. Later they also did the same for more solids (Triangular pyramid, Square pyramid). My aim was, through the nets of the solids, to see how students handle the properties of the above mentioned solids, as also if they could identify a solid by its net and the properties its net "reveals" at the levels 0D/3D, 1D/3D and 2D/3D.

The nets were consisted of hard plastic parts (squares and triangles) which were detachable. The connec-

STUDENTS	M	A	B	D	T
GENDER	Male	Female	Male	Female	Female
AGE	18	16	19	14	17
GRADE	12th	9th	12th	7th	11th
VISUAL IMPAIRMENT	Low vision	Congenitally blind	Low vision	Congenitally blind	Congenitally blind

Table 2: Participants' profile

tions between the parts of the nets were flexible and could be folded in order to create the 3D objects. In a discussion with the students, which took place before the beginning of the teaching experiment I asked them some questions regarding their theoretical background in geometry. I learned that they all had knowledge of the basic plane figures and solids and their properties, from the school course of geometry. My goal was to "challenge" the students to identify the solids during their creation, either through the rotation of plane figures or by their nets.

The first step of the analysis was the transcription of the recordings of the interviews. At first, the recordings of the interviews' discussions were transcribed. Then some critical events in each group were identified and analyzed based on the concepts explicated in the theoretical framework of this study. The critical events were classified as such based on the following criteria: (a) haptic exploration of the figures and students' haptic perception about figures, (b) students' geometrical thinking and spatial perception regarding the task they performed, the figures these included and their properties, (c) the way in which students seem to visualize the plane figures and the solid objects and their properties through haptic exploration and geometrical thinking, (d) the students' gestures while trying to explain what they were thinking/doing or in their attempt to argue about their answers, (e) the students' reasoning processes in order to accomplish the transition from 2D to 3D geometry.

RESULTS

Task 1 – Plane figure rotation

During the rotation task, some students (A, M, D and T) chose to rotate the figure for a while and then stop, making the rotation a picture in their minds. Student B chose to rotate the figure nonstop, in order to identify the solid that was being created. The common factor in all cases was the haptic exploration of the invisible solid that was being created. All the students touched the plane figure during its rotation, as well as the trace that was imaginably left of it on the desk, forming the basis of the solid. All of the students recognized the solids created, moving from a part to the whole of the solid, that is starting from the bottom and then moving to other parts of the solid. The different ways of rotation (temporary or nonstop), and hence visualization and spatial perception of the plane figure's movement, combined with the haptic exploration of the formed solid, led students in different strategies of spatial visualization of the solid created (Table 3). All the visualization processes and the strategies were identified based on students' verbal and gestural descriptions. During the identification procedure the students were always asked by the researcher to describe loudly their thinking processes and also the

ROTATED PLANE FIGURE	ROTATION AXIS	SOLID	STRATEGIES
Rectangle	One of longer edges of the rectangle	Cylinder	– Identification of the base of the solid. Then identification of the whole solid by its net. (students A and M) – Infinite number of identical rectangles in circular array (one behind the other), with common edge the rotation axis. (student B) – Two horizontal disks, joined together by infinite vertical lines or two horizontal disks with height difference. (student D)
Right Triangle	The longer perpendicular edge of the triangle	Cone	– Infinite number of identical right triangles in circular array, with common edge the rotation axis. (student B) – A "cylinder" that narrows gradually towards the top. Instead of a base, the top of the solid is formed to a vertex. (students A, M, D and T)
Disk	One of the diameters of the disk	Sphere	– Infinite number of identical vertical semicircles in circular array, with common part the vertical diameter. (student B) – Infinite number of horizontal circles of different diameters, one above the other. The circle with the largest diameter is in the "middle", whereas moving above and below it the circles have gradually reduced diameters. (students A and M)

Table 3: Strategies of visualization process of solids

ways in which the images of the 3D objects were created in their minds.

Through these strategies we can distinguish the different ways in which students apprehend and conceptually manipulate the plane figures and solids. For instance, the strategy in which students visualize a solid as a composition/array of infinite number of identical plane figures is perhaps a result of their choice never to stop rotating the figure until they identify the solid that is being generated.

The students argued about the way of rotation and the figures they felt being created during the rotation, based on the properties of both the rotating plane figure and the created solid. For example, student B's decision regarding the rotation axis of the disk was based on his knowledge of the infinite number of its diameters. In the case of the cone, the students justified its creation by the rotation of a right triangle because of the existence of the hypotenuse, which "forces the solid to have a pointed top instead of a face top". By the end of the interview, all the students had successfully identified all the created solids.

While thinking or arguing, students used gestures and similes. The similes concerned familiar objects from their everyday lives, which had the same shape as the geometrical solids investigated. The cone was described as a funnel or the orange cone used in football workouts, the cylinder as a milk can and the sphere as a ball, the earth or a "round" egg. Student B and the students D and T (group 2) described the solids' shape in their own words. The use of gestures was particularly strong in describing the rotation of the plane figure and the creation of the solid. Students used gestures sometimes in order to describe what they said (iconic gestures, McNeill, 1992) and other times to highlight some specific features of the figures, like a vertex or an edge (deictic gestures, McNeill, 1992).

Task 2 – Nets of solids

In this task the procedure of visualization was based to a great degree on students' haptic perception. They touched and explored the units of a net and identified their shape and through tactile contact they created visual images in their minds, which allowed them to imagine the solid that would be created by the folding of the net. Also here students used a lot of gestures. While thinking and visualizing the folding of the nets, their thoughts were followed by iconic and beat gestures (ibid.) described the ways in which they were imagining the folding of the net. These gestures helped them get a better understanding of the final shape of the created solid. The students started identifying haptically the figural units of the nets (faces). The basic strategies the students followed to identify the solids, while manipulating their nets, were:

1) <u>Identification based on the properties of the figural units of the solid</u>

At first the students identified the solids based on their figural units, without following the process of the imaginary or real folding of the nets. Thus, the presence of six squares indicated the existence of a cube, while the presence of triangles in the net indicated the creation of a pyramid. For example, in the case of the nets of the pyramids, student D said something very enlightening about her working process: "The pyramids are the only solids having triangles in their nets". This statement emphasizes the importance of knowing the properties of a geometrical shape, which can help a student create connections between geometries of different dimensions, evolving his/her geometrical thinking not only on the level of just 2D or just 3D geometry, but most importantly between them.

2) <u>Identification by the imaginary folding of the net</u>

In this case the students started identifying the figural units of a net, but still that was not the "guide" in finding the solid that corresponded to the net. In order to identify the solid they started to fold the net mentally. At this point one can recognize the important contribution of the process of spatial visualization in geometry and the decisive nature of a student's ability to visualize geometrical shapes dynamically. Students managed to create, in their minds, dynamic images of the net, which they could manipulate and fold mentally thus resulting in the identification of the solid being formed. This is a complex process that requires the student to be familiar with the properties of the shapes and their manipulation, even in the absence of visual contact with the object. In our case, only student B managed to manipulate mentally and in a dynamic way the object that was given to him. After the identification of the net that was given to group 1 of students A and M (cube), I asked them if a change in the net would result again to the creation of a cube. Student M explained that not all nets with six squares result to the creation of a cube, but only those that do not "leave gaps" somewhere in the solid when being folded.

These two strategies may differ. However, both of them are of great importance since each one shows a different aspect of the transition from 2D to 3D geometry, indicating that not all students perceive the geometrical shapes in the same way.

CONCLUSIONS – DISCUSSION

Through the strategies they followed, the students were able to successfully perform transitions from 2D to 3D geometry. The strategies were formed as a combination of both students' everyday-life experiences with objects, both plane figures and solid shapes, and their knowledge regarding the geometrical properties of plane figures and solids from the school subject of geometry. Their tactile experiences of plane figures and solids both from everyday life and school also made a very important contribution to students' visualization processes and reasoning in the transition from 2D to 3D geometrical objects.

The students identified the geometrical solids created, starting from a part and moving to the whole of the solid. This way the students managed to gradually build an image of the object, which was the result of a synthesis of all its individual parts. This procedure was not limited only to the synthesis of an image of a plane figure or a solid, but also continued to the composition of the one through the manipulation of the other. This process is consistent with Duval's view, which supports the partial identification of a solid by its figural units, but is in contrast to the usual, and more holistic, procedure of solids' identification by sighted students.

In most cases students used gestures, such as those mentioned by McNeill (1992), in order to describe something they could not easily express with words or the formal mathematical terminology. Such was also the role of similes used by students. It was often difficult for students to verbally express their reasoning through formal mathematical terminology. Thus, they used similes from their everyday lives, in order to describe the shape or form of a geometrical object or even some of its properties.

In all tasks the visualization process of the geometrical objects and their manipulation by the students was most apparent. The students created images like the ones described by Presmeg (2006), calling them "mental images". These images were sometimes static while other times students assigned to them a dynamic character, by moving them mentally, creating new images. Thus, the contribution of visualization was particularly important both in manipulating and identifying geometrical objects, as also for the transition of geometrical reasoning between geometries of different dimensions. This bidirectional relationship between visualization and geometrical reasoning can help students develop their geometrical thinking.

In the field of didactics of mathematics, no research has yet been done as to the levels of visual impaired students' geometrical thinking. Thus, and mainly because of some parameters that do not exist in the case of sighted students (e.g. lack of real visual images for solids), we cannot be sure whether we are allowed to classify the levels of the participating students' geometrical thinking, using Gutierrez's levels classification (1992). What we can say though, based on these results, is that the strategies followed by the students, and also the way they chose to implement them and result to their responses, show that the students based their geometrical reasoning on the properties of both the plane figures and solids, as also on the relations between these properties. However, realizing the limitations mentioned above, and also the limitations of our research (e.g. limited time and number of students), we cannot but emphasize the importance of further research in this field of mathematics' education, so that we are able to draw clear conclusions.

Agreeing with Healy's view (2012), that if we manage to understand the similarities and differences in practices of blind students in relation to these of sighted students we will then be able to understand even better the correlation between experience and understanding, we think that it would be extremely helpful, not only for visually impaired students but for all students, that the examples these five students gave to us on how to cope with geometrical objects, find their role in the field of didactics of geometry and not be left unexploited.

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Learning geometry through paper-based experiences

Marta Pytlak

University of Rzeszow, Rzeszow, Poland, mpytlak@ur.edu.pl

Teaching of geometry in the elementary school in Poland is carried out in a very static way. Students learn various geometrical concepts in concrete, typical situations. They rarely meet these concepts in less common situations or not directly related to the geometry. In this paper, I will present a part of activities that have been carried out among pupils in the fourth grade of primary school. During these courses, students had the opportunity to take a look at familiar geometric concepts (square, cube, perpendicular, parallelism) from a new perspective.

Keywords: Geometry, primary school education, spatial cards, teaching geometry, 2D and 3D geometry.

INTRODUCTION

In recent years, there has been a lot of discussion in Poland on changing the education system. The results achieved by Polish students in various tests and exams showed that this topic is a lot of catching up to do. The new curriculum for mathematics has pointed out that it is necessary to develop students' competencies, which are important not only during a math class, but also in everyday life. A lot of space was devoted to geometry, which so far has not been significantly present in teaching, especially at the lower levels of education. And often geometrical problems are ignored or treated unfavorably by teachers at a lower level of education (kindergarten education and I-III grades of primary school). Teaching geometry is confined to familiarize students with the basic geometric shapes, i.e. square, rectangle, triangle and circle. In addition, students learn to measure the length of a segment, draw straight lines and segments which are perpendicular or parallel. At the next level of education geometry appears as a separate branch of mathematics, where the student is expected to have a formal geometrical knowledge. This leap from the first to the second level of education meant that for many students geometry has become as a very difficult subject. An additional difficulty is also the fact that geometry requires a much different approach than other areas of mathematics, e.g., arithmetic or algebra. It cannot be "algorithm and routine" and put in rigid frames. Thus, few questions appear: How to help students in the transition between the levels? How to help students develop their geometric intuition, which will be the basis for their future formal knowledge? Trying to find the answer to these questions

I decided to organize a series of activities with the students, whose goal would be to develop their interest in geometry. The activities were organized in such a way so that the student would recognize geometric issues somehow by accident, while having fun.

THEORETICAL BACKGROUND

Geometry is an integral part of our daily life even if we do not realize it. Geometry teaches the basic skills of logical thinking and reasoning. We can observe that in the latest trends in early education a large emphasis is placed on developing the skills needed for a child to explore and understand the world and to cope with different situations of everyday life. The skills that are particularly useful in various situations include analyzing, critical thinking and putting and verifying hypotheses. The tasks of the school according the new curriculum include the care that a child could acquire the knowledge and skills needed to understand the world and the tools the child needs in math skills in real-life and school situations and for solving the problems.

The most important skills acquired by the student in the course of general education in elementary school should be, inter alia, mathematical thinking, comprehension as the ability to use the basic tools of mathematics in his or her daily live and carrying out elementary mathematical procedures. On the higher stages of education mathematics is presented as structured and ordered formal knowledge, with specified chapters of mathematics. This enables the advanced work both on exploring and developing mathematical knowledge and develop formal (symbolic) mathematical language. At lower levels of education (especially in the lower grades of primary school) students are not presented a finished, formal knowledge from different areas of mathematics, but are introduced them to the world of arithmetic and geometry (Hejný & Jirotková, 2006). The world of arithmetic is ultimately structured and governed by clear rules. Individual records and symbols used in this world are read by all in the same way. The presents the situation in the world of geometry is different, as Hejný and Jirotková (2006) write:

> The world of geometry is a community of individuals or small families and there is a large diversity in the linkages between them. From the didactic point of view, arithmetic is suitable for developing abilities systematically, and geometry is more suitable for abilities such as experimenting, discovering, concept creation, hypothesizing and creating mini-structures. (p. 394)

By analyzing the historical development of geometry we can notice that it was accompanied by human in his activities since the dawn of time (much like arithmetic). Initially, geometry was not a theoretical science, but appeared from the need and desire of people to arrange the space around them, solve many practical problems - from construction by travelling to the ornamentation (Hejný, 1990). However, this geometry was the first "scientific field within mathematics" which was created by human. Its significance for the study of the ancient world was great. It had an important role in mathematics. This historical trait also points the way for didactical approaches to teaching school geometry: geometrical knowledge arises by action. Thus, it is important to gain experience, and practical problem solving.

The importance of geometry in the education of children and young people was the topic of many researchers' considerations. There is a belief that geometry can support the overall development of the child's competence in mathematics. Swoboda (2009) has written the importance of geometry in teaching children and adolescents. She emphasizes that a geometrical approach is closer to a child than an arithmetic one and can open doors to the world of mathematics. It is important that "Geometrical cognition starts from a reflection upon the perceived phenomena and in this way correlates with the basic ways of learning among children" (p. 29). In addition, geometry gives the opportunity to develop the mathematical ways of thinking such as generalization, abstraction, perceiving relations and understanding rules.

Although geometry has a great potential to develop mathematical thinking of students in the school teaching it is not treated with due care. It consists of a number of factors. As Karwowska-Paszkiewicz, Łyko, Mamczur and Swoboda (2001) write, one of them is the very limited number of lessons concerning geometry. It is the reason why during geometric lessons some "ready knowledge" is given and students "did not have the opportunity to learn the properties of the figures through manipulating and even if they had it, it was only apparent" (p. 86). Hence, there is no place in this teaching style for problem solving teaching, and there arise difficulties in the connection between the problem, the procedure of solving it and the solution.

To take advantage of the full capabilities of the geometry in the education of children and youth, you need to change the approach to its teaching. Geometry was born out of the action and of humans needs for development and structuring of space around them. Therefore, an important element in the teaching of geometry should be acting. As we can see in Swoboda (2001):

> Action play an important role in the formation of geometrical concept because there is always correlation between concept and the activity addressed to the concept. The object from the real world are perceived as the gestalt. The way of gathering information is perception, but after that the action with the object leads to the verbal description in their properties. (p.151)

So an important didactical issue is how to organize activity in mathematics lessons in such a way as to encourage students to actually participate in the lesson and to give them a chance to creative thinking and discovering mathematics.

The experience plays an important role in the learning process. Specifically writes about this Hejný in his theory of the General Model (Hejný, 2001). Children create their own knowledge primarily based on the experience they have. For a description of these experiences they use language - as it is close to them. The closest experience for a child is the language of gestures. It is the first language, how a child learns. This language is very helpful during learning of mathematics. Just as Cook and Goldin-Meadow (2006) say: "children who produce gestures modeled by the teacher during a lesson are more likely to profit from the lesson than children who do not produce the gestures". Only at a later stage of learning a child meets the formal language of mathematics. It is important that the experience and action in the acquisition of mathematical knowledge appears first, followed by language, and only at the end there is formal knowledge. According to Burton (2009), the human activity is the first and after that mathematics arise. And mathematical language (mathematical concepts, objects and relationship) arise through natural language, and within particular socio-cultural environments, in response to human thinking about quantity, relationship and space.

METHODOLOGY

Data for this paper were collected during classes with third grade students from primary school. It was a series of meetings, whose main purpose was to develop the students' interest and talents of mathematics. Classes took place once a week and last one school hour (45 minutes). Twenty students from the third grade of primary school (10–11 years old) took part in these meetings. They were students, who coped pretty well with school mathematics. They had no major problems with the mastery of the material carried in the classroom. They were students willing to undertake new challenges. During these meetings the students through fun developed their mathematical abilities. Much of the class was devoted to geometry. Students through play, by using paper and scissors, learn about various properties of figures in the plane and in space. The purpose of these meetings was to develop spatial imagination and the ability to perceive relationships and analogies between objects on the plane and space. Also worked on the development of students' mathematical language, and, in particular, to bring students the terms such as parallelism and perpendicularity.

During each class, students had access to colored cardboard, scissors, duct tape, crayons, markers. Classes were recorded with a video camera. After each meeting a protocol was prepared. The research materials consist of the works done by students, videos and chat records.

In this paper, I would like to present the classes concerning geometry called "play with geometry". The main purpose of the course was to develop spatial imagination and students' interest in geometry. In addition, I wanted to examine:

- How will students cope with the creation of three-dimensional models?
- How will they move between dimensions?
- Will they be able to notice the parallel and perpendicular elements in three-dimensional models?
- Will they be able to describe the work made by them in the mathematical language?

Activities under the name of "play with geometry" are divided into two parts. During the first of them students prepared spatial cards from pre-made templates. They cut out, filed and attached elements so that after unfolded their cards they received spatial composition. These classes were analyzed together, as making certain cuts or submit reflected in the final work. Students have tried to capture all the relationships that exist between objects.

After working with ready-made templates it was the time for students own creativity. They were given colored cards and scissors. Their task was to create any space card. Both themes of work and how to perform it completely belonged to students.

This task apparently had little to do with geometry. My aim was to show students that in such seemingly non-mathematics activities mathematical – geometrical ideas can be found. This kind of geometry we can find in our daily life.

Students began working on the worksheet in half. Then they began to mutilate the card. At first their work was very spontaneous and not targeted. After some time, students began to notice that the method of cutting and bending the paper affects how their work. So they began to think over each of the next move, already analyzed the resulting image and then decide on the next move. Very often the first work done was treated as a “training card”. After its execution students looked at it, analyzed it, in order to say what they managed to achieve. Only then they started working on the “final card”. When all the students had finished their work, everyone had the opportunity to present what he did. Also a discussion was conducted, in which all the students participated. During this discussion, the teacher drew attention to the manner of carrying out the cuts by the students. There were such concepts as “parallel”, “perpendicular”. Attention was also drawn to the properties of figures that appeared in the works.

ANALYSIS OF STUDENTS’ WORK

The work of Maks

Maks, a student of average ability, he made a very simple job. In an interview with the teacher noted that his foundation was to cut the “three squares”. Initially, however, he did not quite know what it really means.

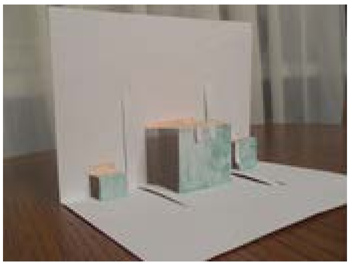

Figure 1: The card made by Maks

Teacher: Maks did the card and he colored it. Please, tell us about your work

Maks: I cut three squares.
Teacher: What does it mean?
Maks: I do not know.
Teacher: [goes to the Maks’ bench, takes his work consists in half and shows the class] Maks cut three squares. But how did you do it?
Maks: I fold it and cut here, here, here, here, here and here. [he shows a pair of scissors, how to carry out cutting]
Teacher: Ok, and what is next?
Maks: I opened it and pull out [he shows the way of making the card]. I watched how it looks and coloured.
Teacher: Ok.. And what were these cuts that created a square?
Maks: [silence]

No one in the class could answer the question posed by the teacher. Everyone knew how to make a “square”, most of the students had it in their work. They could not, however, describe how to construct this object using mathematical language. To help students teacher presented the experience. The teacher took a piece of paper and scissors and made a few cuts. Only then the students began to pay attention to the geometrical relationships.

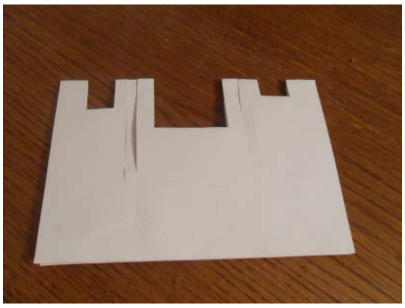

Figure 2: Squares in Maks’s work

Teacher: If cutting can be free? Well, look: I take the card, fold it in half and I will cut it. Here and here [holds the card in two places]. Maks said, that for every square he made two cuts. So I have two cuts and I put the cut piece to the center of my cards [he shows]. Look, did I leave the square?
Students: No.
Teacher: Look, if I want to make a square, the same as Maks did, how should I cut.
Student 1: Well... like a square.
Teacher: And what does it mean „like a square”?
Student 1: Well ... squarely.
Teacher: And what is characterized by square?
Student 2: Because it has all equal sides
Teacher: Well, what else can we say about the square?
Student 3: That is edgy.
Teacher: That is edgy?
Student 2: Because it has all sides perpendicular and the right angle.
Student 1: And it has four sides
Teacher: Very well. So now let’s get it all together: a square has four sides. You said that it has right angle.
Student 2: All sides are equal.

Teacher: Yes, all sides are the same length. And what else can we say about the square?
Maks: [he is looking at his work and pointing to the opposite sides of the cut square] And these are parallel.
Teacher: Great Maks. So what had to be done?
Maks: You have to make two parallel cuts of the same length.
Student 2: Why had not I thought of it?

Square is the first figure, faced by students. It is well known to them, everyone can indicate a square and describe its properties. However, during their work students forgot about the "mathematical" side of a square. In everyday language the square "is such a squared (edgy) thing, which has equal sides". Such a description is sufficient for students at some stage. And this one students applied during the class. The parallelism of the opposite sides and the perpendicularity of the adjacent sides was a secondary issue for them. Both concepts have been widely discussed during the lessons of mathematics. Students had drawn perpendicular and parallel segments. They even marked on the square (using colored pencils) which the sides are parallel and perpendicular.. However, these situations were "pure mathematical". Students know that now they are talking about perpendicularity and parallelism. During the course of creating the spatial cards nobody pointed them that "this is math". They initially did not see the mathematical context in what they were doing. It was for them just a normal execution of simple art work. Only a common conversation with the teacher allowed students to call in an appropriate way what they have made. Only then they began to use mathematical language in their descriptions. These experiences have allowed students to realize new opportunities for creating future cards.

Filip's work

Filip first created a "test card". He worked quickly, did not pay attention to accuracy. When the card was ready, he looked at it, analyzed the arrangement of individual elements, and then created a new card. This time, however, he proceeded very carefully. The effect of his work is presented in picture 3.

Figure 3: Filip's work

The boy was also aware of how it should carry out further cuts to achieve the desired end result. The following conversation with the teacher in an evidence of this:

Teacher: Great job, Filip. Tell us, how did you make it?
Filip: I cut strips
Teacher: And how did you cut these strips?
Filip: Getting shorter. Once a thinner, once a wider.
Teacher: Ok.. And on what more do you pay attention while preparing the work?
Filip: [silence]
Teacher: Well, how do these cuts take place?
Filip: Well ... straight
Teacher: Straight? What does it mean straight, Filip?
Filip: [long break, he looks at his work] Perpendicular to this line [he puts on the edge of the sheet] (...)
Teacher: And how are these cuts are related to each other?
Filip: Parallel.

The boy knew the concept of perpendicularity and parallelism. He could also use in practice the relationship of the perpendicularity and parallelism of objects. However, initially he could not name these concepts. He used the terms "straight, equally". These are the terms taken from everyday life, from everyday language. During creating the cards, it was enough for him. Just in describing what he did, he had to start using mathematical language. Students initially treated the task posed to them as a typical manual-plastic one: I have to make a spatial card. In their work they were rather to the directed to the visual aspect and did not refer to geometrical knowledge (although classes were held in the framework of mathematics lessons). A common discussion on the work made by students made them realize that what they were doing was hidden geometry.

Students were very surprised about that. Often appeared the statements: "oh, here is also a mathematics" or "Can we use the geometry here?" Therefore, in their further work they tried to use knowledge from geometry to create new spatial cards.

CONCLUSION

For 9–10 years old students the classes were in a new form. During the classes the students turned out to be open to new challenges. They approached the problem creatively. The care taken on the details of the created cards can prove the huge interest in the presented subject.

Although the concept of parallelism and perpendicularity were well known for pupils, they had problems with the indication of perpendicular or parallel objects. It seems that there are two different aspects: knowing concept definition and using it in practice. An additional difficulty was also a movement between two planes: the students during cutting worked on a flat plane, and the effect of their work was viewed in three dimensional space. When students were cutting out a "square" they received as a result a "cube".

When discussing the concepts of parallelism and perpendicularity during math class the focus was on two-dimensional geometry. On a flat sheet of paper the students were shown perpendicular and parallel segments and by using a ruler they drew perpendicular and parallel lines. Meanwhile, during the course presented here they "cut out" perpendicular and parallel lines which was a new experience for them.

Hejný (2004) in his theory of the development of student's mathematical knowledge writes that it is very important in the learning process to gain experiences, which are the basis for the creation of formal mathematical knowledge. The more different experiences, the better assimilation of knowledge. To the knowledge of students which is stable, flexible and operational, you need to provide them with as much variety of experiences related to a given concept, recorded in the different planes.

For primary school students the issues connected with geometry are not easy. The classes presented here were an attempt to find a way to present these issues in such a way that students could experience, see. By following the view that it is best to learn by experience, I tried to organize the activities in such a way that students can manipulate the materials given to them and solve the task by themselves. The open question is whether this is the right way to introduce the student to the world of three-dimensional geometry. In my opinion – yes, it is good direction. In my further work I would like to focus on the development of a geometrical environment which will be on the one hand appropriate and conductive for teaching and on the second hand - student-friendly. What should be this geometrical environment? What tools should be used in this environment? What kind of tasks and problems help to teaching and learning geometry?

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Six year old pupils' intuitive knowledge about triangles

Maria Paula Rodrigues[1] *and Maria de Lurdes Serrazina*[2]

1 Universidade de Lisboa, Unidade de Investigação do Instituto de Educação, Lisboa, Portugal, mariapaular@campus.ul.pt

2 Universidade de Lisboa, Escola Superior de Educação de Lisboa, UIDEF, Lisboa, Portugal, lurdess@eselx.ipl.pt

This communication discusses the intuitive knowledge that six years old pupils reveal about triangles. Data is collected through a class group discussion, which was videotaped and transcribed. The results show that pupils articulate visual prototypes with known attributes to recognize triangles. Furthermore they identify nonexamples and its features to recognize this shape. Position didn't seem to be a problema to identify triangles, however, some features, as curved lines or topological properties, appear to be a problema on triangles recognition. Pupils used mainly a partitive classification type but, when they observed and discussed known attributes or properties, they can also use hierarchical classification type.

Keywords: Shapes, properties, classification.

INTRODUCTION

The research presented on this paper reports six years old pupils' intuitive knowledge about triangles when solving tasks, which were based on those presented in (Clements et al., 1999).

This is an exploratory study whose main goal is to get some clues about the intuitive knowledge that Portuguese six years old pupils have about shapes. It is a part of a larger PhD study of the first author, named *Shapes classification: a teaching experience in the early years*. The work reported on this paper intends to discuss what kind of intuitive knowledge Portuguese six years old pupils reveal about triangles.

All participating pupils are six years old and belong to a 1st grade class. The data collected addresses to intuitive knowledge about circles, squares, triangles and rectangles but, in this paper, we only focus on triangles.

CONCEPTUAL FRAMEWORK

When we presented a set of figures in order to identify the shapes which belong it is possible for pupils to reveal knowledge that is related to intuitive character thought, based on visual prototypes, without considering attributes or properties of those forms (Clements et al., 1999). This knowledge can be related to previous experiences and promotes different levels of development (Burger & Shaughnessy, 1986). Also, visual representations, impressions and experiences make up the initial concept image (Vinner & Hershkowitz, 1980).

Intuitive character thought can be related to major theories of concept formation: classical view and the prototypical view (Klausmeier & Sipple 1980; Smith, Shoben, & Rips 1974; Smith & Medin, 1981). According to the classical view, categories are represented by a set of defining features which are shared by all examples. The prototypical view proposes the existence of ideal examples, called prototypes, which are often acquired first and serve as a basis for comparison when categorizing additional examples and nonexamples (Attneave, 1957; Posner & Keele, 1968; Reed, 1972; Rosch, 1973).

Initially, the mental construct of a concept includes mostly visual images based on perceptual similarities of examples, also known as characteristic features (Smith et al., 1974). This initial discrimination may lead to only partial concept acquisition. Later on, examples serve as a basis for both perceptible and nonperceptible attributes, ultimately leading to a concept based on its defining features (Tsamir, Tirosh, & Levenson, 2008). Following this idea, some pupils may recognize shapes supported on the recognition of properties of those shapes and others will articulate visual prototypes with known attributes or properties to identify the same shapes (Clements et al., 1999).

Non-prototypical examples, are often regarded as nonexamples (Hershkowitz, 1989; Schwarz & Hershkowitz, 1999; Wilson, 1990) and specifically, "nonexamples serve to clarify boundaries" of a concept (Bills et al., 2006, p. 127). On the other hand, Fisher (1965) admits that topological properties, mental structures that enable shapes abstraction, such as the configuration or appearance, cannot leave some pupils arrive to the identification of a particular shape, because they can't consider specific properties of that shape.

Regarding to classification, Clements and Sarama (2007) mentioned that pupils tend to a partitive type of classification, where various subsets of concepts are disconnected from each other, into opposition to a hierarchy classification, where the most particular concepts integrates the general ones (de Villiers, 1994).

Tsamir and colleagues (2008) claim that concepts often serve as a means by which people may categorize different things, deciding whether or not something belongs to this class. In other words, one of the functions of a concept is to enable a person to identify both examples and nonexamples of the category.

At a partitive classification process pupils can identify the shapes name without ever having been the opportunity to reflect on their names, attributes or properties and only a small part will be able to provide nonexamples (de Villiers, 1994).

In front of a different figures sets, where the goal is to identify circles; squares; triangles and rectangles, Clements and colleagues (1999) and Sandhofer and Smith (1999) claim that, in order of difficulty, children identify the circle; square; rectangle and triangle.

METHODOLOGY

The exploratory study reported in this paper follows a qualitative interpretative approach (Denzin & Lincoln, 1989). Participants were all six years old and belong to the same 1st grade class, constituted by 21 pupils, of an elementary private school near Sintra, a small village of Lisbon area. All of them attended the kindergarten at the same school and belong to a socio-economical high level. All participants had had informal contact with different shapes during last school year and that fact influenced their intuitive knowledge of shapes.

The study started with four clinical interviews, carried out by the first author to four pupils to test a reworking of tasks used by Clements and colleagues (1999), whose goal was to identify all circles, squares, triangles and rectangles of figures sets. So, the task where children must identify triangles followed two different steps: on a first approach four pupils, two boys and two girls, in different four days, were taken out of the classroom and solved the tasks, as we want to experiment the task with pupils before taking it into classroom. Furthermore, the first four clinical interviews intended, on the one hand, identify the first knowledge about triangles of these four children, and, on the other hand, the given answers served as a starting point to all group discussion. The intention was to create a kind of game where pupils not interviewed should guess whose triangles had been chosen by their colleagues. This game was a motivation for a collective selection of triangles and a discussion about it. These interviews were videotaped. After this first step, the researcher took the task for the classroom and promoted a whole class discussion session, which was also videotaped. The researcher played a kind of game where pupils who did not participate in individual interviews tried to guess which shapes were chosen by the interviewed pupils. The data presented in this paper focus on this whole group discussion.

So, data collection was through a group discussion, by videotaping, where pupils could discuss different ideas, complete or disagree with arguments, creating new reasoning and clarify concepts. This group discussion intended to lead pupils to the possibility of inclusive classification through the construction of shape families that display equal or similar attributes or properties.

The work consisted on a chosen triangles task using a manipulative set of figures, placed in the same position as the one presented by Clements and colleagues (1999), each figure printed on a separate card allowed pupils compare, rotate, overlap, among others. On this set of manipulative figures, pupils had to choose all triangles, justifying their choices. During this stage we intended to understand what kind of knowledge pupils used to recognize triangles: visual prototypes; shape attributes or properties; nonexamples; among others. Besides, their justifications and new questions, related with their answers, we tried to forward them to the construction of shape families to look out to hierarchical classification.

The data were analyzed regarding to the pupils answers, based on visual prototypes, without considering attributes or properties of that forms (Clements et al., 1999), and when they claim that visual representations, impressions and experiences make up the initial concept image (Vinner & Hershkowitz, 1980).

Another aspect we considered on the analysis of data refers to a partitive type of classification influenced by Clements and Sarama (2007) and de Villiers (1994), which mentioned that pupils tend to a partitive type of classification, where various subsets of concepts are disconnected from each other, into opposition to a hierarchical classification, where the most particular concepts integrates the general ones.

Finally, we had in consideration that, probably, triangles, according to Clements and colleagues (1999) and Sandhofer and Smith (1999), will be a difficult shape to identify.

With this kind of methodology we wanted to get some clues about intuitive knowledge of six years old pupils about triangles; what kind of language they use to express this knowledge; understand how large group discussions can lead pupils to a better knowledge of triangles and their properties and, finally, understand if large group discussions facilitate the idea of hierarchical classification.

To preserve the identity of all pupils, on the analysis we used fictitious names initiated by the same capital letter of their real names.

PUPILS' INTUITIVE KNOWLEDGE ABOUT TRIANGLES

As we wrote before, the task where children must identify triangles followed two different steps: the first one we carried out individual interviews with four pupils out of their classroom. With the data collected on these interviews a group discussion session was organized. The researcher played a kind of game with the pupils who had not participated in individual interviews. They should try to guess which were the shapes chosen by the interviewed children. To start the researcher asked to all group which were the triangles they thought had been chosen by their colleagues.

Researcher: In front of these set of shapes, António; Geraldo; Marta and Mara should choose only triangles. Which ones do you think they chose?

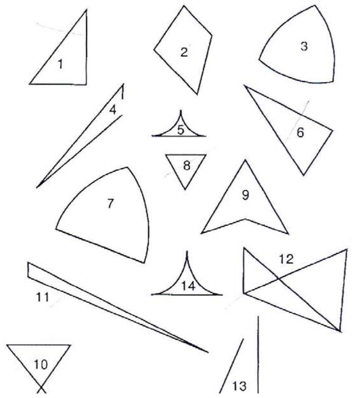

Figure 1: Set of shapes to identify triangles (Burger & Shaughnessy, 1986; Clements & Battista, 1992a)

Augusto: I think they didn't chose number 2. Because it has 4 sides and triangles only have 3 sides.

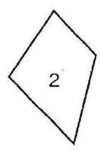

Figure 2: A nonexample of triangle given by Augusto

Some pupils, as Augusto, could recognize triangles identifying a nonexample of them, nominating attributes as the number of sides. Here he seemed regarded non-prototypical examples as nonexamples.

Joel: Picture 2 has 4 sides and 4 angles.
Researcher: So it can belongs to which family?
All group: Rhombus...
Augusto: ... and squares.
Researcher: Why?
Maria Manuela: Because it has 4 sides and 4 angles.
Researcher: So, square and rhombus can be part of which family?
Geraldo: Rectangles.

The group discussions conducted children to observe some attributes or properties and this observation, sometimes, led to comparisons between figures and their properties and offered a larger possibility to identify common attributes or properties that can produce a primary idea of hierarchical classification. However, last answer needs a different kind of work because all mentioned shapes don't have equal attributes or properties and can't be part of same family.

Researcher: What about you Geraldo... which triangles do you think your colleagues had chosen?
Geraldo: Pictures 1, 3, 4, 6, 7, 8, 11 e 13.

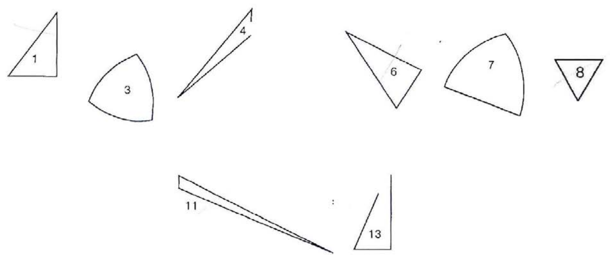

Figure 3: Triangles identified by Geraldo

Analyzing Geraldos's choices we can understand that when he had chosen figures number 3; 4; 7 and 13, his choices were based on topological properties, such configuration or appearance, without consider specific properties of triangles.

However, when he chose figures 6 and 8 he seemed recognize triangles articulating visual prototypes and known attributes. Already, when he identified figure 11 it is possible to say that he had identified some attributes and properties of triangles because the triangle identified by number 11 was a long and narrow scalene triangle which isn't a common representation of triangle represented on an unusual position.

Researcher: Augusto said that picture 10 has 6 sides but, António, you had identified it as a triangle. What do you think now, António?
António: It's a non triangle because it has 6 sides.

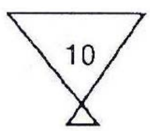

Figure 4: Figure identified as a triangle by António

When children were in group discussion they were encouraged to discuss shapes, attributes or properties, they could compare and integrate new concepts as the one showed on transcription "non triangle", which meant that these group discussions might become richer in terms of concepts. Moreover, the group discussions allowed all pupils participation, sharing, abstraction and reflection about triangles attributes or properties.

Researcher: António, you said that picture 14 is a triangle, but Geraldo, Mara and Marta said that is a non triangle. Who is right?
António: I think I was wrong..
Researcher: Why?
António: Because, now I can observe it has curved lines .
Researcher: So, do you think triangles shouldn't have curved lines?
[...]
Marta: Triangles only could have straight lines because they have 3 angles.
Lívia: I agree with Marta, triangles have angles because it doesn't have curved lines.
Researcher: So, picture 14 is a triangle or, like Marta and Geraldo said, a non triangle?
All group: A non triangle.

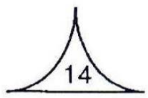

Figure 5: Figure identified as a triangle by António

Once more, group discussion seems to be a place where pupils can reflect about appearance, prototypes, attributes and properties and clarify concepts.

Researcher: António, now, you said that picture 14 is a non triangle because it has curved lines and because of that they can't have any angles.
Can you identify other pictures from the same family?
Mara: Pictures 3 and 7 are not triangles because they don't have any angles.

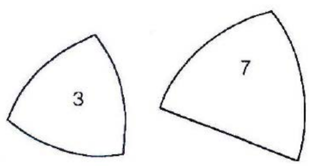

Figure 6: Figures identified as "zero" angles family

Lívia: I think they all can belong to "zero" angles family.

When pupils establish relations between different reasoning and identify necessary shape conditions, inclusive classification may emerge.

Researcher: Marco said that António, Geraldo, Marta and Mara probably had chosen picture 13.
Do you all agree?
Lívia: I didn't because it has open lines. All triangles should have 3 sides, all close. I think picture 13 is a non triangle.

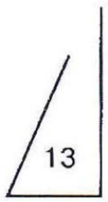

Figure 7: Figure identified as a triangle by Geraldo

During all discussion, Lívia seems to understand what are the necessary conditions to be a triangle and also a necessary condition to be a two dimension shape, when she considered the attribute "close lines".

Researcher: After all this discussion, who wants to tell me what pictures we are sure that represent triangles on this set?
Large group of pupils: Pictures 1; 6; 8 and 11.

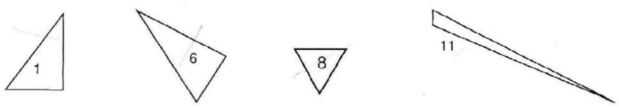

Figure 8: All triangles considered by a large group of pupils at the end of discussion

At the end of group discussion a very large number of pupils seemed to have clarified the concept of triangle and a strong concept image of this shape, while they were observing features of discussed figures and were articulating attributes and properties of triangles.

DISCUSSION

In this exploratory study the pupils articulated visual prototypes with known attributes, for instance, when they identify examples and nonexamples of triangles and were able to justify it, which is according to the results obtained by Clements and colleagues (1999).

In line with Rosch (1973) and other authors, few pupils used nonexamples to identify triangles and justify their choices. But in this study nonexamples served to clarify boundaries of a concept (Bills et al., 2006).

A group of pupils seems to have some difficulties to identify triangles because of some topological properties, as appearance or configuration that prevent the recognition of triangle specific properties. These results are according to those mentioned by Fisher (1995).

Contrary to what would be expected, some pupils recognized triangles independently of the figures position, which may be related with their previous experiences. Burger and Shaughnessy (1986) reveal that experiences and informal contacts with different shapes are very important factors at children intuitive knowledge of shapes.

Finally, we think it's possible to say that when pupils of 1st grade were encouraged to discuss shapes attributes or properties, in group discussions, they can observe differences and similarities and hierarchical classification could emerge (de Villiers, 1994). Furthermore, they seemed to be able to clarify concepts and construct new ones (Tsamir et al., 2008).

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Mathematization of rotation as a didactic task

Ewa Swoboda

University of Rzeszów, Rzeszów, Poland, eswoboda@ur.edu.pl

The inspiration for observation was an idea of getting movement to a static school geometry. In the school curriculum, the only area where you can consciously refer to the movement are questions related to isometrics. I took the premise that the mathematization of movement should be based on student perceptions related to the physical movement of objects. Conducted observation shows that the mathematization of rotation might not be so obvious.

Keywords: Dynamic geometric reasoning, mathematization of physical movement, rotation.

RATIONALE

The process of solving geometrical problems needs an idea of the action. Almost every geometric problem requires focusing attention on certain compounds for their mental transformation. Hence, thinking in geometry requires mental skills to manipulate objects or its fragments. Such approach is visible also in many places of Euclid 'Elements'. Despite the formal expulsion of the movement from Greek geometry, reasoning were based, without doubt, on the imagination of making movement.

In Duval (1998, p. 38), "geometry involves three kinds of cognitive processes which fulfil specific epistemological functions: *visualisation*, (...) *construction*, (...) *reasoning* (...)". I understand this statement very widely: cognitive processes are elements of thought processes, and constructions act as *actions*.

Theories about the formation of geometrical concepts emphasize that the origins of geometrical knowledge proceed without words and rely mainly on visual perception of static phenomena (van Hiele, 1986; Hejný, 2001). Therefore, the problem of bringing students to the ability of making mental transformations I treat as an educational task. In the literature, there is no explicit opinion on what educational level there is possible to create such skills.

Today, while solving geometric problems in school, special computer programs that allow to receive dynamic images are often used (Ferrara & Mammana, 2013). Without doubt, they support not only visualisation, but also construction and reasoning. They facilitate experimentation – students can check certain assumptions, verify the hypothesis. Imagination at the screen shows what happens to the object when changing certain parameters – students can watch the following phases of these changes. Sometimes the student decides what action to perform (like dragging a specific point to make deformation of the figure). In the other situation the whole actions are defined: symmetrical reflection with respect to a particular line, parallel movement to another location, turning about a specific angle relatively to a fixed point. Student sees the overall effect of the change, without going into the mechanism beneath this. These two types of action are different, because they give student the opportunity to experiment in other ways, either through the use of ready-made tools without going into their mode of action, or through planning action based on a conscious determination of conditions governing the changes.

These tools were absent at the time when the geometry was created as a scientific discipline. Mathematicians were able to imagine some of the compounds without seeing them, predict the outcome of actions performed only mentally. How did they develop this ability in them? Can we develop the same in the traditional way of teaching? And maybe today only dynamic computer programs give the opportunity for the creation of dynamic representations of geometric objects?

For me, the starting point for using action in *geometrical thinking* is the problem of turning attention on movement as such. In the Polish school curriculum, the only area where we can consciously refer to the movement in geometry are topics related to isometrics. Isometries are most often introduced as a static relation figure to figure. Names of geometric transformations suggest that isometric transformations are the result of interiorization of rigid movements that take place in a physical space. This trend can be treated as didactic suggestion how to teach isometries at school. Mathematical concepts undoubtedly have a relationship with reality as the result of mathematization. But preparing students to the mathematization of movement raises many questions. Here there are some of them:

— How does the mathematization is done by a young student – shouldn't we attempt to correct some of the dependencies that a child draws attention to?

— Maybe some properties are so deeply rooted in intuitions that the child does not even recognize that they operate? So what is the role of the teacher - what behaviours should he be sensitive about in order not to impose a ready-made mathematical formulas to students, but develop spontaneous knowledge and skills; what to talk about, what to indicate, what to question?

— Don't we commit abuses assuming that the mathematization of movements is based on the experiences gained in physical experiments? A mathematician and a physicist look otherwise on the physical movement of the object, regardless of the fact that both base on their own experiences in physical reality. Both must, therefore, focus attention on different observed phenomena. Which of them are important for making mathematization?

My previous research suggests that students frequently use a rotary movement in solving geometrical problems (Swoboda, 2013), are even are able to bring the configuration of the figures which can be described in the language of mathematical transformations (Jagoda & Swoboda, 2011). The results of these observations, however, do not provide answers on how close the student is able to get in order to underline terms which define such a movement.

Marchini & Vighi (2011), describing their own research aimed on building intuition of geometric transformations among young students, they relates their observations to the Duval theory on different perception of geometric phenomena (Duval, 2006). They emphasize that the mathematical representations of isometrics can be read in two different registers. Mathematical concepts such as parallel shift, rotation, axial symmetry, can be defined both as the static relationships between objects or as transformations. In the light of the Duval theory, this known mathematical fact takes a new didactic interpretation. Not only that the position of the object to the object can function as a relation or as a transformation - perhaps the mental abilities of students mean that some will opt more towards static perception, and others - in a static image can easily see a movement. In Marchini and Vighi's experiment, children spontaneously were able to give the dynamism to the static arrangement of the two objects. In addition, students have used the words from the everyday language which can be treated as the equivalent of parallel displacement, axial symmetry, rotation. To what extent is this interpretation possible for another group of children? How do these results bring us closer to the process of movement's mathematization?

METHODOLOGY

The described research was conducted among 12–13 year old students in two schools in Rzeszów (Poland), in May 2014. So far their geometrical knowledge was associated mainly with geometrical figures. They also had many opportunities to deal with axial symmetry treated as a static relation, although for them there was the only isometric transformation they knew. On the other hand, it is worth mentioning that these students should not use computer animation in the process of learning geometry. Students worked in pairs. They were those students who cope with mathematics without problems and were open in their cooperation with others. The teacher leading the conversation with the students attended the research session as well [1]. At this stage, our goal was to observe and analyze the spontaneous behavior of the students working in the learning environment. That could be the basis for the understanding of isometric transformations on the plane.

Analysis of students' actions should lead to:

— identification of spontaneous student's behavior related to the representation of movement on the plane,

- capture those elements that may be included in the mathematical image of isometrics, and those that should be ousted during the development of mathematical concepts,

- diagnose how the proposed activities can be used as a starting point for talking with students about movements in geometry.

In this article we analyze the work of two students - Eliza and Gabriela. They had to create an animation of a moving triangle. Schoolgirls had a small book (size 10cm × 10cm) with 30 clean sheets for their disposal, to create animations. For facilitate drawing, they got a template of acute-angled triangle. They also had basic geometrical instruments (ruler, set square, compass, protractor) on the table. Girls could draw with a pencil or marker.

Purpose of the meeting was presented to the girls by their colleagues who had previously participated in a similar session. Using ready-made booklet, they showed to the girls how you can get the effect of moving the wings of a butterfly through a quick thumb through books.

A teacher, participating in the study, formulated a task:

Teacher: We'd like you draw a triangle in your booklet and make animations that it changed its position.

A formulation of the request does not dictate the way in which the figure is to be moved. The entire session was filmed. Film and prepared transcripts became the basis for the study.

RESULTS OF OBSERVATION

Determine the type of movement

The girls started their work by determining what kind of movement they will animate. They moved the triangle across the table, alternately offering various moves. All proposals were close to the rotation, and the whole course of this part of the meeting led to clarification how the movement will run. Girls were getting closer and closer to the idea of rotation around a clearly defined point - one of the vertices of the triangle. The first proposal (Figure 1a) – is a circular movement around the closed wheel, with the comment: something like this; the second (Figure 1b) – parallel motion showing by hand on the counter, with the arc trajectory, the third (Figure 1c) – the rotation of the triangle with one fixed point placed rather somewhere inside of the triangle, the fourth (Figure 1d) – rotation of the triangle with respect to one of the vertices.

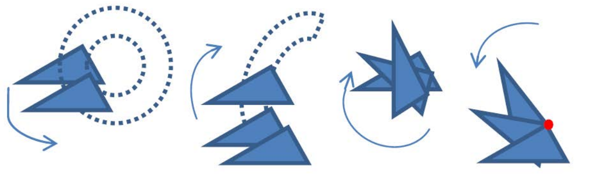

Figure 1: Four approximations leading to determination the type of the movement

The latter choice was commented, and the medium to convey thoughts were the movements:

Eliza: and I like it more like this: *she keeps the triangle with the fingers of one hand and with the other hand she is trying to rotate the figure relative to one vertex, in counter clockwise direction.*

In the next step girls have discussed the initial position of a triangle. Finally they decided (through appropriate arrangement) that the longest side would be parallel to the lateral edge of the page.

This first phase of work revealed rotation as such from intuition and the feeling that it is done with respect to a particular point. This point became more and more clear, it marked the organization of the whole movement. In the next stage, the discussion has been subjected to the initial orientation of the triangle. This position is also taken as the reference for determining the angle of rotation.

The centre of rotation

It seems that girls could not knowingly benefit from the center of rotation, till the end of their work. Although in their intentions triangle had to move in relation to one of the vertices, they do not feel the need to highlight this point – to tick, to discuss, to establish as a point of reference.

The beginning of their work indicates that the location of the triangle in the following slides is very problematic for students. Girls outlined the template of triangle on the first sheet of paper, turned a card in the booklet – and stopped embarrassed. They tried to find a way how to repeat the previous triangle position. Their first idea was to use the parallel shift: one of the girls tried to slip triangle out of the books (Figure 2a), then the second girl turned the page and they tried to push the triangle on this new card (Figure 2b, 2c) repeating its earlier position. Quickly they stated that it is not a good idea because the shift was not parallel and in their movements girls went as far as the "left side" of the earlier page (Figure 2d).

Gabriela found the other solution. She puts the triangle on the left side of the first contour (Figure 3a), closes the book (Figure 3b), turns it on the opposite side so that the triangle in the booklet falls down onto a second card (Figure 3c, 3d). These manipulations can be seen as the use of intuition of the axial symmetry, or folding. The girl uses it for placing a figure onto another plane in the same distance from the edges of the paper.

Later, the next few slides were drawn by using Gabriela's method. Then the girls have perfected their work – noted that on the following pages you can see the outline of the previous triangle position. So she have used this contour as a point of reference, to create a change.

While drawing further slides, girls tried to rotate the template relatively to one of the vertices belonging to the longest side. Sometimes they kept the vertex with their finger, but mostly – after the initial placement of the template on the track they only have tilted the template and thus determined the new position. There were moments when the girls manipulated the template quite freely, and the center of rotation was located somewhere inside the triangle.

The analysis of position of one of the vertices of the triangle through the following slides – that one, with respect to which the movement took place - shows how much this point was unstable. This is evident even in the pictures below (Figure 4, 5) and it is shown on the diagram beside, on which the arrangement (A) marks the position on the first "slide" and (B) – on the last one.

Comment: These first approaches prove that in attempts to perform the movement, the parallel shift and mirror reflection (understand as the tool) are closer to them than rotation. The girls mainly want to reconstruct the earlier position of the entire tri-

Fig. 2 Fig. 2b Fig. 2c Fig. 2d

Figure 2: Following phases of shifting the triangle

Fig. 3a Fig. 3b Fig. 3c Fig. 3d

Figure 3: Gabriela's way to restore the position of the triangle from the previous slide

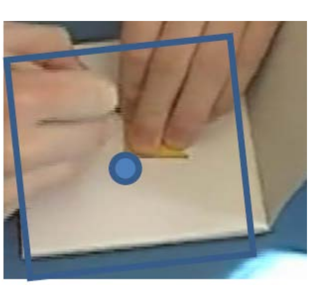

Figure 4

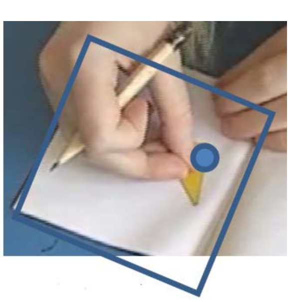

Figure 5

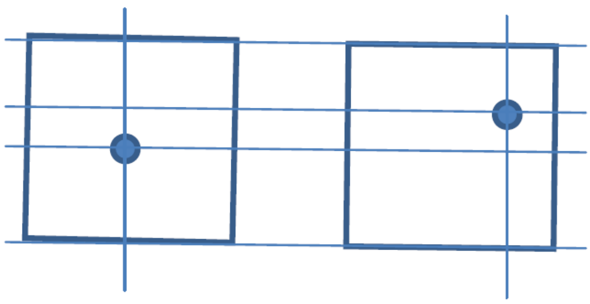

Figure 6

angle. In further work this position will be treated as a starting point for making changes in the position. Despite earlier manipulation on the table by hand, in which rooted movement at one fixed point was visible, they have no idea of the location of the center of rotation. What's more, the center of rotation escapes from their attention, it does not look not as an important constitutive element of movement.

Oriented angle

The first change of the triangle position was held after establishing its earlier position. When the girls dubbed the triangle's location from the first slide, carefully twisted template trying not to change the location of one vertex. Further stages of their work were similar - doubling the "old" position and light twist of template. It was evident that what was particularly important for them was the change of the placement of one of the triangle's sides (the longest - which end by the center of rotation) in relation to its previous position. Such action guided them from the very beginning of the work, which was confirmed in a conversation with the teacher:

Ex. 1

Teacher: What's your idea for the animation?
Students: Wow, that he would be so, so twisted (*both show by the hand the movement, in which it is clear that one vertex is stationary*). So around.
Teacher: So drawing the next steps, the next slides of you animation, you turn your attention to something? What do you pay attention to, because I see that Eliza draws a triangle, then she closes the book
Gabriela: that is, the first we make from that first contour...
Teacher: What, why so?
Students: In order to know *how it was before*, then we *move it a little bit so* (*Gabi is aided with her hands jammed into the table and one arm shows how to turn*)
Teacher: To move it a little further, or to rotate it a little bit?
Students: rotate
Teacher: Oh, do you move it in any way or just only rotate it?
Gabriela: just rotate (*Eliza is busy with another page*)
Teacher: aha.

Even when they discovered that there is an easier way of finding the proper position than the book manipulation, the essence of their work has not changed. The girls noticed that they can use the contour, which is reflected in the following pages. This location was used as the starting point for making changes.

Ex.2

Teacher: What now, Gabi, this facility will consist of?
Gabriela: So now you will not need to bounce it, but here it is already marked, and now only to apply (*she shows a position of the template on the track*) and to twist.

When the motion is realized, the most important is to change the position of one side of a triangle in relation to its position at an earlier slide. This change takes place in a specific way - as a change in slope. With this approach, the angle is understood as a measure, not as a geometrical figure. Thus, it refers to one of the oldest ways of describing the angle. The definition from the first book 'Elements' of Euclid, we have:

> Definition 8. And a plane angle is the inclination of the lines to one another, when two lines in a plane meet one another, and are not lying in a straight-line. [3]

For girls, it is enough 'to skew' a triangle. The existence of a fixed point is pushed aside. Such it looks in implementation, even in an interview the fixed point is marked, supported by gestures.

Size of the angle for the following 'slides' is taken intuitively, although girls try to make rotation rather on the same angle. At no time of their work were they interested by which total angle the triangle is rotated. Girls draw slide after slide, which can be explained by focusing only on the animation of movement; planned movement effect is the main goal of their work.

What you will observe, is a constant angle's orientation. This is evident in many moments of their work. While replacing themselves in drawing (they do it alternately) they remain continuously the same direction of rotation. At some point Gabi, taking the book for drawing, asks Eliza:

Gabriela: In which way it spins? so? (*she shows the movement by turning the template*).
Eliza: Oh, in this way (*shows movement by the hand on the bench*).
Gabriela: Oh (*turns her triangle in the same direction*).

FINAL REMARKS

'Mental rotation' is an issue multilaterally studied by psychologists. Sometimes their studies are used in the didactics of geometry (Steinwandel & Ludwig, 2011). However, these studies recognize the problem from a different side than the one that is presented in this essay. Generally, they are focused on solving tasks which require the mental rotation of three-dimensional figure or the mental movement in space. Our observations are not so much related to rotation usage, but to visualization of rotary motion with regard to its mathematical aspects. In such approach, we intend to realize two aims - the creation of geometrical concepts (here – a geometric transformations: rotation) and the formation of dynamic images in geometry. We are committed that the visualization emphasize those elements that are relevant to the mathematical definition of the rotation and underline the idea of motion.

During this meeting the teacher repeatedly posed the question to participated students, whether the task they perform can be offered to other students during a math class. The girls had no idea what relationship may these classes have with geometry. It is clear that paper geometry (even when learning use of physical models) does not create many opportunities to realize a movement that underlies conducting of the geometric reasoning.

Conducted observation shows that the mathematization of rotation might not be so obvious. The girls do not really have revealed these elements that define the rotation. Their activity gave a fairly imprecise effect of rotation, but did not allow to extract those elements that appear in the definition:

— Centre of rotation - the only fixed point in the transformation - is carried out in a quite fuzzy manner. The clash of what they want to do with what they actually realize shows that there are two different worlds. Action in the physical reality is not translated to the representation of the movement by the drawing. While creating a drawing they don't retain the features designated through physical action, and especially do not respect the stability of the center of rotation. The center of rotation stops being the most important point of reference in changing the position of the object.

— The angle of rotation is treated as a measure of the change of slope of the segment. This change is the essence of the functioning of the movement. It takes place in a weak relationship with the center of rotation. Perhaps this is the clearest element in this movement, showing the change. It seems likely that children are more focused on what is changed, and not on what is fixed. After all – the movement lies in the change.

— Orientation of rotated angle was accented in a very clear way. The initial position and the final position are always indicated. In subsequent stages, the final position moves into initial position - this causes that the movement is smooth, still directed in the same way.

— The measurement of the entire angle was not significant for girls. Only the initial position and fixed side of the triangle was scheduled, which was to define the first arm of the angle. Further work was regulated by the amount of sheets in the booklet – the amount of slides in the animation. Each subsequent turn was held by a small angle – about 5 degrees – but at no stage of their work this value was specified in a conscious way. It can be assumed that for the girls it was enough to have a situation where the following slides' layout was noticeably different from the initial position.

All these conclusions are related to the work of only one pair of students. Therefore they can't be the base for any generalizations. However, it should be noted, that they were students who work well with the school mathematics. For me, collected observations are the starting point for a critical analysis of the other students' work, dealing in the same learning environment.

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ENDNOTE

1. In describing session the teacher was MA Edyta Jagoda.

The meaning of isometries as function of a set of points and the process of understanding of geometric transformation

Xhevdet Thaqi[1], Joaquim Gimenez[2] and Ekrem Aljimi[1]

1 University "Kadri Zeka" Gjilan, Gjilan, Republic of Kosovo, xhevdet.thaqi@uni-pr.edu, ekremhalimi@hotmail.com

2 Unversity of Barcelona, Barcelona, Spain, quimgimenez@ub.edu

In this paper, we try to show that in the process of understanding of isometric transformations, the meaning of isometric transformations is characterized as a function of whole figure to whole figure, as a function of the parts of the figure to the correspondent parts of the figure, and as a function of the set of points of figure to set of points of the same or other figures. This perception of isometric transformation has been observed in an experimental study which enabled us to define and understand different levels of difficulties in recognition of isometric transformation, from which the details are presented in this paper.

Keyword: Geometric transformation, function, teaching geometry, concept images.

INTRODUCTION

Considering low level of geometric reasoning of future teachers observed in the experiences and showed by different investigations (Thaqi, 2009), our interest is the foundations of the professional development of the prospective teacher of mathematical education. For that reason we have to design, plan and implement a practice on learning to teach the mathematics; to analyze elements of the constructions of personal meanings of future teachers about mathematics and, to recognize the difficulties of the students to understand, relate and organize mathematical contents, terms and properties associated to that content. During last years with my colleges we tried to contribute in this process focused concretely in geometrical transformations (Thaqi, Gimenez, & Rosich, 2011; Thaqi & Gimenez, 2012). In this paper we research nature and causes of difficulties in teaching/learning geometrical transformation and the relation of such difficulties with concept images constructed about geometrical transformation.

Some investigations have highlighted the reasons and advantages that provide the study of geometrical transformations (Jackson, 1975; Küchemann, 1980; Jaime, 1993; Harper, 2003; Jagoda & Swoboda, 2011; Thaqi, 2009). A main reason to study the transformation is curricular, since "The transformations are applications of the geometrical functions, and this treatment is fundamental for all the mathematics" (Jackson, 1975, p. 554). Other reason is that the transformations provide geometrical dynamic task. Despite these reasons and advantages that the transformations teaching offers, generally we know that the students show a low level of learning about the transformations (Thaqi, Gimenez, & Rosich, 2011) and we highlight that a program for the education of the teachers must integrate the same objectives that the scholar geometries classes have; the need of additional formation in formal geometry and empathize the conceptual understanding, starting from the analysis of the geometrical environment with conceptual explorations. The study of how prospective teachers build a meaning of the concept of isometrics transformation in process of understanding of geometrical properties and their relation with difficulties is one of the aims of this research.

THEORETICAL FRAMEWORK

Sundry authors have distinguished the everyday concepts (known as spontaneous) and so did the scientific (Piaget, 1970; Vygotsky, 1987). Fischbein (1993) considered that there have been seen three types of conceptual constructions in the investigation: inductive, deductive and inventive building of concept. Later on, the same author tells us that the concepts are the results of accumulated social experiences (Fischbein, 1993). We consider that there has to be built the conceptual meaning in interaction with contexts grounded in the experiences, to later build images and abstractions. Some authors like Sowder (1996) proposed that what characterizes a concept is to state an idea that is given like an answer to non similar stimulation (to varnish the understood like examples). In the opinion of Fischbein (1993) what characterizes the concept is the fact to state a idea, general representation of a class that is based in common characteristics.

The conceptual construction rests on a set of processes of construction, visualization, exploration of properties, elaboration of explanations and classifications, within others. Among more and better experiences we have, the conceptual image gets closer to the concept because, like Vinner and Hershkowitz states:

> to acquire a concept means, to acquire a mechanism of construction and identification through what will be possible to identify and build all the examples of the concept, the same way as that is conceived from the mathematical community [Cited for Jaime & Gutierrez, 1995].

As it concerns to the various investigations about the subject (Jaime & Gutiérrez, 1995; Pearman, 1990) generally they put the manifest that between the Piagetian concept of conservation of the length and the invariance there is a tight relation that they can be saved (Jaime & Gutiérrez, 1995).

The constructs "concept image" and "concept definition" (Tall & Vinner, 1981) will be useful to us also to describe the status of the knowledge of the individual fellow with a relation to a mathematical concept. It is meant to the mental entities that are introduced to distinguish the mathematical concepts formally defined and the cognitive concepts through which they are conceived. With the expression:

> concept image describes that the cognitive structure is totally associated to a concept, that includes the mental images and associated processes and properties (Tall & Vinner, 1981, p. 152).

Jaime & Gutiérrez (1995) state that in the formation of images of a concept that a person has, the experience and the examples that has been seen or used, both in a scholar and extra-scholar context have a basic role. Frequently the examples are few and the students convert them in prototypes. Jagoda & Swoboda (2011) study the process of construction of the concept in rotation spotlighting that "recognition of a specific figure to figure position is only a static image of this relationship, not connected with the movement of one object onto the other". In fact they confirm that the idea of geometrical transformation is necessary to conceive the specific movement that is transforming the initial figure into the final one, through which it is important that such conception stems from mental reflection on the phenomenon of movement. Our position is based on the meaning of the concept of geometrical transformation as a function of (whole) figure to (whole) figure that is the basic level of knowledge about geometrical transformations. It has also been shown by several authors that pre-service elementary teachers have difficulties in determining: (1) the correct attributes of transformation and motion to move an object from one point to another; (2) the results of transformations involving multiple combinations of figures; (3) the use of transformations as mathematically-general operations which require the specification of inputs, but as particular actions, each with given prototypical parameters (Harper 2003).

A recent study concerning prospective teachers' knowledge of rigid transformations (Yanik & Flores 2009) revealed that scholars: (1) started by referring to *transformations as undefined motions of a single object* which is equivalent with *Static Arrangement Figure To Figure* (Jagoda & Swoboda, 2011), followed by (2) using *transformations as defined motions of a single object*, and (3) the understanding of *transformations as defined motions of all points on the plane* which is equivalent with *transformation as function of set of points to set of points*. In our research we will try to explain that precisely these three ways of change of concept images of geometrical transformations enable us to explain and understand the difficulties of the understanding of the concept of geometrical transformation. A head we will show the relationship between the process of construction of the concept of geometrical transformation and the function of the set of the points of figure to the set of points of the (same or) another figure.

METHODOLOGY

The methodology was adapted to several techniques that allowed to approach the construction of the goals of the investigation and allowed to rate it as a theoretical formal study within the interpretative focus. In this way we have elaborated a design in which was combined the own techniques of the studies with case studies. Considering our own experience in the mathematical education of the teachers, we considered adults who acquire a professional scientific knowledge, and our work that has transformed a vision of the processes in the creation of a meaning, we identify that the investigation, action and formation are the three sides of a same methodological triangle. So, the investigation comprises theoretical component with epistemological and cultural character, also an experimental type component. These two parts of the investigation are intimately related giving the product - analysis of the speech and analysis of the personal constructions, to be able to form the conclusions of the investigation.

Participants of the study were 18 students of Faculty of Education in University of Gjilan – Kosovo. This is because two authors of this study have worked in regular lectures with these students in the program of preparing prospective teachers. Participants where purposefully chosen, and voluntary participated for this study. They are 20–22 years old students – from different, rural and urban places, from both sexes and with different studies done in prior studies. A previous curricular-cultural analysis based on textbooks, official curricular proposals and teachers' training materials, of these context is the same as showed in the deep study (Thaqi, 2009), not detailed in this paper.

During the development of the practical sessions dedicated learning to teach geometrical transformations, there has been prepared for every student, work sheets, to be able to have their productions and later analyze them. The quantity of the practical sessions and their length is what we present in the following table (see the main ideas in Table 1).

The results of a final semi structured questionnaire, was the basic data considered in this paper. Data were collected from descriptive notes, reflective notes, interviews and video records. The collected data was analysed using method described by Strauss & Corbin (1998) along with analytic induction. Such a questionnaire is the last step for a more wide developmental study in which group of students have the same training about learning to teach geometrical transformations (Thaqi, 2009).

The activities have been realized in usual classrooms of the Faculty of Education. In the group there were 18 students of 3rd grade of the study program of primary education. Some other questions were added to identify reasoning and specific cultural elements about geometric transformations, ideas about teaching and learning, and about their thinking about future classrooms in teaching geometrical transformations. The students come to the final test after having taken training course (Table 1) about teaching geometric transformations in the school, during spring semester 2013 course. The students were given the question-

Aspect of meaning of geometric transformation	Identified Activities
Isometrics and the everyday life (SI)	Presentation: An experience about isometric transformation - (SIP)
	1.2. The activities about the isometric transformations (SIA)
	1.3. Didactical activity: Presentation - video of teaching symmetry (SID).
Learning the usage and the value of the sources to teach the transformations (SR)	2.1.Presentation: *Scientific article as resource for teacher education* (SRP)
	2.2. Activities about didactical sources and geometrical transformations (SRA)
Projections and shadow (SP)	3.1. Recognition of work about shadow in primary school (SPP)
	3.2. Properties of shadow (SPA)
Reasoning, arguing and justification of geometric transformations (SA)	4.1. Presentation of topics (SAP)
	4.2. Activities about reasoning, proof and justification of geometr. transformations. (SAA)

Table 1: Sets of questions related to mathematical ideas about transformations

naire the last day of the course, and all students have responded to the questionnaire. The issue in focus is the identification of the prospective teachers' concept images and the way they make use of their images and the mathematical definition of certain concepts for geometric transformations that they will find central when they begin their professional life as mathematics teachers. The selection of questions in the questionnaire is closely related to the realized, the same four sessions of didactic practice on learning to teach geometric transformations, in the Faculty of Education in Gjilan.

In the sessions showed in Table 1, there are presented activities that one of the investigation goal was to prepare activities where, the basic knowledge of geometrical transformations are based in the intuitive teaching and experiences about the search, the discovery and the comprehension from the prospective teacher. In this way, the prospective teacher learns the knowledge and geometrical properties from the everyday world aspects constructing them as the concept image. In the first sessions (SI and SR) there is realised the treatment of the isometrics, the development of activities of using various resources. In the session SP there is developing activities to learn how to teach the non-isometric transformations (deformations, projections), while the session SA holds activities of the development of the capacity of arguing, justifying and reasoning in the process of personal construction of the meaning of geometrical transformation.

The students were asked to give an explanation to the following aspects: terminology and type of transformations, properties and relations on transformations, processes of changes, and other aspects about geometric transformations as reasoning, teaching, attitude, etc. We analyzed how the prospective teacher approaches to recognize isometric transformation and construct concept images during the process of a practice of learning isometric transformation; how is the level of recognition of the relationships and the hierarchy between different properties, and levels of difficulties in reasoning about the isometric transformations and communication of the results.

RESULTS

In the analysis of the context (Thaqi, 2009) we have seen that the Program of Geometry in Faculty of Education, talks about the formative teaching, and plans the contents as a set of knowledge and procedures. The study of geometry in this program has as a goal the mathematical knowledge, that basing on the qualified posture as formalist it can be understood as the rules that from some affirmations logically are followed some others. We present now the analysis of the moments of progress or difficulties of the student for future teacher of Primary school, in the process of building the idea of geometrical transformation that figure A is transformed in figure B, the usage of the adequate terminology in every case and identification of different types of transformations. Few students talk explicitly about the isometrics as a transformation that conserve the size and shape. Instead, they do identify the symmetries, rotations and translations as transformation as such property. Anyhow, in some cases, the activity makes the intuitive go ahead the structured knowledge. In that way, when we are in front of the observation of the embroidery, some students show rotation as a unique isometric, since they identify it as the only transformation that acts on the module that is marked (Figure 1). So, the conceptual image of the geometrical transformation is build basing on visual properties (transform=deform), and movement (Isometric=displacement).

Firstly, let´s say that in some cases the transformation is seen as a function of a set of points to the other one, but that function isn´t identified between the posi-

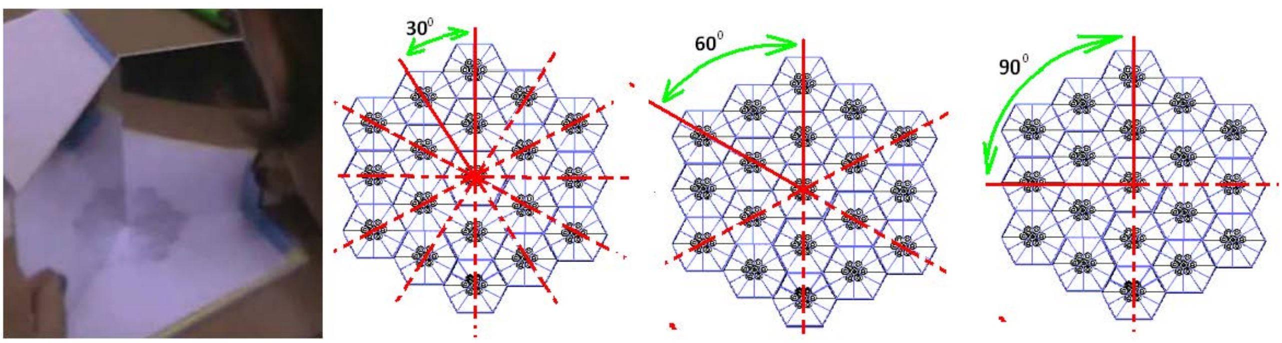

Figure 1: Reproduction embroidery using mirrors

tions of the objects in two different places. This can be explained with the symmetric figures, because it gets established easily the correspondence between the two parts of the figure or object. Instead, for other types of transformations, they should imagine the initial and final position of the transformed figure to be able to establish the idea of transformation as function or correspondence between parts of figure. We got convinced that it´s that way when we analyzed the answer to the problem where there is asked to explain the transformation of the figure A in the figure B, when Blerina (participant of investigation) explains it using transformation of set of points in the figure A on the correspondent points of the figure B. Only one of all participants' talks about isometric transformation as a transformation that conserves the shape and size – conservation of shape and size is the definition of function; while the others identify it as a repetition - which is associated with difficulties on identifying properties of transformations. We find that considering transformation with identification of invariants (form and size) is better level than considering transformation as simple repetitions.

Indeed, in the cases of considering transformation as simple repetition (that is equivalent with considering transformation as a function of whole figure to whole figure) they do not reach to precise the angle of the rotation, around what point or axis, etc. Actually, as the Figure 2 shows, to consider transformation as a function of set of point of figure A to set of points of figure B, they feel the need of naming the vertex of the triangle, and later on, they expresses the functional dependence:

Blerina: .... firstly there has been a displacement of the point A, and during the movement of the point A, the point B and C get the position presented as in the figure.....

In that way, it is constructed the idea of rotation as a function of three points in other three points under the condition:

Blerina: ... so the point C has gone out of the column of the point A one row lower, the point B has gone on point´s A place.

The second step of this process will be to indicate the axial symmetry with the axes of the symmetry instead of the mirror: "...we would see it better if we would imagine a mirror placed in the column of the point A, where the point A is not reflected (moved) and instead, the points B and C are reflected" (explain Blerina). We observe now the findings in the case of the deformations. The activity SAA5 shows the dynamic transformation of a triangle – two stable vertexes and the third one move horizontally inside a segment (Figure 3a). The students have to explain the properties that are observed in this transformation.

About the transformation with the property of invariance of the surface and the variables; it´s interesting for us that the students first identify these properties

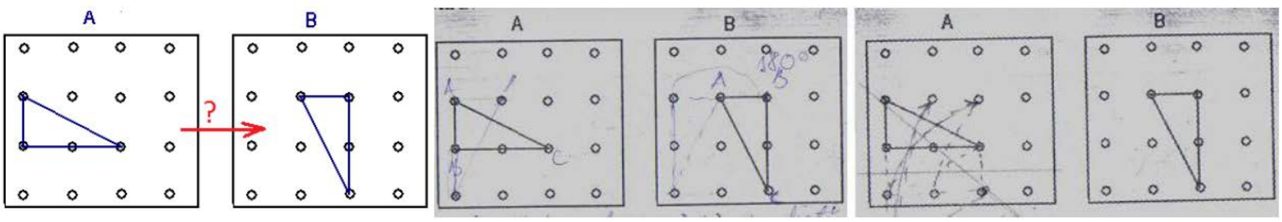

Figure 2: Transformation of triangle as function

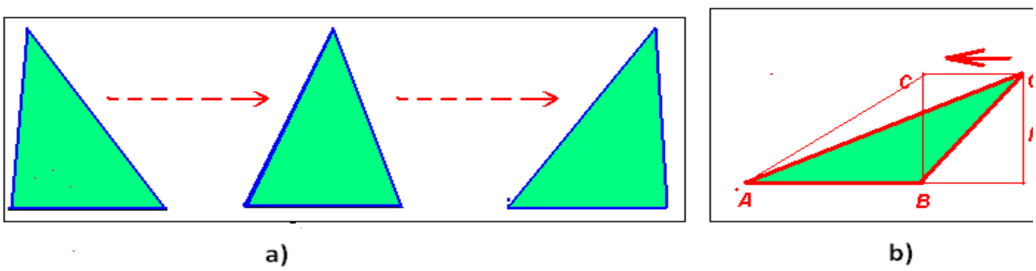

Figure 3: Dynamic transformation of triangle. Invariance and variable

and then justify the announced result. We find that Blerina identifies correctly the elements of the triangle that change and the elements of triangle that don´t change. She identifies the change of the shape, perimeter, position and what does not change like surface, base of the triangle and the height of the triangle. As an illustration we show the part of the dialogue with Blerina, who does a correct justification that the surface of the triangle does not change using the correct symbolization (the formula for the calculation of the surface of triangle: ($S = \frac{bh}{2}$) basing on the definition of the surface of triangle as a product of the base and the height (Figure 3b):

Blerina: The triangle gets converted in a rectangle triangle...
Tutor: What does change in this process? What does not?
Blerina: The height of the triangle doesn´t change.
Tutor: What other changes do we have?.
Blerina: The angles and the sides change and, the surface and height do not.
Tutor: How can you argue that the surface does not change?
Blerina: The surface of the triangle is in function of the base and the height. As said, the base and height does not change and either does the height ... so the surface is constant independently of the position of the upper vertex.

With the development of this and of the activity SAA4 we identify that the dynamic presentation of a deformation makes it possible that the students approach to recognize and build the concept of geometric transformation as a function of variables and constants $S=b{\cdot}h/2$ (this is to identify the elements of deformation that are conserved and variables). In the activities SRA3 there is asked to draw a symmetric image of the figure, having as a symmetric axis the straight line drawn (activity SRA3, Figure 4). The students have mirrors as a didactic resource for their activity. We haven´t noticed in the observation that any of the students didn´t reach to reproduce the symmetric figure from the given one. In the video recordings we have found important to describe the process of reproduction for some students. Before Blerina started to do the construction of the symmetric figure gives the comment: "I draw any figure in the plot of points. Later I draw a straight line in that way that it touches one vertex of the given figure. After that, I count that the straight line has to be the axis of the symmetry. Is it so? .... This means that I do an function of each vertex of the figure counting little squares in the other side of the axis..." Blerina does the symmetry using the properties of the symmetry: the axis, the same distances to the axis and the process of the application point-to-point. It is not needed to determine each point of the figure, she finds the vertexes of the figure and later, she uses the property of the aligned points: aligned points get transformed in aligned points (Figure 4).

High grade students recognize the transformation if they recognize the relevant elements of geometric transformation using the process of the function point-to-point, and the property of the aligned points: aligned points get transformed in aligned points.

CONCLUSION

Analyzing this findings we can say that in the program of mathematics for prospective teachers the main goal of the teaching of geometrical transformation is the informative knowledge, while is necessary program for the teacher education, intended to cultivate and practice the logical reasoning. We consider that the best is an equilibrating education between thinking and acting, or between the cultivated knowledge and

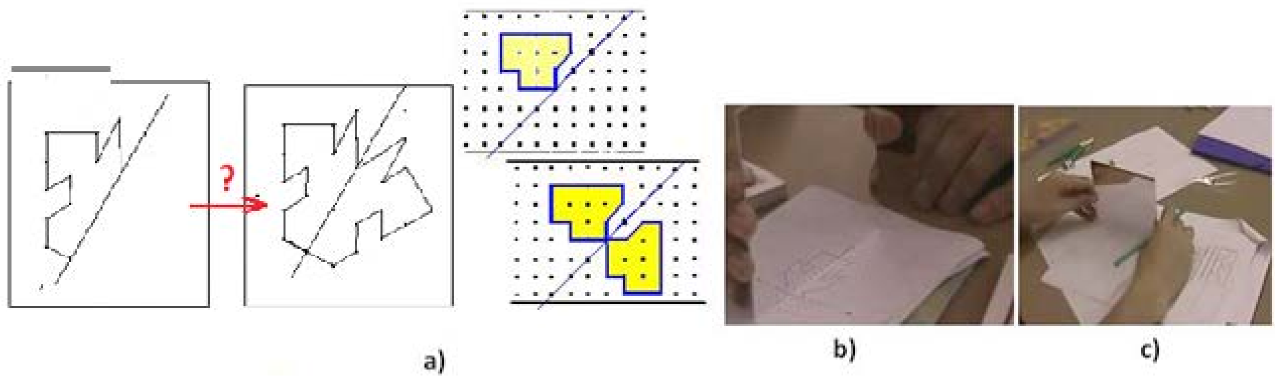

Figure 4: The process of construing symmetric figure

practical knowledge. In the cases that students recognize the transformation as a process of function of points to points, it is easy to identify the functional dependence between positions, the important properties of the transformation such as symmetry axis, vector of translation, center and angle of rotation, etc. and it established the complete concept image about isometric transformation. In cases when students consider isometric transformation as a fold, changing position or repetition of an object or figure, they are confronted with difficulties to establish the important elements of the transformation process. In other words, those who construct the idea of transformation as correspondence between sets of points do not find it difficult to have the complete concept image about isometric transformation, correctly identifying the properties and elements of that transformation as the orientation of the image, the axis of symmetry, the angle of rotation, the translation vector, invariance and variables etc.

ACKNOWLEDGMENT

This research and presentation in CERME 9 is supported by GIZ (Deutsche Gesellschaft für Internationale Zusamenarbait GmbH), office in Republic of Kosovo.

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A meta-classification for students' selections of quadrilaterals: The case of trapezoid

Fadime Ulusoy

Middle East Technical University, Faculty of Education, Ankara, Turkey, fadimebayik@gmail.com

This study aimed to propose a meta-classification for middle school students' selections of quadrilaterals in terms of trapezoid. Data were collected from thirteen seventh grade students via a trapezoid selection instrument and semi-structured interviews. Data analyses were executed by using thematic coding of qualitative research methods by synthesising past and current theories about teaching geometry (figural concepts, concept image-definition, prototypical phenomenon etc.). Meta-classification was characterized considering the types of students' selections, concept images, and errors through seven special categories. This meta-classification shaded light on how students conceive a shape as trapezoid, what they notice their selection procedure and what influences their selections for trapezoids.

Keywords: Concept images, figural concepts, prototypes, quadrilaterals, trapezoid.

INTRODUCTION

NCTM (2000) implies the importance of analyzing characteristics and properties of two-and three-dimensional geometric figures and developing mathematical arguments about geometric relationships. In this regard, one of the basic topics of geometry is quadrilaterals which involve the concepts of rectangle, square, rhombus, parallelogram, kite and trapezoid. Among these 2D-figures, there have been two different definitions of trapezoid in geometry textbooks. While one is that *a quadrilateral with exactly one pair of parallel sides*, another is that *a quadrilaterals with at least one pair of parallel sides*. As the former one is an example of exclusive definitions (e.g. parallelograms are not also trapezoids), latter one is a type of inclusive definitions (e.g. parallelograms are also trapezoids). Since the trapezoids high up in the hierarch of quadrilaterals, the choice of the definition influences the derivation of properties both of trapezoid and parallelograms, rectangle, rhombuses, and squares (Usiskin & Griffin, 2008). Despite the importance of the concept, there are a few studies specifically focused on trapezoid (Manizade & Mason, 2012; Nakahara, 1995; Türnüklü, 2014). Among them, Türnüklü (2014) conducted a qualitative study to determine middle school students' and prospective teachers' concept images regarding trapezoid. She reached that individuals used non-critical properties in non-formal and incorrect definitions and they made overgeneralizations. When considering conducted other studies, it can be claimed that scattered and limited literature related to trapezoid does not give a complete picture about students' conceptions. From this perspective, the aim of this study was to propose a comprehensive meta-classification model for students' selections regarding trapezoid. Thus, this study will put forth more complete and well-coordinated structure shedding light on how students select a shape as trapezoid, what they notice their selections and what influences their selections for trapezoids.

THEORETICAL BACKROUND

In this study, it is necessary to clarify the meaning of essential terms that used when classifying and explaining students' selections, statements, and drawings. All the required terms within the current study were described in the following.

Concept Image/Definition: *Concept image* is the set of all the mental representations associated in the students' mind with the concept name. The image might be nonverbal and implicit. On the other hand, *concept definition* constitutes a form of words which are used to specify the concept (Vinner, 1991). According to this framework, suitable and robust interactions between concept definition and concept image might guarantee the conceptual learning rather than instrumental ones. Unfortunately, learners do not make sense to link between the two elements because there might be irrelevant properties about the concept evoking in students' mind specifically. For instance, the results of some studies indicate that many of students at different grade levels have a concept image of equilateral triangle having a right angle or slanted sides of equal length (Burger & Shaughnessy, 1992; Clements & Battista, 1992). In this sense, if students are encountered limited examples having common figural features of a geometric concept in school or other context, these examples lead to prototypes phenomenon (Hershkowitz,1989).

Prototypes Phenomenon: The *prototype examples* are usually the subset of examples that had the "longest" list of attributes all the critical attributes of the concept and those specific (noncritical) attributes that had strong visual characteristics" (Hershkowitz, 1990, p. 82). Students often see figures in a static way rather than in the dynamic way that would be necessary to understand the inclusion relations of the geometrical figures (de Villiers, 1994). For instance, students receive square is not a rectangle because of their misconception about the length of the opposite sides of rectangle. Consequently, a contradiction between concept images and concept definitions emerges, which may elicit misconceptions in students' mind when classifying quadrilaterals. In the current study context, although a student has learned trapezoid as quadrilaterals with at least one pair of parallel sides, she/he may not admitted parallelogram, rhombus, square, and rectangle as a trapezoid, which clearly asserts the influence of prototype examples on relationship between concept image and concept definition that was structured in student's mind.

Personal/formal figural concepts: Geometrical concepts are characterized as having double nature by two aspects: figural and conceptual (Mariotti & Fischbein, 1997; Fischbein, 1993) similar to the concept image and concept definition (Vinner, 1991) respectively. While figural aspect involves spatial properties like shape, position, and magnitude; conceptual aspect involves abstract and theoretical nature as ideality, abstractness, generality and perfection. According to Fischbein (1993), figural aspect is generally more dominant than conceptual one. For example, parallelograms do not look like a trapezoid, but they are formally trapezoids considering the formal exclusive definition of trapezoid in our context. Based on these ideas, Fujita and Jones (2007) proposed the ideas of personal and formal figural concepts. Formal figural concepts involve formal concept images and definitions in Euclidian geometry. However, personal figural concepts were constituted through individuals' own geometry learning experiences about geometric shapes. For instance, "rectangle is a parallelogram with four right angles" is a formal figural concept definition. Besides, the expression of "a rectangle is a quadrilateral with only opposite sides congruent and four 90° angles" in a student's mind reflects student's personal figural concept.

Undergeneralization and Overgeneralization: Two types of common errors that are exhibited by students have been described in the literature as *undergeneralization* and *overgeneralization* (Klausmeier & Allen, 1978). *Undergeneralization* occurs when examples of a concept are encountered but are not identified as examples. It results when the examples provided in instruction are not sufficiently different from one another in the variable attributes (Klausmeier & Allen, 1978, p. 217). In the context of this study, for example, a student who has experienced only right trapezoids having exactly one pair of parallel sides may not identify trapezoids not having right angle even it has exactly one pair of parallel sides. On the other hand, *overgeneralization* occurs when examples of other concepts treated as members of target concept (Klausmeier & Allen, 1978, p. 217). In the current context, a quadrilateral having no parallel sides and non-equal length of sides or a polygon having more than four sides may be treated as trapezoid because of some reasons as omitting key properties of the concept or focusing only language-related factors.

METHOD

In this study, basic qualitative design methods were utilized to classify students' selections of trapezoids such as semi-structured interviews for data collection and thematic coding for the data analysis.

Participants

Firstly, an elementary school located in the capital city of Turkey was selected in order to determine the participants. At the school, there were two seventh grade classes with 47 students in total. After determining the school, I had an interview with mathematics teachers of the both classes to get information about students' mathematics grades and personal characteristics (e.g., talkative). Furthermore, each class was observed for four hours in order to monitor students' behaviors. Based on the maximum variation sampling, I conducted semi-structured interviews with 13 seventh grade students aged thirteen who were enumerated from S1 to S13. Their achievement levels were categorized according to their average math note belonging to the first and second semester. Semester notes were categorized as 5-5 was *high*; 5-4, 4-5 4-4, 3-4 and 4-3 were *middle*; and 3-3 and lower ones were *low*. According to semester notes, three students (S1, S2 and S3) were attended low achievement level. Five students (from S4 to S8) were attended middle achievement level and five students (from S9 to S13) were attended to high achievement level.

Instrument and data collection procedure

To understand how students select trapezoids in various polygons, researcher generated an instrument of *"Which figures are trapezoid?"*(see Figure 1). The instrument was organized in a manner of containing prototype and non-prototype polygons having different sizes and orientations. While creating the instrument, similar questionnaires about parallelogram in the literature were analyzed (Fujita, 2012; Nakahara, 1995; Okazaki, 1995). Then, a preliminary study was conducted with 86 seventh grade students to understand what they think about trapezoid concept. In the preliminary study, they only defined trapezoid and drew three different trapezoids in a grid paper. After examining the results of the preliminary study, seven possible selection categories were formed (*Note*: Details were explained in the heading of *characteristic features of meta-classification*). According to seven possible categories, figures were added into the data collection tool. Finally, data collection tool was controlled by experts and pilotted with five seventh grade students throughout the interview sessions. Before the selection procedure, participants also asked to make definition and different drawings for trapezoids to support data coming from students' selections of trapezoid.

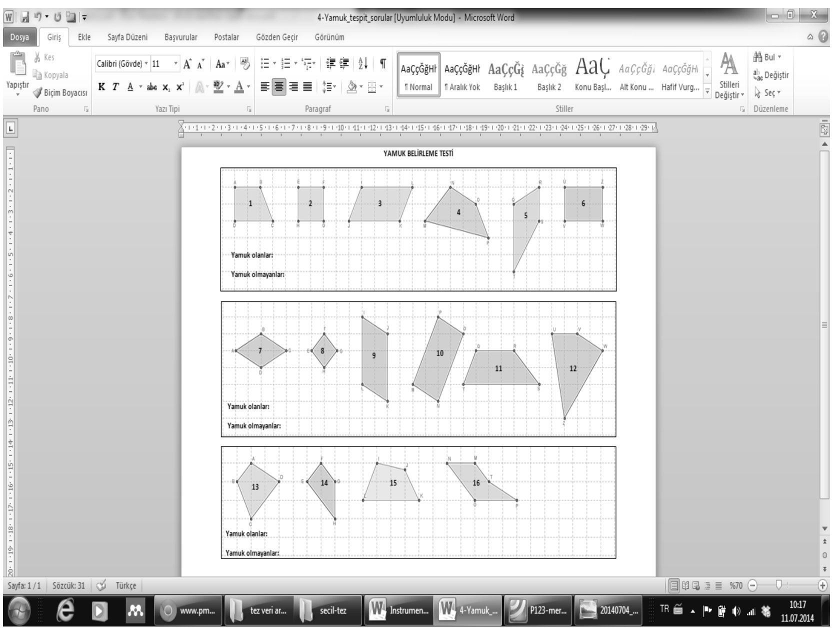


Figure 1: The instrument of "Which figures (1–16) are trapezoid?"

Data were collected via semi-structured individual interviews for in-depth analysis. Average time of an interview was twenty minutes. During interviews, participants explained their selections and the reasons why they select a figure as a trapezoid. Furthermore, they made definition and drawing of trapezoids. All interviews were videotaped and transcribed. To analyse the data, the researcher carefully examined the students' selections, drawings, and definitions. Then, thematic analysis was used to identify, analyse and report the themes in the data. For this purpose, all data were examined by taking account the phases of familiarization with data, generating initial codes, searching for themes among codes, reviewing themes, defining and naming themes, and producing the final report (Braun & Clarke, 2006).

Characteristic features of meta-classification

To propose a comprehensive meta-classification for students' selections of quadrilaterals in terms of trapezoid, I formed seven specific categories after the thematic analysis (see Table 1). The types of categories in meta-classification model were characterized considering the types of *errors*, and *concept images* with the *correctness* of students' selections, drawings, and definitions. To be more precise, the categories show the way in which students made to discriminate trapezoids in various polygons. *Error types* reflect whether students' errors have a character of undergeneralization or overgeneralization for each selection category. For instance, students either made undergeneralization by focusing only right trapezoid and exclusive selections or they made overgeneralization by selecting quadrilaterals with no parallel sides and irregular polygons. Since *types*

Category	Correctness	Image Types	Error types
Cat-1	Correct	Hierarchical	No error
Cat-2	Correct and incomplete	Partial hierarchical	Undergeneralization
Cat-3		Pure prototypical	
Cat-4		Partial prototypical	
Cat-5	Incorrect	Visual	Overgeneralization
Cat-6		Language-based	
Cat-7		Contradictive	

Table 1:* Classification of students' selections for trapezoid

*This table reflects the selections of students who learn inclusive definition of trapezoid in their lessons.

of the students' concept images changed throughout the categories, it was necessary to differentiate the concept images for each category. All other details were given in the following paragraph.

Category 1 reflects learners' correct selections of trapezoid according to hierarchical (inclusive) relation among quadrilaterals. This category indicates an ideal situation including a harmony between the two aspects of a figural concept (Fischbein, 1993). In other words, learners' personal figural concepts were completely and correctly consistent with formal figural concepts (Fujita & Jones, 2007). In *category 2*, learners can connect the correct relations between some quadrilaterals; however, they make their selections by considering partial hierarchical relations (e.g. they only select parallelograms as a trapezoid rather than selecting rhombus, rectangle etc.) even they know the geometric concept in terms of conceptual aspect. This situation supports the idea about the domination of figural aspect (Fischbein, 1993) which can be a result of encountering prototypes in learning process (Hershkowitz, 1990). *Category 3* reflects students' pure prototypical images because they select only all prototypical examples of trapezoids according to exclusive definition even they know the inclusive definition of trapezoid in written or verbal form. As a result, they do not select parallelogram, rhombus, rectangle and square as a trapezoid, which reflects undergeneralization error. According to the students, a square does not figurally a trapezoid, which reflects a conflict that may appear between the figural and the formal constraints as indicated in Fischbein and Nachlieli's study (1998). In a similar vein, *category 4* reflects students' partial prototypical concept images about trapezoid because they think only prototypical examples such as isosceles/right trapezoids, which leads the error of undergeneralization. In summary, *category 2-3-4* show students' personal figural concepts of trapezoid consist of correct concept definition and limited images. In *category 5*, learners make incorrect selections considering only similarity of the shapes in terms of position and visual appearance without thinking formal definition and critical attributes of the figure. This situation causes overgeneralization error because they treat irrelevant figures as a trapezoid. *Category 6* reflects students' language-based images because they think the meaning of the word of "*trapezoid*" in Turkish ordinary language with the meaning of "*oblique*". As a result, they image trapezoid as a figure having more than 4 sides. In other words, they treat some non-examples as examples by extending their knowledge to another context in an inappropriate way, which indicates overgeneralization error. Finally, *category 7* reflects the consistency of students' selections based on the contradiction in their between definition, drawings and selections. To sum, *category 5-6-7* indicate students' personal figural concepts of trapezoid consist of incorrect concept image and concept definition.

RESULTS AND DISCUSSION

In a general sense, results of the current study indicated that although higher level students generally selected shapes according to exclusive relations of quadrilaterals nobody made a selection on inclusive relations without a mistake. In other words, there were no students classified in *category 1&2* in which learners require connecting even partially hierarchical relations among quadrilaterals.

Category 3&4: Generally higher level students (S7-S12-S13) selected trapezoids based on exclusive relations of quadrilaterals and they attended in *category 3*. As a result, they did not think that parallelogram, rhombus, rectangle and square were also a trapezoid, which showed undergeneralization that students made with pure prototypical concept images. Students explained the selection procedure as below:

S12: (By *referring prototype trapezoid shape*) we drew trapezoid as a quadrilateral having only two parallel sides in our lessons.

S13: I remembered trapezoid as a shape having different angles and sides but top and bottom sides must be parallel.

On the other hand, two middle level students' answers (S5 and S8) were placed in the fourth category. Students chose only figures of 1 and 14 as trapezoids. They stated that there must be one pair of parallel sides in a trapezoid and they added unnecessary conditions as having two right angles. To understand their pre-existing concept images, I asked them to construct and to define trapezoid before the selection procedure. Their definitions and drawings also indicated that they had a concept image of trapezoid as a right trapezoid, which was the presence of the undergeneralization due to the restricted prototypical concept image development.

Specifically, the selections attended *category 3&4* showed students' limited understanding based on prototypical concept images of trapezoid (Fujita, 2012; Hershkowitz, 1990). Yet, their selections were interestingly based on exclusive definitions although they learned inclusive definition of trapezoid in their lessons. As a result, students tried to explain the concept based on critical attributes less than required. In other words, they made *undergeneralization*. The reason may be associated with the influence of giving prototype examples of trapezoid in the classroom and textbooks. In the textbooks, while trapezoid was defined a quadrilaterals with at least one pair of parallel sides, prototype trapezoid shapes were heavily given to illustrate the concept. Additionally, when the researcher examined students' notebooks to understand how their teachers explained trapezoid, it was received that teachers gave only prototype trapezoids. To prevent the formation of prototype concept images, it is recommended that teachers need to focus on the definitions by giving examples (e.g. quadrilaterals with at least one pair of parallel sides) and non-examples (e.g. five-sided shape or a quadrilateral with no parallel sides) of the concept (Bills et al., 2006; Petty & Jansson, 1987).

Category 5: Different from the situation in *category 3&4* students (S9, S10 and S11) having high achievement level selected the quadrilaterals with no parallel sides as trapezoid in addition to all prototypes trapezoid shapes in *category 5*. Students noticed only similarity of the shapes in terms of position and visual appearance rather than having at least one pair of parallel sides. Additionally, when researcher asked them to explain the reasons why they selected that shapes as trapezoid, their explanations indicated that they also perceived trapezoid as a quadrilateral having non-equal sides. Students' concept images were limited only visual representation of trapezoid with personal definitions rather than a formation of formal definition and properties. In this way, they reached overgeneralization for special cases by extending their information to special cases in an inappropriate way rather than considering the parallelism property as a critical attribute. At this point, Chazan (1993) proposed various reasons for overgeneralization in mathematics such as use of an insufficient set of examples, prevalent concept images and beliefs. To overcome students' overgeneralizations, the definition and critical attributes of the shape should be stressed. In this regard, using geometry software might be useful since they involve dynamic manipulations such as dragging that preserves critical attributes of the shape in the hierarchical perspective (Erez &Yerushalmy, 2006).

Category 6: Lower (S2-S3) and middle (S4-S6) level students selected an irregular polygon (the shape of 16)

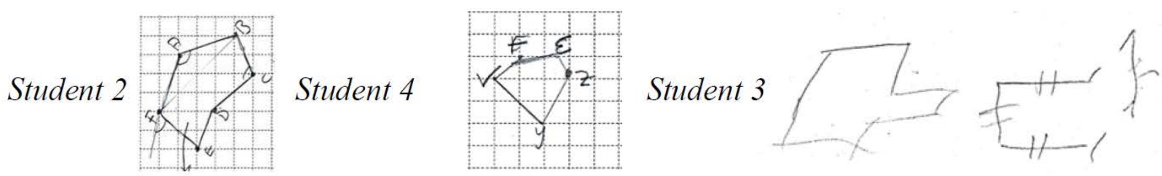


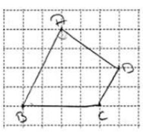
Figure 2: Students' drawings for the trapezoid in category 6

as a trapezoid with language-based concept images, which was supported by the results of similar national studies (Erşen & Karakuş, 2013; Türnüklü, 2014). Students in this category were not able to identify attributes of a trapezoid. Students' drawings were given in Figure 2 to reflect their concept images about trapezoid.

Furthermore, students written and verbal definitions indicated that they did not know the formal definition and properties of trapezoid (e.g. the number of sides) of a trapezoid. In Turkish language, the word of "*yamuk*" is used instead of "*trapezoid*" in all textbooks and teachers' instruction. However, "*yamuk*" is synonym and also means "*oblique*" in Turkish ordinary language. As a result, they made drawings under the influence of linguistic factors rather than focusing on definition and properties of the concept. Because language influences students' knowledge, ability and images about mathematical concepts (Monaghan, 2000; Silfverberg & Matsuo, 2008) teachers and curriculum developers must be more careful about whether they use both necessary and sufficient mathematical and linguistic structure for definitions of the concepts.

Category 7: This category showed the complexity of a lower level student's (S1) concept image and concept definition of trapezoid. Student's selections, definition and drawing of trapezoid contradicted themselves. Although student appeared to make a partial connection with inclusive relations by selecting parallelograms and rectangles as trapezoid, her drawing and definition of trapezoid revealed the imperfect nature of concept images in her mind.

As seen in Figure 3, although student stated there are no parallel sides of trapezoid, she drew a figure having a pair of parallel sides, but $|AB| \parallel |DC|$, and selected all parallelograms and rectangles as trapezoids, which showed student's insufficient knowledge about prerequisite concepts such as parallelism of two lines. Moreover, she drew a figure having four sides, but she selected irregular polygons having more than four sides. As a result, it can be inferred student had contradictive concept images and she did not know what trapezoid and its properties are. This result has reflected the importance of having prior knowledge about basic geometric concepts. Students' inadequate knowledge on geometric concepts may influence their concept images about quadrilaterals because many properties of quadrilaterals are based on basic geometric concepts such as parallelism and perpendicularity (Monaghan, 2000). For this reason, teachers should give more attention to whether their students obtained all required prior knowledge before introducing a new mathematical concept. As a final point, in-depth analysis of this study was limited to examination of thirteen seventh grade students' trapezoid selections. However, the idea of meta-classification of learners' selections can be applied and extended to other classes of quadrilaterals. Thus, the similarity and difference between the structures of meta-classification model may be analysed on the basis of different quadrilateral concepts. Furthermore, it will be more interesting to compare methods of teaching quadrilaterals classification in different mathematics curriculum.



S1's definition: Trapezoid is a figure. The sides of it have different length. It has no parallel sides.

Figure 3: S1's drawing and definition for the trapezoid

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TWG04

Posters

Signs and validation in the teaching of geometry at the end of compulsory schooling in France: A case analysis

Assia Nechache

Université Paris Diderot, Laboratoire de Didactique André Revuz, Paris, France, assia.nechache@hotmail.fr

In the first part of the communication, we present the research question, as well as the theoretical framework and methodology used to answer this question. In the second part, we describe a task given to students and provide a detailed analysis of some extracts from the session we observed. Based on the results obtained from the analysis, the last part provides a practical answer to our research question.

Keywords: *Geometry, geometric work space, validation.*

DESCRIPTION OF THE POSTER CONTENT

Section 1 – Introduction and the research aim

What is the place of signs in the validation process in geometry at the end of compulsory schooling? This question is the starting point for my communication. From the beginning of primary school, students are confronted with various geometric situations, such as: recognition and comparison of geometric forms, and reproduction of figures with drawing tools. Through these activities, they can identify invariants which are directly perceived on figures and then develop a conceptualization of geometric objects through characteristic properties. Then when they enter college, students are initiated to deductive reasoning, and thereafter engage in the transcription of this reasoning as a demonstration.

Section 2 – Research questions

- What is the place of signs in the process of validation in geometry in ninth grade?
- How does the teacher use the semiotic approach in the process of validation?

In French secondary education, teaching and learning validation (reasoning, argumentation, evidence and demonstration) represent a specific central mathematical issue (Balacheff, 1982). This teaching is initiated into the geometry field from the entry to college. Moreover, the teaching of validation is mainly based on the various registers of semiotic representation as a support to help build up a proof. This is where we use the theoretical framework. On one hand, it is necessary to consider the place of signs in the process of validation in geometry. On the other hand, we will analyze how the teacher uses the semiotic dimension in the process of validation.

Section 3 – Theoretical framework

Our analysis is mainly based on the theoretical framework of Mathematical Working Spaces (Kuzniak, 2011) and the use of the concepts of geometrical paradigms (Houdement & Kuzniak, 2006). With this framework, "style" validation adopted by teachers will be described, and particularly the way they integrate the semiotic approach in the discursive and deductive reasoning.

Section 4 – Data collection methodology

Our central question is the use of signs in the process of validation in geometry by secondary school teachers. To answer this question, we have chosen to analyze a ninth grade class session. In this session, students have to solve an exercise (chosen from the textbook used in class) using the concept of Thales' theorem and its reciprocal.

This session analysis is conducted in two steps: an "a priori" task analysis to identify paradigms at stake, as well as the validation process expected by the textbook authors. Secondly, using the geometric working spaces diagram (Kuzniak, 2013) the geometric work done during the activity is identified, allowing to see how the teacher leads the validation.

Section 5 – Results analysis

A specific working was highlighted during the validation process. In fact, the teacher uses the semiotic dimension (in a particular figure) and sets up a "maieutics" didactical contract in order to carry out his teaching project. We also notice that the mathematical work of validation put in place by the teacher himself is different than the one suggested in the textbook. This difference of validation work leads to a misunderstanding and mental blocks among students.

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Development of preservice teachers' content knowledge in geometrical transformations: A teaching experiment

Candas Uygan[1], Nilufer Y. Kose[2] and Dilek Tanisli[2]

1 Eskisehir Osmangazi University, Faculty of Education, Eskişehir, Turkey, cuygan@ogu.edu.tr

2 Anadolu University, Faculty of Education, Eskişehir, Turkey, nyavuzsoy@anadolu.edu.tr, dtanisli@anadolu.edu.tr

Geometrical transformations is one of the basic learning domains of geometry and mathematics teachers' content knowledge has a big impact on construction of middle school students' knowledge in this context. For this reason, the purpose of this study was to research preservice mathematics teachers' content knowledge and to design a teaching process including an appropriate scaffolding tool to provide participants to shift from informal thinking to formal thinking on geometrical transformations. According to results it was revealed that acetate could be designed as an effective scaffolding tool for developing the view of "bijective function", exploring relations, and making generalizations regarding the geometric transformations.

Keywords: Geometrical transformations, geometric habits of mind, preservice mathematics teachers.

INTRODUCTION

Unit of Geometrical Transformations, one of the basic learning domains of geometry, takes part in the Turkish Middle School Mathematics Curriculum including translation, reflection and rotation (MoNE, 2013). However, mathematics teachers' content knowledge about geometrical transformations has a big impact on the construction of middle school students' knowledge at this unit (Li, 2013). Starting from this point, we researched preservice mathematics teachers' content knowledge on geometrical transformations and what kind of scaffolding tools and learning activities can be designed with intent to improve their knowledge in this context.

METHODOLOGY

In this study designed as a teaching experiment research, participants are 28 preservice teachers enrolled in elementary mathematics education undergraduate program at a public university from middle region of Turkey. Two participants, coded as Ayse and Fatma, were selected as focal participants who would attend to clinical interviews among 28 participants.

Teaching experiment and design of a scaffolding tool

According to data obtained from pre-clinical interviews, it was seen that participants explained transformations with informal words as "moving", "folding" and they used trial-error strategies while solving problems. Starting from the constructivist approach, researchers discussed about appropriate learning tools that can provide participants to explore geometrical relations in transformations with informal experiments as "moving", "folding" and "rotating" the materials. Finally researchers decided that an acetate can be used as a scaffolding tool presenting both physical movement and one to one correspondence of points in the context. In addition to this, geometry activities were designed according to Geometric Habits of Mind (Driscoll et al., 2007) and the teaching process focused on visualization of geometrical transformations, exploring "function" meaning of transformations, generalizing strategies and discovering relationships between reflection and perpendicular bisector as well as between rotation and circle (Figure 1).

Figure 1: Using acetate for visualizing and analyzing reflection and rotation

RESULTS

During the posterior clinical interviews held, it was seen that the participants stated the role of "parameters" of geometric transformations within the context of "function". In addition, the participants, in their strategies, referred to their generalizations based on the isosceles trapezoid, radius, chord and perpendicular bisector. According to results, preservice mathematics teachers' content knowledge evolved from movement-based informal thinking to formal thinking in the context of geometrical transformations within teaching episodes including acetate mediated activities. Therefore it was seen that acetate could be designed as an effective scaffolding tool presenting both physical movement and one to one correspondence of points for developing the view of "bijective function", exploring relations, and making generalizations regarding the geometrical transformations.

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TWG05

Probability and statistics education

Introduction to the papers of TWG05: Probability and statistics education

Corinne Hahn[1], Andreas Eichler[2], Sibel Kazak[3], Aisling Leavy[4] and Caterina Primi[5]

1 ESCP Europe, Paris, France, hahn@escpeurope.eu

2 Kassel University, Kassel, Germany, eichler@mathematik.uni-kassel.de

3 Pamukkale University, Denizli, Turkey, skazak@pau.edu.tr

4 MIC-University of Limerick, Limerick, Ireland, aisling.leavy@mic.ul.ie

5 Neurofarba-University of Florence, Florence, Italy, primi@unifi.it

OVERVIEW

TWG gathered 37 participants from 16 countries. 22 papers and 11 posters were accepted.

We started with two ice-breaker activities. The first one allowed participants to introduce themselves. During the second activity, we collected answers to two questions and organized a debate from their answers. The questions and the main answers are gathered in Figure 1.

The papers were organized into five groups, each of which was managed and chaired by one of the co-leaders.

1) KNOWLEDGE AND ATTITUDES (AISLING LEAVY)

This subtheme explored the nature of the knowledge and understandings of statistics held by *particular groups* (middle grade students, pre-service teachers) when engaged in particular tasks and activities (comparing data sets, examining data displays), or communicated in textbooks.

It identified the attitudes of learners towards statistics and identified and/or categorized knowledge and attitudes using a variety of frameworks, scales and perspectives (qualitative content analysis, procedu-

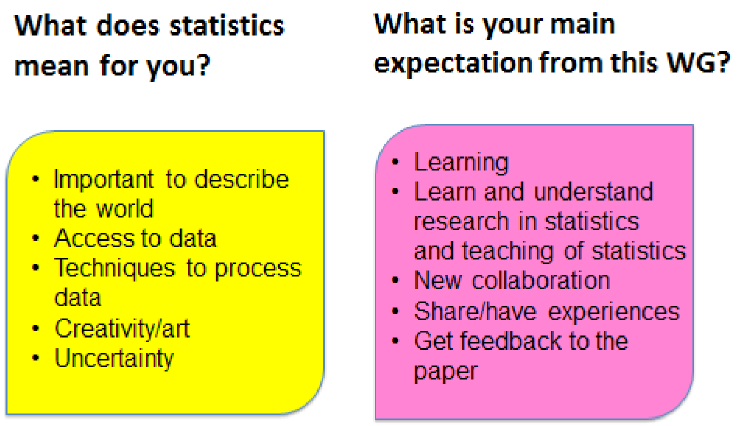


Figure 1: Ice-breaker activity responses

ral vs conceptual distinctions, Vergnaud's Theory of Conceptual Fields, Onto-semiotic approach, SATS).

2) PROBABILITY EDUCATION (CATERINA PRIMI)

This subtheme was about the relation between probability and statistics.

It addressed the questions of probabilistic reasoning and statistical literacy:

— How to integrate the different theoretical perspectives?

— When to start to teach probability?

— Which content is more suitable for different age groups?

A major question was the question of the use of simulation to teach probability and of its strengths and weaknesses.

3) INFORMAL STATISTICAL INFERENCE (SIBEL KAZAK)

The third subtheme was about informal statistical inference.

The discussion of the papers in this session focused on the key elements supporting students'/teachers' reasoning that leads to making informal statistical inferences based on data (cf. Makar, Bakker, & Ben-Zvi, 2011). These included:

— Knowledge of statistics: Statistical concepts, i.e., sampling, sample representativeness, sample size, uncertainty

— Knowledge of context given in the problem/task

— Design of learning environment: Task (i.e., use of physical objects, open-ended investigations), computer tools (i.e., TinkerPlots), students' age/level, teacher-student interaction, student-student interaction

— Transition from informal to formal statistical inference

4) STOCHASTIC THINKING AND TEACHERS (ANDREAS EICHLER)

The perspective of teachers in terms of teachers' knowledge and beliefs was the main topic in the fourth subtheme. This topic is important when researching the daily practice of teaching statistics and probability. Thus, teachers' knowledge and beliefs strongly impact on their instructional planning, their classroom practice and also impact on their students' learning. Not surprisingly, several papers at CERME9 refer to the teachers' knowledge and beliefs. Three papers concern the teachers' knowledge and beliefs before the teachers' classroom practice referring to teachers' knowledge about variability (Jacob, Lee, Tran, & Doerr), simulation approaches for informal inference (Lee, Tran, Nickel, & Doerr), and teachers' strategies for fostering decision making (Gonzales). One paper focuses on the relationship of teachers' espoused beliefs and the teachers' classroom practice (Bakogianni). Many important questions came out from the debates. For example, since research often yield shortcomings in teachers' knowledge, a challenge of research in mathematics education could be to develop a clear concept of what teachers have to know. Based on such a concept it could be possible to describe further the considerable corpus of the teachers' knowledge.

5) STATISTICS, CONTEXT AND REALITY (CORINNE HAHN)

The four papers presented in this group raise the question of the critical dimension of statistics education. These papers explore how we interpret statistical information from authentic contexts (newspaper articles mainly). The research works presented were conducted with a variety of audiences: teachers (Ozen & Cakiroglu), high school students in mathematics classrooms (Sturm & Eichler) or biology (Plicht et al.), tertiary students (Monteiro et al.).

They showed the influence of beliefs and affective aspects in the interpretation of statistical information. They raise the question of what can be considered as statistical as well as the question of the authenticity of the activities.

CONCLUSION

In this working group, we challenged current frameworks and perspectives on statistics education research.

Some important issues emerged from the discussions: The role of technology in statistical teaching and research, the relationship between statistics and probability, the goals for teacher education:

(What are the big ideas in teaching statistics?), the question of attitudes, motivation and efficacy in teaching statistics, the role of context in statistics, the question of methods and approaches to research in teaching statistics ...

We also anticipated directions of change: New role of technology, new topics (e.g., Big data...)

TWG05

Research papers

Studying the process of transforming a statistical inquiry-based task in the context of a teacher study group

Dionysia Bakogianni

University of Athens, Athens, Greece, dbakogianni@math.uoa.gr

Statistical inquiry, although beneficial for developing students' statistical thinking, constitutes a quite demanding task for the mathematics teacher. In this paper, I focus on the transformation of a statistical inquiry-based task. In the context of a study group of five secondary mathematics teachers, I investigate the various stages that the task passes through (set up phase, implementation phase, reflection) and explore factors that seem to frame the process of transformation. The emerging factors related to teachers' familiarity with the content and the teaching of statistics, students' prior statistical knowledge, classroom reality issues and issues related to the stochastic context of statistical problems are presented and further discussed.

Keywords: Statistics education, secondary mathematics education, teaching practice, task transformation, statistical enquiry cycle.

INTRODUCTION

Last years, statistics has increased its role in mathematics education (e.g. NCTM, 2000). The stochastic nature of statistics and the complexity of the underlying concepts and procedures require teaching approaches that promote active involvement of the students, personal investigation, experiences with meaningful contexts and explorations with real data (Jones et al., 2004).

Regarding the particularities of the statistical content, many researchers have turned their lens on the characteristics of teachers' knowledge related to the teaching of statistics (e.g., Burgess, 2007). Although the interest on teachers and teachers' knowledge for teaching statistics is growing rapidly in the statistics education research field, we still know very little on how this knowledge is transformed in the actual teaching. Rowland and his team (Rowland et al., 2009) define the *transformation* of knowledge as the ability of teachers to transfer what they know in a way that can be appropriate and accessible to students and they set transformation as one of the four categories in the *knowledge quartet* of mathematics teaching. The study of the process of transformation can contribute not only to the process of students' learning but also to the professional development of teachers. This study aims to get insight into the process of transformation in the context of teaching statistics and explore factors that seem to frame this process.

THEORETICAL BACKGROUND

Statistical tasks that promote inquiry in the classroom, engage students in a dynamic context where they explore, negotiate, make interpretations, search for solutions, communicate their results and proceed to informal inference. Such tasks are considered as essential learning experience for the students (Garfield, 1995).

Chapman (2013) argues that the way teachers transform what they know into tasks for their students, determines students' opportunities to engage in a meaningful way with mathematical activities and develop a conceptual understanding of concepts and procedures. Stein and Lane (1996) state that, the structural characteristics of a mathematical task (features, cognitive demands) can be changed through the process of transformation, often failing to achieve its learning potential. According to their findings, the transformation process of a mathematical task can be affected by various factors, such as teachers' learning goals, teachers' knowledge related to the content and to the students, factors related to the classroom norms and students' behaviour, which alter the high-demanding character of the task to a low one.

Regarding statistical inquiry tasks, Makar (2008) refers to some teaching skills that are almost rare in mathematics classroom but remain essential for statistical activities. Among these skills are: the ability to cope with ambiguity and uncertainty, the re-balance between teacher guidance and students' independence, the recognition of opportunities for learning in unexpected outcomes, flexible and creative thinking, deep understanding of disciplinary content and tolerance for periods of noise and disorganization. Considering the particularities of the statistical context, my aim is to study the process of transformation in the case of inquiry-based statistical tasks and identify factors that seem to affect the learning potential of such tasks.

I see the transformation process in the context of a study group. Many researchers emphasize the utility of study groups as form of professional development for teachers and refer to how the educational community can benefit from the research of study groups in order to get insight into how teaching practices are developed and changed (e.g., Arbaugh, 2003). I also explore the potentiality of this context to study the process of transformation in the case of statistical inquiry tasks, and investigate factors that seem to frame this process.

THE STUDY

In the context of a study group, 10 secondary mathematics teachers and two researchers worked together voluntarily for a period of two academic years (2012–2014). The groups' work focuses on the teaching and learning of statistics through a sequence of (a) deepening in a specific topic (b) design a task for the classroom-*design phase* (c) implement the task in the classroom-*implementation phase* (d) reflect on this experience-*reflection phase*. The role of the researchers in the study is mainly to finalize the groups' meetings' agenda and support the teachers by giving them particular tasks, materials and resources. Collaboration and interaction among the teachers was encouraged in all phases of their work.

In this paper I focus on five teachers who worked as a team on the design and implementation of the statistical investigation cycle in the mathematics classroom.

The participants of the study

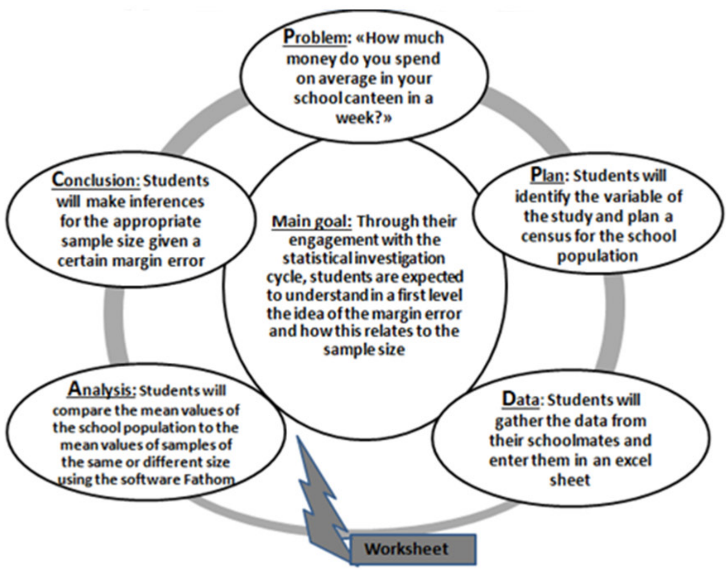
All five teachers have a mathematics degree and postgraduate studies in mathematics education. Dinos, is a practising teacher with 10 years of teaching experience while the others, Sofi, Cloe, Anna and Ersi are novice teachers with little teaching experience. Although the practising teacher has no particular experience in teaching statistics, he is quite familiar with the statistical content due to personal involvement in statistical methods, while novice teachers' involvement with the statistical content is limited to their university experience. Due to the lack of teaching experience and to their unfamiliarity with the statistical content, the four novice teachers consider Dinos as a team leader.

The work of the group

The five teachers were asked to design a task for the classroom and implement it. For the design, the teachers were supposed to build on the former phase of the study groups' work in which they had focused on sample and sampling notions and their relation to the concept of probability. For this phase the teachers met seven times, each meeting lasted about two and a half hours. The author was present in all meetings, and the wider group met in the four of the seven meetings (10 teachers and 2 researchers), exchanging views and giving support to one another.

The task was intended for the 8th Grade students with no background in statistics. The main learning goal, as set up by the teachers, was to help students understand in a first level the idea of the margin error in statistical results and how it relates to the sample size. The teachers decided to engage students with all the steps of a statistical investigation cycle based on the PPDAC (Problem-Plan-Data-Analysis-Conclusions) model (see Wild & Phannkuch, 1999) transformed as seen in Figure 1. As seen in the picture, a worksheet was designed between the steps of Data and Analysis. This worksheet aimed to introduce students to the concepts of mean value, sample and margin error that were necessary for the last steps of the statistical investigation.

The task was implemented in Dinos' classroom. Although all the members of the team were present in the implementation, Dinos had the leading role while the others (including the author) were mainly observers, taking field notes and supporting the various stages of the task. The implementation covered 5



Mathematical aspects

- Ability to calculate the mean value
- Ability to construct and read tables and graphs
- Ability to find from a given set of numbers all possible combinations of given size subset
- Conceptual understanding of the concept of probability and connecting it with the sample concept

Stochastic aspects

- Understand sources of variability in a sampling method
- Organize and manage real data
- Making interpretations of the data displays in relation to the reference problem
- Use statistical tools to answer to posed questions
- Make (informal) inferences for the reliability of a given sample

Figure 1: The investigation cycle as performed by the teachers and the related skills

didactical hours (1hour for PPD steps, 2hours for AC steps and 2 hours for the worksheet between).

After the implementation the group met two more times for feedback and reflection. The first time the discussion was based on their impressions regarding students' reactions and an overall assessment for the teams' work. In the second meeting, the teachers were asked to analyze more systematically the characteristics and the cognitive demands of the designed task and to evaluate the consistency with the initial goals and demands in the implementation phase. The group had a three-hour meeting discussing on their analysis and reflecting on their practice.

The data of the study consist of the audio and video recordings from the seven meetings' regarding the design phase, the field notes from the implementation and audio recordings from the two meetings of the reflection phase.

Analysis of the data

The first analysis of the data was conducted in collaboration with the teachers. We used the Stein's & Lane's (1996) framework for the relationship between instructional tasks and students' learning outcomes. Stein & Lane identify three stages that instructional tasks pass through: (a) instructional task as represented in curricular/instructional material (b) instructional task as set up by teachers in the classroom and (c) instructional task as implemented by students in the classroom. In each of these stages, they define two interrelated dimensions, the task features and the cognitive demands. The task features refer to characteristics of the task that are related to the engagement of students' reasoning, thinking and sense making, such as the extent to which the task lends itself to multiple solution strategies, the extent to which the task encourages multiple representations or the extent to which the task demands explanations from the students. The cognitive demands refer to the kind of thinking processes contained in the task, such as use of complex, non-algorithmic thinking or use of procedures with connection to underlying conceptual ideas.

Following this framework as a guide of analysis, we identified the task's features and cognitive demands and analyzed them in the set up and the implementation stage. For the set up, based on the audio recordings of the groups' meetings and the final worksheet intended for the students, we made a characterization of the extent to which these features and cognitive demands were supported by the design of the task. This characterization was mainly on a Small/Big note. Afterwards, using the field notes from inside the classroom, we studied the consistency of the design in the implementation phase. For example, if we note as Big the extent to which the task demands explanations from the students, we then explored if, through the implementation, this demand remained in a big extent or declined or supported more than expected. For every change identified between the set up and the implementation we further discussed possible factors that may influence such a change. The analysis of the task was part of the study's group work.

	Features and Cognitive Demands as identified in the task designed by the teachers	Extent of demand in the various steps of the PPDAC cycle SET UP PHASE					Extent of demand in the various steps of the PPDAC cycle IMPLEMENTATION PHASE				
		P	P	D	A	C	P	P	D	A	C
FEATURES	multiple representations				B					√	
	explanations from students	B	B	B	B	B	√	√	√	–	–
	active engagement (self-action, decision making, exploration, interpretation)	S	S	B	S	S	√	√	+	–	–
COGNITIVE DEMANDS	use of statistical tools in order to answer a question				B	B				√	–
	connection to mathematical concepts and procedures				B	B				√	–
	recognition of stochastical aspects		B	B	B	B		√	+	√	√
	connection to the context of the problem	B	B	B	B	B	√	√	√	–	–
	use of statistical thinking and reasoning			B	S	S			√	–	–
	Explanation of the used symbols: B: Big extent of demand S: Small extent of demand	√ : the extent of demand was consistent to the design – : the extent of demand was declined +' : the extent of demand was enhanced									

Table 1: Features and Cognitive Demands in the various phases of the task

In a second analysis, using the transcriptions of the groups' discussions from the design and the reflection phase, I focused on the process of transformation. The data were analysed with respect to the two dimensions as explored in the first part of the analysis, namely tasks' features and cognitive demands, focusing on factors that seem to frame the observed consistency or shift on the various elements of the task.

FINDINGS

The results from the first analysis, which was conducted in collaboration with the teachers, are described on Table 1.

As seen on the above table, in the set up stage the teachers aimed at large extent features and cognitive demands that set the task as a high-level activity for the students.

The fact that the teachers worked as a team during the design phase combined with the fact that all of them supported the implementation phase, were conditions that could lead to the assumption that the demands of the activity would remain in a high-level through the implementation, contrariwise, things proved to be different. In the most cognitive demanding parts of the PPDAC, the most of the task demands show a decline in the extent to which they were sustained. Since the analysis of the features and demands of the task is not central in this paper, a more detailed description of this phase's results is available in our research report, regarding teacher's reflection on the of PPDAC in the mathematics classroom (see Efstathiou et al., 2014).

The group's discussions from the set up and reflection phase helped me to get insight into factors that seem to frame the transformation process. The factors emerged from the analysis regarding the two dimensions under study, tasks' features and cognitive demands, were grouped in terms of whether they indicate consistency or shift. In Table 2 there is a summary of the main factors as appeared in each group.

Shifts on tasks' features	Shifts on tasks' cognitive demands: Shifts on mathematical demands	Shifts on tasks' cognitive demands: Shifts on stochastic demands	Maintenance of learning potentials
• students prior knowledge • time limitations • lack of access in a computer room so that students can experiment on an appropriate software tool	• students' prior knowledge • difficulty in the management of alternations from the mathematical to the realistic context • inappropriate ammount of time	• overemphasis on mathematical ideas • lack of balance between teacher's guidance and student independence • inappropriate amount of time alloted to the various parts of the task • disconnection of the reference context • inappropriateness of tasks and examples • time limitations	• strengthening of teachers' subject matter knowledge • teachers' flexibility and availability to act in the moment • collaboration among teachers

Table 2: Factors that frame the process of transformation in the case of a statistical investigation cycle

Shifts on task features

On the left column of the table, I refer to factors that seem to cause a decline into the features of the task through the process of transformation. The first factor that seems to be essential is the prior knowledge of students. The fact that students did not have any previous statistical knowledge, resulted in the teachers' decision to limit students' engagement in several parts of the task at the set up phase (e.g., on the decision of the statistical variable or the method of the investigation). The same factor compared with time limitations led to a decline on giving students space for explanations and active engagement. Moreover, in our case students did not have access to a computer room and this proved to be a crucial factor. According to the design of the task, the teachers were supposed to counterbalance this restriction by giving time to students to discuss and negotiate throughout the activity on the software, but due to time limitations there was no appropriate time for interaction and discussion causing a further decline in the engagement feature. Characteristically, Dinos admitted in his a posteriori analysis: "We didn't prevent students from being observers. I tried to engage them by asking questions, but the time was too limited, I am not sure that they could easily follow what was happening, the majority of them probably couldn't."

Shifts on task's cognitive demands

The factors regarding cognitive demands were separated into two subcategories, the cognitive demands related to mathematical issues and the cognitive demands related to stochastic issues (see Figure 1). The mathematical issues of the tasks refer to the first two demands as displayed on Table1 while the stochastic refer to the other three. The lack of students' prior knowledge in statistics resulted in dedicating time for constructing the necessary background for the goals of the task. Nonetheless, students seem to have coped well with the mathematical demands when they worked on the mathematical context but had difficulties when they needed to transfer their mathematical knowledge to the reference context of the problem. Moreover, due to time limitations there was not enough time for the teachers to make the appropriate connections and students remained to trust more their intuitions than their mathematical knowledge to make conclusions for their problem.

The cognitive demands related to stochastic issues proved to be the most challenging to maintain in a high-level during the process of transformation. First, the teachers' overemphasis on the mathematical ideas seems to act as barrier in their aim to help students develop statistical thinking. Characteristically, Dinos, in the discussion through the reflection phase, admitted: "In my view I could say that we succeed in helping students connect the sample notions with the concept of probability, but, the development of stochastic thinking proved to be difficult. I am not sure that we succeed in that."

In addition to this, another factor that seems to play a crucial role is the disconnection of the reference context of the problem. The disconnection of the reference context results in disorientating students from the underlying problem and they tend to think more in terms of what is right and what is wrong and less in terms of what is reasonable and acceptable. Although the teachers tried to restrict the effect of disconnection, they didn't spend adequate time on the necessary connections causing a decline in the cognitive demands of the task. The allocation of available time in the various parts of the task, is also an emerged factor in the transformation process. The above extract is indicative of the teachers' reconsidering the time allocation:

Dinos: If I were to implement it again, I would spend most of the time on the experimentation with the software. Students need to explore themselves, to see, to explore more and more samples. This is how they can start thinking in a different way.

Cloe: I totally agree with Dinos. We spent a lot of time in talking about combinations and we lost our goal. The point is the statistics. It is important for students to have the opportunity to make their own explorations. I don't think that the way we used the software was rather beneficial for the students. They needed more time in working on the software.

In the extract above, another factor is apparent, the balance between teacher guidance and student independence. The restriction of students' independence was something that results on a decline of the statistical thinking demand from the set up phase, and due to time limitations a further decline was caused in the implementation phase. The teachers seem to develop an awareness of the difficulty to encourage students' statistical thinking, and also an appreciation for the role of the exploration and personal engagement with data in this respect.

Another factor is the appropriateness of chosen examples. In this case, the teachers, in order to help students understand the variety of possible samples in a given population and understand better the meaning of the margin error of the mean in a sample with a given size, gave students an example of population with size 5 and asked them to explore all possible samples with size 3, but according to Dinos reflection: "this example gives students the impression of a certainty that does not exist in bigger populations, actually framing their stochastic thinking."

Finally, as in the previous category, time limitations seem to be dominant in this case as well. Students proved to need much time in order to develop statistical thinking, time for more explorations, more discussions and inquiry processes, but the time available was not the appropriate. For their attempt to challenge students in the way they use proportional reasoning when they think about samples they discussed:

Sofi: I think that we manage to make them conflict with the idea of proportionality.
Dinos: Yes, we actually did it, but, then what? You could see that there is nothing to come after. When they rejected the proportionality, they answered almost randomly. The development of an alternative thinking, more distributional, needs time, it requires more data explorations by the students themselves.

Although the teachers in the design phase had considered proportional reasoning as a potential obstacle, they didn't anticipate students' difficulty in proceeding to distributional reasoning. This difficulty appears to be unexpected and hard to handle.

Maintenance of learning potentials

In this last category I included factors that seem to act beneficially in the process of transformation. The first factor is the strengthening of teachers' subject matter knowledge. In the phase of the task design, the teachers discussed a lot on the statistical concepts and procedures, they read textbooks and research reports regarding the investigation cycle and they asked for help and clarifications in many instances. This route helped them to identify specific learning goals, to recognize the underlying concepts in the various steps of the PPDAC and to support challenging learning opportunities for the students. Moreover, the collaboration among the teachers seems to act as catalyst for the process of transformation. The interactions among them and the feeling of support in every step of the design and the implementation, seem to help them form and accomplish high-levels of statistical activity for the students. It is indicative that in the reflection phase, all the teachers mentioned that they wouldn't try such an activity if they didn't have the support of their colleagues.

Another factor that seems to be essential is teachers' flexibility and availability to act in the moment. In our case, the fact that the task was mainly implemented by Dinos, who is an experienced teacher with familiarity to the statistical content, had a great impact in the task's features. For example, in the extract below the teachers discuss on the questions that would be posed to the students in the part of the data analysis:

Anna: This part requires a lot of guiding by the teacher. Let's write down the questions step by step.
Sofi: We don't need to be so strict with the questions. It will be better to be more open, to give more space to students.
Anna: Don't we write the questions?
Sofi: Don't forget that Dinos is the one who will manage this. Dinos can handle it.

From this extract, we can see a feeling of insecurity regarding open questions. The fact that the team members seem to trust Dinos' ability to act in the moment affected the engagement of students during the implementation.

CONCLUSION

This study helped us to get insight into the transformation process of a statistical investigation task. Regarding the context of statistics, the management of the uncertainty proved to be, not only a difficult learning goal to achieve, but also a significant teaching challenge. The factors that appeared to affect the transformation process constitute an amalgam of teachers' subject matter knowledge, teachers' skills and abilities to confront with uncertainty and manage classroom discussions and teachers' knowledge of students in terms of students' prior knowledge and of ways students can be assisted in developing stochastic thinking as well. Many of the required skills and abilities, although determinant in statistical inquiry as mentioned by Makar (2008), are rare in mathematics classrooms. The collaboration among the teachers seems to encourage them to support statistical inquiry and promote high cognitive demands in the mathematics classroom. Through collaboration and interaction, teachers have the opportunity to understand deeper the statistical concepts and procedures, to define specific learning goals regarding the statistical content, and to be aware of learning difficulties that are related to them. Moreover, the support among the teachers in the implementation phase, and the collaborative management of classroom, can result in an increase regarding the use of open questions and inquiring strategies by the teachers.

On the other hand, an overemphasis on mathematical ideas and disconnections of the reference context of the problem, can result in a decline of the stochastic demands of the task. Furthermore, limitations regarding the time allocation in the various parts of the task, the use of inappropriate examples and the imbalance between teacher's guidance and students' engagement, can also cause a decline in the task's cognitive demands. Other factors that appeared to play a crucial role on shifts in the characteristics of the task are students' unfamiliarity with statistical concepts compared with the limited time available for exploration and development of conceptual understanding. Lack in access of statistical software, that allows students to interact with the data, is an additive obstacle to helping students develop stochastic thinking.

Stein's and Lane's (1996) framework proved to be useful not only as a tool for exploring the transformation process but also as a tool for teachers' reflection. The analysis of the characteristics of the task and the investigation of shifts through the various stages that instructional tasks pass, seemed to help the teachers evaluate their practice, identify sources of complexity and consider alternative strategies to develop more effective teaching.

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Gender differences in attitudes toward statistics: Is there a case for a confidence gap?

Francesca Chiesi and Caterina Primi

University of Florence, Florence, Italy, francesca.chiesi@unifi.it

Female students tend to underestimate their abilities and to have more negative attitudes toward quantitative disciplines when compared to male students. In teaching statistics this concern has to be taken into account since it may create an obstacle for learning. The aim of the present study was twofold: to test if women had less confidence and more negative attitude than men regardless their actual abilities, and to investigate if achievement in statistics was affected by these factors controlling for gender. Results showed that women did not differ in their abilities but showed less confidence and more negative attitudes when compared to men. Moreover, confidence and attitude played a role on achievement in women but not in men. The importance of enhancing attitudes toward statistics in female students was discussed.

Keywords: Statistics education, attitudes toward statistics, gender differences.

INTRODUCTION

Since statistical literacy is a key ability expected of citizens in information-laden societies, and is deemed a necessary component of adults' numeracy and literacy (Gal, 2002), it is crucial to develop students' statistical learning. To accomplish this goal, i.e., to provide students with tools for understanding data-related arguments, building intuition about data, and making reasoned judgments and decisions in their professional and private lives, statistics has been included into a wide range of university programs. Thus, in many countries students progressing towards a degree other than statistics have to pass at least a compulsory statistics exam, and concerns related to teaching and learning statistics are internationally shared (for a review see, e.g., Zieffler et al., 2008).

In teaching statistics in postsecondary education gender might be a relevant factor since, although all students experience stress and difficulties in learning statistics, female students are more likely to feel uneasy in dealing with this discipline. Indeed, they tend to underestimate their abilities and to have more negative attitudes toward quantitative disciplines when compared to male students. For instance, referring to mathematics, it has been shown that women' self-efficacy is consistently and significantly lower than those of men (e.g., Pajares & Miller, 1994; Stevens, Wang, Olivarez, & Hamman, 2007) regardless their actual ability (Quest, Hyde, & Linn, 2010). This phenomenon, called the *confidence gap* (Sadker & Sadker, 1994), emerges during the high school years and impacts on the subsequent scholastic/academic choices, i.e. female students tend to avoid scientific secondary and postsecondary degrees and prefer non-mathematical ones (e.g., Halpern et al., 2007).

Nonetheless, female students enrolled in non-mathematical degrees such as Psychology, Education, and Health Sciences encounter statistics courses in their programs. Thus, it is likely that their *confidence gap* might affect their approach to a discipline like statistics. Indeed, it has been widely demonstrated that, along with cognitive components (e.g., mathematical knowledge and general scholastic background), non cognitive factors play a determinant role in learning statistics (Chiesi & Primi, 2010; Harlow, Burkholder, & Morrow, 2002; Nasser, 2004; Schutz, Drogsz, White, & Distefano, 1998; Tempelaar, Van Der Loeff, & Gijselaers, 2007; Tremblay, Gardner, & Heipel, 2000; Wisenbaker, Scott, & Nasser, 2000). Among the non cognitive factors, large attention has been paid to the attitudes toward statistics that include a self-confidence dimension (e.g., the trust in one's own knowledge and skills when applied to statistics) along with measures of feelings concerning statistics, and beliefs about the usefulness and the difficulty of statistics.

Starting from these premises, the present study aimed to investigate the relationship between attitudes toward statistics and achievement in male and female psychology students attending introductory statistics courses. In detail, the aim of the present study was twofold: to test if females showed more negative attitudes toward statistics (and, specifically, less self-confidence) than males, and to investigate if attitudes affected their achievement. These relationships were investigated controlling for the students' actual ability, i.e., mathematical basics deemed necessary for introductory statistics courses.

As refers to the first aim, literature on gender differences in attitudes toward statistics reports contradictory results. Some authors reported that men expressed more positive attitudes toward statistics than women (e.g., Auzmendi, 1991; Tempelaar & Nijhuis, 2007). Others found no gender differences (e.g., Estrada, Batanero, Fortuny, & Díaz, 2005; Judi, Ashaari, Mohamed, & Wook, 2011; Martins, Nascimento, & Estrada, 2011; Schau, Stevens, Dauphinee, & Del Vecchio, 1995; Wisenbaker et al., 2000). Some others have reported more positive attitudes for women (e.g., Mahmud & Zainol, 2008; Rhoads & Hubele, 2000). Assuming that those differences might be partially related to the sample characteristics (engineering students, economic students, psychology students, pre-service teachers), and referring to the above mentioned literature on the *confidence gap*, we hypothesized that psychology female students had more negative attitude toward statistics than their male counterpart, and, specifically, they show less confidence in their knowledge and skills when applied to statistics.

From a different perspective, i.e., the "stereotype threat" theory (Spencer, Steele, & Quinn, 1999), it could be argued that since psychology female students are not confronted with many male students, they should have higher confidence and more positive attitudes. Indeed, the "stereotype threat" refers to the concern that is experienced when one feels at risk of confirming, as self-characteristic, a negative stereotype about one's group. Given the stereotype concerning gender and math ability - that propose that women have less mathematical aptitude than men-, several studies suggest that gender differences in math performance occur in environments in which gender identity is salient, e.g., a class with a majority of male students is sufficient to create a threatening environment for female students. Nonetheless, in this stage of our research, we were interested in measuring students' competence and attitudes at the beginning of the course in order to highlight the way in which the students start to deal with statistics. As such, in line with previous studies (e.g., Chiesi & Primi, 2010; Nasser, 2004; Schutz et al., 1998) we assume that their initial attitudes do not depend on the current educational context, but on their previous experience with quantitative disciplines during the high school. Additionally, as reported by Hyde and colleagues (2008) and Quest and colleagues (2010) for mathematics, we hypothesized that, regardless gender similarities in abilities, women had less confidence and overall more negative attitude toward statistics than men.

Concerning the second aim, we investigated the relationships among mathematical competence, attitudes and achievement in men and women. Referring to literature on cognitive and non cognitive factors influencing statistics achievement, we hypothesized that mathematical knowledge has an effect on achievement (e.g., Chiesi & Primi, 2010; Harlow et al., 2002; Schutz et al., 1998; Tremblay et al., 2000; Wisenbaker et al., 2000), and, referring to the literature on female students' attitudes toward the quantitative disciplines (e.g., Hyde et al., 1990; McGraw et al., 2006; Pajares & Miller, 1994; Stevens et al., 2007), we hypothesised that attitudes, and especially self-perceived competence, might impact on women' performance differently than in men.

METHOD

Participants

Participants were 179 psychology students enrolled in an introductory statistics course at the University of Florence in Italy. The course was compulsory for first year students that represent the majority of the sample (91.3%). The course was scheduled to take place over 10 weeks, at 6 hours per week (for a total amount of 60 hours). It covered the usual introductory topics of descriptive and inferential statistics, and their application in psychological research. During each class some theoretical issues were introduced followed by examples and exercises. Students were requested to solve exercises by paper-and-pencil procedure (no computer package was used), and then solutions were presented and discussed.

Participants' age ranged from 19 to 54 with a mean age of 22.0 years (*SD* = 5.26). Female students were 113 (*mean age*= 21.9, *SD*=4.77) and male students were 68 (*mean age*= 22.7, *SD*=6.24). All students participated on a voluntary basis after they were given information about the general aim of the investigation (i.e., collecting information to improve students' statistics achievement).

Measures and procedure

In a previous study (Chiesi & Primi, 2010), we provided evidence that some mathematical basics are needed for introductory courses and to measure them we developed the *Prerequisiti di Matematica per la Psicometria (PMP)* scale (Galli, Chiesi, & Primi, 2011). The contents were defined on the basis of the basic mathematics abilities requested to solve descriptive and inferential statistics problems. The PMP is composed of 30 multiple choice (one correct out of four alternatives) questions including fractions, set theory (inclusion-exclusion, and intersection concepts), first order equations, relations (between numbers that range from 0 to 1 and numbers expressed in absolute values), and probability (base-rates, independence notion, disjunction and conjunction rules). Fractions are employed both in descriptive and inferential statistics tasks (e.g., to compute the standard deviation, as well as the *t* or *z* values). Equations are required, for instance, in the standardization procedure and in regression analysis. Establishing relations between numbers is necessary to compare the computed and critical value in the hypothesis testing. Set theory principles help to understand probability rules, and basics of probability are the prerequisite of the hypothesis testing. A single composite, based on the sum of correct answers, was calculated (range 0–30).

Attitude toward statistics was measured administering the 28-item version of the *Survey of Attitudes toward Statistics (SATS)* (Schau et al., 1995; Italian version: Chiesi & Primi, 2009). We chose SATS since it was proved to be invariant respect to gender (Hilton, Schau, & Olsen, 2004), i.e., equally suitable for male and female respondents, and because it assesses four attitudes dimensions including a self-confidence dimension. In detail, *Cognitive Competence* subscale (6 items) measures students' attitudes about their intellectual knowledge and skills when applied to statistics (e.g. "I can learn statistics"); *Affect* subscale (6 items) measures positive and negative feelings concerning statistics (e.g. "I feel insecure when I have to do statistics problems"); *Value* subscale (9 items) measures attitudes about the usefulness, relevance, and worth of statistics in personal and professional life (e.g. "Statistics is worthless"); *Difficulty* subscale (7 items) measures students' attitudes about the difficulty of statistics as a subject (e.g. "Statistics is a complicated subject"). The scale contains Likert-type items using a 7-point scale ranging from *strongly disagree* to *strongly agree*. Responses to negatively scored items were reversed, and then scores were obtained for each subscale, with higher ratings representing more positive attitudes. In the present sample, Cronbach's alphas for the four subscales were: *Cognitive Competence* = .76, *Affect* = .80, *Value* = .74 and *Difficulty* = .65.

Students were administered the SATS and the PMP in this order during the first day of class. The questionnaires were introduced briefly to the students and instructions for completion were given. Answers were collected in paper-and-pencil format and the time needed to complete them ranged from 25 to 40 minutes.

To measure achievement, we employed a midcourse test developed to monitor learning during the course and administered toward the end of the fifth week of the course. The test was composed as follow. Students were given a data matrix (3–4 variables, 10–12 cases) and referring to it they had to solve two problems (e.g., report frequency and percentage distributions, construct a two-way table, draw graphs, compute central tendency, spread and association measures) by paper-and-pencil procedure without the support of a statistics computer package. Additionally, they had to answer two open-ended questions (e.g., to define the measures of central tendency, to interpret the meaning of z values or percentiles). All the items pertained to contents covered in class. The test was timed (1 hour) and books and notes were not allowed to be used. For each problem the score ranged from 0 to 3: 0 = totally incorrect or not solved; 1 = partially solved; 2 = almost solved; 3 = completely solved. For each question the score ranged from 0 to 2: 0 = totally incorrect or no answer; 1 = partially answered; 2 = correctly answered. Two assistant teachers, preliminary trained, scored the tasks. The scores were aggregated in a single measure (range 0–10).

RESULTS

Gender difference in mathematical knowledge, attitudes, and achievement. All descriptives are reported in Table 1. No gender differences were found in mathematical knowledge ($t(177) = 1.10$, $p = .271$) and in the *Value* scores ($t(158) = -0.45$, $p = .97$). In contrast, differences were found in *Cognitive Competence* ($t(158) = 2.05$, $p < .05$, $d = .31$), *Affect* ($t(158) = 3.68$, $p < .001$, $d = .55$), and *Difficulty* scores ($t(158) = 2.13$, $p < .05$, $d = .32$) indicating that men were more confident about their own capabilities, had more positive feelings toward the discipline, and deemed the discipline less difficult than women. Concerning achievement, differences between male and female students were not statically significant ($t(155) = 1.38$, $p = .17$)

Gender difference in the relationships among mathematical knowledge, attitudes, and achievement. We investigated if mathematical knowledge and attitudes were related to achievement looking at the Pearson product-moment correlations separately in male and female students (Table 2). Males' achievement was related to mathematical knowledge whereas it was not related to attitudes toward statistics with the only exception of a small correlation with the *Cognitive Competence* dimension. Females' achievement was related to mathematical knowledge and to attitudes toward statistics with the exception of the *Value* dimension.

Since in both gender groups mathematical knowledge was related to attitudes and achievement, the correlations between them might be biased. Therefore, partial correlations were computed controlling for mathematical knowledge. For the male students the correlation with the *Cognitive Competence* score was not significant ($r = .04$, $p = .83$) once the effect of mathematical knowledge was controlled. Instead, for the female students the correlations between achievement and *Affect* ($r = .24$, $p < .05$) was still significant, as well as the correlations between achievement and *Cognitive Competence* ($r = .37$, $p < .01$).

Mathematical knowledge, Affect, and Cognitive Competence as predictors of achievement in female students. To establish the relative impact of mathematical knowledge, *Cognitive Competence*, and *Affect* on achievement in women, regression hierarchical analyses were run (Table 3). In the first step, the mathematical knowledge, and, in the second step *Cognitive Competence* and *Affect* were added as predictors. The

	Males		Females	
	M	SD	M	SD
PMP	23.71	4.34	22.90	4.94
SATS-Cognitive Competence	31.59	6.67	29.70	5.43
SATS-Affect	26.00	6.93	22.27	6.63
SATS-Value	46.00	8.20	46.05	7.54
SATS-Difficulty	25.91	5.62	24.30	4.49
MT	6.34	2.25	5.77	2.19

Table 1: Means and standard deviations of mathematical knowledge (PMP), attitudes toward statistics (SATS) subscales, and achievement (midcourse test = MT) for Males and Females

	Males					Females				
	1	2	3	4	5	1	2	3	4	5
1. PMP										
2. SATS-CC	.40**					.35**				
3. SATS-A	.13	.81**				.33**	.72**			
4. SATS_V	.22*	.54**	.55**			.07	.39**	.44**		
5. SATS-D	.34*	.66**	.74**	.38*		.26*	.55**	.68**	.25*	
6. MT	.46**	.22*	.05	-.06	-.01	.52**	.48**	.36**	.07	.25*

** Correlation is significant at the 0.01 level (2-tailed) * Correlation is significant at the 0.05 level (2-tailed)

Table 2: Intercorrelations between mathematical knowledge (PMP), attitudes toward statistics (SATS subscales: A= Affect; CC= Cognitive Competence; V=Value, D= Difficulty), and achievement (midcourse test = MT) in Males and Females

two dimensions of attitudes toward statistics were kept separate to avoid multicollinearity due to the high correlation between them (r = .72) that makes difficult to estimate the single contribution of each one when kept together in the same analysis. Results showed that the mathematical knowledge was a significant predictor (F (1,79) = 29.35, p < .001) that accounted for the 27% of the variability in achievement. Adding separately at the two dimensions of attitudes, results showed that the model was significant both when *Cognitive Competence* was included (F (2,78) = 22.78, p < .001), and when *Affect* was included (F (2,78) = 17.53, p < .001). In both cases, they contributed significantly to explain achievement along with mathematical knowledge. Nonetheless, *Affect* explained only an additional 4% whereas *Cognitive Competence* accounted for an additional 10% of the variability in achievement.

DISCUSSION

The current study aimed at investigating the interplay among previous competence, attitudes, and achievement in statistics and taking into account gender-related differences. In detail, the first aim was investigating gender differences in attitudes toward statistics and the relationships with mathematical competences. As expected (and in line with Auzmendi, 1991; Tempelaar & Nijhuis, 2007), when compared to men, women were less confident about their own ability in dealing with statistics, perceived it more difficult, and had more negative feeling about the discipline. However, the present results shows that these differences were not related to different mathematical knowledge since any gender differences were detected in mathematical basics deemed necessary for introductory statistics courses. The second aim concerned the predictive role of mathematical knowledge and attitudes toward statistics on achievement. Confirming previous results, the *Cognitive Competence*, *Affect* and *Difficulty* components, but not the *Value* one, were related to achievement (Tempelaar et al., 2007; Wisenbaker et al., 2000), as well as mathematical knowledge (e.g., Chiesi & Primi, 2010; Harlow et al., 2002; Schutz et al., 1998; Tremblay et al., 2000; Wisenbaker et al., 2000). Concerning gender differences, as expected, gender induced changes in the relationships between achievement and its predictors. More in detail, mathematical knowledge was the only significant predictor for men, whereas – along with mathematical knowledge – the *Affect* and *Cognitive Competence* attitude components had an additional effect on performance for women. That is, low self-perceived abilities accounted for worse achievement, and more negative affect was associated to lower achievement (and vice versa). In particular, *Affect* had a smaller effect than *Cognitive Competence*.

Given these findings, it becomes important to identify methods for counteracting female students' tendency to underestimate their competence and to have negative feeling toward the discipline. In this way it could be possible to promote a better approach to the discipline and, as a consequence, a better performance. Specifically, it might be useful to arrange activities during the course in which students could realize that they can master the topics, develop confidence, perceive the subject easier, and reduce negative feelings toward the discipline. Thus, future researches might be conducted collecting repeated measures of attitudes from the beginning to the end of the course in order to monitor changes that might be due to the course itself and to specific activities implemented by teachers and tutors.

The present study has some limitations that we have to take into account when interpreting the results. First, we used the midcourse test score as indicator of achievement. For the purpose of the present investigation, we deemed this measure as an adequate indicator but it could be interesting to take into account the final examination's grades to better ascertain the role of attitudes on general achievement. In doing that, it should be necessary, as stated above, to monitor the changes in the attitudes that could occur during

Predictors	β	t	p	R^2	R^2 *change*	*F change*	p
Step 1: *PMP*	.52	5.42	<.001	.27	-	-	-
Step 2a: *PMP*	.45	4.52	<.001	.31	-	-	-
+ *SATS-A*	.21	2.10	<.05		.04	4.43	<.05
Step 2b: *PMP*	.40	4.20	<.001	.37	-	-	-
+ SATS-CC	.33	3.48	<.01		.10	12.09	<.01

Table 3: Hierarchical regression analyses on statistics achievement for Females. PMP = mathematical knowledge, SATS-A = Affect, SATS-CC = Cognitive Competence

the course, i.e., to have a measure of the student's attitudes at the end of the course, just before to take the final exam. Additionally, given the relevance of the self-confidence dimension (i.e., the trust in one's own knowledge and skills when applied to statistics) more attention should be paid in investigating more in detail this aspect, for instance using instrument to measure specifically the student's confidence in solving successfully typical statistic tasks. Finally, the present research was conducted with Italian psychology students and this may limit the generalizability of the current findings. Thus, future investigations should be conducted with different student populations to provide further evidence.

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Enactive metaphoric approaches to randomness

Daniela Díaz-Rojas and Jorge Soto-Andrade

CIAE and Department of Mathematics, Faculty of Science, University of Chile, Santiago, Chile, sotoandrade@u.uchile.cl

Our work aims at developing means to facilitating the access to stochastic thinking, especially for non-mathematically oriented learners. To this end, we draw on metaphoric and enactive approaches to the teaching and learning of randomness. More precisely, we report on a challenging didactical situation implemented in various classrooms, with students and prospective and practicing teachers, concerning problem posing and solving in the context of randomness that is approached through enactive metaphoring. The findings suggest that this sort of approach allows non-mathematically oriented learners to make sense of and abduct otherwise inaccessible mathematical notions and facts.

Keywords: Enaction, metaphor, randomness, embodiment, waiting time.

INTRODUCTION

We are concerned about facilitating the access to stochastic thinking and its practice and appreciation by the learners as one way to make sense of the world. We are especially interested in approaches that might be meaningful and helpful for "general" non-mathematically oriented students in school, college and university.

The approaches we develop relay on metaphoring, enaction, embodied and situated cognition. Our main hypothesis is that most students – especially those with no special mathematical skills – can think mathematically if they *enact* first suitable didactical situations, involving problem posing and solving. Here enacting is meant in the most literal sense, as when enacting a role, bodily, on stage. In this way they may notice and "see" facts or relations that they may have trouble seeing in an abstract or symbolic setting. We put this into practice here in the context of probabilistic and statistical thinking, with challenging tasks like figuring out the expected waiting time for success in a dichotomic success-failure random experiment. Of course even learners with no probabilistic background can tackle this sort of task in a pure experimental statistical way, calculating the average waiting time for an increasingly larger number of repetitions of the experiment. Many of them may become sensitive nevertheless to the fact that they are calculating blindly without being able to anticipate, i.e., to "see" beforehand, what value the experimental average will be close to. We also aim at fostering the development of this sensitivity as an antidote to the common misconception of mathematics as just rote formula applying and calculating. These approaches have been tested with students and teachers with various backgrounds, ranging from elementary and secondary school students to university students majoring in science and humanities and to prospective and in service elementary and secondary school teachers.

THEORETICAL FRAMEWORK: METAPHORS, DIDACTICAL SITUATIONS AND ENACTION

Metaphors in cognitive science and mathematics education

Widespread agreement has been reached in cognitive sciences that metaphor serves as the often unknowing foundation for human thought (Gibbs 2008; Soto-Andrade 2014) since our ordinary conceptual system, in terms of which we both think and act, is fundamentally metaphorical in nature (Johnson & Lakoff, 2003)

We find remarkably theory-constitutive metaphors that do not "worn out" like literary metaphors and provide us with heuristics and guide our research (Boyd, 1993; Lakoff & Núñez, 1997). We might even claim metaphorically that a theory is just the unfolding of a metaphor! Recall the "tree of life" metaphor in Darwin's theory of evolution or the "encapsulation metaphor" in Dubinsky's APOS theory (Soto-Andrade, 2014). In what follows we will use the metaphorical approach as a meta-theory to describe the other theoretical frameworks we will use.

In mathematics education proper it has been progressively recognized during the last decade (Araya, 2000; Chiu, 2000; English, 1997; Johnson & Lakoff, 2003; Lakoff & Núñez, 2000; Presmeg, 1997; Sfard, 2009, Soto-Andrade, 2006, 2007, 2013 and many others) that metaphors are not just rhetorical devices, but powerful cognitive tools, that help us in building or grasping new concepts, as well as in solving problems in an efficient and friendly way. See also Soto-Andrade (2014) for a recent survey. We make use of conceptual metaphors (Lakoff & Núñez, 2000), that appear as mappings from a "source domain" into a "target domain", carrying the inferential structure of the source into the one of the target, enabling us to understand the latter, usually more abstract and opaque, in terms of the former, more down-to-earth and transparent. Our metaphoric approach to the learning of mathematics emphasizes their "poietic" role, that brings concepts into existence ("reification" in the terms of Sfard, 2009). For instance we may bring the concept of probability into existence when, studying a symmetric random walk on the integers, we see the walker *splitting* into 2 equal halves instead of going equally likely right or left (Soto-Andrade, 2013).

Didactical situations

The theory of didactical situations (Brousseau, 1998) might be described as an unfolding of the *emergence metaphor* for mathematical content: mathematical concepts or procedures we intend to teach should emerge in a suitable challenging situation the learner is enmeshed in, as the only means to "save his life". No real learning is possible if mathematical concepts "come out of the blue" or are "airborne" from Olympus.

Metaphors play an important role in didactical situations, that we describe with the help of a "voltaic metaphor": Key metaphors are likely to emerge, as sparking voltaic arcs, in and among the learners, when enough "didactical tension" builds up in a didactical situation for them. Typically this happens when students try hard and long enough, interacting with each other, to solve a challenging problematic situation. Suddenly a key metaphor to solving the problem emerges in one – or several – of them. Of course, to have the students sustain and endure the necessary didactical tension, is not an easy task in most classrooms...

Enaction

An unfolding metaphor for enaction is Antonio Machado's poem (Thompson, 2007; Malkemus, 2012):

"Caminante, son tus huellas el camino, y nada más; caminante, no hay camino, se hace camino al andar" [Wanderer, your footsteps are the path, nothing else; there is no path, you lay down a path in walking].

Indeed Varela had already metaphorized enaction as the laying down of a path in walking (Varela, 1987, p. 63), when he introduced the enactive approach in cognitive science (Varela, Thompson, & Rosch, 1991). In his own words: "The world is not something that is given to us but something we engage in by moving, touching, breathing, and eating. This is what I call cognition as enaction since enaction connotes this bringing forth by concrete handling" (Varela, 1999, p. 8).

Enaction in mathematics education may be traced back to Bruner (1953), who introduced it as "learning by doing". In fact he described *enactive representation* of a domain of knowledge (or a problem therein) as a set of actions appropriate for achieving a certain result, in contrast with *iconic representation*, where summary images or graphics are employed, or *symbolic representation*, based on symbols and their syntax. Later Bruner's ideas were successfully implemented and diffused via Singapore's CPA (Concrete-Pictorial-Abstract) methodology. For recent significant theoretical and practical developments of enaction in the field of education, which highlight the role of the teacher as an enactive practitioner acting in situation and prompt us to focus on the ways of being that can be fostered in the classroom rather than just monitoring the specific mathematical knowledge generated, see Masciotra, Roth, & Morel (2007) and Proulx & Simmt (2013).

ENACTIVE METAPHORIC APPROACHES TO MATHEMATICAL NOTIONS: THE EXPECTATION OF A WAITING TIME

We have discussed elsewhere (Soto-Andrade, 2013) an enactive metaphoric approach to the case of a symmetric 2D random walk (Brownie's walk). Here we will address the case of an important and ubiquitous family of random variables, to wit *waiting times*, and their expected values. The simplest case is that of the waiting time for success in a dichotomic success-failure random experiment, with success probability p and failure probability q. We specialize here to the simplest case $p = q = ½$, that can be modeled, or *metaphorized*, by tossing a coin and waiting for heads.

Methodology

The task (rather a problematic situation) proposed to the undergraduate students described below, was to figure out how long (how many tosses) they had to wait to get heads when tossing a coin. They had not been exposed to probability and statistics at the University and they were invited not to recall what they had heard about probability and statistics at school (where the subject is badly taught anyway). So they tackled this "impossible question" essentially bare handed. Notice that figuring out a sensible answer for our impossible question with no definite answer, is part of the task. We have then an "open task" that becomes stepwise more precise through the interactive work of the students (see a priori analysis below), accordingly with the enactive approach where problems are not "out there" waiting to be solved but are co-constructed by the cognitive subject and the world (Varela, 1987, 1999). It is our aim that the students learn to explore when tackling a problematic situation and then to conjecture and "see" beforehand ways of solution instead of blindly calculating the sum of an infinite series or computing an average over many repetitions.

Our experimentation regarding this task was carried out in the classroom with:

a. 1st year University of Chile students majoring in social sciences and humanities, from 2011 to 2014 (1 semester mathematics course, averaging 60 students per semester).

b. 25 University of Chile students enrolled in an optional one semester course in Post Modern Mathematics, majoring in mathematics or in pedagogy in physics and mathematics, in 2012.

c. 40 University of Chile prospective physics and mathematics secondary school teachers (one semester probability and statistics course) in 2014.

Students were observed by the authors during interactive 90 minute work sessions. They did group work splitting usually into two groups of no more than 20 each.

We describe now the mathematical analysis and the *a priori* and *a posteriori* analyses of this experimentation in the sense of didactical engineering (Artigue, 2009).

The mathematical situation

Notice that the experiment of flipping a coin until you get heads can be looked upon as a symmetric random walk on a truncated binary tree (you begin at the root and turn right if you get heads and to the left if you get tails, stopping when you get heads the first time), so that the question "How long will I have to wait for heads?" becomes "How long will it take the walker to get to one of the absorbing ends of the tree?" So flipping a coin or walking on the tree, are each one a metaphor for the other.

Since the random variable T = "waiting time for heads when flipping a coin" takes values n with probabilities $1/2^n$, its expectation E(T) is given by the series $\sum_1^\infty \frac{n}{2^n}$. Adding terms diagonally one can show that this series coincides with the geometric series $\sum_0^\infty \frac{1}{2^n}$, whose sum is 2.

The challenge we address below is whether the students could enactively "see" this result without calculating the corresponding series (symbolic approach, rather unfriendly for many) or averaging over an increasing number of repetitions of the experiment (statistical approach, also unfriendly for many). Notice that most students, and teachers, do not usually see another way to tackle the problem, besides the theoretical-symbolic one and the empirical-statistical one!

The didactical situation: Tentative script and a priori analysis of the enactment

The teacher flips a coin once and asks for interesting exploratory questions. Very likely some students suggest to flip it again. Various interesting questions may arise. In particular, if the teacher gets, say, 2 or 3 tails in a row, students may begin to wait for heads. Eventually the whole class may get interested in the question: How long has one to wait for heads?

Different answers may come up, the level 0 answer being: "Nobody knows, only Jesus knows!" Other answers are expressed in gestural language. Some students may suggest experimenting. Each of them flips a coin until he or she gets heads. They realize the variability of their waiting times. Some may find that the situation is hopeless. Others, more positively minded, may suggest to average. They average their waiting times and find, say, 1,7 or 2,6. So what?

This stirs usually a lot of discussion. Some students suggest further experimenting. Other make guesses, like "the average should tend to 3". One natural question is what average are we likely to get if we entice all students on campus to do the experiment?

Some become tired of experimenting and begin to look for a more theoretical approach. Eventually they draw the corresponding possibility tree (a truncated binary tree) and assign probabilities with the help of a hydraulic or pedestrian metaphor, as in Soto-Andrade (2013). Recall that in the first one we visualize the possibility tree in vertical position, root at the top, pour a litre of water at the root and let it drain evenly downwards. Probabilities are metaphorized as quantities of water. In the second one a suitable number of pedestrians (a power of 2 in this case) starts at the root splitting evenly at each junction. Probabilities appear now as proportions (ratios) of pedestrians reaching a node in the tree. This is easy, but then they realize that to calculate the theoretical average, i.e. the expected waiting time, they have to calculate the infinite sum $\sum_1^\infty \frac{n}{2^n}$, that is rather unwieldy for them. Some students begin to intuit that there should be something like a limiting value for the average (if all Chinese waited for heads...) or either an "ideal average". It is not clear for them however how to pin it down. The teacher prompts the students to suggest other approaches. If no new ideas arise, the teacher may suggest to *enact* the situation, all together.

More precisely, she suggests that all students stand up in a circle and each waits for heads when flipping a coin. She asks then how they would calculate the *average waiting time*. The students suggest the obvious way that entails asking each one of them how many times he had to flip the coin to obtain heads, add all these waiting times and divide by the number of players. The problem arises then however that this procedure does not allow them to guess or to estimate neither the experimental average waiting time nor the "ideal average waiting time". It is nevertheless clear for most of them that "ideally" half of them should get heads at the first flip. Then among those who failed, ideally one half will get heads at the second flip. This is an interesting idea that actually comes up from some students, but leads them to an infinite sum all the same (although simpler to evaluate than the sum above). So the students are still motivated to look for a friendlier approach.

At this point the teacher might suggest that to ask every player how long she had to wait for heads is a bit cumbersome. The question arises as to how could the students proceed in a friendlier and more concrete way, so that they really "see" what has happened to each one of them (notice the switch to a non verbal cognitive style). After some minutes thought at least one student suggests: flip several coins, one after the other, instead of just one! All appreciate this bright idea and begin to flip one coin after another (eventually the teacher has to lend coins to some students). After a while, each of them has a group of coins in front of her. If nothing happens, the teacher may ask them: what do you see? Some may say: "not very much, just a bunch of coins on the floor". But others remark quickly: "there is just one head in front of each of us". Other recall that to calculate the average we should count the total number of coins on the floor and divide by the number of players. Said that way however, the result is not easy to estimate beforehand. Then usually a few students realize that they will be dividing the total number of coins by the number of coins showing heads. But they "know" which is the "ideal ratio" here: if I see 17 coins showing heads on the floor, I would have expected "ideally" 34 coins in all. Of course there might be 37 coins instead. But this shows that "ideally" the average of all waiting times should be 2. After this breakthrough, usually the teacher invites the students to keep silent and quiet for a little while (one minute, say), in an introspective attitude, and to visualize the whole picture. After that she may prompt them to draw an image of the whole enactment (the circle of players, each with a bunch of coins in front of him...), so as to enter the iconic register. And then, according to the mathematical profile of the students, she can prompt them to formulate their conclusions in symbolic language.

An interesting fact that we have observed is that for most mathematicians this enactment *is* a proof while for most secondary school teachers it is not! Putting the whole situation on its head, we could even say that we have found an enactive metaphorical proof that the infinite sum $\sum_1^\infty \frac{n}{2^n}$ adds up to 2. We claim that this is closer to real mathematics that the usual purely symbolic, abstract and axiomatic approach, thst is just one genre among many possible ones (Manin, 2007; Soto -

Andrade, 2014). Finally we should remark that also the equality $\sum_{1}^{\infty}\frac{n}{2^n} = \sum_{0}^{\infty}\frac{1}{2^n}$ may be gleaned from our enactment: when students realize that they should collect all coins lying on the floor, the teacher may prompt them to suggest different ways to do it. Usually some students come up with the idea of the "common pot" (as in a soup-kitchen), like the one squatters occupying a school in protest would organize (a familiar experience for many students). The idea is that to collect all the coins, the teacher should ask first everyone to put one coin into the common pot. All can do that. Then she would ask for a second coin. Ideally only half of the players are able to do that. And so on. This shows immediately that the ideal average of waiting times coincides with the sum $\sum_{0}^{\infty}\frac{1}{2^n}$, hence the equality of the two infinite sums above.

The didactical situation: A posteriori analysis

The idea of flipping a new coin after each failure in getting heads came up as easily in students majoring in humanities and social sciences as in students majoring in mathematics or pedagogy in physics and mathematics. The fact that the number of students in the circle coincides with the number of coins showing heads did not come up very quickly, but we saw no significant difference between mathematically oriented students and humanists. Some of them realized this after contemplation of an iconic representation of their enactment, others did it after staring hard at the coins on the floor. A lot of discussion emerges as to how to register their results.

The relationship between the experimental average waiting time and the ideal one emerged rather slowly. Estimates of the experimental average of waiting times for millions of people varied in a significant way between students a and b, c. Recall that the former had previously flipped a coin 100 times and registered what they observed (not just the final result, but the whole process!). So they realized that the simplest avatar of randomness already creates shapes that look like mountain ridges and stock exchange charts. They were less prone than students b and c to think that in we cannot say anything sensible or even approximate about random experiment results. They also "saw" more easily that the ideal average waiting time should be 2, the inverse of the ideal relative frequency of heads when you flip many times a coin.

In all groups of students those who intuited that the ideal average waiting time should be 2 (out of statistical experimentation) related this immediately with the inverse of the ideal relative frequency of heads when flipping a coin many times. All students conjectured quickly after this experience that the ideal average waiting time for "ace" when tossing a dice should be 6. Further work in more advanced symbolic mode was easier with students b and c that had a more intensive mathematical training. Sometimes a game emerged after this enactment, that may suggest another approach to the expected waiting time: the teacher gives each student as many coins as necessary to get heads flipping them one after another. When a student finally gets heads, the game is over and he keeps all flipped coins. The natural question is: How much should the teacher charge for playing this game, so that it becomes a fair game?

DISCUSSION AND CONCLUSIONS

Crossing a priori and a posteriori analyses we see that several years of traditional mathematical training (students b and c) did not make a significant difference in performance in an enactment like the one we report here: students a, who come directly from high school, usually with a poor relationship to mathematics, did at least as well as students b and c, when trying to figure out enactively the value of the ideal average waiting time for heads. In fact they did even better regarding their intuition of the behaviour of the experimental average waiting time for an increasing number of flippers. We conjecture that this phenomenon is due to the fact that – in contrast to students b and c - they had made the enactive experience of flipping a coin 100 times and registering the whole stochastic process, realizing its relationship with everyday shapes like mountain ridges and stock exchange charts. This suggests that in some sense enactive experiences may concatenate and interact in a feedback loop in the life story of the learners, as suggested in Varela, Thompson and Rosch (1991): "cognition is not the representation of a pre-given world by a pre- given mind but is rather the enactment of a world and a mind on the basis of a *history* of the variety of actions that a being in the world performs" (emphasis is ours).

We also observed that learning enactively is not a one man (or woman) show, it is a collective social undertaking, that may be seen upon in some cases as an avatar of swarm intelligence. We have noticed, especially in group a, that among students reacting remarkably faster and better than the average in enactive situations, a high percentage of come from alternative Montessori or Waldorf schools.

We should remark that our enactive approach poses some significant challenges: teachers and students need to be able to play tightrope walkers and transit seamlessly between different cognitive styles (from symbolic to enactive particularly). Special support and dedication is needed for those who come from more formal and robotic school systems, so that they can progressively adapt to new ways of working and approaching mathematical experience and knowledge. If this sort of "cognitive therapy" is not provided (by a devoted highly qualified teacher assistant, for instance) the risk arises of a severe stratification in the classroom, that could demotivate many students. We see then that significant situational intelligence, in the sense of Masciotra, Roth, & Morel (2007) is required from teachers as enactive practitioners.

On the other hand, as pluses of our approach we have observed that is very often highly motivating for the students, involves surprise (a key factor for learning, according to Peirce and Freire), knowledge is constructed in a cooperative way in the classroom, space is given for questions to emerge from the students in experiential situations, instead of answers being given to them before they have a chance to ask the questions first (Freire, 2011; Tillich as cited by Brown, 1971; Mason, 2014) and finally, it fosters participation so that students become protagonists of their learning.

As open ends, we may mention, among others: the didactical study of histories of enactive experiences of the learners; the relation between enaction and intuition (first steps in this direction may be found in Diaz-Rojas, 2013); the systematic study of the emergence of enactive metaphoring in suitable didactical situations; the enactive exploration of the real meaning of being mathematical in the classroom in the sense of Mason (2014), research on curriculum reshaping motivated by the enactive approach in mathematics education.

ACKNOWLEDGEMENT

Funding from PIA-CONICYT Basal Funds for Centres of Excellence Project BF0003 is gratefully acknowledged.

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Aspects of students' changing mental models when acting within statistical situations

Andreas Eichler[1] and Markus Vogel[2]

1 University of Kassel, Kassel, Germany, eichler@mathematik.uni-kassel.de

2 University of Education Heidelberg, Heidelberg, Germany

In the qualitative part of a larger research project aiming to investigate students mental modelling of statistical situations we found aspects of a statistical situation that provoke intended but also not intended (conceptual) changes of students' mental modelling. To describe these changes, we firstly outline the results of our former research followed by a brief discussion of our theoretical framework and our method. Then, we discuss results from the perspective of changing mental models.

Keywords: Statistical mental models, conceptual change, interview study.

INTRODUCTION

The contemporary educational research on statistics follows mostly the approach of intervention studies (Jones & Thornton, 2005). The amount of this research has grown in recent decades, since statistics become a crucial part of the curriculum of stochastics (statistics and probability) in many countries (Batanero, Burrill, & Reading, 2011). By contrast research referring unschooled students' acting and reasoning within data-centred statistical situations of uncertainty is sparsely developed (Eichler & Vogel, 2012). For this reason, we started a research project aiming to analyse non-schooled students' modelling and reasoning about statistical situations. The main aim of this research is to gather more information about young students' primary statistical thinking as an empirical based starting point for statistics education in school.

In this paper, we refer to the qualitative part of our research including interviews with young students, who act within statistical situations. For this, we outline our theoretical framework. Then, we briefly outline the main findings of our previous work and derive the research questions of this paper followed by a short report of our methodological approach. Finally, we give an account of research results, and particularly of both students' mental modelling when reflecting on comparable statistical situations given in paper-pencil-task, and a (conceptual) change of mental modelling when handling those situations in reality afterwards.

THEORETICAL FRAMEWORK

Students' conceptions are not always compatible with the intended statistical conceptions. This finding can often be explained by the influence of (naïve) prior conceptions and non-accomplished processes of their reconstruction. The mental model theory (Johnson-Laird, 1983) and a mental model oriented conceptual development approach (Vosniadou, 2002) meet the demands to capture this problem from a theoretical point of view. We outline briefly those aspects of our theoretical framework that are relevant for this paper.

Mental models

Johnson-Laird (1983, p. 156) states: "A mental model [...] plays a direct representational role since it is analogous to the structure of the corresponding state of affairs in the world – as we perceive or conceive it." From the perspective of information processing Schnotz and Bannert (1999) describe mental models as being constructed individually according to a task and its requirements within a specific situation. By this, they conclude that a mental model represents the structure as well as the function of the modelled situation that could be described as follows:

— Structure: An essential process of mentally modelling a situation's structure is recognising the observable or not observable physical objects of the situation, as well as the relationship of these objects that an individual identifies as mainly impacting on the situation. Given data are also to be seen as being part of a situation's structure because they represent results of a process having passed. For example a die, the characteristics of the die, the player throwing the die, the ground where the die will roll and even data representing results of the die having been thrown before are characteristics of the situation that an individual has to process when he build up a mental model of the situation.

— Function: Concerning the dynamic aspect of mental models, i.e. the function, Seel (2001) suggests that, when coping with demands of a specific situation, an individual constructs a mental model in order to simulate relevant aspects of the situation by anticipating possible results. The functional aspect of statistical situations is directly reasonable, since it represents a qualitative simulation of the situation in terms of making a prediction, for example of those numbers a die will show in the next throws. In general, such mental simulations cannot result in quantitatively exact conclusions – for example when a die with an unsymmetrical shape is given – but in qualitative ideas about reasonably expectable outcomes of such simulations (De Kleer & Brown, 1983). These "qualitative simulations" (ibid., p. 155) require sense making about the system or process that should be simulated, its constituent components and their relationships.

Conceptual development and change in terms of mental model theory

With regard to conceptual development mental models play different important roles: They serve as aids in the construction of explanations, are mediators in the interpretation and acquisition of new information, and provide tools that allow experimentation and theory revision (Vosniadou, 2002; Van Dooren et al., 2006).

Conceptual change in terms of mental model theory involves a validation of the model that necessitates a moderately changed or a fundamentally renewed model of a situation (Vosniadou, 2006). Posner and colleagues (1982) identify a disadvantage of existing models concerning new facts or phenomena as well as the existence of alternative models that seem to be – evidently or not – more promising to explain new facts or phenomena. In some sense an individual's conceptual change equals the acquirement of new pieces of knowledge. However, it is not necessary that this new knowledge eliminate the "old" knowledge (Vosniadou, 2006). Thus, students who conduct a conceptual change may construct different mental models of structural similar situations that were provoked dependent of the specific situation or task (Siegler, 1996). Because such different mental models occupy an autonomous role (Morgan & Morrison, 1999) they have not be consistent in an objective sense but they have to be experienced as being useful concerning their explanatory power in a subjective sense.

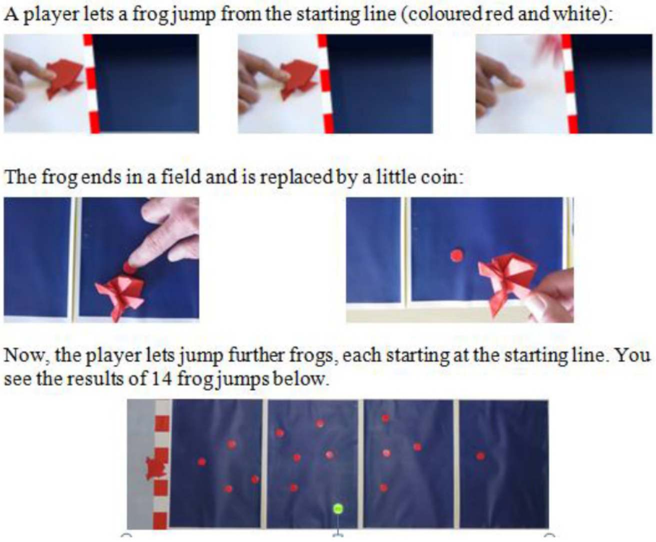
The situational perspective of mental model theories is closely connected with the approach of situated cognition: "Situations might be said to co-produce knowledge through activity. Learning and cognition, it is now possible to argue, are fundamentally situated." (Brown, Collins, & Duguid, 1989, p. 32) Amongst other theoretical approaches, the conceptual change approach refers fundamentally to situativity and mental model theory (Vosniadou, 2002). For this reason, we regard the change of a student's mental model from the perspective of conceptual change.

PREVIOUS WORK

Theoretical aspects

In our research we use the mental model approach for conceptual framing the aspects of students' modelling within specific statistical situations (Vogel & Eichler, 2014). Because we are especially interested in what Fischbein (1975) calls students' primary intuitions concerning statistics situations of everyday life we reduce ourselves to the descriptive mode (cf. Prediger, 2008) of mental modelling (referring to the interpretations students actually use) and leave out the prescriptive mode (referring to the mathematical interpretations intended to be learned). As an example, the situation of a frog jump is described in Figure 1.

The characteristics of the structure and the function of the mental model concept are provided in the situation of frog jumps: The structure consists of non-human objects (the frog, the fields, the starting line and so on), human objects (the player) and the relations among the objects. Further, the data (the shims on the fields; cf. Figure 1) are a part of the structure representing previous results of the situation. The function of a mental model is implicitly incorporated in the



The frog task:

The frog ends four times in field 1, six times in field 2, four times in field 3, and one time in field 4.

Make an estimation to the following experiments and justify your answer:

How often the frog will end in field 3 by 100 (in field 3 or in field 4 by 1000) jumps?

Figure 1: The situation of the frog jump

situation, i.e. a mental simulation of the situation, e.g. resulting in estimation for the next jump.

The situations and the mentioned characteristics of mental models allowed us to derive a theoretical model of task complexity with three influencing factors (Eichler & Vogel, 2012): The tasks could provide all the objects that potentially impact on the situation (or not), the tasks could provide data (or not), and the tasks could require a request of a mental simulation (or not).

Empirical aspects

Using the results of a questionnaire survey, the complexity model could be empirically backed up in a quantitative way (ibid.). Additional qualitative analyses of students' responses yield interesting insights in students thinking (ibid.): For example, students tend to show a bigger variation of mental models when there were situations to be judged in which random processes were apparently, at least partially, influenced by human. This assumption can be underlined for example by an overview of the analyses of 74 written student responses concerning a task using rolling dice on the one hand, and a task using jumping paper frogs (cf. ibid.). In the so-called die task no student mentioned the hand of the person which throws the dice whereas in the frog task (as well as in other tasks which not comprise dice) human influence was considered in different elaborated ways.

RESEARCH QUESTIONS

The research question referring the qualitative part of our research is a result of our previous findings and future aims.

In this qualitative part, the students work with the concrete material that is shown in the statistical situations in the tasks (e.g., the frogs, the fields and the starting line). Besides other (primary) goals of this part, it aims to disclose possible changes of students' mental modelling in specific statistical situation. This aim seems to be necessary to focus on how to engage students to change their mental models, e.g. as a result of an intervention. For this, beyond the mental model concept we use the conceptual change approach (see above). We are aware that the time span provided for the students to work within the statistical situations in our research might be not sufficient to get evidence for conceptual changes. However, if a student changes his mental model concerning a statistical situation this change could be interpreted as a beginning of a conceptual change that potentially took place on the long run.

We derive our research questions from these considerations: Confronted with a statistical situation (like the frog jump) a student could build up an initial mental model of this situation and conduct mental simulations only based on this model. When a student tackles the situation in reality and works with the concrete objects afterwards, it is possible that the student changes his mental models. Based on our previous work we are especially interested in the following two questions:

— When a student changes his notion of the objects from hardly to strongly impacting on the results of a situation of uncertainty (or rather the other way around), in which way will he or she express this by arguing and/or acting?

A student changing his notions of the impact of the objects could potentially also change his notion of the process of data generation and, accordingly, the function of the situation. In concrete, we ask:

— When a student start with the notion of a deterministic impact of one or more objects on the results of a situation and change this notion to a perception as a random situation (or rather the other way round), in which way will he or she express this by arguing and/or acting?

METHOD

The sample of the qualitative part of our research project consists of about 60 students grouped in pairs. We use pairs of students because we want them to solve a statistical problem situation together by arguing, discussing (cf. Dreher & Dreher, 1982) and thus, externalizing their ongoing mental modelling during this collaboration. 24 students (12 pairs) of this sample worked with the situation of the frog jump as it is shown in Figure 1. The students (grade 6; age 11–13) come from schools of urban and rural regions in Germany. We collected data in videotaped interviews with pairs of students lasting about 20 minutes and consisting of three steps:

1) Each student was asked to solve one of two tasks representing similar versions of the situation of the frog jump. In this step the students were asked to work alone and without any intervention of the interviewer.

2) The students were asked to explain their tasks and their solution to each other, and, further, to find consistent solutions for each task. The interviewer had the task to moderate the students' discussion and prompt the students to continue their explanations until the other student understand this explanation.

3) The interviewers used the physical objects of the situations represented in the tasks. The students were asked to re-enact the situation of the frog task and to conduct the experiment of the frog jumps. After this step, the students were asked to modify their collective solution.

The interviews were transcribed verbatim. The analysis of transcripts follows a coding approach close to Grounded Theory (cf. Kuckartz, 2012; Strauss & Corbin, 1998).

RESULTS

We firstly illustrate the diversity of students' mental models referring the structure when they explained their solutions in the pairs (step 2) after working alone on the paper and pencil tasks (step 1). This diversity replicates qualitatively the diversity that we found in our quantitative survey (Eichler & Vogel, 2012).

Chris: "If I let the frogs jump in the same way as showed on the picture then I will hit this field as well."
Gerd: "The frog is not able to jump that far."

Both students stem from different groups. The students mostly mentioned the player (human object) or the frog (non-human object) to have the main impact on the situation. Only few students also refer to the fields and the border between two fields. In all cases, the notion of the structure presupposes the mental simulation of the situation resulting in a prediction for the next jump.

According to results of former studies (ibid.), the students' rationales for a prediction of the next frog jumps differ considerably. Differences refer to the relevance and the number of situations' information represented by objects or data that the students used in their rationales. We want only to illustrate these differences by quoting the responses of two students. Both students predicted a certain field for the next jump of the frog and afterwards justified their prediction:

Ernest: "I made it in this field, because in the second field are the most of the coins [he points to the sheet with the task]."
George: "The frog is heavy. He will crash to the ground easily."

When the pairs of students were asked to conduct the experiment of the frog jumps (step 3), it was striking that all students were immediately able to re-enact the situation and to conduct the experiment as it is represented in the task. This result is crucial for our analysis since there is no evidence that the students misinterpreted the situation presented in a task on a paper sheet.

However, when the students experimented with the frogs, in some cases signs for a potential (conceptual) change concerning the situation's evaluation can be reconstructed. We illustrate exemplarily a change in mental modelling concerning the pair of Eva and Franka (in comparison to Figure 1, in this case, the tasks include a slightly different distribution of coins or rather frog jumps):

> Eva: "I believe that the frog will end 70 times by 100 attempts in the third field, because he ended in this field 7 times by 10 attempts. Thus, I have only added a 0, because concerning 100 there is also a 0 more at the end in comparison to 10."

Before they conducted the experiment, Eva and Franka seem to identify the frog (non-human object) to be mainly impacting on the situation. It is not decidable, if the students mentally model the situation randomly or in a deterministic way.

When dealing with the frogs, they changed their mental modelling by judging the human object (the player) instead of the non-human object (the frog) as having the main impact on the situation.

> Franka: "It depends of the force. If you let the frog jump with little force, the frog will not jump as far if you let him jump with more force."

After the first experiments, both students feel certain that they are able to produce any desired results. The reason of variation in the results seems to be determined only by the intention of the students. Thus, the mental simulation seems to be mostly of deterministic nature.

After a period, in which the students experiment several times with the frogs the answers of Eva and Franka provide indications of another (conceptual) change. Firstly, they re-changed their judgement of the object as mainly impacting on the situation and, further, changed their mental simulation from a more deterministic to a more stochastic way:

> Franka: "I think they [the frogs] approximately will jump equally."
> Eva: "I think this depends on fortune."
> Franka: "Yes. I think that there would not been many exceptions from that the frogs will jump equally. At most a little. I think the frogs will jump equal distances."

Like these both students, other pairs of students also seemed to change their mental models one or more times to yield a fit of the model and the situation. However, most of the students appear to consolidate one mental model at the end of the interview. A categorizing of all student pairs' answers yields the following change type (for which Eva and Franka are representatively standing):

> Change type I: From a deterministic model of a situation of uncertainty to a stochastic model that includes a pattern ("the frogs will jump equally") but also a first idea of variation ("fortune", "exceptions" from equal jumps).

If we hypothesise that these changes of mental modelling are the beginning of a conceptual change in the long run, the trajectory of changes is desirable. Referring the structure of the mental model this type includes a change from the human to the non-human object as mainly impacting factor and a change from a deterministic to a stochastic mental simulation.

Of course, the latter interpretation leads to a hypothesis that has to be proven in further situations, or similar argumentation have to be shown by other pairs of students. Actually, few of the pairs of students showed a change of mental modelling like Eva and Franka. In contrast, more pairs of students showed changes that are in contrast to the change type described above. We illustrate this other type with the case of Hubert and Ian.

> Hubert: "He will jump in this field because [he points to the sheet with the task] here are only four, here are much [coins], and here is only one."

Firstly, the frog was considered as mainly impacting on the situation thereby the data were taken into account. After jumping the frog several times, Ian and Hubert stated the following:

Ian: "I think I know it. The more I press at the back the farther the frog jumps."
Hubert: "But if you press to far at the end it does not work." […]
Ian: "The jump depends on who it makes."

Both students jumped the frog again and again resulting in different lengths of the jumps. They changed their position and made further frog jumps. They commented their results from time to time, consolidating that the jumper (the human object) has the main impact on the situation. They ended with the following assertion:

Ian: "The most important thing is, how I hold the frog and how I make it [the jump]"

Both students initially tended to favour a stochastic model of the situation based on an informal data analysis. When they jumped the frogs, each of the students showed a considerable variation of length that could potentially evoke a notion of a jumping distribution including a centre and variation. However, these students as well as other pairs of students represent another change type than Eva and Franka.

> Change type II: From a stochastic model of a situation of uncertainty that includes a pattern and a first idea of variation to a deterministic model.

This result condensed in the second change type was surprising on the one side and was not intended on the other. Following Fischbein (1975) this change type potentially show a trajectory from an appropriate primary intuition to a misleading secondary intuition.

DISCUSSION AND CONCLUSION

In this paper we reported the qualitative part of our research into young students' changing mental models when they act within statistical situations. The aim is to gather information about initial mental models of young students judging a statistical situation and changes of these models when they can tackle the situation. In an educational point of view the changes to sufficient mental models concerning statistical situations are especially interesting when they are stable over time. For this reason, we refer beyond the mental model concept also to the conceptual change approach. Of course we do not claim that working within a certain statistical situation will be followed by a conceptual change, this is never possible to state in general. But we consider reconstructed changed mental models as a potential initiation of a conceptual change that potentially endures. Under these theoretical conditions we derived two aspects, to which students' mental model changing possible refer to: the extent to which certain objects impact on the results of a statistical situation and the preference concerning a more deterministic or a more stochastic estimation of the situation.

In the qualitative interview study reported in this paper we found indications for both of these theoretically derived aspects of mental model changes. These indications were illustrated and discussed along the frog task, which was exemplarily presented in this paper. The analyses of students' responses and actions concerning this task's statistical situation yielded two change types: from a deterministic dominated interpretation of the situation to a stochastic dominated one (this was the intended one) and vice versa. Beyond the reason for getting deeper insights in young students' statistically mental modelling we conducted this qualitative interview study to generate hypotheses for the following research steps. In this point of view we assume based on our findings that the impact of the human-factor should be reduced in the statistical situations, with which the students are confronted when introduced to statistics. At the moment we investigate this assumption in more detail by changing the context of the frog task to a (in sense of the statistical situation) structural equivalent task using toy cars. The results of this current study will also be reported on the 9th Congress of European Research in Mathematics.

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Preservice teachers' statistical reasoning when comparing groups facilitated by software

Daniel Frischemeier and Rolf Biehler

University of Paderborn, Paderborn, Germany, dafr@math.upb.de, biehler@math.upb.de

Comparing groups is a fundamental skill preservice teachers are supposed to gain after attending a statistics course at university level. Preferably these activities are embedded in the well-known PPDAC-Cycle and contain the exploration of real and motivating data. Adequate software such as TinkerPlots may support learners when exploring data and carving out differences between distributions of numerical variables. In this article we want to present first results of a study on statistical reasoning of preservice teachers while doing group comparisons with TinkerPlots.

Keywords: Preservice teacher education, statistical education, group comparisons, software use, tinkerplots.

INTRODUCTION

Comparing distributions of numerical variables lies in "the heart of statistics" (Konold & Higgins, 2003, p. 206). A question which may motivate a group comparison is for example "In which respect do men and women differ regarding their income?". This could be also posed in the daily-life media, so we see the importance of such an activity, not only for students but also for upcoming teachers. Preferably questions and activities are embedded in a cycle (like PPDAC, Wild & Pfannkuch, 1999) where students are enabled to generate their own statistical questions and hypotheses, to design a questionnaire for collecting data, to analyse their data and to draw conclusions from it. When analyzing large datasets, the use of adequate statistical software becomes inevitable. TinkerPlots (Konold & Miller, 2011) yields many requirements for statistical software for use at school and university level. At the one hand TinkerPlots can be seen as educational software supporting learners to learn data analysis, on the other hand it can be seen as a tool for exploring multivariate data. Additionally, the teacher can also use it as demonstration medium in the classroom. From our research point of view, we are primarily interested how preservice teachers compare groups with TinkerPlots. In this article we want to introduce a framework to rate preservice teachers´ skills when comparing groups in large datasets with TinkerPlots. Therefore we designed a course to deepen preservice teachers´ statistical knowledge and conducted an interview-video-study to evaluate in which way the participants are able to compare distributions with TinkerPlots in large datasets. First results of this study will be reported at the end of this article.

LITERATURE REVIEW

Comparing groups is an important domain in statistics education research: There are several empirical studies which concentrate on rating learners' statistical reasoning when comparing groups. We can differentiate between at least five different ideas in these studies (not in order).

A first idea is set by Watson and Moritz (1999) who investigated Australian 3–8 graders when comparing two data sets. The participants were given two data sets with test scores of two classes and were asked which class has done better in the test? The distributions of the variable test score was displayed as a stacked dot plot and the interview protocol offered different types of group comparisons, e.g. distributions which differed in the number of cases, in variation or in skew (Watson & Moritz, 1999, p. 151). The learners responses were rated via SOLO taxonomy where the responses where rated "unistructural", "multistructural" and "relational" and distinguished between the comparison of equal and unequal sizes. One major result of their study was, that students in higher grades tend to reason proportionally rather than younger students (for further details see Watson & Moritz, 1999, p. 153). A second main idea is displayed by Makar & Confrey (2002). They conducted a one semester professional course for preservice teachers including group comparison tasks with Fathom and developed a "taxonomy for classifying levels of reasoning when comparing two groups" to evaluate their participants reasoning from an interview task (concluding the course regarding to comparing groups). The participants were asked to compare two distributions of test scores of two schools given as a stacked dot plot in Fathom (with no use of software itself). A crucial point in their analysis was in which way inferential terms (like "evidence" and "significance") were used in the comparison process of the participants and in which way the participants draw conclusions from samples. A third idea of research on comparing groups is given by the work of Biehler (2001) and (2007). Biehler gives a normative view on comparing groups and expresses which elements shall be included in a "good" group comparison. He mentions that *p*-based [1] and *q*-based [2] comparisons (Biehler, 2001, p. 110) might offer intuitive strategies for students and he also emphasizes an interpretation of the difference in the skewness of distributions as a possible comparison element (Biehler, 2001, p. 101). There is also an idea (fourth) which covers the use of software when comparing groups. Biehler (1997, p. 175) has set up a cycle and gives an overview on four phases "statistical problem", "problem for the software", "results of software use" and "interpretation of results in statistics" which are run when doing data analysis with software. Maxara (2009) designed a framework for evaluating learner´s software competencies when simulating chance experiments with Fathom. This is not directly related to group comparisons but nonetheless adaptable for evaluating leaner´s competences when using software (such as TinkerPlots) when comparing groups. Overarching for the fifth idea might be the work of Pfannkuch and colleagues (2004; 2006; 2007). In these research papers a framework for evaluating learner's competencies when comparing two distributions was developed. Since Pfannkuch (2007) is a succeeding study of Pfannkuch and colleagues (2004) and Pfannkuch (2006), we want to refer to Pfannkuch (2007) only. In this empirical study she gave a boxplot comparison task (see Pfannkuch, 2007, p. 157) to "Year 10"- students. They were given two boxplots, asked to compare them in the sense of making three statements to explain differences or similarities between the distributions. Pfannkuch (2007, p. 159) had a look on different statistical aspects that were used by the participants and set up categories for the evaluation of statistical reasoning elements when comparing two distributions by boxplots. On a structural level she distinguished between "summary", "spread", "shift" and "signal". Then she rated each statement regarding to its quality: "point decoder" (level 0), "Shape comparison describer" (level 1), "shape comparison decoder" (level 2) and "shape comparison assessor" (level 3). Main results of this study were that the students mostly refer to summary and spread elements, but neglected elements on shift and signal. Furthermore they tended to stay mostly on the describing and decoding but not on the assessing level when pointing out differences and similarities between both distributions.

All in all, we can derive three dimensions having an influence on the group comparison process from the literature review: *Software cycle when comparing groups* (Biehler, 1997), *competence of using software (TinkerPlots) when comparing groups* (Maxara, 2009) and *"statistical reasoning" when comparing groups* (Watson & Moritz, 1999; Biehler, 2001; Makar & Confrey, 2002; Pfannkuch et al., 2004; Pfannkuch, 2006; Biehler, 2007 and Pfannkuch, 2007). In this study the work of Pfannkuch seems to be the most interesting aspect: Watson and Moritz (1999) deal with given data and given distributions (but in datasets with a small amount of cases) and a focus on counting strategies and proportional reasoning. Makar & Confrey (2002) have had their crucial research point of interest on how learners draw conclusions from samples to a population while comparing samples. Pfannkuch offers an open framework which firstly structures learners' outcome in regard to the statistical element used and secondly rates this in form of quality. Since working with software offers a broad spectrum of statistical elements (e.g. center, spread, shift, etc.) which can be used in group comparisons even in large datasets, the framework of Pfannkuch with enrichment of Biehler´s (2001) suggestions (skewness, *p*- and *q*-based comparisons) seems to offer possible and adequate comparison elements when comparing groups and a solid basis for evaluating the outcomes of learners when comparing groups. These ideas and aspects motivated us to design a course for the education of preservice teachers in statistics in which we want to teach the comparison of groups (with TinkerPlots) with the elements (such as center, spread, etc.) described above.

COURSE "DEVELOPING STATISTICAL REASONING WITH TP"

The authors of this article have designed a course for preservice teachers called "Developing statistical reasoning with using the software TinkerPlots" (Frischemeier & Biehler, 2012) in the sense of the design based research paradigm (Cobb et al., 2003). In this course, which goal is the development of statistical content (but not pedagogical) knowledge, the participants go through the whole PPDAC-cycle (Wild & Pfannkuch, 1999) which includes analysing self-collected data with TinkerPlots and writing down findings in statistical reports. In the analysing section the participants got to know about how they could compare distributions via different aspects (such as center, spread, shift (see Pfannkuch, 2007) and skewness, *p*-based- and *q*-based-comparisons (see Biehler, 2001)) with TinkerPlots. At first they were taught to identify differences between distributions (regarding to center, spread, etc.), then they were told to interpret these differences. A norm set by us was to work out as many differences regarding to center, spread, etc. between both distributions as possible. At this stage we firstly want our participants to work out as many differences as possible and to interpret them. In a next step, not reported in this paper, the participants are asked to synthesize their findings (e.g., in form of writing a statistical report). For further details see (Frischemeier & Biehler, 2012).

RESEACH QUESTIONS

Since the ability of handling a large set of real and multivariate data is important for upcoming teachers in statistics, we want to investigate how preservice teachers explore large datasets and compare distributions with TinkerPlots. In this article we want to concentrate on the "statistical reasoning" component of preservice teachers when comparing groups with TinkerPlots only, so two research questions arise: Which group comparison elements (which were taught in our course - such as center, spread, shift, etc.) are used by the participants when comparing groups? How is the quality level of these group comparison elements used by the preservice teachers?

DESIGN OF THE STUDY

As part of the Ph.D. study of the first author, an interview-video study was designed in which the participants were asked to compare two distributions with TinkerPlots in pairs of two. For the selection of task we chose a task which deals with the exploration of the income distributions of male and female employees, which has got a lot of publicity in Germany with regard to gender biases in monthly incomes. The dataset taken from the German Bureau of Statistics was imported in TinkerPlots and contains 861 cases and more than 20 variables (such as gender, monthly income, region, kind of employment, etc.). This was drawn as a random sample out of 60,552 which itself was sampled at random (stratified) from the population of all German employees. Furthermore we handed out a TinkerPlots file containing the dataset and an exercise sheet where the participants were asked to make notes on it. After motivating the problem of a "gender difference income" with a newspaper article, the task for the participants was: "*In which way do the men and women differ regarding their income? Carve out differences in both distributions!*" Some impulses which differences can be carved out between the distributions can be found in (Biehler & Frischemeier, 2015). The video study was two-phased adapted from the design of Busse & Borreomeo-Ferri (2003). In phase 1, the "working phase", the participants work on the task in pairs and were forced to communicate to each other while doing the task. Figure 1 shows a TinkerPlots graph which displays the differences between both distributions using boxplots (and the mean) produced in TinkerPlots by participants during the working phase.

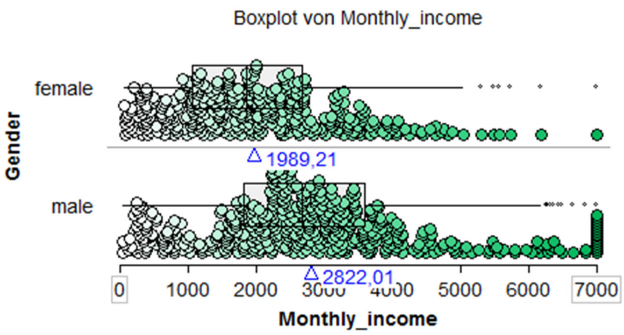

Figure 1: TinkerPlots Graph with boxplots and mean

In this phase there was no intervention by the interviewer (first author of this article). In phase 2 cognitive processes of the pairs should be revealed via "stimulated recall".

DATA, PARTICIPANTS AND METHODOLOGY

All in all 14 participants (7 pairs) took part in the study. All of them were preservice teachers for mathematics for primary and secondary school at the University of Paderborn and all of the participants attended the course "Developing statistical reasoning with TinkerPlots". The interviews were done 4–6 weeks after the last session of the course "Developing statistical reasoning with TinkerPlots". The participants were asked to work on the task in teams of two while they and their screen activities were video recorded. TinkerPlots files, exercise sheets and video recordings were also collected. The communication and action with TinkerPlots was transcribed. Our goal is to evaluate the whole communication of the participants regarding to their statistical reasoning elements when working on the task. So a large amount of data is to handle and there is also a need for an evaluation procedure which is comprehensible. The qualitative content analysis in the sense of Mayring (2010) "can be well applied in the case of having the intention to analyse a huge amount of transcribed data" and follows the "reduction of the huge amount of data in form of an analysis via category systems" (Kohlbacher, 2006). One main goal of this procedure is "to filter out a particular structure from the material" (Kohlbacher, 2006). Mayring (2010, p. 63) has pointed out different streams of the qualitative content analysis, such as "structural", "explicating" or "summarising". We want to structure the group comparison elements of the participants first and then scale (evaluate) them. The sequence of a structured-scaled qualitative content analysis (Mayring, 2010, p. 101) starts with choosing the analysis units (Step 1). In this case the analysis units are the transcribed data from the communication of the pairs when the participants were working on the task. As a second step the dimension of analysis is set up (step 2) – in our case this is *statistical reasoning when comparing groups*. In a further and third step we determine the characteristics of "dimensions of analysis" (Step 3) in form of setting up a category system, which is seen as the basis of this method where coding rules and key examples are given for an exact assignment between coding and data material and therefore define categories for evaluating the transcribed communication of the participants (Step 4). These categories can be defined deductively, inductively or mixed (Kuckartz, 2012, p. 62).

FRAMEWORK FOR COMPARING GROUPS

We used this construction of categories in the sense of Kuckartz (2012, p. 62) for our purpose. The aim was firstly to structure the transcribed data regarding the statistical elements used for group comparison and secondly to evaluate the quality of each element. So we took into account the elements "summary", "spread", "signal" and "shift" of the framework of Pfannkuch (2007) in the sense of a deductive approach. Since our participants were asked to find differences between both distributions and since they have had the possibility of a free data exploration with TinkerPlots, we have had to modify the categories of Pfannkuch (2007) for our purpose. Having a look on Biehler's (2001) normative point about group comparisons, where the comparison of skewness and *p*/*q*-based-comparisons also plays a huge part in the comparison process, we decide to add the elements "skewness", "*p*-based" and "*q*-based" to our framework. The further step sees an inductive refinement of the categories (Step 5) in the sense of Kuckartz (2012, p. 69). In this step 5 we have

	High quality	Medium quality	Low quality
Center	Measures of center (mean, median) are compared in a quantitative way and are interpreted.	Measures of center (mean, median) are compared in a qualitative way and are not interpreted.	Measures of center (mean, median) are compared in a wrong way.
Spread	Measures of spread (IQR) or informal descriptions of spread (such as "density", "close") are compared and interpreted.	Measures of spread (IQR) or informal descriptions of spread (such as "density", "close") are compared and not interpreted.	Spread is compared with inadequate measures (like range) and is wrongly interpreted.
Shift	Shift between both distributions is quantified correctly (with comparing the position of the middle 50% or with comparing non-corresponding numbers)	Shift between both distributions is described in a qualitative way.	Shift between both distributions is worked out in a wrong way.
Skewness	Skewness of both distributions is described correctly and the differences between the distributions are interpreted correctly.	Skewness of both distributions is described correctly but not interpreted.	Differences of skewness are worked out in a wrong way.
p-based	p-based differences are identified and interpreted	p-based differences are identified but not interpreted	p-based differences are worked out in a wrong way
q-based	q-based differences are identified and interpreted	q-based differences are identified but not interpreted	q-based differences are worked out in a wrong way

Figure 2: Definitions of codings

gone through the data with our deductively developed categories and refined them inductively. After this process, we added the elements "center" instead of "summary" because we wanted to concentrate on the comparison of mean and median and not on the comparison of all summary statistics and we have left out the category "signal" which is a special element for boxplot comparisons but not necessarily for group comparisons in general. We did not focus on the construction of plots in TinkerPlots, we just focus on working out as many differences with TinkerPlots as possible. Mostly standard displays, also primarily used in our course, like stacked dot plots, histograms and boxplots were used. All in all we finally have the following elements for our analysis: "center", "spread", "shift", "skewness", "*p*-based" and "*q*-based". We see these elements as our categories (see Figure 2).

As we see in Figure 2 we also generated for each of these categories the ratings "high quality", "medium quality" and "low quality" to evaluate the quality of the use of the elements in the comparison process. Generally we coded a group comparison element used by our participants with a "high" quality, if the difference of the distributions was worked out quantitatively and was also interpreted (in the idea of Pfannkuch's category "assessor"). An element is coded in the sense of a "medium" quality, if the difference is at least worked out on a qualitative level ("X is higher than Y") but not interpreted. Finally a "low"-quality code is given if the difference is worked out in a wrong or in an inadequate way. For illustrating the definition of codings we want to give examples (see Figure 3) arisen from our data.

When having set up the category system, we chose (in a sixth step) a word as minimal coding unit and a phrase as maximal coding unit. With this agenda we have coded the transcripts of four of the seven pairs so far. In this paper we refer only to codings belonging to the working phase, the codings of the transcripts of the stimulated recall phase are not reported here. If codings of passages were unclear, we had a look in the video to clarify the situation. A further step (step 7: Revision of codings) included the discussion of the codings with an independent researcher and in the following the revision of categories and definitions of categories. Finally a frequency analysis of the occurrence of the several categories was made (step 8: frequency-analysis of occurrence of steps).

RESULTS

Let us have a look which group comparison elements were used by the teams and how well they did in using them when working on the task with TinkerPlots.

At first we can say that we have 23 codings in total. The codings of the elements center, spread, skew and shift are at least at a medium quality level. All *p*-based comparions were rated with medium quality. There has been no team using *q*-based comparisons. All in all

	High quality	Medium quality	Low quality
Center	The men earn 29,5% more than women on average. (L & R)	The mean of men is higher than the mean of women (H & I)	No example.
Spread	The middle 50% of men spreads more than the middle 50% of the women. (H & I)	The Interquartile Ranges of the distributions are almost identical. (C & M)	No example.
Shift	The median of the distribution of men is almost equal to the first quartile of the distribution of women. (H & I)	The middle 50% of men are shifted right compared to the middle 50% of women. (H & I)	No example.
Skewness	The distribution of men seems to have some peaks but the distribution of women seems to be right skewed, so there might be more women earning little money compared to men. (L & R)	Here [distribution of salary of women] we can find a peak at 400€…the men [distribution of salary of men]…okay there is also a peak, but it is not so high. (L & R)	No example.
p-based	No example.	10% of the men earn more than 5000€, only 2% of the women earn more than 5000%. (S & L)	No example.
q-based	No example.	No example.	No example.

Figure 3: Key examples of codings of the group comparison elements

Overall	High quality	Medium quality	Low quality	Overall
Center	1 (33%)	2 (67%)	0 (0%)	3 (100%)
Spread	2 (40%)	3 (60%)	0 (0%)	5 (100%)
Skewness	2 (50%)	2 (50%)	0 (0%)	4 (100%)
Shift	3 (60%)	2 (40%)	0 (0%)	5 (100%)
p-based	0 (0%)	6 (100%)	0 (0%)	6 (100%)
q-based	0 (0%)	0 (0%)	0 (0%)	0 (100%)
Overall	8 (35%)	15 (65%)	0 (0%)	23 (100%)

Figure 4: Overwiew of all codings related to group comparison elements

Hilde & Iris	High quality	Medium quality	Low quality	Overall
Center	0	2	0	2
Spread	1	1	0	2
Skewness	0	0	0	0
Shift	3	0	0	3
p-based	0	1	0	1
q-based	0	0	0	0
Overall	4	4	0	8

Conrad & Maria	High quality	Medium quality	Low quality	Overall
Center	0	0	0	0
Spread	0	2	0	2
Skewness	0	0	0	0
Shift	0	1	0	1
p-based	0	0	0	0
q-based	0	0	0	0
Overall	0	3	0	3

Ricarda & Laura	High quality	Medium quality	Low quality	Overall
Center	1	0	0	1
Spread	1	0	0	1
Skewness	1	1	0	2
Shift	0	1	0	1
p-based	0	2	0	2
q-based	0	0	0	0
Overall	3	4	0	7

Sandra & Luzie	High quality	Medium quality	Low quality	Overall
Center	0	0	0	0
Spread	0	0	0	0
Skewness	1	1	0	2
Shift	0	0	0	0
p-based	0	3	0	3
q-based	0	0	0	0
Overall	1	4	0	5

Figure 5: Codings (group comparison elements) distinguished by pairs

we can say that all of the statements and conclusions which were done by the pairs are at a high (35% of the codings) or medium (65% of the codings) quality level. Let us now have a look on the codings distinguished by pairs.

Hilde & Iris use amongst others center and spread elements - both of their statements using the group comparison element center were on a medium quality, one of their two elements regarding spread are on a high, the other on a medium quality. The conclusions regarding the comparison of shift of the both distributions are all on "high" quality. They also offer a medium quality *p*-based comparison, but they do not use a comparison of skewness or a *q*-based comparison. Conrad & Maria do not show any high quality statements in the whole solving process of the task. They offer at least three statements at medium quality. Conrad & Maria only use spread and shift elements but no center elements. They also do not use any skewness element neither do they use a *p*-based or a *q*-based comparison. Having a look on the codings of Laura & Ricarda we can say that they use every comparison element except *q*-based comparisons. They use center and spread (both in high quality) to compare the distributions and also shift and *p*-based comparisons (all in medium quality). Sandra & Luzie do not use center, spread or shift elements at all and work out differences from both distributions using skewness elements (one in high quality, one in medium quality) and *p*-based comparisons (all (3) in medium quality).

CONCLUSIONS

The statements of the pairs offer a broad variety of the use of comparison elements and none of the teams show low quality group comparison elements when working on the task. Most of the statements relating to the codings of "medium" quality could have been improved with the addition of an interpretation of the differences. Nevertheless we have to report some shortcomings which occurred: The amount of codings (overall: 23) is low. That means that all four teams made 23 comparison statements in total. On a first view this aspect is not necessarily negative, but in the course we have set up the norm that a group comparison should include as many investigations as possible. In this task there could have been found several differences along all aspects (center, spread, skewness, shift, *p*-based and *q*-based), so we finally expected some more codings relating to the comparison of both distributions. Whereas Hilde & Iris and Laura & Ricarda made eight respectively seven group comparison statements, Conrad & Maria only made three of them. *Q*-based comparisons were not used at all by the teams, although they played a big role when e.g. comparing boxplots in our course. Comparisons of skewness of both distributions were only done by two teams and apart from Hilde and Iris, the shift between both distributions was not worked out adequately. *P*-based comparisons were all done without interpreting the differences and are therefore all rated on a medium quality. As a further step in research we will take into account our findings from all three dimensions and search for relations between them. With these findings and the re-design of the course in mind we might conclude that our norm "to work out as many differences between two distributions as possible" should be made more explicit. A data analysis scheme, which structures the data analysis process and gives hints of possible comparison elements, might support learners when comparing groups. Furthermore we might conclude that there should be a closer focus on interpreting differences between the distributions regarding center, spread, shift, etc. This may be done with contrasting adequate and non-adequate examples in regard to comparisons via center, spread, etc. Additionally we might reemphasize comparing groups with q-based comparisons in an upcoming course.

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ENDNOTES

1. Comparisons of two distributions of numerical variables are called p-based, if for x the relative frequencies $h(V \leq x)$ are $h(W \leq x)$compared. So in p-based comparisons a specific argument can be given (for example: 10 hours) und the proportion of cases which are equal or larger than 10 hours is compared in both groups. (see Biehler, 2001, p. 110)

2. Comparisons of two distributions of numerical variables are called *q*-based, if for a proportion p between 0 and 1 the matching quantiles of the variables V und W, $q_V(p)$ with $q_W(p)$, are compared. With $q(p)$ we mean the quantile regarding to p. For $p = 0.5$ this is a comparison of medians. (Since the comparisons of medians is also included in the category "center" we do not want to include this special case for $p = 0.5$ here). (see Biehler, 2001, p. 110)

An investigation of understanding of preservice elementary mathematics teachers (PEMT) about data displays

Kurt Gamze

Middle East Technical University in Ankara, Faculty of Education, Ankara, Turkey, gkurt@metu.edu.tr
Mersin University, Faculty of Education, Mersin, Turkey

Understanding to what extent preservice teachers are capable of conceptual and procedural knowledge of data displays is the aim of this study through analyzing a part of data collected during an independent study course. Change in the middle school curriculum in Turkey necessitates the study of examination of understanding of PEMT about statistics. Therefore, this study is significant in supporting the needs of teacher educators as well as it contributes to the consequences of curriculum efforts. In order to possess an understanding of statistical concepts for preservice mathematics teachers, they must have both conceptual and procedural knowledge (Hiebert & Lefevre, 1986), which is the main concern of this study.

Keywords: PEMT, data displays, conceptual knowledge, procedural knowledge.

INTRODUCTION

In our revised curriculum, which started to be instructed in middle-level schools in Turkey in September 2013, statistics has been the most emphasized subject. It was considered as a separate learning area named as data handling and it was included in all grades from the 5th grade through the 8th grade. However, the content of probability was reduced compared to previous curriculum, and it is placed into the 8th grade level only with a basic understanding of probability. These changes in the new curriculum could be identified as reflection of Moore's (1997, as cited in Biehler, Ben-Zvi, Bakker, & Makar, 2012) recommendation emphasizing that curriculum needs more statistics and less probability while leaving the deeper conceptual knowledge to the high-school level. The earlier studies showed that PEMT have less comprehension of statistics and probability compared with the other learning areas of curriculum, that I,; they found probability and statistics subjects difficult to teach especially because of their lack of content knowledge in probability and statistics (Quinn, 1997; Stohl, 2005). Contemporary efforts are addressing the same issue as well, so that teacher education should be enhanced through giving an attention to statistics and probability teaching for mathematics teachers (Stohl, 2005; Jones & Thornton, 2005).

Change in the elementary school curriculum necessitates the study of examination of understanding of PEMT about probability and statistics. It should be understood whether preservice elementary teachers have both conceptual and procedural levels of understanding of probability and statistics in order to teach them (Star, 2005). Thus, this study is significant in the above needs of the Turkish mathematics education literature as well as it contributes to the consequences of curriculum efforts and will be a light for future considerations of this issue. This study is part of a relatively larger study which was conducted by the researcher in spring semester in 2014. The research questions examined here are as follows: (a) to what extent are PEMT capable of conceptual and procedural knowledge of data displays? (b) what are the main strengths and weaknesses of PEMT in data displays?

REVIEW OF RELATED LITERATURE

Statistical knowledge for teaching can be interpreted under the framework of mathematical knowledge for teaching. This framework has two main dimensions for mathematical knowledge for teaching: first, subject matter knowledge which includes common content knowledge (CCK), specialized content knowledge (SCK) and knowledge at the mathematical horizon; second, pedagogical content knowledge which includes knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of curriculum (Hill, Ball, & Schilling, 2008). From the statistics point of view, CCK is considered as computing and interpreting the most frequent measures of central tendency; SCK is considered as special for teaching as which is best for which statistics term; horizon knowledge is considered as working on populations will eventually emerge the working on samples, for example. For the second dimension, KCS can help teachers to catch the common strategies which students use in developing students' statistical reasoning; KCT deals with the content-specific strategies like knowing how to explain arithmetic mean as a fair share or as a balance point; and knowledge of curriculum can help teachers about structural properties that a curriculum possess (Groth, 2012). Therefore, Groth (2012) has developed a framework for combining above terminology and suggested the figure in his paper.

Based on the efforts in conducting the course which Groth (2012) was teaching, namely as Statistical Knowledge for Teaching (SKT), he has developed the framework for SKT, while adding two new constructs to the statistical knowledge for teaching framework, one of which is key developmental understandings and the second one is pedagogically powerful ideas. Key developmental understandings were defined as "cognitive landmarks in the learning of fundamental ideas needed to understand content" (Simon, 2006, as cited in Groth, 2013). Pedagogically powerful ideas can be defined as ideas that occur as the result of transforming key developmental understandings into ideas that facilitate students' learning of the key developmental understandings. Groth (2013) in his hypothesized framework relates these two dimensions with the other existing dimensions of mathematical knowledge of teaching.

According to above framework, subject matter knowledge needed for statistics and probability teaching is the concern of this study and its sub-dimensions such as conceptual and procedural knowledge types should be integrated to it as well. The terms were introduced by Scheffler (1965), but expanded by Hiebert and Lefevre (1986) and Star (2005), where "conceptual knowledge is characterized most clearly as knowledge that is rich in relationships, like a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (p. 3). They also categorize the conceptual knowledge as primary and reflective. The primary level of conceptual knowledge is formed as in "the relationship connecting the information is constructed at the same level of abstractness (or at a less abstract level) than that at which the information itself is represented" (Hiebert & Lefevre, 1986, p. 5). The reflective level of conceptual knowledge is constructed in 'a relationship which requires a higher, more abstract level than the pieces of information they connect' (p. 5). Apart from conceptual knowledge, Hiebert & Lefevre (1986) also explains the procedural knowledge in two types: "one kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols; the second kind of procedural knowledge consists of rules or procedures for solving mathematical problems" (p. 7). In order to develop an understanding of statistical concepts for PEMT, they must have both conceptual and procedural knowledge (Hiebert & Lefevre, 1986). Ball (1988) described the subject matter knowledge similar to above; she names the procedural knowledge as substantive knowledge, which refers to "understanding of particular topics, procedures, concepts and relations among them" (p. 4), and secondarily, knowledge about mathematics is named in place of conceptual knowledge. Hence, this type of categorization fits with the above expressions and summarizes our theoretical framework.

METHODOLOGY

This study has a phenomenological approach which is categorized by one of the qualitative research designs (Creswell, 2007) since researcher tries to understand the shared experiences and understandings of a group of participants on data displays and their weaknesses/strengths as a phenomena.

In order to investigate its research questions, data were collected through face-to-face interviews (later transcribed verbatim). 23 participants from PEMT were interviewed voluntarily, 12 of whom are 4th year students and the rest are in their 3rd year of elementary mathematics teacher education. They have already attended to a statistics and probability course (which is a must course in their undergraduate education) in their second year and it includes the probability and statistics subjects in a very advanced and a theoretical form. Apart from that, six participants had taken the "methods for statistics teaching" (which is a fourth year must course in one of the universities studied) before the instrument and they specified especially that they learned statistics through that course.

In order to investigate the understanding of PEMT's knowledge of statistics and probability subjects framed by elementary school curriculum, an instrument was used. It involves open-ended questions as well as multiple-choice items which measure both procedural and conceptual types of knowledge required for understanding of statistics and probability. Specifically, it involves statistics questions related to measures of central tendency (mean, median and mode), measures of variation (range) and data displays; and probability questions related to probability of a basic event, certain, impossible and equally-likely events, theoretical and experimental probability. Some items are presented at the end of the paper as Appendix. The instrument is organized as most of the questions were taken from two tests written by Diagnostic Teacher Assessments in Mathematics and Science developed at the University of Louisville (CRiMSTeD- Center for research in Mathematics and Science Teacher Development) with their permission to use. There is also one question taken from Jacobbe's (2007) dissertation. These assessments have established high levels of reliability and validity (Bush et al, under review, as cited in Jacobbe, 2007). Since the instrument was implemented to the participants as face-to-face interviews, it is necessary to mention about the interviews summarized below.

Each interview is composed of three sections. In the first section, participants were asked questions related with their choice of being elementary mathematics teacher, interest of mathematics and mathematics teaching, subjects which they know as best and least covered in middle school mathematics curriculum, comments on change on curriculum regarding the statistics and probability subjects, interest of learning/using technological tools/materials needed for teaching and comments on test items at the end of the interview. In the second section, participants were directed to questions related with the measures of central tendency, measures of variation and probability, such as 'how do you define mean?' or 'what does mode of a group of data tell you?' in procedural and conceptual knowledge levels. Thus, the findings of this part were the results of test items according to the subjects. In the last part, participants were given the test and they were expected to solve open-ended questions as orally. Implementing the instrument as in this way provides us in order to learn how preservice mathematics teachers understood probability and statistics, more specifically their conceptual and procedural way of understanding on these subjects.

The last two sections consider the content knowledge of participants while taking its different types as procedural and conceptual into consideration comparatively. That is, these sections compensate for each other in order to investigate the understanding of participants on the subjects which analyzed.

The analysis of the data gathered from first and second part of the interviews was done according to themes and codes specified before the implementation. The evaluation of the data gathered from the instrument was performed according to a rubric which was prepared for only open-ended items. The evaluation of open-ended items was done in only three categories: full response, incomplete response and wrong response. A full response addresses that participant gives the best explanation using the right terminology and expected logical foundations in order to rationalize the subject whereas an incomplete response addresses that participant does not give a fully satisfying response or s/he cannot rationalize his/her response. A wrong response means that participant gives a completely wrong response without making any logical explanation or rationalization by using his/her understanding of the subject.

FINDINGS

Although the methodology part mentions about the parent study of the study framed here, only the findings for data displays will be presented in order to answer the research questions outlined above. The instrument included 22 items and 11 items were associated with data displays. The responses given to each item and the subject related with the item are given below in the table:

Most of the participants did not know about most of the graphical representations which directed as questions through the instrument. For example, all of the participants do not seem to have any idea about box-and-whiskers plot and they couldn't catch the me-

Question	Type	Related Subject	Ratio of Achievement
Item 1	Multiple Choice	Stem-and-leaf display	2 correct, 1 wrong, 20 of 23 have no information about topic.
Item 2	Multiple Choice	Graphical representations of data	2 correct, 5 have no information about the subject, 16 of 23 are wrong.
Item 3	Multiple Choice	Scatter-plot	13 of 23 are correct.
Item 4	Open-ended	Box-and-whiskers plot	All of them have no information about topic, and they analyzed graphic with their own understanding, and sometimes come up with correct answer. Hence, they answered incomplete.
Item 5	Open-ended	Biased graphic displays	2 of 23 responded full, 1 of 23 responded false, and 20 of them responded incomplete.
Item 6	Open-ended	Circle graph	13 of 23 responded full, 2 of them responded wrong, 8 of 23 responded incomplete.
Item 7	Multiple Choice	Line graph	10 of 23 are correct.
Item 8	Multiple Choice	Line graph	22 of 23 are correct.
Item 9	Multiple Choice	Frequency table	21 of 23 are correct.
Item 10	Open-ended	Line graph, bar graph, categorical variable	10 of 23 responded wrong, 10 of 23 responded full, 3 of 23 responded incomplete.
Item 11	Open-ended	Frequency distribution, mean, median, mode, range, normal distribution, data displays	8 of 23 responded incomplete, 15 of 23 responded full.

Table 1: Findings per item with respect to the subject and type

dian and percentiles from the representation given in the item 4 (in Appendix). Most of the participants do not know of stem-and-leaf display, either, as realized from the findings of item 1 (in Appendix) in the test. The item 2 is another item which has only 2 correct responses. Item 2 was asking the best description of the distribution of achievement test scores for all 4th graders in a school while choosing among scatter plot, box-and-whiskers plot, line graph and circle graph. This finding is expected since box-and-whiskers plot is not known by the participants so as in item 4.

The scatter-plot question (item 3) has a half success by the participants. Most of the participants cannot relate each dot with two axes onto the graph, many of the correct responses were given with an unclear explanation. Although, all of the participants could realize the difference between two different graphs with the same data, they couldn't explain its consequences correctly and their responses show a misunderstanding or inadequate knowledge of biased graphical data displays. Item 6 asks for reasons of mistakes made by a student who draws a circle graph of a given set of data and how to overcome those mistake. Most of the participants could easily found the mistake, however, nearly half of them could present ways to overcome it.

Item 7 (in Appendix) is having one of the most wrong respondents which is also an unexpected result compared with the result of other line graph question, item 8. Item 8 asks the true alternative for the difference of average salaries taken by university or high school graduated workers and it includes two line graphs on the same display. Item 9 presents the frequencies of data as a rotated bar graph and asks the true alternative among the other 3 false alternatives like in the previous item. The responses for this item are mostly correct and we can say that the most achieved items which the participants responded with a higher rate than others were the 8th and 9th items. Another item which nearly half of the participants responded correctly is the item 10 (in Appendix) since they could realize the importance of having a categorical variable (like in this item) for a bar graph in this item. The item 11, which was taken from the Jacobbe's (2007) dissertation, is the longest question in the instrument. Nearly two-thirds of the participants responded as full, the rest of them could neither grasp the dot plot display nor explain their choice with a satisfying rationale.

DISCUSSION

The findings of this study show some aspects mentioned in the theoretical framework for preservice mathematics teachers' understandings described by Groth (2013), Hiebert & Lefevre (1986) and Ball (1988). It can be said that the questions directed to participants during the interview, they could not show their conceptual knowledge about data displays since their answers were mostly in procedural knowledge base. In general, PEMT has a high achievement in procedural level of knowledge for data displays.

It is also worth to mention that the responses given to the items in the instrument shows also resemblance to the findings through interviews which investigates the understanding of the statistical concepts, specifically measures of central tendency and measures of variation. For definitions of mean, median and mode, participants seem to not know the difference between calculations and meanings of them, as emphasized before (McGatha, Cobb, & McClain, 1998, as cited in Jacobbe, 2007). This can be discussed that they don't know the foundations under mean, median and mode and can be explained as there is a difference between knowing how something is calculated and knowing why it is (Hiebert & Lefevre, 1986).

Moreover, based on the findings of the interviews, it can be claimed that participants has a high achievement in procedural level of knowledge for measures of central tendency. They mostly know concepts; but, most of the participants have difficulty in answering questions necessitating conceptual knowledge, which are connected with the subjects of meanings of measures of central tendency as related with data displays, difference or relation between those. It can be argued that participants have not enough ability to connect what they know about measures of central tendency and the associated data displays; besides, they have not a higher-order comprehension needed for knowledge answering to the questions (Ball, 1988; Hiebert & Lefevre, 1986; Groth, 2013).

When we consider the possible reasons of why conceptual knowledge of PEMT have been less-developed compared with procedural knowledge, the courses offered for teacher candidates during their university education are like 'recipe-type' or 'rule-bound' courses which only deals with the calculations and lead preservice teachers to memorize the subjects while underestimating the logic behind it, as Shaughnessy (1992) stressed out previously. He claims also that preservice teachers lack of opportunity to develop their stochastic reasoning in university courses due to their misunderstandings about statistics. Nearly half of the participants have stressed during interviews that they feel themselves not knowing very well about statistics although they have taken a course namely as "statistics and probability" which they took in their 2nd year. The other half of the students have mentioned that they have a course about "teaching probability and statistics in elementary level". However, unless they learned about statistics very well, they do not feel to be able to teach it. Hence, they first need to know it, as they expressed and eventually corresponds with the arguments of Shaughnessy (1992). Although it was not one of the interview questions, most of the participants mentioned about their complaints about "statistics and probability" course. They emphasized that the course have not included any practical implementation to a real-life example, but was mostly theory-based and proof-based. Therefore, most of them specified that they could not learn much about statistics or probability subjects in this course, for example, they did not meet even with a boxplot display during the course.

In brief, this study discussed the understanding of PEMT for statistics, specifically the issue of data displays. Findings implied that content knowledge assessed by the items in the instrument have two dimensions, procedural and conceptual knowledge, as discussed clearly by the researcher previously in the review of related literature part above (Hiebert & Lefevre, 1986; Ball, 1988; Groth; 2013) and corresponds to the framework which was bounded.

The implications of this study will be enlightening to the future research for the understanding of PEMT in Turkey. The discussion of the findings can have an impact on teacher education programs in the universities. While some universities have currently specific courses related with pedagogical content knowledge for statistics and probability including content knowledge needed for those subjects as well, some of them have not. As specified before, PEMT are not ready for teaching statistics because of lack of statistical knowledge (Greer & Ritson, 1994). Therefore, as the participants of this study stressed out that they should learn statistics very well while learning methods of teaching statistics. Having such consequences, this study can have positive influences on the development of elementary mathematics education programs, and might affect the perspectives of teacher educators, who are responsible for educating the teachers, as well.

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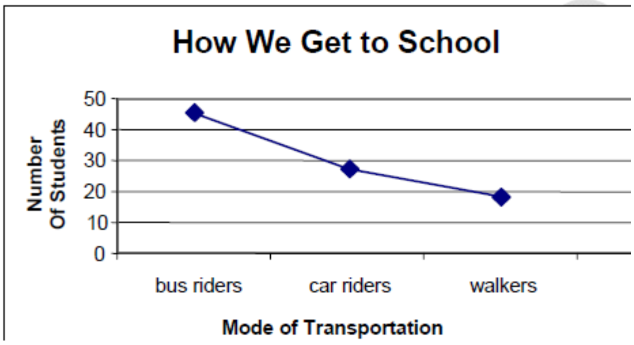
APPENDIX: SAMPLE ITEMS FROM THE INSTRUMENT

Item 7: Which of the following best describes a line graph?
a. A graph that visually represents the median, the quartiles, and the smallest and largest values of a data set.
b. A graph consisting of a horizontal number line with data points represented by X's.
c. A graph consisting of points, one for each item being measured. The two coordinates of a point represent the measures of two attributes of each item.
d. A graph with a vertical and horizontal axis that is primarily used to show changes over time.

Item 10: A student collected data from 90 fourth grade students about how each student traveled to and from school. The student created the following graph. How would you respond to this student?

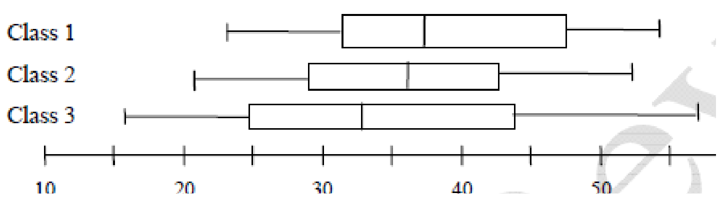


Item 1: The students in a sixth-grade class were timed to the nearest second for a run around the school's gymnasium. The times for the class are listed below in a stem-and-leaf plot. Which of the following is true?

2	4 8
3	1 3 4 5 6 7 8
4	1 2 3 7 7 8
6	0 1 4 8 9

a. The lowest time was 28 seconds
b. Half of the students had times above 41 seconds
c. The highest times was 60 seconds
d. 50% of the students had times below 38 seconds

Item 4: The box-and-whiskers plot below represents the test scores of three classes on the same test.
a. Which class performed the best and which class performed the worst.
b. Provide justifications for your choices with data from the box-and-whiskers plots.



Regression in high school: An empirical analysis of Spanish textbooks

María M. Gea, Carmen Batanero, Pedro Arteaga, J. Miguel Contreras and Gustavo R. Cañadas

University of Granada, Faculty of Education, Granada, Spain, mmgea@ugr.es

The aim of this study was to analyze the presentation of regression in Spanish school textbooks aimed at Social Sciences students. In a sample of eight textbooks we analyzed the problem-fields and suggested procedures, as well as the associated concepts, which are classified according to the way they are introduced and the properties associated with them. Our results suggest that these textbooks mostly reduce regression to lineal regression, present many properties at an operational level without deep discussion of their meaning, and there is a great variety between the textbooks. Some suggestions to improve the presentation of this topic are included.

Keywords: *Regression, textbooks, high school, mathematical objects.*

INTRODUCTION

Correlation and regression are fundamental statistical ideas, due to their usefulness to model phenomena in different fields (Engel & Sedlmeier, 2011). Previous research is mainly focused on students understanding of correlation and describes several misconceptions (Estepa & Batanero 1995; Estepa, 2008; Zieffler & Garfield, 2009). There has not been, however, much interest in the way the topic is taught or presented in the textbooks, even although textbooks are important educational tools. From the official curricular guidelines to the teaching implemented in the classroom, an important step is the written curriculum reflected in textbooks (Herbel, 2007). The selected textbook is an important part of teaching and learning mathematics at secondary and high school level, since the presentation of the topics and the problems proposed provide the main basis of why the topic is taught (Shield & Dole, 2013). Moreover, in other mathematical topics, textbooks receive increasing attention from the international community; see for example Fan, and Zhu (2007).

The aim of this research was to analyze the presentation of regression in high school Spanish textbooks aimed at Social Sciences students. It is part of a wider project where we compare the way in which correlation and regression are presented in the textbooks in Spain. Due to length limitation we only present here a part of our analysis; complementary results were published in (Gea, Batanero, Cañadas, & Contreras, 2013), and (Gea et al., 2014).

THEORETICAL BACKGROUND

According to Rittle-Johnson, Siegler and Alibali (2001) conceptual and procedural knowledge are the two extremes in a continuous; conceptual knowledge is more flexible and includes the implicit or explicit understanding of a domain and its interrelationships. Sfard (1991) suggests that a concept is a construct which corresponds to the mathematical universe and distinguishes two types of definitions of the same: structural (where concepts are introduced by describing their essential conditions or properties) and operational (when it is described by a formula) definitions. In our analysis we will take into account both types of definitions and will also describe in detail the properties associated with regression.

The division between conceptual and procedural knowledge is analyzed in the onto-semiotic approach to mathematics education (Godino & Batanero, 1998; Godino, 2002) where mathematical activity plays a central role. Knowledge in this framework is modelled in terms of systems of practices, in which different types of objects intervene when a subject faces the solution of a given problem. "Object" is understood in a broad sense for any entity which is involved in mathematical practice and can be identified separately. More specifically in our analysis we will consider the following types of mathematical objects: problem-situations; procedures; concepts and properties.

Previous research

In spite of the relevance of the topic, previous research suggests poor results in people's understanding of correlation. For example, Chapman and Chapman (1967) described *illusory correlation* where people are often guided by their theories rather than by data when estimating correlation. Estepa (1994) defined several misconceptions including the *deterministic conception*, where the student only considers whether functional dependence exists. Sánchez Cobo (1999) described students' failures in ordering correlation coefficients and in interpreting the relationship between the correlation coefficient and the slope of the regression line. As regards research on textbooks, Sánchez Cobo (1999) classified the definitions of concepts presented in 11 textbooks published in the period 1977–1990 as procedural, structural or a mixture of both. Lavalle, Micheli and Rubio (2006) analyzed the concepts and procedures included in 7 high school textbooks from Argentina.

As stated in the introduction, this paper is part of a more comprehensive project. In (Gea et al., 2013) we analyze the problem-situations used to contextualize correlation and regression in eight Spanish textbooks and in (Gea et al., 2014) the symbolic, verbal and graphical language used in this topic. The current paper complements these publications with the analysis of concepts, properties and procedures included in the study of regression in the same textbooks, which were not included in these papers. In each of these types of objects we analyzed all the categories that are relevant for the teaching of the topic.

METHOD

The sample consisted of eight mathematics high school textbooks aimed at Social Sciences students and that were published just after the current curricular guidelines were introduced (MEC, 2007). These particular textbooks have a wide diffusion in Spain due to the publishers' prestige and are still used in the schools (See Appendix). We performed a content analysis (Neuendorf, 2002) of the chapters devoted to correlation and regression with the following steps: a) following an inductive and cyclic procedure we first categorized all the different mathematical objects explicitly or implicitly included in the chapter (the different types of problems, concepts, properties and procedures); b) For each of them we analyzed the way in which they are described or used in the textbooks. In this paper we only describe the mathematical objects linked to regression; as the remaining results have been published elsewhere (Gea et al., 2013; 2014).

PROBLEMS AND PROCEDURES

Anthony and Walshaw (2009) reported on the different types of tasks that have been analysed in mathematics education research, that include problems focused on specific mathematical content; problems that promote mathematical modelling; others that require discussion of aspects that vary; those that ask students to interpret and criticise data and those that prompt sense making and justification of thinking. Gea and colleagues (2013) identified two main types of problems in the study of regression in which one of several of the above type of tasks may be combined: a) Fitting a model to the data requires sense making and criticism of data; mathematical modelling, discussion of variation and justification of thinking, b) Using the model to predict a value of the dependent variable is a more computational type of task; however, much of the time judgement of the goodness of fit involves criticism, decision making and justification. Both types of problems appear with different frequency in all the books; each of these types of problems range from 20% of all the problems posed in the chapter in [H4] to 35% in [H6] 1. To solve these two types of problems, the books introduce the following procedures:

P1. Fitting the least squares line. Lineal regression by the least squares method is included in all the books, with two different procedures to compute the line of best fit: a) Equation of the line which includes the gravity centre (point whose coordinates are the mean for the two variables); and b) General expression for the lineal function:

Regression line $Y = f(X)$
a) $y - \overline{y} = \mathrm{b}_{yx} \cdot (x - \overline{x})$ b) $y = a + b_{yx} \cdot x$

Regression line $X = f(Y)$
a) $x - \overline{x} = \mathrm{b}_{xy} \cdot (y - \overline{y})$ b) $x = a' + b_{xy} \cdot y$

1 All the books also introduced the analysis of the sign and strength of correlation between the variables as a previous problem as well as problems related to graphical representation of bivariate data. We are not including here these problems that were analyzed in (Gea et al., 2013).

P2. Fitting other regression models. Only [H8] includes the regression towards the median developed by Tukey, which is a robust method in the presence of outliers as well as variable transformation to fit exponential and polynomial models.

P3. Computing the determination coefficient and interpreting the goodness of fit. Only a few textbooks introduce the determination coefficient r^2 as a measure of the goodness of fit; in all of them the coefficient is previously defined.

P4. Prediction with the line of best fit and interpreting the goodness of prediction. All the books propose some tasks where the students have to estimate the response variable *Y* given a value for the explanatory variable *X*. Some of them ([H1], [H2]) suggest that when the correlation coefficient is strong it is possible to use the same line to predict *X* from a value of *Y*. Since this property is not general (as there are two different regression lines) students could make an incorrect generalization. Some books qualitatively evaluate the goodness of fit by comparing the estimated and observed values for isolated data. This informal method is not reliable, as the reliability of estimation depends on the goodness of fit (given by the determination coefficient) as well as on the closeness of the estimated value to the centre of the distribution of the explanatory variable.

In Table 1 we observe that all the books include the least square line, as well as its use for prediction, while only one includes the Tukey line and a few the evaluation of the goodness of fit. A missing point is the use of informal "eye fitting" methods that are useful to build the students' intuitions and can be implemented with applets (e.g., //docentes.educacion.navarra.es/msadaall/geogebra/figuras/e3regresion.htm). Results are better than those by Lavalle, Micheli and Rubio (2006); where only half the books include the line of best fit and 60% of the books include prediction activities.

CONCEPTS AND PROPERTIES

In our analysis we found the following concepts linked to regression:

C1. Dependent (response) and independent (explanatory) variable: While correlation is symmetric, regression is asymmetric; for this reason we should discriminate the response and explanatory variables (Estepa, 1994). Only a few books make the distinction explicit although in other books it is implicit when they introduce two different regression lines.

C2. Model of fit. The idea that the regression line is only a model (and therefore does not exactly coincide with all the data) is only implicitly introduced; a couple of books make this idea explicit: "*When there is strong correlation between X and Y the analysis of regression helps to find a mathematical function as a model to fit the data. This function can be a straight line, a parabola, exponential…*" ([H4], p.226).

C3. Line of best fit (linear model). All the textbooks include the definition and explanation of the minimum squares method (in an informal way); however, only a few of them justify the utility of the model to estimate values of *Y* in situations where the variable is difficult to measure. Moreover, as in Sánchez Cobo (1999), few texts highlight the predictive utility of the regression line.

C4. Regression coefficients. Since there are two possible lines of best fit (depending on which is the explanatory and response variable) there are two different regression coefficients, but only a few books make this explicit. They also include the interpretation of these coefficients: "*The line that minimizes the sum of residuals* Σd_i^2 *is given by the following expression:* $y = \overline{y} + \sigma_{xy}(x - \overline{x})/\sigma_x^2$. *The slope* σ_{xy}/σ_x^2, *is the regression coefficient*" ([H1], p. 230).

Procedures	H1	H2	H3	H4	H5	H6	H7	H8
P1. Fitting the least squares line	x	x	x	x	x	x	x	X
P2. Fitting other regression models								X
P3.Computing the determination coefficient and interpreting the goodness of fit					x	x		
P4. Prediction and interpretation (line of best fit)	x	x	x	x	x	x	x	X

Table 1: Regression analysis procedures included in the books

C5. Goodness of fit. Coefficient of determination. These concepts help the student understand the meaning of regression; however they are only included in [H5] and [H6], where the accuracy of the model is identified with the accuracy in the prediction for any particular point. This is not true, as the accuracy is higher when the point approaches the centre of the distribution of the explanatory variable.

C6. Non lineal models of fit. Regression is a general method for understanding relationships between variables (Moore, 2005), and therefore it is necessary to introduce different models of fit; however, only a few textbooks implicitly define non lineal models, where [H5] and [H6] define them explicitly.

We summarize the definition of concepts in Table 2, where we observe the predominance of adding examples to the definitions; usually the texts include scatter plots with the line of best fit added to show the residuals. The line of best fit is introduced in all the books and is generally defined both in a structural and operational way (Sfard, 1991). The presentation is very similar to that in Sánchez Cobo (1999). Other definitions (regression coefficients, dependent and independent variable, goodness of fit and non linear models) are missing in some textbooks or are only defined in an operational way. The definitions are introduced in different orders; sometimes the examples are followed by the definition and vice-versa. In the same way the order to introduce operational or structural definitions also varies.

Properties

The textbooks add different properties to the definition of the concepts or to relate different concepts, as described below:

P1. Least squares property. Most textbooks explain that the regression lines make the sum of residuals from the points to the line minimum. Usually the explanation is only visual (a formal deductive proof is avoided).

P2. Two different regression lines. Most of the books implicitly remark that there are two different regression lines and part of them warns the students of the danger of using an inadequate line to make a prediction. Two books ([H2] y [H8]) do not remark on this property. This omission may reinforce the deterministic conception of some students (Estepa, 1994), since in deterministic dependence there is only one algebraic expression (function) to express the dependence.

P3. Percentage of variance explained (r2). The determination coefficient measures the goodness of fit. Some textbooks also analyze its interpretation as the percentage of variance explained by the regression line: "*(r^2x100)% is the percentage of variance of Y explained by the value of X*"([H6], p.185).

P4. Estimation using the regression line. The regression line serves to predict the value of response (*Y*) given a value of the explanatory variable (*X*). The books implicitly indicate that these estimates are only approximations. They insist that, contrary to functional dependence, there are several values of *Y* for a given value of X, and the regression line provides the average of all these values: "*These estimates are approximations and involve a probability; it is probable that when $x = x_o$ the value of y is approximately $y(x_o)$.* ([H1], p.230).

P5. The regression line crosses the distribution centre of gravity, a property only included in half the books in the study by Sánchez Cobo (1999).

P6. Estimates are more accurate for values closer to the centre of gravity. However some books only judge the reliability of estimates by the value of the correlation coefficient.

Concepts	H1	H2	H3	H4	H5	H6	H7	H8
C1. Dependent and independent variable					O			O
C2. Model of fit			ES	SE				
C3. Line of best fit (linear models)	ESO	EOS	SO	SOE	SOE	SOE	SOE	SO
C4. Regression coefficients	O	SO		O				
C5. Goodness of fit.					SOE	OE		
C6.Non linear models					SOE	SOE		

E = Examples; O = Operational definition; S =Structural definition

Table 2: Concepts linked to regression and type of definition

P7. Reliability of estimates and sample size. It is included only in a few books: *"The estimate accuracy increases with the number of data; the regression line computed with few data has little reliability even if r is high"* ([H5], p. 260).

P8. Strength of correlation and angle of regression lines: This angle varies from perpendicular lines (independence) to only one line (perfect linear dependence).

P9. Regression line, covariance, correlation. Covariance and correlation are interpreted as regards the closeness of the points to the regression line. Their sign is related to the slope in the regression line: "*Depending on the position of* (x_i, y_i) *as regards* $(\overline{x}, \overline{y})$, *the product* $(x_i - \overline{x}) \cdot (y_i - \overline{y})$ *will be positive or negative. If many points are close to a line with a positive slope, most of these products are positive and the covariance and correlation are positive"* ([H1], p. 228). These properties were not found in the books analyzed by Sánchez Cobo (1999).

P10. Product of regression coefficients. Some books suggest that the product of regression coefficients is equal to the square of the correlation coefficient: r^2. This property was found in most books in Sánchez Cobo's research (1999) but is found in only two books in our study.

PP11. Correlation and reliability of estimates. Most textbooks relate both concepts: "*The higher the correlation coefficient r, the higher the reliability of estimates: when r is close to zero, there is not much sense in doing an estimation; as r approaches to 1 or -1, the real values will approach our estimates; when r = 1 or r = -1, real values and estimates coincide"* ([H4], p. 226).

In Table 3 we summarize the properties of regression included in the books. There are great differences between textbooks, because while some of them ([H4], [H8]) hardly describe any of the properties analyzed, others ([H3]) include almost all of them. The most frequent property is the least squares, the existence of two different regression lines, estimation with the line, centre of gravity crossing the line, and relationship of reliability in the estimate, centre of gravity and correlation coefficient. Globally the books introduce a rich set of properties of regression. We remark that some textbooks do not include the properties P8 and P9; this omission may reinforce the students' failures in interpreting the relationship between the correlation coefficient and the slope of the regression line described by Sánchez Cobo (1999).

DISCUSSION AND IMPLICATIONS FOR TEACHING

Our results suggest little changes in the presentation of regression in the high school textbooks, as regards the analysis by Sánchez Cobo (1999), although the books studied by this author were published between 1977 and 1990. Our analysis complement that research and that by Lavalle, Micheli, & Rubio (2006), because neither of these previous studies analyzed the properties linked to regression, in spite of the fact that Sfard (1991) considered that the properties are an essential part of the concepts. This study also complements our previous studies: (Gea et al., 2013), where we analyzed with more detail the problem-situations used to contextualize correlation and regression, and (Gea et al., 2014) where we described the symbolic, verbal and graphical language used in this theme.

	H1	H2	H3	H4	H5	H6	H7	H8
P1. Least squares property	x	x	x		x	x	x	x
P2.Two different regression lines	x		x	x	x	x	x	
P3. Percentage of variance explained (r2)					x	x		
P4. Estimation using the regression line	x	x	x	x	x	x	x	
P5. Regression line and centre of gravity	x	x	x		x	x	x	x
P6. Reliability of estimates and centre of gravity	x	x	x		x	x	x	x
P7. Reliability of estimates and sample size			x		x			
P8. Strength of correlation- angle of regression lines	x		x				x	x
P9. Regression line, correlation, covariance	x		x		x	x	x	
P10. Product of regression coefficients	x		x					
P11. Correlation coefficient – reliability of prediction	x	x	x	x	x	x	x	

Table 3: Properties of regression

Although all the textbooks introduce linear regression and propose methods to compute the line of best fit and make predictions with the same, only a few of them introduce procedures to compute the determination coefficient and its interpretation as a percentage of explained variance; however these properties are introduced theoretically with no practical applications or procedures related to the same. Similar to the study of Lavalle, Micheli, & Rubio (2006), only a few textbooks introduce examples of non linear regression, even when a few propose tasks where different models would be preferable. This coincides with Sánchez Cobo's research (1999) where some textbooks introduced examples of non linear regression without discussing these models.

We found few definitions of the concepts linked to regression, apart from the definition of the line of best fit. This fact could be explained because these textbooks use most of the available space for this theme in the study of correlation (computing correlation coefficients and interpreting its sign and strength), as we are shown in (Gea et al., 2013). Although correlation is no doubt an important concept, it doesn´t make much sense that the books devote so much space to its study if this study is not completed with the study of regression; the need to fit a model to the data is the reason to study correlation between variables; the isolated study of correlation is useless. We therefore recommend reinforcing the study of other concepts linked to regression, such as model, centre of gravity, regression coefficients, and parameters of the line of best fit (slope; coordinate at the origin).

In general, these textbooks cover many important properties of regression; the variety of concepts linked through these properties suggests the high semiotic complexity of the topic, due to the richness of the associated epistemic configuration. However these properties are mostly introduced at an operational level (Sfard, 1991), that is, present the property by a formula or procedure, with too excessive emphasis on computation and giving little relevance to the meaning and interpretations of these properties, which are scarcely introduced at a structural level. For example, the textbooks introduce the computation needed to determine the regression lines (formula); but there is no deep discussion of the meaning of the parameters in the lines (slope, ordinate in the origin). The relation of the regression line with the correlation coefficient is only established in an operational way to study the reliability of prediction, with no connection, for example of the sign of correlation and the slope of the regression line. In the same way there is no discussion of examples where a low correlation coefficient may be associated to a strong non linear dependence; for example a parabola. It is also important to emphasize explicitly the existence of two different regression lines; that may sometimes be very close when r is close to +1 or -1, but may be very different in the general case. Many times the difference is only implicit.

When comparing the different books, [H5] and [H6] are far more complete than the other books, as they present all the procedures for the linear model; they are the only books that define the goodness and non linear models (although they do not include the procedures for these models) and introduce the majority of properties analyzed.

ACKNOWLEDGEMENT

Project EDU2013-41141-P (MEC) and group FQM126 (Junta de Andalucía).

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APPENDIX: TEXBOOKS ANALYZED

[H1]. Colera, J., Oliveira, M.J., García, R., & Santaella, E. (2008). *Matemáticas aplicadas a las Ciencias Sociales I*. Madrid: Grupo Anaya.

[H2]. Arias, J. M., & Maza, I. (2011). *Matemáticas aplicadas a las Ciencias Sociales 1*. Madrid: Grupo Editorial Bruño.

[H3]. Anguera, J., Biosca, A., Espinet, M. J., Fandos, M.J., Gimeno, M., & Rey, J. (2008). *Matemáticas I aplicadas a las Ciencias Sociales*. Barcelona: Guadiel.

[H4]. Monteagudo, M. F., & Paz, J. (2008). *1º Bachillerato. Matemáticas aplicadas a las Ciencias Sociales*. Zaragoza: Edelvives (Luis Vives).

[H5]. Martínez, J. M., Cuadra, R., & Heras, A. (2008). *Matemáticas aplicadas a las Ciencias Sociales. 1º Bachillerato*. Madrid: McGraw-Hill.

[H6]. Bescós, E., & Pena, Z. (2008). *Matemáticas aplicadas a las Ciencias Sociales. 1 Bachillerato*. Vizcaya: Oxford University Press España.

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Mathematics teachers' conceptions of how to promote decision-making while teaching statistics: The case of Japanese secondary school teachers

Orlando González

Hiroshima University, Graduate School for International Development and Cooperation, Hiroshima, Japan, ogonzalez@hiroshima-u.ac.jp

The role of informed decision-making in today's society is very important at personal, professional, societal, and even educational levels. Thus, mathematics teachers are required, now more than ever, to provide students with appropriate tasks having the potential to promote decision-making skills. This article reports on the features and competence demands of tasks that are thought to promote decision-making skills by a purposeful sample of twelve Japanese secondary school mathematics teachers. A qualitative analysis on the collected data revealed some commonalities between the answers, as well as correspondences and discrepancies between what the participants and statistics educators think a task with the potential to promote decision-making is.

Keywords: Decision-making, teachers' conceptions of decision-making, statistical tasks, Japanese secondary school mathematics teachers, statistics education.

INTRODUCTION

Statistics has become very important at all levels of citizenry in today's society, in which large amounts of data are available to almost everyone. Thus, to be part of modern society in a competent and critical way, citizens need to be able to interpret such data in a broad sense, and understand the variability, dispersion, and heterogeneity which cause uncertainty in interpreting information, in facing risks, and in making decisions. In the particular case of the latter, many statistics educators, curriculum developers and international agencies around the world agree on the increasing importance for students to gain competence in using, handling and interpreting data to inform decision-making at personal, professional, and societal levels (Garfield & Ben-Zvi, 2008).

The last reform to the Japanese Course of Study echoes these ideas. In fact, the "difficulty … in description-type problems that require thinking, decision-making, and representation", reported in the 2006 Programme for International Student Assessment (PISA) study report about Japanese students by the Organization for Economic Cooperation and Development (OECD), is the first motivation to that reform (MEXT, 2008, p. 1; MEXT, 2009, p. 1). As a result, in the particular case of secondary school level, the latest Japanese Course of Study emphasizes – in the mathematical domains "Practical Use of Data" at junior high school, and "Analysis of Data" at senior high school – nurturing the attitude and ability to purposely process daily-life data, capture its trends and features, and make decisions based on such analysis (MEXT, 2008, 2009). Thus, on account of the significant place held by fostering decision-making skills in the Japanese secondary school mathematics curricula, teachers must be able to carry out themselves data-based decision-making, as well as to design instruction aimed to develop students' decision-making skills.

Despite the attention given to students' development of decision-making skills and attitude in the latest Japanese Courses of Study, teaching guides, and even the 2007 amendment to the School Education Law, what decision-making is, and how to promote the skills related to it, are not explained in any of those official documents. Therefore, teachers are left to determine by themselves how decision-making could be promoted in their students, which raises particular concern, due to the reported need for appropriate training in statistics education in the preparation of mathematics teachers in Japan (Isoda & González, 2012; González, 2014).

Then, in order to shed light on how Japanese mathematics teachers conceptualize decision-making and its promotion, the following research questions are addressed:

(1) What kind of tasks do Japanese secondary school mathematics teachers regard as having the potential to promote decision-making?

(2) What knowledge and skills do Japanese secondary school mathematics teachers believe to be associated with the promotion of decision-making?

THEORETICAL BACKGROUND

Decision – What is it?

A decision is defined as "the broader process within which a choice among specific options will be made" (Brown, 2005, p. 1). Through this process, the decision-maker is ultimately able to determine what action to take, by identifying a choice – i.e., selecting among previously-identified options (Brown, 2005, pp. 1, 236–237).

Decision-making situations demand from the decision-maker to determine a course of action, typically while considering uncertainty and data variability (Gal, 2004). In fact, a decision-maker operates, at once, in all the four dimensions of statistical thinking identified by Wild and Pfannkuch (1999).

Before choosing a particular course of action, a decision-maker has to engage in the many phases of the decision-making process, which are identified below (Wild & Pfannkuch, 1999; Gal, 2004, p. 43; Arvai, Campbell, Baird, & Rivers, 2004; Edelson, Tarnoff, Schwille, Bruozas, & Switzer, 2006):

Definition: here decision-makers define the specific decision that has to be made, as well as a broad set of end objectives in the context of the impending decision. This phase is the equivalent to the "Problem" step of Wild and Pfannkuch's (1999) statistical investigative cycle, known as the PPDAC cycle.

Planning: during this phase, the identification, design, and choice of optimal ways to use resources – i.e., means to achieve end-objectives – are determined. The choices must be a set of appealing and purposeful alternatives from the objectives previously defined. Here, the identification of constraints and considerations as basic criteria for the decision is also fundamental, since some choices might be eliminated based on failure to meet constraints, while others may be ranked based on how they fare on considerations. This phase is not only connected to the "Plan" step of the PPDAC cycle, but also with strategic type of thinking, and the "Generate" step of Wild and Pfannkuch's (1999) statistical interrogative cycle.

Data: the "Planning" phase tends to be followed by a recalling and seeking of information, as well as by collecting statistical data relevant to the achievement of end-objectives. This phase is connected to the "Data" step of the PPDAC cycle, to the "Seek" step of the statistical interrogative cycle, and to the recognition of need for data (Wild & Pfannkuch, 1999).

Evaluation: here, decision-makers must assess the implications of different choices for the decision, as well as the impact on the different stakeholders involved in the decision on hand. This phase is connected to the "Criticize" step of the statistical interrogative cycle, as well as to the strategic type of thinking, and being logical.

Weighing impact: during this phase, decision-makers weigh the impacts of the different options on stakeholders based on their own values. Thus, in this phase decision-makers bring in their values and see how different values can lead to different decisions. This phase is connected to both the "Analysis" and "Criticize" steps of the PPDAC and statistical interrogative cycles, respectively.

Making and justifying a decision: during this phase, decision-makers finally select the course of action that better addresses their objectives, in the light of the decision-makers' constraints, considerations, assumptions, value systems, judgment of probabilities, stakeholder impact, etc., and provide an informed justification for such a decision. This phase is connected to both the "Conclusions" and "Judge" steps of the PPDAC and statistical interrogative cycles, respectively.

Decisions – What types are there?

In the present study, decisions will be classified in four types: three of them – i.e., personal, professional and civic – were identified by Brown (2005, pp. 5–7), while the last type – i.e., object-related – is proposed by the author.

Personal decisions: those that decision-makers make on their own behalf; for example, about a career path or going on a date.

Professional decisions: those that decision-makers, as professional individuals and specialist decision aiders, make on behalf of others in a work capacity, such as in business management or medical practice.

Civic decisions: these are decisions made on a public – i.e., not personal – issue, such as when decision-makers, as citizens, take a private position on someone else's – e.g., government's – choice, for which they have no direct responsibility.

Object-related decisions: those that decision-makers make about parameters or particular features of the statistical objects – i.e., decisions regarding language situations, concepts, propositions, procedures and arguments (Godino, Ortiz, Roa, & Wilhelmi, 2011, p. 277) – involved in a given statistical problem. An example could be a decision about what measure it would be appropriate to employ to decide what the best design for a paper plane is – e.g., decide whether to use the distance travelled, accuracy, or time spent airborne to determine which design is the best.

Decision-making while studying statistics – How to do it?

According to Godino and colleagues (2011, p. 274), a basic statistical problem concerning decision-making is contextualized in a real situation; is driven by uncertainty; involves specific statistical practices – such as randomization, collecting sample data, transnumeration, data reduction, and using statistical models – ; and leads to the emergence of specific representations, concepts, procedures, properties and arguments. Several statistics educators – e.g., Makar & Fielding-Wells, 2011 – recommend to engage students in decision-making through making them experience the complete statistical investigation cycle. This makes absolute sense, because it is possible to match every phase of the decision-making process to steps of the PPDAC cycle.

METHODOLOGY

Data-collection instrument and participants

In order to address the research questions of this empirical study, an assignment-like survey was designed, asking respondents the following open questions:

1) From a textbook, teacher's guide, student workbook, internet, academic journal, or other type of source, choose a task or activity that, in your opinion, would promote decision-making skills in your students in secondary school mathematics when you teach contents in the mathematical domains "Practical Use of Data" or "Analysis of Data". You may also develop a task or activity by yourself.

2) Attach a copied or printed version of the chosen task or activity, and write down its source.

3) Briefly explain why, in your opinion, the chosen task or activity has the potential to promote decision-making skills.

At the time of writing this article (September 2014), the data-collection process – which began in mid-July 2014 – was ongoing. Thus, in this paper it is reported a preliminary analysis of the data gathered from the first twelve teachers who voluntarily and anonymously responded and mailed back the survey booklets. The respondents are part of a purposeful sample of 25 Japanese secondary school mathematics teachers who participated in a national academic meeting on mathematics education in Japan. Five of the respondents were working at junior high school, while the remaining seven were working at senior high school. The respondents were between 24 and 63 years old, and they had between two and forty-one years of teaching experience – with seven of them with at least 13.

Data analysis

During the initial phase, all the questionnaire answers were translated from Japanese into English by the author of this paper. Also, they were read repeatedly in order to gain an overall impression, as well as to identify commonalities among the participants. A "bottom up" approach to coding was initially used to analyze the tasks' features and the participant's reasons for choosing such tasks. This grounded form of analysis ensures that the themes or categories extracted were, in fact, grounded in the data and hence reflected the participants' own knowledge base and conceptions regarding decision-making. The author reviewed all the given answers to the three questions and identified answers that occurred frequently in the data. Such answers appearing to contain similar content were initially given the same code, and each code was further analyzed to find true meanings within their text. A process of reduction and clustering of categories, which were refitted and refined, followed, resulting in summary groupings (or "clusters") of themes sharing common meaning.

RESULTS

Tasks' features – What kinds of tasks are thought to promote decision-making?

The participants used a variety of sources to select their tasks. The most commonly used source was the Internet, used by four teachers – i.e., by T2, T7, T11 and T12. Three teachers – i.e., T3, T5 and T6 – referred to research reports or academic journals; used activity books, mathematics textbooks or exercise books – i.e., T4, T7 and T11 – ; and developed the chosen task by themselves – i.e., T1, T9 and T11. Two teachers – i.e., T1 and T8 – chose problems from the National Assessment of Academic Ability and Learning Situation, conducted yearly by the National Institute for Educational Policy Research of Japan. Finally, one teacher – i.e., T10 – selected a task he learnt at the prefectural training program.

From the qualitative analysis performed on the collected data, several task features were identified, including the number of solution strategies, number and kind of representations displayed, or if the task was set in a real-life context. The result of sorting and clustering such task-based features is shown in Table 1.

From Table 1, it seems the all the participants agree that a task intended to promote decision-making should engage students in statistical investigations, connect different statistical concepts, be set in a re-

		TEACHER											
		T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
FEATURES	Number of choices offered by the task	2	0	0	0	0	2	0	0	2	0	0, 2	2
	The task explicitly requests students to think of several possible solutions / to solve the problem in different ways	Y	Y	N	N	N	Y	Y	N	Y	Y	N	N
	The task invites students to engage in open inquiry and investigation	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y&N	Y
	The task required to connect different statistical concepts	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y
	The task is a multi-step one, comprised of several mini-tasks	Y	N	N	N	N	Y	Y	Y	N	Y	Y	N
	The task explicitly asks students to communicate / justify their procedures	Y	N	Y	N	Y	Y	Y	Y	Y	Y	Y	N
	Different types of statistical representations in the task	1	0	1	1	1	0	1	1	2	1	4	0
	The task includes the use of manipulatives	N	N	N	N	N	Y	N	N	N	Y	N	Y
	The task is set in a real-life context	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y&N	Y
	The task can be solved in several ways	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y&N	Y
	Type of decision requested by the task (Pe = personal; Pr = professional; C = civic; O = object-related)	O	C	C	Pe	C	Pe	O	O	Pe	O	O	Pe
	Environment in which the task is supposed to take place (I = indoors, O = outdoors)	I	I&O	I	I	I	I	I	I	I	I	I	I

Table 1: Features of a task with potential to promote decision-making, according to the information provided by the participants on the questionnaire

al-life context, and have multiple ways to be solved. This is in line with many previous studies on the instruction of decision-making at school level – e.g., Arvai et al., 2004; Edelson et al., 2006; Edwards & Chelst, 2007; Garfield & Ben-Zvi, 2008; Godino et al., 2011; Pfannkuch & Ben-Zvi, 2011.

Since 11 out of 12 teachers chose a task to be carried out completely inside the classroom, it seems that the majority of participants consider that promoting decision-making should be done within the limits of the mathematics classroom. By doing so, students might miss the opportunity to extend their knowledge beyond the limits of the classroom, into the real world, which is needed to be statistically literate – cf. Garfield & Ben-Zvi, 2008.

Nine participants shared the idea that a task with the potential to promote decision-making must explicitly ask students to provide arguments justifying their decisions. In fact, this feature is the ultimate phase of the decision-making process. Many researchers – e.g., Edelson et al., 2006; Edwards & Chelst, 2007 – recommend teachers to give substantial guidance when introducing the decision-making process to students, in order to provide scaffolding for them to learn how to do well-informed and justified decisions by themselves.

Half of the respondents seem to consider that a task intended to promote decision-making should be a multi-step problem, or should explicitly request finding multiple solving strategies. Thus, regarding these pair of task features, this sample of respondents did not lead toward a particular one. In relation to being a multi-step problem, some teachers added questions to the original tasks in order to assure students of considering multiple solving strategies. For example, after challenging students with a problem from the 2012 National Assessment of Academic Ability and Learning Situation for Grade 9, T1 added the following step: "Listen to your friends' ideas. At the end, whose idea do you agree with? (Do not mind if it is your own idea.) Please, also write down the reason." In this way, T1 is not only handling classroom communication, but also facilitating discussion and negotiation, which is fundamental in any statistical investigation (cf. Makar & Fielding-Wells, 2011).

Also, it is also noticeable from Table 1 the lack of an overwhelming consensus about which type of decisions should be requested in a problem intended to promote decision-making. Five teachers chose tasks requiring an object-related decision; personal decisions were required in the tasks selected by four teachers; and three respondents demanded civic decisions in their problems. In this regard, specialists say that, although decision-making skills are not specific to any particular type of decision, developing such skills has been done mainly in the context of personal and civic ones, to which students can most immediately relate (cf. Brown, 2005, p. 155; Edelson et al., 2006). Moreover, by just requesting object-related decisions, the completion of the task seems to depend more on students' statistical content knowledge, and hence it is missed an opportunity to create and strengthen a link between the learning of statistics and students' actions in real situations.

Reasons for choice – What competence aspects are associated to decision-making?

From a grounded analysis of the reasons given by teachers about why their chosen tasks have the potential to promote decision-making, six category clusters of competence aspects that participants seem to associate with decision-making were identified. Such results are shown in Table 2.

Many trends in teachers' answers were noted from the results in Table 2. For example, 10 out of 12 participants associated decision-making with skills related to mathematical and statistical literacy, by using expressions such as "mathematical grounds", "handling of information and data", "knowledge about statistical ideas", "ability to read data properly", "ability to grasp data trends", and "practical use of multiple statistical representations". This is in agreement with what many statistics educators have previously reported on this matter: that decision-making heavily depends on skills related to statistical literacy such as understanding, explaining and quantifying the variability in the data (e.g., Gal, 2004; Pfannkuch & Ben-Zvi, 2011).

Only 3 out of 12 participants expressed that decision-making requires the enactment of personal or societal values, such as social fairness. In this respect, it is important to highlight that two of these teachers – i.e., T5 and T9 – selected tasks requesting either a personal or a civic decision. This is in agreement with previous studies on decision-making – e.g., Arvai et al., 2004; Edelson et al., 2006 – , which point out that creating a set of appealing and purposeful alternatives, as well as making thoughtful and high-quality decisions, needs the application of decision-maker's values and technical information.

Just 3 out of 12 of the respondents highlighted that decision-making involves an opportunity for students to build their own criteria or rules for decision, which is emphasized in the literature as one of the main characteristics of the decision-making process (cf. Brown, 2005; Edelson et al., 2006; Garfield & Ben-Zvi, 2008, p. 277).

The fact that decision-making involves engagement with a familiar real problem was pointed out by only two teachers. This could be a likely reason for having five teachers selecting tasks requesting object-related questions to students. According to many authors (e.g., Brown, 2005, p. 155; Edelson et al., 2006), students relate more easily to realistic decision-making scenarios, particularly those in which they are asked for personal or civic decisions.

Nine out of the 12 participants pointed out that decision-making involves engagement with different steps of the open-ended approach. These teachers mentioned aspects such as "coming up with a diversity of ways of thinking", "generating alternative designs", "dealing with problems related to uncertain events", "opportunity to decide about parameters and methodologies", and "opportunity to determine what properties would be good to examine". The number of teachers falling into this category is not surprising at all, because in Japan the lessons usually involve mathematical activities using ill-defined questions, which are often solved by engaging with the open-ended approach method. In fact, "dealing with the openness" during the decision-making process is a main feature of it (cf. Edelson et al., 2006; Edwards & Chelst, 2007; Makar & Fielding-Wells, 2011).

Finally, only 2 out of the 12 respondents explicitly related the decision-making

process to social and inter-personal processes such as discussion, communication, argumentation, persuasion and negotiation to build consensus and common understanding. This result was quite unexpected, since a typical Japanese mathematics lesson has the "takuto" and "neriage" phases, in which students present their solutions or ideas to the whole class, and discuss the validity and pertinence of the proposed ideas, respectively. Many researchers (e.g., Arvai et al., 2004; Edwards & Chelst, 2007) are of the idea that the decision-making process is much stronger with discussion and feedback, and therefore recommend that students should at first brainstorm on their own a list of objectives related to the decision, and then convene to discuss them in groups and with the rest of their class, in order to overcome particular biases related to individual and group thinking, and then to

		TEACHER											
		T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12
ASPECTS	Decision-making involves opportunity to build students' own decision criteria	✔	✔	✔	✘	✘	✘	✘	✘	✘	✘	✘	✘
	Decision-making involves personal or societal values	✔	✘	✘	✘	✔	✘	✘	✘	✔	✘	✘	✘
	Decision-making demands from students to make use of their own mathematical and statistical literacy skills	✔	✔	✔	✔	✘	✔	✔	✔	✔	✔	✔	✘
	Decision-making involves engagement with different steps of the open-ended approach	✔	✔	✘	✔	✘	✔	✔	✔	✘	✔	✔	✔
	Decision-making involves engagement with a familiar real problem	✘	✔	✔	✘	✘	✘	✘	✘	✘	✘	✘	✘
	Decision-making requires inter- personal processes such as discussion, communication, argumentation, negotiation, and collaboration	✘	✘	✔	✘	✔	✘	✘	✘	✘	✘	✘	✘

Table 2: Competence aspects associated with decision-making, according to the information provided by the participants on the questionnaire

effectively undertake the decision-making process in the classroom.

CONCLUSIONS

In general, this study found how the surveyed Japanese mathematics teachers conceptualize decision-making and the tasks intended to promote it. A majority of the respondents selected tasks embedded in a real-life context, having multiple ways of solving, requiring the connection of different statistical and mathematical ideas, meant to be carried out entirely inside the classroom, involving a partial or complete engagement with the statistical investigation cycle, and requiring to communicate and justify procedures and solutions. All these features are, indeed, characteristics of tasks able to promote decision-making skills indicated by previous researches.

The information about how the participants conceptualize decision-making was obtained from the different cognitive aspects identified in teachers' explanations of why they chose their tasks. The majority of participants related to decision-making aspects such as engagement with steps of the open-ended approach, and practical use of mathematical and statistical literacy skills. These features of the decision-making process are in line with those identified in the specialized literature. However, other main characteristics of decision-making – e.g., engagement with a familiar real problem, formulation of personal decision criteria, and engagement with particular social processes – were not explicitly noted by most of the surveyed teachers. Also, just two teachers explicitly considered affective aspects of decision-making, such as the enactment of personal and societal values. Due to the relevant place promoting decision-making in students has in the Japanese mathematics curriculum, it seems that secondary school mathematics teachers need to improve their professional knowledge about what decision-making is and how to promote it, in order to make them appreciate decision-making as a statistical rather than a mathematical work.

Realization of the potential of any task to promote decision-making heavily depends on teachers' statistical knowledge for teaching (González, 2014). Thus, examining how teachers may implement their chosen tasks seems a natural next step for this study.

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Secondary education students' understanding of sampling

Gilda Lisbôa Guimarães and Tâmara Marques da Silva Gomes

University Federal of Pernambuco, Recife, Brazil, gilda.lguimaraes@gmail.com, tamara_msg@hotmail.com

This study aims to identify what students in the 5th and 9th grades from Secondary Education (10 to 14) understand of sampling. Semi-structured, person-to-person audiotaped interviews were carried out with 40 students (20 from each grade), to whom questions were asked concerning some different skills related to sampling. It was noted that, besides the differences between the years of schooling, there was no significant difference in association with the students' performance. This study shows that, albeit all difficulties, students in the 5th grade are able to answer the sampling aspects.

Keywords: Sampling, sample, population, statistics education.

STATISTICS EDUCATION

Statistics data have been present in various different social situations, thus becoming increasingly essential to understand them so that we can consciously exert our role in society. Due to this fact, statistical literacy has been important to Basic Education, since it enables students to develop their critical, investigative, and reflexive attitude. This posture is greatly necessary for the current society characterized by data dissemination (Campos, Jacobini, Wodewotzki, & Ferreira, 2011).

In the Brazilian curriculum, the National Curriculum Parameters (NCP) have outscored that, in the early years (9 to 10 years old), statistical content should be looked upon in the classroom to enable students "to analyze relevant data regarding knowledge and establish the highest number of relation among them by using mathematical knowledge to interpret them and critically assess them". (Brasil, 1997, p. 48). By the 3rd cycle (11 and 12 years old), abilities will need to be developed by students to build sample space and show success possibility of an event by using reasoning (Brasil, 1998, p. 75).

After analyzing Probability and Statistics teaching in the Mathematical Curriculum for Secondary Education, Lopes (1998) took into account the importance of these themes for students' formation as they provide for interdisciplinary approach, experiments, and exploration of random events, which lead to a change of the deterministic model characterized in mathematics.

Among the several statistical concepts that must be worked with students, sampling should not be overlooked. It is part of the investigative cycle of a research and helps to elaborate teaching systematic situations.

For data collection, it is essential to figure out who and how many individuals should be investigated, and how to approach them and put the questions to them in such an efficient way as to gather the most possible representative data. At the moment of data analysis and interpretation in statistical research, researchers must take into consideration the way the data are selected, the methods are applied, the variables analyzed, so that they can understand what is being investigated in context and then compare them to other situations. Sampling is noted implicitly and explicitly through all of these steps. Therefore, an approach using these steps along with mathematical teaching may contribute to important reflections upon the use and the importance of this concept for statistical activities.

It's important to show that this article doesn't provide a discussion of the various types of samples, but investigates the importance of working the concept of sampling from the early years of schooling so that students understand the relationship between the variability of the sample and the generalizability of the data for the entire population.

SAMPLING

The sampling concept is one of several statistical concepts to be developed with students, as part of the investigative cycle research. According to Stevenson (1981), sampling is the part of the group being examined, and the entire group is called population or universe. To him, "the purpose of sampling is to make general assumptions about the whole group, without examining every and each of its elements" (Idem, p. 158).

Some authors as Rubin, Bruce and Tenny (1990), Garfield (2003) and Innabi (2006) have mentioned difficulties students and lay people alike face when tackling statistics to understand the basic concepts of sampling. Besides, when pondering on the rigor of the samples, these people may be neglecting the quintessential factors of their representativeness, namely, size and variability.

Ben-Zvi, Makar, Bakker and Aridor (2011) noticed that by using a sequence of activities with 11-year-old students, in which an increase of the size of the samples had been identified, they got stimulated to think about the relations population-sample. The analysis of the individuals' inferential reasoning showed a development in points of view on what may be concluded from a small sample (contradictory and full of uncertain inferences) and with large ones, which favor the use of informal inferences.

The same way, during a didactic sequence applied to 12-year-old children, Gil and Ben-Zvi (2010) noticed that the students' first explanations, after being questioned about the rigor of the information based on the sample, reflect a conflict between random sample perception and reliability in their inferences. However, along with the activities and discussions, Gil and Ben-Zvi undertook the development of conceptions about sample and sampling, as students tried to infer from a random sample on population. Ideas at random and random sampling were partially understood and used by students at that age group. They were able to understand the implications in sampling representativeness and their variability. However, they were not able to understand their relations.

Pfannkuch (2008) also noticed that, when carrying out a study aiming to identify conceptual growth on students from New Zealand about sampling variability, 14-year-old students started to build some relations about sample variability upon studying samples of different sizes. It was concluded that the comprehension of sampling concepts became even more complex due to limited knowledge of other concepts, such as distribution and variability. Nevertheless, by involving students in contexts composed of discussions over sampling, students were able to show development regarding ideas of sampling variability and to relate population and sample.

The study of Pfannkuch (2008) ratifies the idea that the aspects approach linked to sample and population concepts, such as representativeness, sampling variability, inference, and distribution, when carried out with interrelation, facilitates understanding and building of sampling.

Given the importance and the needs to develop skills related to statistical reasoning, a research realized by Gomes (2013) was carried out to verify what students in the 5th and 9th grades of schooling understood of sampling. This article show some questions about this study emphasizing the definition of the sampling concept, an example of sampling, selection, size, purpose and representativeness of the sample, and population definition.

METHODOLOGY

Our aim is what students of 5th and 9th grades understand about sampling? Forty students in the 5th grade (10 years old) and 9th grade (14 years old) of Secondary Education took part in this study, 20 of each group. These groups belong to the final grade of each level of the Brazilian curriculum. Semi-structured, person-to-person audiotaped interviews composed of 10 questions on sampling were carried out as follows by one of the research. Based on the answers given by the students categories were created observing and valorizing the diversity of responses on each question: not respond, incorrect, partially correct and correct. A comparison of these grades allows observing the teaching of these concepts in school has provided a meaningful learning.

RESULTS

Regarding sample concept, students were encouraged to answer two questions: "*What do you think sample means*?" and "*To find out which candidate running for* *mayor of Recife has a greater chance to win the elections, researchers interviewed a sample of one thousand voters. What does this sample mean in this case*?" Most of the students did not answer or answered incorrectly these two questions about sample definition (33 individuals). It was noticed that the word "sample" was mostly associated with the verb to show (they are homographic in Portuguese). This association shows the influence of regional linguistic characteristics in the formulation of concepts, provided that this word is many times applied as synonym of 'to show' by people that live in Pernambuco, as shown in the following example:

> Student 1: Amostrar alguma coisa, um objeto (*To show something, an object*.)
> Student 2: Quando a pessoa pode mostrar alguma coisa.(*When someone can show something.)*

A great number of appropriate answers for the first question from students in the 9th grade were observed. The difference in the years of schooling is really meaningful ($X^2 = 4.329$, gl 1, $p< 0.037$). It was also common among the appropriate answers examples to be given in lieu of defining the concept of sample, as seen in the following example:

> Student 37:Uma amostra de sangue que as pessoas tiram.(*A sample of blood people collect.)*
> Student 40: Eu acho que é uma pequena parte de tudo. *(I guess it is a small part of everything.)*

In the second question, the students who gave appropriate definition for sample were those who defined sample as a part of all voters in Recife.

> Student 34: Eu acho que essa tá significando uma parte dos eleitores pra saber o que eles preferem. *(I guess it means a part of the voters to know what they prefer.)*

At this question, although the 9th graders had shown a greater number of appropriate answers, unlike the previous question, there was no significant difference in the years of schooling (X^2 1.558, gl 1, $p < 0.212$). These data imply that the same concept has different ways of tackling its definition, that is, by approaching directly or by a context.

When students were once again asked to give an example of sample, most of them (30 individuals) did not exemplify nor presented incorrect examples. A great deal of the given examples did not show any association with statistics. In fact, they were linked to the concept of sample defined by the students at the first question.

> Student 01: Eu quero lhe amostrar (mostrar) meu caderno, aí eu amostro à senhora. *(I want to show you my notebook, then I show to you.)*
> Student 14: Eu pegar esse lápis e lhe amostrar (mostrar). *(I take this pencil and show you (to show).*

Students in the 9th grade gave significantly better answers than students in the 5th grade (X^2 4.800, gl 1, $p < 0.028$). In studies carried out by Rubin, Bruce and Tenny (1990), it was shown that several answers from the students had been grounded on their own personal experiences. Because of that, we believed this question would have been quite easy for them to respond. We found daily situations in which *sample* was used, such as: free sample of products, or else blood or saliva sample for DNA test. Yet, it was clear the participants did not relate these examples to the applied question.

Nevertheless, the appropriate answers presented by the students in the 5th grade show that it is possible for the students at this age to learn this concept. Those who defined this concept partially or totally correct proved to understand sample as part of a whole.

> Student 10: Por exemplo, um suco, uma amostra de suco. *(For instance, some juice, a sample of juice.)*
> Student 32: Aqueles vidrinhos de perfumes ou aqueles papeizinhos com perfume quando a gente passa e entregam pra gente. (*Those little bottles of perfume or those pieces of paper sprayed with perfume given to us when we pass by the stores.)*

Concerning the sample selection, two questions were asked chiefly to understand whether students were able to list important criteria in order to select a sample. During the first activity, students were encouraged to come up with a strategy to select a sample, as being the most possible representative of the studied population. Thus, the following situations were introduced: "*A researcher wanted to know the students' favorite snack in the public schools of Recife. As he could not interview all students, he decided to interview only two hundred of them. How should he choose these students in order to have a better scope on their preference*"?

The performance of most participants, once again, was below the expected. The number of appropriate answers was higher among participants in the 9th grade. However, there was again no significant difference in the years of schooling (X^2 0.476, gl 1, $p < 0.490$). By suggesting the way of sample selection, these students showed similar answers to those found in Rubin, Bruce and Tenny's (1990) studies, with students in High School (14 to 17 years old). The participants presented variable models in association with sample selection so that it could represent the expected population, as long as their answers were grounded on personal intuition.

Answers with at least one aspect referring to probability sample were taken as correct; in other words, those that may be used with the entire population. In this kind of answer, it was common for the participants to suggest that the sample be chosen by drawing the lots, characteristic of random sampling, as described below:

> Student 28: Ele dividia pelas escolas que tem no Recife e sorteava. *(He divided by schools in Recife and drew the lots.)*

The second question about the sample selection showed a hypothetical context: "*Five friends wanted to know approximately how many books the people who live in their neighborhood read a year. As the district had about 10.000 residents, they couldn't to interview every one*". Participants were asked to indicate the most appropriate sample among five options: 100 residents who frequent de local library (large and biased); 100 residents of the district (large and unbiased); 10 residents who frequent de local library (small and biased); 10 residents of the district (small and unbiased); men, women, boys and girls (without information about size and sample selection criterion). Thirty-one students gave inappropriate answers, supporting their answers without taking into account whether the sample had characteristics of the studied population or a bias of selection. Once again, there was no statistical difference between grades analyzed (X^2 1.290, gl 1, $p < 0.256$).

In the same question, answers were analyzed taking into consideration, this turn, size in association with sample selection. Although there has been a difference between grades, it was not shown to be significant (X^2 3.750, gl 3, $p < 0.053$). The results achieved revealed a more appropriate performance of the participants, since twenty-three participants had their answers classified as partially correct, and only one as correct.

Answers were considered partially correct when students took into account the larger number of residents, disassociating population with the number of individuals of the sample as well as one of the criteria for their representativeness.

> Student 1: Porque a opção um tem mais moradores. (*Because option one has more residents.)*
>
> Student 13: [...] A opção 2. Porque tem mais pessoas. [...] *(Option 2. Because it has more people.)*

The single participant with a correct answer was a student in the 9th grade, who emphasized the size of the sample as one of the important factors for its representativeness. Likewise, although a smaller sample could also represent the population of the mentioned example, a larger number of individuals would be more appropriate.

> Student 34: [...] porque o três pegou só 10. Vai dar, mas vai dar muito pouco para saber do bairro todo. *([...] because the number three had only 10. It will do, but it will not be sufficient to learn about the entire neighborhood.)*

Regarding the purpose of using samples, participants had to answer the following question: "*To find out which candidate running for mayor of Recife has a greater chance to win the elections, researchers interviewed a sample of one thousand voters. Why do you think they used a sample and not all voters in Recife*"?

It should be pointed out that, despite the fact that a large number of participants could not produce a definition, nor provide a correct exemplification, when led to a situation of research by using samples and asked to explain why using them, almost half of the students (16 individuals) answered appropriately. Among these participants, 8 (eight) were students of the 5th grade, who explained their answers based on the convenience of using samples, as well as declared to have skills to broaden their knowledge on the purpose of using samples, as answered the student 34:

> Student 34: Se tiver 2 milhões de habitantes no Recife e ele pegou mil já dá pra ter pelo menos uma ideia. (*If there are 2 million inhabitants in Recife and he got a thousand, it is enough, at least, to have an idea.)*

There was no significant difference between the levels of schooling (X^2 0.000, gl 1, $p < 1.000$). Results achieved by this question highlighted the importance of activities with different contexts, involving sample concept to enable learning.

Still, another aspect verified has to do with sample representativeness. It is important to remember that it had been assessed indirectly in other questions; however, being representativeness the main purpose of sampling, it had already been used directly in specific questions. A situation was presented to the students, in which a research would be carried out in a school assigned beforehand. They were asked whether to select individuals for a research in order to represent the school, it would be more appropriate to draw students of all years of schooling or rather only a class would be enough. More than half of participants responded incorrectly (22 individuals). When answers were classified as appropriate or inappropriate, it was clearly proved that the difference in the performance of 5th and 9th graders was not significant (X^2 2.558, gl 1, $p < 0.110$). Some students chose the sample without justifying the reason for the option; others presented justifications unrelated to the sample representativeness; and there were those who inappropriately used aspects associated with sampling, as Student 25 explains:

> Student 25: Melhor (só de uma turma). *(It is better (only one classromm).*
> Researcher: Por que melhor? *(Why is it better?)*
> Student 25: Porque vai ser mais rápido para entrevistar eles. *(Because it is going to be quicker to interview them.)*

Better posed answers indicate that, if encouraged, students of different ages would be able to develop the skills needed to list eligible criteria of a sample. For justifications partially correct or correct, the relevance of sample variability was implicit.

> Student 27: Pegar de uma sala só é pior, professora. Não é melhor escolher da escola toda que tem mais variedade?! Da mesma sala vai mostrar só daquela. *(Collecting from only one classroom is worse, teacher. Isn't it better to choose form the entire school that has more variability?! From the same classroom it will show only from that one.)*

This comes to show that, although students could elaborate a more formal and complete answer, they noted that, to validate a sample, it must have some specific features of the population.

To investigate the understanding of population, two questions were asked. The first question raised the idea of population as a group of people. Thus, it was included in the previous situation about the candidate running for mayor of Recife, the following question: "What would the analyzed population be?" More than half of participants (25 individuals) responded appropriately; however, more students in the 9th grade responded in an appropriate and significant fashion (X^2 5.227, gl 1, $p < 0.022$). The correct answers narrowed the population as to target variable, in other words, those who could vote.

> Student 29: Os eleitores. Do Recife. *(The voters. From Recife.)*

The second question took population as a group of objects: "If a research intending to find out how long computers of a specific brand last were carried out; what type of population would be analyzed in this research?" In this situation, most of them responded incorrectly (35individuals) without any significant statistical difference within the grades studied (X^2 5.227, gl 1, $p < 0.022$). Five students, three in the 5th grade, answered correctly, proving to understand that population not always are people, but the whole being researched, as it is clear in the answer of student 06:

> Student 06: Os computadores da marca. *(Computers of a brand.)*

This understanding shows that even more complex definitions could be more simply worked in the early years of schooling.

Thus, it is important that these students' performance differences be discussed during activities that involve the same definition. As Ben-Zvi, Makar, Bakker and Aridor (2011) state that a variety of situations encourage students to think about the population-sample relations.

CONSIDERATIONS

Investigating different skills with the same students enabled us to identify what skills are more easily understood by them. Similarly, when we verify partial understanding of these different skills, we have some suggestions to put into use during the teaching-learning process, starting with the situations understood by the students in order to build more structured and complex knowledge.

This research showed that despite the great difficulty of the students to understand concepts related to sampling, students from the 5th grade already show to understand concepts associated with sampling in some situations. The identification of this students' aptitude make us to ratify that it is necessary to rethink of what schools can and should offer to the students. The learning ability of these contents from the early years, as shown by the individuals of this research, outscores the idea that studies on sampling should not be limited to the last years of schooling; rather, it must seek strategies to improve these skills in order to find positive changes since the initials levels of education.

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Rolling the dice – exploring different approaches to probability with primary school students

Markus A. Helmerich

Universität Siegen, Siegen, Germany, helmerich@mathematik.uni-siegen.de

This paper focuses on a probability project and shows how primary school students develop probabilistic thinking according to different approaches to probability. In the project students explore the rolling of "odd dice" and compare the winning chances of these dice. By experimenting with the odd dice and comparing their structure to achieve insight into winning strategies the students have to deal with different approaches to probability, especially the subjective and frequency interpretations. A specific work-flow for processing such projects is presented.

Keywords: Probability, frequentist approach, subjective approach, learning environment, classroom project.

INTRODUCTION

This study wants to make a contribution on how teaching of probability in primary school could be laid out, by introducing a mathematical project on "odd dice", which is open for an experimental approach and rich discoveries on probabilities beyond the standard material in stochastics. Furthermore a specific project work-flow for the design of the classroom interaction and for the analysis of the learning process as a diagnostic tool is presented.

THEORETICAL FRAMEWORK: APPROACHES TO PROBABILITY

There are mainly three different approaches to the concept of probability which are crucial for teaching probability at primary schools and aiming at a broad and reasonable understanding of probability for students. Since you only have limited access to an axiomatic interpretation of probability in school mathematics it is important to show alternative interpretations which connect to students' previous knowledge and everyday experiences, even appreciating subjective ideas and conceptions and relate them to mathematical views on probability.

Subjective approach

The subjective approach to probability is the one you will encounter first when teaching probability in school mathematics. "We identify probabilities with degrees of confidence, or credences, or 'partial' beliefs of suitable agents" (Hájek, 2012). Almost all children gathered experiences with statements on probability in the context of chances in games e.g. rolling six dots with dice in the game "Ludo". The subjective approach is fully loaded with individual experiences, naïve ideas and with personal preferences to the point of superstitious beliefs like lucky numbers (see Büchter et al., 2005). Children hold certain ideas and conceptions and that is what teachers have to deal with for successfully teaching stochastics. By setting up activating learning environments such pre-experiences of students could be made explicit for the learning process. The teaching problem is that some of the students' conceptions do not match with the mathematical concepts. To overcome this gap it is necessary to get students into a reflection on concepts.

Frequentist approach

The frequentist approach to probability will help to develop a broader understanding of probabilistic processes. It defines a probability of an event as the limit of its relative frequency in a large number of trials, according to the law of large numbers. In primary schools one often deals just with counting absolute frequency in relation to a fixed number of trials, e.g., rolling the dice 100 times and then counting the number of occurrences of the pips. This empirical approach suits well with activating learning environments in which students are able to determine probabilities by random experiments. Students will evaluate their experiments with tally tables.

Classic approach

At last the classic approach to probability will be considered as a mathematical sophisticated approach which has a profound theoretical meaning but is only accessible in a limited way in primary schools. The probability of an event is given by its ratio of the number of cases favorable to it, to the number of all cases possible. This requires that none of the cases occur more than any other i.e. all cases are equally possible. This approach goes back to a definition by Laplace from 1814 in his Philosophical Essay on Probabilities. Since fractions and rational numbers are only available in a very limited way in primary school it will not be possible to apply this approach in-depth. But the basic idea could be used to count the favorable cases and compare them to a fixed number of possible cases. A special case of the classic approach is the geometric interpretation of probability, e.g. looking at the faces of the dice and – assuming that all faces are the same size – the probability for each face will be the same.

"ODD DICE" – A PROBABILITY PROJECT

The teaching material "Spürnasen Mathematik" [mathematics sleuths] consists of a box with mathematical projects on arithmetic, geometry and stochastics aiming at an open, activity-oriented learning process, accompanied with working books for a systematical training and learning process. To ensure rich experiences with mathematics and linking-up with situations of everyday life the projects are starting always with a hands-on activity for children, followed up by a step by step systematization and formalization with mathematical tools and mathematical language. This specific approach is chosen to increase the motivation to deal with mathematical contents and tools, and to see the potential of mathematics to describe everyday life situations, to solve real life problems and to provide a language to communicate and compare information and data. The mathematical tasks and the project work are open according to Peschel (2007) in the classroom format and social form in which the children will work on the projects (e.g. most project tasks could be carried out individually, pair or group work), the organization of the learning process (e.g. the children can choose from different materials to work with), and the conceptual opening (e.g. the children are encouraged to solve the problems on their competence level and in their way of mathematical thinking). Therefore the projects have a high potential for differentiation in the learning process and the resulting products. This classroom research report will focus on the project named "Only by Chance" classified as a stochastics project. This projects aims for an exploration of dealing with chances and probabilities in different contexts.

One part of the project looks at "odd dice" and the odds to win by playing dice and the corresponding probabilities. The odd dice are given by their cube nets (see Figure 1). These four coloured dice are an adaption of the "Miwin dice" (Winkelmann, 2012) and do not show the usual dots of a dice and equally probable occurrence of each side of the dice, but a special arrangement of the numbers 0 to 6 as shown in Figure 1.

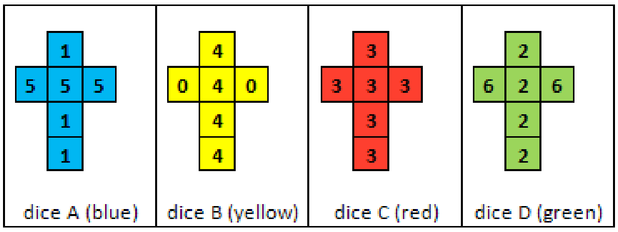

Figure 1: Cube nets of the odd dice for the probability project

In the following I will outline the project tasks for the students, some of the learning outcome objectives and possible solutions of the problems. (Remark: In the teaching material "Spürnasen Mathematik" all the tasks are gathered on a task instruction card. Since the material is in German the original card is not shown.)

Task 1

The first task of the project asks the students to explore with prepared odd dice, which numbers on the dice will occur most frequently. After making a guess, the students roll the dice and take track of the occurrences with a tally sheet (see Figure 2). This will help students to get started with the project and become more familiar with the odd dice, since they probably haven't been working with such dice beforehand. The students learn to estimate the possible outcomes of throwing the dice, and are encouraged to experiment and set up a series of dice rolling. By documenting their results in tally sheets they will learn a specific technique of data representation.

Figure 2: Suggestion for documenting the occurrences

Task 2

The second task is to roll the dice in a little game of dice for two students using the red and green dice. The student throwing the higher number will win and gets one point. The game will stop when one student has gained 10 points. Again the students should make an assumption first and then use a tally list to write down the gaming results, and observe which dice is winning the game. It is necessary to give the students enough time to explore the different dice and to analyze which dice is better. The game and a systematic representation of the game result will lead to a reasonable good assumption of the better dice.

Task 3

In the following task the students should play the dice game over and over again with two different dice in order to find "the best" dice, using one of the various suggested documentation styles for their assumptions and their results. With a systematic approach for testing all combinations of dice – this is encouraged by the proposed documentation forms – the students have to explore, communicate and argue to find the best dice of all. A demonstrative and helpful tool for comparing the dice are the tables showing for each pair of dice the possible events and marking the winning dice (see Figure 3).

These tables give insight into the winning probability of each pair of odd dice. For example have a look at the top left corner of Figure 3. There you can see dice C (red) and dice D (green) with all possible cases of rolling dice results. Each table cell states (by colour and letter) which dice is the winning dice for a certain case. You can clearly see, that the red dice will win in more cases than the green dice (to be precise in 24 out of 36 cases), thus making the red dice the better choice for the game. Evaluating all tables you will end up with a diagram shown in Figure 4. There is no dice with exclusively outgoing or incoming arrows in the diagram, therefore none of the dice could be considered 'the best', but you always find one dice which is – on the long run – better than (or at least as good as) a certain chosen dice. This will not imply you are winning in every turn of the game, but if you as second player in the game choose the "right" dice according to the diagram you will enlarge your winning odds and on the long run with many follow-ups of the game win the game more often.

Task 4

By using proposed forms of documentation the students will result in a good overview of the winning odds of the dice and be able to answer the last task. In the final task the students are requested to discuss a given statement of student Jonas (illustrated in the teaching material), that you will always win the game if you take the second turn to pick one dice, and to argue on the question, why these dice are called "odd". Here the students are trained to take a close look at

dice D (green) \ dice C (red)	3	3	3	3	3	3
2	C	C	C	C	C	C
2	C	C	C	C	C	C
2	C	C	C	C	C	C
2	C	C	C	C	C	C
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice A (blue) \ dice B (yellow)	0	0	4	4	4	4
1	A	A	B	B	B	B
1	A	A	B	B	B	B
1	A	A	B	B	B	B
6	A	A	A	A	A	A
6	A	A	A	A	A	A
6	A	A	A	A	A	A

dice A (blue) \ dice C (red)	3	3	3	3	3	3
1	C	C	C	C	C	C
1	C	C	C	C	C	C
1	C	C	C	C	C	C
5	A	A	A	A	A	A
5	A	A	A	A	A	A
5	A	A	A	A	A	A

dice D (green) \ dice A (blue)	1	1	1	5	5	5
2	D	D	D	A	A	A
2	D	D	D	A	A	A
2	D	D	D	A	A	A
2	D	D	D	A	A	A
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice D (green) \ dice B (yellow)	0	0	4	4	4	4
2	D	D	B	B	B	B
2	D	D	B	B	B	B
2	D	D	B	B	B	B
2	D	D	B	B	B	B
6	D	D	D	D	D	D
6	D	D	D	D	D	D

dice B (yellow) \ dice C (red)	3	3	3	3	3	3
0	C	C	C	C	C	C
0	C	C	C	C	C	C
4	B	B	B	B	B	B
4	B	B	B	B	B	B
4	B	B	B	B	B	B
4	B	B	B	B	B	B

Figure 3: Comparison of the odd dice

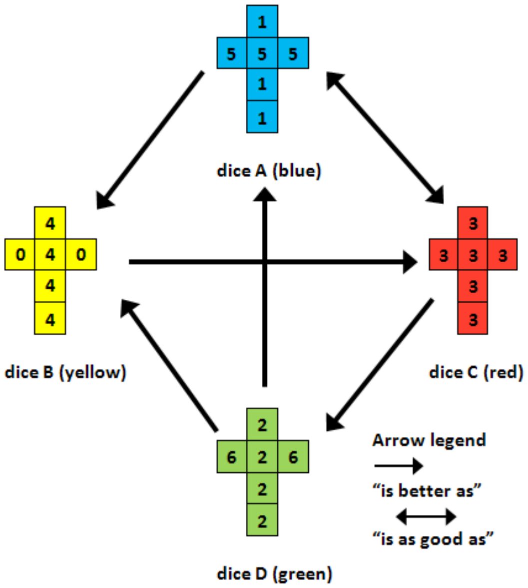

Figure 4: Diagram showing the relation between the odd dice

the situation: Jonas' remark just states that there will be a reasonable choice for a dice in response to a chosen dice in the first draw. But there is no "best" dice at all. This finding is an explanation for the naming of the odd dice and linking these odd dice to the widely known "rock, paper, scissors" game with an analogous relation between the three possible events (there is always one beating another, but no move to win in every case).

RESEARCH QUESTIONS AND DESIGN OF STUDY

The following empirical findings will describe which approaches to probability are used by students working on the project "odd dice" and how their understanding of probability could be developed to a broader approach by working on the tasks concerned with odd dice.

The probability project learning environment was applied in a primary school at Eitorf near Siegen (Germany) in a third grade class with 23 kids (11 girls, 12 boys, age ranging from 8 to 10 years old, 5 with special needs). The students were chosen because they were used to work on open projects since it is the usual learning environment in this school. This was an important criterion to observe if and to what extent it is possible to make students understand a specific mathematical content like probability in such a learning environment setting. The project "odd dice" was processed by two groups with each four students and a third group with two students in about 2 hours. The student's documents and observations of a trainee teacher (Nelia Kasemir who documented the students' results in her final thesis, see Kasemir, 2013), are the basis for an interpretative approach for the in-depth data analysis. The trainee teacher was very familiar with this class and basically played the role of managing the working and learning process according to the work flow presented below (see Figure 5). Only in the first plateau phase some examples for probabilities of events were presented and in the later plateau phases, students were encouraged to share their intermediate results.

For evaluating these project products and learning and teaching processes, a work flow for mathematical projects of the teaching material "Spürnasen Mathematik" (see Helmerich & Lengnink, 2013, and Lengnink, 2012) could be used, following the "Think,

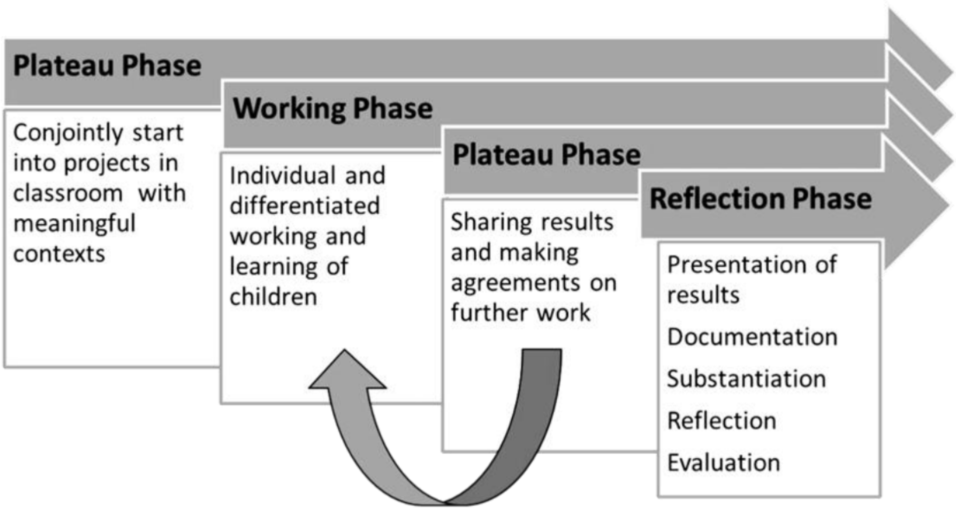

Figure 5: Work flow for mathematical projects in classroom interaction

Pair, Share" method (for example, presented in Barzel et al., 2007, p. 118).

The children are introduced to learning objectives and possible tasks in the project, before starting off with individual and group work. The students of third grade already acquired competence in dealing with probability in the sense of assigning several everyday-life situations to a probability continuum chart reaching from "impossible", over "unlikely" and "rather probable" to "certain". Before the odd dice project they worked on probabilities of usual, regular dice and the urn model and projects on probability of drawing a certain colour out of a package of chocolate beans, and the probability of letters in words and de-coding strategies in Caesar code applications. Within a sharing circle the pre-knowledge was re-activated. The recurrence of plateau phases, where children share their ideas and preliminarily results marks the important issue of reflecting the process and give teachers the opportunity to adjust the children's work for the next working phase.

ANALYSIS OF THE WORKING PROCESS AND RESULTS OF THE STUDENTS

This phase model will be used to analyze the learning process and to show different approaches to probability as seen in the students' documents. In the first phase the students' previous knowledge on and experiences with probabilities are activated by collecting situations in everyday life in which the term 'probability' appear. The students are asked to find a definition for probability and state some events which in their opinion are probable, impossible definite. The most common paraphrase for a "probable event" used, was something "that could happen, but doesn't have to." With this preparation the students start with their group work on different probability projects.

In the second phase the students started off with their work on the project tasks. The students mainly worked together in pairs of two, which were formed at random. With the first task of the "odd dice" project two approaches to probability are activated: the students are supposed to estimate the number of occurrences of the dice faces which activates subjective views. Some students decided to analyze the dice of their favorite colour or the dice with their lucky number on it. The only notion before starting the rolling experiment was on dice C (red) with exclusively threes on each face. It was obvious for the children that this dice will produce the event '3' all the time. Other estimations were not made. With rolling the dice the frequentist approach came into action. It seemed clear that rolling the dice several times will show, which results one can get with each dice. However, some students restricted their tally sheet only to some numbers on the dice, so the actual distribution of the events is not represented but could be extracted from the notes. For example Simon (see Figure 6) rolled all the dice ten times, so you can calculate the number of occurrences of the other dice rolling results. Final remarks on the probabilities or a location in the probability continuum were not made at this stage.

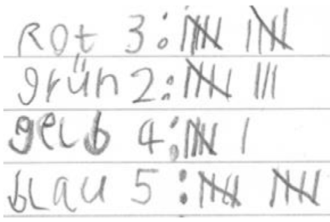

Figure 6: Simon's tally sheet on task 1 (marking the results for the red ("rot"), green ("green"), yellow ("gelb") and blue ("blau") dice)

The students stepped on to task 2 and the comparison of dice. Interestingly some students started the game using both the same coloured dice (see Figure 7). This could be explained by the need to become more familiar with these odd dice and to set up a situation ensuring a level playing field. In these cases the students achieved as expected narrow outcomes in their game.

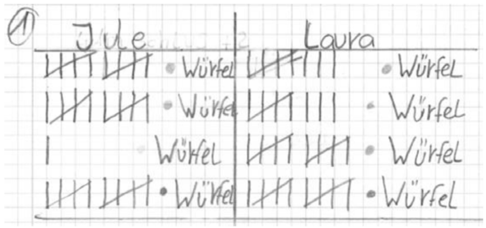

Figure 7: The table of Jule and Laura ("Würfel" means dice)

To overcome this strategy and to get to the actual task it was necessary to compare the first results in a plateau phase. Thereby Jule and Laura were encouraged to proceed with the comparison of dice. In Figure 8 the results of the comparison of the blue and the yellow as well as the red and green dice are shown in a conjoint table.

In this task the difficulty occurred that the results of the game did not represent the expected insight into the relation of the dice since the game stopped

Figure 8: Jule and Laura playing the dice game ("blau" = blue, "orange" = red, "gelb" = yellow, "grün" = green)

with one student gaining ten points. To get over this problem, it might have been helpful to have a systematic look at the dice and their faces and draw tables as shown in Figure 3. Due to the limited time for the project during class this was not worked out in detail. But the students nevertheless tried to figure out whether there is always a dice for winning the game like Jonas suggested. In this process it is interesting to state that the students did not carry out the systematic comparison according to the classic approach to probability but fall back on subjective views. Looking at Simon's documentation of task 4 (see Figure 9) reveals that Simon (in the table abbreviated to "Sim") plays the game with Saheb (short "Sah") by choosing a pair of dice, roll them, record the result and play again with a new pair of dice. This approach to focus on the single outcome of the game could be characterized as a uni-structural thinking (Shaughnessy, 2007; Watson & Kelly, 2004). Thinking in relations and an understanding enriched by relational conceptions was not achieved of Simon. This emphasizes the challenge of teaching probability, to master the mental step from a subjective approach to the more elaborated classic approach of probability.

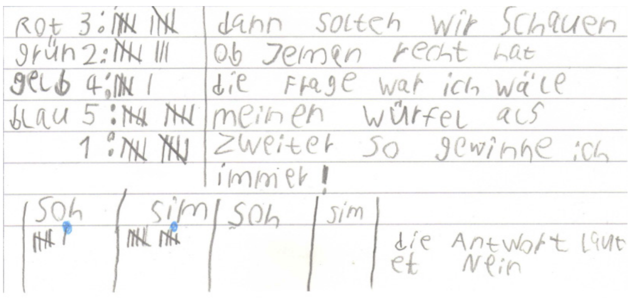

Figure 9: Simon's argumentation on task 4: "then we should have a look if someone is right: the question was: I choose my dice second, thus I will always win! The answer is no."

DISCUSSION AND CONCLUSION

Such mathematical projects like the one on odd dice make it possible to let students explore different approaches to probability in an activating, experimental way. The project even shows the potential not only to stick to the subjective and frequency interpretations of probability but to merge these aspects to the classic approach. The broader view on probability is especially inherent in the plateau phases when students have the opportunity to share their work achieved so far and could be enriched with some new ideas and strategies, also from the teacher, for further project work. In higher grades this project on odd dice could be unfold to a broader mathematical analysis of the odd dice using the classic approach to probability and calculating and comparing the probability for each number on the dice. This might be an approach to an extension of the probability concept to non-Laplacian experiments and the law of large numbers, since in the game situation with a limited number of dice rolls the experiment does not always lead to the theoretically expected outcomes. The project is a contribution to the fundamental mathematical idea of "data and chance" as it is stated in the national standards for mathematics teaching in Germany (e.g., KMK, 2004). It combines the idea of data collection by rolling the dice with the ides of chance by investigating the probabilities of the dice. This study was the starting point for a broader investigation of mathematical ideas in stochastics in primary school teaching. It just gives some insight on the thinking and reasoning of primary school students, but shows the potential of such projects for diagnostic approaches in research of learning processes.

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Students' informal inference when exploring a statistical investigation

Ana Henriques and Hélia Oliveira

Universidade de Lisboa, Instituto de Educação, Lisboa, Portugal, achenriques@ie.ul.pt

This paper reports on preliminary results of a study aiming to identify informal inference aspects that emerge when grade 8 students explore a statistical investigation using the software TinkerPlots for data handling. Examples from students' work on one task in a sequence, designed to engage students in posing statistical questions about meaningful phenomena, in collecting and representing data and finally in making data-based inferences, illustrate how informal statistical inferences emerge. The results provide suggestions for further research and some educational implications are drawn.

Keywords: Informal statistical inference, statistical reasoning, statistical investigation, tasks, TinkerPlots.

INTRODUCTION

Students' ability to reason about data and to use them effective and critically for prediction and decision-making is now a priority in statistics education, meeting the call for having statistically literate and informed citizens in our data-driven society (Batanero, Burrill, & Reading, 2011). Informal statistical inference (ISI) is widely recognized as an important foundation for statistical reasoning, given its potential to deepen students' understanding of the purpose and utility of data in meaning making of the real-world, aspect that has been neglected in statistics education (Makar, Bakker, & Ben-Zvi, 2011; McPhee & Makar, 2014). Hence, it is important to rethink the nature of students' statistical school experiences in order to consider how to best develop their informal ideas of inference. Reflecting these concerns, this paper reports on preliminary results of a study that aims to identify informal inference aspects that emerge when grade 8 students explore a statistical investigation using the software *TinkerPlots* (Konold & Miller, 2005) for data handling.

INFORMAL STATISTICAL INFERENCE

Informal statistical inference is described as a reasoned but informal process of "making probabilistic generalizations from (evidenced with) data that extend beyond the data collected" (Makar & Rubin, 2009, p. 83). For these authors, three key principles are fundamental to ISI, wherein the first is particular to the process of inference and the latter two are specific to statistics: (i) generalization (predictions, parameter estimates, conclusions) that extends beyond describing the given data; (ii) the use of data as evidence for generalizations; and (iii) the use of probabilistic language for describing the generalization, including reference to levels of certainty about the drawn conclusions. Moreover, deriving conclusions that apply to a universe beyond the data leads naturally to the need of providing persuasive arguments based on data analysis. Accordingly, Ben-Zvi (2006) argues in favour of an integration of both informal inference and informal argumentation when aiming to develop students' statistical reasoning in rich learning contexts.

References to ISI appear in curriculum documents (e.g., Common Core State Standards Initiative, 2010; Franklin et al., 2005; National Council of Teachers of Mathematics [NCTM], 2000), as well as in recent research in statistics education. The research advocates the development of students' informal ideas of statistical inference from the earliest years of schooling as it is a known area of difficulty for older students when formal ideas are later introduced (Ben-Zvi, Makar, Bakker, & Aridor, 2012; Makar, Bakker, & Ben-Zvi, 2011). However, ISI should not be taught to students as an entity in itself but rather it would be preferable to focus the instruction on reasoning processes that lead to inference (Makar & Rubin, 2009). Thus, one possible approach to foster the emergence of students' inferential practices is to embed those processes in a data analysis cycle, like the PPDAC (Wild & Pfannkuch, 1999). Presenting statistics as an investigative process to solve real-world problems may be quite motivating for students. They can engage in posing their own statistical questions (hypothesis) about a meaningful phenomenon, designing and employing a plan to collect appropriate data, selecting adequate graphical or numerical methods to analyse the data, and finally drawing data-based conclusions and inferences that relate the results interpretation to the original questions (Franklin et al., 2005; McPhee & Makar, 2014). Due to their nature, statistical investigations often provide a distinctive context for observing students' conceptual ideas about statistical reasoning, namely fundamental processes like variation, transnumeration, evaluating statistical models and integrating contextual and statistical features of the problem (Wild & Pfannkuch, 1999). At the same time, they may involve students (even the younger ones) in fundamental components of informal inference, such as decision making and prediction (Makar & Rubin, 2009; Watson, 2008). For teachers, the use of investigations also provide knowledge that can be used in the design, implementation, and assessment of instruction in statistics and data exploration (Henriques & Oliveira, 2013), since they incorporate domain-specific knowledge of students' statistical reasoning. Accordingly, Pratt and Ainley (2008) argue that inferential reasoning and statistical investigations cannot be separated.

Additionally, the huge development of technology has provided teachers and students with new tools to explore the ISI in rich and meaningful contexts, including through the broader process of statistical investigation (Ben-Zvi et al., 2012). The educational community has been leading a number of studies on the affordances provided by dynamic statistics learning environments, such as the *TinkerPlots* (Konold & Miller, 2005), for making inferential reasoning accessible to (even young) students, with very encouraging results (Ben-Zvi et al., 2012). These studies illustrate how the use of such software, in combination with appropriate curricula and instructional settings introducing ISI, may help students to develop a strong conceptual base on which to build later a more formal teaching of inferential statistics.

RESEARCH QUESTION

While the referred perspectives and studies have begun to shed some light on important aspects of ISI, research results are still insufficient to guide an effective implementation of instructional settings that support students' informal notions of statistical inference. This paper addresses the following research question which has been used to structure the study and the analysis of the collected data: *What aspects of informal inference do grade 8 students reveal when exploring a statistical investigation using the dynamic statistics software TinkerPlots for data handling*?

THE STUDY

Background

This study arose in the context of a developmental research project (DRP), conducted by the two authors, aiming to provide opportunities for professional development in statistics for basic education mathematics teachers in Portugal. The DRP is part of a larger research project – *Developing statistical literacy: Student learning and teacher education* (Oliveira, Henriques, & Ponte, 2012) and strives to understand of how sequences of instructional activities, based on the use of the software *TinkerPlots*, promote students' statistical reasoning. In Portugal, formal statistical inference is reserved for high school courses and, traditionally, grade 8 (13–14 years old) students are not exposed to ISI methods. The eleven basic education mathematics teachers, who accepted the invitation to participate in the DRP, are effective and innovative teachers, in the sense of including inquiry in their mathematics classes and became interested in learning how to use an instructional approach aligned with the recent curricular trends in school statistics (Franklin et al., 2005; NCTM, 2000).

Participants and methods

Research studies involving teaching experiments are especially powerful because they enable researchers and teachers to trace students' individual and collective development in statistical reasoning during instruction. The study followed a teaching experiment design (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) where a sequence of tasks and classroom implementation conditions are improved through iterative cycles. The results reported here come from the study's first cycle which was carried out during the second term of the school year of 2013/2014 in one class of thirty grade 8 students (20 boys and 10 girls, which are referenced by their first names' initial letter). The class teacher was one of the participants in the above referred DRP. Data collection included students' written work, screen records of their work with *TinkerPlots* (using *AutoScreenRecorder 3.1 Pro*) and audio and video records across the lessons. A qualitative and interpretative analysis of the data, using Makar and Rubin's framework (Makar & Rubin, 2009), provided insights about the characteristics of ISI that emerged in students' work on a statistical investigation.

Lessons context and tasks

A classroom culture which promotes inferential reasoning requires students addressing problems in an environment that encourages collaborative norms, statistical concepts (formal or informal) and one complex problem context where they face conflicts with their knowledge and beliefs about the world (Makar, Bakker, & Ben-Zvi, 2011). In all lessons, students worked both autonomously in pairs and participated in whole class discussions, in order to create a classroom culture that supported questioning and the use of evidence.

Participant teachers in DRP and the authors jointly selected and adapted or produced tasks and materials for a sequence of lessons aligned with the Statistics Reasoning Learning Environment principles defined by Garfield and Ben-Zvi (2008). Since in Portugal, inference is not included in mathematics syllabus for this age group, the sequence of tasks was not explicitly designed to focus on inference. However, these tasks, using meaningful contexts and real data, provided opportunities to engage and simultaneously support students in several aspects of informal inferential practice. *TinkerPlots* software (Konold & Miller, 2005) was used in all tasks, as a tool for data handling, since it is easy to use and provides a dynamic learning environment to support the development of students' statistical reasoning. Students can create and explore, in ways that they choose, the potential of different graphical representations and statistical measures, by using the drag-and-drop facility or a set of informal operators (e.g. separate, stack, and order). Taking advantage of the software's tools the students can also engage in goal-directed activities that may lead to further insights in their data examinations.

Two preliminary tasks (tasks 1 and 2) were applied to build students' skills that they would need for conducting a statistical investigation and for making inferences, since they were not familiar with this kind of activity. Tasks 1 and 2 engaged students in exploring real data using *TinkerPlots*, while developing key statistical concepts of syllabus as well as their reasoning. Since this was the students' first contact with the software, task 1 was structured and oriented by a set of questions leading them to interpreting contexts, exploring data (available from *TinkerPlots* database) in several ways and using various representations (which were discussed regarding their adequacy to give evidence). Finally they used their knowledge of the data to answer one initial question and to make predictions. Task 2 involved students in comparison of distributions using simulations provided by the *TinkerPlots* tools.

This study focus on task 3 – *The human body: a study in school*. Following the previous activity, this task engaged students in a statistical investigation (Wild & Pfannkuch, 1999) to discover more about the students in their school. Students were required to use their previous knowledge about the context and statistical concepts and processes (e.g., understanding the need of data, graphical representations, distribution, variability and sampling) to make informal judgments and predictions about the school population, based on data collected in their class. Finally, students were asked to explain their reasoning, integrating persuasive data-based arguments in their conclusions.

RESULTS: THE EMERGENCE OF STUDENTS' INFORMAL INFERENCE

In this section, a description of students' work in the context of a statistical investigation provides an opportunity to envision the role of this activity in the emergence of students' informal inferential practices.

Making generalizations beyond the data

This component of informal inferential reasoning emerged in students' work in different moments of the task. The teacher introduced the task in the whole class with an initial question aimed at having students collecting data and using them to make predictions and draw conclusions about a population beyond the collected data: "How would you characterize middle school students [grades 7, 8 and 9] of your school regarding some of Vitruvius' measures such as height, foot size and arm span?". In the subsequent whole class discussion, it was clear that students understood the utility of the samples in the inferential process, to draw conclusions, because they showed no intention to survey the entire school population to get the information they needed to answer the question. They said: "It is best to conduct a survey" (E) and "To randomly select [students] from grade 7 to grade 9" (R). Although they were not unanimous about how the sample could be collected and used to find out the characteristics of all school students, they stressed randomness as an important condition to ensure the representativeness of the sample. Due to time limitations, students agreed to collect data from their own class, although recognizing the lack of representativeness of the sample.

After data collection, the teacher also asked them to formulate possible interesting questions (hypothesis) about the phenomena in study and to predict how their questions might be answered, first without providing them with data. This was proposed to later confront students' anecdotal answers with data-based evidences, pushing them to understand the need of data. Some students reveal inability to think about the data as an aggregate and to full make sense of the questions' intention to help them to answer the initial question in the task. Some of the emerging questions/hypothesis such as "What is the biggest value for the arm span?" (W&F), "What is the percentage of students with more than 160 cm?" (B&R) or "Does the highest student has the bigger shoe?" (B&R) reveal a deterministic nature and did not prompt students to think beyond their data. As expected, some of these students justified their predictions based on guessing and simple observation: "Based on [the observation of] my colleagues, I think that the average height of the students [boys] is 165cm" (M&S). Although these findings were consistent with the difficulties observed in other studies (e.g., Meletiou-Mavrotheris & Paparistodemou, 2014), students had just collected their own data and this activity might have diverted their attention from using inferentially the data. However, the majority of students seemed to accept the existence of variability and formulated general questions/hypothesis such as "Boys tend to be bigger than girls" (T&D), "Do boys and girls have similar arm spans?" (D&B) or "I think that maybe the arm span has a higher value for boys" (B&B). These statements show students' attempts to find one global description that accounts for variability between the groups. These claims could be considered a progress if compared to the ones in the previous questions, which are mostly local. However, they still focus on describing the class. These predictions also contained elements of uncertainty, which reveal that students were beginning to adopt a statistical perspective of trend.

After the exploration of the collected data (from the class), using the *TinkerPlots*, each group wrote a report predicting what they might find about the height, foot and arm span, more generally, for the whole school, and explaining what evidences they had to support their conclusions. These reports aimed at pushing further the students to see their own data as evidence for making inferences about whole school, building their ability to think "beyond the data". As occurred with young students (Watson, 2008), their initial focus on describing their own class data did not restrict them from creating more global interpretations of data. In fact, students' responses suggest that many of them did see the data collected in their class as useful evidence for their predictions. The excerpts below show that some students assumed that data collected in the class would have very similar properties to the data from the whole school, and generalized the results: "Considering the whole school students, the boys' height and arm span are bigger when compared with the girls, based on our class" (A&E) and "Boys tend to be bigger than girls, based on our class data" (A&N). It is interesting to notice that in their predictions students used probabilistic language and were attentive to the distribution and not to specific measures as they were in the formulation of the initial questions. This finding may suggest that students associated the generalization to uncertainty and that previous work in organizing and describing their own data supported them to go further in the inferential reasoning by helping them to improve their thinking of the data as an aggregate and to consider variability. Even though they have found the average height for their class, their predictions went beyond that to include uncertainty, adopting the statistical perspective of trends that are generally true but still have exceptions: "On average, boys are higher. But we also observe boys with 1,4 [m] who are shorter than girls (...)" (W&F). On contrary, a noteworthy number of students predicted that there would be some differences, as observed in the following claims: "The average height of the school students probably will increase due to students' natural growth" (M&S), "Boys will be higher than girls, so the average will move towards the boys' value because, probably, there are more boys than girls in school" (T&D) or "Although some differences in the height may occur, the average height will be higher, depending on the grade" (J&D).

Using data as evidence for generalizations

Making predictions about the whole school from their data was evident across the groups of students. Initially, students explored the data collected in their class, based on graphical displays created as they use the *TinkerPlots*. Given that students have not been oriented toward a specific graph, it was interesting to observe that they created a variety of graphical displays and adequately used them to represent and interpret their own data to get evidence for their claims, which shows their meta-representational competence (English, 2014). Dot plots were the most popular representations to gather evidence for answering deterministic questions focused on individual features, but most students also created boxplots to compare the two genres regarding the variables in study, as the ones shown in figure 1. These students' graphical displays were accompanied, respectively, by claims such "Students' average height is 162,7" (S&P), "The percentage of students with more than 160 cm is 55%" (B&R) and "Boys are higher than girls. Boys' average is 163 and girls' average is 160 cm" (E&S).

Students started comparing groups by noticing special values within distributions (such as the average or the maximum) and then moved to think about between-group variability. The comparison activity, based on their representations, also supported their claims about the whole school, as they said: "Boys tend to be bigger than girls, based on our class data" (A&N) or "We have based on our class which has more boys than girls and they are higher" (D&B). Moreover, their claims are not based only on data but also on their knowledge about the context which is familiar to them. For example, one group states "I think that for the whole school we will find something similar because our class is the 8th grade and that is the "median" of the middle school [it includes grades 7, 8 and 9]" (R&R). The exploration of a sample (data collected in the class), guided by the initial questions and facilitated by the visualization provided by graphical representations created in *TinkerPlots*, supported the students' involvement in the subsequent generalization of the observed characteristics and the use of such data as evidence.

Using probabilistic language to describe generalizations

Students' responses, presented in previous sections, suggest that they recognized that their predictions were tentative since they included, in their claims, several elements of uncertainty expressed by the terms "probably", "maybe" or "something similar" instead of equal. They also used the term "tend to be" showing that they started to adopt a statistical perspective of trend. In all these cases, the uncertainty was expressed qualitatively, without confidence levels or margins of error (Dierdorp, Bakker, van Maanen, & Eijkelhof, 2012), as expected taking into account the students' statistical experience. Nevertheless, this component of ISI was the less evident in students' work, suggesting the need of specific work regarding probabilistic language.

DISCUSSION

In this paper, we described some examples of students' ISI that emerged through an informal data-based approach, rooted in the statistical investigation cycle (Wild & Pfannkuch, 1999). Despite the reported findings are limited to one task in one class, those examples give us some insight into the students' capabili-

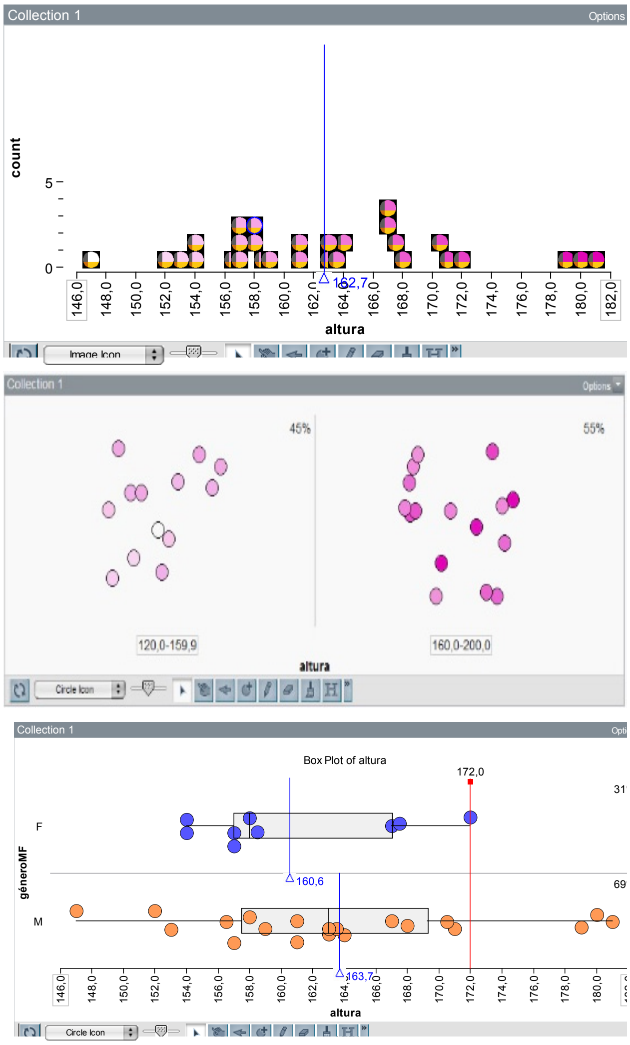

Figure 1: Students' graphical displays

ties and the challenges they faced in making informal statistical inferences when experiencing statistics as an investigative process.

All students, even in the less successful groups, were able to demonstrate some aspect of informal statistical inference during the lessons. Students drew their conclusions based on the data they had collected from their class and often used the data to make inferences about an unknown population (whole school). Few students used probabilistic language for describing their generalizations but even those who made it had difficulties in including references to levels of uncertainty. These results highlight the need of working on probabilistic language issues, helping students to evolve from a deterministic perspective of inference to include uncertainty in their statements.

There are also some evidences from the study that the software *TinkerPlots* was used by the students as a reasoning tool (Ben-Zvi, 2006) providing opportunities for structuring and displaying data in ways they choose and understand. However, this was not the focus of this study, and although students' interaction with the software in the previous tasks seems to have contributed to their meta-representational competence (English, 2014) the complex role of technology in supporting ISI deserves further discussion. These encouraging findings do suggest that the adoption of a statistical investigation approach in a technological environment has the potential to make ISI accessible to students, as observed in other studies (Ben-Zvi et al, 2012).

Finally, many studies show that several skills are required for the emergence of students' practices of ISI, such as: articulating or predicting from observations, recording and organizing data using invented methods and working with aggregates and variability (e.g., McPhee & Makar, 2014). The proposed sequence of tasks in this DRP appeared to have supported students' development of such skills but they still need to be re-examined before they are used in a new teaching experiment. Further analysis of students' interaction within the groups, which were not considered in this study due to time and space limitations, may convey new and deeper insights on their reasoning, as well as to contribute for the reformulation of the tasks and their implementation conditions.

AKNOWLEDGMENTS

Study supported by national funds through FCT – Fundação para a Ciência e Tecnologia, Project Developing statistical literacy: student learning and teacher training (contract PTDC/CPE-CED/117933/2010). We thank Ana Isabel Mota who assumed the role of research assistant for her extensive work in data collection.

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Improving teachers' reasoning about sampling variability: A cross institutional effort

Bridgette L. Jacob[1], Hollylynne S. Lee[2], Dung Tran[2] and Helen M. Doerr[3]

1 Onondaga Community College, Syracuse, United States, jacobb@sunyocc.edu

2 North Carolina State University, Raleigh, United States

3 Syracuse University, Syracuse, United States

The research reported here uses common items to assess statistical reasoning of teachers enrolled in a graduate-level education course to evaluate their reasoning about sampling variability. In particular, we discuss key aspects of a purposeful course design aimed at improving teachers' learning and teaching of statistics, and the resulting different ways of reasoning about sampling variability that teachers exhibited before and after the course.

Keywords: Statistics, sampling, variability, reasoning, assessment.

Given the strong attention to statistics in the secondary curriculum in many countries (e.g., England Department of Education, 2014; CCSSI, 2010), many teacher professional development efforts and graduate courses include more opportunities for secondary teachers to develop their statistical reasoning and to learn pedagogical strategies for teaching statistics. Researchers have investigated the statistical knowledge needed for teaching using various frameworks (e.g., Burgess, 2011; Noll, 2011). Each of these frameworks has identified teachers' own statistical reasoning as a foundational aspect of their ability to teach statistics.

Our research is situated within the collaborative design and implementation of a graduate course across two institutions focused on teaching and learning statistics. Our design work is largely influenced by Pfannkuch and Ben-Zvi's (2011) recommendations for designing experiences to develop teachers' statistical reasoning. Though our courses aim to develop teachers' understanding for *teaching* statistics, in this paper we report on teachers' reasoning related to sampling variability, without regard to their understanding of how to teach others about sampling variability. To help assess the impact of the experiences we designed in our courses, we used both qualitative and quantitative data sources. The focus in this paper is to examine how teachers' reasoning about sampling variability changed.

Understanding sampling variability has been established as difficult, but key, in one's overall statistical reasoning (e.g., Shaughnessy, 2007). Two aspects of sampling variability are reported in this paper: representations of sampling variability and the effect of sample size on variability. Researchers have found that the understanding of variability from expected values among samples improves with experience and age (e.g., Watson & Kelly, 2004). The role of sample size in variation from expected values has also been the focus of much research. Concerning the effects of sample size, researchers have reported that students, and teachers, use equiprobable reasoning in determining the likelihood of two events without considering differences in the sample size of the two events. (e.g., Watson & Callingham, 2013). In a meta-analysis, Noll and Sharma (2014) discussed the Hospital Task, one of the four items reported here, which has been used to assess reasoning about the effect of sample size on parameter estimation. Since the original version presented by Kahneman and Tversky (1972), it has been revised and implemented with students from approximately 10 years of age through college with the predominant response being the equiprobable response. In 2013, Lee, Doerr, Arleback, and Pulis reported that after experiencing an earlier version of our graduate course, teachers still exhibited difficulties with sampling variability, including a tendency to apply equiprobable reasoning by ignoring the effect of sample size on variability. Our reflections on those findings led to the current work.

COURSE AND PARTICIPANTS

Following from the results of Lee and colleagues (2013), a team of four instructors from two institutions began conceptualizing ways to improve a graduate course focused on teaching and learning statistics. Related to sampling variability, suggested improvements included readings and discussions targeted to draw attention to students' tendency for equiprobable reasoning. These suggestions also included purposeful task design using technology tools for data exploration. The team met weekly via videoconference for an academic year (2013–14) to design a 15-week course, and to discuss issues and alter plans as the course was taught in Spring 2014. The course consisted of opportunities for teachers to engage in statistical investigations with real data and tasks designed to develop their understandings of distribution, samples and sampling distributions, and inferential statistics, especially using randomization approaches. The course used the dynamic software *Fathom* (Finzer, 2005) and *TinkerPlots* (Konold & Miller, 2011), as well as online applets and resources such as StatKey (lock5stat.com/statkey). The course included readings and discussions about (a) the nature of statistical reasoning, and (b) students' learning and reasoning related to the aforementioned topics. Software tools were used to support teachers' learning by allowing them to flexibly explore graphical representations, easily compare data sets, and make changes to data in displays to explore conjectures. The software provided the simulation tools necessary to create representations of a population, a sample, and an empirical sampling distribution. Given the research on students' struggle to understand sampling distributions (e.g., Saldanha & Thompson, 2014), we saw these representations as critical for developing teachers' knowledge of sampling variability.

Across institutions, the course served a variety of graduate students (n = 27, 8 in Course1 and 19 in Course2). Participants consisted of one pre-service teacher (5th year senior), six pre-service and 11 in-service teachers in masters programs, one full-time masters student in mathematics education, seven doctoral students in Mathematics or Mathematics Education (three currently teaching in post-secondary contexts), and one doctoral student with interests in statistics education. Twenty-one participants were female and six were male, with six participants for whom English was a second language. Most participants had completed the equivalent of an undergraduate major in mathematics, with all but two having had at least one course in statistics. Henceforth we refer to course participants as teachers.

DATA SOURCES AND ANALYSIS

One source of data was participants' responses to a statistical concept inventory constructed to align with our course goals, content, and experiences. On the first day of class and during the final week of the course, all participants completed a 20-item multiple choice test with items in five categories: distributions (5 items), comparing distributions (3 items), probability (2 items), sampling variability (7 items), and formal inference (3 items). Eleven items were drawn from validated instruments (delMas, et al., 2007; Garfield, 2003), with seven selected from the ARTIST database (apps3.cehd.umn.edu/artist), and two items adapted from research (e.g., Watson & Kelly, 2004; Zieffler et al., 2008). The 20-item test was agreed upon by instructors during the planning phase to ensure items had content validity to measure concepts to be addressed in the course. For two of the four sampling variability items we highlight in this paper, teachers were asked to justify their choices.

After the course, semi-structured interviews were conducted with selected teachers (n = 14) across institutions to understand changes in their reasoning and perceptions of what might have influenced those changes. Interviews (45–90 minutes) were audio or video taped. Interviewees were purposely selected because of trends in their responses to items on the test. For example, some were selected because they had improved from an incorrect response to a correct response on several items, while maintaining incorrect responses on other items. During the interview, participants were shown an item, given time to reread, told which choice (A, B, C, etc.) they had selected on the pre and post-test, and then asked about their reasoning. Based on responses, the interviewer asked questions prompting them to elaborate about their reasoning.

Descriptive statistics and t-tests were used for the 20-item assessment to document the change in teachers' performance on the pre and post-test, both overall and on individual subscales. Teachers' responses to the two open-ended items and responses of the 14 teachers interviewed were open coded to identify emerging themes to gain insight into the teachers' reasoning about the changes in their responses.

RESULTS

Analysis of the pre and post-test showed significant improvement in teachers' overall scores (out of 20), with a mean increase of 1.84 points (s.d.=1.98). Strong gains were found in the items related to sampling variability, with a mean increase in scores (out of 7) of 1.3055 (s.d.=1.19). In this paper, we report on changes in teachers' reasoning for two key categories: (1) the effect of sample size on the likelihood of outcomes, and (2) representations of sampling variability.

Two items, Brown Candies and Two Hospitals, pertained to the effects of sample size on the likelihood of outcomes from a sample. At the beginning of the course, about half of the teachers correctly answered each of these items. Two additional items, Sample Means and Sample Proportions, asked teachers to reason about expected variability in a distribution of sample statistics when sampling from a given population distribution. For both of these items, a larger proportion of teachers were able to correctly respond at the beginning of the course, 78% and 70% respectively.

Effect of sample size on the likelihood of outcomes

Table 1 illustrates the distribution of responses for the Brown Candies item. There was a major shift to 83.2% responding correctly on the post-test. Most notable was the decrease in the number of teachers choosing the equiprobable response (E).

By the end of the course, the teachers gained a clearer understanding that smaller sample sizes have greater variability than larger sample sizes, thus resulting in small samples being *more likely* to have larger deviations from the expected percentage of 50% brown candies. The following is representative of teachers' responses when asked to explain their reasoning about the Brown Candies item.

> The large [bag] is more like 50%, the small [bag] is more unlikely because of smaller sample size. For example you flip a coin, you have more chance to have 8 and 2 whereas you flip 200, you are more likely to get 50%. (Teacher 27)

The single teacher to choose 'B', the larger bag having more variability, on the post-test, chose the equiprobable response 'E' on the pre-test. However, during the interview, she realized she had chosen in error and stated her reasoning:

> Maybe at first when I answered it, at first I think I didn't have any idea about sample size. But after we learned something about sample size in class and how it will affect, you know, like the variability... [reading the problem] So Sam is the one having a larger, a large family sized bag? So that implies that large family sized bag will have a large sample size? And that implies that we should have less variability? I think I was wrong. So it should be 'C'. (Teacher 1)

This teacher appeared to have difficulty with the complex terminology in the item rather than a misunderstanding of the underlying concept. Three teachers

Brown Candies: A certain manufacturer claims that they produce 50% brown candies. Sam plans to buy a large family size bag of these candies and Kerry plans to buy a small fun size bag. Which bag is more likely to have more than 70% brown candies?		
	Pre-Test	Post-Test
Sam, because there are more candies, so his bag can have more brown candies.	0.0% (0)	0.0% (0)
Sam, because there is more variability in the proportion of browns among large samples.	3.7% (1)	3.7% (1)
Kerry, because there is more variability in the proportion of browns among smaller samples.	**51.9% (14)**	**83.2% (23)**
Kerry, because most small bags will have more than 50% brown candies.	3.7% (1)	0.0% (0)
Both have the same chance because they are both random samples.	40.7% (11)	11.1% (3)

Table 1: Results of the Brown Candies Item

chose 'E', the equiprobable response, on both the pre and post-test. One teacher explains:

> I was thinking about this idea of flipping a coin... flipping it head, the next time you flip, it is 1 over 2. So ... making an analogy to this it says that a certain manufacturer claims that they produce 50% brown candies... So it doesn't really matter whether the bag has ten candies or a thousand candies... there is 50% chance I mean 50% of them... would be brown. So long as the bag contains candies from this manufacturer. (Teacher 7)

This teacher is equating samples in the large and small bags of candies to tossing a single coin rather than a series of coin flips in the long or short run.

For the related Two Hospitals item, teachers needed to reverse their thinking by choosing the hospital that was *less likely* to record a high percentage of female births. Teachers were also asked to write about their reasoning for this item on the test. Table 2 shows results for the Two Hospitals item in which gains were made in teachers' correct responses. Again, we saw a decrease in the number of teachers choosing the equiprobable response (C) and an increase in the correct choice (A).

Of the 22 teachers answering this correctly on the post-test, the responses below are representative of their thoughts about sample size and variability.

> So Hospital B is more likely to have 80% or more. And then I also thought about the numbers, if you're doing 50 births a day, 40 girls out of 50 seems like a lot compared to 8 out of 10. (Teacher 3)

> The larger hospital will have less variability from the expected value of 50% boys and 50% girls. (Teacher 17)

These teachers illustrated an understanding of the relationship between sample size and variability. In the open-ended responses, four teachers also referred to the actual number of births, stating 8 out of 10 female births was a likely outcome for the smaller hospital but 40 out of 50 female births was unlikely for the larger hospital. During interviews, six teachers noted the conceptual similarity between the Brown Candies and Two Hospitals items and that they had to "reverse" their thinking for the latter.

Of the five teachers choosing incorrectly on the post-test, three of them indicated the smaller hospital was less likely to record a high percentage of female births (B). During interviews with two of these teachers, they indicated they had misread the problem on the post-test. "Yeah 'A', the big one. The reason is because there is more variability... [laughs]. The answer is wrong but my explanation is correct. There is more variability in the small sample." (Teacher 6) Both teachers had responded instead to which hospital would be *more likely* to record 80% female births.

The remaining two teachers responding incorrectly on the post-test chose the equiprobable response. One of them chose the equiprobable response on the pre and post-test, as he had done for the Brown Candies problem. His open-ended response on the post-test revealed his reasoning, "Each birth is independent of the other births and there is a 0.5 probability that each birth (independent) of others would result in a boy or a girl." (Teacher 7) Both teachers demonstrated the difficulty of dispelling the notion of equiprobability of events with small and large samples.

Representations of sampling variability

The Sample Means and Sample Proportions items assess teachers' understanding about sampling distribution by asking teachers to predict variability from expected outcomes, using different representations.

Two Hospitals: Suppose about half of all newborns are girls and half are boys. Hospital A, a large city hospital, records an average of 50 births a day. Hospital B, a small, rural hospital, records an average of 10 births a day. On a particular day, which hospital is less likely to record 80% or more female births?

	Pre-Test	Post-Test
Hospital A (with 50 births a day)	**51.9% (14)**	**81.5% (22)**
Hospital B (with 10 births a day)	14.8% (4)	11.1% (3)
The two hospitals are equally likely to record such an event	18.5% (5)	7.4% (2)
Not able to determine based on given information.	14.8% (4)	0% (0)

Table 2: Results of the Two Hospitals Item

The Sample Means item provides a graphical display of the population distribution with the mean and standard deviation, and asks teachers to choose the most likely dotplot of five sample means of size 10. The Sample Proportions item gives the population proportion numerically and asks which set of five proportions from random samples of size 20 is most likely.

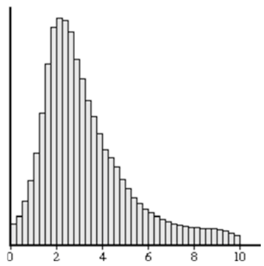
Table 3 shows the distribution of teachers' responses on the pre and post-test for the Sample Means item. Initially 78% of the teachers were able to identify reasonable variation from expected. This number increased by 11% after the course. Further examining the open responses and interviews gave insight into their reasoning.

Seven of the 14 teachers interviewed were able to eliminate too little variation (response a), and too much variation (response b), indicating a strong sense for reasonable expectations in variation. For example, Teacher 15 stated, "Here with only five samples, I think 'a' would be too perfect. I would throw out 'b' because the chance you have 10 values with the mean of 8.5 would be very slim, if even ever [pointing to dot above 8.5 on dotplot]". Four of the teachers were able to distinguish between the population, samples and sample means, and the sampling distribution. "I try to grasp my head around five students with 10 values, so that this dot represents the mean of 10 values, not just one value I picked out" (Teacher 15). In addition, they could incorporate the sample size into the expected variability of the sampling distribution. As an example, Teacher 11 replied:

> Sample 10 seems to me not a large sample size set of data so I am certainly expecting... if I take 1000 values, I am convinced that the sample mean should be put together. With 10 values I am suspicious. That's possible but less probable. And by exactly the opposite reasoning, 10 seems big enough that I see so much variation that I see. No way I can quantify... that is too much variation of sample 10. That dot [pointing to right dot in 'b'] I should have many values over here [pointing to right tail of the population distribution] that's simply outrageous; I believe if it is 1 sample [pointing to 'b'].

When reasoning about the population, samples, and the distribution of sample means, the teachers either went back to the graph of the population to estimate the means, or used the numerical statistics given to evaluate the possibility of the sample means.

Sample Means: The distribution for a population of measurements is presented at the right. The mean is 3.2 and the standard deviation is 2. Suppose that five students each take a sample of ten values from the population and each student calculates the sample mean for his or her ten data values. The students draw a dotplot of their five sample means on the classroom board so that they can compare them. Which of the following dotplots do you think is the most plausible for the one they drew on the board?



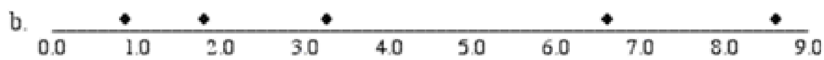
	Pre-Test	Post-Test
	14.8% (4)	3.7% (1)
	7.4% (2)	7.4% (2)
	77.8% (21)	88.9% (24)

Table 3: Results of the Sample Means Item

> I think the new SD should be between 1/2 and 1. I don't know if 10 is large enough, but with large number of sample, the sampling distribution is approximate a normal distribution so I think if the mean is here [point to about 3.2 in 'c'] so other will be 1/2 SD from the mean and other is about 1 SD within the mean. (Teacher 18)

> It is about 76%, then 2SD of 95%... I am looking for most of the things within that. This is like 8.5, it is way out of range. This [pointing to 'a'] does not have enough of deviation within 1 SD, they cramp together. 'C' seems more within the 2SD. (Teacher 30)

These responses reveal teachers' reasoning about sample means using the sampling distribution and the Central Limit Theorem.

Three teachers who got this item wrong on the post-test did not show a robust understanding of population, sample, and sampling distribution. For example, one teacher seemed to believe the distribution of sample means should resemble the population distribution "I think 'b' is making the most sense because I can see the skewness to the right." The sampling distribution remained complicated for her even though the course focused extensively on that construct.

Table 4 shows the distribution of teachers' responses on the Sample Proportions item. While teachers (70.4%) began our course with a good intuition about variation from expected, almost all correctly responded to this item after taking the course.

Teachers' interview comments indicated they were able to eliminate wrong options based on their sense of variability from expected. For example:

> I eliminate 'C' because of the 5% and 95%. If I know 35% of the candies are yellow, I know it is not impossible but, I just don't see someone picks 20 candies all but one being yellow and I know there are enough candies in there just 35% of 1000, it could happen. I just, 20 candies is not enough to see the perfect 35% every time. One kid might see it; not every kid might see it. (Teacher 13)

For the teachers who chose an incorrect response, they seemed to be using either equiprobable reasoning or the thinking that anything can happen.

Reasoning across Items

At the beginning of the course, only 10 of 27 teachers (37%) answered both the Brown Candies and Two Hospitals items correctly. Of the 13 teachers choosing incorrectly on the Brown Candies item, the predominant response chosen by 11 of them was 'E', the equiprobable response (Both have same chance because both are random samples). Of these 11, only four teachers chose the equiprobable response for the Hospital item as well. Another common misconception that larger sample sizes have greater variability was demonstrated by one teacher for the Brown Candies item and four teachers for the Hospital Problem.

After the course, 20 out of 27 teachers (74%) correctly responded to both items; a marked increase from the pre-test. Of the remaining seven teachers, three chose an equiprobable response to one of the items, repeating their error from the pre-test. There was only one teacher who exhibited an equiprobable misconception for both items on the pre-test and the post-test.

For the Sample Mean and Sample Proportion items, before the course 16 out of 27 (59%) teachers correctly answered both items. By item, 21 judged the variation

Sample Proportions: Imagine you have a barrel that contains thousands of candies with several different colors. We know that the manufacturer produces 35% yellow candies. Five students each take a random sample of 20 candies, one at a time, and record the percentage of yellow candies in their sample. Which sequence below is the most plausible for the percent of yellow candies obtained in these five samples?		
	Pre-Test	**Post-Test**
30%, 35%, 15%, 40%, 50%.	**70.4% (19)**	**92.6% (25)**
35%, 35%, 35%, 35%, 35%.	14.8% (4)	3.7% (1)
5%, 60%, 10%, 50%, 95%.	3.7% (1)	0% (0)
Any of the above.	11.1% (3)	3.7% (1)

Table 4: Results of the Sample Proportions Item

in a graphical format correctly whereas 19 selected a correct response for the question concerning variation in a numerical format. By the end of the course, almost all teachers correctly answered both items (24/27, 88.8%).

DISCUSSION AND CONCLUSIONS

Overall, the teachers improved their understanding about sampling variability, in particular the relationship between sample size and variability, and variability from expected. This could be attributed to the extensive focus on statistical investigation and many experiences with simulations in which attention was drawn to expectations from a population distribution, collecting samples and sample measures, and discussing the distribution of sample measures. A few teachers still had difficulty on these items, corroborating prior findings that sampling distribution and sampling variability is complicated to understand (e.g., Saldanha & Thompson, 2014).

The results of this study also confirm that equiprobable reasoning can be misapplied in reasoning about samples of different sizes, and that this reasoning may become more stable for some teachers (e.g., Watson & Callingham, 2013). We observed that for two of the teachers, even with intensive experiences with variability, they still held a deterministic understanding of probability. This might be rooted in their early exposure to theoretical probability that they need to revisit and re-evaluate in order to build up a robust understanding.

We also observed that teachers could develop a sound understanding about sampling variability and reason correctly about an item, yet still choose a wrong answer. This resulted from a misreading of a problem or a misunderstanding of a particular word. Also, for the Sample Means item, teachers could give a correct answer, reasoning with a sense of variability from expected without necessarily understanding the relationship between a population, samples, and the sampling distribution. Thus, we are concerned that this item may be more useful for measuring understanding of variation from expected values rather than sampling distributions.

This study adds to the sparse literature related to teachers' reasoning about statistics, focusing on their understanding of sampling variability. It illustrates how a carefully designed graduate-level course in teaching and learning statistics improves teachers' understanding of important statistical concepts. In particular, the focus on statistical investigation and reasoning experiences, and the emphasis on a simulation approach for inference, seems to improve teachers' knowledge about sampling variability.

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A Bayesian inspired approach to reasoning about uncertainty: 'How confident are you?'

Sibel Kazak

Pamukkale University, Denizli, Turkey, skazak@pau.edu.tr

This paper focuses on a task design that is aimed at eliciting young students' reasoning about uncertainty as it relates to their personal degree of confidence through a Bayesian inspired informal inferential reasoning about chance games. This Bayesian inspired approach is described and discussed based on preliminary analyses of data from a teaching experiment in a designed-based research study. With this approach, beginning by a hypothesis (or prediction) about the fairness of a game and revising it based on new information appear to come natural to the students. The change in strength of their personal level of confidence seems to vary by the conflicting results obtained by playing the game, the size of the data collected, and the multiple computer simulations conducted using TinkerPlots software.

Keywords: Uncertainty, probability, inferential reasoning, task design.

INTRODUCTION

Making decisions and inferences based on data is part of everyday life. In statistics, statistical inference deals with "drawing conclusions about populations or processes based on sample data" (Zieffler, Garfield, delMas, & Reading, 2008, p. 40) by means of certain techniques, such as confidence intervals and hypothesis testing. As argued by Engel and Erickson (2013) it is yet hard to grasp the logic behind statistical inference for students. In order to develop the foundation for the formal ideas involved in statistical inference early on, informal statistical inference has become the focus of the statistics education research (Rubin, Hammerman, & Konold, 2006; Makar & Rubin, 2009). As a relatively recent concept, informal statistical inference refers to a way of making conclusions about a population or process from which the data come by using statistical processes (Pfannkuch, 2006). In addition, the underlying reasoning process leading to informal statistical inference is called informal inferential reasoning (Makar, Bakker, & Ben-Zvi, 2011).

In recent research studies, there has been a great emphasis particularly on informal inferential reasoning to help students develop deeper understanding of statistical concepts, ideas, and processes (Ben-Zvi, Aridor, Makar, & Bakker, 2012; Jacob & Doerr, 2013; Makar, 2014; Paparistodemou & Meletiou-Mavrotheris, 2008). One of these ideas at the heart of informal statistical inference is uncertainty because drawing conclusions beyond the data about a wider population requires an articulation of uncertainty (Makar & Rubin, 2009) and probabilistic justifications (Rossman, 2008). When it comes to teaching probability as part of statistics, a current debated topic is to introduce Bayesian thinking at school level because of its applications in realistic situations and its closeness to how people actually reason about uncertainty (Chernoff, 2014; Nilsson, Blomberg, & Ärlebäck, 2014; Martignon & Erickson, 2014). Therefore more research into supporting young students' development of Bayesian thinking is needed.

The aim of this paper is to describe a Bayesian inspired approach in a task design for eliciting students' reasoning about uncertainty as it relates to personal degree of confidence. First, I outline the ideas behind this approach for task design, which is followed by a brief description of the study and the task. Then I focus on how these ideas are explored during the task as a way to articulate students' reasoning about uncertainty with some preliminary analyses of students' reasoning.

THEORETICAL BACKGROUND

Informal statistical inference and inferential reasoning

In an informal statistical inference, a common underlying reasoning process involves "assessing the strength of evidence against a claim" (Rossman, 2008, p. 7) based on observed data. This type of informal inferential reasoning known as Fisherian approach can be described as follows (Rossman, 2008): formulating an initial hypothesis (or null hypothesis); evaluating that if the hypothesis were true, observed data would have been very unlikely (i.e., intuitively computing a p-value); and rejecting the initial hypothesis based on the very small p-value. According to Rossman, students yet do not appear to use this common reasoning naturally when making statistical inference.

Alternatively, another approach to statistical inference involves a Bayesian perspective based on the subjectivist interpretation of probability. A main distinction between Bayesian inference and Fisherian approach is drawing conclusions based on a subjective or personal assessment of uncertainty. So the reasoning process from a Bayesian approach includes starting with a priori probabilities associated with a hypothesis based on a personal belief and updating these probabilities in the light of new information or data (Rossman, 2008). Albert (2002) argues that the Bayesian reasoning is more intuitive than the Fisherian perspective in statistical inference and better reflects the commonsense thinking about uncertainty in everyday life.

Reasoning about uncertainty

Both in school mathematics curricula and research on students' understandings of uncertain events, games and experiments involving chance devices, such as coin, dice, and spinners, are quite widely used. Games of chance provide a rich context for children to explore random situations, to notice the unpredictability of outcomes and to see the need for probability (Cañizares, Batanero, Serrano, & Ortiz, 2003) while making decisions under uncertainty. The notion of fairness in game situations has also been recognized as a motivating and productive area of inquiry for students investigating probability and uncertainty (Pratt, 2000; Watson & Moritz, 2003; Stohl & Tarr, 2002). For example students can build on their intuitive ideas of fairness to evaluate whether each player has an equal chance of winning or whether each possible outcome is equally likely in different games involving coins and dice.

These studies on fairness in chance games mainly focus on the concept of probability from classical and frequentist approaches. There is a lack of research on young students' conceptions of subjective probability, which is closely related to the Bayesian reasoning mentioned above. One of the earlier studies both relevant to games of chance and subjective perspective to probability (Huber & Huber, 1987) suggests that young children are able to use personal knowledge or beliefs when comparing the likelihoods of chance events in the contexts of sports and gambling. It is also noted that children's subjective probability evaluations tend to be more stable in the gambling context because the objective probabilities could be assessed also through the sections in the spinner device used in the task (Huber & Huber, 1987).

OVERVIEW OF TASK DESIGN

The purpose of the task design was to elicit students' reasoning about uncertainty in the context of informal statistical inference about the fairness of chance games. A previous study using an earlier version of this task suggested that students' reasoning about uncertainty was inherent both in the chance games and in personal degree of confidence (Kazak, Fujita, & Wegerif, 2013). The task then was revised to closely look at students' reasoning about uncertainty as it relates to personal degree of confidence, which is the focus of this paper. To build on students' intuitive and informal inferential reasoning, a Bayesian inspired approach was adopted to design the Matching Tokens Game task in which students were asked to make probability assessments and state their level of confidence in judging the fairness of a game. This approach enabled students to articulate uncertainty while evaluating the fairness of different chance games by making an initial hypothesis and expressing their confidence in the likelihood of a particular game actually being fair or not, and then by revising both their hypothesis and level of confidence with new information obtained from the data through physical experiments and computer simulations. The underlying process is usually viewed consistent with people's way of developing intuitions based on learning from their experiences and revising their beliefs as new information is acquired (Falk & Konold, 1992).

Study background

The task design is part of a design-based research study. As described by Cobb, Confrey, diSessa, Lehrer, and Shauble (2002), this study involved an iterative design through planning, testing, and revising conjectures about students' learning and ways of supporting their learning of domain specific content, i.e. probability and statistics. The research cycle involved designing instructional materials and a learning environment that supports the desired learning goals, conducting teaching sessions, and retrospective analysis. Three iterations in local schools in Exeter, UK were conducted as part of the larger research study. In these teaching experiments, students worked in groups of two or three on a joint activity. Each group was given worksheets and videotaped while working around a computer. The task described here was tested and revised based on the earlier iterations. The empirical data used to explain a Bayesian inspired approach in this paper are from the last iteration with eleven 10–11-year-old students: group A (Justin, Owen, Matt), group B (Taylor and Sam), group C (Meg, Julie, and Jailyn), and group D (Maya, Eleanor, and Jena).

Task and tools

The Matching Tokens Games involve randomly drawing a token from each bag shown in Figure 1. For example, in Game 1 one bag has 3 red tokens and 1 blue token and the other has 1 red token and 3 blue tokens. To play the game, a token will be randomly drawn from each bag. If both tokens are the same color, students win. If they are different (mixed) color, teacher wins. The question posed to students before playing the game is whether the game is fair or not. These four games were chosen and sequenced based on the students' responses on the previous two iterations of the task. Students investigated each game one after another in the given order.

Through adapting a Bayesian inspired approach this game context was introduced to students in a specific structure involving three phases and ten questions. As seen in Figure 2, in the prediction phase students began by formulating a hypothesis about the fairness of Game 1 based on their personal knowledge/belief. Then on a scale (0–10) they evaluated how confident they were about the un/fairness of the game initially based on the explanation they were asked to give in question 1. In the game-playing phase, students working in groups collected as many data as they wanted by playing the game with the given bags and recording their results and used the results to update their initial hypothesis as well as their level of confidence if needed. In the modeling phase, they built a computer model to simulate the game results and to collect more data, and again revised their previous hypothesis and level of confidence in the light of new information.

TinkerPlots software (Konold & Miller, 2011) was used as a modeling tool in this study. The Sampler tool in *TinkerPlots* allowed students to build their own chance models using a variety of devices (i.e., mixer, spinner, bars, stacks, curve, counter) that can be filled with different elements to sample from. It also enabled students to collect outcomes and carry out a large number of trials quite quickly. For instance, to build a model of Game 1 in *TinkerPlots* (see Figure 3) group B used two connected mixer devices, one with

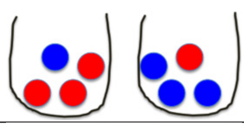
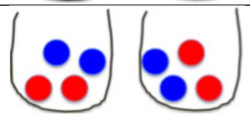
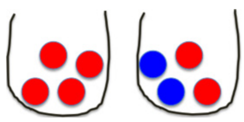
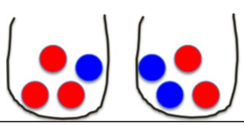
<u>Game 1</u> Bag one: 3 red tokens, 1 blue token Bag two: 1 red token, 3 blue tokens	
<u>Game 2</u> Bag one: 2 red tokens, 2 blue tokens Bag two: 2 red tokens, 2 blue tokens	
<u>Game 3</u> Bag one: 4 red tokens Bag two: 2 red tokens, 2 blue tokens	
<u>Game 4</u> Bag one: 3 red tokens, 1 blue token Bag two: 2 red tokens, 2 blue tokens	

Figure 1: Games given on each worksheet

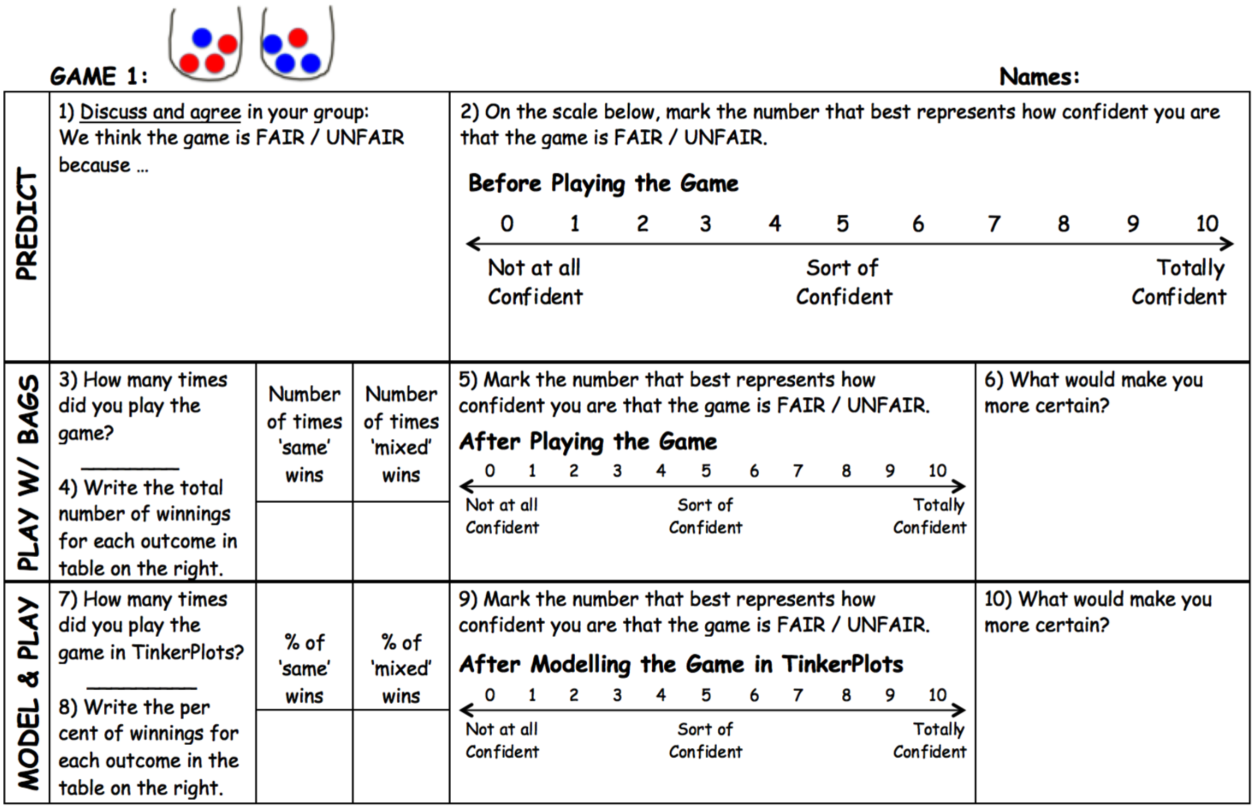


Figure 2: Student worksheet for Game 1

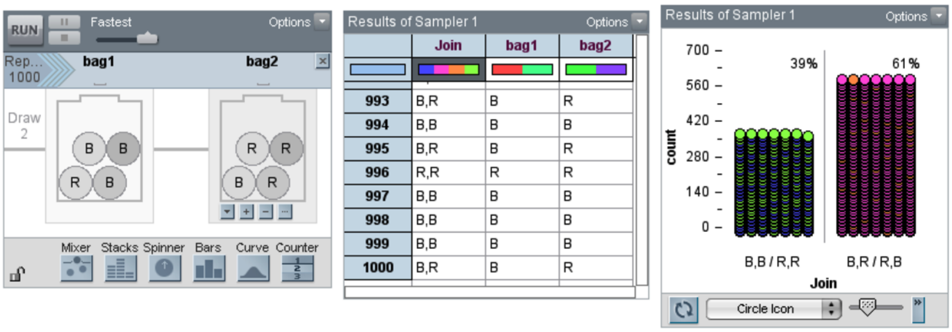


Figure 3: Computer model of Game 1

three red (R) and one blue (B) balls and the other with one red and three blue balls, representing the number of red and blue tokens in each bag. Repeat number is set to 1000. After randomly being selected from each mixer, the outcomes of each trial are displayed in the results table for 1000 repetitions. In the graph, the individual outcomes, 'B,B' and 'R,R' then 'B,R' and 'R,B', are combined into a bin by dragging one into the other to display the percentage of "the same color" and "the mixed color" outcomes respectively.

CHARACTERISTICS OF A BAYESIAN INSPIRED APPROACH TO REASONING ABOUT UNCERTAINY

In order to develop students' reasoning about uncertainty through informal inferential reasoning, the task design involved a Bayesian inspired approach to informal inference in the context of chance games. Each game investigation starts by having students state their initial hypothesis about the fairness of the game with an explanation and then rate their level of confidence about the un/fairness of the game on a scale from 0 to 10. In order to test their initial predictions students play the game to collect some data and record their results on the worksheet (questions 3 and 4 in Figure 2). Judging the fairness of these games, except Game 2, were found to be highly counter-intuitive based on the students' initial responses in the previous iterations of the task (Kazak, Wegerif, & Fujita, 2014). So, students are expected to update their initial hypothesis and level of confidence based on new evidence from the game results. Subsequently students build a model of the game in *TinkerPlots* to collect more data to reevaluate their previous hypothesis and confidence level through computer simulations. Next I describe the phases of the student investigations characterized by the adopted Bayesian inspired approach with some analyses of students' responses on the worksheets.

Forming a hypothesis with a level of confidence

Based on their current understanding of probability students began by stating their initial hypothesis about fairness of each game and explanation for it. These students had previous experience with simple events, but the ones described in the games required an understanding of combined outcomes. Thus they relied primarily on their intuitive ideas, which generally led them to an incorrect judgement. The explanations given for their predictions indicated that they seemed to focus either on the symmetry in the total number of red and blue tokens in the bags with an additive reasoning in Game 1 or on the likelihood of simple events in each bag, i.e., the chance of getting a red token from a bag in Game 3 and Game 4. Unlike these counter-intuitive games (1, 3 and 4), Game 2 was consistent with the students' intuitions. Although their predictions about Game 2 were correct, their reasoning was problematic. They expected that the symmetry in the combined bags (equal number of red and blue tokens) would generalize to the combined outcomes.

Starting with students' initial predictions about the fairness of the games was an essential part of the Bayesian inspired approach used in the task. When dealing with uncertainty, we often draw upon a variety of evidence, but particularly personal knowledge or belief or past experience in the absence of empirical results or theoretical knowledge. From the Bayesian perspective this is the basis for subjective probability. Since these beliefs can change based on new evidence, it is important to assess personal degree of confidence in the initial hypothesis or prediction and look at how it changes over the course of gathering new relevant information. Each group's responses on the worksheets showed that students were likely to be more confident about their initial hypothesis in Game 2 that was more intuitive. The initial confidence level in Game 2 was on average 9.4 (out of 10) for all groups while the average level of confidence in Game 1, Game 3, and Game 4 was 8.9, 8.4, 8.8 respectively. These values can be argued as indications for how much students are willing to rely on their personal beliefs or knowledge without additional evidence. Since the fairness judgment for the games 1, 3, and 4 were less intuitive to them, none of the groups correctly identified whether the game was fair or not initially.

Using information based on experiment data to update level of confidence

To test their initial predictions, each group played the game as many times as they wanted. The number of times that the games were played, i.e. the number of trials, tended to be small and varied among groups, from 5 to 30. Students mostly did not incline to revise their initial beliefs based on the game results unless they believed that they had contradicting results. For example, group A changed their first predictions in Game 1 (based on "same wins=11", "mixed wins=19") and Game 3 (based on "same wins=2", "mixed wins=3"), and group B modified theirs in Game 1 (based on "same wins=5", "mixed wins=15") and Game 4 (based on "same wins=5", "mixed wins=5") after playing the game. These changes in predictions involved switching to 'unfair' in Game 1 and to 'fair' in Game 3 and Game 4. The changes in the level of confidence seemed to vary depending on how much they were convinced by the actual game results. While the level of confidence dropped for Game 3 and Game 4 by -4.5 and -1 points respectively, there was no adjustment for Game 1 in both groups.

Using information based on simulated data to update level of confidence

To further test their initial or current updated hypotheses, each group built a model of the games using *TinkerPlots* to quickly collect more data. Naturally the number of trials in computer simulations became higher, ranging from 100 to 100000. However, students mostly seemed to find 100 trials large enough to base their decisions. The simulated data results generally seemed to help students update their current hypotheses. When asked what would make them more certain in questions 6 and 10 (Figure 2), some groups tended to suggest conducting 'more tests', i.e., running the simulation again with the same number of trial. Indeed group A and group C carried out a few more simulations with the same number of trials using *TinkerPlots* and recorded the results. At the end of the task, the groups arrived at the correct judgment about the fairness in the majority of the games after running multiple tests using computer simulations. Moreover, there was a positive increase in the level of confidence overall during the simulations. While the average change in confidence level in the case of switching to a right judgment was +0.6, this was doubled (i.e., +1.2) when students already had the right judgment from the game results. This result can be an indication of the effectiveness of additional evidence to support current hypothesis and personal belief.

CONCLUSIONS

The main premise of a Bayesian inspired approach to reasoning about uncertainty is that students' informal inferential reasoning about the fairness of the chance games is closely associated with their personal degree of confidence. Starting with a hypothesis about whether the game is fair or not and revising it based on new information come natural to students. The task design described in this paper is intended to help this process be more systematic by scaffolding students' reasoning step by step in each of the three phases (Figure 2). This way students do not only change their predictions about the fairness of the game based on data they collect but also update the level of their confidence which is linked to their certainty level.

The preliminary analysis offered in this paper suggested that students' first 'intuition-based' hypotheses about the fairness of the games mainly led them to wrong judgments initially particularly in the counter-intuitive games. However, both their conjectures and level of confidence on them tended to improve as they collected more data. More specifically, conducting a large number of trials and multiple tests using *TinkerPlots* modeling tool allowed students to change their beliefs about the fairness of the games and the strength of their confidence. Moreover, obtaining 'surprising' results in the game playing phase generally helped students revise their initial predictions while their level of confidence seemed to decrease to some extend. Students however tended to show more confidence in their personal belief about the fairness of the game when it was confirmed by 100 or more data collected during the modeling and simulation phase of the task.

Overall, the study shows the importance of integrating the personal beliefs, level of confidence, and empirical results from experiments and simulations in reasoning about uncertainty when making inferences about the fairness of chance games. The approached used in this task suggests a way to bring Bayesian thinking to the school level and makes it accessible to young students as well. Further research on this approach using clinical interviews with more individual groups might be useful to deepen our understanding of how students' experiences based on game results and simulation results affect their beliefs about the fairness of the games, personal degree of confidence, and strategies in the long run.

ACKNOWLEDGEMENT

This research was supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme. The author thanks the STATSTALK project members, Rupert Wegerif and Taro Fujita.

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Simulation approaches for informal inference: Models to develop understanding

Hollylynne S. Lee[1], Dung Tran[1], Jennifer Nickell[1] and Helen Doerr[2]

1 NC State University, Raleigh, North Carolina, USA, hollylynne@ncsu.edu

2 Syracuse University, Syracuse, New York, USA

Simulation approaches to inference have gained prominence in statistics education. In this paper we combine a theoretical analysis of the ways researchers and curriculum developers have represented models of simulation processes with an empirical analysis of the ways teachers represent models of the process they believe would be helpful to students. Our results cumulate in suggestions for a more explicit framework for using a simulation approach to inference.

Keywords: Teachers, modelling, technology, informal inference.

INTRODUCTION

The meat of doing statistics is making inferences about data. Researchers suggest introducing inference informally first and then transitioning to the procedures of formal inference (e.g., Zieffler et al., 2008). In building up informal inferential reasoning, a simulation approach can be an important tool to help students build a deep understanding of the abstract statistical concepts involved (Burrill, 2002; Maxara & Biehler, 2006). Indeed, Cobb (2007) suggests that educators can help students develop an understanding of inference through the "three R's: randomize data, repeat by simulation, and reject any model that puts your data in its tail" (p.12). Incorporating hands on experiences and dynamic statistical software can allow students to visualize the statistical process in order to develop an understanding of the inference process (Budgett, Pfannkuch, Regan, & Wild, 2013). Simulation approaches have been used in several curricula efforts in the US at the collegiate level, and new national standards in the US suggest such an approach for high school students. Researchers have reported modest results in improvement of students' understandings of inference through collegiate curriculum that use a simulation approach (e.g., Garfield, delMas, & Zieffler, 2012; Tintle et al., 2012).

A simulation approach seems to be an appropriate way to help students develop statistical inference conceptually. However, while the "three R's" process may seem simple, understanding all parts of a simulation is conceptually complicated. In fact, even students who know *how* to conduct a simulation, may not have a robust understanding of why they are conducting a simulation, what is being simulated, and how to make appropriate conclusions based on a simulation. Thus, *our research aims to consider, from a theoretical and empirical perspective, how teachers could and should represent the processes involved in using a simulation approach that could assist students in better understanding the general usefulness of such an approach in inference contexts.*

Our work is situated in a models and modelling perspective on teaching and learning mathematics as articulated by Lesh and Doerr (2003). In this perspective, models are systems of elements, relationships, operations, and rules that can be used to describe, explain or predict the behaviour of some other familiar system. Thus, we are very interested in how models (and modelling) can be used to improve teachers and students' understanding of a simulation approach to inference.

MODELS USED BY OTHERS

Many have engaged in research and curriculum development over the past 15 years, focused particularly on understanding inference and simulation approaches. For the purposes of our paper, we will highlight the work of several researchers whose models seem to build from one another. In 2002, Saldanha and Thompson reported that, when students can visualize a simulation process through a three-tier scheme, they develop a deeper understanding of the process and logic of inference. This scheme is centered around "the images of repeatedly sampling from a population, recording a statistic, and tracking the accumulation of statistics as they distribute themselves along a range of possibilities" (p. 261). The diagram in Figure 1 is meant to explicitly draw attention to the multiplicative relationship between a population, sample(s), and a distribution of sample statistics. Thus, the diagram can serve as a model for the process of repeated sampling. Their work also had students experience and attend to three levels in the sampling process: 1) randomly draw items to form a sample of a given size and record a statistic of interest, 2) repeat Level 1 process a large number of times and accumulate a collection of statistics, and 3) partition the collection of statistics to determine what proportion lies beyond a given value.

Several researchers have built from Saldanha and Thompson's multi-tier scheme and the models and modelling work of Lesh and Doerr (2003) for creating models that can assist students when using simulation approaches, or resampling techniques, for inference (Garfield et al., 2012; Lane-Getaz, 2006). Lane-Getaz (2006) offered the Simulation Process Model (SPM). This process includes three tiers: population parameters, random samples, and distribution of sample statistics. The SPM resembles Saldanha and Thompson's (2002) model and verbal description of three levels, but uses more explicit language in the diagram itself. The first tier is to describe the population distribution as the beginning of the simulation process, then, random samples are drawn from the population, and a sample statistic is selected related to the simulation process for Tier 2. In the last and third tier, the distribution of the sample statistics is formulated, and used to evaluate the likelihood of the event happened in the original problem (Figure 2). Lane-Getaz described how she used the SPM as an organizer to help students understand the general process of inference. She then adapted the SPM to specific examples used in her course so that students can see how the model frames the simulation process used in different contexts.

In line with Lane-Getaz's suggestion, Garfield and colleagues (2012) used a models and modeling approach in the design and research of the CATALYST curriculum (Catalyst for Change, 2012). Figure 3 shows their three-level framework including specifying a model, samples and numerical summary measures, and distribution of the numerical summary measures applied to one task (Cereal Boxes). They, too, advocated using a general structured diagram with students to organize their thinking about the general simulation process and for specific problems.

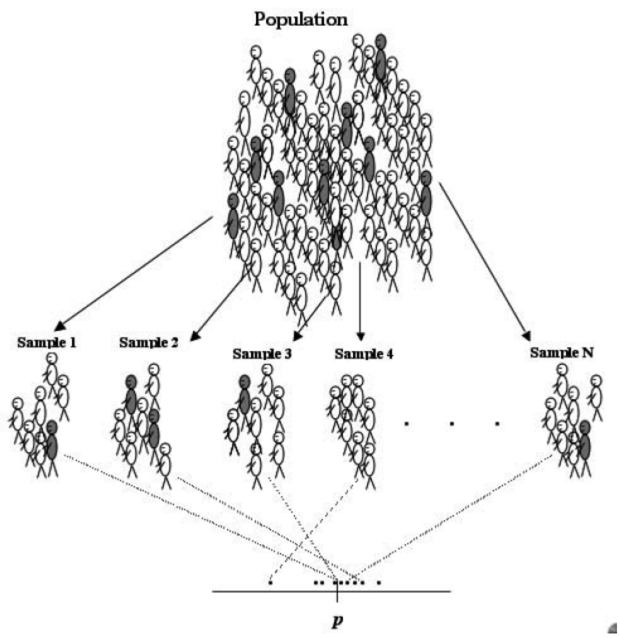

Figure 1: Model for sampling conception (Saldanha &Thompson, 2002, p. 267)

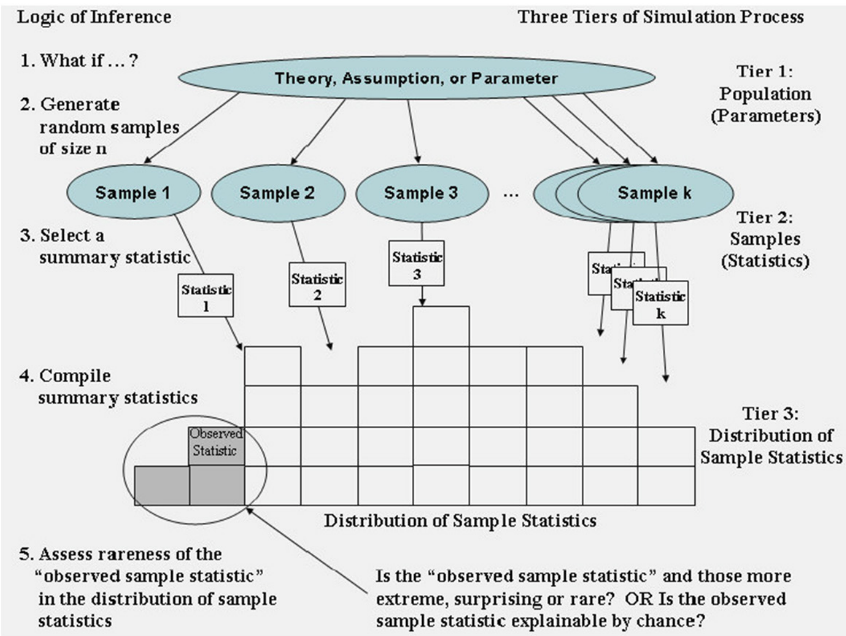

Figure 2: Simulation Process Model (Lane-Getaz, 2006, p. 280)

All the models shown in Figures 1, 2, and 3 include three tiers/levels for the simulation process, including population, samples, and sampling distribution, even though these terms are either implicitly or explicitly used. However, there are some differences. For example, Saldanha and Thompson (2002) implicitly allude to a statistic of interest in their diagram but explicitly refer to it in their verbal description of the three-level process. The statistic of interest is explicitly referred to in the other two diagrams (Figures 2 and 3).

Our understandings of the literature on simulation approaches to inference and the representations used by others, informed our design of tasks to use with teachers in a graduate course on teaching and learning statistics. What follows is a description of the course, participants, and several tasks that took a simulation approach to inference. The set of tasks served as a model development sequence (Lesh et al., 2003) that enabled us to support the development of teachers' understanding of a simulation approach to inference while also revealing and eliciting their thinking. As teachers (and students) are learning about simulation techniques, we cannot assume that they fully understand the computer representations of that process and the underlying randomization and sampling that is occurring. Hence, this paper focuses on a task presented to teachers to elicit how they would help students understand the simulation process. Specifically, we wondered: *What can we learn from teachers' visualizations of the components of a simulation process for drawing an inference to suggest a general framework (model) that could assist learners?*

COURSE AND DATA COLLECTION

A team of four instructors from two institutions met weekly via videoconference for an academic year to design a 15-week course offered at each institution, and to discuss issues and alter plans as the course was taught. The course consisted of opportunities for teachers to engage in statistical investigations with real data and tasks designed to develop understandings of distribution, samples and sampling distributions, and inferential statistics, especially using randomization approaches. The course used the software *TinkerPlots* (Konold & Miller, 2011) and applets (e.g., http://lock5stat.com/statkey). The software provided simulation tools needed to represent a population, a sample, and a distribution of sample statistics.

The simulation tasks used in the course were adapted from typical ones used by popular introductory statistics materials that use modeling and randomization approaches (e.g., *Paul the Octopus task*, Lock et

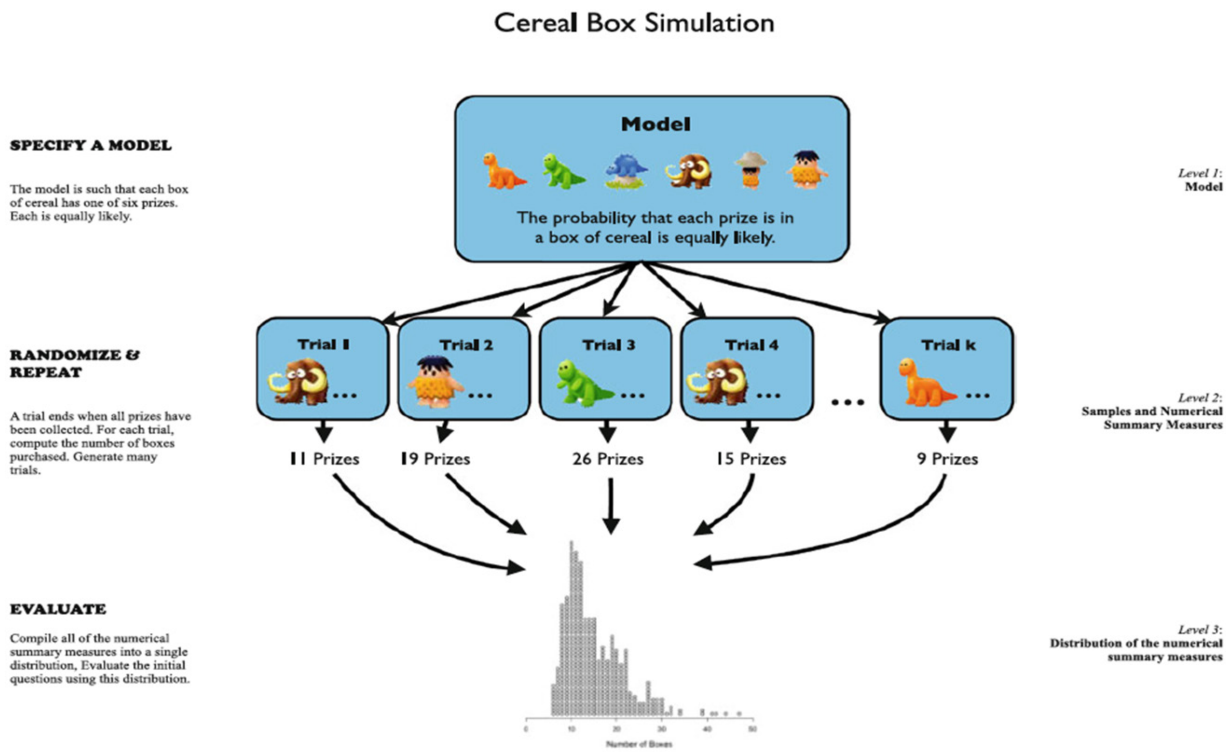

Figure 3: CATALYST modeling and simulation process (Garfield et al., 2012, p. 890)

al., 2013; *Dolphin Therapy task*, Catalysts for Change, 2012). Teachers had also been assigned two articles (Lane-Getaz, 2006; Lee, Starling, & Gonzalez, 2014) to read in which authors used diagrams (Figure 2, and another similar to Figure 1) to illustrate the simulation approach used in the tasks discussed in the articles.

After two simulation tasks (*Paul the Octopus* and *Dolphin Therapy*), the instructor summarized the models that were designed and the resulting simulation process used. The intent of this summary was to help make connections across the two simulation tasks and generalize the processes used to develop an overall conception of the modeling and simulation process. After this summary, and before the assigned readings had been discussed, the instructor used the following task to allow teachers an opportunity to express their developing conceptions of the simulation process in terms of how they would help their students to understand the process. Teachers formed nine groups of 2–3 to create diagrams on a large poster. The exact wording of the task posed to teachers was:

Suppose you were going to use a randomization approach with your students to help them use a simulation (with physical objects or computer models) to investigate if an observed statistic is likely to occur, or not unlikely to occur. Draw a diagram you could use to help students understand the general process used for applying randomization techniques for solving these types of tasks.

Across institutions, the course served a variety of graduate students. This paper focuses on the 19 participants from one institution, since this diagramming task was not completed at the other institution. The 19 participants consisted of one advanced standing preservice teacher (5th year senior), three preservice teachers enrolled in an M.A.T. program; 10 teachers in a masters program while currently teaching students in secondary or tertiary contexts; and five PhD students in Mathematics or Mathematics Education, three of whom were currently teaching in tertiary contexts. Fifteen teachers were female and four were male, with two teachers for whom English was a second language. All but one teacher had completed at least a first level course in statistics. Henceforth we refer to course participants as teachers.

ANALYSIS OF TEACHERS' DIAGRAMS

Five of the nine diagrams, created by teachers in our course are shown in Figure 4 (4a, 4b, 4c, 4d, 4e). These diagrams are representative of the collection of diagrams and were selected to illustrate points made in this section. Though we began our open coding informed by literature and the ways others had represented sampling and simulation approaches (Figures 1, 2, 3), we will explicate how our analysis of teachers' self-created diagrams led us to identify aspects that may be more or less salient for teachers, and perhaps other learners. What follows is a description of the major themes we identified in the representations/descriptions in the diagrams that indicated to us that teachers' had a strong (or incomplete) understanding of the simulation process. It is these themes that are shaping our vision for ways to be more explicit in our modeling processes when using a simulation approach to inference.

Representations in Level/Tier 1 (Population/Problem)

Lane-Getaz (2006) presented the process of using simulation to develop the logic of inference starting with a question in mind, "what if", to investigate a problem (see Figure 2). In this step, students need to specify a "theory, assumption, or parameter" for further sampling. In the Model level of Garfield and colleagues (2012, Figure 3), there are more explicit unpacking of the real world cereal box into statistical terms (six equally likely prizes). We consider this step as crucial in creating a model of the real-world problem. The purpose in the modeling process is to express the problem of interest in mathematical/statistical terms that include a set of assumptions (e.g., likelihood of an event occurring). Six teachers make this aspect of the modeling explicit in their diagrams. For example, the top row of "steps" in Figures 4a and 4e rudimentarily addresses the importance of creating a model of the real world problem. Figure 4c shows that the group of teachers decides to use a coin flip as a model of mom's reaction (yes or no) to whether we can have a party. Implicitly, this coin flip model makes the assumption that probability of a head/tail (likely assumed to be 0.5) is congruent to the probability of mom's response of yes/no. In Figure 4d, teachers also elaborate steps needed to model a real world problem by stating, "determine parameter of interest, determine assumptions for proportion(s), and simulation model (based on assumptions)".

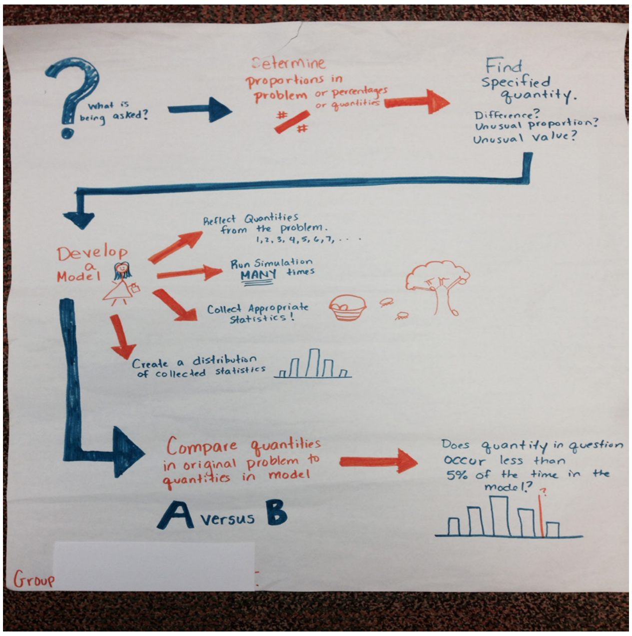

Figure 4a

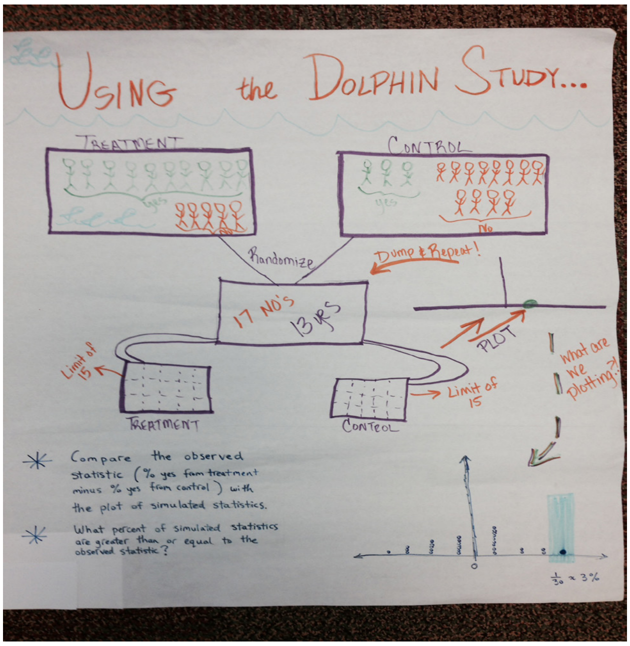

Figure 4b

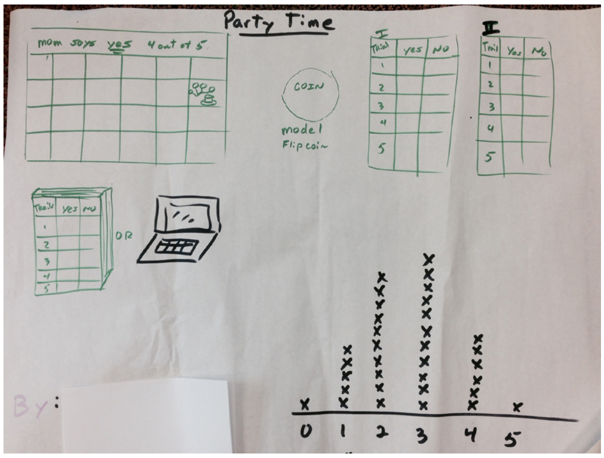

Figure 4c

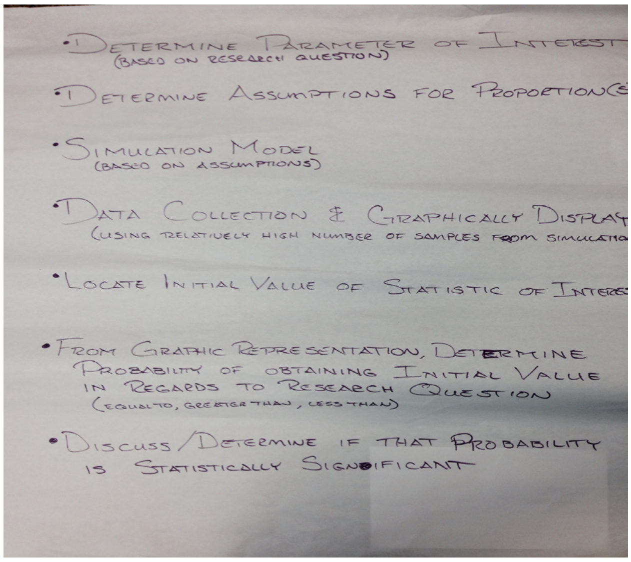

Figure 4d

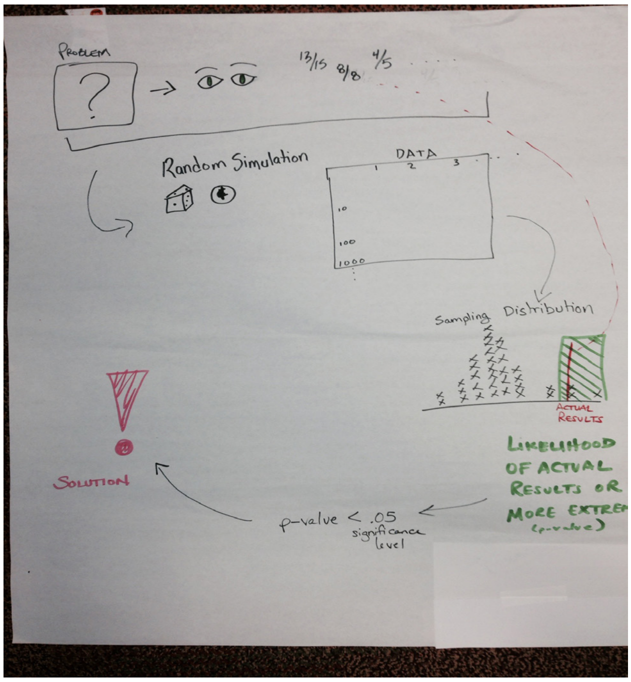

Figure 4e

Figure 4: Five diagrams created by teachers

Emerging from our teachers' diagrams, we noticed that they often attended to identifying the statistic of interest in the original problem that would be later used in the simulation process and for decision making. For example, the posters in Figures 4a, 4b, and 4e explicitly mark or indicate a quantity of interest form the original problem and refer to it later in the simulation process as the statistics to collect and that one needs to locate that statistics in the distribution of sample statistics for decision making. Thus, we found that six diagrams emphasized the process of creating a model for the real world problem and attended to the importance of the original statistics of interest in the problem. This level of detail does not appear explicitly in the models shown in Figures 1, 2, and 3.

After the model *of* the real-world problem is formulated, one needs to develop a model *for* the simulation process. In this modeling step, one decides which types of random-generating devices (physical or technological tools) can be used to accurately represent the random selection or assignment in the problem. When conducting the simulation, it is important to consider the assumptions behind the use of the tools to align with those made in the questions of interest. For example, when using a coin to model the chance of success in Mom saying yes to a party (Figure 4c), an assumption inherently made is that the coin is fair or the probabilities of a head and a tail are equal and equal to 0.5. Similarly, if one decides to use a simulator tool, it is important to specify the probability distribution and how it matches the assumptions in the problem (See Figure 4d "Simulation Model"). We also see in Figure 4b that teachers represented the process of combining all participants together and reassigning to groups for the *Dolphin Therapy* example.

Representations in Level/Tier 2 (Randomize and Repeat)

Teachers had very different ways of representing the process of generating a random sample of size n and computing a statistic, repeating this process k times, and collecting and displaying statistics from all k samples. To begin with, the notion of a random sample (or randomization process) was not always explicit. While posters in Figures 4b and 4e indicated random sampling, the poster in Figure 4c may have implied the random process by noting the "coin flip". The notion of a single sample of size n, repeated k times was also not always well represented. For the posters using a specific example (Figures 4b and 4c), the n that was noted in the pictures matched the problem context (n=15 in each group, and n= 5 responses from Mom). However, the teachers that drew or described a general process did not explicitly state the need for k samples of size n, though phrases such as "many samples" "high number" and the pictorial diagrams in Figures 4c and 4d implied a repeated process. As deliberately pointed out in the Lane-Getaz's (2006) diagram (Figure 2), "the samples of size n" are important to distinguish from k samples (often many) in sampling. Both n and k are critical parameters in designing and running a simulation, especially since they are often inputs required in software such as *TinkerPlots*.

All teachers explicitly or implicitly indicated that the simulation process including recording, collecting, and graphing a statistic of interest from each sample. However, the level of detail or pictorial representation of this process varied greatly. Since these diagram were meant for teacher to express a representation they could use to help their students understand this process, we were certainly left wondering whether they really understood the randomize, repeat and collect phase.

Representations from Level/Tier 3 (Empirical Sampling Distribution)

Seven diagrams included an image of an empirical sampling distribution in their diagram, with an exception shown in Figure 4d. Many also explicitly drew or indicated that the original statistic from the problem context should be located in the distribution (Figures 4a, 4d, 4e) and used to assess likelihood that the original statistics would occur under the assumed model of random selection or assignment. Only six diagrams explicitly indicated where to look in the empirical sampling distribution and how to estimate a probability (proportion) of the actual observed event by examining the tail(s) of the distribution (Figures 4a, 4b, 4e).

SUMMARY AND SUGGESTIONS

As we reflected on the diagrams constructed by our teachers and compared these with the diagrams for simulation processes discussed in research literature, we saw the need to propose aspects of a simulation approach that should be made much more explicit for learners and teachers. One major distinction we suggest is that more attention needs to be given to the modelling process. We feel that there is a two part modelling process that should be made explicit. This modelling process is similar to the emergent models that Gravemeijer (1999) purports can assist in transitioning from a real context to formal mathematics. The first is to create a local specific *model of* the real world context in statistical terms. The second is creating a *model for* the simulation process that can be used to generate random samples. The second model for the simulation process is more general because it can be applied to many problems. Most previous works have combined these two aspects into a single "model" or population level. We also suggest being more explicit concerning building a distribution of sample statistics, using the distribution to reason about the observed statistic, and making a claim about the chance of that observed statistic occurring. We encourage others to consider making explicit the following aspects:

Level 1. Population: Create a MODEL OF the real-world problem

- *Make assumptions to build a mathematical/ statistical model of the problem – determine a null hypothesis*
- *Specify the observed statistic and the statistic of interest*

Level 2: Simulation Model: Create a MODEL FOR simulation process

- *Choose appropriate tool(s) (physical/technological) for the problem that aligns with the assumptions made in creating the model of the real problem*

Level 3. Samples and Statistics: Randomize and repeat

- *Draw a random sample of same size n and record relevant statistic*
- *Repeat random sample k times (large number) and collect statistics from each sample*

Level 4. Empirical Sampling Distribution: Examine how statistics vary

- *Build a distribution for the recorded statistics*
- *Locate the original observed statistic in the sampling distribution*

Level 5. Final Decision: Making inferences from models

- *Use proportional reasoning to evaluate the likelihood of the event happened.*
- *Decide if the observed statistic and those more extreme are explainable by chance.*

We maintain that a simulation approach framework could help support scaffolding, and eventually abstraction, for how a simulation approach can be used for inference. It is also important for students to experience various models in specific situations. Such pedagogical approaches have been advocated for by many, and used successfully in work such as Garfield and colleagues (2012) and Podworny and Biehler (2014).

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Analysing teachers' knowledge about sampling using TinkerPlots 2.0

Maria Niedja Martins[1], *Carlos Monteiro*[2] *and Carolina Carvalho*[1]

1 University of Lisbon, Lisboa, Portugal, marianiedjamartins@campus.ul.pt, cfcarvalho@ie.ul.pt

2 Universidade Federal de Pernambuco, Recife, Brazil, cefmonteiro@gmail.com

The conceptualization of sampling is crucial to understand statistical data. However, the teaching about sampling is not generally emphasised in school curriculum. This study investigated how teachers understand about size and representativeness of samples using TinkerPlots 2.0 software. The study was comprised of two sessions. A semi-structured interview and a familiarization about the basic use of TinkerPlots were developed during the first session, and participants engaged in three tasks on sampling using TinkerPlots during the second session. As a result, the teachers began to consider aspects of the variation of data to determine when representative samples were involved in TinkerPlots. The ability to select samples and analyse them seemed to contribute to improve their understanding about sample size and representativeness.

Keywords: Statistics education, primary school teachers, TinkerPlots.

INTRODUTION

The recognition of the influence of statistical data in the current society demanded the inclusion of this topic in national curriculum of many countries (Monteiro & Ainley, 2004). Several studies suggested that statistics education can provide bases to students develop abilities to argue and counter-argue information, understand the generation of statistical data and make informed decisions based on their analysis (Gal, 2002). Therefore, statistical knowledge is essential for critical reflective and participatory citizenship (Carvalho & Solomon, 2012).

An important knowledge that enables citizens to understand critically statistical data is related to conceptualization of sample and sampling. Bolfarine and Bussab (2005) conceptualize sample as any subset of a given population, and sampling as a technique of selection of such subsets. Innabi (2006) argues that in order to analyse the representativeness of a sample is necessary to know whether the sample is large enough and has the variety present in the population. It is recommended to increase the sample to ensure the variety of the population can be better visualized. However, sample sizes from a homogeneous population tend to be smaller, because such samples will have less variability.

To understand the conceptualization of sampling is crucial consider how the data were chosen, what methods are employed for the selection of these cases, what features and prioritized variables, so we can understand other contexts in which the information can be applied (Saldanha & Thompson, 2002; 2007). Therefore, the understanding about sampling seems to be essential as curriculum school content, therefore it is very relevant for teachers who teach statistics.

Although, the teaching about samples and sampling is fundamental to base the practices of statistics, it needs to be more emphasised in school curriculum (Watson, 2004). Recently, several studies investigated the conceptualization of sample and sampling among students from different levels. However, it is also important to investigate such situations among teachers who are going to approach such curriculum content (Martins, Monteiro, & Queiroz, 2013).

Several studies investigated the developing of understanding about sample and sampling using computer based tasks. For example, Manor, Ben-Zvi, & Aridor (2013) conducted a study that engaged students in designed instructional activities using computer modelling and simulations of drawing many samples. According to those authors, the research tasks enabled the students to think about sampling as a process when analyses are associated with samples.

Baker, Derry and Konold (2006) involved young students in two experiments about center and variation. In one of situations they used *TinkerPlots* (Konold & Miller, 2011) to develop a task in which students can get engaged in an inferential game. According to these authors "the inferential approach acknowledges that students with their teachers have to take part in the social practice of reasoning (p. 2)". When the students were comparing two distributions, they should realize that they needed of certain concepts to reach a conclusion on the distributions were different or not. The students could come to the conclusion that the concept of average was important to identify these differences. Therefore, the uses of certain concepts involved in a game of give and ask for explanations.

These studies suggest that it seems to be important the selection of several samples. The *TinkerPlots* offers the possibility to explore the relationships between data and chance (Konold & Kazak, 2008), since it is possible to perform simulations of samples and populations.

Delmas and colleagues (1999) used a computer environment in which the students could simulate several samples of different sizes and visualize the distribution of the values of a statistic. The results suggested that students indicated that larger samples should produce a statistical distribution similar to their population of origin. According to those authors it is possible that the students had the intuition that the average is a point within the population, and that gather more averages it will have a distribution very similar to the population.

In this paper we discuss some aspects of a study that investigated teachers' knowledge about sample size and representativeness. The aim of this study was to explore possible computer based tasks which can help teachers to understand those important aspects about the sample and sampling.

METHODOLOGY

This was a qualitative exploratory study that followed an interpretative approach. The research was conducted in a rural public school located of a municipality of Metropolitan Region of Recife (RMR), Brazil. The choice for this school was based on a survey conducted by GPEME - Research Group on Mathematics and Statistics Education (Carvalho & Monteiro, 2012), which identified 85 public schools in the RMR which had computer labs, and investigated how those labs were used.

There were two research sessions to collect the data. The first session was comprised of an individual semi-structured interview in order to have information about teaching experiences, as well as to identify their levels of understanding about the concept of sampling. The interview questions were based on a sample questionnaire used in the studies of Watson, Collis and Moritz (1995), Watson and Moritz (2000) and Watson (2004). These studies developed tasks associated with questions about sample, representativeness of small and large samples, sampling, and media news about sample surveys with inadequate statistical basis.

At the first research session, we also develop a familiarization with the *TinkerPlots2.0*. The researcher presented different functions of *TinkerPlots* to the teachers, including those to handle the database and produce graphs. This familiarization was carried out because the participants did not know about the software, and it was expected that they had certain autonomy to use the *TinkerPlots* during other research sessions.

The second session was comprised of three tasks using *TinkerPlots*. These tasks were about representativeness, size and type of sample. Therefore, in this paper due lack of space, we report examples taken only from analysis of task 1 and 2.

The study was developed with four female teachers. Due to lack of space in this paper, we report aspects of research data from one participant. For this report, her name was changed. Suzy was 30 years old, and she had 5 years of experience as a teacher. She uses the computer every day to search contents related to her teaching activities and to access emails. Suzy has university degree in Education. However, she said that never had any specific learning on sampling, and she did not know about *TinkerPlots* or other educational software for teaching Statistics. In this paper we do not discuss the data collected from the semi-structured interview.

Task 1

The aim of this task was to know if the participants understood that increasing a sample, could have better accuracy of inferences about the population, since the variability of the population would be better visualized.

Figure 1 shows a copy of screen with 625 cases (fish) of a *TinkerPlots* database called Fish Population. Each case had a numerical code, and information about the type and size of fish.

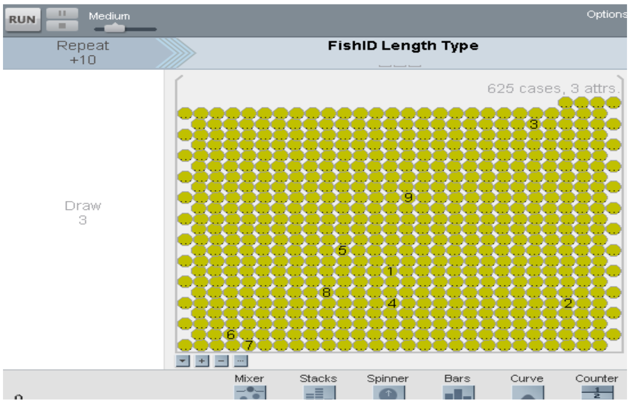

Figure 1: TinkerPlots Screen with 625 cases of Fish Population database

We asked the participant to read the following situation, which was based on the *TinkerPlots* resources:

> A certain fish farmer bought some genetically modified fish of a company with the promise that they would grow more than the non GM fish. In order to check whether GM fish grow more, the fish farmer joined GM fish with other fish that he used to have in a tank in which totalized 625 fish. After the total growth time of fish, the fish farmer gradually withdrew each fish from tank, and measured each one. From the data analysis in TinkerPlots, indicate which type of fish had greater length. Did the fish farmer make good deal?

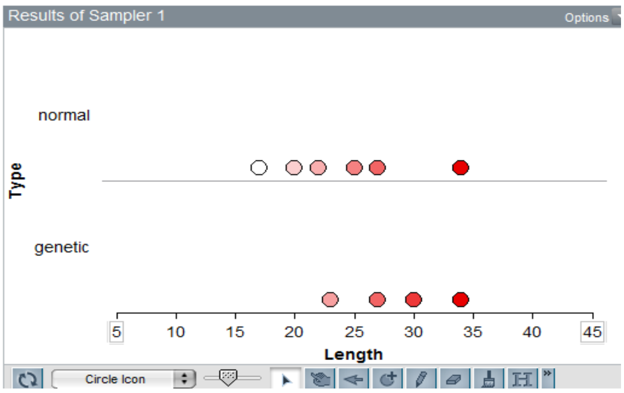

Figure 2: TinkerPlots Screen with fish population database, n = 10

During participants' analyses of task 1, we took initially samples from 10 cases, and then it was increased based on their indications. The participants should infer which population had bigger fish interpreting a graph similar to the Figure 2.

For each new inclusion of cases in the sample, we asked the participant to informally rate her confidence level in a scale from 0 to 10. Therefore, rate 10 should be teacher's maximum confidence. This procedure aimed to make explicit their understanding about changes on their own analyses (Prodromou, 2011).

Task 2

The second task was based two TinkerPlots databases: MysteryMixer1 and MyisteryMixer2, which were comprised of only one variable that is *number*. The Figure 3 presents a database used in this task.

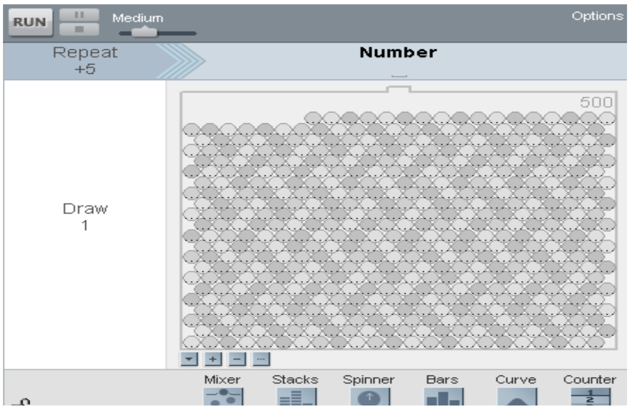

Figure 3: TinkerPlots Screen with MysteryMixer1 database N = 500

The database had 500 cases disposed in the simulator ranging from 0 to 100. This second task aimed to identify whether the teachers could reach a conclusion on a small sample. Therefore, the participants should identify clusters of samples and infer them to the population, using the smallest possible sample. To ensure this, we engaged the participants in a fictional situation about costs of sample survey:

> You have a limited amount of money to conduct survey on numbers. Each selection of five cases of this survey you should pay R$1,00. Your task is to identify a range in which all numerical values are repeated. You need to spend the least amount

of money possible, but you need to be quite sure about your answer.

In each task, the first author, acted as researcher asking questions to make more explicit the teachers' considerations about the data, and assisting them in the selection and manipulation of *TinkerPlots* tools.

The *Camtasia Studio 7.1* software was used to record on video the participants' speeches, their gestures and manipulations developed in the computer screen while solved Tasks 1 and 2. The transcriptions of audio records generated protocols which were base to the data analysis.

RESULTS

The analyses of participants' response suggested aspects of their understandings about the relationship between size and representativeness of samples.

Task 1

During the development of task 1, when we increased sample size the participants informally rated their level of confidence about their inferences. Table 1 shows the Suzy´s rates during this task.

Suzy gave a low confidence rate in their conclusions about small samples at the beginning of this task. The following extracts exemplify their arguments, when interpreting a graph similar to Figure 4.

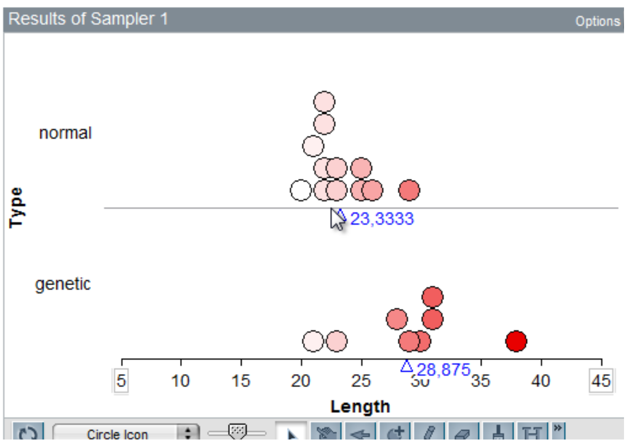

Figure 4: TinkerPlots Screen with fish population database, n = 20

Suzy: For me, this amount is not significant. It's because... well, the first time we had (a sample with) just over 2% (of the population) and now we have just over 3% (of the population). I think 3% is not significant value to buy something to put in a bowl and make a test. If so .. I would find significant 6% ... 10% ... just great! But to do a test ... to say ... (3%) I think very little.

Researcher: Right. But then, you see ... looking over here, can you observe who is showing a larger size?

Suzy: The GM.

Researcher: But, you are saying that perhaps this may not be significant for the rest?

Suzy: Exactly!

This fragment of Suzy's speech suggests that the teacher relates sample size to population size, and expressing that she considered the sample too small to make an inference. Suzy was unsure to make an inference, although that she identified a trend of genetically modified fish were larger.

Another extract from Suzy's protocol indicates that she was analysing the sample, and questioning the data variation in the samples, because she did not know the exact amount of fish for each population, since task 1 does not give this information.

Suzy: Does it [TinkerPlots] say the amount that it puts [in the sample]?

Researcher: No. It does not say the amount of one and another... whether it has more GM fish or normal ones. But, do you are absolutely sure that these here [GM] will continue growing?!

Suzy: It´s because, look... 12 and 8 [amount of fish for each type]. We do not know the amount per type of fish that he put here [in the population].

Researcher: What does that mean?

Suzy: That these results may change here because these data can be very different from there [population].

Teacher	S1	Confidence	S2	Confidence	S3	Confidence	S4	Confidence
Suzy	10	0	20	0	100	8	150	10

Table 1: Informal levels of confidence about the increasing of sample sizes

Suzy seemed to be concerned about possible errors due to small samples. We can infer that she was concerned about the variation, because only taking cases at random from the population would not ensure that the values of sample were identical to those of the population of origin.

In addition, Suzy developed the strategy of seeking patterns by analyzing the distribution of data and trends in the samples. This was reflected in her assigned increasing levels of confidence to her inferences, since Suzy could confirm in each sample a tendency on genetically modified fish to be larger. The need to use the concept of average also emerged, according to view in Figure 5. Suzy's response seemed to be influenced by the average value in the different samples:

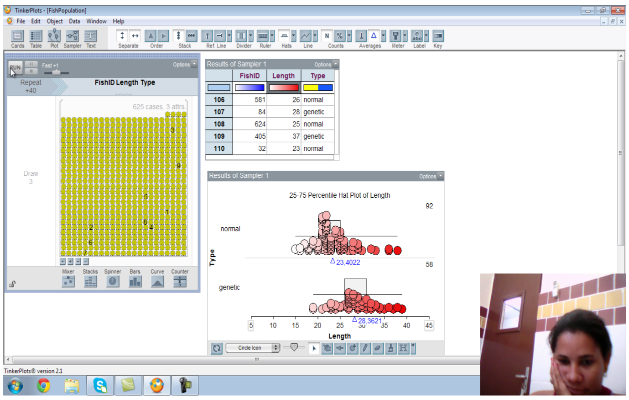

Figure 5: TinkerPlots Screen with fish population database, n = 150

Researcher: The average showed something to you?

Suzy: It shows that I'm correct! Because like this, it did not have changes. If the average had very fluctuated, there would be worrying. However, it remained constant to the extent that we have been getting more information.

Researcher: So it helped?

Suzy: It is. Now I have a 100% certainty.

Task 2

Suzy reached her answer with a smaller sample, and justify the response based on the idea of homogeneity of sample.

Researcher: Why did you find easy to say a response with lower number of cases?

Suzy: I think was because that issue of the group that I told you... because it was concentred in the group ... and ... don't know more.

Researcher: Concentred in the group? What do you mean?

Suzy: So... lets I say... don't know. I thought so... to the extent that we were taking... I thought, should not have 50%, then I was dropping, 25%. And with this there, I did far less than the percentage that I thought at first. And, to the extent that I was taking (cases from the simulator), and that I was doing, the concentration kept constant. Then, I did not need to take all this data, I had focused on to analyze a bigger percentage... I believe this happens because the information is contained like that, in that group. It is not one thing mixed. I think that's it. I just do not know to explain, but I understand.

Suzy's strategy to be able to generalize the results of the sample to the population focused on the analysis of the trend of data in successive samples. She quickly realized that the curve where was concentrated most of the data remained constant even when the sample grew and relied on it to provide a final inference.

Another strategy that also seemed important for Suzy to choose a representative sample was associated with hypothetical costs of sample. The following extract from Suzy's interview exemplifies how this aspect was relevant to her analyses.

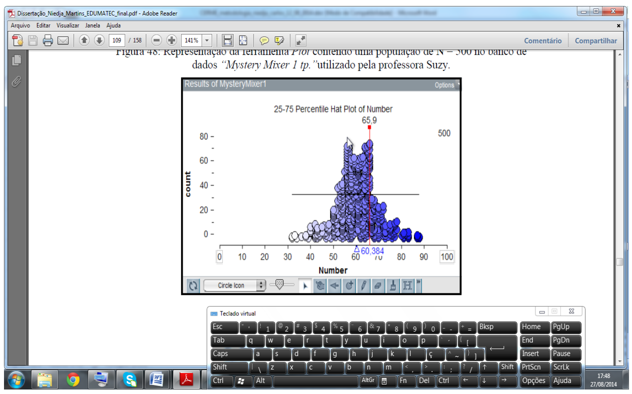

Figure 6: Graph produced by Suzy using MysteryMixer1 database

Researcher: [after show the graph with all cases according to Figure 6] Was close to what you said?

Suzy: I said 50 and 63. It was close!

Researcher: Were you pleased with your survey? With the amount you spent to give this approximate answer?

Suzy: Bargain, right?! [Laughing] So... Do you know one thing that I am worry about? In taking the part [sample], the bigger amount possible. And then, with the result, I can see that perhaps with smaller percentage, I could already tell you the answer. I could have been spending less because the sample was already confirming where it was [the cluster].

Suzy made inferences from 15 cases sample when she was interpreting the MysteryMixer2 database. The analyses of interview protocols suggested that this reduction was due to the fact that the teachers identify that the data of the samples were homogeneous.

CONCLUSIONS

The results of this study suggested that the teachers presented different ideas about sample when they analysed heterogeneous samples. This result corroborates the idea that the sampling involves different statistical concepts and ideas, and that inconsistencies of these notions can influence how a person perceives the representative samples.

In task 1, the analysis of variance of cases of homogeneous samples and the hypothetical cost to the sampling seemed to be the main influences on determining the appropriate sample size to make a final inference. One explanation for this result is the possibility that the teachers had to see the increasing of the samples and to compare the trends showed in *TinkerPlots* representations.

Therefore, the situations in which teachers can compare distributions may be potentially important to understand the tasks with the sampling; as seen in the study (Ben-Zvi et al., 2011) who found that the use of increasing samples can easily identify and recognize patterns representative established through comparison.

From the results of this study, further research is necessary to explore the autonomy of teachers to use software like *TinkerPlots* in order to build understandings of statistical concepts and also because teacher education in statistics software seems to have a gearing effect on eventual student learning of statistical ideas (Pratt, Davies, & Connor, 2011). In addition, it is crucial to investigate how this knowledge constructed from their interaction with software can motivate reflective situations to explore new ways to teach statistics.

ACKNOWLEDGEMENT

This research was supported by CAPES Foundation, Ministry of Education of Brazil.

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Intuition about concept of chance in elementary school children

Jasmina Milinković

University of Belgrade, Teacher Education Faculty, Beograd, Serbia, milinkovic.jasmina@yahoo.com

The paper deals with initial development of stochastic reasoning in children. After short overview of the theories of development of stochastic reasoning, we focus our attention on intuition about probability. In the second part of the paper we report results of an empirical research on intuition about the concept of chance across grades 4th to 7th. The analysis of findings provides evidence of preconceptions about probability of undetermined events. These preconceptions are in conflict with formal theory but support Siegler's rule based model. We argue that we should take into account pupils' intuition about the concept of chance as well as other prerequisite mathematics concepts (fractions, proportions, etc.).

Keywords: Chance, intuition, Siegler's rules, stochastic reasoning, misconceptions.

INTRODUCTION

Probability is an area of mathematics contrary to math logic, paradoxical and counterintuitive (Kapadia & Borovcnik, 1991). How do we gain our initial understanding of events with uncertain result? How do we transform our initial conception of chance? Can we expect development of concept of chance despite of lacking opportunities to learn? We endeavoured to address these questions in order to find when the right moment is to introduce these topics in school curriculum. The early years of schooling in Serbian school system are considered as preliminary for empirical introduction of fundamental (and essentially abstract) concepts in mathematics such as relations, functions, equivalence, probability, and the like. The decision when we start to deal with a new class of concepts is associated with the decision on how we want to deal with them.

Our knowledge begins with experience. Experience provides basis for intuitive knowledge. Generally, intuition is expected to support (formal) learning. Kant notices that "... even our empirical knowledge is made up of what we receive through impressions and what our faculty of knowledge...supplies for itself (1929, pp. 41–42). Fishbein (1975) defines intuition as an autonomous cognitive activity based on individual experience. Intuition is general and immediate, which allows direct impact on the reasoning about situation. We can speak about two natures of intuitive reasoning in mathematics learning: the intuitive acceptability and intuitive anticipation. Intuition eligibility gives a sense of the sufficiency of the evidence of an assertion or obvious truth of certain facts.

The main aspects of intuitive knowledge about the probability of events are belief in the regularity of occurrence in the environment and (consequently) their predictability. Intuition is developed naturally as a consequence of experiencing the stochastic nature of the environment. Fischbein (1975) points out that humans need to rationalize and to predict leads us to search for regularities in phenomena. He noted that acceptance of intuition as a subjective conviction carries the conviction of self-proved (obviousness) and inner coherency. Chernoff (2008) discusses the dual classification of probabilities "belief-type" and "frequency-type". We may say that probability estimates are an expression of intuition about the relative frequency. In the case of intuition about the frequency of events, people tend to make judgments on the basis of time-limited past experience, which they estimate to be adequate.

Key theories about development of stochastic reasoning give us clues about appropriate time to start dealing with the concept of uncertainty. To begin with, Piaget and Inhelder (1951) claimed that at the operational level child finally becomes able to detect causal relationships that lead to the differentiation of unpredictable events. They recognize two sources of reasoning about probability: previous experience and mathematical constructions. Fischbein, on the other hand, believed that without formal education children cannot reach the operating concept of probability. He believed that intuition about probability is formed by the age of 14 or 15. Fischbein and his associates' study (Fischbein, Pampa, & Manzato, 1970) indicated a developmental leap of sixth grade pupils, but also the possibility of achieving a similar level of skills in younger trained pupils. Finally, Shaughnessy (1992) identifies four levels of conceptual development of stochastic reasoning linked to levels of formal mathematics education: (1) non-statistical level, (2) naive statistical level, (3) "on the horizon" statistical reasoning, and (4) pragmatic level. On the first level, reasoning is based on beliefs, deterministic understanding of phenomena, on causal inference or generalizations based of one case. The second level is characterized by conclusions using various heuristics such as representativeness, availability as well as by shifts in thinking about randomness and chance. The next level of reasoning is achieved when person has ability to apply normative models on simple problems, identify differences between reasoning based on intuition and on the basis of a mathematical model as well as knowledge of different mathematical representations of the concept of chance. Finally, the pragmatic level implies deep understanding of the mathematical model, the possibility of comparing different representations, the ability to apply a normative model and knowledge of the limitations of individual stochastic models.

Along the line of Piaget's theory comes Siegler's "rule-assessment approach" to cognitive development. His research encompasses the development of stochastic reasoning. (Yet, unlike Piaget, he does not advocate across domains cognitive developmental levels.) Siegler claims that cognitive development may be described as "acquisition of increasingly powerful rules for solving problems" (Siegler, 1981, p. 3). According to him, children first generate a series of alternative rules based on rational task analysis, previous empirical work and similar activities. Next, a set of problem types yields to patterns of correct answers and errors for children following each of the rules. Finally, if there is a theoretical prediction for a certain type of comparable problems, the asynchrony may be identified and cause changes in reasoning. Scholz (1996, pp. 301–302) reflects on Siegler's study which included variants of card games and drawing an object from the urn. He reports on Siegler's description of the pathway of cognitive development through process of decision making primarily based on implementation of (new) rules. There were four rules: 1) always choose urn with a larger number of favourable outcomes, 2) if the number of favourable and unfavourable outcomes are the same, elect the urn with fewer unfavourable outcomes, 3) the difference between the number of favourable and unfavourable outcome is calculated for each urn and the one with the greater difference is selected, and 4) the ratio of favourable and unfavourable outcomes is the election criteria (Scholz, 1991, p. 246). The rules were determined in relation to the dominant dimension of a favourable outcome and subordinate dimension of adverse outcomes. The researchers noticed that pre-schoolers only applied the first rule while the children in the lower grades applied the fourth rule and less the third rule while completely ignoring the second rule. Generally, the predictions of the respondents from 3 to 20 years, agreed with the first and fourth rule.

Initial research in the area of development of probabilistic reasoning dealt mainly with intuition, including misconceptions that we have about uncertain events as a result of growing up (not education). Here, we mention the research of Piaget and Inhalder (1951), Tversky and Kahneman (1982), Fishbein (1983), Green (1982), Hawkins and Kapadia (1984), and Nisbet and Williams (2009). Unlike, Kazak and Confrey (2006) for example, who claim that the results of their study conducted among 9 year old children supports the idea that confronted with various tasks with "chance settings" children could develop a quantitative perception of probability. In recent review of the research in probabilistic reasoning Schlottmann and Wilkening conclude that the contemporary research move boundaries for understanding concept of probability for earlier ages prior to instruction but does not provide understanding of the implications of these preconceptions (Schlottmann & Wilkening, 2011). Our paper offers a small contribution in this matter.

METHODOLOGY AND RESEARCH FINDINGS

The study was conducted among pupils in Serbia, where no probability and statistics topics has been a part of the state curriculum for elementary schools. Our research sample consisted of 392 children, Grade 4 to 7 (11 to 14 year old). We have observed lessons in 16 classes in 3 different school. All three schools are located in the centre of a large city (with about 2 million inhabitants) and a short distance away. Schools were selected based on the principle of similarity of children population, to eliminate factors of education or lifestyle that may have an effect on the intuitive understanding of statistical concepts of interest to us. The fact is that this population may be described as prone to reading the daily newspapers or watching shows on TV (which often use statistical data). The study was performed in regular classes and respected regular composition of classes. The teachers used lesson plans developed by researcher. The researcher made field notes during the lessons. The researcher and the teacher had meetings to discuss what was happening in the classroom.

Students observed and analysed through class discussion different situations that might provoke thinking about the concept of chance. Here, children were prompted to express their beliefs about simple game like situation with undetermined result. In the activity which is in focus of this paper, pupils discussed how likely is to get a red cube out of the boxes containing red and white cubes.

The teachers were expected to probe pupils' intuitive reasoning about chance. Their role were of a moderator. The teachers were not supposed to provide theoretical background, give correct answers or leading clues. They asked pupils either to conduct an experiment and analyse the data or to study a scenario describing results of this experiment.

There were three boxes with different numbers of white and red cubes. In the first one, there was 1 white and 1 red cube. In the second box, there were 9 white and 1 red cubes, and in the third, 2 white and 2 red cubes (Fig 1).

Box	Content
First box	□■
Second box	□□□□□□□□□■
Third box	□□■■

Figure 1: Three boxes

We quote and analyse pupils' logic during class discussion. The discussion started with the experiment of pulling a cube from the first box containing 1 red and 1 white cube.

Teacher: What do you think, which colour you are going to get out of the first box?
Milos: I'll get red because that colour is in charge.
Olga: I'll get red because I have to get it.
Angela: Red, it's a beautiful colour.
Goran: I think I'll randomly draw a white.
Jovan: We're sure to a draw some.
Zoran: I think I'll pull the red one, although the chances are fifty-fifty.

Milos, Olga and Angela considered that red was supposed to happen because of their desire to get a red cube. It could be discerned that those children believed that they somehow could affect the result of the pull. We could identify non-statistical level of reasoning among substantial number of children. Zoran and Goran apparently were aware of concepts such as "randomness" and "(equal) chance" but it did not prevent them from having "non-statistical" judgments. Significantly, we could observe confusion between what they believed and what they thought they were supposed to say. Successively, the same question was posed regarding the second and the third boxes. Children's judgments about the second box with 9 white and 1 red cube showed more sophisticated statistical views of children.

Ksenia: White, because there are more whites.
Luka: I'll get white. There are many more (white cubes), and they are more likely.
Obrad: I will pull both colours. Maybe I'll draw some more, but you never know that.
Danko: If we pull a cube for fifty times from the first box we could get, for example, 3 red and 47 white cubes, because anything is possible.

Ksenia and Luke ground their expectations in the principle of "the more favourable outcomes, the better chance." (Siegler's Rule 1). Unlike them, Obrad reasoned that the result of next drawing could not be predicted. Even more, he stretched his conclusion as he stated that "we cannot talk about any predictions what so ever." He was not the only one who believed in that. Danko, had similar thoughts about impossibility to predict results. Pupils who conducted the experiment, by the time of dealing with the third box, have already begun to change their opinion about the predictability of the outcomes on the basis of their experience with two prior cases. Some pupils simply concluded that they could not predict what they were going to get in the next trial.

Finally, teacher asked pupils to compare the chances to draw a white cube out of different boxes. Most children thought that the chance to get out red cube from the first box is bigger than from the second box (as if using Siegler's rule 2, comparing the number of unfavourable outcomes). But, then came the challenge of comparing chances to pull a red cube from the first box and the third boxes.

Marina: It's the same thing.
Mitar: Fifty, fifty. (No explanation what this means).
Goca: It is the same as 1 to 1 and 2 to 2.
Janko: Similar.
Jovan: They are approximate (numbers).
Milan: It is more likely if there is two plus two cubes.
Uros: The number of cubes in the box is important.

Note that pupils learned about equivalent fractions and about proportions in the 5th Grade, prior to this study. Goca was a 7th grade pupil (age 14). Similarly to Marina and Mitar, half of 7th grade class agreed with Goca that 1:1 and 2:2 are the same. But, the rest of children have not been convinced that the chances were the same. Milan's and Uros's answers prompted us to continue discussion. Younger pupils, from the 4th grade, were prone to offer incorrect answers. The teacher was provoked by these answers to test pupils' belief.

Teacher: Suppose you are offered a reward if you pull a red cube. Which box would you like to be given, the first one or the third one?
Janko: It is easier to draw a red cube from the third box because it had two cubes. And to me it would be easier.
Igor: I would like (to pull out of) the third one. Chances are bigger in the third. Because, there are twice more.
Dragana: The chance to win as well to lose are doubled when pulling from the third box.
Janko: It is easier to pull out a red cube from the first box because there are only two cubes in the box.

Only a small number of pupils thought that it does not matter whether it is drawn from the first box or from the third box. In the process of making judgment, the most common way of reasoning was similar to Igor's response, favouring the box with more favourable outcomes which is in accordance with Siegler's rule 1. But Janko used modified Siegler's rule 2 to pick a case with smaller number of unfavorable outcomes. We should mention that the teacher extended discussion to better understand pupils' perception. Somewhat surprisingly, pupils did not change initially opinion after prolonged discussion. For example, after Igor's response, the teacher inquired delicate questions.

Teacher: But there are also two white cubes in the third box. Does it matter to you?
Igor: No.
Teacher: Ok. If you are promised to get a reward if you pull a white cube, would you like to pull a cube from the first or from the third box?
Igor: From the third one. There are more whites in it, too.

When comparing the first and third case, most pupils agreed with Igor. Again, they followed up the Siegler's rule 1, comparing the number of favourable outcomes.

CONCLUSION

We have acknowledged in this study the existence of intuition about concept of chance in elementary school children. The episodes we have chosen to present demonstrated that pupils from age 11 to 14, have had formed certain preconceptions (and misconceptions) about chance, prior to any instruction. We remarked that pupil's answers indicated different levels of primary intuition of phenomena with an uncertain outcome. Our study provides evidence that such intuition develops regardless of lacking content matter in formal learning. The preconceptions about chance are in conflict with formal theory but partially support the Siegler's model of intuitive reasoning about chance. Our analysis unveiled that most of the time children displayed non-statistical level of reasoning or tended to rely only on Siegler's rules 1 and 2. Their preconceptions led them toward the simplest and naïve analysis of the "chance situations". The findings allow us to conjecture that grasping other mathematics concepts (such as fractions, proportions, etc.) are considerably important for understanding probability. The mastery of these concepts could influence children's advancements in formal stochastic reasoning and therefore should happen before learning about stochastic reasoning. The design of our study did not provide conditions for deeper analysis of pupil's preconceptions (as would e.g. one-to-one interview with a particular child). But we believe that pupils' preconceptions should be accounted for when planning initial formal learning of the concept of chance. Our proposal should be examined further.

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Making sense of statistical and probabilistic information in the media texts: Pre-service teachers' critical thinking processes

Mehtap Ozen and Erdinc Cakiroglu

Middle East Technical University, Faculty of Education, Ankara, Turkey, ozmehtap@metu.edu.tr

This study aimed to investigate the critical thinking processes that pre-service middle school mathematics teachers utilize when they intensely engaged with the media text based on statistical and probabilistic information. Data were collected through in-depth interviews with four pre-service middle school mathematics teachers in a public university. The findings of the study pointed out that pre-service middle school mathematics teachers progressed through different critical thinking processes, including comprehending, making connections, inferring ideas, critiquing, and self-reflecting about the information given in the newspaper article.

Keywords: Critical thinking, statistical literacy, media texts.

INTRODUCTION

Probability and statistics are regarded as the domains interrelated with critical thinking, which has a crucial role in scientific and social contexts, especially in newspaper articles. Newspaper articles present statistical results about various social and scientific issues such as health, finance, education, and culture and address these results by using numbers, probabilistic statements, or representing them with graphs or tables. Journalists or researchers, however, may release misleading information or use vague language of probability and statistics. To cope with such misleading information, people should make sense of probability and statistics in these contexts and think critically about them (Gal, 2004; Watson, 2006).

In the school context, most of the students do not have enough knowledge of statistical concepts to interpret media texts and blindly adopt one's one-sided ideas or information without questioning, which could prevent them to make efficient decisions about their life (Watson, 2006). To help students in this sense, teachers themselves need be able to think critically about such information. However, teachers' lack of knowledge, as well as lack of critical perspectives could be an obstacle in providing such a help to students (Watson, Callingham, & Nathan, 2009). In mathematics teacher education programs, pre-service teachers complete statistics and probability courses. However, such courses do not usually stress critical use of statistical and probabilistic information in the contexts beyond the school. In this sense, this paper reports an investigation of pre-service mathematics teachers' uses of critical thinking processes to make sense of statistical and probabilistic information when they intensely engaged with a media text.

Critical thinking was conceptualized by various researchers (Ennis, 1985; Facione, 1990). Ennis (1985) defined critical thinking as "reflective and reasonable thinking that is focused on deciding what to believe or do" (p. 45) and conceptualized as a combination of certain cognitive skills (e.g., judging the credibility, analyzing arguments, drawing conclusions, or clarifying ideas) and dispositions toward critical thinking (e.g., being disposed to open different alternatives, to be informed, or to search for alternatives). In a further study, Facione (1990) worked on the conceptualization of critical thinking by forty experts in this subject. These experts had a consensus on two dimensions of critical thinking; cognitive skills (interpretation, analysis, evaluation, self-regulation, inference, and explanation) and affective dispositions (truth-seeking, open-minded, analytical, systematic, confident in reasoning, inquisitive, judicious). In the project of Jones and colleagues (1995), the framework proposed by Facione (1990) was reviewed by faculty, employers and policy makers to decide which aspects of critical thinking are more important for college students. The conceptual frameworks of critical thinking proposed by Facione (1990) and Jones and colleagues (1995) could be applicable to various subject matter domains and contexts (Ten Dam & Volman, 2004). In the current study we made use of their frameworks as a starting point and a baseline in order to identify indicators of critical thinking processes in the data obtained from in-depth interviews through which the participants were intensely working on a media text that involve statistical and probabilistic information. It is important to note that we focused on the cognitive dimension of critical thinking rather than affective dimension in this study. Previous studies pointed out that critical thinking is transferable to different subject matters or contexts. On the other hand, it is a challenge for teachers to enable their students to transfer such critical thinking processes to the contexts beyond school. To provide transferability of critical thinking, newspapers or other media texts could be used in teacher education programs (Halpern, 1998). In the context of statistics, recent research suggested to use tasks that require thinking about complicated educational issues (Osana & Seymour, 2004), newspaper articles (Watson, 2011); tasks that require statistical literacy based on infusion approach (Aizikovitsh-Udi & Kuntze, 2014). In line with these studies, the current study investigated pre-service mathematics teachers' uses of critical thinking processes while intensely working on a media text that involved statistical and probabilistic information.

METHOD

Participants

The study was conducted with four pre-service middle school mathematics teachers in the fourth year of Elementary Mathematics Education Program (EME) of a public university in Ankara, Turkey. The EME program is a four year undergraduate program in which the enrolled students are trained to be mathematics teachers of grade levels 5 to 8 in middle schools. In this program, courses named "Introduction to Probability and Statistics" and "Research Methods" are required for all students. Participants were selected among 38 pre-service middle school mathematics teachers in the fourth year. Participants were selected on the basis of their potential to provide rich data. To select the participants, all of the fourth year students were given a newspaper article that was not used in the main study and were asked to write down a critical reflection about the statistical information in it. The participants were selected randomly among the ones who could produce significant reflections, had tendency to use valid quantitative procedures and mathematical language, as well as could detect points to criticize in the article.

Data collection

The major data source of the study was in-depth interviews. Participants were asked to read the newspaper article about cheating partners, claiming men are better at detecting a cheating partner than females. It was published in the Mercury newspaper in Tasmania ("Cheat radar better tuned in men, study finds", 2008) and proposed by Watson (2011) to be used in educational settings. The newspaper article includes some probabilistic and statistical statements that participants may pay attention while trying to make sense of the given results (Table 1). During the interviews participants were asked to think and reflect about the following main questions: What is the main idea of the newspaper article? What conclusions did researchers reach? What conclusions could you draw from the text? How could the researcher conduct the study reported in the newspaper article? (e. g. how to select sample, how to collect and analyze the data, how

Statement 1:	The results, published in New Scientist, show 29 per cent of men admitted they had cheated compared with 18.5 per cent of women.
Statement 2:	Researcher Paul Andrews said men were better at judging fidelity than women. 'Eighty per cent of women's inferences about fidelity or infidelity were correct, but men were even better, accurate 94 percent of the time' Dr. Andrews said.
Statement 3:	Men were more likely to catch out a cheating partner, picking up on 75 per cent of the reported infidelities compared with 41 per cent discovered by women.
Statement 4:	Men are better at detecting a cheating partner than females, and they are more likely to suspect infidelities that do not exist.

Table 1: Some of the probabilistic and statistical statements in the newspaper article

to reach reported findings) How would you evaluate reported findings? What do you think about generalizability of the reported statistics in the newspaper article? In addition, they were asked what they understand from four probabilistic statements in the newspaper article (Table 1). The duration of each interview was approximately forty five minutes and interviews were audio and video-recorded. Data were collected in the 2011–2012 spring semester.

Data analysis

The data were coded in order to identify expressions of the participants that indicate their critical thinking process. To determine possible indicators of critical thinking, we made use of the frameworks of Facione (1990) and Jones and colleagues (1995) as a starting point and a baseline in order to identify indicators of critical thinking processes in the interview data. To be precise, data were analyzed and searched for instances and processes in their thinking by making use of the frameworks suggested by Facione and Jones et al. as a base line. In some cases, certain dimensions of these frameworks could not be completely matched with any part of the data in this study. Thus, to make the dimensions of these frameworks more suitable with our data, we adapted and restated them as the data codes and categories of the current study, without making major alterations in their conceptual meaning. For example, the critical dimension of "explanation" in the original framework of Facione was excluded in the current study, since it was not observed in the data. On the other hand, other dimensions of critical thinking such as interpretation, analysis, evaluation, and self-regulation were included, but revised and their explanations were restated to make them more suitable with our context. As a result of data analysis, five interrelated processes of critical thinking were identified (Table 2).

FINDINGS

Critical thinking about the bases of reported findings in the newspaper article

Bases of reported findings refer to the background of the study in the newspaper article, some of which are not explicitly given in the article such as selection of sample, data collection, data analysis, or reporting of the findings. Such information about the article was one of the dimensions that the participants reflected critically. They mostly attempted to use critical thinking processes of comprehending, critiquing, and self-reflecting. Regarding sampling, one participant (Ali) recognized the essential role of the sample and the need for sample to be representative in critiquing credibility of the study. He interrogated the extent to which sample size of the study is enough to make accu-

Comprehending
Identification of the main idea of the text (e.g. identifying extraneous ideas in the text)
Organization of the contextual information (e.g. making use of graph, diagram, or table to organize the contextual information)
Clarification of the information (e.g. defining the ambiguous or vague terms)
Making Connections
Examining link between ideas (e.g. identifying closely related statements)
Identification of claims or arguments in the newspaper article (e.g. determining whether author states reason for supporting his claim)
Inferring
Examining evidence (e.g. seeking the background information or issue that needs to be addressed)
Proposing alternatives (e.g. suggesting plans with the consideration of their pros and cons)
Drawing conclusions (e.g. figuring out new meaning by making use of clues)
Critiquing
Detecting misleading information (e.g. detecting inconsistencies or author's exaggerated generalization)
Recognizing factors of credibility (e.g. appreciating sufficiency of information such as sample, data collection, or analysis processes)
Self-Reflecting
Expressing one's own strengths and weaknesses of own thinking process (e.g. rereading sources to make sure that one has not overlooked important information; ask themselves questions about their beliefs or attitudes)
Making corrections or revisions when they realized their mistakes or misunderstandings

Table 2: Indicators used to code critical thinking processes of the participants

rate inferences about the study. He tried to support his evaluation by considering possible effect of extreme values in the data of a study with small sample size on drawing accurate conclusions from the data. On the other hand, the other participants just restated sample size of the reported study.

Regarding data collection, all participants focused on the issue of what was measured and how it was measured by clarifying the questions that were asked to the subjects of reported study. Meltem, for example, attempted to clarify question of "whether they [subjects in the study] had ever strayed" reported in the newspaper article, stating "...I mean, thinking of cheating in the past, [I thought that it was] a question such as 'Have you ever cheated?' The article could have just said like this: the young couple could have been informed that this study was about their current relationship." She stipulated the condition of "current relationship", which makes the meaning of question narrower and removing the ambiguity of the question that might lead readers to think about subjects' current relationship or relationships in the past. It was only Ali, who reflected a different critical thinking process by critiquing misleading statement in the newspaper article, stating "Are men better confessors or do men deceive [their partners] more, it is unclear, some might deceive [their partners] and say they didn't; that's why, I think this may not give an idea about who deceives more." He thought about the validity of the argument of "29 per cent of men admitted that they had cheated compared with 18.5 per cent of women" in the newspaper article with the consideration of the possible bias in measurement in which subjects might give misleading information about cheating of their partners.

Another finding was that all participants were in the process of comprehending by rethinking the categories in the article while considering about data analysis procedure of the study in the newspaper article. İrem was thinking about one of the statements in the newspaper article; that is "Eighty per cent of women's inferences about fidelity or infidelity were correct, but men were even better, accurate 94 per cent of the time."(Figure 1). By this table she analyzed women's incorrect inferences about their partners' fidelity. In this process, she organized the possible conditions to comprehend how the researchers could reach to the conclusions reported in the newspaper article and related raw data to percentages as a summary statistics.

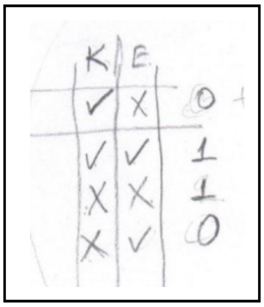

Figure 1: İrem's organization process regarding data analysis

Another finding was related to participants' thoughts about results or conclusions reported in the newspaper article. While thinking about the main idea of article, they did not raised any concern about the results in the article that presents only correct inferences of men and women. They restated the statements in the newspaper article as the main idea of the text and did not consider men or women's wrong inferences about their partners, which do not exist in the article. Moreover, while thinking about the results of the article, all of the participants attempted to critique the results in the article. However, their judgments were mostly subjective. Melek, for example, recognized the difference between the results given in the newspaper article to determine the reliability of the results or conclusions:

> If numerical data are compared, it was found that 80 of women's inferences were correct but 94% of men were right in these inferences. There is 14% difference; below [pointing the last paragraph of the article] there is much higher difference. It can have a difference of 75%; in the other one it can detect 41%, so that's why I thought the test is really reliable.

She assumed observed differences in the results are large enough, especially in the case of numerical values of 75% and 41%, which are given in the statement of "Men were more likely to catch out a cheating partner, picking up on 75 per cent of the reported infidelities compared with 41 per cent discovered by women." in the newspaper article. However, her judgment was subjective, indicating she might not be aware of statistical and practical significance of the results reported in the article. Supportively, in the process of critiquing of results and conclusions, two of the participants made self-reflection by reflecting their own thinking processes. Melek, for example, explained her subjective assessment in deciding if the study is reliable or not in the following:

> [...]when assessing the test, I think I'm adding my own opinions a little too; but, for instance, when considering its reliability, I'm looking at the claims made at the beginning and the numbers below, I'm comparing them. So even if I am not doing calculations, may be because it fits my line of thought a little, I mean I believe in it more.

In summary, regarding the background of the study in the newspaper article, participants were mostly in the processes of comprehending, critiquing, and self-reflecting. During these processes, they mostly focused on the existing information (e.g. sample size and questions asked to subjects in the study) in the article. Only few of them attempted to interrogate information about the background of the study, which does not exist in the newspaper article.

Critical thinking about the reported statistics

This part includes participants' critical thinking processes about descriptive or summary statistics (percentages and probabilistic statements), which already exist in the newspaper article. Participants reflected different critical thinking processes such as comprehending, making connections, inferring, and self-reflecting. One of the main findings was that participants dealt with clarifying the meanings of Statement 2 and Statement 3 (see in the data collection part of this study). For example, İrem had confusion with the meaning of "fidelity" concept and tried to define the terms of fidelity and infidelity:

> I think I don't know the meaning of the concept 'fidelity'. I can't distinguish these two conditions [Statement 2 and Statement 3]. I think predicting [fidelity] correctly means when they say that they don't think their partner cheated on them and actually they [their partner] hadn't; and predicting infidelity correctly means when they say that their partner definitely must have cheated on them and their partner had done so.

After developing an idea about these terms, she reasoned through proportionality and calculated the number of female and male who made correct inferences about her/his partner's fidelity or infidelity, stating "162 women predicted correctly whether or not their partner cheated on them. And I understood that 190 men accurately predicted whether their partner cheated on them." Although she expressed her difficulty understanding the difference between Statement 2 and Statement 3, she did not make clear the difference between them. When asked what she understood from the Statement 3, she stated "41 per cent of 203 couples; so, 83 women detected that their partner cheated on them.", which can be considered as an evidence that she did not recognise the condition of cheating immediately. After that, she realized Statement 1 in the newspaper article, which gives information about the number of people cheating their partners. She made connections between two related statements (Statement 1 and Statement 3). In this process, she overviewed the newspaper article and read statements again if she overlooked anything, which can show us her self-monitoring process. Then, he corrected her mistake, stating "men noticed 75% of the cheatings done by their partner. I mean, it seems that 75% of cheating partners were noticed". Surprisingly, Ali and Melek have such a recursive process of thinking in a similar way.In addition, during this process Ali and İrem constructed a table or diagram to organize the findings of the study reported in the newspaper, which also shows their making connections among reported statistics in the newspaper article. On the other hand, Melek and Meltem could not go further, which could be due to the fact that they could not make explicit the differences between the statements in the newspaper article, especially Statement 2 and Statement 3 whereas Ali and İrem advanced their categorization of the reported findings by examining closely related statements and drawing new conclusions from the newspaper article (see Figure 2).

Another finding of the study was that they had difficulty in detecting misleading statements in the newspaper article, especially regarding Statement 4, which require understanding of the conditional probability. For example, İrem tried to critique the Statement 4, which is "Men are better at detecting a cheating partner than females, and they are more likely to suspect infidelities that do not exist". In this critiquing process, İrem dealt with the clarification of the Statement 4. She proposed two alternative meanings for the concept of "being suspicious": Male/Female says that his/her partner is cheating, Male/Female says that his/her partner is cheating when his/her partner is not cheating in real world. The ambiguity about the meaning of this concept and having difficulty in conditional probability might have prevented her to draw an improper conclusion about Statement 4 and partially critiquing it, stating as "I think it [Statement 4] is wrong...1.4% [3/203] and for women, 2.8% [6/203].

I think this is the opposite of [Statement 4]. If we consider the first meaning [Male/Female says that his/her partner is cheating], it becomes true…it depends on the meaning of "being suspicious". If we think the second meaning that I believe in, women are more suspicious, that is, anyway women are suspicious unnecessarily."

Critical thinking about the generalizability of the reported findings

During interviewing, participants were encouraged to think about the generalizability of the reported findings. While thinking about this issue, all participants attempted to critique the arguments reported in the newspaper article with recognition of relevant factors to determine if it was generalizable to population or other similar contexts. In this process, they differ from each other by focusing on different factors such as sample size, sampling method, and cultural factors. Two participants (Ali and İrem) also reflected the process of inferring (examining evidence) regarding what background information about sample characteristics needs to be addressed to critique generalizability of the reported findings. For example, İrem discussed generalizability as following:

> Well, I don't know if 203 couples are enough. I don't think it can be generalized. My usual opinion, you can't imagine something big from a small sample. If I ask each of the 203 men or if I get 58 men, in this case, will only 23 of 58 of all their wives predict correctly? It seems that this will not be correct all the time…You know, different results will be obtained from different samples; well, here the 203 couples don't have any characteristic features anyway. I mean, where do they live, in which country, I don't know how long they have been married; maybe there are many influential factors. It has only mentioned that they are young couples […]

İrem was not sure about to what extent sample size is sufficient for generalizability from deterministic point of view, stating "you can't imagine something big from a small sample." Conversely, Meltem made somewhat immediate comments regarding generalizability and did not provide enough evidence to support his evaluation. She considered factor of the sample size of the study enough to generalize the conclusions and tended to relate the generalizability of the study to the clear presentation of the findings in the newspaper in the following quote:

> 203 is actually a good number; in statistics when we, for example, carry out a study, we say it's a good result when it is over 30, or 100, for instance. Well, 203, compared with that, is good, that's why…it can be generalized because everything is clear […]

In summary, all participants attempted to critique findings in the article when they were asked to think about the generalizability. However, they did not reflect comprehensible reasoning to make sound assessment about the generalizability.

FINAL REMARKS

One of the main conclusions arising from this study is that pre-service mathematics teachers mostly focused on the existing information rather than on the missing or misleading information in the newspaper

İrem	Melek	Ali

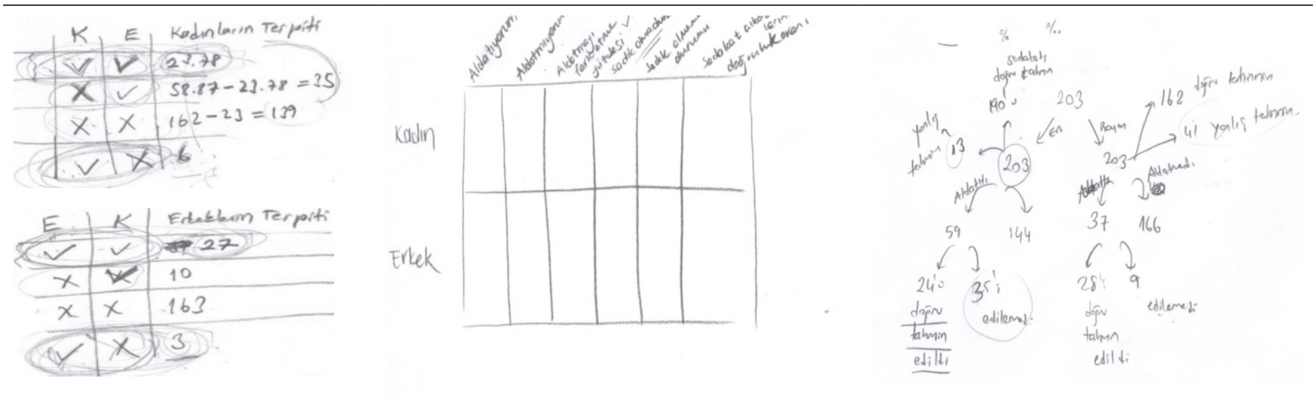

Figure 2: Three participants' thinking processes about the reported findings

article, which might have prevented them to detect one-sided arguments (Watson, 2006). Moreover, they mostly attempted to comprehend the statements in the newspaper article by clarifying their meanings and make immediate comments or overgeneralizations like "it is good" or "the results of the study is generalizable" without enough evidence to support their ideas, which contradicts with the nature of critical thinking that requires skeptical thinking, and inquiry on the basis of evidence (Facione, 1990). Another important conclusion is that they had difficulty in comprehending conditional probability statements, which might prevented them to make appropriate inferences and critique the reported findings in the newspaper article. These results are consistent with the findings of Ozen and Cakiroglu (2013). This indicates their lack uses of critical thinking processes regarding conditional probability in the media texts even though they have already studied about this concept in their statistics courses. This study contributes to our understanding pre-service mathematics teachers' engagement in the media contexts, which might lead us to reconsider the content of statistics courses in the teacher education programs regarding how these courses really address the issue of uses of critical thinking processes regarding the statistical and probabilistic information in real life contexts. Media texts could be used as a mediator to contribute their critical thinking process in designing of statistics courses in which they could be encouraged to think about both proper and improper examples of newspaper articles about diverse topics, rather than just focusing on the computational procedures.

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An interview study on reading statistical representations in biology education

Christine Plicht, Markus Vogel and Christoph Randler

University of Education Heidelberg, Heidelberg, Germany, plicht@ph-heidelber.de

In an interdisciplinary research project of mathematics and biology it is focussed on how graphs, charts and diagrams are used in biology classes. The aim of this research approach is to get a better understanding of how students read statistical representations used in biology schoolbooks. Therefore, an interview study was conducted. In this paper the theoretical background and some empirical findings of this study will be presented. The results refer to the reading activities of students and yield a concept of reading comprehension which contains influencing factors that were identified. The paper concludes with a discussion of the results and a short outlook on following steps of the research project.

Keywords: Reading statistical representation, interview study, statistic education in biology.

INTRODUCTION

Graphs, charts and diagrams are omnipresent in everyday life, but also in classrooms a main tool to illustrate relations and structures. Reading and creating graphs is an essential part of mathematics education and it demands semantic and syntactic requirements, for encoding the information given in the representation and to engage the context of it. Students have to focus on syntactic requirements and necessaries like: What are the necessary parts of graph or diagram? How to read the data points? Which conclusions can you draw from it? But not only in mathematical lessons this kind of representation are used and you can also focus more on semantic aspects than concerning about the syntactic elements. For example biology uses diagrams as an instrument to visualize data and context specific correlations of biology (Kattmann, 2006). Therefore, there are curricular requirements to illustrate data on measurable parameters with linguistic, mathematical or visual design elements (e.g., KMK, 2005).

The project "Statistische Representationen im fachübergreifenden Unterrichtskontext von Mathematik und Biologis" (engl. "statistical representations in an interdisciplinary approach of mathematic and biology in school") at the University of Education in Heidelberg aims to explore the use of statistical representations at the interface between mathematic and biology. The interview study that is described and discussed in this paper is one of three steps to the analysis of reading comprehension of graphs, charts and diagrams being used in biology education in schools. Preliminary there was an inquiry on statistical representations in schoolbooks resulting in a classification on charts, graphs diagrams. A summary will be presented before the paper focuses on the interview study. Afterwards there is a short outlook on the last step of the project, which focuses more on examine reading comprehension of representation in actual biology classes.

THEORETICAL BACKGROUND

Charts, graphs and diagrams imply syntax and semantic information. In addition to information you get by dealing with data points, there is other information representing the background of the data. To access this information you have to relate it to its inner context (Friel, Curcio, & Bright, 2001). Therefore, reading and interpreting data is a process consisting of various components. Curcio (1987) differentiated a level model ("Reading the Data"-Model) with different degrees of requests: Two levels, *reading the data* and *reading between the data* are mostly related to syntax features and focus on reading and comparing data points to identify trends and turning points. In contrast, another level, *reading beyond the data*, considers more the context to employ further predictions. This model is complemented with a fourth level, *reading* *behind the data* (Shaughnessy, 2007). In this level information is taken with account to the specific dataset context, such as the collecting of the data or existing comparable data, to explain variation of the data.

Other relevant work about diagrams with biological content proposes a structural model of diagram competences for diagnosing the ability to construct diagrams (Lachmayer, 2007). Furthermore, there are studies about mistakes by science students when constructing diagrams (Kotzebue, 2014). The integrated model of text and picture comprehension by Schnotz and colleagues (2002, p. 390) focus more on how "descriptive" and "depictive" representations are proceeded and based on this how knowledge is gained from representations in general.

Charts, graphs and diagrams represent relations, have an iconic character (Bakker & Hoffmann, 2005) and are constituted as part of a depictive system that follows rules (Hoffmann, 2005). In the present research, charts, graphs and diagrams are defined as *statistical representations visualizing data* and in particular, data from biological settings. This definition serves the purpose to investigate representations in a biological setting and is framed in appropriate width to allow for applications in a learning environment of this research approach.

A CLASSIFICATION OF DIAGRAMS AS (CONCEPTUAL) BASIS FOR FURTHER WORK

The first step to gather information about reading comprehension of statistical representation in biology courses requires investigating the representations individually. For a systematic and qualitative analysis of graphs, charts and diagrams in biology schoolbooks, we did a classification to get an idea which representations are being used at all and to make assumptions about influencing factors for reading them (Plicht, 2013). This classification is also used for a conceptual distinction in the further research, differentiating the graphs, charts and diagrams when using them as research objects.

The diagrams used in a biological context can be divided into those which

a) focus mostly on structure and data
b) focus mostly on context and biological background.

A second high-level classifying characteristic is the diagram design. There are several levels how graphical elements are an integrative part of the representation. For an example see Figure 1: the cows and the milk cans are graphical element in a chart. They are part of the representation, but not necessary needed for accessing information from the chart. After the inspection of over seventy representations in school books an assumption arose concerning the use of graphical elements: an increased number of included graphic elements affects how a diagram or graph is read, because it can change the semantic interpretation or change the syntactic proportions of scale and size. In Figure 1 it is obviously that the size milk cans do not fit to the numeric value of the data.

RESEARCH QUESTIONS

Only little research focuses on the connections between mathematics and biology with their specific requirements in education - especially in reading comprehension of graphs, charts and diagrams including the involvement of a specific context with its semantic extent. Based on the result of the developed classification, in the next step the reader and the reading process of different kinds of diagrams, charts and graphs were included in the investigation. With these considerations, it was necessary to formulate research questions that capture the dimensionality of reading statistical representations in context of the connection between mathematical and biological education.

The following research question emerged from these requirements:

— How do children read statistical representation with biological content?

— What are the influencing factors when they read charts, graphs and diagrams?

These research questions were investigated with a qualitative research design that leaves room for open-ended analysis with an explorative character.

METHODOLOGICAL APPROACH

The interview study that emerged from these questions with an insight into the methodological approach and the analyzing process will be described here before the results are presented and discussed.

Sample and data collection

Twelve interviews were conducted with students (aged 10–12 years; from secondary school). To get a pleasant atmosphere and make it easier for the students to answer they were interviewed in pairs for about thirty minutes (Selter & Spiegel, 2001, p. 106). The interviewer questioned the students about three to four statistical representations; the interviews were videotaped and transcribed afterwards. We selected the test samples (graph, charts and diagrams), varying in design and material content based on the developed classification, from common German biology textbooks. An interview guideline for the half-hour conversation with open-ended questions served as a dialogue support and further questions regarding the four levels of reading the data (Shaughnessy, 2008) were used to help the students read the graphs. The interviews were to be conducted as openly as possible to capture the original ideas and interpretations of the children. It was important for the researchers to hear the children's spontaneous interpretations and to leave room for what else they had to say on the subject. For example these questions were used to start and carry on the interview:

— What is the diagram about?

— What do you see there?

— Is there anything else you notice?

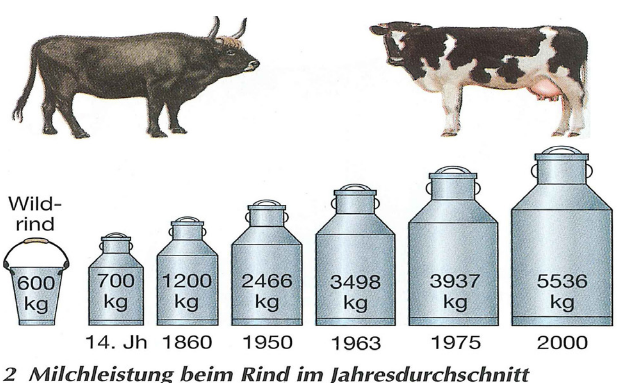

Figure 1: Chart about milk production (Source: Erlebnis Natur & Co 3, Schroedel) that was used in the interviews

Data analysis

The analysis was based on the Grounded Theory Methodology (GTM; Strauss & Corbin, 1990) with an open inductive approach to find categories to generate hypotheses about reading statistical representations.

In the GTM there are three steps of coding before finding a general theory (Strauss & Corbin, 1990). The coding process in this project was performed on a computer with MAXQDA 10. The first step, the *open coding* was supplemented with memos and working hypotheses. To become aware of the reading process and starting to describe the phenomena, sentences and words of the transcript became codes. After generating several codes they were linked and set in relation in a circular work process during the *axial coding*. There the main categories were formed and the subcategories arranged to them. As a result of the last step, the *selective coding*, two different category systems and a concept of reading comprehension (Figure 2) that combined them was developed and will be presented in the following chapter.

TWO DIFFERENT CATEGORY SYSTEMS

Analysing the transcripts with the GTM revealed two viewing directions, which have been studied in detail. On the one hand, the analysis specified what children do while reading diagrams. Four main categories were explored that describe the *activity of reading* and interpreting statistical representations. On the other hand, categories were formed and identified as *influencing factors* to the reading comprehension related to the abilities of the child.

Areas of reading activities

1) READING
2) REASONING
3) APPLYING
4) JUDGING

READING[1]: Children READ, when they refer to or express information directly from the diagram. This includes reading single and multiple data points, the reading of labels or mentioning the general topic of

1 Note that the main category READING is different and more restricted than the competence in reading or reading comprehension in general.

the diagram. Describing the content or the graphic elements is also a subcategory of READING.

> Cw4, 12y.[2]: It is about, how it, so, there is a wild cow and it has, so, it brings 600kg milk per year and you see here in 1400, 1860 and then so on and goes on higher.

REASONING: Children REASON, when they justify their interpretation of correlation or data of the graphs, charts and diagrams. Their interpretation can be supported by the representation itself or from their prior knowledge ideas about the content. Children also REASON about the background or causes of the data collection.

> Aw1, 10y.: Yeah, because I think, because they [the cows] have always been more and more cultured, so they should give more milk.

APPLYING: Children apply the graph or diagram when they consider the meaning of it for themselves personally or in general. They draw conclusions from the graph like projecting the data in the future and consequences of that. These statements usually come from referencing to (their own) life and the environment.

> Aw1, 10y: The cow has become pretty old.

JUDGING: This category includes positive or negative statements of children that JUDGE the diagram itself or the data behind it. Beside the presentation they also valuate and discuss the date collection. They even doubt the correctness of the data. These judgments are mainly justified by reference to their own experiences.

> Bm5, 11y.: It is obvious that here is somehow a huge distance between the year 1400 and 1860. I don't quite understand why there is such a huge distance.

Individual differences as underlying factors driving reading comprehension

During the analysis the question arose what factors driving reading comprehension might be. There were several codes that did not describe the activity of reading but seemed to describe how the students read and what they need to understand it. This included codes about skills children used to read the charts and diagrams. These skills could be, for example, mathematical and biological abilities and experience from their own world. These codes do not describe individual actions, but rather the abilities or tools children used to read. They were labelled during the process with the title "individual abilities of the children". These are possible influencing factors who affect reading graphs, charts and diagrams.

These codes and categories contain the interdisciplinary competences students apply when working with a diagram. We differentiated between *mathematical skills for* reading the syntax of the diagram, and *biological skills* to assess the content. Moreover, the specific *domain knowledge* could further influence how students read and understood diagrams. Other concepts students used were everyday ideas or *naive concepts* to explain the data. They tried to fit their experiences or imagination and the data together even if that was contradictory to the presented data. These categories can be summarized as *influencing factors stemming from the subject*, the reader of the diagram, and affect how the diagram is read.

CONCEPT OF READING COMPREHENSION AND OTHER INFLUENCING FACTORS

The results of the analysis with the GTM revealed two different category systems: The *activities fields of reading* and the *influencing factors stemming from the subject*.

There is a clear connection between these two category systems: The statements depend on the background of the student. For example, they JUDGE the diagram by using their everyday ideas or own experiences.

> CW4, 12y[3].: No, that's not quite right, because the/

> CW5, 11y.: Yes, that's too much...

> CW4, 12y.: ... for a person for a daily water use. Because *my* laundry is not washed every day.

2 All quotes were original in German, the authors translated them for this paper and refer to Figure 1.

3 The quote is from an interview about a chart concerning the daily water consumption.

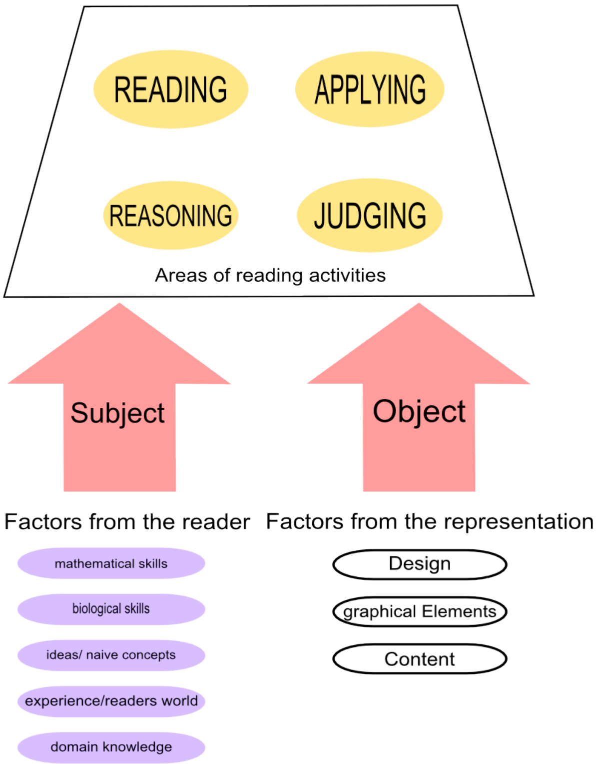

Figure 2: Concept of reading comprehension

Considering the previous work - the analysis of the graphs, charts and diagram in school books - it appears that the influencing factors concerning the readers might not be the only factors that affect reading comprehension. There are features of the object that also determine the reading comprehension. The classification showed relevant criteria to divide the objects and we concluded that the design containing the use of graphical elements and the content define the diagram itself. The content determines whether students can REASON when drawing conclusions; the design and syntax can help to READ data points. Therefore, the content, the graphic elements and the type of diagrams may also be *influencing factors stemming from the object* for the activity fields. Combining these results yields a concept of reading comprehension (Figure 2) with two kinds of influencing factors (subject and object) that affect the areas of reading activities.

DISCUSSION

In the analysis an open and unprejudiced access generated the results concerning a deeper understanding of how children read statistical representation with biological content. Nevertheless, there are other known concepts (e.g., Shaughnessy, 2007; Shah, 2002) how to read data and good scientific practice especially in a qualitative research design demands a sceptical glance on theories that were developed in a different context with different research questions. The "Reading the Data"-Model was introduced earlier and it has to be discussed, with this model as an example, how we coped with other concepts concerning the developed one.

It was not the goal to use or confirm the "Reading the Data"- model of Curcio/Shaughnessy in this study. Therefore, in this study, previous concepts were not used as a basis for coding. Rather, we employed an inductive procedure to find concepts. A post-hoc comparison of our results with existing theories such as "Reading the Data"-model indicate overlaps. The "Reading the Data"-model aims to consider the context of the representation, but it does not specify how the context might affect the reading comprehension and it is not very explicit how the context is involved in the whole reading process. However, there is a close connection between our work and this model. It provides a basis that could be considered in the present research and the similarities of the two models are therefore not surprising. It is even possible to find the levels of data reading in the concept of reading activities. Obviously, the levels *reading* and *reading between data* are included in the category READING. The other levels can be found in phases where students REASON, APPLY or JUDGE (with) the diagram. When reader check with their own experience or make a strong reference their world to REASON the data, that's in the level *reading behind the data.*

In contrast to the "Reading the Data"- model the concept of "the areas of reading activity" is not build on levels, but rather application-oriented. It is possible to describe the main categories of the reading activity as either object or context focused. READING is more focused on the object, JUDGE, APPLY and JUSTIFY is more focused on the context of the representation.

In this study the reading comprehension was not examined in an authentic education environment. Statistical representations are used in schools in a situational context where teachers control the setting. As teachers choose the representation and the content of the diagram and prepare the classroom setting in which the graphs, charts and diagrams are used they determine the domain knowledge and the professional skill of the children that affect the activities of reading. So in an authentic educational environment teachers can control the influencing factors stemming the students and the representation.

FURTHER RESEARCH

Our aim is to get more information about how statistical representation could be used in a learning environment effectively and to get further information how to apply them in biology classes constructively. Therefore, we must understand how students read diagrams with a biological content and especially schoolbook diagrams that are used authentically. After investigating diagrams in a non-school setting to get a conceptual basis and used as distinction for the discussed interview study in this paper, the next step is to investigate the use of graphs, charts and diagrams in actual classroom situations. Our results from this study and the classification, indicating factors influencing diagram reading will be used to design a classroom study to get further information on how students actually learn from statistical representations.

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Interest in statistics: Examining the effects of individual and situational characteristics

Maria Anna Donati[1], Caterina Primi[1], Francesca Chiesi[1] and Kinga Morsanyi[2]

1 University of Florence, Department NEUROFARBA, Florence, Italy, caterina.primi@unifi.it

2 Queen's University, Belfast, United Kingdom

Statistic lessons are often considered difficult and unpleasant. One reason for this is that many learners do not find statistics intrinsically interesting and engaging. Nevertheless, interest in a particular teaching session does not only depend on the characteristics of learners, but also on the characteristics of the session itself. The aim of the present study was to investigate the relationship between learner's characteristics (i.e., individual interest and intrinsic motivation) and situational interest. Participants were university students who participated in a tutorial activity. The results provided support for a moderated mediation model which showed that the relationship between individul and situational interest was dependent both on students' intrinsic motivation and the perceived appeal of the activity. The discussion underlines the importance of the interaction between individual and situational factors in the process of teaching statistics.

Keywords: Learning statistics, individual interest, intrinsic motivation, situational interest, moderated mediation model.

INTRODUCTION

Statistics is commonly viewed as a difficult and unpleasant topic and students often perceive statistics courses as a burden, they sometimes fail to pass the exams and, as a result, some of them might even abandon their academic and professional aspirations. Indeed, individuals are not always interested in what is being taught as part of their course (see Matarazzo, Durik, & Delaney, 2010), and this is especially true for domains that are unappealing or feared by students (e.g., Middleton & Spanias, 1999; Zeidner, Roberts, & Matthews, 2008). Low interest in a learning activity might also undermine students' persistence and performance (e.g., Durik & Harackiewicz, 2007). As a consequence, it is desirable that research focused on improving statistical education identifies variables that can promote or undermine statistical interest.

THEORETICAL FRAMEWORK

In the educational literature, several studies have focused on the concept of *interest* in college subjects . Recent reviews (see Linnenbrik-Garcia, Durik, Conley, Barron, Tauer, Karabenick, & Harackiewicz, 2010; Matarazzo et al., 2010), described two different types of interest: *individual interest* and *situational interest*. *Individual interest* is an enduring predisposition to engage in certain content domains, and it is accompanied by concentration and positive feelings (Hidi & Renninger, 2006). Once developed, individual interest is relatively stable across time and situations. High individual interest involves high levels of knowledge, experience, and the attribution of high value to the domain of interest (Hidi & Renninger, 2006; Renninger, 2000). In contrast with individual interest, *situational interest* is a short-term state of focused attention and affective engagement (Matarazzo et al., 2010), which might be triggered by the characteristics of the learning situation (Hidi & Harackiewicz, 2000).

In the domain of mathematics, Durik and Harackiewicz (2007) found that situational interest was positively influenced by individual interest in mathematics. Moreover, these authors found that an important process which was related to both individual and situational interest was *intrinsic motivation*, i.e. the desire to engage in an activity for the value inherent in doing it (Deci & Ryan, 1985). According to

Harackiewicz and Sansone (1991), the experience of intrinsic motivation during task engagement reflects the importance that a person places on competent performance, refers to the extent to which an individual feels focused on and absorbed in an activity during task engagement, and concerns an individual's self-assessment of competence while performing an activity.

Interest (individual and situational) and intrinsic motivation are closely related (see e.g., Deci, 1992; Ryan & Deci, 2000). Thus, it is important to understand the underlying mechanisms of how students become involved and interested in their courses. To the best of our knowledge, until now no study has investigated individual interest in statistics and its relationship with intrinsic motivation and situational interest in the context of a statistics class . The aim of this study was to fill this gap by investigating how individual interest and intrinsic motivation were associated with situational interest in the domain of statistics. Specifically, our research questions were the following.

First, we wanted to investigate the relationships between individual interest, situational interest, and intrinsic motivation in statistics. We hypothesized that there would be strong, positive correlations between these characteristics. Additionally,

we wanted to test a mediation model explaining the underlying mechanisms by which these individual characteristics are related to each other. We hypothesized that the relationship between individual interest and situational interest would be partially mediated by intrinsic motivation; that is, that higher individual interest would be directly related to both greater intrinsic motivation and greater situational interest in the learning situation (Durik & Harackiewicz, 2007), but there would also be an indirect link between individual interest and situational interest through intrinsic motivation.

Second, we wanted to test a model explaining the underlying mechanisms by which a characteristic linked to situational factors, i.e. the *perception of the appeal of the activity* or, in other terms, the extent to which participants like the activity, interacts with the other variables in the mediation model explaining the relationships between individual interest in statistics, intrinsic motivation, and situational interest in statistics. Specifically, given that Durik and Harackiewicz (2007) found that making the teaching materials appealing by using nice and colourful pictures had a positive effect on situational interest in mathematics, we hypothesized that the perception of the appeal of the activity, which was elicited by making the teaching materials as appealing as possible through the inclusion of pictures, images and colours, would have a significant positive effect on situational interest in the activity. We also predicted an interaction between intrinsic motivation and the appeal of the activity. In particular, we aimed to verify that the relationship between intrinsic motivation and situational interest was moderated by students' perception of the appeal of the activity. To provide evidence for these hypotheses, we tested a moderated mediation model in which individual interest in statistics affects situational interest in a statistic learning activity through intrinsic motivation, and this mediation effect is moderated by students' perception of the appeal of the activity.

METHOD

Participants

The participants were 127 psychology students attending the University of Florence in Italy, who enrolled in an undergraduate introductory statistics course. The participants' age ranged from 19 to 44 (Mean=20.44, *SD*=3.19). Most of the participants were females (79%). Students participated on a voluntary basis and they received course credit for their participation.

Materials and procedure

Participants were invited to engage in a statistics tutorial activity during one of the lectures of their introductory statistics course. The activity, which was introduced in the academic year 2013–2014 (and took place in November 2013), was conducted by a trainer who was different from the course lecturer. The activity started with the explanation of the phenomenon of *collective statistical illiteracy*, defined as a widespread lack of understanding of health statistics in society, and, ultimately, the tendency to draw invalid conclusions regarding the meaning of statistical information without noticing (Gigerenzer, Gaissmaier, Kurz-Milcke, Schwartz, & Woloshin, 2008). To illustrate this phenomenon, a real-world example of collective statistical illiteracy regarding birth control was presented[1]. Then, the activity was organized in explanation of specific critical arguments particularly biased in health statistics, i.e. absolute and relative risk in medical fields and conjunctive and conditional probability of epidemiological data presented in contingency tables. The explanation was conducted using power point slides with imagines, animations and graphical examples. It was followed by demonstration of some exercises regarding real-life situations about the medical field. Each student received an individual workbook which contained, along with the exercises, a series of scales developed for the purpose of this research.

The following scales were developed for the purpose of this research.

At the beginning of the session, individual interest in statistics (IIS) was measured using a seven-item scale that tapped into the general evaluation of statistics (e.g., "I find statistics enjoyable"). Participants indicated from 1 (*strongly disagree*) to 7 (*strongly agree*) the extent to which they agreed with each statement. A total score on the scale was calculated so that high scores corresponded to high levels of IIS. The internal consistency of this scale was good (Cronbach's α = .89).

During the activity, when students were presented with practice exercises, they were invited to assess their intrinsic motivation (IM) in solving them. IM was measured using a twelve-item scale measuring self-reported competence valuation (e.g., "It is important to me that I perform well"); task involvement (e.g., "I got caught up in doing this exercise") and perceived competence (e.g., "I think I did well in the exercise"). The scale had a 7-point Likert response scale (from 1 = *strongly disagree* to 7 = *strongly agree*). Cronbach's α was .86 indicating good internal consistency. A total score on the scale was calculated so that high scores corresponded to high levels of IM.

1 "In October 1995, the U.K. Committee on Safety of Medicines issued a warning that third-generation oral contraceptive pills increased the risk of potentially life-threatening blood clots in the legs or lungs twofold—that is, by 100%. This information was passed on in "Dear Doctor" letters to 190,000 general practitioners, pharmacists, and directors of public health and was presented in an emergency announcement to the media. The news caused great anxiety, and distressed women stopped taking the pill, which led to unwanted pregnancies and abortions (Furedi, 1999)." (Gigerenzer et al., 2008, p. 54).

Finally, after the activity, situational interest (SI) and perceived appeal (PA) of the activity were assessed. SI was measured through an eight-item scale (Cronbach's α = .91) referring to the student's specific interest in the ongoing learning activity (e.g., "I have found this activity very interesting"). PA was measured by a four-item scale (e.g., "I liked the slides very much") (Cronbach's α = .93). For both scales, participants indicated from 1 (*strongly disagree*) to 7 (*strongly agree*) the extent to which they agreed with each statement. A total score on the scales was calculated so that high scores corresponded to high levels of SI and PA.

In line with the usual course lessons, the total duration of the activity was two hours.

RESULTS

Relationships between individual interest in statistics, situational interest in statistics, and intrinsic motivation: To analyze the relationships between individual interest in statistics, situational interest in statistics, and intrinsic motivation, correlations between the variables were calculated (Table 1). Situational interest was significantly and positively correlated with individual interest in statistics and intrinsic motivation. Moreover, there was a significant positive correlation between individual interest in statistics and intrinsic motivation.

To evaluate the adequacy of the hypothesized mediation model explaining the underlying relationships between individual interest in statistics, intrinsic motivation, and situational interest in statistics, we tested the extent to which the relationship between individual interest in statistics and situational interest was mediated by intrinsic motivation. Specifically, we verified whether individual interest in statistics had both a direct and an indirect effect on situational interest in the statistics tutorial activity through

	1	2	3
1. Individual interest	-		
2. Situational interest	.40**	-	
3. Intrinsic motivation	.34**	.58**	-
M	38.88	43.19	125.38
SD	9.50	8.75	18.25

Table 1: Summary of Intercorrelations, Means, and Standard Deviations for Scores of the individual interest, situational interest, and intrinsic motivation

intrinsic motivation. We used the INDIRECT macro for SPSS (Hayes, 2013), which tested the hypothesized mediation model using the bootstrapping procedure (with 5000 bootstrap samples) to estimate the 95% confidence interval (95% CI; for more details, see Preacher & Hayes, 2008). The bootstrapping procedure is considered to represent the most reliable test for assessing the effects of mediation models (Hayes & Scharkow, 2013). As shown in Figure 1, the mediation model was estimated to derive the total, direct, and indirect effects of individual interest in statistics on situational interest in the activity through intrinsic motivation. We estimated the indirect effect of individual interest in statistics on situational interest in the activity, quantified as the product of the ordinary least squares (OLS) regression coefficient estimating intrinsic motivation from individual interest in statistics (i.e., Path a in Figure 1) and the OLS regression coefficient estimating situational interest in the activity from intrinsic motivation controlling for individual interest in statistics (i.e., Path b in Figure 1). A bias-corrected bootstrap 95% CI for the product of these paths that does not include zero provides evidence of a significant indirect effect (Hayes, 2009; Preacher & Hayes, 2008). Results showed a significant positive direct effect of individual interest in statistics on situational interest in the activity (point estimate = .19, 95% CI = [.02, .35]). Moreover, results showed a significant positive indirect effect of individual interest in statistics on situational interest through intrinsic motivation (point estimate = 0.16, 95% CI = [0.06, 0.31]).

To evaluate the adequacy of the hypothesized moderated mediation model explaining the role of the perceived appeal of the activity in the above described mediation model, we tested the extent to which perceived appeal had a positive main effect on situational interest and whether the mediation of intrinsic motivation between individual interest in statistics and situational interest in the statistics activity was moderated by the perception of the appeal of the activity. In order to verify this model, we conducted a moderated mediation analysis as suggested by Preacher, Rucker, and Hayes (2007). Similarly to the mediation analysis, this analysis tested the hypothesized model using the bootstrapping procedure (with 5000 bootstrap samples) to estimate the 95% confidence interval. As shown in Figure 2, the results confirmed a significant positive direct effect of individual interest in statistics on situational interest (point estimate = 0.16, 95% CI = [0.04, 0.28]). The results also showed a significant main effect of the perception of the appeal of the activity on situational interest in the statistics activity (OLS coefficient=2.35; *SE*=.65, *p*<.001) as well as a significant interaction between intrinsic motivation and the perception of the appeal of the activity on situational interest in the statistics activity (OLS coefficient=-.01; *SE*=.01, *p*<.05). Given the evidence for this interaction, we estimated the conditional indirect effect of individual interest in statistics through intrinsic motivation on situational interest at various levels of the perceived appeal of the activity. We found that the indirect effect of intrinsic motivation on situational interest was significant only for low (point estimate = - 0.11, 95% CI = [0.03, 0.23]) and medium (point estimate = 0.08, 95% CI = [0.03, 0.16]) levels of perceived appeal of the activity. The effect was not significant for high levels of perceived appeal of the activity (point estimate = 0.04, 95% CI = [-0.01, 0.12]).

Thus, the results suggested that the positive indirect effect on situational interest through intrinsic motivation (controlling for individual interest in statistics) was significant for low and medium levels of perceived appeal of the activity but not for high levels of perceived appeal. In other words, among students who perceived the activity as poorly or averagely appealing, results showed significant differences in

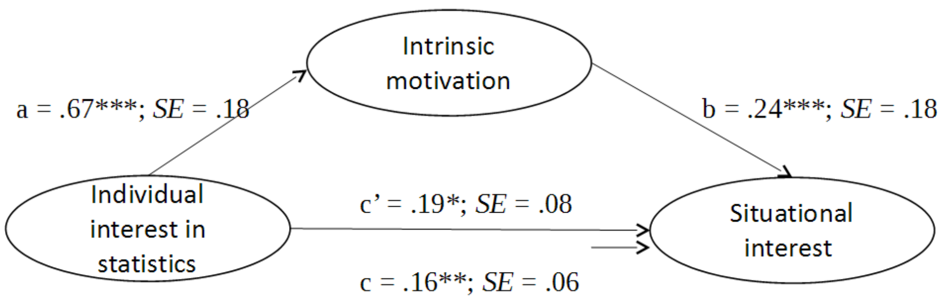

Figure 1: Path coefficients for mediation analysis on situational interest in the statistics activity. Dotted line denotes the effect of individual interest on situational rest in the statistics activity when intrinsic motivation is not included as a mediator. a, b, c, and c' are unstandardized ordinary least squares (OLS) regression coefficients. *p < .01 **p < .001 ***p<.001

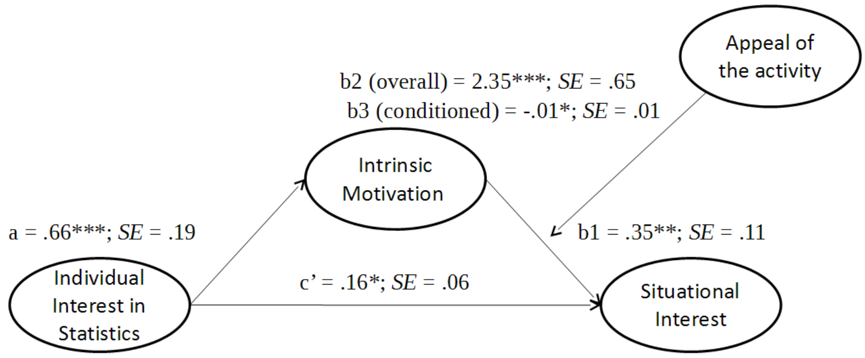

Figure 2: Path coefficients for moderated mediation analysis on situational interest. a, b1, b2, b3, and c' are unstandardized ordinary least squares (OLS) regression coefficients. *$p < .05$, **$p < .01$, *** $p < .001$

situational interest between those with low, medium, and high motivation. By contrast, among students who perceived the activity as highly appealing, there was no significant difference on situational interest between those with low, medium, and high motivation. This meant that if a statistics activity is perceived as highly appealing by the students, it is likely that they will perceive the activity also as highly interesting, regardless of their motivation.

DISCUSSION AND CONCLUSION

Given the relative lack of studies on interest in the statistics domain, the present work was aimed at investigating the relationship between individual interest and situational interest in the statistics domain. Specifically, our aim was to develop and test a model explaining how specific features of the teaching session, as perceived by students, act in concert with some individual differences in individual interest, intrinsic motivation, and situational interest. Overall, three important findings emerged from this study.

First, individual interest was found to have a direct effect on both situational interest and motivation in the statistics domain. This result is in line with Durik and Harackiewicz's (2007) earlier findings in the mathematics domain. Second, our study is the first to provide evidence that intrinsic motivation mediates the relationship between individual interest in statistics and situational interest in a statistics activity. More specifically, greater individual interest in statistics appears to be related to greater intrinsic motivation, which, in turn, is related to a greater likelihood to show interest in the activity. Third, our results suggest that the indirect effects of individual interest in statistics on situational interest through intrinsic motivation is moderated by the perceived appeal of the activity. In other words, the extent to which intrinsic motivation mediates the relationship between individual and situational interest, interacts with how the student perceives the appeal of the activity. Specifically, if a student perceives a statistics activity as highly appealing, even if he/she was poorly motivated to participate in the lesson, he/she is likely to be very interested in the statistics activity.

These results have important implications. First, these results indicate that intrinsic motivation plays an important role in the relationship between individual interest and situational interest in statistics, Thus, interventions aimed at increasing students' interest in statistics activities could be focused on improving intrinsic motivation, for example by highlighting the importance of statistics in everyday life and in the profession of a psychologist. Second, as our study shows that students' perception of the aesthetical appeal of teaching materials moderates the effect of intrinsic motivation on situational interest, educators who are interested in finding ways to involve the highest possible number of students (including the less motivated ones) in statistics learning, could aim to prepare teaching materials with the highest possible level of esthetical appeal.

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Affective exhibition during the interpretation of statistical data

Tamires Queiroz[1], Carlos Monteiro[1], Liliane Carvalho[1] and Karen François[2]

1 Federal University of Pernambuco, Recife, Brazil, tamires_nq@hotmail.com, cefmonteiro@gmail.com, lmtlcarvalho@gmail.com

2 Free University of Brussels – VUB, Brussels, Belgium, karen.francois@vub.ac.be

In recent years, research on the processes of teaching and learning of Statistics emphasises that the interpretation of data is a complex process that involves cognitive and technical aspects. However, the interpretation of statistical data is a human activity that also involves affective aspects. However these aspects were not sufficiently investigated. This paper discusses some elements from an empirical pilot study that explores the idea of affective expression during the interpretation of statistical data by undergraduate students of statistics and pedagogy. Although the participants had different academic background their interpretations of media statistics data did not follow patterns that were influenced by the university courses they attended.

Keywords: Affective exhibition, interpretation of statistical data, statistics education.

INTRODUCTION

Several studies on statistics education have given evidence that the interpretation of data is a complex activity which is associated with several aspects. Gal (2002) argues that *statistical literacy* consists of two types of components: cognitive and dispositional. The cognitive components, such as mathematical and statistical knowledge, are related to reasoning aspects that enable readers to technically interpret data. The dispositional components refer to more subjective elements related, for example, to beliefs, attitudes and emotional aspects of the individual that is reading the data. Based on Gal´s perspective, readers can discuss and communicate their conclusions about statistical data using their expertise on specific area, as well as their personal and social experiences.

Arcavi (2003) emphasises that the interpretation of (statistical) data is not the result of a simple decoding of graph components. Although Arcavi is mainly working on mathematics learning he also refers to allied disciplines (e.g., data handling or statistics) and the way of data representations and date graphing (Arcavi, 2003, p. 217).

Since "we live in a world where information is transmitted mostly in visual wrappings, and technologies support and encourage communication which is essentially visual" (Arcavi, 2003, p. 215) human being is encouraged to interpret visual representations. He emphasises the complexity of the phenomenon of interpreting visual representation, e.g. graphical display. It is not only related to what comes "within sight", but we are also encouraged and aspire to 'see' what we are unable to see. Referring to Goethe he explains the quote "We don't know what we see, we see what we know" (Arcavi, 2003, p. 230) stressing the last part of the expression "We see what we know". Arcavi argues that the same visual objects may have different meanings in different contexts. He therefore proposes to classify three types of 'difficulties', see cognitive, sociological and cultural.

Lima (1998) and Monteiro (1998) suggest that during the interpretation of data, the way readers use their mathematical and/or statistical knowledge is a complex aspect which is not the result from only one aspect, such as their academic background. Lima (1998) analyses the interpretation of data developed by designers and mathematics teachers. The author concluded that the interpretations of the participants were different in the way they read the data, however both groups were similar in the use of mathematical knowledge during the interpretations. Monteiro (1998) investigates the processes of interpreting graphs of printed media by a group of businessmen with different academic backgrounds and a group of economists. The author did not identified differences in relation to the strategies of problem solving, although the group of economists tended to produce more estimates in their interpretations.

Monteiro (2005) develops the idea of *critical sense* during the process of interpretation of statistics media graphs that is related to mobilisations and balance of several elements. The term mobilisation (Monteiro & Ainley, 2003) is related to the possibility of re-using or re-sourcing (Adler, 2000) previous knowledge and experiences during the process of interpretations of media graphs. This mobilisation seems to be a process in which readers explore the data, confronting it with their own perspective, and their previous experiences related to the data interpreted. However, the process of mobilisation in interpretation of media graphs does not 'naturally' happen. In order to mobilise their previous knowledge and experiences to interpret a media graph, readers need to establish a certain level of engagement in the task, which then leads to the articulation in which they make a recontextualisation of the knowledge and experiences mobilised, comparing them to the data. The reader also needs to balance different elements. Therefore, there is no direct application of knowledge and experiences for the process of interpretation. This complex process of mobilise and balance different elements during the interpretation of statistical data displayed in the graph is called *critical sense*.

McLeod (1992) did an extensive review of the literature in mathematics education that addresses the affective domain. The author states that, among other factors, most of these studies did not impact mathematics education because they were focused only on stable aspects of affectivity. In other words, these studies were more concerned with the products and not with the processes involved. Most studies reviewed by McLeod seem to conceive narrowly affectivity, merely investigate more specific concepts such as beliefs and attitudes. In addition, the studies did not make a link between affective and cognitive factors. Therefore, similar to those studies that investigated only cognitive aspects, the studies that addressed only affective aspects seem to have no impact on the learning and teaching of mathematics curriculum content. McLeod suggests that beyond a deepening of theoretical questions about the definition of affectivity and their relation to cognition, studies need to be based on research approaches that combine quantitative and qualitative methods.

Based on this literature review we can conclude that Gal (2002) provided statistical literacy principles comprised of cognitive and dispositional elements. Arcavi (2003) emphasised the importance of people's social and cultural background in the interpretation of data, which is also exemplified by Lima (1998) and Monteiro (1998). Finally, Monteiro (2005) added the notion of 'critical sense' based on statistical literacy principles, however emphasising the complex interrelation between components and processes involved. Research on the specific affective influences in statistics interpretation is still lacking. Looking at the field of mathematical literacy, affectivity has the rather narrow meaning of attitudes and beliefs. With the concept of affectivity we will include the dispositional elements that are take part at the interpretation of statistical data.

In the field of statistics education, it is necessary to develop a similar literature review that investigates the number of studies focused on the affective aspects, and how they can make impact on the teaching and learning of school statistics. Besides we need empirical evidence based on quantitative and qualitative approaches to better understand the interpretation of statistical data.

RESEARCH QUESTION

The research question in this exploratory study is if the affective expressions of students in situations of interpretation of statistical data are related to the students' background in the teaching and learning of statistics. Therefore we investigated the interpretations of students from two different backgrounds in the teaching and learning of statistics, viz. (i) bachelor in statistics and (ii) degree in pedagogy. We expected that the differences related to the type of course in which the participants were enrolled might influence their affective expressions on their interpretations.

METHODOLOGY

This pilot study was a qualitative investigation based on standardized open-ended interviews. In order to investigate aspects about the affective expression and statistics literacy, we invited first year students from two different university courses from the same Brazilian Federal University: two undergraduate students from an education course (P1 and P2) and two students from a statistics bachelor course (S1 and S2).

We chose a group of students from education because they are pre-service primary school teachers, and also because in this course has disciplines, such as Psychology which addresses affective aspects. The first year education students already attended those disciplines. The choice for students from bachelor in statistics was because this course has a curriculum focused on disciplines such as probability, data analysis, and stochastic phenomena. The first year students already had such disciplines and they will not have any discipline that approaches the affective aspects as part of their course. The first author contacted the students in their classroom and explained the research. The students interviewed were volunteers. Data collection was conducted in November and December 2013. The Pedagogy students P1 (57 years old, female) and P2 (33 years old, male) completed two disciplines that approaches affective aspects associated with teaching and learning. They did not attend any statistics course during their first year. The statistics students S1 (17 years old, female) and S2 (18 years old, female) completed five specific disciplines related to statistics.

DATA COLLECTION

The data collection was developed from individual standardized open-ended interviews which were composed by four tasks related to statistics data from different publications. Statistical data had the following topics: (1) on mammography examination, (2) on traffic accidents, (3) on life and health insurance, and (4) on high school students´ handguns. (Due to lack of space in this paper, we report on 2 and 4).The main reason to choose these cases was that the topics had a certain level of polemic. We expected that this type of data would motivate participants to make comments related to technical aspects as well as to their emotional reaction to the data. During the interviews, each task was presented to the student on a printed sheet. The researcher read the specific questions for each task. Interviews were recorded, and transcribed. The protocols were originally in Portuguese, fragments for this paper were translated by the authors.

Task 2 was comprised of questions about a line graph (Figure 1) that shows the percentages of deaths among Brazilian youth population caused by traffic accidents between 1998 and 2008. This graph was originally published in a report from the Brazilian ministry of justice (Waiselfisz, 2011).

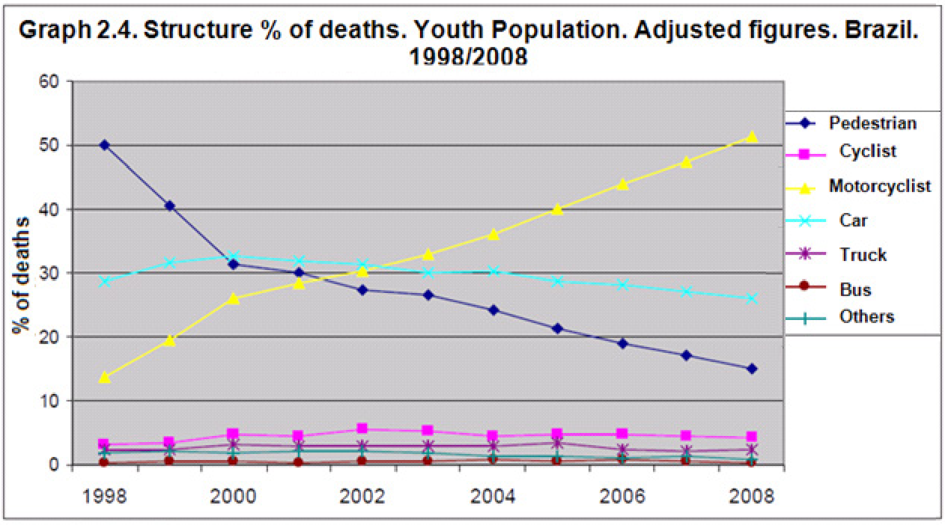
Interview questions related to task 2 (Figure 1) were:

1) What can you conclude from the results presented in this graph?

2) If you could ask a question to whoever built this graph, would you do? Which one(s)?

3) What would you say? What elements which would you emphasise?

4) What do you think these data are between 2008 and 2013? Why do you think that?



Source: SIM/SVS/MS

Figure 1: The percentages of deaths among youth Brazilian from 1998 until 2008

Task 4 was used by Watson (2005). It is a fragment of media news about the use of handguns by high school students:

> About 6 in 10 United States high school students say they could get a handgun if they wanted one, a third of them within an hour, a survey shows. The poll of 2,508 junior and senior high school students in Chicago also found 15% had actually carried a handgun within the past 30 days, with 4% taking one to school.

Interview questions related to task 4 were

1) What can you conclude about this research?

2) What do you think about the data collected in this research?

3) Do you have any questions or comments about this research? Which one(s)?

THE ANALYSES OF PARTICIPANTS' RESPONSES

The transcriptions of audio records were organized in protocols which were base to the data analyses. The participants' responses were analysed by a categorisation which emerged from a qualitative approach. Initially, we categorised their responses for each task and item. In order to discuss aspects from the data collected in the following sections we exemplify with some extracts from the interviews.

TASK 2 – DEATHS AMONG BRAZILIAN YOUTH POPULATION

In order to answer the first item of this task (What can you conclude from the results presented in this graph?) all participants made initially general comments related to the graph itself. However, they also made some observation which could make explicit their personal reaction to the data displayed. The extract from the interview with P1 can exemplify this:

S1: Here, the person [who reads the graph] realizes that motorcyclists [figures] rose so much. The pedestrians, I think, they became more conscious in relation to that, before used to be something that nobody cared. Nowadays have footpath, have road sign, have zebra crossing, have more... much care. While, the motorcyclists [figures]... is getting worst day by day. While the cars has a decrease but it is not much. While the ones [figures] of pedestrians remain all in one pattern, a little less.

The second question (If you could ask a question to whoever built this graph, would you do? Which one(s)?) motivated P2, S1 and S2 to ask questions related to the survey itself. They asked questions about the data collection, sampling and where the survey was conducted. Those questions indicated concern about the reliability of the research, e.g. S2 asked how the data were collected, as illustrated below:

Researcher: If you could ask a question to whoever built this graph, would you do?
S2: I would do... How the data was collected ... Only.
Researcher: What do mean?
S2: Kind of... If it was a street survey... Asking how many accidents you have in one year or if it was a survey... For example... Within the IML [Institute of Legal Medicine] which is the agency responsible for deaths.

The questions formulated by participants suggested that they were mainly concerned about technical aspects related to the data. Maybe they needed this type of information to be more comfortable to make other comments about the data. In some way, we could say that their questions were also expressions of their feelings about the data, because they made explicit some scepticism about the data displayed on the graph.

The responses to question 3 (What would you say? What elements which would you emphasize?) were complementary to those of question 2. The questions motivated the students to make more observations related to the data displayed. Generally, the participants now responded as a rereading of the graph. They pointed to specific figures (as they did when answering question 1). The most frequent responses were related to the increase in deaths of motorcyclists, followed by references to the decrease in pedestrians' deaths. P1 developed a more extensive response, questioning more explicitly, as illustrated in the following extract of her interview:

Researcher: Ok. It is... if you were going to comment to someone about this graph, which points would you think that should be important to discuss about it? What are the points that you would emphasise, that you would discuss more about?

P1: I think that it would be really to make relations between these categories... of pedestrians, motorcyclists and drivers. Why these categories are most affected in the traffic? Look at the difference of others, of the trucks, of the cyclists... Cyclists also do not have an incentive, don't they?... for the use of cycleway. But, it would be to make relation between these categories.

We can observe that P1 emphasises the importance of analysing the graph as a whole, as P1 attributes importance to the relationships between the categories. P1was interpreting beyond the data displayed when she was referring to specific issues which are part of her daily experiences and observations (e.g. when she talked about the situation of cyclists). P1 seems to be preoccupied with the effects on people rather than the performance effect of the figures.

From the analyses of protocols we identified more variety of participants' responses to the fourth question (What do you think these data are between 2008 and 2013? Why do you think that?). All participants justified their answers based on *contextual reference* (Monteiro, 2005), that is when they contextualise the data displayed on the graph making references which are related to their formal knowledge in different areas and their opinion.

On the one hand, P2 and S2justified their answers talking about a possible increase in numbers of accidents based on information that they had from two media reports.

Researcher: In the case, these results were from1998 until 2008, right? What do you think these data are between 2008 and 2013?

S2: I think that tends to increase the number of deaths of cyclists [but really referring to motorcyclists], well... for my knowledge of the world, right? Because... kind... the rates of IPVA [Brazilian tax on the ownership of motor vehicles], these things have decreased and more people are buying cars and... by this graph, it tends to increase even [the number of accidents].

On the other hand, P1 and S1 were more positive, they referred to the effectiveness of dry law that prohibit people to drive after drink any amount of alcoholic. The following extract gives an example of this type of approach:

Researcher: In this case, this... This report was made from 1998 to 2008. What do you think these data are between 2008 and 2013?

S1: Guy, I think it must have dropped. Slowly, but it is dropping. Our conscience is more... we have the dry law then we are taking more care, aren´t we? I believe that is a little bit better than two years ago.

From our analysis of the question 4 responses, we can infer that these participants also expressed different feelings about the same data displayed. These different affective expressions certainly are related to individual aspects from those who interpret the data. Hence there is also evidence that the interpretation of statistical data is composed of affective elements which need to be considered.

TASK 4 – THE USE OF HANDGUNS BY HIGH SCHOOL STUDENTS

Most of participants' responses to the first question (What can you conclude about this research?) tended to express feelings about the survey. The following extract from P1 interview is an example:

P1: I was shocked with the facility of armaments and weapons that the American population has. Because it is not just at school, no, any citizen, isn't it? Have in their home one or two weapons. ...The poll of 2,508 junior and senior high school students of the first and the last year of high school students... ...said that 15% of them had a gun in the last 30 days, with 4% having taken to school... [reading the task]. From these 15%, 4 [%] led to the school, didn't they? Other day I was commenting on that... so... some of them have their jets, their imported cars, their helicopter; others live picking up litter, don't they? And, for the world to arm itself is also very fast. It's just you do your atomic bomb, to make intimidations with each other. It's shocking. I find shocking.

P1 tended to interpret the news report basing on an emotional reaction to the theme, although she considered the statistical figures related to the poll. S1 and S2 interpretations were quite similar to P1 when they responded this question. We can infer that the theme related to this task was more explicitly polemic, and may have influenced these participants' interpretations. Only P2 had a more descriptive reading of the data which did not seem to have had any personal reaction about the news report. As we can see the excerpt from his interview:

P2: I conclude that... it was done a piece of research, a poll... that from 2.508 high school students from the United States... it was detected that 60% of these students say they could get a gun if they wanted. Then, moreover, this 60%... it is... a third of these students could get a gun within an hour, and moreover, it... yet... students from first and last year of high school [rereading the task]... that means... it has here... at first, the research was done with high school students. Making a correction! And... this research has reached that percentage of 6 out of 10 could have access. After that, it was done a research investigating this number of students 2.508, of the final years of high school and among these ... it is... it was found that 15[%] of them carried a handgun... carried a handgun in the last 30 days and 4% of them had already led gun to school.

In this part of the interview, P2 was trying to understand the details about the procedures related to the data collection, and other details of the news data. His concerns to specific aspect of the news did not allow him to question or expose his point of view.

The second question of task 4 intended to explore more specifically the participants' interpretation about statistical data involved (What do you think about the data collected in this research?). The participants' responses seemed to be complementary to their comments on the first question. Therefore, P1, S1 and S2 who did not make observations about the figures, responded here by making observations concerning quantitative data as well as expressing the personal point of view, e.g. the following extract form the interview:

Researcher: What do you think about the data collected in this research?

S1: I found it very serious. Because if among... 2,500 students, 15% of them have revolver, that is too many... if 4% can take it to school, imagine how many people within that school have a revolver. Any time something happens... Nowadays, in the ways those things are... any... "step on somebody's toes", you are already assaulting somebody... this is a very dangerous thing and has to have drastic measures.

On the other hand, the interpretation of P2 was more explicitly related to his personal opinion about the data.

P2: These data here, it shows and proves about the reality of a country that has a policy of well open access to weapons, right? And at the same time, there is no much control and no much oversight, right?

Finally, the third question was an opportunity to the participants to make final remarks about the data (Do you have any questions or comments about this research? Which one?).

P2 and S1 questioned about the survey sampling, the sites where the poll was taken and how the research was done, pointing out some possible biases. According to them, the information provided about the survey was insufficient to answer their questions, however, none of them expressed they didn't trust the research. The comments of P1 and S2 were based on their opinions about the situation, taking into account their knowledge about social, economic and political aspects which might be related to the theme, and which could eventually influence the interpretation of the data. These two participants did not ask for further information.

FINAL CONSIDERATIONS

This study explored aspects of the exhibition of affective components in the interpretation of statistical data. Specifically, the empirical research data was generated from situations in which the participants interpreted statistical data. The participants were students engaged in different processes of teaching and learning related to affectivity and statistics. Students of Pedagogy attended courses only related to affectivity, and students of statistics had approached various contents related to data analysis, and they did not attend courses that comprise the theme of affectivity as curriculum content.

The research tasks were associated with statistical data of controversial themes. Our expectation was to propose research situations that could provide possibilities for the participants to present more personal expressions during their interpretation of the data. We also expected that participants' academic background could influence how they would interpret and to express their affectivity in relation to the data.

The analyses of protocols suggest that the interpretation of statistical data is a dynamic process, which do not follow predictable patterns. Participants responded in different ways during their processes of interpretation to most of the questions. They mostly expressed their opinions and feelings in responding to the questions, or at some moments they mixed their objective analysis related to the statistical knowledge with the subjectivity of their impressions about the data. Even more technical oriented questions (see task 2 question 2 and task 4 question 2) can be interpreted as expressions of feelings (e.g. scepticism) or expressions of a personal point of view.

In line with the theoretical investigations on interpreting statistical data, these preliminary data gave evidence that the process of interpretation is a complex process consisting of cognitive and dispositional components. Based on the preliminary research findings, we have no clear evidence that these components were determined by the curriculum background of the students (pedagogical versus statistical). However our findings make clear that a broader discussion about the processes of data handling and the interrelation with affective aspects is an important issue in the further development of teachers' understanding of statistical literacy. Further investigations have to reveal possible differences between first year students and students in a final stage of the courses.

ACKNOWLEDGEMENT

This research was supported by CAPES Foundation, Ministry of Education of Brazil.

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Individual concepts of students comparing distributions

Susanne Schnell and Christian Büscher

TU Dortmund University, IEEM, Dortmund, Germany, susanne.schnell@tu-dortmund.de, christian.buescher@tu-dortmund.de

While distributions were long understood as "fundamental given of statistical reasoning" (Wild 2006, 10), recent research uncovered students' difficulties in acquiring the underlying conceptual structure and making statistically sustainable inferences when comparing distributions. Research so far describes informal strategies such as using 'clumps' as productive when learning to compare distributions. However, more insights are needed regarding why some of these strategies are chosen in certain situations in order to fully relate students' informal strategies to statistical concepts and measures in task design. This paper aims at clarifying the students' reasoning behind what to focus on for comparisons. We will present empirical snapshots from students in grade 8 (13–15 years old) who focus almost exclusively on absolute frequencies of dots and specifically determined intervals for comparing distributions.

Keywords: Statistical reasoning, comparing distributions, design research.

INTRODUCTION

Comparing empirical distributions has a high value for statistics education: "Concepts and judgments involved in comparing groups have been found to be a productive vehicle for motivating learners to reason statistically and are critical for building the intuitive foundation for inferential reasoning" (Ben-Zvi, 2003, p. 1; also Bakker & Gravemeijer, 2004). A combination of descriptive and inferential reasoning is needed in order to make sense of differences and commonalities of two (or more) distributions and go beyond the data at hand. Thus, this activity is an important part of data analysis. The necessary concepts and insights are "multifaceted" (Ben-Zvi, 2003) as properties of and between distributions have to be considered. The aim of this paper is to explore students underlying informal resources and rationales as well as conceptual difficulties as starting points for establishing a suitable learning environment.

PROPERTIES OF AND BETWEEN DISTRIBUTIONS

From a normative perspective, comparing distributions statistically requires students to perceive a distribution as an "organizing conceptual structure with which they can conceive the aggregate instead of just the individual values" (Bakker & Gravemeijer, 2004, p. 148). Wild (1999) calls a distribution a lens, through which variation is looked at by "set[ting] aside case labels" (p. 11). This short descriptions already points out that a distributions is in fact a net of different intertwined concepts: Centre, spread, density and skewedness are *properties of* a distribution and constitute its shape (ibid., Ben-Zvi, 2003). "The concept of distribution has a complex structure, but this concept is also part of a larger structure consisting of big ideas such as variation and sampling (…). [One can] deal informally and coherently with all these big ideas at the same time with distribution in a central position." (Bakker & Gravemeijer, 2004, p. 149).

These properties can be approached formally (e.g. calculating arithmetic mean, mode and median as measures for the centre), but also in more phenomenological and visual ways (e.g., determining intervals with high density, gaps and clusters; cf. Pfannkuch et al. 2010 for the visual approach). For clarification, this paper uses 'properties' to refer to statistical concepts and 'features' to more visual aspects of a distribution.

Inherent in the statistical concept of distribution is the necessity to not only focus on single data points or small groups (so-called local view; Ben-Zvi & Arcavi, 2001), but to perceive a distribution as a whole, allowing to "search for, recognize, describe and explain general patterns in a set of data" (so-called global view;

Ben-Zvi & Arcavi, 2001, p. 38). Especially the latter is fundamental to statistical reasoning, but also challenging for students to acquire (ibid.).

When comparing two or more distributions, properties have to be put *in relation between* the distribution, adding further relative insights such as overlap, shift and unusual features (e.g. outliers; Pfannkuch et al., 2010; cf. Ben-Zvi, 2003 for comparing measures of variation within and between groups). The comparison can then also allow for new insights into the peculiarities of the initial distribution: For instance, looking at a set of temperatures from July 2014 on the mountain Zugspitze could become an indicator for the effects of global warming when put in relation to the distributions of the 1900s.

INDIVIDUAL APPROACHES TO COMPARING DISTRIBUTIONS

It is not surprising that this complex interplay of concepts is challenging for students: Recent research points out that taking a global view on distributions rather than focussing on single data points or groups is especially challenging (Ben-Zvi & Arcavi, 2001; Bakker & Gravemeijer, 2004). Problems persist even after instruction in statistics (Ben-Zvi, 2003; Konold et al., 1997): students who are familiar with formal measures such as mean and median for single distributions do not make use of them when comparing distributions (e.g., Watson & Moritz, 1999; Konold et al., 1997). As Konold and colleagues (1997) argue, this might indicate a lacking understanding of averages as properties that represent a distribution.

However, some informal strategies were repeatedly shown, which offer productive starting points for structuring learning pathways: Focussing on visually remarkable aspects of distributions (e.g., represented as dot plots), learners make use of informal concepts such as 'clumps', 'hills' or 'chunks' to describe and compare distributions (Bakker & Gravemeijer, 2004; Konold, 2002; Cobb, 1999). Konold (2002) for instance describes how students use ranges of data in the heart of the distribution ("modal clumps"), which he interprets as vehicles for describing the centre (average) and at the same time the variation of data points. Bakker & Gravemeijer (2004) show that students divide given distributions in three groups (low, middle and high), which are then interpreted in the given context and compared. They understand this as steps from a local to a more global view.

While many studies reproduced the use of informal descriptors such as 'bumps', it remains open what underlying rationale guides students in choosing or dismissing features such as modal clumps in situations involving the comparison of distributions. Understanding *why* certain foci are chosen to compare might provide further insights into how task design has to be structured to promote development of statistical reasoning. These questions call for research on the micro-level and reconstructing step by step the individual concepts activated by students and the foci they take on the distribution.

METHODOLOGY AND DESIGN OF THE CASE STUDY

The presented study is part of a larger design research project using the methodological framework of topic-specific Didactical Design Research (Prediger & Schnell, 2014; Prediger et al., 2012), which has two intertwined aims: (1) designing a teaching-learning arrangement to facilitate the acquisition of the concept of distribution by comparing distributions and (2) deepening the understanding of the processes of conceptual development on an epistemological level. The design research is conducted by iterative cycles of design experiments, consisting of closely related phases of (re-)structuring learning goals, (re-)constructing the teaching-learning arrangement, conducting and analysing the design experiments and developing local theories. By combining process-oriented analysis and construction of teaching-learning arrangement, this framework provides for the need of research on the micro-level outlined above. Situated early in the research process, this first design experiment cycle aimed to explore how students in German middle schools (informally) compare frequency distributions (represented as stacked dot plots) and identify individual approaches as starting points for task design. Specifically, the research aimed at finding answers to the following questions:

- (RQ1) Which individual concepts do students use to compare distributions?
- (RQ2) Which rationales guide students in choosing certain foci for comparing distributions?

Data collection

To investigate the complex processes of comparing distributions, we conducted and videotaped design experiments (45 to 60 minutes) in a laboratory setting (cf. Cobb et al., 2003) with three pairs of students, aged 13 to 15. To make sure that students were familiar with distributions as a prerequisite for the activities, we chose students who had learnt about box plots in class a few weeks before the experiments. With statistics playing only a small role in German mathematics education, the students learned to construct boxplots and interpret them in a course over six lessons. The focus was on formal methods, e.g., for determining the five parameters. There was only limited attention given to informally examining variation in terms of centre and spread. The students were not familiar with stacked dot plots. Guiding the experiments was the July climate task, comparing stacked dot plots of temperature on a mountain in July in different years (Figure 1).

Students were tasked with comparing the temperatures in July on the top of the Zugspitze in the years 2002, 2004 and later 2007 in order to determine the warmest month. Although the graphs were created in Tinkerplots, the students at this point had only access to the plots printed on paper to encourage informal statistical reasoning without focussing on pre-given measurement and tools. Later in the experiment, the students were also given boxplots that had to be matched with the according dot plots.

Data analysis

The fine-grained analysis is conducted under an interpretative paradigm using the framework of Vergnaud's Theory of Conceptual Fields (Vergnaud, 1996). To give insight into students' individual concepts, we adapted the theoretical construct 'concept-in-action', which is defined as "categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the world into distinct elements and aspects, and pick up the most adequate selection of information" (Vergnaud, 1996, p. 225). The reconstructed concepts-in-action are symbolised as ||...|| and can provide for different functions: They can be the guiding category in *how* to compare distributions such as the ||*absolute frequency*|| *of dots under zero* (see episode 1 below). Furthermore, we found concepts-in-action which guide the students in *why* they choose certain aspects to compare, such as an ||*individual representativity of chosen intervals for the specific properties of a distribution*|| (see episode 2 below).

Concepts-in-action are not necessarily in line with normative mathematical ideas but guide the students' individual process of making sense of the situation. They are shown through action (ibid.) and can be uncovered through interpretation of the students' behaviour.

The in-depth analysis is so far limited to the case of Annika and Bastian; preliminary analysis showed that the other pairs are comparable in terms of focussing on visual features of the dot plots and determining absolute frequencies, but were less able to explain their reasoning behind certain actions and communicate their ideas and strategies. In the first step of the analysis of our data, we reconstructed the nature of this case from video, identifying crucial episodes of the students' reasoning process. These scenes were transcribed verbatim and annotated by both researchers separately. The goal of the analysis was to infer a) students' individual concepts-in-action when

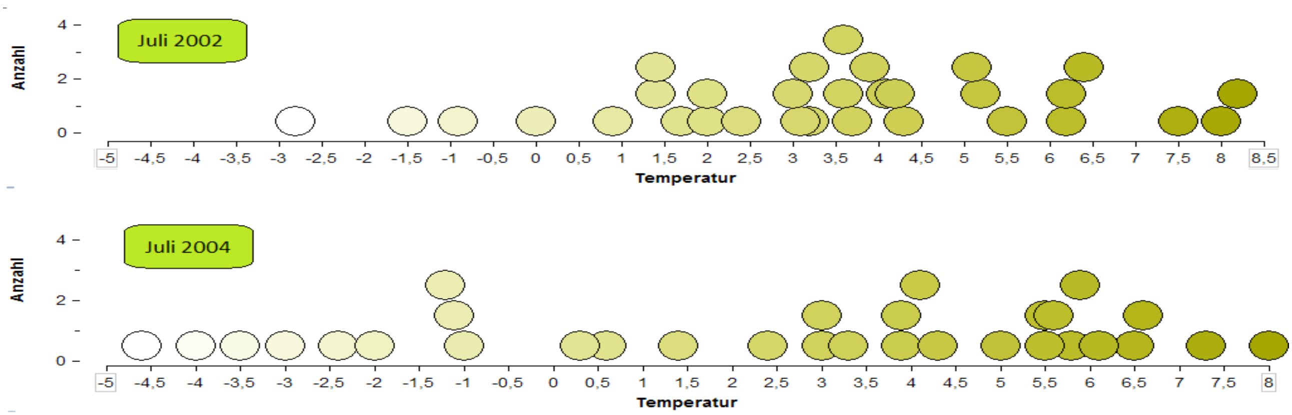

Figure 1: The stacked dot plots of the July climate task (July 2002 and 2004)

comparing the given distributions and b) students' underlying reasons for choosing specific foci on the distribution(s). The results of the analysis were then compared and discussed until a consensus on the interpretations was reached.

EMPIRICAL RESULTS

Annika and Bastian finished a lesson on statistics using box plots, but are unfamiliar with the representation of stacked dot plots. In line with other studies, they use exclusively informal methods to compare the presented distributions of temperatures. While they mention that it would be convenient to "have the arithmetic mean" or "use box plots" for "a better comparison", they are not making an effort to generate them. When presented with boxplots in the end of the interview, they have no trouble interpreting and matching them with the dot plots. This activity is rather superficial though and stays on a level of formal procedures rather than connecting insights acquired in the informal analysis of the dot plots. As mentioned above, this is in line with recent research. We will thus focus exclusively on the comparing activities concerning the dot plots.

Throughout the interview Annika und Bastian make use of various concepts-in-action guiding for comparing the distributions, which we present in Table 1 (RQ1). The concepts-in-action of ||*density*||, ||*value*||, and ||*spread*|| only appear rarely, as Annika and Bastian mostly focus on the ||*absolute frequency*|| and ||*position*|| of groups of dots. For this, they create so-called "sections" (intervals with groups of dots) within the distribution; this activity is mostly guided by the visually perceived 'hills' (a modal interval), which the students call "agglomeration area" (a geographical term, which might indicate that they also take the density into account). In Figure 2, we marked the sections, which the students address verbally or by gestures[1]: They first focus on the visual hill in 2002 (see Figure 2, section 1_{2002}). The number of dots in this interval is then compared with the number of dots in the same interval in 2004 (Figure 2, section $1_{2004;}$ episode 1 below gives more details on this comparison activity). Section 2 is defined by the agglomeration area in 2004; section 3 consists of the 'most right dots' (also right of the border of section 2) and section 4 is defined as left of the agglomeration area of 2002. In other situations in the interview, the students also make use of the scale and the context by looking at dots under or around 0°C or "dots in the colder interval".

The students focus almost exclusively on such groups of data points in a local view. However, the in-depth analysis uncovers that the students have features of the whole distribution in mind when comparing sections, as we will show in the next segment.

Episode 1: Comparing and equalising sections of distributions

25 minutes into the experiment, Annika is summarizing previous arguments of her and Bastian in favour of calling July 2002 the warmer.

1 Annika: If these were 11 dots in the agglomeration area *[circle in section 1_{2002}]*, and here are 8, then *[points to section 1_{2004}]*. Then that's a difference of 3 [...]

2 Annika: But .. Bastian says here *[points to section 2]* is also a difference of 3. So here in the agglomeration area *[in section 2_{2002}]* are 3

1 Dotted lines, circles and section numbers added by authors for clarification; circled are the previously identified 'agglomeration areas'; numbers indicate the order in which the sections are addressed by the students.

Concept-in-action	Activity
||Absolute frequencies of dots in certain intervals||	Comparing the (difference of) the amount of data points in chosen intervals
||Density in certain intervals||	Comparing the number of dots in relation to the width of the interval
||Position of certain (intervals of) dots||	Comparing the relative position of data points by 'left of'/'right of' or 'higher'/'lower than'
||Value of certain (intervals of) dots||	Comparing the temperature values of data points
||Spread of dots in certain intervals||	Comparing how spread out the data points are

Table 1: Concepts-in-action activated in comparing activities

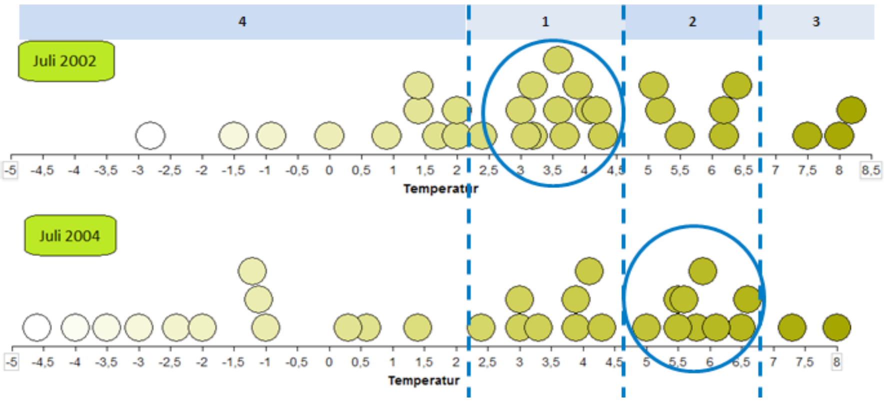

Figure 2: Annika & Bastian's sections when comparing the distributions

fewer and here in this agglomeration area *[in section 1_{2004}]* are also 3 fewer, so to say. Thus, it is even again, Bastian said.

In line 1, Annika compares the ||*differences of absolute frequencies*|| starting with the visual hills (circled in Figure 2): Each distribution's agglomeration area has three dots more than the corresponding section of the other distribution. Not activating a concept of ||*values*||, Bastian's reasoning as repeated by Annika seems to be that the differences between the distributions thus even out for these two sections (line 2). Therefore, the agglomeration areas alone might not be suitable to determine the warmer month and – according to Annika – other features have to be taken into account:

3a Annika: Thus I argued that – up here *[points to section 3_{2002}]* has one dot more; they are higher, too.
3b [cont'd] And all these dots *[points to section 4_{2002}]* are much more spread downwards *[points to section 4_{2004}]*. That is why in my opinion 2002 is warmer.

Annika now compares the distributions by the ||*absolute frequency*|| and ||*position*|| *of dots with highest value* (maxima; section 3, line 3a). In line 3b, her gesture pointing at the dots in section 4 of 2002 and saying "all these dots are much more spread downwards [in 2004]" seems to indicate that she assumes the same absolute frequency for the sections. Thus, it is the ||*spread*|| in combination with the ||*position*|| of the dots which makes the difference and lets her back up her argument that 2002 is warmer.

We call Annika's strategy 'equalising': When agglomeration areas are not useful for comparison as they are 'equal' in terms of absolute frequency, the other intervals have to be taken into account. In regard to RQ1, this episode highlights how the students use different concepts-in-action to compare distributions. Concerning RQ2 and the question of underlying rationales, we uncover how Annika seems to choose the outer intervals of the distribution and concepts-in-action other than ||*absolute frequencies*|| because she perceives the agglomeration areas – in this case of 'equal absolute frequencies' – as not helpful for the comparison.

Episode 2: Discussing the width of intervals for comparing

The determination of sections plays a crucial role throughout the students' comparative actions. At first, Annika and Bastian choose them spontaneously according to the perceived visual features of the distributions. When after 36 minutes the third distribution is discussed, they are asked to give a general rule for comparing distributions. Annika explains her strategy as "count how many dots are in the lower section and how many are in the higher section and compare them", stating that it is not useful to look only at the agglomeration areas as was established in episode 1. The interviewer then prompts the students to consider the choice of sections explicitly:

4 Researcher: And these sections you pick out, where do they come from? [...]
5 Bastian: What you just said about counting the dots *[points at Annika]*; I would definitely use the same distances. That means always 5, so to say: minus 5 to 0, 5 to 8.5 or 10 *[draws imaginary vertical lines through all distributions, indicated with dotted lines in Figure 3]* [...] because you always need equal sections [...]

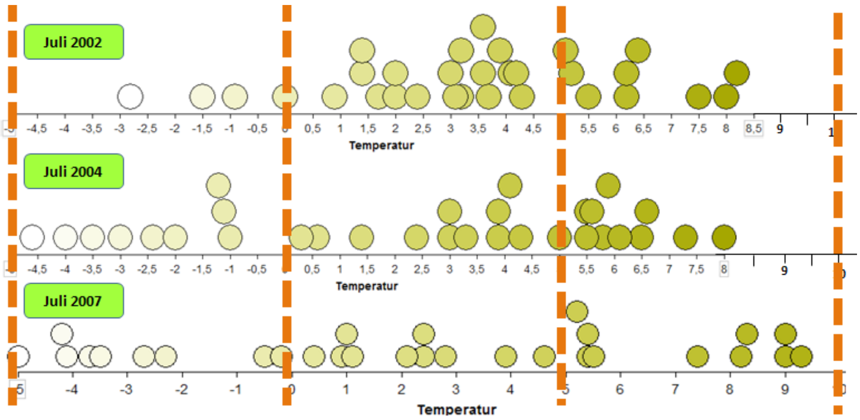

Figure 3: Bastian's proposed sections

6a Annika: Yes... *[5 sec]* I have to think about that again. *[looks at distributions, 27 sec]*

6b [cont'd] Well, I'm kind of undecided. On the one hand, I think that it makes sense. On the other hand I think why not take sections of different size? But then I think, then you can't compare it that well with the other sections, but can compare it still with the sections of the other years *[moves hand vertically over the three distributions]*. [...]

6c [cont'd] Well here, from minus 5 to 0, we simply use the step of 5 now. [The step] to [plus] 5 is not fitting though, because these dots *[points to dots just above 5° in 2007, circled in Figure 3]* are close to those under 5. [...] and that is why it is sometimes better to use different sections, because then points still just belong to it and are not already in another section.

In line 5, Bastian proposes an approach of choosing sections within a distribution due to a fixed width of the interval of 5. Annika however seems torn between the ideas of fixed and dynamic interval width (line 6b). To her, there seem to be instances where interval width can be chosen arbitrarily (from -5 to 0, line 6c), and where interval borders have to respect features of the distribution (from 0 not to 5, line 6c). Her reason "these dots are close to those under 5" might refer to the visual impression of 2007: the group of five dots around 5° (circled in Figure 3) are separated from others by gaps and thus form a visual unit. We interpret this as an indication of an underlying concepts-in-action: To compare different distributions, one has to take the gaps and groups into account. Thus, sections have to represent the specific visual features of a distribution, which we call an individual concept-in-action of || *representativity of a distribution's features*||.

In line 6b, Annika utters the underlying reason for her conflict: Annika seems to explicitly differentiate between comparing sections *within* a distribution and comparing sections *between* distributions: Sections of different widths (as in Figure 2) are worse for comparing them within a distribution, but due to their ||*individual representativity*|| better for comparing them with other distributions. This indicates a global view on the distribution which is Annika is dealing with by the informal approach of determining absolute frequencies in sections of different width.

CONCLUSION AND OUTLOOK

Consistent with literature (e.g., Bakker & Gravemeijer, 2004), the students organised the data through visual features such as modal clumps. The partition of the data however did not necessarily follow the structure of low-middle-high, but was informed by complex interplay of various concepts-in-action. The empirical snapshots show that these students are mostly focusing on the ||*absolute frequency*|| instead of the position of certain features in relation to each other and values of data points. Elaborate strategies such as 'equalising' combine different concepts-in-action and create individual rules of which features to compare in certain situation. The choice of sections in which absolute frequencies are determined are guided by an individual concept-in-action of ||*representativity*|| which does allow seeing the characteristic properties of one distribution but might be an obstacle to put different distributions in relation with each other.

The episodes shown in this paper are not intended to highlight deficits of the lessons students had for acquiring underlying concepts underlying box plots. Rather, the intention is to give insights into students' hidden individual concepts and reasons in order to understand the rationality of their activities. The presented reasoning processes have a lot of potential for a deep understanding and statistical reasoning, as a rich repertory of (in itself mathematically sustainable) concepts is activated and consciously combined.

Even though this phenomenon was so far discovered in the design experiment with only one pair of students, we take its possible impact on learning pathways seriously: In the next design experiment cycle, we will make room for students to explicitly address the question if and in how far it is necessary to represent a distribution's features for comparisons. We will carefully consider tasks to guide students in the shift from focusing on absolute frequencies to ordinal views of the relative position of features. Furthermore, our further research aims at uncovering other concepts-in-action guiding students in comparing distributions.

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Changing beliefs about the benefit of statistical knowledge

Alexandra Sturm[1] and Andreas Eichler[2]

1 University of Education Freiburg, Freiburg, Germany, sturmfr@ph-freiburg.de

2 University of Kassel, Kassel, Germany, eichler@mathematik.uni-kassel.de

In this day and age, statistically-based information is nearly omnipresent in daily media like newspapers or TV. Therefore, people need skills to read and interpret such information adequately. Due to the fact that previous research shows that sometimes statistical knowledge was improved without seeing the usefulness of this knowledge, we focus on improving both statistical knowledge and beliefs about the benefit of statistical knowledge. We examine whether a short-term (two hours) intervention has measurable influence on knowledge and beliefs of students at school and university, who have different mathematical foci. In this paper we discuss the design of our study and mainly focus on the intervention.

Keywords: Statistics, beliefs, attitudes, intervention study.

INTRODUCTION

The ability to read, to understand and to judge statistical information adequately has become increasingly important in our information society for everyone. The National Council of Teachers of Mathematics [NCTM] (2000, p. 48) highlights this significance in its 'Principles and Standards for School Mathematics': "The amount of data available to help make decisions in business, politics, research, and everyday life is staggering." Accordingly, the NCTM emphasizes that statistics skills are necessary for students "to becoming informed citizens and intelligent consumers" (ibid.).

There is a wide consensus that the ability to properly interpret quantitative data and, thus, to properly interpret statistical data in daily life is based on knowledge elements, but also on beliefs about the importance of statistics for society or the own life. For example, Wallman (1993) describes the term statistical literacy as "the ability to understand and critically evaluate statistical results that permeate daily life, coupled with the ability to appreciate the contributions that statistical thinking can make in public and private, professional and personal decisions" (p. 1). Emphasizing knowledge elements as part of statistical literacy seems to be self-explaining. Further, Gal (2004, p. 69) suggests dispositional elements including "the willingness to invest mental effort" to be a second part of statistical literacy. It can be assumed that beliefs and attitudes have consequences on using knowledge learned before: Schau and Emmioglu (2012, p. 86) suggest "that students who leave their statistics courses with negative attitudes are unlikely ever to use what they have learned. That is, they will not intelligently and literately use statistics in their professional and personal lives or in any educational venture".

Accordingly, an educational goal all over the world is developing statistical literacy including both knowledge elements and dispositional elements (Shaughnessy, 2007). But research shows that students being schooled in statistics before could have improved knowledge, but not improved beliefs (Schau & Emmioglou, 2012; Eichler, 2011). Therefore, a possible assumption is that statistical literacy is only sustainably developed if both parts, i.e. knowledge elements and dispositional elements, are developed appropriately.

As a consequence of the discussion above and taken into account that research shows that even adults are predominantly in a struggle with handling statistical information (Gal, 2004), our research project aims to investigate the relation of knowledge elements and dispositional elements when developing statistical literacy. For this reason, we developed an intervention aiming to improve both knowledge elements and dispositional elements. Students' beliefs, their perception referring the significance and benefit of statistics for both society and their own life, are a main focus of our research. We investigate students' perception referring to different samples. Firstly, we investigate students at university that use mathematics in a different way: Students of mathematics education, students of health education that have to apply mathematics and students of pedagogy that do not use mathematics in their university studies. Each subsample will comprise at least 60 participants. Further, we will investigate students at school (grade 11, age 17) in a second step of our study.

In the first part of this report, we outline the main constructs of our research by describing a model of statistical literacy and by specifying the construct of beliefs. Subsequently, we present a specific statistical topic which we address, i.e. theorem of Bayes, and partly the problems of the area of 'risk communication' respectively 'health literacy' used in our intervention. Afterwards, we outline such a statistically-laden situation and its visualization before discussing methods aiming to investigate knowledge and beliefs as a part of statistical literacy. Finally, we present first results of the intervention with 118 students of health education.

A MODEL OF STATISTICAL LITERACY

We use the construct of statistical literacy to describe students' ability to cope with statistical-laden situations. For describing statistical literacy, we primarily refer to the model of Gal (2004, p. 51; cf. Table 1). Gal's model describes both knowledge elements and dispositional elements as constituent parts of statistical literacy, similar to Wallman (1993). The left side of the model comprises five components which are briefly presented in the following, starting with the three non-mathematical and non-statistical aspects. Literacy skills are necessary to perceive information through an oral or written text, whereas context skills are necessary to perceive a certain context in which data are produced. Critical questions include the ability to be aware of possible manipulations in reports that are based on statistics. Further, Gal (2004) distinguishes between mathematical and statistical knowledge. We avoid this distinction in our research approach and subsume mathematical knowledge to statistical knowledge, although it's possible to differentiate these components by defining certain mathematical procedures as parts of a specific statistical knowledge (Gal, 2004).

Since we focus especially on the dispositional elements of statistical literacy, beliefs and attitudes, we will briefly outline our understanding of these constructs to discuss the right side of Gal's model.

BELIEFS AND ATTITUDES AS ELEMENTS OF STATISTICAL LITERACY

Following Hannula (2012), beliefs and attitudes are parts of mathematics-related affect. We understand the term belief as an individual's personal conviction concerning a specific subject, which shapes an individual's way of both receiving information about a subject and acting in a specific situation (Pajares, 1992). Although sometimes beliefs are understood as stable, we are aware that stability is no inherent and definable characteristic of beliefs (Liljedahl, Oesterle, & Berèche, 2012). In contrast to rather cognitive beliefs, attitudes embrace the more affective part of mathematics-related affect (cf. Hannula, 2012). According to McLeod (1992, p. 581), attitudes could be defined as "affective responses that involve positive or negative feelings of moderate intensity and reasonable stability". For example, beliefs about the benefit of statistical knowledge can be measured by items as "statistics is necessary to understand decision making in society", because it's an indicator of an individual conviction and, thus, a belief. By contrast, agreeing "I like statistics" indicates a favor towards an object, statistics, and, thus, an attitude (cf. Eagly & Chaiken, 1998).

Knowledge elements of statistical literacy	Dispositional elements of statistical literacy
Literacy skills	Beliefs and Attitudes
Statistical knowledge	Critical stance
Mathematical knowledge	
Context knowledge	
Critical questions	

Table 1: Aspects of statistical literacy

Concerning beliefs, which are our main focus in comparison to attitudes, we further distinguish between beliefs towards the world and beliefs towards the self. For example, it is possible that a student believes that statistics provides a benefit for the society in a global sense on the one hand, and evaluates further his ability to use statistics in his own life on the other.

Research referring to stability of beliefs in mathematics education shows that positive influences on beliefs are partly very rare (Eichler, 2011; Maaß, 2010). Therefore we discuss in the following ideas which we used in our intervention to address positive beliefs towards statistics.

PROMOTING KNOWLEDGE AND DISPOSITIONAL ELEMENTS

Principles of the intervention

Principles for the design of our intervention are based on possible reasons why students' do not appreciate the benefit of statistics for society or their own life. Firstly, it is possible that statistics is not part of the curriculum (cf. Burrill, 2011). A second possible reason is that statistics is part of the curriculum, but teachers do not teach it, because of a self-estimated lack of time or feeling uncomfortable with statistics (Eichler, 2011). Another possible reason is that statistics is taught, but in an inappropriate way of teaching: using not-application-oriented contexts and problems (we point out an example in Figure 2).

In our intervention, we try to meet requirements for application-oriented contexts that potentially highlight the relevance of statistics for students sustainably. Further, we formulated three requirements for an appropriate subject matter that

1. emphasizes the benefit of statistics for both society and, in particular, individual's life;
2. focuses on an issue that is not common for students;
3. focuses on an issue for that exist elaborated strategies for designing a potentially effective short-term intervention.

In our opinion, the Bayes' theorem fulfills all these requirements.

Firstly, the Bayes' theorem is existent in daily media (cf. Figure 3) which is in our opinion an indicator for emphasizing the benefit of statistics for the society and an individual.

Further, concerning the second requirement, the Bayes' theorem or rather Bayesian thinking is not commonplace (Sedlmeier & Gigerenzer, 2001) and it is taught at school rarely. Thus, this subject potentially gives evidence about the benefit of statistics in a field in which adults without training in Bayesian thinking mostly fail to give correct estimations of probabilities (Sedlmeier & Gigerenzer, 2001).

Finally, as a consequence of findings in educational research, there are different strategies to improve understanding referring Bayesian problems. E.g., Sedlmeier and Gigerenzer (2001) found that representing the statistical information in a problem as natural frequencies increases the rate of correct estimations. Further there is evidence for the efficiency of two visualization-forms in short-term interventions: The tree with natural frequencies (ibid.; Wassner, 2004) and the unit square (Bea, 1995).

The intervention

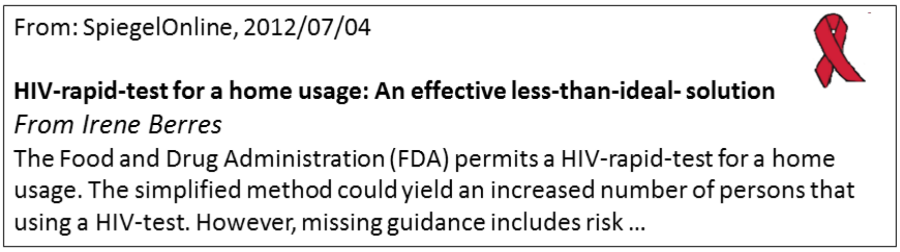
As mentioned above, we laid emphasis on an authentic and application-oriented context. An analysis of textbooks showed a considerable amount of less authentic contexts. For example, the question in the task referring Bayes formula shown in Figure 2 seems

Excercises

A car dealer recieves cars from two different factories (A and B). 65% of the cars from factory A and 75% of the cars from factory B have no failure in the guarantee period.

What is the probability that a car was produced in factory B given a failure in the guarantee period?

Figure 2: A task with a context that is not authentic


From: SpiegelOnline, 2012/07/04

HIV-rapid-test for a home usage: An effective less-than-ideal- solution

From Irene Berres

The Food and Drug Administration (FDA) permits a HIV-rapid-test for a home usage. The simplified method could yield an increased number of persons that using a HIV-test. However, missing guidance includes risk ...

Figure 3: Headline in a German online-newspaper (translated)

to be irrelevant, since the car dealer could know the answer of this question without using Bayes' formula.

By using actual newspaper articles and leaflets (Figure 3), we tried to present the students an authentic situation, i.e. a situation that could potentially be a real problem for the student in their future life.

This material should also contribute to recognize and emphasize the significance of the content. Based on the appropriate examples of Boer (1993), Wassner (2004), Pinkernell (2006) and Beckmann (2013), the intervention involves three main-contents: HIV-Testing, breast-cancer-screening and prenatal screenings.

A second characteristic of the intervention should be its comprehensibility. For this, we use natural frequencies additional to relative frequencies or rather probabilities for the representation of quantitative information in the tasks. Natural frequencies simplify the Bayes' theorem for students, who have only to relate numbers to each other instead of calculating a complex formula that entails three multiplications. The research of Gigerenzer and Hoffrage (1995) shows that representing statistical information as natural frequencies increases the ability to solve Bayesian problems. A possible reason for this effect is that the nested sets structure becomes more salient. Nonetheless, in newspaper articles, TV and other everyday situations probabilities are often used, so that we didn't want to drop probabilities.

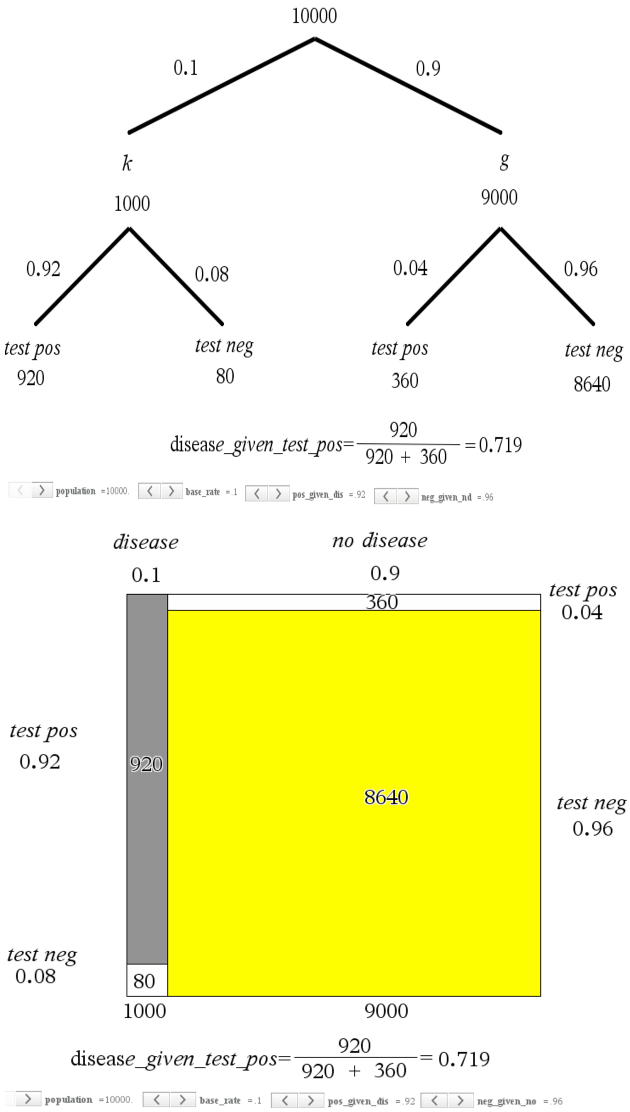


Figure 4: Tree diagram and unit square with natural frequencies

We further use the tree diagram (Wassner, 2004) and the unit square (Bea, 1995) with natural and relative frequencies that we illustrate in Figure 4 for a fictive situation of a disease-test.

Both diagrams are helpful to apply and understand the Bayes' theorem (Sedlmeier & Gigerenzer, 2001). While the advantage of the tree diagram is visualizing the chronological sequence of the given information, the advantage of the unit square is visualizing the proportions. This aspect seems to be especially helpful in cases when parameter were changed (Eichler & Vogel, 2010). Furthermore, we avoided using formal Bayes' theorem and difficult terms like sensitivity or specificity to support the comprehensibility.

As mentioned above, the three main-contents of our intervention mainly refer to a diagnosis of a disease (HIV-Testing, breast-cancer-screening and prenatal screenings) and the problems could be solved by using Bayes' theorem. In each problem, a base rate of a disease or an infection and conditional probabilities representing the right-positive rate (probability of a positive test result given a disease) and representing the false-positive rate (probability of a positive test result given no disease) have to be arranged according to Bayes' theorem to compute the probability of disease given a positive test. We think that a further interesting content are doping-tests. They have a high significance in sports and provide a great basis for modeling, because the base rate is unknown (estimated 20–35 %, cf. Pitsch, 2009) and the sensitivity- and specificity-values of used doping-tests are not published by WADA (World Anti-Doping Agency), so they must also be estimated (Pitsch, 2009). But in respect on this complexity it's not appropriate for our short-term intervention.

The tasks potentially promote a critical stance as a crucial part of statistical literacy. For example, possible critical questions could refer to the following issues:

— What happens, if the sensitivity of the test would be better (the probability to get a positive test result given the HIV infection)?

— What happens, if the test is used in another country like Germany (the base rate for HIV is 0.4% in US, but 0.1% in Germany)?

— What happens if the specificity of the test is not as good as the producer of the test indicates (the producer indicates that a person that is not infected gets in 99.98% of the cases a negative test result).

Other questions that integrate context in a broader sense could be: "Why could a confirmatory test be necessary?" or "Should the German administration permit the approval of HIV-rapid-tests for home usage? Which reasons are arguments against it, which do support the approval?"

As the context like disease could be potentially significant for students, we hypothesise that particularly critical questions could improve the students' beliefs referring the relevance of statistics for both society and the students' own life.

First results of the ongoing study

In November 2014, 118 students of health education (age: mean=21.72, sd=2.9) were assessed pre- and post-intervention and after two week follow-up. Their beliefs and attitudes were measured by a questionnaire which comprised e.g. two components of the Survey of Attitudes Towards Statistics (SATS-36©): value and interest of statistics (Schau, 2005). The range of coefficient alpha values for the interest-component varies from .868 to .879 for the different measurement times. The scale shows thereby a high internal consistency. The mathematical and statistical knowledge elements of statistical literacy were assessed by Bayesian-situation-tasks. Students were asked to estimate the right percentages of those people who are infected given a positive test result. As expected, students showed a low performance before the intervention and a high performance after the intervention for these tasks.

DISCUSSION AND CONCLUSION

There seems to be a discrepancy between the enormous relevance of statistics in our society and the poor relevance of statistics often assigned by students and adults. It is on the one side possible that these students (or adults) have little statistical knowledge and, hence, do not appreciate the benefit of statistics for both, society and own life. However, research shows that even students with considerable statistical knowledge assign statistics little or rather no relevance outside school mathematics (Eichler, 2008).

As a consequence to these findings the main aim of our research is to investigate the relation of developing knowledge elements and developing dispositional elements of statistical literacy. For this reason a main challenge of our research is to develop an intervention that potentially improves both elements of statistical literacy, i.e. knowledge elements and dispositional elements. We provided the example of the HIV-test that potentially fulfills three requirements of the intervention that we defined. Firstly the HIV-test represents an authentic context that is controversially discussed in daily newspapers and official statements. Further the context of Bayesian problems is not common for most of the students or adults and, thus, could serve as a new example showing the significance of statistics. Finally, there exist elaborated and empirically proved ways to teach Bayesian thinking that facilitate an intervention that is not linked to a regular statistics course at school. In this paper we provided the tree diagram and the unit square connected with natural and relative frequencies as possible strategy to represent the information in a Bayesian problem appropriately.

To investigate the mentioned connection of the development of knowledge elements and dispositional elements we vary the population in different samples. Since we regard students at school as possible future audience for our intervention, we regard also students at university that show different characteristics that are, from a theoretical perspective, important referring the status of statistical literacy: The first sample consists of prospective mathematics teachers who potentially have a considerable amount of mathematical knowledge as part of statistical literacy. The second sample consists of students of health education who potentially hold context knowledge referring to the HIV-test and who potentially need this knowledge in their professional careers. Students of both samples could either show positive beliefs or negative beliefs about the relevance of mathematics or statistics. However, first results show that it seems to be possible to influence the statistical literacy referring to his dispositional elements in a short term intervention as our investigation referring the students of health education imply. We expect to present further results of the impact of our intervention described in this paper at the conference.

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TWG05

Posters

Understanding the knowledge demands of teaching statistics: Insights gained from examining practice

Aisling Leavy

Mary Immaculate College, University of Limerick, Limerick, Ireland

This research examines the knowledge demands placed on 73 pre-service teachers who are observed as they plan, teach and re-teach data lessons in classrooms. Problems of practice are identified and categorized using the Ball, Thames and Phelps (2008) subdomains of common content knowledge (CCK), specialised content knowledge (SCK), knowledge of content and students (KCS) and knowledge of content and teaching (KCT). The results provide insights into the specific knowledge demands placed on early career teachers when teaching data and statistics and identifies the ways in which these knowledge demands are revealed as pre-service teachers engage in Japanese Lesson Study. The results illustrate that development of understandings in one knowledge subdomain can motivate and impact learning in another subdomain. These interrelationships were found to exist both within and between the domains of content and pedagogical content knowledge.

Keywords: Teacher knowledge, statistics, teacher education, lesson study.

BACKGROUND

The publication presents the research findings on the knowledge demands of teaching elementary school data lessons and illustrates how particular stages of the research process (Lesson Study) provides insights into specific subdomains of knowledge required to teach statistics.

THEORETICAL PERSPECTIVE

This research uses the Ball and colleagues' (2008) practice-based theory of content knowledge for teaching to identify the mathematical knowledge needed for teaching primary level statistics. These insights into the knowledge components of statistics teaching were generated from looking at *teaching in action* in classrooms and are motivated by the belief that the knowledge teachers need to teach well is embedded in practice (what Cochran-Smith and Lytle (1999) term 'knowledge-in-practice'). Hence the study uses the classroom as the unit of analysis. It is through using the mathematics lesson, and its enactment, as the focus of pedagogical and mathematical inquiry, that insights into particular knowledge demands placed on teachers when teaching statistics are revealed.

METHODOLOGY

The use of lesson study (Lewis & Tsuchida, 1998) provides an avenue to reveal knowledge demands as they arise in the context of planning and teaching primary level data handling lessons. This paper reports on the findings from three years of research carried out with 73 Irish pre-service primary teachers , detailed findings can be sourced from the original publications (Leavy, 2015; Leavy, 2010).

SUMMARY FINDINGS

- The research revealed was the complexities of teaching statistics for early career teachers in particular in terms of the wide and varying knowledge demands placed on them.

- The knowledge needed for teaching primary level data needed to be more flexible, robust and interconnected. Evidence of knowledge needs arose in the pedagogical knowledge subdomains of *knowledge of content and students* and *knowledge of content and teaching*.

— Addressing knowledge demands as they arose in one knowledge domain lead to development of understandings in the other. Attention to pedagogical knowledge (in this case *knowledge of content and teaching*) lead to knowledge developments in 'pure' content knowledge (in this case *specialized content knowledge*).

— Tackling the questions and misconceptions posited by the children, and reconsidering teaching activities and representations in light of these questions, precipitated pre-service teachers in really tackling and unpacking (Ma 1999) their own content knowledge understanding of statistical concepts.

— Lesson study as a valuable site for examination of the knowledge demands of teaching statistics.

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Developing statistical literacy: The case of graphs with preservice teachers

Ana Paula Canavarro

University of Évora, Évora & University of Lisbon, Research Unity of Institute of Education, Lisbon, Portugal

The poster identifies the dimensions of graph exploration and analysis of 18 pre-service primary teachers using Tinkerplots. Although some students use the software to explore different graphs and reveal some creativity, the majority seems not to "understand data", nor to "think about data".

Keywords: Statistical literacy, dimensions of graph exploration/analysis, TinkerPlots.

FOCUS AND RELEVANCE

This study focuses on the statistical literacy of pre-service primary teachers, namely in what concerns their use of statistical graphs for interpretation of real life situations. It is our purpose to analyse the graphs they create and use when they are asked to represent and to give meaning to particular situations based on numerical data and to identify the dimensions they reveal in what concerns the exploratory data analysis in the context of the graphing environment TinkerPlots: Dynamic Data Exploration. Statistics technologies offer extended possibilities to statistics education of all persons of the society. It is very important that future teachers learn how to use statistics software not only to draw graphs but also to improve their statistical literacy, in which data and graph sense are essential aspects.

BRIEF THEORETICAL FRAMEWORK

Statistical literacy is a key issue in the orientations of the current mathematics curricula. It comprises the capacity of interpreting, critically evaluate and make judgments concerning statistical information and messages based in data or chance phenomenon (Gal, 2003). Research shows that both students and teachers experience many difficulties with statistical literacy, namely when the situations in study appeal to a more sophisticated use of statistical knowledge, associated with the interpretation and attribution of meaning to statistical information, expressed as numbers or graphs. These difficulties also arise when the situations require assessment and critical judgment of conclusions derived from statistical studies (Shaugnessy, 2007).

The use of software that allows exploratory data analysis can help to further extend the analysis of a given situation. Fitzallen (2013) developed a framework for analysing the way in which the students worked within learning environments like TinkerPlots to create and interpret graphical representations, based on previous studies by Friel, Curcio, and Bright (2001), Moritz (2004), Pfannkuch and Wild (2004), and Shaughnessy (2007) in relation to graphing and graph sense-making. She proposes four interrelated dimensions for analysing the graph exploration of the students when using Tinkerplots: "Generic knowledge", "being creative with data", "understanding data", and "thinking about data" (Fitzallen, 2013).

METHOD

We adopted the design of a case study of pre-service primary teachers. Data were collected in the context of their classes of Didactics of Mathematics, where these students were asked to produce written responses concerning the analysis of different situations involving data exploration and analysis. One example is the study of the weight of backpacks of children. Data analysis was based on Fitzallen's framework.

PRESENTATION OF RESULTS

In the poster we include some written responses of the pre-service teachers, showing the diversity of graphs they produced to represent the situations of the tasks.

CONCLUSIONS AND IMPLICATIONS

Although some pre-service teachers used Tinkerplots to explore different graphs and revealed some creativity, the majority seems not to understand data, nor to think about data. This stresses the importance of including technology in teacher training.

ACKNOWLEDGEMENT

This study is supported by national funds by FCT – Fundação para a Ciência e Tecnologia through the Project Developing Statistical Literacy (contract PTDC/CPE-CED/117933/2010).

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Statistical variability: Comprehension of children in primary school

Erica Cavalcanti and Gilda Guimarães

Universidade Federal de Pernambuco, Education Center, Recife, Brazil, erica_cavalcanti@yahoo.com.br, gilda.lguimaraes@gmail.com

This study examined the understanding of 48 Brazilians students of 2nd and 5th grades (seven and ten years old) of statistical variability of data in bar charts. A Piaget's clinical interview was conducted involving four activities of variability: description or explanation of the variability; representation of variability; prediction results from the variability of the data; comparison between data sets. Students showed ease in recognizing endpoints, but did not make predictions based on what they had observed. The representation of the variability was shown to be an important factor in data interpretation. Make comparison between data sets was complex for most students. Therefore, it is necessary to promote interrelationship among different aspects in order to make students reflect on the data and predictions.

Keywords: Data variability, teaching of primary school, graphs.

CONCEPT OF STATISTICAL VARIABILITY

The variability concept is essential to statistics, since there would be no need to do statistics if the data does not vary. Despite the centrality of the concept, few studies systematically investigated conceptions of students from primary schools, with activities involving different aspects of variability with the same students. According to Garfield and Ben-Zvi (2005), a deep understanding of the concept of variability requires the exploration of its components since the early years, which justifies the need of the present study. Aspects of such components were selected and explored in four activities with students of public schools in Recife - Brazil. These activities were adapted from previous studies of Watson & Kelly (2002), Watson (2009) and Kader & Perry (2007).

The first activity consist of a bar chart about children's arrival at the school and explored the explanation of variability, mode, prediction from the mode and comparison of categories. In the second activity, manipulated cards were offered to the students. These cards represented a number of books read by the children. They were asked to explain variability and make predictions. The third activity was asked to compare quantitative data sets presented in bar graphs and in the fourth activity was asked to compare qualitative data.

A description or explanation of the variability shown in graphic was possible for most individuals in the 5th grade, but not for individuals in the 2nd grade ($t (46) = 3.93$, $p < 0.001$). When the activity discussed representation of data explanation, both groups were able good performance.

The representation of the data variability was also significantly easier for students in the 5th grade ($t (46) = 2.75$, $p = 0.009$). It was noted that only individuals in the 5th grade were able to organize cards by drawing a bar graph.

Although students in the 5th grade had shown a significantly superior performance compared to individuals in the 2nd grade when asked to identify the mode on the graph ($t (46) = 2.75$, $p = 0.009$), at the moment of the prediction from the mode, the performance was as low as it was for the group in the 2nd grade. Regarding the prediction of results in situation of representation of variability (from the highest point), half of students in the 5th grade were able to prove them appropriately, with a significant difference to the 2nd grade ($t (46) = 2.17$, $p = 0.035$).

Finally, there is the aspect related to comparison between data sets. For both groups, these activities were complex, regardless the type of data presented to be qualitative or quantitative. Kader & Perry (2007) argued that the comparison of qualitative data (categorical) would be something intuitive. Thus, a better performance from students in the 5th grade could be expected in the comparison between these sets, but that was not confirmed. Hence, this may suggest that even in the case of qualitative data, comparisons between sets are not as intuitive as they could imagine.

This study has as main contribution the analysis of different aspects related to variability investigated with the same students, showing degrees of difficulties within the aspects observed. We emphasize the need to lead students to establish the relationship between different aspects of variability so that they are able to make prediction results from what they observe.

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Difficulties in learning statistics: An experiment with young children

Susana Colaço

Escola Superior de Educação do Instituto Politécnico de Santarém, Santarém

Universidade de Lisboa, Centro de Investigação Operacional, Lisboa, Portugal, susana.colaco@ese.ipsantarem.pt

The poster presents the main difficulties of 2nd grade students in an experiment which involves tasks of collecting, organization and analysis of data. The results show that the main difficulties of these students are related to representing the data using charts and their interpretation. Using real contexts may facilitate the interpretation of data but may provide the existence of misinterpretations, derived from the students' own experience.

Keywords: Statistics, learning, first grades, charts.

LEARNING STATISTICS IN THE FIRST GRADES

The teaching of statistics in the early years must take into account that the collection of data should be meaningful to the children and a set of decisions must be taken, by the children, to represent the data and to be able to interpret it (Pereira-Mendoza, 1986). Gil & Ben-Zvi (2011) claim that knowing and understanding the real contexts of a statistical task plays an important role in students' performance. Also, Langrall, Nisbet, & Mooney (2006) say that this knowledge is important to engage students in statistical tasks. The same authors, on the other hand, report that in some situations conflicts exist between the knowledge about the real context and statistical data.

Several studies investigated difficulties in graphing and interpreting graphs. Curcio (1987) presented three levels of students' comprehension, namely, reading the data, reading between the data and reading beyond the data. Additionally Friel, Curcio, & Bright (2001) identify critical factors that appear to influence the graphs comprehension. Moreover, these authors mentioned the importance of context in graph comprehension.

METHOD

This poster presents one experiment of collecting, organization and analysis of data performed in a 2nd grade class. The goal was to understand what were the main difficulties of the students during this experiment, from the initial question: Will we weigh more than last year? And understand to what extent the real context may or may not be a facilitator in this process.

The main instruments of data collection consisted of two lessons that involved Collecting, Organizing and Data Analysis (CODA), observation, video recording and documents produced by students.

RESULTS

The experiment, began with the contextualization of the initial question. It was the first time that these students worked with CODA. Several difficulties were identified in these experiments: 1) data collection: recording of current weight rounded to one decimal place; confusion on how to register the data when faced with more than one answer; 2) data organization: were unable to choose a proper scale for the bar graph; 3) data analysis: when discussing the data represented several questions arose – confusion regarding the number of answers when compared with the number of participants.

DISCUSSION

It was found that using a real context with which the participants identified and sometimes resorting to their own experience ended up making the formulation of the problem and the planning of data collection easy to understand. On the other hand, misinterpretations were introduced due to the real context/experiences that were not explicit in the data, for example confusion between height and weight. These "conflicts" between the data and real context are similar to what is described in Langrall and colleagues (2006) and Ben-Zvi, Aridor, Makar and Bakker (2012).

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An obstacle to teaching hypothesis testing to future managers: The plurality of conceptions related to the concept of variable

Roxane Jallet-Cattan[1] *and Corinne Hahn*[2]

1 ESSCA & ESCP Europe

2 ESCP Europe, Hahn@escpeurope.eu

The teaching of hypothesis testing is an important part of business statistics curricula but it is most of the time reduced to its operational dimension. We claim that other historical and systematic dimensions should be better taken into account. We have focused our research on exploring the teaching and learning of hypothesis tests for the difference between two means. In our poster, we present the results of the first stage of our research: we have studied how this test was introduced in business statistics books and how it was integrated into the structure of each book.

Keywords: Hypothesis testing, business statistics, variable

For the past few years in France the teaching of statistics has been reduced in many areas of higher education, particularly in management sciences. It is often replaced by learning to use technological "black boxes", reducing the teaching of statistics to learning a set of techniques and emphasizing what Fabre (2010) calls the operational dimension of knowledge. We claim that it is important to take into account the other two dimensions, historical and systematic, as well.

THE NOTION OF HYPOTHESIS TESTING

We have focused our research on statistical hypothesis testing. We have known for a long time that it is a difficult notion (for an extensive review see Batanero, 2000). Some authors have recommended abandoning it in favor of confidence interval (Cumming & Finch, 2005). Misconceptions by students and professionals or statisticians, have been widely studied (see for example Batanero, 2000, Falk & Greenbaum, 1995, Sotos et al, 2009). It seems to us that most of these misconceptions can be explained by the primacy given to the operational dimension. History tells us that the commonly taught hybrid form incorporates elements of Fisher's theory and elements of Neyman-Pearson's. This dual epistemology, rejected by some elsewhere as impossible to combine, is rarely made explicit and leads to confusion between pvalue and significance level (see Hubbard et al, 2003, Lehman, 1993).

SYSTEMATIC DIMENSION

Beyond this extensively documented error, it seemed that the systematic dimension, i.e. the entry into a structured corpus of knowledge, also posed problems given the different statuses that some concepts can take in descriptive statistics and inferential statistics. Some authors challenge the very notion of the existence of two different areas within statistics (Konold and Pollatsek, 2002). We feel that to not identify descriptive statistics leads to deny the complex epistemology of the discipline. Standard deviation can be viewed either as an indicator of the intensity of noise (inferential statistics) or as an indicator of the amount of information (descriptive statistics). The question is how we link the two conceptions.

THE RESEARCH

We have focused our research on exploring the teaching and learning of hypothesis tests for the difference between two means. In our poster, we present the results of the first stage of our research: we have studied how this test was introduced in 8 business statistics books and how it was integrated into the structure of each book. A central element seems to be the way the author "gives meaning to the letters," in the words of Malisani and Spagnolo (2009), and incorporates the concept of variable in its different forms. To perform the structure analysis, we relied on the typology proposed by these authors and have adapted it to the field of statistics.

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Students' beliefs about statistics: Lessons from a journalism statistics class

Dun Nkhoma Kasoka

University of Malawi, The Polytechnic, Zomba, Malawi, dkasoka@poly.ac.mw

The poster summarises and presents findings, in part, of my classroom study on students' beliefs about statistics. Specifically, the poster presents some statistical evidence of the existence of students' beliefs about statistics and how these changed over the study period towards greater disagreement with the belief that statistics was just a 'subject' forced on students studying for a diploma in journalism. The students became more appreciative of statistics in journalism by the end of the study period.

Keywords: Statistics, beliefs, journalism.

INTRODUCTION

The application of statistics in the society today is common and obvious. In particular journalists use statistics for handling, interpreting data and decision making. Unfortunately, most diploma in journalism students at the Polytechnic seem unaware of the obvious role of statistics in their profession. This can be attributed to several factors such as teaching methods, beliefs and attitudes, modes of assessment and students' background. For students, beliefs and attitudes shape their cognitive developmental processes and their ability to learn statistics (Bartsch, 2006; Ramirez, Schau, & Emmioğlu, 2012; Schau & Emmioğlu, 2012). Griffith, Adams, Gu, Hart and Whitehead (2012) observe that learning and success in statistics are influenced by not only cognitive, demographic and pedagogical factors, but also by "affective and attitudinal factors among students" themselves (p. 46). As argued by Ramirez, Schau and Emmioğlu, positive attitudes help learners use what they have learnt hence journalism students' attitudes and beliefs are significant in their learning of statistic. Furthermore, Kesici and Erdogan (2009), and Torner (2002) suggest that when students hold and use motivational beliefs, their level of success increases. While considerable research has been carried out on students' beliefs in Malawi, the focus has mostly been on primary and secondary school mathematics.

THE STUDY

The study involved 102 diploma in journalism students in three cohorts who took an introductory statistics module. A questionnaire was constructed based on theoretical considerations and previous studies (Bond et al., 2012; Ramirez, Schau, & Emmioğlu, 2012). The questionnaire had 18 items covering students' beliefs about the nature of statistics, teaching and learning methods, students' statistical competency, the role of lecturer, and importance of statistics. All the students completed the 'statistics beliefs' questionnaire twice; at the begging and end of the semester. Component analysis, correlation and reliability tests were performed. Results from the analysis are presented as factors to be retained, loading for each factor, correlation between factors and internal consistency of reliability.

RESULTS AND DISCUSSION

The analysis showed that there were four factors that could be extracted; Lecture and learning statistics, usefulness of statistics, statistical competency, and statistical excellence. Furthermore, the analysis of loading on each item strongly converged on the first and second factors. However, there was an imbalance of items among the factors. This suggests a need for improving the questionnaire. The first factor comprised of seven items that formed a cluster of two components of beliefs about how statistics is learnt and the role of lecturer. The observation is in line with Schau and Emmioğlu (2012) classification of students' attitudes towards statistics. Item weighting showed that the role of the lecturer in the learning processes is seen as very supreme. The students strongly believed that 'drilling' is a very significant.

In conclusion, data analyses suggest that the questionnaire was fairly good in describing students' beliefs from which lessons can be drawn about how to approach statistics classes for journalism students at the Polytechnic. However, more work is needed so that the questionnaire comprehensively accounts for its theoretical underpinnings. Further rigorous analysis and improvement of the questionnaire are required to make it a more reliable and comprehensive instrument.

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Attitudes to statistics and affective expressions in use of graphs developed by primary school teachers

Maria Niedja Martins and Carolina Carvalho

University of Lisbon, Institute of Education, Lisbon, Portugal, marianiedjamartins@campus.ul.pt, cfcarvalho@ie.ulisboa.pt

This study aims to investigate the attitudes towards statistics and affective expressions in use of graphs developed by primary school teachers. The study participants will be 206 primary school teacher of Pernambuco - Brazil. The approach of this investigation is based on a mixed methodology of data collection and analysis, composed of three complementary datasets: questionnaires, attitude scale and interviews. For the quantitative analysis, we will use one-dimensional parametric and non-parametric methods and factorial analysis. To do the analysis of the teachers' interpretation will be used the content analysis.

Keywords: Attitudes towards statistics, affective expressions, primary school teachers.

INTRODUTION

The conceptualization of attitude is investigated by increasingly number of studies in education field (Gleitman, Fridlund, & Reisberg, 2011). For example, in mathematics education and statistics education, studies investigate the close relationship between negative attitudes and low academic performance. According to Gal, Ginsburg, and Shau (1997), the main concerned is to analyse the relationships between affectivity and learning in these two areas from the study of students' attitudes. However, attitude towards statistics scales usually disregard affective expressions that may emerge during the interpretation of statistical data.

Evans (2000) argues that when people interpret graphs, they can express pleasurable and/or painful feelings related to previous experiences of their lives. Authors such as Monteiro and Ainley (2010) suggest that statistical literacy is comprised of these affective elements. Others authors argue that the previous experiences and feelings about the topics related to the data can also be a negative influence during the process of interpretation (Cooper & Dunne, 2000).

The aim of this study is to investigate the relationships between attitudes, statistical knowledge of primary school teacher and their interpretation of graphs. We also intend to investigate which elements might influence the mobilization of teachers' affective expressions during their selection of graphs to work in classrooms.

METHODOLOGY

The study will be conducted with teachers of municipal schools in the Metropolitan Region of Recife - Pernambuco, Brazil, with a sample of 209 teachers. The methodological approach is mixed, with questionnaires, Likert Attitude scale and interviews. The study will be composed of three complementary datasets: questionnaires, attitude scale and interviews. The questionnaire items will be related to individual information's' and graphical knowledge. The attitude scale will have a 4-point Likert format scale, with 10 positive statements and 10 negative, developed and validated by Cazorla and colleagues (1999).

Based on the results, we will select 20 teachers who have more negative attitude towards statistics; and 20 teachers who have more positive attitudes towards statistics. In the third stage, semi-structured individual interviews will be held with those 40 selected teachers. They will discuss data represented by graphs which present data about relevant topics as well as usual and unusual representations published in the media.

For data analysis, we will use SPSS and NVivo. The statistical analyses with SPSS will be useful to identify possible differences and similarities between the situations presented in order to emerge affective expressions from participants' interpretations. The NVivo will be utilised in the analysis of qualitative data from interviews with participants.

ACKNOWLEDGEMENT

Research funded by CAPES from the Brazilian federal government. Process number: 104514-8

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Analysis of teachers' attitudes towards statistics

José Alexandre Martins[1], Assumpta Estrada[2] and Maria Nascimento[3]

1 UDI/IPG, IPG, Guarda, Portugal, jasvm@ipg.pt
2 Universidad de Lleida, Lleida, Spain
3 UTAD, CIDTFF, Vila Real, Portugal

Changes within teaching of statistics in elementary education in Portugal, lead to this research on the attitudes of teachers towards statistics. This study focuses on the measurement and characterization of attitudes towards statistics of Portuguese teachers from the 1st and 2nd cycles of basic education (ages 6 to 11). With this work we hope that paths may be devised to introduce an attitudes' pedagogy and the interventions to prevent negative attitudes and/or correct them, thus positively contributing to the professional development of teachers as well as to students' success in statistics education.

Keywords: Attitudes, statistics, teachers, basic education.

INTRODUCTION

Nowadays statistics is recognized as a key knowledge area with gradually integration in the mathematics curriculum, and in particular in Portuguese basic and secondary school levels. Nevertheless, despite the curriculum guidelines, there are factors that may endanger its implementation. On the other hand, in Portugal, research of attitudes towards statistics has not being done, and there was no research about Portuguese teachers' attitudes towards statistics. So, the main objective in this study was to do a 1st assessment and characterization of Portuguese teachers' attitudes towards statistics, of basic education. The specific objectives are: Studying this attitudes as a global measure, as well as in their components; Determine if there are significant differences between these attitudes in the two cycles; Inquire about the existence of significant relationships between teachers' attitudes and the demographic and school training variables.

METHODOLOGY AND RESULTS

The research method was a mixed study, with a stronger quantitative component, and a concurrent analysis for the countries comparisons. The instrument used is the Estrada's Scale of attitudes towards statistics, EAEE (Estrada, 2002; with a fully English presentation in Martins, Nascimento, & Estrada, 2012). It is an attitudes five-point Likert scale with 25 items that was specially design for teachers and presents good psychometric properties. In a cluster sampling from 3 Portuguese regions, 1135 teachers were surveyed resulting in 1098 valid questionnaires. The sample has a variety of cases relevant for the study. For the quantitative analysis the descriptive statistics were computed as well as one-dimensional parametric and non-parametric methods, multidimensional clusters analysis and factorial analysis. To do the analysis of the teachers' explanations the content analysis was used (Martins et al., 2012). This study confirmed the high internal consistency of the instrument with a Cronbach alpha of 0,869. The multidimensional aspects of the EAEE emerged. Teachers' attitudes towards statistics were positive. This study highlighted in a positive way the cognitive and social components and in a less positive way the behavioural and instrumental components. The comparison with others studies reinforced the admissibility of these results (Estrada, Bazán, & Aparicio, 2010). Attitudes towards statistics of these teachers: were not significantly related with their gender and were significantly related with the cycle of teaching, the teaching experience, their training area, the training in statistics and the teaching of statistics.

CONCLUSION

In conclusion, teachers have a clear conception that statistics is useful and they value its role in the citizens' daily life and they also have a clear sense of the importance of including it in current curricula. At the same time they did not emphasize collaborative work and didn't refer sharing statistics difficulties with other teachers. Outside the school, teachers do not see statistics as a tool in their own daily life and they express a feeling of disbelieve towards the use of statistics and the information in television. As actions to improve attitudes toward statistics we mention the improvement in training (initial and in-service) at several levels and the promotion of the collaborative work amongst teachers. It is also important to guarantee the reinforcement of statistical education in schools, from the early years, as well as in the training of future teachers and of in-service training in statistics for teachers.

ACKNOWLEDGMENT

This work was sponsored by the Project EDU 2013-41141-P (MICIIN, Spain), by FCT/MEC through national funds (PIDDAC) and co-sponsored by FEDER through COMPETE - Programa Operacional Fatores de Competitividade within the PEst-C/CED/UI0194/2013 project and by the PEst-OE/EGE/UI4056/2014 UDI/IPG from UDI/IPG financed by FCT, Portugal.

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The development of informal inferential reasoning via resampling

Jeffrey A. McLean and Helen M. Doerr

Syracuse University, Syracuse, USA, jamcle01@syr.edu

This study focuses on the development of secondary and tertiary students' informal inferential reasoning while engaging in data driven sampling and resampling activities. Through the use of hands-on manipulatives and simulations with technology, the participants will construct empirical sampling distributions in order to investigate the inferences that can be drawn from the data. Data collection and analysis will begin during the fall of 2014. Results and initial analyses will be presented in this poster.

Keywords: Statistics education, resampling, modelling.

RESEARCH TOPIC

A trend in statistics education is the shift from a focus on theoretical distributions and numerical approximations into an emphasis on data (Cobb, 2007). Cobb asserted that many statistics curricula are outdated and based on how statistics could be learned prior to the computing power of modern times. Technology is now capable of collecting many samples nearly instantaneously and this advance in technology should have an impact on the statistics curriculum. New curricula for introductory statistics courses, such as the CATALST (Change Agents for Teaching and Learning Statistics) curriculum at the tertiary level (Garfield, delMas, & Zieffler, 2012), emphasize the ideas of data creation, exploration and simulation with methods of sampling and resampling. This study aims to continue in the direction of such curricula and investigate how students develop their reasoning of resampling methods, the reasoning that develops as students move from using sampling methods to resampling methods, and the reasoning that is revealed and supported by moving from use of hands-on manipulatives to computer simulations during sampling and resampling activities.

THEORETICAL FRAMEWORK

The focus of analysis for this study will be the models of sampling and resampling that the students will create while engaging in a model development sequence (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). This study defines models as "conceptual systems … that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviors of other system(s)" (Lesh & Doerr, 2003, p. 10). Teaching and learning from a modeling approach shifts the focus of an activity from finding an answer to one particular problem to constructing a system of relationships that is generalized and can be extended to other situations (Doerr & English, 2003). Students' mathematical models are useful for research since they provide a means for investigating students' developing knowledge.

DESIGN AND METHODOLOGY

This study is a mixed methods case study with an intervention and a pretest, post-test, and delayed post-test. I will collaborate with two introductory statistics instructors (one each at the secondary and tertiary levels) to create an instructional unit that will consist of two model development sequences. During the unit, I will videotape four focus groups of students, collect student work from all participants, and videotape whole class discussions and presentations. Select students from the focus groups will participate in interviews to discuss their thinking while participating in the instructional unit. I will analyze the pretest, post-test, and delayed post-test quantitatively to investigate changes in students' understandings. The development of participants' reasoning collected from the videos, student work, and interviews will be analyzed with qualitative methods in order to construct the development of models of inferential reasoning used by participants.

POSTER PRESENTATION

The poster will discuss details of the two model development sequences, including how the hands-on manipulatives and simulations with technology were used. Student work and the initial analyses of their models of inferential reasoning will also be presented.

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Core competencies for the professional use of applied statistics in business administration, how to promote them in the classroom?

Vanessa Serrano Molinero and Lucinio González-Sabaté

University Ramon Llull, IQS School of Management, Barcelona, Spain, vanessa.serrano@iqs.edu, lucinio.gonzalez@iqs.edu

The main purpose of this paper consists in determining the competencies for the professional use ofstatistics, highly appreciated to obtain a job in the field of Business Administration. A second objective is to propose a short activity in an Applied statistics class to identify the level of the students' top-rated skills and promote them. Applied statistics is understood as an instrumental subject of statistics taught in numerous degrees that do not seek to train professional statisticians. As a consequence, it may not be separated from the context of its degree.

Keywords: Competencies, applied statistics, business administration.

Batanero (2002) highlighted the important role that statistics carried out in the development of modern society and the need to introduce statistics at school arose. In accordance with the European Space for Higher Education (ESHE) and the market itself, it is necessary to train the future users of this discipline in competencies as well as instructing them in the usual contents. In order to achieve this, it is essential to determine which competencies are the most relevant in the professional environment.

The model proposed by Serrano, Puig & González-Sabaté (2010), consisting of 27 competencies, has been taken as the starting point. In order to simplify this model a validation technique through triangulation (Ghrayeb, Damodaran, & Vohra, 2011) was used together with an adaptation of the Delphi method in two rounds in three groups of experts: Business Administration graduates with professional experience, Statistic's teachers and Human Resources (HR) specialists. According to Astigarraga (2010), the Delphi method is based on the systematic use of intuitive judgment issued by a group of experts. The method would thus consist in asking experts questions by successive questionnaires in order to highlight convergences of opinions and deduce possible agreements. After this, the initial model was simplified to a new one composed by 12 competencies. Later, a final classification was elaborated using the opinions of a new group of 66 HR professionals who were contacted thanks to the IQS Business Alumni. In result, the skill that seems to be most important is how to "Interpret the results", defined as the ability to interpret the results obtained in a study in order to use them in their context. The next two competencies are "Decision-making" and "Critical thinking".

In the learning process based on competencies, activities reflecting the conditions in which they will be put into practice are necessary (Villardón Gallego, 2006). Nevertheless, due to the limited amount of time the subject of statistics usually has at its disposal, a short activity has been proposed for the proper development of the top-rated skill. In this case, the students were invited by the teacher to write a short report that summarized the information contained in a box-plot graph related to the income of a company. A sample of students of the degree in Business Administration carried out this activity during the academic year 2013/14. The results obtained were compared with those proposed by the teacher, thus highlighting the fact that both the students and the teacher develop the skill through the activity. The ideas that arose from the exercise were classified in four typologies: description, calculation, errors and interpretations. Students were informed of the main ideas drawn in order to improve their ability to interpret results in future activities. Most of the students explained the graph in statistical terms but decontextualized from the situation described. However, just a small number of them tried to interpret the results observed within the framework of the statement. Very few students made calculations to give consistency to their explanations. Lastly, only one student identified an error in the graph shown; this is related also with the "Critical thinking" skill. In addition, some errors appeared in students' reports. The teacher concluded that was necessary proposing new activities to improve the skill "Interpret the results".

ACKNOWLEDGEMENT

Fundings: Funded by the project EDU2012-39312-C03-01 of the Spanish Ministry of Economy and Competitivity

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TWG06

Applications and modelling

Introduction to the papers of TWG06: Applications and modelling

Susana Carreira[1], Berta Barquero[2], Gabriele Kaiser[3], Thomas Lingefjard[4] and Geoff Wake[5]

1 University of Algarve, Faro, Portugal, scarrei@ualg.pt
2 University of Barcelona, Barcelona, Spain, bbarquero@ub.ed
3 University of Hamburg, Hamburg, Germany, gabriele.kaiser@uni-hamburg.de
4 University of Gothenburg, Gothenburg, Sweden, Thomas.Lingefjard@gu.se
5 University of Nottingham, Nottingham, United Kingdom, geoffrey.wake@nottingham.ac.uk

THE OVERARCHING THEMES OF TWG06 PRESENTATIONS

The contributions that formed the working basis of the Thematic Working Group on Applications and modelling were characterized by a strong diversity in the topics and issues that were addressed. The group's research field has thus shown to be both active and fruitful whilst maintaining its distinctive facet of being inclusive in terms of the different theoretical, methodological and philosophical perspectives taken by its researchers.

One of the overarching themes illustrates just how important it is to debate and theorize the very notion of mathematical modelling and the concept of mathematical application, namely the question of *perspectives and conceptualisations on modelling* as an important issue addressed by the group. Although it may be useful to emphasise differences and distinguish between modelling problems and mathematical application situations, there are also proposals to blend modelling and applications. The concept of model becomes in both cases a keyword and in general the distinction refers to the need to create a model not available at the outset as opposed to using a model that is previously known. However, there are perspectives for which a common conceptualisation of modelling requires assuming that there are no completely independent mathematical models nor totally disconnected models and that modelling means not only create models, but also integrate and coordinate models somehow contradicting the idea that applications are simply a limited portion of the modelling cycle.

Another emphasis in the discussion refers to areas that might be considered as neighbouring mathematical modelling and potential connections that eventually can be found between mathematical modelling, problem solving, project work and Ethnomathematics. One way to link these different areas can be found in the fact that all these fields are associated to certain types of teaching practices with common characteristics, namely, discovery teaching, inquiry based learning and generally views that put the individual learner actively involved in seeking solutions to real world situations based on mathematics.

One open question stemming from the work that was presented in the group's discussions is whether it will be possible to move towards a comprehensive theory about the teaching and learning of modelling and applications and if such an aim is desirable. Indeed, one other important issue that has much impact in relation to this argument is the growing emphasis on mathematical modelling in the implicit and explicit rhetoric across countries, as portrayed by the most recent frameworks from the OECD's PISA study. Some reluctance towards a pseudo-modelling reflected in certain problems and the need to account for the cyclical nature of the modelling process as a distinctive characteristic, stress the importance of keeping in mind the contrast between mathematical modelling as a pedagogic activity and its professional counterpart.

This topic leads to another overarching theme concerning *the nature of modelling tasks* for school mathematics. The need to converge in a vision on what to consider as good modelling tasks has often been raised. A seemingly relevant item is directly associated with the degree of difficulty of modelling tasks and problems.

Results of research studies indicate that much rests on the students' previous experience, background and pre-knowledge when it comes to modelling tasks; however, it is considered important that modelling tasks reflect authenticity and bring the opportunity to seek, investigate and explore. It is also relatively clear that modelling tasks are still rare in daily school lessons, in part due to the fact that teachers have a restrained attitude towards implementing modelling tasks, sometimes based on the perception that they are too difficult for many students. One of the suggested and analysed difficulties lies on the student's need to picture or imagine a particular real situation, which can be described as creating a situation model. Ideas on the wording of the problems and on how this variable may be of significance were offered and may induce further thinking on the study of task variables in modelling problems.

Modelling contexts and environments was also a developed theme in the working group. On the one hand, the importance of using specially tailored problems for specific areas of knowledge was advocated, as in the case of engineering studies. Emphasis was given to the fact that students often value the specificity of the problems addressed and their relevance to other disciplines in a particular field. On the other hand, it was endorsed the educational wealth of multidisciplinary approaches in allowing integrating tools from different disciplines and subjects, namely mathematical modelling, especially when developed within inquiry teams that are more able to generate questions of real interest.

On the theme *didactic issues in mathematical modelling*, one of the queries was the transposition of the concept of mathematical modelling, as it is usually described in academic literature, to textbooks. This transposition seems to be marked by a poor representation of a holistic view on the modelling process and by leaving aside the critical sense that is inherent to the modelling process. Fragmentation of the model development into a step-by-step model construction tends to prevail and there seems to be a tendency to divide the model into mathematical and non-mathematical parts. Apparently, the widely disseminated design of the modelling process largely supported by studies and research contributions suffers considerable distortions on the route to the textbook and ultimately to the teacher.

Along with this question, the role of mathematical knowledge followed the examination of the differences between experts' and novices' modelling approaches. For some presenters, the differences can be traced by the kind of representations and strategies used by each of them, in close liaison with the mathematical knowledge that they differently dominate. But for others, those differences in fact set the need to clearly understand how mathematical modelling is related to mathematics as a science, and to the many mathematical practices appearing in society outside school. And this implies considering variable features of the modelling activities for the class, namely the design of the activity, its goals, organization, autonomy of the students and adaptation to the local constraints.

One of the more prevalent themes within the participants' contributions – *students' modelling activity* – was investigated in several directions and with different foci. Although a review of this diversity is not easy, it is possible to highlight the importance given to the idea of mathematisation in several papers. For example, this strand appeared with respect to the importance of making assumptions when tackling a mathematical modelling problem. This interpretive flavour of the mathematical modelling activity is also linked to the need for interpretation of the mathematical model back into the original real context. Students' contextual approaches to mathematical notions, concepts and results were widely illustrated at various times, including situations involving experimentation with daily objects and artefacts. To a large extent, the viewpoint of contextual modelling came up strongly represented in a number of studies conducted with students of lower secondary school. In what concerns studies involving upper secondary and tertiary levels, the students' modelling activity was mainly analysed in terms of its relationship to cognitive structures and competing models, formal and non-formal concepts, understanding of scientific concepts and everyday interpretations, and more generally to the mathematical framing of the real world situation.

In the sphere of teacher education and professional development, two themes were discussed: *issues in learning to teach modelling* and *teaching practices and beliefs*. For these themes, investigations have emerged based on different training models, such as the 'lesson study' and the 'study and research path for teacher education'. In both cases, a particular attention has been placed on the planning of the lesson and on the design of the modelling tasks. The planning of the activities to be carried out in classes of mathematical modelling seems to be a central factor in the shaping of teaching and learning and has a great influence on questioning the usual ways of teaching, namely content-centred practices, and the official curriculum. On the other hand, teaching mathematical modelling in school requires the teacher to realise that many of the steps of the modelling process, such as the simplification of the real situation, are not simple matters and cannot be undervalued.

In regard to teachers' beliefs on mathematical modelling, methodological approaches to obtain data in international comparison studies were presented and examined. As for the teachers' practices, it was stressed the strong influence that beliefs have on practices but it was also highlighted the dilemmas that can be quite evident in mathematical modelling lessons. For example, the demands the teacher faces when getting substantially different models from students and also the need to find appropriate ways of dealing with possibly inadequate models that may emerge during the course of a class.

CONCLUDING IDEAS

The conclusions that emerge from the group's work point to three broad categories: various signs of diversity and plurality of views; some elements of uniformity; and major elicited questions.

The diversity

The diversity is easily detectable in the foci and research questions pursued by the group members in their studies. For example, a quick list illustrates this variety of interests and research aims: to deepen and clarify the meaning of modelling and model; to learn more about the design of modelling tasks; to scrutinize the pedagogical resources in the teaching of mathematical modelling; to see how modelling is translated to learning and to competencies development; to analyse forms of teachers' professional development around the implementation of modelling tasks and to better realise the teachers' beliefs about modelling in teaching practices across countries.

There is much diversity in the theoretical frameworks used by researchers in the field: the range covers many theories and perspectives from the more cognitive oriented to the more epistemological/anthropological approaches.

With regard to the school levels represented in empirical studies, there was also some diversity, although it should be noted an absence of studies in primary school.

Finally, it is evident that there is plurality in the visions and arguments of researchers from different countries about the purposes of including modelling in the mathematics curriculum and in regular classroom activities.

The uniformity

In contrast with the above mentioned tendency of diversity there are also several signs of uniformity through the studies presented in TWG06. In terms of methodological approaches, for example, there are predominantly qualitative and small-scale studies. It further appears that the modelling cycle and the modelling phases represent the most common conceptual tools in informing the research. There are also shared concerns related to the contexts of mathematical modelling problems and the authenticity of the tasks. Finally, there are prevailing references towards the assessment of mathematical modelling activity, particularly the external and internal assessment of performance, competence, effectiveness, student's involvement, etc.

Major questions

Given the collection of papers and the discussions developed in the working group, we identified a set of big questions that may help us to put in perspective the future development of the TWG and of this area of research.

Some of these big questions immediately capture the diversity of theoretical perspectives represented in the research of the TWG. This diversity, as already pointed out, ensures an enriched agenda and extends the scope of the research but it also has consequences for impact because of diversity in the likely resulting studies and results. Therefore, the following questions and sub-questions point to the importance of reflecting on this particular factor:

1) How to cope with diversity within the research community and how is such diversity enabling/

disabling the growth of research on mathematical modelling in education?

2) How to induct young researchers into the complexity of the field (theories, terminology, contexts of tasks,...)?

3) What is the influence and the tension between the theoretical approach and the nature of the modelling tasks in mathematics teaching and learning?

— At the level of design/selection;

— At the level of implementation;

— At the level of analysis of empirical data.

To conclude, another question emerges from the awareness of the importance of broader and larger scale studies that may systematize or add confidence to the more localized studies grounded on small experiments or particular cases:

4) How to facilitate the transformation of local practices with mathematical modelling into large scale integration of modelling at all school levels?

To this respect, ideas were suggested in the contributions offered, namely concerning efforts to be invested in the initial and continuing training of teachers, renewed attention to resources to support teachers and strengthening the interaction between researchers and practitioners.

TWG06

Research papers

Mathematization and modelling of physical phenomena: Analysis of two proposals

Jose Benito Búa Ares[1], Maria Teresa Fernández Blanco[2] and Rubén Figueroa Sestelo[2]

1 IES Ramón Cabanillas, Cambados, Spain, bua@edu.xunta.es

2 University of Santiago de Compostela, Santiago de Compostela, Spain, teref.blanco@usc.es, ruben.figueroa@usc.es

The manifold factors involved in mathematical modelling render this activity a complex process. The teaching of mathematical modelling assumes, at least in part, such complexity, though adding some elements inherent to a didactic environment. This paper presents two proposals for the classroom which have led to interesting conclusions, both from the point of view of the construction of mathematical knowledge (in this case focused on the content of the curricular topic of functions) and from the process itself of drafting the models, as well as suggesting some reflections on our own teaching practice.

Keywords: Modelling, functions, GeoGebra, physical phenomena, upper secondary.

INTRODUCTION

In a synthetic way, Blum and Niss (1991, p. 39) characterize a mathematical model as a triple:

> To be a bit more precise, a mathematical model can be viewed as a triple (S, M, R), consisting of some real problem situation S, some collection M of mathematical entities and some relation R by which objects and relations of S are related to objects and relations of M.

Likewise, they make a distinction between mathematization and modelling:

> While mathematization is the process from the real model into mathematics, we use modelling or model building to mean the entire process leading from the original real problem situation to a mathematical model.

Thus, mathematical modelling presents a situation in which the real world (identified with the extra-mathematical) relates to something that is mathematical (identified with the intra-mathematical). Hence, work takes place at the heart of mathematics that is closely connected to the initial extra-mathematical situation or problem, where all such elements are intensely interrelated by means of complex processes.

In line with the degree of importance attributed to each one of the present elements (the processes that take place and the aims intended and assigned to such processes), different approaches and perspectives appear in the way that modelling is taught and learnt (Borromeo, 2006; Kaiser & Sriraman, 2006; Kaiser, Sriraman, Blomhøj et al., 2007). A broad variety of schemes result from here (e.g., Perrenet & Zwaneveld, 2012), which describe the elements present in modelling and the relationships established among them, sometimes distinguishing between which elements are most appropriate for researchers, teachers, and students (see, e.g., Blum & Borromeo, 2009). Such schemes undertake the characteristics of a modelling cycle.

We use the cycle of Blum and Leiss (2007) which is divided in seven steps: 1 Constructing; 2 Simplifying/Structuring; 3 Mathematising; 4 Working mathematically; 5 Interpreting; 6 Validating; and 7 Exposing.

The aim of this paper is to illustrate the complexity of the relations established by the students between the extra-mathematical and the intra-mathematical worlds, and between the mathematical model itself and the mathematical model interpreted in the context of the situation from which the model issued (Blum & Leiss, 2007).

Our work shows an example of modelling that prompts students to obtain data that is subsequently processed with a computer, enabling to obtain the model, which take the form of a function in one variable (see, e.g., Lingefjärd, 2011). Once such a function is obtained, the aim will focus on analysing the students' answers to questions about the interpretation of the mathematical model against the original real context, the relevant concepts, notions and basic knowledge on functions, and about knowledge that is not directly related to functions.

METHODOLOGY AND DESCRIPTION OF THE ACTIVITIES

The two activities we describe below were proposed during two consecutive years (academic years 2010–2011 and 2011–2012) to students of the course Mathematics I in the 1st year of Science and Technology Baccalaureate (aged 16–17 years), as a voluntary practice to be performed after the regular class hours. Participating students of the first year are designated as A1,..., A13, and participating students of the second year as B1,..., B12 (total of 25 students). The activities were performed one after the other, with a two-week interval in between. Students were distributed into working groups G1, G2, etc., (3–4 students per group) and their work was recorded in audio and video.

As a previous step to the proposed activities (two weeks before), the students learnt how to use GeoGebra to obtain the analytical expression of an adjustment function starting by dumping the data from a table as points in a Cartesian plane (adjustment of a function from a set of points on the plane using sliders).

We divide the development of both activities (herein after *Spring* and *Oil and water*) in three different phases: (Phase 1) data collection at the laboratory, (Phase 2) data dumping and obtaining the adjustment function using a computer and (Phase 3) asking questions about the obtained model. At the end, the students answered two questions about their opinions on the activities and the phases into which these had been divided.

In the following, we describe the different phases.

Phase 1. The first phase of the activities was proposed with the following statements:

> *Spring*: "Let's study how a spring stretches when we hang a weight from it. Take all the data you consider necessary, and do it the way you think is best."
>
> *Oil and water*: "Let's study how the diameter of an oil slick on water varies when we add more oil. Take all the data you consider necessary, and do it the way you think is best."

Each group had their own materials to collect the data: weights, weight stand, spring (different in each working group), tub, detergent, oil, 2-ml, 5-ml and 10-ml syringes, rulers, tape measure. They had to decide how many data should be collected and the way to do that.

Phase 2. The second phase was proposed in the following manner in both activities:

> "Dump the data you have obtained at the laboratory into the computer and try to obtain a function by adjusting the data using the GeoGebra program."

Each group was provided with a computer and a photocopy containing the graphs for the fundamental functions: affine, quadratic, inverse proportionality, square root ($f(x)= k \cdot \sqrt{x}, k \in R$), exponential with bases greater than and less than 1, logarithmic with bases greater than and less than 1, sine, cosine and tangent. As in the precedent phase, students did their work in an autonomous and open way. The adequate function for the case of *Spring* is the linear function $f(x)=ax$ or affine function $f(x)=ax+b, a,b \in R^+$, depending on whether only the length of the stretched spring is measured or the full length. In the case of *Oil and water*, the adequate function is $f(x)= k \cdot \sqrt{x}, k \in R^+$.

Phase 3. In the third phase, the questions asked about both models were proposed so that each student would reply individually and in writing, except for the last three questions about the model *Oil and water.* These three questions represented the previous necessary step to apply the model to a hypothetical real situation, i.e. the last phase of modelling the behaviour of oil on water. Below we reproduce the questions asked to students. When analysing their answers, we will only refer to some of them.

Spring

1) What type of function have you obtained? Interpret the result. 2) Which variable is dependent and which is independent in the function? 3) In the function you

have deduced, is there any parameter? If the answer is yes, what does it mean in the experiment you are conducting? 4) How much does the spring stretch with 370 g of weight? 5) Which weight corresponds to a length of 21 cm of the spring? 6) What length of the spring do you obtain from the function if you do not put any weight on the stand? Interpret your result. 7) According to the function you have obtained, is it possible to stretch the spring indefinitely? Interpret this result, though taking into account the specific experiment you have conducted. 8) Try to deduce how to calculate the weight applied if you know the length of the spring. 9) Do you think the function you have obtained describes well the behaviour of a spring to which a weight has been attached? 10) The obtained functions are different. What do you think the reason for this is?

Oil and water

1) The function describing the data is always expressed as $f(x)= k \cdot \sqrt{x}$, k being a constant, x the amount of oil in ml, and $f(x)$ the diameter in mm. k is different depending on each case. In this respect, k varies. Would it be appropriate to call it a 'variable'? Why? Would you use another name? Why do you think k varies in each case studied? 2) One data is the diameter. How would the function change if we had used the radius instead of the diameter? 3) One data is the diameter. How would the function change if we had used the area of oil on water instead of the diameter? 4) What function would we obtain if we represent the diameter on the x-axis and the amount of oil on the y-axis? (Or the area and the amount of oil).

Application of the *Oil and water* model to a hypothetical real situation:

Figure 1: Hypothetical image of an oil spill (its original size has been reduced)

The photograph below is a satellite view of an oil spill off the coast of Cambados.

1) Determine the scale of the photograph. 2) Determine the area of the polluted surface. Use any instruments and knowledge that you consider necessary or suitable. 3) Apply the model you have obtained to determine the amount of fuel in the spill.

In order to determine the scale of the photograph, the students were provided with a photocopy of a nautical chart on a scale of 1:30,000 of the same coastal area, and with material for technical drawing (ruler, compass, set-square, etc.).

ANALYSIS OF STUDENTS' ANSWERS

Here we highlight some general considerations about the development of the activities:

The working environment was relaxed in all the phases, the available time was enough and there were no delays. During the phase of data collection and obtaining the adjustment function all the students participated actively, deciding the distribution of tasks by consensus and assuming the relevance of performing correct measurements and adjustments. The students called the teacher on very few occasions, and most of the questions were related to mere technical issues, which led us to assume that they were committed to perform the work autonomously.

The time employed is one of the obstacles mentioned about the introduction of modelling either in school or university (see, e.g., Blum & Niss, 1991). Regarding this issue, no time limit was fixed to perform the successive phases. The time used by each group to obtain the data table was in no case longer than 18 minutes, in the case of *Spring*. In the case of *Oil and water* it varied to a greater extent, ranging from 30 to 55 minutes. Some groups took longer due to the difficulty of pouring oil on water using a syringe. The time employed for adjusting the function using the computer did not exceed 15 minutes in any group. Most of the time was consumed entering the data from their tables into the computer. The time used to answer the questions posed about the obtained model was approximately the same: 35 minutes in the case of the spring, 40 minutes in the case of the first question and the discussion about *Oil and water*, and 70 minutes in the case of the application of the oil and water model.

Data table and adjustment function

The number of data and the titles of columns each group assigned in their data table varied considerably: in the case of *Spring* the number of data varied from 9 to 32; in the case of *Oil and water* they obtained between 9 and 23 data pairs. The groups referred the measurements they made in the first row of the table as Weight and Length (groups G1, G2, G3 and G6) or *x* and *y* (group G5). Group G4 did not write anything to describe the columns. So, students identified the real-life variables (Weight, Length, etc.) with mathematical variables (*x* and *y*) because they wrote data using the usual representation of a data table of a function. So, we may consider that this phase includes the steps 1 Constructing, 2 Simplifying/Structuring and 3 Mathematising (Blum & Leiss, 2007).

In the case of *Spring*, group G1 (Figure 2) obtained the equation of a straight line determined from two data from their table, and another group (G5) modified the parameters a and b in the equation of a straight line on the plane, $y=ax+b$. The remaining groups performed a similar procedure using the functional expression $f(x)=ax+b$. In the case of *Oil and water* all the groups provided a function of the kind $f(x)= k \cdot \sqrt{x}$ as the solution, with different values for k in each group. So, we could say that, when each group inserted the appropriate function, they had previously decided the functional variables x and y (identified with weight and length and volume and diameter, respectively) and the necessary parameters they had to use. Therefore, this second phase should include the steps of 3 Mathematising and 4 Working mathematically (Blum & Leiss, 2007). Since they had previously identified real-life variables with mathematical variables, the mathematical model and result (the obtained function) should be interpreted as a real model and result (5 Interpreting).

Answers to the questions about the obtained model

63.6% of the students did not identify the variables correctly (question 2 about *Spring*; Figure 3a). Also, 95.5% of the students did not recognise the parameters (question 3 about *Spring*).

In the first question about the model for *Oil and water*, 52.2% of the students characterized k as a variable (Figure 3b) and 34.8% as a constant. 16% of the students initially asserted it was a variable and subsequently contradicted themselves and sustained it was a constant.

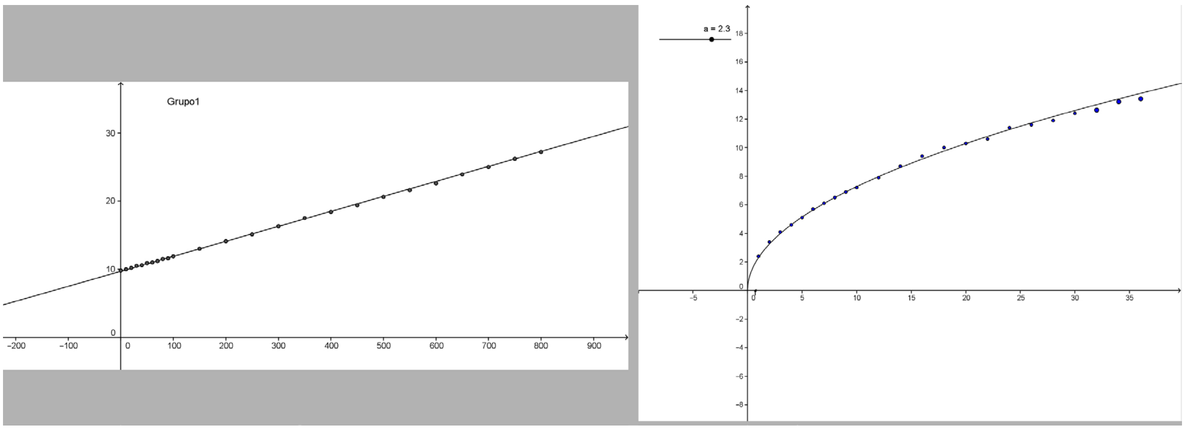
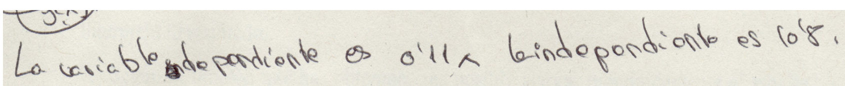
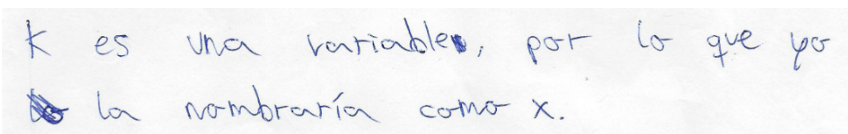


Figure 2: Data and functions obtained by group G1 [y=0.02x+9.7; f(x) = 2.3 √x]



Translation:
Dependent variable is 0'11x the independent one is 10'8.

Figure 3a: Variables and parameters



Translation:
k is a variable, so I would call it x

Figure 3b: Variables and parameters

So, regarding the variables and parameters, students did not identify the variables correctly and made confusion between the dependent and independent variables and the parameters.

In *Spring* they had obtained a function that enabled to obtain the length given the weight (mass); however, surprisingly, in questions 4 and 5 half of the students used the rule of three in one or both questions (Figure 4). Only 27.3% of the students used the function in both questions and another one used a rule of three and the function, without questioning themselves why the obtained results were different. Therefore, they associated the graph of the function (straight line) with direct proportionality, and direct proportionality with the constant rate of change.

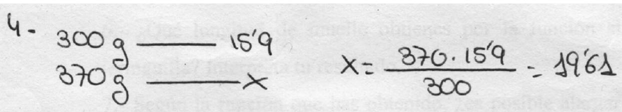


Figure 4: Use of the rule of three

Sometimes, students designated the function as an equation and the variables as unknowns (Fig 5).

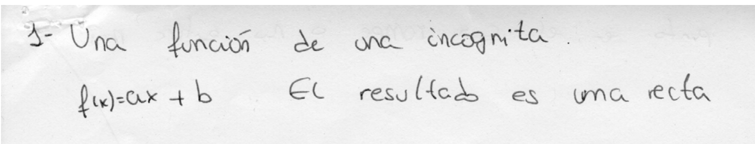
In questions 5 and 8 for *Spring*, the students did not mention the inverse function in their answers, and they performed the calculations with the function expression as if it were an equation which they must *solve* to obtain the unknown. In the discussion about oil, the first-year students almost immediately reached the conclusion that the function requested in the fourth question was the inverse function. One of the second-year students, after a few minutes, provided the way to calculate this function without realising it was the inverse function. Another student (B8) reached the conclusion that it was the algorithm they learnt to obtain the inverse function:

> Student B8: It's the inverse of the function. That is the inverse. When we look at the composition. Change *x* for *y* and solve for *y*.

Regarding the discussion about the adequate expression for the radius, there are differences from one academic year to the other. For example, the first-year students discussed for six minutes whether the adequate expression is $\frac{f(x)}{2}$ and about the consequences of dividing only one member of an equality by a number. The second-year students discussed for 10 minutes whether the expression of the radius should be $\frac{2.1}{2}\cdot\sqrt{x}$, $2.1\cdot\frac{\sqrt{x}}{2}$ or $\frac{2.1\cdot\sqrt{x}}{2}$. Neither the first-year nor the second-year students denoted the new functions they obtained in the discussion about *Oil and water* in a different way, and they always designated the results as *f(x)*.

Regarding the application of the *Oil and water* model to a hypothetical case of a pollution spill, we highlight the following results:

- One fourth of the students determined a value close to the right one for the scale. The remainder either used an inadequate system (55%), or failed to determine the scale (20%).
- Only two students achieved a correct calculation method to estimate the value of the spill area on the photograph. 55% of the students drew a circle on the picture and measured the radius or the diameter, in a clear relation to their observations of the behaviour of oil during the first phase.
- The students had many difficulties in the application of the model: they did not remember basic formulae to calculate areas, they confused the names of geometric figures, they misused the units of area or volume, etc.
- In no case they interpreted both images (map and photograph) as a similarity, so they failed to use the relationship between areas and volumes of similar figures.
- From all the students, only two achieved to provide a volume for the spill following a calculation system that may be considered acceptable.



Translation:
A function of one unknown f(x)=ax+b. The result is a straight line

Figure 5: Function equality as an equation and identifying the expression with its graph

Opinions and valuations of the students

From the analysis of the opinions and valuations by the students, we highlight below the aspects that we consider as the most relevant.

83.3% of students stated that the modelling for *Oil and water* was the one they liked most or found most interesting, often mentioning the discussion as a very positive way to learn from their peers and exchange ideas among themselves (41.7%).

Regarding the phases, the students considered all of them important, with special mention to the data collection at the laboratory (62.5%) because this makes the model becomes *their* model. One third of the students highlighted the application of the *Oil and water* model as interesting and enabling to apply the model to a real, socially problematic situation, and so they valued the model obtained as being useful.

Overall, they considered it very important to perform this kind of activities (that they sometimes referred to as *experiments*), however, at the same time, they thought that this kind of activities had nothing to do with *mathematics teaching*. They acknowledged that the kind of teaching they receive is based on learning algorithms.

In order to illustrate the comments above, we present a sample of the interviews made to the students:

Student B6: (...) I didn't know how to measure an oil slick or spill on water and, for example, that was one of the things I enjoyed most, wasn't it? Because you ask yourself: how is it possible to measure that on water? And then, you even see how the shape is formed and everything.

Student B3: (...) working on a problem is finding the solution and the same goes for equations. But here it's all about finding the data and then you have to find out something else; it's different, these are different steps you have to take with your own observations and collecting data. In class they give them to you, you don't have to do it for yourself.

Student A1: (...) You can't spend all the time doing experiments, you must have class.

Student B3: Well, in maths, when you have class it's basically doing exercises and learning what you have to know. This is something on the side that helps but it has nothing to do with the class, at least for me it has nothing to do with maths, with the class, I mean, because it's something different.

Student A1: (...) Because the problem with maths is that some people see it as something they're never going to use in their lives. A lot of people say: why would I want to know how to solve an equation? Why do I want to know about what a function is? I don't know, if you actually see it in real life then at least it should stir your curiosity.

CONCLUSIONS AND ELEMENTS FOR DISCUSSION

The issue that we consider essential is that the students succeeded in obtaining a model (a function) at the end of the second phase. However, their answers to the questions in subsequent phases lead us to think that the mathematising process (step 3, modelling cycle, Blum & Leiss, 2007) has been performed in a very limited manner. Consequently, it could be questioned whether the obtained function actually represents a mathematical model and result (Blum & Leiss, 2007). However, if we see only the mathematical result obtained by the students, it is possible to think they completed at least four of the seven steps of the modelling cycle. Therefore, when we propose to students to develop a mathematical model in an autonomous and open way, it is necessary to set questions related to important mathematical concepts and notions during the modelling process and not just at the end of it.

Moreover, for this result to become a solution to the originally proposed problem, students should interpret the mathematical result they obtained as a real result (step 5, Blum & Leiss, 2007). In fact, two functions coexist in both modelling processes with different domains and paths (the mathematical function and the function that relates mass to length and volume to diameter).

We must also add to the above the differences between both academic years regarding the development of the discussion as well as the problems that arose in the application of the *Oil and water* model. Modelling introduces some elements of uncertainty about what the teacher may expect to encounter in the classroom, such as the students' answers. The difficulties that arise during the process are not entirely predictable and consequently teaching becomes more open (see, e.g., Blum & Borromeo, 2009). Moreover, both models start from extra-mathematical situations inherently linked to physical laws and magnitudes of certain relevance (difference between mass and weight, Hooke's Law, behaviour of fluids of different densities, Archimedes' Principle, surface tension, etc.). Therefore, a series of questions arise that the teacher has to consider; the answers to those will determine a different kind of modelling and, consequently, a different modelling cycle. To give just a few examples: one could opt for omitting the data collection phase (by considering it to pertain to experimental science) and focus the modelling on such aspects perceived as more appropriate to mathematics, or we could also opt for using the obtained model to introduce some magnitudes and laws of physics; the degree of difficulty of obtaining the adjustment function could be increased by not providing a photocopy of the graphs for the fundamental functions; the square root may be presented on the photocopy as $\sqrt{x}$ or as $k\sqrt{x}$, etc. Likewise, answers to the questions raised to the students could be approached in a joint discussion; or by distributing the students into smaller, independent discussion groups; or individually in writing.

Finally, if we focus on more general objectives, modelling can be proposed as a means to introduce key concepts and notions about functions, or also to detect difficulties and obstacles in students who have already studied such content previously.

Therefore, modelling introduces some elements that force teachers to make previously meditated and reflected decisions about their teaching methods (see, e.g., Doerr, 2006). Such decisions will affect the kind of modelling proposed, the ways in which it is developed, the answers of students and the steps of the modelling process.

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At the core of modelling: Connecting, coordinating and integrating models

Jonas Bergman Ärlebäck[1] and Helen M. Doerr[2]

1 Linköping University, Department of Mathematics, Linköping, Sweden, jonas.bergman.arleback@liu.se

2 Syracuse University, Department of Mathematics, Syracuse, USA

This theoretical paper introduces the notions of connecting, coordinating and integrating models to analyse and reflect on how models are created and developed. We define, discuss and apply these constructs to some theoretical perspectives in the present modelling discourse. We draw on an example from a model application activity within a model development sequence to illustrate these constructs. Our hope is to spark a discussion that will enhance our understanding about the nature of mathematical modelling and the teaching and learning of, and through, modelling.

Keywords: Modelling, theory, model development sequences, variation theory.

There are many different perspectives on modelling that researchers can adapt when studying the teaching and learning of mathematics (Kaiser & Sriraman, 2006). One commonly made distinction in the literature is the one between modelling and application. Often this characterisation is based on a philosophical stance about the nature and relation between 'mathematical knowledge' (mathematics) and 'knowledge about the rest of the experienced world' (reality) (Blum, Galbraith, & Niss, 2007). Stillman (2012, p. 903) describes this distinction as follows:

> With applications the direction (mathematics → reality) is the focus. "Where can I use this particular piece of mathematical knowledge?" The model is already learnt and built. With mathematical modelling the reverse direction (reality→mathematics) becomes the focus. "Where can I find some mathematics to help me with this problem?" The model has to be built through idealising, specifying and mathematising the real world situation. Both types of task have their place in school classrooms.

Research on modelling perceived in the latter characterisation often uses, or is based on, an idealized conceptualization of modelling as a cyclic process (e.g., Blum & Leiβ, 2007; Borromeo Ferri, 2006).

The role and function of applications and modelling in the teaching and learning of mathematics is also an important dimension in the on-going discussion – is teaching modelling a goal in itself, or is modelling a vehicle for teaching and learning mathematics? Both modelling as described by Stillman as well as most modelling perspectives represented by a cycle diagram of the modelling process include, or at least point out, the potential driver in modelling for teaching and learning mathematics.

Another perspective on modelling is the models and modelling perspective (Lesh & Doerr, 2003b), which provides a coherent framework to think about multitude aspects involved in teaching and learning. Central in the models and modelling perspective (MMP) are the students' previous experiences and knowledge, and how contexts are chosen and used in modelling tasks. Whereas perspectives aligning with the cyclic view on modelling (c.f. Blum & Leiβ (2007) and others) clearly are modelling according to Stillman's distinction, it is not in our view possible to situate the MMP in this dichotomy. Rather, MMP blends applications and modelling to form a modelling-based pedagogy, and through this use of modelling and applications students arguably learn both mathematics and mathematical modelling.

Although our overall research interest aims to better understand the teaching of mathematical modelling, and the teaching of mathematics through modelling, in this paper we ask more fundamental questions about the nature of modelling. We also wish to initiate a discussion about the nature of mathematical modelling, as well as teaching and learning mathematics through modelling, by examining the constructs of connecting, coordinating, and integrating models.

Before addressing the notions of connecting, coordinating and integrating models, we will discuss some of the central ideas in the models and modelling perspective. Our initial thinking is based on our work within this perspective, and the examples used for illustrational purposes are from this context.

THEORETICAL FRAMEWORK

In the models and modelling perspective, "*[m]odels* are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviours of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently. A *mathematical* model focuses on structural characteristics ... of the relevant systems" (Lesh & Doerr, 2003a, p. 10, italics in original). It is by engaging in learning activities that students' models are developed, modified, extended and revised through "multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and reconstructed by the learner" (Doerr & English, 2003, p. 112).

A well-established line of research within this perspective has focused on model eliciting activities (MEAs) in multiple contexts with learners from primary school through university (see references in Ärlebäck, Doerr, and O'Neil (2013)). MEAs are activities where students are confronted with a problem situation in which they need to construct a model in order to make sense of the situation. There are six well-established principles for designing MEAs. The six design principles are: the reality (or sense-making) principle; the construction principle; the self-evaluation principle; the documentation principle; the simple prototype; the generalization principle (Lesh & Doerr, 2003a; Lesh, Hoover, Hole, Kelly, & Post, 2000). However, isolated MEAs can fall short of supporting students developing a generalized model that can be used and re-used in a range of contexts (Doerr & English, 2003). What is needed are multiple structurally related modelling activities offering multiple opportunities for the students to explore, apply and test relevant mathematical constructs in different situations and contexts. This is the idea and function of model development sequences (Doerr & English, 2003; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003).

Model development sequences begin with a MEA to confront the student with the need to construct a model to make sense of a problem situation. The MEA is then followed by one or more model exploration activities and model application activities (see Figure 1). Model exploration activities (MXA) focus on the underlying structure of the elicited model in the MEA with special attention to the use and function of different ways to represent the elicited model. The initially elicited model is further developed by examining the strengths of various representations and ways of using representations productively. Model application activities (MAA) engage students in applying their model to new situations and contexts, thereby refining their language for interpreting and describing the context.

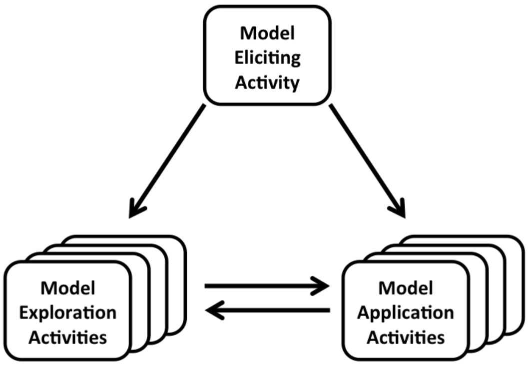

Figure 1: The general structure of a model development sequence

When students work through the model development sequence, they engage in multiple cycles of descriptions, interpretations, conjectures and explanations, resulting in iteratively refining and developing their models. In this process, interacting with other students and participating in teacher-led class discussions are key practices for facilitating this development.

A model development sequence focusing on the average rate of change

We now turn to briefly describe a model development sequence focusing on average rate of change consisting of one MEA, two MXAs, and two MAAs (see Figure 2). For a more detailed description see Ärlebäck, Doerr and O'Neil (2013). From this point an onwards, references to the particular activities in this sequence will be indicated with an asterisk (*). In the MEA* of this sequence, students analysed their own bodily motion along a straight line. They experimented with motion detectors attached to graphing calculators generating position vs. time graphs, created linear graphs based on written specifications, replicated the motion behind given position vs. time graphs, and generated written descriptions of how they moved. Within this context, students' initial thinking and models about function values (position), average rate of change (average velocity), sequences of differing values of average rate of change (sequences of differing average velocities) and the relationship between these quantitates were elicited.

In the two MXA*s the students explored various representations to describe and interpret changing phenomena using their emerging model of average rate of change. Using two different computer environments, the students generated motion of animated characters by creating velocity vs. time graphs, created position graphs from velocity information, investigated how the average rate of change of a function is represented as table values, graphs, and equations, and explored representations of exponential growth and decay.

In the two MAA*s that followed, the students used their models to make explicit interpretations, descriptions and predictions about the behaviour of, first, the intensity of light with respect to the distance from a light source, and second, the voltage drop over a fully charged discharging capacitor in an simple resistor-capacitor circuit. The MAA*s gave the students opportunities to use their models in different contexts (one having distance rather than time as the independent variable), and to work with phenomena with negative rates of change.

A model application activity – the light lab

In this section, we will briefly describe the Light Lab model application activity (MAA). The overall structure of the six tasks in this MAA is shown in Figure 2.

The first pre-lab task focused on the students' intuitive and initial models about how intensity varies depending on the distance from a point light source. In a one-dimensional scenario of an approaching car the students sketched graphs of how the intensity of the car's headlights varied depending on the distance to the car, and described how light disperses from a point source in terms of light rays.

In the second task, students used a point source of light to collect 15 measurements of light intensity data at one cm intervals from the source. In part one of the lab, students made scatter plots of their data and wrote descriptions of how the intensity of the light changed with respect to distance from the light source, and compared this relationship to their predictions from the first pre-lab. The students also calculated, described and plotted the average rate of change of the data in one cm intervals, and created rate graphs of the calculated average rates of change.

The second pre-lab introduced an inverse square model for how the light intensity varies with distance from a light source. Using four images representing light intensity indicated by number of dots per square inch at different distances, students determined the intensity at given distances from the light source.

In the second part of the lab, students determined a function fitting their collected data, explained their work, and analyzed the average rates of change of their function using the difference quotient; they calculated and graphed the average rates of change of the function, and described and interpreted the graphs of the average rates of change values. We will use these tasks from within this model application activity to examine our notions of connecting, coordinating and integrating models.

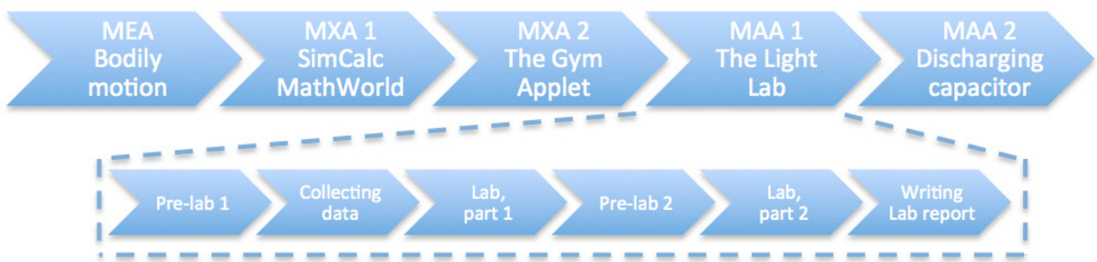

Figure 2: The model development sequence (top) and the six Light Lab tasks (bottom)

CONNECTING, COORDINATING AND INTEGRATING MODELS

We take *connecting models* to mean the establishing of a relationship between two or more previously unrelated models. As used here, connecting models captures the realization that previously isolated and unattached models, potentially from different disciplines or subject matters, partially overlap or have points of interest in common that in a given problem solving situation seem productive to explore. Connecting models also means taking the initial steps in identifying and delimiting the models being connected in the first place. Connecting models can thus be thought of as a united set of models together with some initial ideas and rationale for why these models might productively be considered together (see Figure 3). The reasons and rationale for connecting the models can be intuitive, tentative or speculative in nature, and to a large extent will depend on the modeller's previous experiences and knowledge. An example of an activity connecting models is the first pre-lab in the Light lab MAA*. Here, students' intuitive and initial models about how intensity varies depending on the distance from a point source of light in terms of previous experiences of car's headlights and how light disperses from a point source in terms of light rays (knowledge from previous courses in physics) are elicited. The juxtaposition of these questions implicitly suggests that it might be productive to consider these conceptual systems (or models) together. Another example of an activity connecting models from the Light lab MAA* comes from the second pre-lab which introduces a new representation (density of dots at varying distances) to be examined and considered along with the set of models the students are currently working with.

When a set of models has been connected, the *coordinating of the models* is the successively more systematic exploration and pursuit of the overlaps and points of interest in common across the models. The coordination of the models transforms the set containing the previously disjoint models into a set of more aligned models. This is a process entailing gradually mapping out where, and becoming clearer about how, the models partially overlap. The goal for moving beyond having a connected set of models and striving towards a coordinated sets of models is to facilitate the latter set of coordinated models as a whole to function in a more intertwined and concerted way when put into action and applied to the problem solving situation at hand. This is achieved by creating one or multiple bonds between the models, and successively making these clearer and stronger (see Figure 3). The coordinating of models is a multi-dialectic process where the specifics of all of the models in the set are important. The process of coordination results in the set of models becoming more aligned or (perhaps) more alienated. We note that the process of iterative refinement in a models and modelling perspective includes the sorting out and filtering of ideas, where some constructs (or models) are rejected as others become more aligned and subsequently integrated. A consequence of an unsuccessful coordination can give rise to the possible exclusion of the alienated model(s) or the rejection of the whole set altogether. An example of an activity facilitating students to coordinate models is the first part of the Light lab MAA* (Lab, part 1). Here the students are urged to keep thinking about their initial ideas and models elicited in the first pre-lab alongside representations (scatter plots, descriptions, rate graphs) of their collected real data and calculated values of the average rate of change over one centimeter intervals.

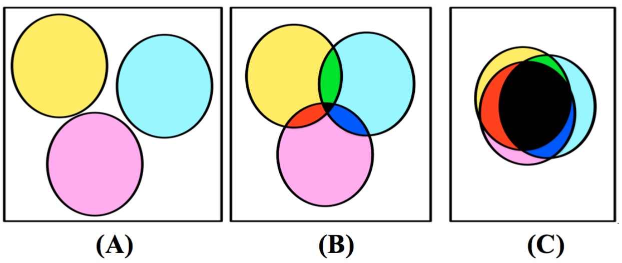

Figure 3: A conceptualization of connecting (A), coordinating (B), and integrating (C) models

Integrating models means merging the models so that the set now is conceived as one model in its own right (see Figure 3). Coordinated models are said to be *integrated* when the level of coordination is so high that further coordinating of the models in principle does not change the understanding and function of the newly merged models. A central feature of an integrated model is that it is self-contained. In the Light Lab MAA*, the students write a final lab report. The goal for the student is to present an integrated model of how intensity varies depending on the distance from a point light source. The purpose of the report is to reflect the students' integrated understanding of light dispersion on a qualitative and quantitative level using their collected data, average rate of change, rate of change graphs, the difference quotient, and a spherical light emission model.

Having introduced and exemplified the notions of connecting, coordinating and integrating models, we now turn to discuss how these constructs might be used and applied to current discussions about the nature of mathematical modelling, as well as the teaching of, and through, modelling.

MEAs, MXAs, and MAAs

Generally within a model development sequence, an MEA elicits the students' initial thinking about a problem situation, as that thinking is externally represented. This means that the students start to structure the problem solving situation by trying to identify and connect models that can work productively in the specific situation. An example of an MEA functioning to connect models is the MEA* where the students are working with motion detectors to make sense of bodily motion along a straight path. In this activity, the students are placed in a situation that exposes them to multiple representations through which to make sense of the situation. Centred on their own bodily motion, students are offered an opportunity to form and connect an initial set of models, consisting of their previous ideas and models together with the representations introduced in the MEA*, which then can be furthered explored and applied.

Generally within a model development sequence, the MXAs focus on supporting the students in developing the models elicited in the MEA by examining different representations: graphs, symbols and algebraic representations, tables, everyday language, manipulatives, embodied and animated motions as well as students' self-constructed representations. Regarding representations as models in their own right (Lesh, Post, & Behr, 1987), MXAs can then primarily be seen to be about the coordination of models with occasional elements of connecting models if new representations or models are introduced during the activity. Typically, this is done by either exploring communalities of the models connected in the MEA, but the emphasis of a MXA is really on connecting, coordinating and using representations. The two MXA*s in the model development sequence on average rate of change are good illustrations of this. In these two activities students use computer programs that provide access to digital environments and work on tasks that explore different representations of key ideas in their own right as well as the relationships among these.

The main purpose of MAAs is to provide students with new contexts and situations where they can apply their developing or previously developed models. However, to do this some sort of connection has to be established mapping the models of the students to the context of the problem situation at hand. In other words, the context of the problem situation and the models of the students first have to be connected. Then, in order to ensure the adequacy, legitimacy and the proficiency of this connection, it has to be coordinated, integrating the specifics of the new situation and context with the model being applied.

Applications, modelling, and the modelling cycle

Returning to Stillman's (2012) distinction between applications and modelling, it is now possible to argue analogously that connecting, coordinating and integrating models are necessary and fundamental processes both for applying models "already built and learned" (p. 903) and when "model has to be built through idealising, specifying and mathematising the real world situation (p. 903). The important difference between applications and modelling is not the point of departure (the mathematics vs. the real world) per se; rather, the difference manifests itself in the amount of effort and time spent on coordinating the context of problem situation and the models of the modeller before one ends up with an integrated model for adequately make sense of, and use for, the situation.

Regarding modelling ideally conceived as a cyclic process (e.g., Blum & Leiβ, 2007, and others), connecting, coordinating and integrating models provides a more dialectic and dynamic conceptualisation of the processes involved. For example, each of the transitions in cyclic perspectives on modelling, such as the modelling cycle used by Blum and Leiβ (2007) and Borromeo Ferri (2006), e.g., transitions between *real situation – mental representations of the situation – real model – mathematical model*, have to be subjected to sequential connecting, coordinating and finally developed into an integrated model to tackle the prevailing situation. In addition, the different transitions in terms of connecting, coordinating and integration should not be thought of as carried out sequentially, but rather as processes fundamentally evolving simultaneously, nested and organically. From this perspective, the typically one-way pointing arrows indicating the transitions in cyclic conceptualizations of modelling (e.g., Blum & Leiβ, 2007, and others) are misleading when trying to understand the complexity of, first, the modelling process in its own right, and second, and more importantly, the teaching and learning of modelling as well as teaching and learning mathematics through modelling.

THE SELF-SIMILAR NATURE OF TEACHING MATHEMATICS THROUGH MODELLING

Lesh and Doerr (2003b) discuss the connection between MMP and complexity theoretical ideas. One foundational idea of complexity theory is the principle of self-similarity, that structure and pattern repeat itself at different scales (Davis & Simmt, 2006). We wish to argue that this is also the case for the structure of model development sequences and the activities therein (at least for the MXAs and MAAs). The structure of the model development sequence at large and the activities that constitutes the sequence naturally varies from design to design, but using the notions of connecting, coordinating and integrating, the self-similarity of the two levels starts to stand out. In this paper we have tried to illustrate this point by providing examples from the model development sequence on average rate of change and the tasks in the Light Lab MAA*. We have argued that the two pre-labs in the Light Lab MAA* are activities connecting models, a characteristic of an MEA. In other words, Pre-lab 1, focusing on the students' initial ideas in the Light Lab MAA* (see Figure 3), has the function of an MEA. The elicited models and ideas are then subsequently coordinated in part one and part two of the lab under the support of an activity of data collection and a connecting activity (Pre-lab 2). However, the two parts of the lab (Lab, part 1&2) both function as MXAs with the primary goal of supporting the students in developing their model based on work on representations. The last task of the Lab, when students write a report, really aims at pulling the whole Light Lab MAA* together in an integrating sense. Looking at any MXA in a model development sequence, a similar argument applies: models need to be connected to set up the activity before the coordinating of the models can start. This self-similar aspect of model development sequences and the activities within occurs because when teaching mathematics through modelling, it is a necessity (by definition!) to bring in extra-mathematical contexts and situations.

DISCUSSION

One of the main points we have tried to emphasis in this paper is that regardless what theoretical stance on modelling is adapted, the three processes of connecting models, coordinating models, and integrating models are fundamental in all types of modelling situations. In our opinion, these three constructs capture and acknowledge the dialectic and complex nature of creating and developing models. Constructs like this might be the first steps towards a common conceptualization of modelling that bridges the research field so that we might better understand, coordinate and summarize research findings from different research traditions based on different perspectives.

By applying the three notions of connecting, coordinating and integrating models to the model development sequences, MXAs and MAA respectively, the self-similarity between the sequence and the MXAs and MAAs within the sequence came to the fore. The implication of this structural finding for teaching and the design of tasks might be profitably considered in future research.

While thinking about how models really develop and are formed in terms of connecting, coordinating and integrating models, we also found strong similarities to the principles of variation theory. The four types of variation within variation theory discussed by Marton, Runesson and Tsui (2004), *contrast*, *generalization*, *separation*, and *fusion* especially seems to resonate with the notions put forward in this paper: connecting models – contrast variation; coordinating models – generalization and variation of separation; integrating – fusion. To further investigate this, and what more variation theory has to offer, seems a promising way to continue the research initiated in this paper.

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A study and research path on mathematical modelling for teacher education

Berta Barquero[1], Marianna Bosch[2] and Avenilde Romo[3]

1 University of Barcelona, Faculty of Education, Barcelona, Spain, bbarquero@ub.edu

2 University Ramon Llull, Barcelona, Spain

3 CICATA-IPN, Ciudad de México, Mexico

Considering the general problem of integrating mathematical modelling into current educational systems, this paper focuses on the ineluctable step of the professional development of teachers. Within the framework of the Anthropological Theory of the Didactic, the use of study and research paths for teacher education (SRP-TE) was recently proposed as a means to combine a constant practical and theoretical questioning of mathematical modelling school activities. After presenting the rationale of our proposal, we will illustrate the phases of the SRP-TE design and some preliminary results with the implementation of an on-line course for in-service secondary school teachers in CICATA-IPN (Mexico).

Keywords: Mathematical modelling, anthropological theory of the didactic, study and research paths, teacher education, on-line course.

INTRODUCTION

There exists an extended agreement about the necessity to foster the teaching of mathematics as a modelling tool and to enrich the study of contents at school through the development of inquiry processes. At the same time, many investigations highlight important objective difficulties that hinder any proposal of implementing modelling and inquiry as normalised activities in current educational systems (Burkhardt, 2008; Kaiser & Maaß, 2007; Doerr, 2007; among others). Many of these constraints are related to what has recently been called the 'monumentalistic' paradigm (Chevallard, 2012), which rules in many of our teaching systems, where mathematical contents tend to appear as 'works to visit' more than tools to provide answers to questions. To approach this far-reaching problem and help move towards the new paradigm of 'questioning the world', recent research carried out in the framework of the Anthropological Theory of the Didactic (ATD) proposes a new teaching device called *study and research paths* (SRP) based on the long-term inquiry of generating problematic questions (Barquero et al., 2008, 2013). However, designing and locally implementing new devices is not enough to ensure their long-term viability. Among many challenges, an important one is related to teachers' professional knowledge and competences, and furthermore to the mathematical and didactic infrastructures that need to be at their disposal to face this change.

The research project we are presenting starts from some particular cases of SRP that have been designed, locally implemented and analysed in previous research at preschool, primary, secondary and tertiary educational levels. Our purpose is to explore how these SRP could be used in professional development programmes for teachers. The aim is twofold: on the one hand, to identify teachers' professional problems in their day-to-day activities, and establish possible ways of approaching them; on the other hand, to enrich teachers' mathematical and didactic experiences with inquiry and modelling processes which will later be used as a source to introduce the didactic tools for the analysis and the questioning of any kind of teaching and learning processes. In this paper we will present the framework, principles and phases for the design of what we call SRP for Teachers Education (SRP-TE), together with some preliminary results of one of its first implementations.

STUDY AND RESEARCH PATHS FOR TEACHER EDUCATION

Previous research on the ATD concerning the problem of teacher education (Cirade, 2006) has made the following contributions (Bosch & Gascón, 2009). First of all, it is assumed that teacher education programmes should consider the difficulties and quandaries affecting the development of the professional activities of teachers and locate them at the core of the educational proposals. It is clear that the different tasks teachers should carry out are more or less specific to the content to be taught. However, an empirical study on more than 7000 questions posed by in-service teacher-students (Cirade, 2006) shows that many of these questions have an essential mathematical dimension. In other words, many of the problems teachers face are related to mathematics and, particularly, to the didactic transposition process and the specific mathematical knowledge to be taught (Chevallard, 1985). This issue has a clear connection with research on Pedagogical Mathematical Content (Ball & Bass, 2000), which we will not take into consideration here.

Secondly, starting from teachers' professional problems may help introduce didactic knowledge as a tool to approach them in a motivated way. Didactics of mathematics thus appears as a tool to solve problems instead of a set of (more or less dogmatic) theoretical developments to be known. At the same time, didactic knowledge should also appear as a tool to pose new problems and particularly to question the prevailing teaching proposals, including the curriculum and the pedagogical organisations.

Last but not least, it is important to note that many of the problems teachers face are still open problems for the mathematics education research community. Said problems need an important process of reformulation before they can be approached from a founded perspective. In fact, the more general question of what kind of knowledge has to be made available to teachers and how to help teachers develop it still is (and will always, to a certain extend, remain) an open question.

In order to take these considerations in teachers' development programmes into account, we propose to implement what we call 'study and research paths for teachers' education' (SRP-TE) as a way to provide teachers with pertinent (theoretical and practical) tools to nourish and sustain their professional development. They consist of the following five stages:

1) The starting point is an open question that comes from the teaching profession itself and is related to a given piece of knowledge to be taught: how to teach proportionality, algebra, integers, linear regression, etc.? This question is initially approached searching information and documentation available, including results from research, official curriculum guidelines and innovation proposals.

2) The second stage consists in presenting a *study and research path* (SRP) similar to what could exist in an ordinary classroom and is related to the professional question approached. The SRP can actually have been implemented in previous investigations or may simply have been designed by researchers for this purpose. Teacher-students have to follow the SRP as if they were students, under the supervision of educators.

3) The third stage is devoted to the analysis of the teaching process just followed. Three main phases are distinguished: (a) the *mathematical analysis* of the work done, including the elaboration of a reference epistemological model describing the modelling process involved (Bosch & Gascón, 2006); (b) a didactic analysis of the process, including a description of the differences between the contract established during the SRP to manage the modelling process, compared to the usual school didactic contract centred on the transmission of contents; (c) a more general study of the viability of SRP, including the identification of the institutional conditions and constraints affecting the development of modelling practices in school settings.

4) The fourth stage consists in *designing a SRP* based on the one previously followed and analysed, adapted to a given group of students. This design should be based on the analyses of the previous stage: sequence of mathematical questions to be posed to the students; sharing of responsibilities between teacher and students to pursue the questions; teaching devices to ensure the viability of the SRP.

5) The final stage of the SRP-TE, if possible, corresponds to the *implementation and a posteriori analysis* of the SRP designed. The same didactic tools made available at stages 3 and 4 are again supposed to play an important role: not only to provide some provisional answers to the question that was at the origin of the whole process ('How to teach ...?'), but also as a means to anal-

yse other possible alternative answers, as those found in stage 1.

The hypothesis of our research is that SRP-TE may contribute to the considerations previously presented in the following sense:

— *A tool to question mathematical contents to be taught.* The carrying out of a SRP (stage 2) provides a specific form of epistemological analysis of the content at stake (what we call a *reference epistemological model*) that helps approach the problematic mathematical dimension of the problem: what 'modelling' is, how the inquiry process can be described in terms of sequences of questions approached instead of contents used, how this sequence provides possible rationales to the contents at stake, etc. This reference epistemological model is a crucial tool to get rid of the transparency of school mathematical contents and to start questioning it.

— *Release teachers from the usual way of doing and teaching mathematics at school.* The mathematical activity developed in stage 2 is clearly different from (even if partially compatible with) current school activities and does not assume all the constraints of traditional teaching. This raises new questions during stages 3 and 4 to describe the mathematical and didactic activities carried out and to adapt them to real school settings. SRP-TE thus appears to be a good tool for detecting the institutional constraints hindering inquiry and modelling activities at school.

— *A fair contract between teachers and teacher educators.* Since in a SRP the teacher assumes the role of supervisor, there is no problem if the professional question taken as the starting point of the SRP-TE is an open question in research: teacher educators are not supposed to provide definitive answers (which do not exist) but help student-teachers approach the question by critically accessing the materials available.

Our research project wishes to explore to what extent these hypotheses can be confirmed and what changes or adaptations are suggested, using different implementations of SRP-TE as empirical basis. We are here presenting a single case of a SRP-TE for in-service secondary school mathematical teachers.

A SRP-TE ON SALES FORECASTING FOR IN-SERVICE TEACHERS

In Autumn 2013, a SRP-TE was experimented in an on-line course for in-service secondary school teachers coordinated by the CICATA-IPN centre (Legaria, Mexico) as part of a postgraduate programme in Mathematics Education. The course was led by a team of six teachers, three from CICATA-IPN and three from Spain, all of them researchers in mathematics education. The authors of this paper were all part of the team. In this case, the SRP-TE took the problem of teaching mathematical modelling at secondary school as the initial question. It was initially formulated as follows:

Q_0: How to analyse, adapt, develop and integrate a learning process related to mathematical modelling in our teaching practice? How to institutionally sustain a long-term learning processes based on modelling? What difficulties should be overcome? What teaching tools are needed? What new questions arise?

For four weeks, these issues were approached through a SRP on sales forecasting, considering four activities corresponding to the last four SRP-TE stages introduced in the previous section. There were 15 participants, all of them in-service secondary school teachers. They were supposed to spend 80 hours on the SRP-TE for five weeks: one week for each activity and one week for the final report.

The SRP on sales forecasts that was at the basis of the SRP-TE had previously been designed and implemented at university level and also at upper-secondary level (Serrano et al., 2010). In other words, we took an already experimented SRP, with a previous mathematical and didactic a priori design and some material concerning its implementation and a posteriori analysis. Students were informed of it and were invited to review some published works in the third phase of the SRP-TE. More concretely, the first activity (*Activity 1*) proposed the *Resolution and analysis of 'Forecast sales of Desigual'* with the main aim of letting participants experiment a SRP similar to the one experimented. Participants were asked to 'live' it like mathematical learners or apprentices. They had to act like a team of mathematical consultants and had to provide an answer to a request from Desigual (a Spanish fashion brand), which wanted to have an in-depth study on 'how to predict the evolution of several variables (see Figure 1): weekly sales in several of their shops, evolution of their benefits or of new national and international shop openings, etc.'

Participants were organised in five teams of three consultants each, combining individual work with group work (using the on-line forums of the CICATA virtual campus and Skype). They first had to act individually and propose their own answer to the question (phase 1). They later had to share and contrast their proposals with their partners (phase 2). Finally, in phase 3, they were asked to prepare and present a final report together, providing some answers to Desigual's request and defending it as the best proposal for the project. The final answer had to be accompanied by an analysis of the process followed by the team, including the difficulties encountered.

In *Activity 2*, the participants were asked to prepare a 'lesson plan' based on the mathematical work previously carried out in *Activity 1*. The situation proposed was that they were supposed secondary school teachers that had planned to implement the activity of 'Forecasting Desigual sales' in their classroom. Due to a cultural trip with other students, they had to ask another teacher to replace them. They were asked to write a brief and easy to read lesson plan including all the necessary elements for the substitute teacher to carry out the lesson/s. Like in the previous activity, participants first had to prepare and individual proposal, then share their proposal with the rest of their team and agree on a final common lesson plan. This activity was supposed to provide a first spontaneous answer of the teacher to the question: 'How to teach a modelling activity based on Activity 1?' in terms of a teaching proposal designed.

Activity 3 consisted in the experimentation of the participants' own design of the activity with a group of students. The participants had to individually assume the role of the teacher and implement the initial phases of the lesson plan proposed in Activity 2. With this purpose in mind, they had to elaborate a more detailed design, a more in-depth a priori analysis (phase 1), experiment their proposal (phase 2), finish with the a posteriori analysis (phase 3) and prepare a brief 'experimentation report' (phase 4).

Finally, *Activity 4* was devoted to a joint analysis and final revision of the lesson plan with the aim of proposing a new version taking into account both their own experience and the experience of their teammates. In particular, the difficulties found in the implementation of the modelling activity (a posteriori analysis) were supposed to highlight the constraints related to the normal implementation of this kind of teaching proposals and the possible ways to overcome them.

The supervision of the teacher educators during the SRP-TE consisted of the following. By way of feedback to the team discussions in the forum and to the activities (reports, lesson plans, etc.) submitted, the course staff progressively introduced some didactic tools to support the mathematical analysis of activity 1: notions of model and system, criteria and ways to characterise the models provided, ways of comparing them, etc. At the end of activity 2, as a means to carry out the didactic analysis of the spontaneous teaching proposals, some publications about SRP were provided: Serrano and colleagues (2010) and Chevallard (2012). Between activities 3 and 4, the educators prepared a guideline with the main sections of the a posteriori analysis of a SRP, including some examples of its mathematical description, some criteria to describe the didactic organisation and some elements of the conditions produced, the constraints faced and the global evaluation of the teaching process. They also provided an assessment grid for the final report and a questionnaire about the development of the course to be answered individually at the very end of the

9) Evolución de las ventas semanales de las camisetas one-print desde el 15 de febrero de 2010 en la tienda de las Ramblas 136 (Barcelona):

Semana	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11
Camisetas one-print vendidas	233	112	118	130	116	151	159	173	175	230	253

10) Evolución de las ventas semanales de las camisetas one-print desde el 1 de marzo de 2010 en la tienda de la Rambla de Catalunya 140 (Barcelona):

Semana	1	2	3	4	5	6	7	8	9	10
Camisetas one-print vendidas	100	101	107	115	125	140	140	164	194	210

Figure 1: Initial worksheet of Desigual's request to the consultants

course. All the material produced by the students was gathered during the course, especially the students' discussions in the forums (including the teacher educators' interventions), the students' questions raised (in the forums or by mail), the partial and final reports and their answers to the questionnaire.

THE 'LESSON PLAN' AS A CRUCIAL TOOL

Given the fact that this was the first on-line course based on SRP-TE, the results found are mainly related to the organisation of the course, the weaknesses encountered and the possible way to overcome them in further implementations. We will only present those concerning the function of the 'lesson plan', which appears to be a central element of the SRP-TE.

According to the design of this SRP-TE, activities 1, 2 and 3 were mainly based on the teachers' mathematical and professional knowledge. The function of the lesson plan is to provide shared teaching materials to support the analysis in a triple dimension: (1) as a description of the initial modelling-based activity on forecasting sales; (2) as a teaching proposal spontaneously designed by the teachers according to their professional knowledge and adapted to the usual institutional school conditions; (3) as the support of a real (partial) teaching and learning process. It is mainly in activity 4 where new types of didactic knowledge are needed to provide a critical analysis of the mathematical and teaching processes followed. In this section, we present how the lesson plan was used in the SRP-TE, its productivity and limitations.

In spite of some initial difficulties, the teacher-students quite easily dealt with activity 1 and experienced a specific mathematical 'unguided' work based on modelling and the inquiry of an open question close to the paradigm of questioning the world. In activity 2 (lesson plan), many teachers fell back on the usual didactic contract based on the learning of contents (as opposed to the study of open questions) and searched a school mathematical subject related to sales forecasting (such as 'linear regression' or 'function graphs') to teach it. They then prepared a list of mathematical techniques needed to answer the question previously provided to the students. For example in the lesson plan of teacher A, the teaching proposal was based on the presentation of different time-series forecast techniques, such as Gompertz (S-shape) curves.

In this lesson plan, the teacher is supposed to play the traditional role of teaching a repertoire of mathematical techniques the students should learn before proposing a forecast, as if the students could do nothing without it. However, other teachers respected the open character of the study process. For example, in the lesson plan proposed by teacher B, the students were asked to find an answer to the forecast question without any previously established strategy:

> In the next classes you will be employed by the company to make a short and long term forecast for each of the variables of the file 'Problem of Sales Desigual.pdf'.

Two tasks were proposed to guide the students through this new kind of work:

> Activity: Read the information provided by Desigual:
>
> 1) Which are the variables on which Desigual provided information? What types of variables are they? How does variation can be described for each of these variables?
>
> 2) Search for the following information: What articles do Desigual shops sell? Where in Spain are Desigual shops located? And outside Spain?

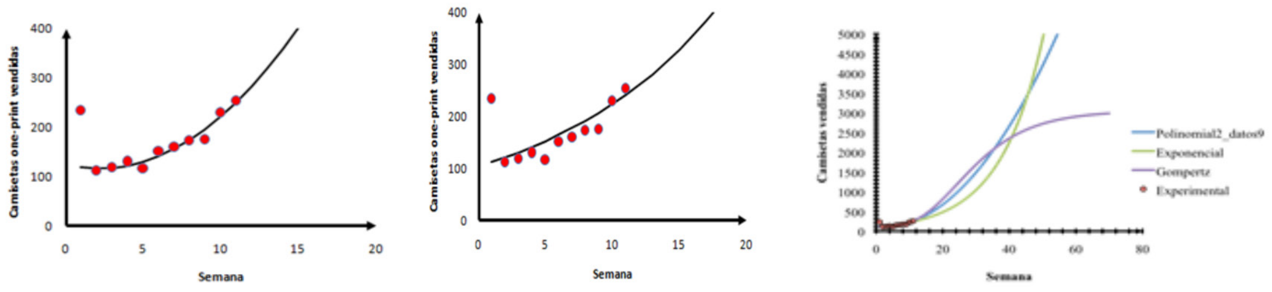

Figure 2: Mathematical models proposed by teacher A

Teachers A and B worked in the same team when preparing the first version of the joint lesson plan. In this case, the team ended up with a proposal similar to the proposal of teacher B (paradigm of questioning the world), but it could also have been the other way round. In any case, the reasons put forward and the discussions carried out in the forum appeared to be highly interesting material for the teachers' initial professional knowledge to be changed (especially the assumptions, reasons and criteria used to support their decisions). This knowledge was related to the school institutional constraints and was enriched with new didactic tools that enabled it to evolve. The a priori elements, progressively made accessible by the educators, consider the lesson plans as an initial empirical basis that was enriched during the teaching implementation in activity 3.

All this work eventually turned into a revision of the initial lesson plan now including the results of the experimentation. At this point, the supervisors proposed a guideline to organise the didactic tools provided in relation with the three previous activities. The mathematical analysis corresponded to activity 1. At this stage, the epistemological elements used to describe the modelling activity were completed with some examples of SRP descriptions in terms of sequences of questions and answers (see Figure 3).

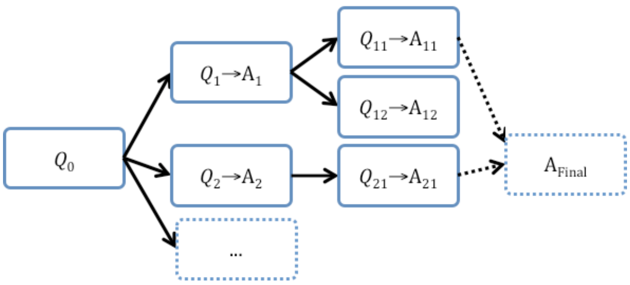

Figure 3: Sequence of questions and answers to describe a study and research path

This description starts with an open question (Q_0) and leads to different ways to formulate new sub-questions (Q_i) and obtain partial answers (A_i) until arriving at an acceptable final answer (A_{Final}) (also partial but provisionally considered as definitive). For instance, the team of teachers A and B proposed a description starting with the following sequence (the questions in italics being added by the authors):

Q_1: How to forecast the sales of the company, given some time-series real data?

A_1: Fit different functions to the real data, choose the best function and evaluate it in the future periods considered using appropriate software.

$Q_{1.1}$: What software to use: Excel, Geogebra, R?

$A_{1.1}$: Geogebra and Excel, which are the ones the participants know well.

$Q_{1.1.1}$: Do both tools provide the same results?

$A_{1.1.1}$: In the first calculations, the results obtained were different.

$Q_{1.1.2}$: How to explain the differences? [unapproached question]

$Q_{1.2}$: Which theoretical tools can be used? Where can they be found?

$A_{1.2}$: Document "Time-series. Least squares fit" provided by a participant.

$Q_{1.2.1}$: Should we use this sophisticated material to forecast the sales of an item that has increased from 100 to 500 units in ten weeks' time?

$A_{1.3}$: Elementary functions, 'trend line' option using Excel, 'fit line' using Geogebra.

$Q_{1.3.1}$: Which model fits the data best?

$A_{1.3.1}$: The best model is the one with the fewest errors.

$Q_{1.3.1.1}$: What types of errors are there? Do they lead to the same results?

$A_{1.3.2}$: The best model is the one with the highest R^2.

$Q_{1.3.2.1}$: What is R^2 and how is it related to the errors?

This schema of questions and answers, called the 'mathematical skeleton' by the educators, help the participants describe the elements of the teaching proposal designed, both those which effectively appeared in the process implemented and those left out. Based on this epistemological analysis, the guideline elaborated by the educators proposed certain elements to carry out a didactic analysis of the teaching and learning process (activities 2 and 3) experimented, using the notion of didactic organisation and focusing on the sharing of responsibilities between teacher and students during the development of the modelling process. We will not describe the SRP-TE any further. To this short description, we will simply add that the documents given to the participants as complementary reading were of crucial importance.

CONCLUSIONS

To sum up, let us stress what we consider at this very initial moment of our research, to be the main contributions of the theoretical framework used, ATD. An important characteristic of a SRP-TE is to locate the questioning of the mathematics content to be taught (here, a modelling activity) and of the traditional didactic organisations prevailing in our current schools at the heart of the teachers education programme. In the case here presented, the first stage of the SRP-TE (searching information and documentation available) was not developed. This clearly appears as a weakness, since it would most certainly give rise to interesting discussions about different ways of interpreting modelling as a school content, the ambiguity of the official guidelines regarding this matter and the variety of proposals existing in different countries and even within the same country. Another characteristic of a SRP-TE is to nourish the questioning of the mathematical content to be taught through the 'in vivo' performance of a mathematical activity based on a previously designed SRP. The role-playing technique used was a good choice and it worked well, in spite of some logical difficulties at the beginning for the participants to enter the new contract: they all initially hesitated between acting as secondary school students or as real mathematicians, but ended up playing their role. Of course, another main contribution of the ATD is the kind of methodological tools provided by the educators to help participants describe the content to be taught and the didactic organisation of the teaching and learning process, an aspect of our research that has just been outlined here, especially with respect to what concerns the description of the knowledge to be taught, the new responsibilities assumed by both teachers and students, and the institutional constraints found.

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Mathematical modelling, problem solving, project and ethnomathematics: Confluent points

Maria Salett Biembengut

Pontifícia Universidade Católica do Rio Grande do Sul, Porto Alegre, Brazil, maria.salett@pucrs.br

This paper presents a documental study about the confluent points among mathematical modelling, problem solving, project and ethnomathematics as methods of research and mathematics teaching. As a result, the study has shown that there are elements that bind these methods structurally together as research methods. Starting from the fact that education should promote knowledge this study provides evidence for these methods. Thus in each one of them, it is required knowledge from the student about the topic of the problem-situation; method to delineate the phases to go through; technological instruments to support the solution; aesthetics for the expression; and ethics in the communication of the resulting ideas.

Keywords: Mathematical modelling, problem solving, project, ethnomathematics.

INTRODUCTION

The concepts of modelling, problem solving and project permeate the most varied contexts. It is known that concepts arise from the human mind and are developed by necessity. According to Boutinet (1990), this ability to model the 'imagined thing' is what drives people to conceive projects. As concepts guide people's daily actions, they have been at the core of educational curriculum reforms in mathematics and have nourished several thematic developments, among them three topics that are confluent with Mathematical Modelling (MM): Problem Solving, Project and Ethnomathematics. Despite being independent topics, they instigate us to identify confluent procedures in research and teaching. Two questions steered this study: (1) what are the confluences among Modelling, Problem Solving, Project and Ethnomathematics? and (2) in what circumstances can the involved phases be integrated into pedagogical practices of MM in Education?

METHODOLOGICAL PROCEDURES

In order to answer the research questions, I collected data in the literature about Modelling, Problem Solving, Project & Ethnomathematics. I sought to gather a deep understanding of the authors' propositions as well as attempted to identify the commonalities among the topics. I also subdivided the expression about each topic into two parts: scientific research method and teaching method. This study may be considered documental, since the data comes from publications of authors who are referenced in academic literature. In this context, it is essential to highlight that in the process of identifying confluences among the topics, different identifications may coexist. Additionally, it is crucial to bear in mind that the expression in this article is not neutral, since it carries along the conceptions implicit in the works analyzed.

MODELLING, PROBLEM SOLVING, PROJECT AND ETHNOMATHEMATICS

Most of the problem situations people experience require them to make use of representations to solve them. These situations are inserted in a sociocultural context and sometimes exceed the needs of subsistence. As a result, such situations prompt people to design ways to solve problems and to learn new knowledge in order to solve them. When one of such designs involves the 'desire to know' about something, a research method is required to accomplish the goal. If this 'knowledge' aims to solve a problem situation for which the available data are not sufficient for the person to use an existing model, there is the need for Modelling. However, if this 'knowledge' aims at constructing knowledge, at explaining how a person or a group from a social culture elaborates a mathematical model or makes use of this model in their activities, the method is called Ethnomathematics.

Mathematical modelling

Modelling is the process involved in the development of a model in any field of knowledge. The essence of this process emerges in a person's mind when a genuine doubt/circumstance instigates her to find the best way to solve, understand, create or improve something. The route of scientific research is delineated when someone models a problem situation using mathematical concepts. According to Biembengut (2014), this research happens through three connected phases: (F1) understand and grasp – recognize and become familiar with the subject; (F2) understand and explain – formulate the model and solve the problem situation; (F3) articulate meaning – interpret the solution, validate, explain the process and results.

As MM is in essence a research process, it has been advocated as a process for mathematics teaching. The goal of MM in Education is to teach students the contents of the curriculum departing from a topic, formulating a project with a proper design and guiding students to conduct research within the limits of the school structure. The teacher organizes the project following the modelling steps by promoting (E1) understanding and grasping (topic presentation, provoking questions, selection of adequate issues to develop the content); (E2) understanding and explaining (presentation of the data, suggestion of hypotheses, presentation of the content and examples); (E3) expressing meaning (model formulation, requiring students to solve the problem, result evaluation). The purpose consists in providing students with opportunities to research about topics of their interest while they learn the curricular contents.

Problem solving

A problem emerges when the person realizes that there is a gap between the situation she does not know how to solve and her desire to solve it. The solution may come from an idea, in a heuristic approach: a set of rules and methods that leads users to solutions and discoveries. A person may solve problems using different heuristics, and each heuristic may be more appropriate or less for each kind of problem. Authors such as Wallas (1921), Dewey (1922) and Polya (1981) are recurrently cited as references in heuristics for Mathematical Education.

Dewey (1922) proposed five phases: identification, definition, plan, execution and survey; Wallas (1921), four stages: preparation, incubation, illumination and verification; Polya (1981), four: understanding, plan, execution and retrospection. As these heuristics are related, four non-linear phases may be proposed: (F1) identify the problem situation: become familiar, collect data, identify relationships – preparation; (F2) establish procedures: look for similar problem situations, map, identify the knowledge required; (F3) solve the problem: implement strategies for a plausible solution – illumination; (F4) evaluate results and validate them: check the validity of the result – retrospection.

As these procedures are requirements for any type of activity, they are part of the pedagogical proposals in Mathematical Education. Researchers such as Schoenfeld (1985), Gage and Berliner (1992) consider that helping students to solve problem situations is the reason why we should teach Mathematics. Problem situations should require students to review basic facts, identify unknowns, look for meanings in the unknowns, (re)learn math operations, understand relationships between operations and their implications, formulate, solve and argue whether solutions are compatible.

In light of the exposed, it is possible to prescribe a three-step method for Problem Solving in pedagogical practices: (E1) proposition – proposing a problem situation to students that require them to decompose the data, identify the unknowns; (E2) ideation – guiding them to perceive mathematical relationships and formulate a plan; (E3) synthesis – guiding them to apply and evaluate the resulting data. These steps allow students to combine their ideas to the extent that they are encouraged to learn about issues of their interest.

Project

A large number of people have needs that exceed the solution of practical problems. Anything may inspire people to know, do or have something, an impulse that leads one to design ways to achieve solutions. People may delineate a project to solve a problem related to physical survival as well as when they are interested in learning about something. Boutinet (1990) suggested essential steps in this process: (F1) diagnosis – setting the purpose and learning about the available resources; (F2) outline – describing means to obtain data and evaluating; (F3) strategy – identifying how to organize and classify the data; (F4) execution – performing actions and evaluations; (F5) analysis – judging the results and evaluating the situation.

Similarly, the project heuristics is a proposal for Education. Izard (1997) states that school projects encourage active participation from students; provide them with motivation and challenges; encourage them to ask questions, to learn about some topic, to observe their context. Projects also reveal students' interests in the formulation and description of the data as well as their skills in interpreting and evaluating the results. Three steps are proposed as a method to include projects in pedagogical practices: (E1) preparation – instigating students to know about a topic/issue, look for data and learn about the topic; (E2) development – guiding students to ask questions about the topic/issue, describe the data encountered, identify concepts, formulate and understand the circumstances; (E3) projection – guiding students to interpret results, analyze them and identify other facts. I believe that one of the purposes of education is achieved when students are offered with projects that enable them to expand their knowledge, to learn about their surroundings, making them feel valued.

Ethnomathematics

D'Ambrosio (2001) sustains that all social cultures have a legacy of knowledge, behavior and rules that they seek to convey to generations, making it possible to perpetuate cultures. Many social cultures have created and developed instruments to explain, understand, and learn about things. Each culture has developed unique mathematical techniques arising from their needs. D'Ambrosio calls Ethnomathematics the art or technique to explain, know or understand how a person or a group generates mathematical knowledge, makes use of it in their business, and organizes and transmits knowledge to others. It is the study of concepts, traditions and mathematical practices of a person or a sociocultural group. The researcher goes deep into the person/community's culture by observing, asking questions, among other means.

Research on Ethnomathematics, for D'Ambrosio (2001), consists of three phases: (1) develop methods from practices and solutions; (2) develop theories from methods; (3) innovate from theories. Ethnomathematics procedures involve the need to: (F1) explicit the fact – recognize the fact, become familiar with the practices and the solutions presented by a person/group; (F2) present the explanatory assumption – analyze facts, methods, practices and solutions, establish conceptual principles and formulate explanations; (F3) indicate other facts – interpret, validate the pragmatic model from the perspective of the person/group, describe and check whether theory resulted from the method; (F4) complement the fact – identify whether innovation resulted from theory ↔ method ↔ practices and solutions.

In Ethnomathematics, the focus lies on recognizing the mathematical actions and knowledge of people, resultant from the needs and experiences these people have. Using people's activities and knowledge in pedagogical practices may best contribute to the students' learning process, since they may interact with problems and solutions, and experience the culture. The Ethnomathematics method for pedagogical practices consists of three stages: (E1) interaction – help students to get acquainted with the facts from the activities, to interact with the person/group and to recognize practices and solutions; (E2) explanation – encourage students to understand practices and solutions, to identify mathematical concepts, to stipulate phases, to learn and make explanatory assumptions; (E3) indication – guide students to interpret the assumptions, to describe or verify whether the method or theory used resulted in innovation.

Ethnomathematics has as its source the activities that a person/group performs and the knowledge they have resulting from the needs of their daily lives. It assumes dynamic sharing between activities and knowledge; values popular knowledge as a means of contributing to knowledge production; and proposes education as a process in which knowledge is transmitted across generations.

CONFLUENT POINTS

Modelling has elements that converge with Problem Solving, Project and Ethnomathematics, depending on the desire to know. As Education should promote knowledge, instigate the desire to know, I believe one of these methods should relate to the desire to teach.

Confluent points in the 'desire to know' as a research method

Modelling is used when we need to solve a problem situation for which the available data are not sufficient to apply an existing model to obtain the solution, or when we need to (re)create, improve something. To confront the MM procedures with those of Problem Solving, Project and Ethnomathematics, for research purposes, the similar points consist of: delimitation of the problem situation, theoretical framework, hypotheses/assumptions, development, implementation, interpretation and evaluation of the solution.

MM is the area of research for the development/creation of a model for the solution of a particular problem situation or to support other applications and theories. Ethnomathematics is the area of research that seeks to understand a person/group's knowledge related to their activities. When someone intends to conduct research, a project with a proper design is required to accomplish such a goal. In light of these definitions about the desire to know, I may conclude that in all these areas, the researcher is guided by the scientific methodology following common steps to produce knowledge; be it something original, be it a complement to what already exists.

It can be stated that a research design, be it in Modelling or Ethnomathematics, has as its source a problem situation, a desire to know. To solve a problem, we direct our steps through other areas of knowledge, fact that allows the researcher to expose events, enunciate models of relationships between things and facts. As they are part of the scientific research pathway, they cannot be denied in the school context as methods of teaching and research, since they give students the chance to design projects, be them in Modelling, Problem Solving or Ethnomathematics.

Confluent points in the 'desire to teach' – teaching method

Modelling in Education is a method for teaching research in Basic Education. To confront the procedures of Modelling with those of Problem Solving, Project and Ethnomathematics, for teaching purposes, the 'desire to teach' point is similar: proposition of a topic or problem situation, review or presentation of data, procedures for resolution and evaluation of results. For instance, the teacher may use the context of an apple plantation to teach modelling. She may begin the class by talking about the context in the Brazilian state of Santa Catarina. She may present some data, such as the fact that apple trees are planted in line and that for that reason, there is a particular spacing among them. She may ask students: what is the ideal distance between one tree and the next one if we aim at producing more apples? To help them answer the question, the teacher models the data by teaching the students the required mathematical contents. At the same time, the students study the ethno-mathematic aspect of the problem and start a project. They seek to understand how the producer uses farming "models" (his knowledge might possibly have been acquired through experience, or inherited from his ancestors). Students can research about the topic by talking directly to the producer and observing his work. Drawing on the data collected, students can attempt to verify whether or not the producer received technical orientation (help from specialists), whether he modified his practice, and hence, his mathematical conceptions.

I conclude that the essence of the four themes resides primarily in involving each student in the interaction among real elements present in the environment in relation to the topic in question. Be it in Modelling for Education, be it in Ethnomathematics, students will have a problem situation as a starting point, and to solve the problem they will delineate a project to guide them. Moreover, to outline a project, they ought to have knowledge beyond the traditional school topics. I believe that when students feel themselves capable of solving a problem situation that requires going through the steps of Modeling or Ethnomathematics they learn the curricular contents along the process, and teachers will feel encouraged in their desire to teach.

FINAL REMARKS

As a researcher, I hope I have not committed any kind of improbity by articulating the principles of each of the themes. I hope this work can be improved and contribute to teaching and research in education. This study had as its main guiding question the confluent points among Modelling, Problem Solving, Project, Ethnomathematics and the desire to know research method and the desire to teach teaching method. The art of these themes lies in guiding students to understand the environment in which they live as well as in the potential to put these topics into practice: by translating results into suitable language for general comprehension. The procedures, even when implemented as extracurricular activity and only by a few group of teachers and students, can contribute to improve the knowledge of those who are involved in the process. Taking the real value of Education into account, it is worth considering one of the themes, or all combined in the classroom, since they offer students the opportunity to ask questions, identify and learn about facts, accumulate experiences, become more attentive to events, become enchanted with solutions to problems, and feel free to express their knowledge.

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Taxonomy of modelling tasks

Wolfgang Bock, Martin Bracke and Jana Kreckler

University of Kaiserslautern, Kaiserslautern, Germany,
bock@mathematik.uni-kl.de, bracke@mathematik.uni-kl.de, kreckler@mathematik.uni-kl.de

When teaching mathematical modelling it becomes essential to be able to construct modelling tasks of a similar difficulty for projects and exams. In order to be able to compare these tasks, an evaluation scheme concerning the difficulty becomes necessary. In this paper, we introduce an extension of a model developed by Bock, W., Bracke, M., Götz, T. & Siller, H.-S. (2014) which is based on a model from Eyerer & Krause (2012) for comparing difficulties of projects between industry and school. This is used to evaluate the difficulty of a modelling task. The theoretical model of Bock and colleagues (2014) is made applicable by means of a software tool: Taking into account all relevant dimensions of modelling, a measure of difficulty is calculated based on the normalized covered area in the constructed multi-dimensional model. This allows a comparison of the difficulty of several modelling tasks.

Keywords: Taxonomy, authentic problem, modelling.

THEORETICAL BACKGROUND

Mathematical modelling is part of the educational standards in many different countries; moreover it is named by the German Education minister Conference (KMK, 2012) as one of the general mathematical competencies that need to be taught and learnt. “By mathematical modelling competence we mean being able to autonomously and insightfully carry through all aspects of a mathematical modelling process in a certain context” (Blomhøj & Jensen, 2003). A modelling process can be described as a modelling cycle, see, for example, (Blum & Leiß, 2007) or (Kaiser, 1995); compare also (Ackoff, Arnoff, & Churchman, 1957) for similar cycles in Operations Research. In order to teach mathematical modelling, both concepts as well as good modelling tasks need to be developed. Following a definition of Blomhøj & Kjeldsen (2006), a good modelling task must fulfill the following seven criteria. A good modelling task should

… be understandable and reasonable,

… give an appropriate challenge for an independent work,

… be authentic and include authentic data,

… be open for interesting modelling results,

… be open for critics to the model,

… lead to representative modelling activities and

… challenge the students appropriately to work with concepts and methods that are relevant for their mathematical learning.

In most cases mathematical modelling in school, at university or in industry is group work. To be able to compare student results regarding different modelling tasks during class or in exams, an evaluation scheme for their level of difficulty becomes necessary.

Cohors-Fresenborg, Sjuts, and Sommer (2004) have developed a model to determine the level of difficulty of PISA-tasks. The focus of their model lay on the cognitive processes necessary when solving the tasks. The four criteria *linguistic complexity*, *cognitive complexity*, *formalization of knowledge* and the *handling of formulas* were defined to be the criteria affecting the difficulty of tasks. Each criterion was divided into three levels of difficulty 0, 1 and 2. The complexity of a task was then defined to be the sum over those levels achieved in the four criteria.

The difficulty of modelling tasks is much harder to determine as many different dimensions play a decisive role. Reit (2014) has developed a model determining the difficulty of modelling tasks based on thought structures of different solution approaches. The model is based on the assumption that different mathematical models and solutions to the same task require different knowledge and mental activities. Based on cognitive load theory, the approach acts on the assumption that parallel thought structures are more difficult than sequential thoughts. The level of difficulty is then described as the sum of the factorial of the single levels.

The problem of the above model is that student solutions have to be available a priori. In many problem settings this is not the case. Especially for authentic or real world problems (see, e.g., Bock & Bracke, 2013) possible student solutions are hardly predictable. These problems are classified by a client structure and are of full generality.

Definition: An authentic problem is a problem posed by a client, who wants to obtain a solution, which is applicable in the issues of the client. The problem is not filtered or reduced and has the full generality without any manipulations, i.e. it is posed as it is seen. A real-world or realistic problem, is an authentic problem, which involves ingredients, which can be accessed by the students in real life.

With these problem settings another aspect makes a taxonomy of the tasks hard. The problem can be posed to students from primary school as well as to students from university, since every learning group is using their individual methods. This makes a general classification dependent on competences and the respective pre-knowledge. Since general industrial projects for mathematical modelling are of a high dimensionality concerning the use of competencies, Eyerer and Krause (2012) developed the spider web method to illustrate difficulty of tasks in the TheoPrax method. TheoPrax is a method which focusses on project teaching with real industrial projects. Here, industrial partners give tasks to students who then have to write an offer to the industry to obtain the job. The project is then worked out by the students. The industrial partner is obliged to cover the costs and also to finance workshops for the students if the offer of the project is acceptable for them. The financing part also has to be planned by the students.

The idea of the Eyerer model is to use a spider web diagram, comparable to Figure 1, to grade a project. This model was adapted by Bock, Bracke, Götz, & Siller (2014) to measure how teachers rate the difficulty of certain modelling tasks. For this purpose eight dimensions and corresponding levels of complexity for each of the dimensions were chosen. Using this system teachers and supervisors rated several modelling problems and their ratings were illustrated by spider web diagrams. If we want to compare the difficulty of two modelling tasks we have to compare two spider web diagrams: Let us assume that we get diagrams *A* and *B* as in Figure 1 (the diagrams are for the modified nine-dimensional spider webs introduced later in this paper).

We would now like to compare the two tasks *A* and *B* regarding their difficulty – of course relative to our special situation (time frame, learning group, ...) and needs. This seems to be nontrivial because of the multidimensional nature of the data and different weights we may have for each of the dimensions. In the following paragraph we first extend the model developed by Bock and colleagues (2014) by one dimension which is relevant when dealing with authentic modelling tasks. In order to compare the difficulty of modelling tasks we propose a *measure of difficulty* which can be easily calculated using a software tool. Moreover, the rating of modelling tasks as well as the computation of the

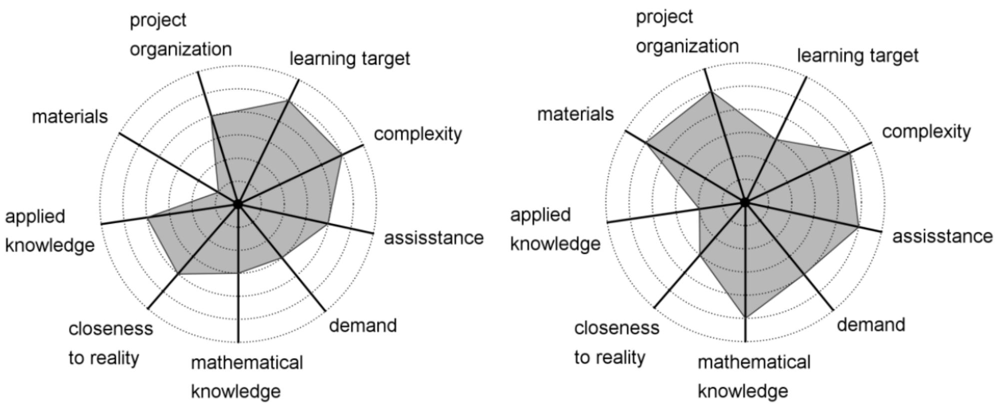

Figure 1: Comparison of two star diagrams for modelling tasks *A* and *B*

new measure are made easily applicable by a newly implemented tool.

THE MODEL

The aim of the following model is to describe and compare the difficulty of modelling tasks from the viewpoint of one individual teacher or supervisor. Note that the measure of difficulty for different teachers using the model to evaluate the difficulty may vary. This is due to the different background and experience teachers have. Also with growing experience and competencies the value may vary in time. However on a small time scale, in the opinion of the authors, the score value is stable.

Assume that the difficulty of a modelling task depends on many different dimensions. These dimensions also could be adapted, extended or reduced according to which aspects the score will focus on. *To be able to apply the model it is essential that the person using it can account for practical experiences with mathematical modelling tasks.* This is at first necessary to be able to estimate the different scales of the dimensions and secondly to obtain an intuition for their interplay. In a small pilot study a group of teachers participating in a modelling week was asked to rate different modelling tasks using the following model. For some teachers this turned out to be quiet difficult as they were missing some essential knowledge and experience concerning modelling.

Bock, Bracke, Götz, and Siller (2014) identified the following eight dimensions affecting the complexity of modelling tasks:

1) Project organization
2) Learning target
3) Complexity
4) Assistance
5) Demand
6) Mathematical knowledge
7) Closeness to reality
8) Applied knowledge

Each of these dimensions was divided into six levels of complexity, where 1 describes the easiest and 6 the most difficult level. For each dimension the possible answers are categorized, where the numbers are assigned to the respective competence levels. An example is given for the dimension *complexity:*

1) solution approach is clear
2) one-sided methods (e.g. only programming, geometry…)
3) alternative solution approaches possible
4) data set is too big or insufficient
5) solution requires variety of methods
6) alternative solution approaches in combination with many methods necessary

All eight dimensions, each consisting of six complexity levels, are illustrated in a diagram in the shape of a star (compare Figure 2). Each dimension is pictured as a ray emitted from the centre of the star. The length of each ray is divided into six equal parts. The first ring defines the easiest complexity level 1, the most complex level 6 is reached at the end of each ray.

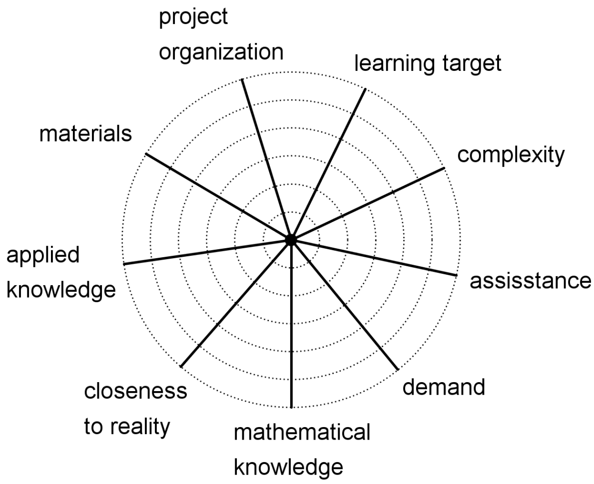

Figure 2: Extended model: nine-dimensional star

Considering a given modelling task, its complexity is rated for each dimension separately and marked on the correspondent ray. The marks of all rays can then be connected and the enclosed area calculated (compare Figure 3). The size of the enclosed area symbolizes the difficulty of the task and therefore gives a possibility to compare the difficulty of modelling tasks. The larger the area, the more difficult the task. Note that the diagram shows also the directional weights of the individual dimensions, which can be used to test the tasks according to the pre-knowledge and competencies of the students.

Due to the authors experiences the model has been developed further by incorporating the dimension "*materials*" to the model with the following score levels:

1) Computer/Laptops with internet connection are available for research the whole time

2) Computer/Laptops with internet connection are available for research temporarily only
3) A subject-specific library is available for research
4) Selected books and journals are available for research
5) Some information selected by the teacher is available for research
6) There is no possibility for research

This dimension takes into account that the complexity of a modelling task does also depend on the amount of research that is possible during the process of finding a solution. An extra ray for the dimension *materials* was added to the model (see Figure 2). The model was then implemented in a new software tool such that ratings of modelling tasks can easily be evaluated and compared. Of course, rating a modelling task according to the named dimensions always depends on the specific target group and the individual project settings.

With the help of the implemented tool the area which is formed by connecting the neighboring score levels is computed and normalized by the maximal area, i.e., if all dimensions have maximal score. Thus the output can be interpreted as the percentage of the task compared to a task of maximal difficulty and delivers a number between 0 and 1 indicating the difficulty of the investigated task. We will call this number the *measure of difficulty (MOD)*. The closer MOD is to 1, the more difficult the task.

Definition: (i) Let M be the area of the convex hull spanned by the complexity ranking of 6 in each dimension. Let I be the measured area of the enclosed area of a rated modelling task. The measure of difficulty (MOD) of the modelling task is then equal to: MOD = I/M and takes values in the interval (0,1].

(ii) Two modelling tasks T1 and T2 are said to be of equal complexity of fineness e if

$$|MOD(T1)-MOD(T2)|<e.$$

A difficulty that arises when calculating the MOD is the fact, that the area differs for the same rating with different arrangements in the sequence of the dimensions. This problem was solved by calculating not only the area but the mean value of the areas over all permutations of arrangements.

EXAMPLE

As an example consider the following setting: A group of students from 11th grade of a German secondary school is supposed to work in small groups on the authentic *Airline Problem*. During the time of the modelling activity (4 h) the students have access to computers and internet.

Airline Problem: The time a plane is on the ground is time in which the airline is making no money! Therefore the airline is interested in a system for the boarding of a plane such that the time the plane is on the ground is minimized.

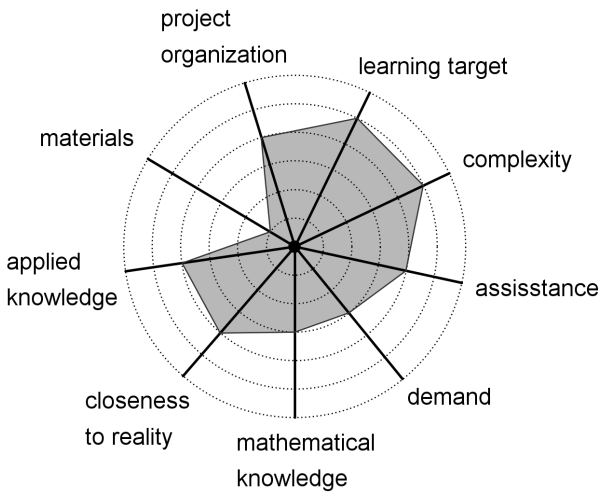

Figure 3: Example: Rating a modelling task to calculate the MOD

The authors would rate this problem according to the model as in Figure 3. This rating was developed by taking the average value of the individual ratings of the authors. The precise categories apart from the material dimension can be found in Bock and colleagues (2014). For the *Airline Problem* the MOD is calculated to be 0.3688. This rating can be retraced in the following way.

The dimension *materials* is rated depending on the individual setting set for the project work planned. In our example, level 1*"computers/laptops with internet connection are available for research the whole time"* describes the situation planned by the teacher.

Project organization is rated with complexity level 4 *"with great difficulties, risk to fail is controllable."* This is justified by the fact, that the Airline Problem has a very open formulation which leaves several decisions and estimations to the students. Still, these difficulties can be overcome and a solution can be obtained by a simple simulation with chairs and a stopwatch.

The *learning target* is described in level 5, *"new combinations of different techniques"*, which is set as the desired outcome while working on the problem. Hence, the dimension *complexity* is rated with 5,*"solution requires variety of methods"*.

The *assistance* given in the above setting can be described within the meaning of level 4 *"teacher or external tutor supporting in wide steps"*. The rating of this dimension is depended on the individual support teachers are planning to give their students and can be varied for each realization of a modelling project.

The *demand* on the Airline Problem can be rated by level 3, saying that the *"demand (is) alternatingly increasing"*. This marks a medium level of difficulty to the task. The *mathematical knowledge* in this example is based on the *"recognizing (of) missing knowledge in detail"* (level 3) while the *applied knowledge* requires the *"researching and arranging (of) missing information and correlations"* (level 4).

Finally, the Airline Problem has a *"high correspondence to reality"* which leads to complexity level 4 in dimension *closeness to reality*.

POSSIBLE EXTENSIONS

If one wants to lay more focus to certain dimensions, the model can be extended to a weighted model. This can be dependent on the background and formulation of an individual task. Depending on the situation, some of the nine dimensions carry a greater weight than others and therefore influence the complexity of a modelling task more than others.

For example, in the described setting of the *Airline Problem* computers and internet are accessible but play a minor role for the finding of a good solution. In practice, the authors noted that students for example simulated themselves the time to sit down on a seat while blocked with the help of chairs in the room. With a very limited time horizon there is no gain of having access to computers or internet. Therefore, the dimension of material can be weighted less than the other dimensions in this case.

Of course, also the model of maximal difficulty has to be adapted. This can be implemented by modifying the length of the individual rays. A dimension which is considered to be less important in its effect on the difficulty of a modelling task is assigned a shorter ray than more important dimensions. For this, an exact assignment of weights and the proportional change in the length of the corresponding rays still needs to be formulated.

Possible modifications are also the adding and deleting of certain dimensions from the diagram. But this has to be done carefully since too few dimensions are not reflecting the whole difficulty of the modelling task while also too many dimensions make the tool inconvenient and unclear.

SUMMARY AND OUTLOOK

In this paper, we presented the extension of a model developed by Bock, Bracke, Götz, & Siller (2014) based on (Eyerer & Krause, 2012) evaluating the difficulty of modelling tasks. The extended model considers nine dimensions with six complexity levels each affecting the difficulty and is illustrated as a star with nine rays (see Figure 2). To calculate the newly defined measure of difficulty (MOD) of a modelling task, the task is rated with respect to its complexity level in each of the nine dimensions. The ratings are marked in the nine-dimensional star and connected to calculate the enclosed surface area which leads to the definition of the MOD (see Figure 3). A new tool was implemented such that ratings of various modelling tasks can easily be compared by their value of MOD. The closer the value of MOD is to 1, the more difficult the modelling task. In further research the validity of the model should be analyzed and an exact ranking for the values of MOD defined. This could be done by comparing the MOD values with student solutions and the correlation between those. Up to now, this model represents a subjective rating. It is therefore necessary to undertake an empirical study to validate the model by comparing theoretical ratings with the feedback given by students working on the respective modelling projects.

The model of a multi-dimensional star with rays of equal or weighted length can also be used to evaluate self-assessments of participating students or to assess the expectations of students regarding modelling or other activities. For this, different dimensions and items need to be defined. Bock and colleagues (2014) developed a questionnaire to investigate the self-assessment of students. In the context of modelling days at the University of Kaiserslautern, Germany, Kreckler developed item formulations to compare expectations and conclusions of students concerning the project days. A small sample of students was tested and evaluated. The answers for expectations and conclusions were marked on the rays and compared for each student. With this an evaluation regarding the categories *exceeded expectations*, *fulfilled expectations* and *expectations not fulfilled* for the single dimensions was made possible.

Of course, it may not be correct to assume that every dimension is of equal weight. To overcome this, practical empirical studies are planned, in which the weights of the individual dimensions are obtained via a fitting. For this, also a survey for the students is in preparation. The aim is to find a benchmark model for the measure of difficulty.

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Conceptions in France about mathematical modelling: Exploratory research with design of semi-structured interviews

Richard Cabassut[1] and Irene Ferrando[2]

1 Strasbourg University, LISEC-EA 2310, Strasbourg, Paris 7 University, LDAR, Paris, France, richard.cabassut@unistra.fr

2 Universitat de València, Dpto. de Didáctica de las Matemáticas, València, Spain

From ongoing research, we present the design of interviews on conceptions about modelling in order to better know the relations between students, teachers, researchers, authors of resources and decision makers in the field of mathematical education in France. The theoretical framework combines theories on conceptions, on modelling and on anthropological approach. The methodology uses bibliography, exploratory data analysis, comparative approach and interviews. We describe how a review of the literature, an exploratory data analysis of existing questionnaires and interviews are used to design these semi-structured interviews.

Keywords: Modelling, interview, France, teacher, conception.

DESIGN OF INTERVIEWS ON CONCEPTION ABOUT MODELLING

Philipp (2007) points that teachers' conceptions play a key role in the teaching and learning of mathematics. Kaiser (2006) remarks that German "teachers and their beliefs concerning mathematics must be regarded as essential reasons for the low realisation of applications and modelling in mathematics teaching". CERME has reported several research on teachers' conceptions about modelling as we will show later in the review of the literature. We found only one research, (Cabassut & Villette, 2011), dealing with French teachers' conceptions: it was limited to French primary school teachers attending a teacher training course on modelling. *The problematic of our ongoing research* is the following: What are conceptions about modelling in France? What are their relations with the practice of modelling? The answers to these questions will help to understand differences and similarities with conceptions in other countries and to produce resources taking into account French conceptions: for example to produce resources considering difficulties or obstacles expressed in French conceptions. The *focus of this paper* is at the beginning of the exploratory phase of this ongoing research. We describe the design of interviews and we expose how Chevallard's levels of didactic codetermination will help us to analyse the French conceptions. Hypotheses and results about French conceptions are not the focus of this present paper and will be produced in another paper. *The results of this exploratory phase proposed in this paper* are the theoretical framework, the review of literature and the design of interviews. Let us now describe our theoretical framework.

CONCEPTIONS ABOUT MODELLING

In many researches, different terms such as beliefs, emotions, attitude... are used to refer to similar concepts. We will adopt the definitions given by Philipp (2007), particularly: "Knowledge – beliefs held with certainty or justified true belief [...]. Conception – a general notion or mental structure encompassing beliefs, meanings, concepts, propositions, rules, mental images, and preferences". The general term "conception" allows us to avoid the debate on the differences between "beliefs" and "knowledge" and to concentrate on the interaction between teachers' conceptions and teachers' practice.

Maaβ (2006, p. 115) expresses: "Modelling problems are authentic, complex and open problems which relate to reality. Problem-solving and divergent thinking is required in solving them". This conception and other ones collected in the review of the literature are considered in the questionnaire but it is not the focus of this paper to discuss them. The extract of the interviews in part 6 shows how these conceptions are taken into account and how the future answers could express them.

ANTHROPOLOGICAL APPROACH

To help analysing these conceptions, we use Chevallard's levels of didactic codetermination as suggested in Barquero, Bosch & Gascón (2011, p. 1938). At the *level of mathematic discipline*, we ask questions about mathematic conceptions and practices in order to observe relations with modelling conceptions and practices, and to point what is specific to modelling and what is general to mathematics. For example in the interview, some open questions are: "What are mathematics for you? What is mathematic teaching for you? How do you teach mathematics? How easy is it?" Examples of question with Likert scale: "from my point of view, in my mathematic lessons, school mathematics is the memorizing and application of definitions, formulas, mathematical facts and procedures." About practice with a Likert scale: "In my mathematic practice it is important to do open problem solving". At the *level of domain*, we can consider modelling as a transversal domain, covering different mathematical domains. We ask questions about modelling conceptions (already mentioned in part 2) and its practice. At the *level of pedagogy* for example we ask "In my mathematic teaching practice is it important to do small group work?" At the *level of school*, in the biography part, we ask in what type of teaching institution (primary school, general, technical or vocational secondary school, tertiary education) the teacher works. To consider the *level of society* the semi-structured interviews will be adapted to different types of members of French mathematical educational sphere, in order to conjecture how teaching, training, resource and decision making contribute to the conceptions and practices of modelling. The *French mathematical educational sphere* gathers students to become mathematics teachers from different types of school, researchers, authors of resources and decision makers (mathematical supervisor, curriculum adviser, etc.) in the field of mathematical education. At the *level of civilization,* Cabassut and Ferrando (2013) show how PISA and European parliament recommendations are explicitly mentioned in French common base of knowledge and skills that encourage modelling. We adopt a comparative approach involving others countries (Switzerland, Spain and Germany) to conjecture variations between different cultures.

The role of the different levels of didactic codetermination resulting of the semi-structured interviews will be interpreted with help of other studies on modelling. At this point of our research we just prepare the way to collect all the interviews and are waiting for their results to be analysed through the grid of levels of determination.

METHODOLOGY OF EXPLORATORY RESEARCH

We will describe now the four components of our exploratory methodology, method of interviewing, review of literature, exploratory analysis of existing data and comparative approach, which will all help us to design the interviews.

Semi-structured interview

Among the different ways of collecting conceptions about modelling, we choose semi-structured interviews proposed by Bernard (2006, p. 212). For every topic within the interview, we ask at first some open questions without directive contents to enable the interviewed person to express freely his or her mind without direction given by interviewer. The topics are selected from a review of modelling research literature. In a second part, the answers are completed with directive closed questions in order to cover entirely the topics. This second part will be the base of a future written questionnaire that will be produced for confirmatory research. The questions are selected from a review of modelling research literature: LEMA project (Maaβ & Gurlith, 2009), Schmidt (2011), Borromeo and Blum (2013). Some questions are adapted from inquiry based research: Dorier and Garcia (2013), Engeln and colleagues (2013). They often use the 5 levels Likert scale.

The semi-structured interviews are particularly adapted to researchers, authors of resource and decision makers because they are not so numerous in comparison with students or teachers, sometimes it is difficult to contact, and they have often time constraints that make harder to get a written answer to a questionnaire from them. Finally, after the exploratory research, we will produce a questionnaire to test the hypotheses. A cluster analysis of people answering the questionnaires could be produced, delivering for every cluster a representative paragon of the cluster. An additional clarifying semi-structured interviews could be made with clusters paragons in order to make the results of the analysis of the questionnaires answers clearer. This shows how interesting semi-structured interviews are.

Review of literature

The review of literature concerns the two last CERME proceedings, the last three years of Educational Studies in Mathematics review, and of Zentralblatt für Didaktik der Mathematik, the proceedings of ICTMA 14 and other papers found on the internet or known by the authors. These review deals with research papers on conceptions about modelling in the mathematical educational sphere. We would like to remind the reader that we did not find research on conceptions of modelling specific to France, and that every collected research studied a specific population (for example primary school teacher or prospective secondary teacher) and none studied the conceptions of researchers, authors of resources and decision makers.

Exploratory analysis of existing data

The LEMA project described in Cabassut and Mousoulides (2009) has developed a teacher training course on modelling. During the project, two different kinds of data were produced. First, 83 teachers attended a LEMA pilot training course and have answered a questionnaire before and after the training in order to evaluate the training course. The questionnaire is well described in Maaß and Gurlitt (2009). Secondly, 3 French teachers have been interviewed after the LEMA training course to evaluate the impact of the training course. We propose an exploratory analysis of these data to explore these teachers' conception about modelling in the way suggested by Tukey (1977): we use descriptive statistic to observe existing data without preconceptions and without formulating hypotheses in advance. We do not use inferential statistics to test, in a deductive way, hypotheses formulated in advance on data to be collected for the research: that will be the job of confirmatory analysis.

For the 83 teachers, the questionnaire is composed of questions (variables) with multiple-choice qualitative answers and questions with quantitative answers. We split the variables in two parts: the biographical variables (country, age, gender, type of school…) and the active variables on conceptions about mathematical modelling. We will process a cluster analysis (as described in Cabassut & Villette, 2011). For every cluster we will consider the splitting active variables. A *splitting active variable* is an answer for which the percentage of the answer in the cluster is very different than in the whole population. Every cluster will be described through these splitting variables. The biographical variables are also split by clusters. Some of them are *split variables* when the percentage of the answer to this question in the cluster is very different than in the whole population. The splitting and split variables are essential because they enable to explain heterogeneity and differences that are not shown by aggregated statistic indicators.

From the previous cluster analysis, three French primary school teachers belonging to different clusters were interviewed at the time of the LEMA project. This interview will be mentioned in this paper as *LEMA interview*. The LEMA interview is analysed as suggested in Creswell (2007, pp. 156–157) for grounded theory approach: transcription and reading, reduction to different categories and development of relations between categories.

It is not the focus of this paper to explain how the answers to the questionnaire will be analysed. But we can mention that a cluster analysis similar to the previous one will enable the description of every cluster with biographical variables (country, age, experience, job and education), mathematic conception variables, modelling conception variables, and variables about difficulties by modelling teaching. Interpretation of these clusters through levels of determination and additional clarifying interviews with clusters paragons will complete this analysis.

Comparative approach

We will also adopt a comparative approach as suggested in Cabassut (2007) to observe the levels of determination of modelling conceptions in different countries. The levels of determination can appear explicitly (in the questions or in the interview) and implicitly in the possibility to compare the answers between countries. The interview guide has to be formulated clearly for each country of the project: we will be aware about countries cultural references and translation issues. For example the expression "inquiry based approach" is not translated word by word: we use "démarche d'investigation" in French and "problema de investigación" in Spanish.

In the chronology of designing the semi-structured interview, we started from an existing questionnaire of LEMA project that was modified in a dialectic process with every new review of papers, results of exploratory analysis of existing answers to LEMA questionnaire, results of LEMA interview, and discussions between the two authors of this paper. As resulting process, the different parts of semi-structured interviews are defined. Then, the general questions introducing each part for the non-directive part of the interview are formulated and completed. Thus, for the directive part, the corresponding questionnaire is completed. The guide is adapted for every category of *mathematical educational sphere*. We choose to present now every part by illustrating it with questions and their justifications from literature and existing analysis.

DESIGN OF SEMI-STRUCTURED INTERVIEW

Research shows that biographical questions, within the first part of interview, play an important role in the teaching of modelling. Cabassut and Villette (2011) show that country, age and type of school are split variables. Borromeo and Blum (2013) show differences between teachers who studied mathematics as a subject or not; Borromeo and Blum (2013) and Kuntze (2011) identify the influence of experience in teaching mathematical modelling, what Cabassut and Villette (2011) identify as split variable. Dorier and Garcia (2013) point the importance of initial training.

The second part deals with the conceptions and practice about mathematics and can be introduced by the following questions "What are mathematics for you? What is mathematic teaching for you? How do you teach mathematics? How easy is it?". As pointed by Maaβ and Gurlitt (2009) "teacher's knowledge and beliefs about the nature of mathematics [...] influence how they design or select tasks, plan, implement and evaluate their lessons". Also Lee (2012) pointed "the significant impact of teachers' knowledge and beliefs on their interpretation and implementation of curricula and daily teaching practices". We keep the splitting questions of LEMA dealing with mathematical beliefs (Cabassut & Villette, 2011).

An interviewee from LEMA interview mentions that the difficulty related to heterogeneity is not specific to modelling problem but general to every mathematical problem. Another interviewee asserts that there are no more difficulties in modelling situations than in ordinary ones. Another one mentions group management difficulties. For this reason we ask questions about difficulties in mathematic, to be compared with the difficulties about modelling.

The third part of the interview deals with conceptions and practice about modelling. If the interviewed person does not understand the word "modelling" and to enable him to go on with the interview, we suggest a possible definition based on Maaβ definition mentioned on previous part on modelling conception. Borromeo and Blum (2013) are also defining modelling and LEMA questionnaire suggests an example of modelling task. About practice, Borromeo and Blum (2013) show the influence of experiences in teaching mathematical modelling concerning barriers and motivations. They point also the importance of training on modelling, which can only be learned effectively if there are teachers that have competencies in this field. An interviewee from LEMA interview expresses needs to create a lot of modelling situations to be used during teaching. In following part 6, the extract of interview guide is related to conceptions and practice about modelling.

The last part deals with difficulties about modelling collected from international research to observe how these are perceived in the French *educational sphere*. We selected six topics: time, assessment, lesson organisation, context, students' involvement and resources. This selection is explained by the following reasons. Schmidt (2011) and Borromeo and Blum (2013) find time, assessment and resources as the three main difficulties by German teachers. An interviewee from LEMA interview mentions he needs time to keep the modelling process in class. Additionally, Borromeo and Blum show that "for 50% of the teachers, "time" is seen as a barrier". But when it is a barrier for inexperienced teachers, it is not the case for experienced teachers; they observe the same difference about material. From the questionnaire of Engeln and colleagues (2013) on inquiry-based learning, we adapted some items to modelling: for example "to find modelling problems for the class I use textbook?". Concerning assessments, Borromeo and Blum point that "teachers who did not study mathematics see here a barrier to teach modelling and for the other teachers "assessment" is a strong motivator". About lesson organisation, Borromeo and Blum show that lesson-planning is relevant for the teacher but it is not necessarily seen as an obstacle; for several teachers it is a motivation. Cabassut and Villette (2011) find splitting variables about lesson organisation (for example, to design the lesson or to help student). An interviewee needs to prepare a lot for a modelling lesson. About the context, Borromeo and Blum point "systemic obstacles (such as expectations of parents, scientific associations and other pressure groups, regulation examinations)". An interviewee from LEMA interview mentions that the syllabus is not an obstacle and can be interpreted in his mind. Sometimes "the resistance does not only come from teachers but also from students or maybe even parents or the society as a whole" (Dorier & García, 2013). The topic about students' involvement is based on Borromeo and Blum's questionnaire. This part about difficulties is illustrated in the following extract of the interview guide.

EXTRACT OF THE INTERVIEW GUIDE

This extract is from parts 3 and 4 of the interview guide for teachers.

Part 3 Conceptions on modelling

Do you know what means "modelling problem"? (Yes/No)

If yes go on. If no go to item « A possible definition of a modelling problem »

Open question

For you what is a modelling problem at school?

Complementary questions

For each of the following statements mark how much you agree (agree, rather agree, neutral, rather disagree, disagree):

- a modelling problem at school is related to real world
- a modelling problem at school deals with authentic data
- for a modelling problem at school, there is only one solution
- a modelling problem at school is an open problem
- a modelling problem at school requires an inquiry based approach
- pupils have to be familiar with the context of the modelling problem

A possible definition of modelling problem

We consider a possible definition of modelling problem and other terms.
A modelling problem is:

- a problem (no immediate answer),
- open (it could be necessary to make some parts clearer: questions, data, hypotheses, mathematical solution, way to find a solution...)
- complex (many steps to solve the problem, some data have to be searched, discovery of a new type of solution...)
- related to real world (the context of the problem is related to reality on the contrary of some abstract mathematical problems)
- using mathematics and inquiry based approach to find a solution. How much do you agree this definition? (agree, rather agree, neutral, rather disagree, disagree)

Modelling practice

Open question

Do you use, in your teaching, problems that are open, complex, related to real world, and solved with mathematics and inquiry based approach? How? What resources do you use?

Complementary questions

How often do you use the following types of problems in class? (often, rather often, rather not often, often)

- Modelling problems
- Problems to be solved with inquiry based approach
- Authentic problems from reality.
- Problems with data interpretation
- Open problems
- Complex problems
- Problems with more than one solution

Resources

To find modelling problems for the class I use: (often, rather often, rather not often, not applicable because I don't use modelling)"

- textbooks
- internet
- exchanges with colleagues
- exchanges with pupils or students
- other resources

Mention the other resources

Part 4 Facilities and difficulties of modelling
Open question
Is teaching of modelling easy or difficult? Why?

Complementary questions
How much do you agree (agree, rather agree, neutral, rather disagree, disagree)?

Time
It takes too much time to prepare modelling task for teaching.
The work on modelling tasks in the classroom is very time consuming.
It takes too much time to assess modelling tasks.
When I teach modelling, I have not enough time left for other learning content.
It is difficult to estimate how long it takes to solve a modelling task.

PERSPECTIVE OF CONFIRMATORY RESEARCH

With the interview guide, several interviews will be addressed to different members of *educational sphere*. In a next step of research, the results of these interviews will help to formulate research hypotheses and to adjust the final questionnaire. Then, a large-scale questionnaire will be processed to confirm hypotheses. It is difficult to get answers to a questionnaire from some members of *mathematical education sphere*. In this case the questionnaire will be replaced by an interview. “Finally, the question of how culture-dependent task-specific convictions of mathematics teachers related to modelling are merits attention in corresponding comparative research” (Kuntze, 2011). It is why all this research includes a comparative component to study the role of the different levels of determination when the context varies. This confirmatory research and its analysis will be the last step of our study.

ACKNOWLEDGEMENT

This research is funded by Strasbourg University and the Mission of IDEX and also by the Spanish Ministerio de Economía y Competitividad under research EDU2012-35638 project.

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Assessing the best staircase: Students’ modelling based on experimentation with real objects

Susana Carreira[1] *and Ana Margarida Baioa*[2]

1 University of Algarve, Faro and University of Lisbon, Research Unit of the Institute of Education, Lisbon, Portugal, scarrei@ualg.pt

2 Group of Schools D. Manuel I – Tavira and Centro Ciência Viva de Tavira, Faro, Portugal, ambaioa@gmail.com

In this study, we look at the use of experimental activities for developing mathematical modelling, considering those as consistent with the Models and Modelling Perspective and Realistic Mathematics Education. The experimentation with several staircases elicited a central model based on linearization. Such model stems from the key idea of average step and constant slope and allows mathematical generalization.

Keywords: Mathematical models, realistic mathematics education, models and modelling perspective, experimental activities.

INTRODUCTION

The use of experimental activities in mathematics is a way of providing students a practical approach to real world problems with a particular emphasis on mathematical modelling. It encourages students to collect data, to interpret them and to develop reasoning and mathematical communication through discussion and presentation of findings. Experimentation with real and material daily objects in mathematics learning puts the real phenomenon as a central part of students’ work, pushing them to understand how it works, to address it from a mathematical point of view, and to look for ways in which it can be “driven” mathematically, while incorporating real experience and everyday-life knowledge.

Unlike what happens in experimental sciences, it is far from common the inclusion of activities in mathematics classes where students are invited to examine and study material objects and artefacts from the real world in search for mathematical ways of interpreting the surrounding everyday reality. However, this may represent an interesting possibility to address mathematical modelling from a distinctive perspective, namely by representing an opportunity to investigate the role of experimentation in the development of mathematical models in school learning (Bonotto, 2003; Halverscheid, 2008).

It is possible to explore the mathematics involved in everyday life by asking about the appropriate size of the steps of a staircase. In the image beside (Figure 1), one can see a diagram of a staircase profile retrieved from a website on housing and accessibility where the ‘step geometry’ for best preventing falls, especially from elderly people, is suggested.

The activity of studying a number of different staircases in the surrounding environment was seen a good context to engage 9th graders in significant mathematical modelling, namely entailing mathematical ideas such as gradient, slope, similarity of triangles, trigonometric ratios and others. Our research aim is to understand the effects of experimental work with artefacts on students’ development of mathematical ideas and concepts through modelling tasks. In particular, we intend to see how the idea of slope is con-

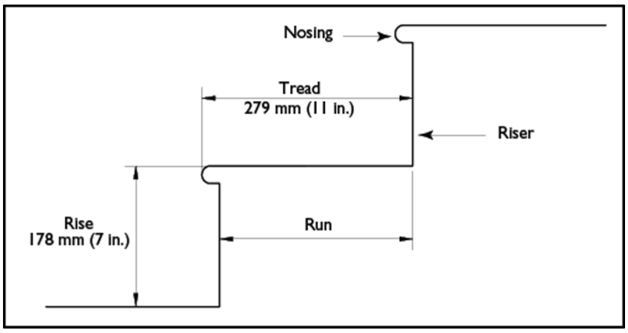

Figure 1: Suggested dimensions for the steps
(Source: http://www.cmhc-schl.gc.ca/en/co/acho/acho_012.cfm)

ceptualised throughout the activity of examining and assessing staircases.

THEORETICAL FRAMEWORK

The theoretical framework draws on both the theory of Realistic Mathematics Education and the Models and Modelling Perspective, by taking experimentation with real artefacts as a hinge which connects and articulates the two theories. This is carried out under the perspective of networking theories that bear significant intersecting concepts in spite of their different roots and foundations: in particular, both theories conceptualize learning in mathematics in terms of constructing and developing mathematical models of the real world.

The model development perspective

In the Models and Modelling Perspective (MMP), the term *model* is the key to explain students' learning of mathematics. A model is conceived as a system of elements, relationships between elements, and operations that describe or explain how the elements interact (Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Zawojewski, 2007). To become a model, the system has to be useful for describing, representing, interpreting, explaining or making predictions about a phenomenon and its behaviour.

In this perspective, modelling activities are important for students to reveal their various modes of reasoning, as well as for the development of conceptual systems and for creating efficient ways of representing structural aspects of the situation.

> The focus of model exploration activities is on the underlying structure of the elicited model and especially on the strengths of various representations as well as on how to productively use these different representations (Ärlebäck, Doerr, & O'Neil, 2013, p. 941).

An essential aspect of MMP is the recognition that solutions to problems usually involve several modelling cycles where the descriptions, explanations and predictions are gradually refined while solutions are being revised or rejected based on their interpretation in the real context.

Model eliciting activities are the kind of activities that provide students with real-life situations where mathematical thinking is necessary for successfully dealing with problems and achieving solutions (Lesh & Lehrer, 2003). The kind of problems proposed is meant to engage students in bringing forth their personal experience and informal knowledge. It is expected that more than one adequate solution is proposed and the use of tools, resources and collaboration are highly endorsed.

Students are presented a real situation from which a problem is originated. For solving it, they usually start to simulate the real situation through experimentation, then they have to model the situation (mathematizing it through diagrams, tables, symbols, relationships between variables, equations...), and finally they generate mathematical results and obtain one or more solutions to the initial problem. Throughout this process students perform micro-cycles of mathematization, i.e., within the modelling cycle, the steps are not all sequentially performed but rather there are forward and backward movements within the cycle itself.

The problem of finding a real model of a convenient staircase seems to comply with this perspective. The real situation of going up and down a ladder is something very common in daily life and it incorporates empirical knowledge and personal experience. There are staircases in many places, both indoors and outdoors, which makes experimentation, simulation, and actual experience with the artefact easy to accomplish. In addition, there is a clear purpose in electing a convenient staircase: everyone has felt, at some point, that there are stairs which aren't easy to climb or descend. Finally, a staircase is an object intended to join by successive steps different levels of a site and the mathematical concept of slope is immediately present in the relationship between the run and the rise of a step. Deciding on the best staircase involves forms of representing and expressing the real world situation, it comprises problem solving (or decision making), brings a specific purpose to modelling, and the resulting model can be developed, modified, and generalised.

The realistic mathematics education theory

Two main ideas that are central to the theory of Realistic Mathematics Education are *mathematization* and *guided reinvention*. Contextualized situations aim to generate meaningful experiences for students that will bring forth the implicit mathematics through a mathematization process. By working on contextualized situations to achieve contextualized solutions, students gradually develop mathematical tools that progressively lead to higher levels of mathematical thinking.

The principle of guided reinvention requires contextualized problems to be well chosen, in order to engage students in the development of strategies that generate informal solutions. The informal way to get solutions is a starting point for formalization and generalization, which is referred to as progressive mathematization (Gravemeijer, 1994; 1999). The process of reinvention is set in motion when students use everyday language (informal description of phenomena) to develop informal or formal mathematical ways of conceptualizing solutions to real problems.

In RME, the starting point of mathematics teaching should be experientially real to the student. The mathematization process will lead to: explore the situation; locate and identify relevant mathematical elements; schematize and visualize patterns, and develop a model that integrates mathematical concepts. It is expected that students subsequently apply these to other realistic situations, and in doing so, reinforce and strengthen their mathematical knowledge.

The models allow students to work at different levels of abstraction, making possible that even those who have difficulty with more formal notions make progresses and create strategies for solving problems (Gravemeijer & Stephan, 2002). The term model refers to models of situations and to mathematical models that are developed by the students at different abstraction levels.

Four levels of models are described in the design of experiences based on the RME theory. At the situational level, domain-specific knowledge and situational strategies are used within the context of the situation, taking into account the knowledge and experience often obtained out of school; at the referential level or in the production of *models of*, the models are closely related to the situation described in the problem; at the general level or in the production of *models for*, mathematical strategies dominate over the reference to the context; and at the formal level, one works with conventional procedures and notations (formal mathematical language) without the support of the context of the situation (initial model).

One of the features that stands out from activities grounded on the RME theory is the mathematical concept(s) that the student is expected to develop. This is usually apparent when reading and interpreting the situations presented, which sometimes include a request to find a generalization. This generalization is not the only goal but it is surely a goal to be reached throughout continued work on this type of activities.

From the point of view of RME, the situation of deciding on a convenient staircase motivates students to work actively on the context. The mathematization process is expected to start with realistically considering the actual situation of climbing and descending staircases. A referential model would probably start to grow from sketches of different staircases, leading to the idea of step as a covariation between horizontal and vertical distance. A model of the covariation might be developed in close relationship with the particular features of a staircase and in terms of how it fits human body (the size of the foot and the range of a person's step). Ultimately a general model of a staircase may be developed in terms of relevant dimensions and this entails a mathematical model for the slope of a straight line.

Experimentation and artefacts use in mathematics

In this study, we have in mind the rationale offered by Bonotto (2003, 2010, 2013) and by other researchers (e.g., Alsina, 2007; Halverscheid, 2008) for the use of experiments in school mathematics. The main idea is that handling and manipulating objects and cultural artefacts existing in students' daily lives is an important way to connect mathematics with reality.

> ... an extensive use of suitable artifacts could be a useful instrument in creating a new link between school mathematics and everyday-life with its incorporated mathematics, by bringing students' everyday-life experiences and reasoning into play (Bonotto, 2010, p. 21).

It can be argued that mathematical modelling involves a comprehensive cycle that stands on its own as a mathematical approach to solving real problems. However, both MMP and RME are theories that seek to connect modelling and mathematizing with the learning of mathematical ideas and the development of mathematical meaning. They both promote activities where realistic situations require some kind of informal knowledge and the unfolding of some mathematics by acting upon and representing real phenomena. They both place mathematics as the ultimate target of the modelling activity but they both acknowledge that models have to be evaluated as to their agreement with the real world and revised if needed.

Experiences with cultural artefacts are more than a context for the formulation of problems; they are the realistic contexts in which data, informal constructions, conjectures and testing occurs, that is, they become the real world.

> Experiments related to mathematics find their natural place in the framework of mathematical modelling because they represent the 'rest of the world' for which mathematical models are built (Halverscheid, 2008, p. 226).

Experimental activities with real objects as the basis for mathematical modelling are grounded on the following facts: (1) students have the opportunity to "learn by doing" as they manipulate and experiment, conjecture and validate; (2) working with concrete materials is a form of questioning mathematically the properties of objects; and (3) inquiry through experimentation is reflected in thinking and mental models and becomes a means to develop an understanding of mathematical models.

Quite often the argument that manipulatives are only useful in elementary schooling is set forth. In contrast, we claim that mathematical modelling with experiments and artefacts may synthesize the connection between two relevant theories – RME and MMP – for the inclusion of mathematical modelling in mathematics learning.

METHODOLOGY

Our study concerns a classroom activity implemented in two classes of 9th grade students (14–15 year-olds), one with 20 students and the other with 23 students. Throughout the teaching intervention several other activities that involved experimentation with cultural and everyday objects were proposed. The activities were performed in groups of 4 to 5 students in each class, totalling ten groups. The groups were video-recorded during the classes; the transcribed dialogues and the written reports of the groups were collected and analysed.

Qualitative content analysis of the data was developed and episodes were selected to document students' models and the conceptual structures underlying such models.

The task

The real problem consists of establishing the convenient dimensions of a staircase for a house building. To initiate the activity, the examination of particular cases was proposed as a means to define the convenience criteria; measuring, recording and sketching were also conducted, which entailed the identification of variables and relationships between them (Figure 2). Therefore, students went out of the school and walked in the city to experience many of the stairs that are part of the outdoor architecture, from the oldest to the newest ones.

The activity was divided into four parts. The first is the introduction to the topic under study. The second, called "From experience…" is a hands-on experimental phase performed with everyday objects (staircases) aiming at developing informal models and collecting data. The third, called "… to the model" consists of a mathematical study of the data obtained in the experimental stage, and one of its goals is the creation of a local mathematical model and possibly a more general model that can apply to wide-ranging situations. Finally, a written report is required to document the experimental situation, the assumptions made, the strategy used, the results obtained, and the evaluation of the proposed solution.

Figure 2: Staircases in the city and students collecting data

DATA AND RESULTS

The model of the average step and constant slope

During the fieldwork, all students experienced up and down five staircases with different characteristics, in order to classify them from 1 to 5, corresponding to a scale from less convenient (1) to more convenient (5). They also measured the run and the rise of several steps in each of the staircases and recorded them on their worksheets. The five staircases were proposed by the teacher within a relatively limited area around the school and fitting the time available for working on the outside.

One observation made by several groups was that the steps in certain staircases did not maintain the same dimensions, with variations of a few centimetres on the treads and on the mirrors. The subsequent discussion was on the possibility of standardizing the sizes of the steps on each staircase considered. The idea of taking the average of the lengths of the treads and the lengths of the mirrors was one of the suggestions. One group realised that the sum of the treads corresponded to the total run the same way as the sum of the mirrors was equal to the total rise. Then, dividing by the number of steps they got the average step dimensions and started to analyse the ratio between them. Another group preferred to measure the pitch line (the hypotenuse) for a sequence of k steps and found the total rise by adding up the mirrors of the k steps. Then using Pythagoras obtained the total run and finally divided the run and the rise by k to get the average step dimensions.

Therefore the model used by most of the groups for the staircase was a model of constant slope regardless the variability of the steps. By considering homogenous steps, students were making the assumption that each step corresponds to a unit of change. No matter the steps were irregular, the inclination was consistent in every step. Based on the ideas that students proposed, which evolved around the steepness of the staircase, the notion of constant slope was being developed as schematized in Figure 3. Even though they have not mentioned this ratio as the slope of the pitch line, their reasoning involved the idea of inclination of a straight line.

In their reports, students showed their representations of the staircases and their evaluations in terms of convenience for climbing and descending. To their qualitative evaluation they associated the ratio of the vertical change (mirror) to the horizontal change (tread), even though they did not define this ratio as the slope of the pitch line. In their discussions it became consensual that there was a distinctive ratio for the most convenient stairs, as in the following dialogue:

> (After reviewing the classification made and the relations found)
>
> Student M: So the conclusion is that the tread should be twice the mirror so that the stairs are convenient!
>
> Teacher: But can the tread be of any length as long as it's twice the mirror?
>
> Student D: A bit larger than the size of the foot.
>
> Student M: About 30 cm.

An extract of this group's report shows how they assessed a few staircases (Figure 4).

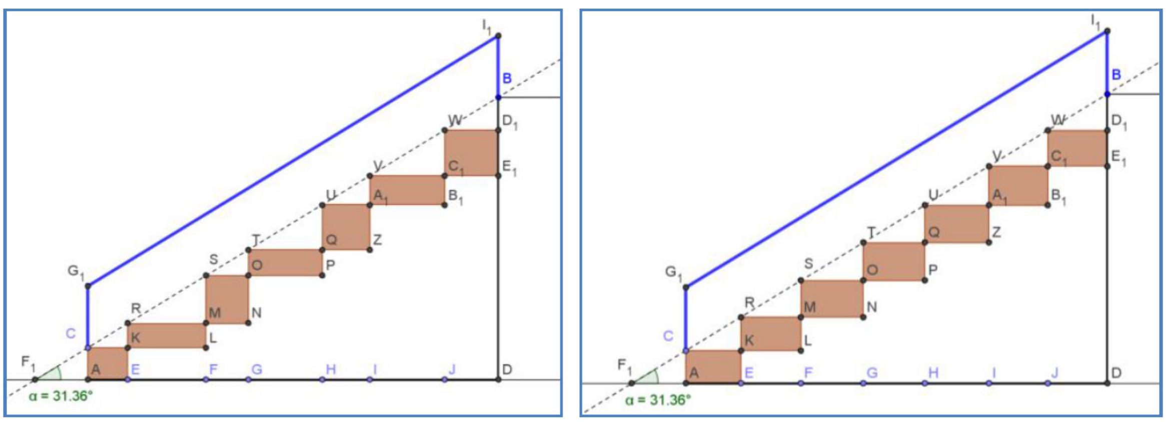

Figure 3: The variation of the steps doesn't change the slope of the staircase

Escadinhas do Castelo: O espelho é aproximadamente metade do cobertor, por isso as escadas são cómodas.

Escadas da Praça da República: O cobertor é aproximadamente 2,5 vezes maior que o espelho.

Escadas das Portas do Castelo: O cobertor é aproximadamente 4 vezes maior que o espelho, mas como é exagerado, as escadas não se adaptam ao nosso andar.

Stairs of the Castle: The mirror is approximately half the tread, therefore the stairs are convenient.

Stairs of the Republic Square: The tread is approximately 2,5 times longer than the mirror.

Stairs of the Castle Doors: The tread is approximately 4 times longer than the mirror but this is too extreme and therefore the stairs are not convenient for our climbing.

Figure 4: Excerpt of a report with the slope as a criterion for convenience

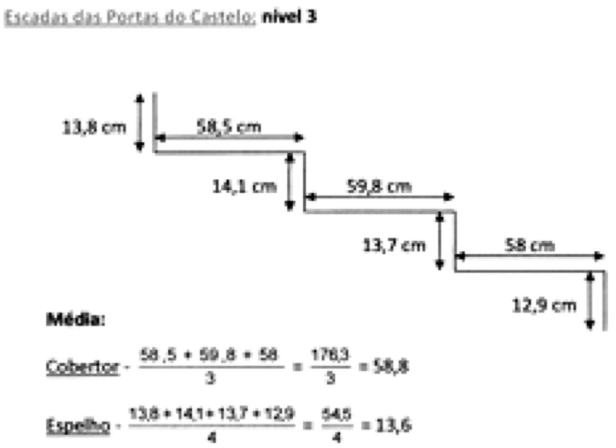

Figure 5: Model of the stairs of the Castle Doors (level 3) and the average step

Another piece of the report shows the idea of using the mean of the dimensions of a sample of steps to find out the dimensions of the average step by assuming a constant slope for the staircase (Figure 5). This was a common strategy in other groups, as a way of dealing with the irregular steps.

The conclusion of this particular group about a convenient staircase was summarised as follows: "We have found that the tread should be approximately twice the mirror so that the stairs are convenient, and the tread would have the approximate length of the foot of most people, therefore measuring between 30 and 40 cm, depending on being indoors or outdoors stairs. As to the mirror, it should measure between 15 and 25 cm".

DISCUSSION AND CONCLUSIONS

The mathematics curriculum for grades 7–9 covers the study of the linear function and stresses the meaning of the parameters in the equation $y = ax + b$, in particular through its graphical representation. The meaning of the parameter a is handled qualitatively in terms of the slope of the line. The linear function is also related to the idea of constant rate of change. However, this conceptual model brings the difficulty of associating a rate between quantities to a geometric notion.

The task of finding a convenient staircase led to the observation that real stairs in the existing surroundings are not perfectly regular stairs. It also showed several facts clearly important to the concept of linear variation and to its translation into a geometric model. In fact, a plausible real model of a staircase is coherent with the notion of regularity, that is, all the steps having the same dimensions. However, it is not necessarily so in real life. Yet, the idea of a constant slope is a good initial model of a staircase. In other words, the average slope of the steps corresponds to the slope of the staircase, and both are consistent with the rate between the rise and the run.

From the point of view of the MMP, the activity of choosing a convenient staircase elicited a relevant mathematical model. The profile of a staircase is consistent with the constant ratio between the length of the tread and the length of the mirror thus suggesting how steep the staircase is. It is a robust and mathematizable idea which makes the basis of the mathematical model of linear variation (or slope):

$$\frac{(\Delta y)_1}{(\Delta y)_1} = \frac{(\Delta y)_2}{(\Delta y)_2} = \dots = \frac{(\Delta y)_i}{(\Delta y)_i}$$

for any number of steps i, and

$$\frac{\sum_i (\Delta y)_i}{\sum_i (\Delta y)_i} = \frac{(\Delta y)_i}{(\Delta y)_i}$$

Its translation into the geometric constant rate of change is well represented by the variable but proportional dimensions of the steps as in the following graphs, showing different changes in run and rise (Figure 6).

Regarding the theory of RME, the mathematization of the idea of a regular staircase matches an informal model of linearization, leading to calculating the mean of the dimensions of the different steps. Then,

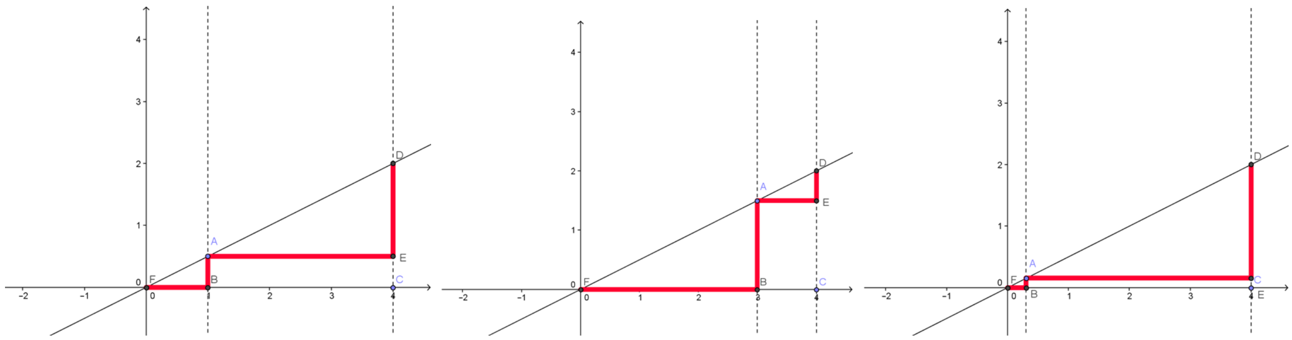

Figure 6: Variable but proportional dimensions of the steps (equal slope)

the relationship between the average rise and the average run develops into a gradient: the constant rate at which the value of the rise changes with respect to the change of the run, or the 'slope of the staircase'. Therefore the slope has a connection to the steepness of the stairs as expressed by the ratio of the mirror to the tread. A model of a convenient staircase (referential model) was characterized by the students as having a ratio of 2:1 for tread to mirror. This is a *model of* a convenient staircase but it is also a model that can be easily transformed into a *model for* any staircase and likewise for the formal constant rate of change.

Both theories allow seeing mathematical modelling as a process that simultaneously creates a simplification of reality, introduces a mathematical point of view and leads to interpreting reality through mathematics and interpreting mathematics through reality.

The experimental, practical and direct way to address the problem through examination of several outdoors real stairs proved to be of vital importance. In the real world stairs are uneven and linearization emerged as a sensible mathematization tool.

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Competing conceptual systems and their impact on generating mathematical models

Jennifer A. Czocher

Texas State University, San Marcos, USA, czocher.1@txstate.edu

Using a models and modelling perspective, this paper examines instances of competing conceptual systems within a problem situation for an engineering undergraduate. The goal is to explore how conflict impacts the framing of a mathematical model and to present evidence that the assumptions the modeller makes are influenced by his mathematical framing of the problem context.

Keywords: Mathematical modelling, tertiary education.

Research on mathematical thinking often focuses on pointing out or classifying student misconceptions. Fewer studies have examined how an individual makes sense of the problem context, though the student's interpretation may differ from what the task writer intends (Stillman, 2000). Sense making in modelling tasks is worth examining explicitly because student work leading to errors or incorrect models may be attributed to mathematical misconceptions rather than to the student's interpretation of the task.

It is unclear how an individual's framing of a modelling task is tied to the generation of a conventional mathematical representation, especially in a case where there are multiple possible framings. Each framing comes with its own set of assumptions, variables, parameters, and even constraints against which the model and its predictions must be checked. In a group work setting, Lesh & Doerr (2003) identified the production and resolution of "competing conceptual systems" as part of model creation and Lesh, Doerr, Carmona, & Hjalmarson (2003) classified the cognitive conflicts that accompany competing conceptual systems. But what is the interplay between mathematical framing and conceptual systems?

Drawing on a models and modelling perspective, this paper closely examines instances of competing conceptual systems for a problem situation within a single individual in order to explore how the mathematical framing impacts the creation of the mathematical model. I argue that model construction is influenced by the mathematical framing adapted or derived from the individual's interactions with the problem context and that additional sources of knowledge, beyond mathematical and contextual, are necessary for resolving the conflicts that arise from competing conceptual systems.

THEORETICAL PERSPECTIVE AND DEFINITIONS

The theoretical perspective adopted here draws on the models and modelling perspective (MMP) (Lesh et al., 2003) and a characterization of modelling as a process (the mathematical modelling cycle or MMC) (Blum & Leiß, 2007). The theories are taken together in order to operationalize *mathematical framing* (defined below) and examine its connection to the conventional mathematical representation constructed to represent the real life situation being modelled.

In the most general terms, a *model* is a simplified representation of a system. A *mathematical model* has three components: a situation in the real world, a mathematical representation, and an invertible relationship between the two constructed by the modeller (Blum & Niss, 1991) that preserves structural characteristics as mathematical properties. A key component is identifying appropriate structural characteristics that can be put into correspondence with appropriate mathematical structures and concepts.

The MMP emphasizes the usefulness of mathematical concepts, that models (and not solutions) are the important products of modelling tasks, and that an individual approaches a task with an initial interpretation of the task. The process of formulating a well-posed mathematical problem can be summarized as *framing* – how an individual renders a problem setting, giving it context, determining which facts and relationships are relevant, and which rules are usable for reasoning (Schwarzkopf, 2007). The framing process can be operationalized in terms of the MMC, which is briefly introduced below.

Blum & Leiß (2007) decomposed the mathematical modelling process into six stages of modelling construction supported by six transitional activities. The first three stages of model construction are: the situation model (an understanding of the problem; a conceptual model of the problem), the real model (an idealized version of the problem with simplifications and assumptions), and the mathematical representation (in conventional mathematical terms). The stages are connected by activities. The situation model is brought about by an individual forming an *understanding* of the problem. During *simplifying/structuring* the modeller identifies assumptions, variables, parameters, and conditions that reduce the potentially messy problem to an idealized real model. For example, an object falling from a height might be thought of as an object in free fall and a freebody diagram could be drawn where the relevant forces from Newtonian mechanics are identified. Next, *mathematizing* occurs where the individual represents the real model (in essence, a collection of assumptions and constraints) in conventional mathematical terms. This is the mathematical representation from Blum & Niss's (1991) definition.

Cognitive conflicts, in the Piagetian sense, often arise as the modeller attempts to frame the problem. Lesh, Doerr, Carmona, & Hjalmarson (2003) described three kinds of cognitive conflicts that arise as conceptual models develop: within-model mismatches (incongruence among aspects of representational media), model-reality mismatches (when predictions do not match reality), and between-model mismatches (incongruence between ways of thinking about a problem). For example an object falling from a great height could be modelled using the algebraically based kinematics equations. We know from mechanics that these equations hold only when air resistance is negligible. In a situation where an extended body is falling a great distance the effect of air resistance is not negligible. If we used kinematics equations to predict the velocity of the body at time , the prediction might differ significantly from an actual measurement. This is an example of a model-reality mismatch. Debating whether to use kinematics equations or more accurate differential equations to model the situation would be an example of a between-model mismatch.

The three mismatches are not mutually exclusive but are evidence of competing conceptual systems and highlight where the individual is considering alternative framings of the task. In this paper, I refer to the mismatches with regard to local mathematical model construction rather than global mathematical knowledge construction. The results are presented as a set of illustrative vignettes purposefully selected for their ability to demonstrate and explain the impact of framing on model construction.

METHODOLOGY

This study followed a case study logic found in social sciences (Walton, 1992) where the guiding principle was to provide evidence that challenged dominant ideas about the modelling process. Specifically, the goal was to provide insight into how competing conceptual models in order to argue that framing is idiosyncratically tied to the modeller's experiences. The phenomenon to be illustrated, *competing conceptual systems* "coalesced in the course of the research through a systematic dialogue of ideas and evidence" (Ragin, 2004, p. 127). The case is presented through a set of illustrative vignettes purposefully selected for their ability to demonstrate and explain the impact of framing on model construction.

Data were collected from a series of seven task-based interviews with four undergraduate engineering students enrolled in differential equations. Tasks were designed in order to evoke the mathematical modelling cycle (Blum & Leiß, 2007) and in accordance with guiding principles on openness of the problem statements (Lesh & Zawojewski, 2007; Maaß, 2010). The participants were selected in order to maximize variety in their approaches to modelling tasks and so they had a range of mathematical strength. The participant whose work is reported in this paper, Trystane, did not score top marks in his mathematics or engineering classes. His work on the Falling Body Problem (described below) was selected to share because within one task he exhibited all three kinds of cognitive conflict.

The interviews were video recorded and transcribed. Cognitive conflicts were identified in two ways. First, the transcripts were coded according to the transitions in the MMC and the occurrences of the first three transitions were examined to be sure that framing was taking place. Second, transcripts were summarized as thick descriptions (Geertz, 1973) of the mathematics used by each student during each task. Introduction, changes, or adjustments in mathematical framings were tracked. Throughout the reduction, my focus was on the student's mathematics and on its structural ties to the real world situation presented by the problem statement. This analysis through writing produced a second-order model (Steffe, 2013) of the student's mathematics during modelling. Instances were interpreted as competing conceptual frameworks when the student was debating among mathematical framings or among assumptions that would simplify the problem but would require different mathematical framings.

The Falling Body Problem, for which Trystane's work is presented below, is a first-year physics or calculus problem solvable by kinematics if one assumes that there is no wind resistance. Otherwise, a differential equation is necessary to model the falling body's velocity.

On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin, TX died from injuries sustained after jumping from a second floor window to escape a fire at his home. What was his impact speed?

ANALYSIS AND RESULTS

The vignettes presented below are based on the thick descriptions but reference the transcripts from a problem common to a first year kinematics course, *The Falling Body Problem*. Trystane's work on this task was selected to illustrate cognitive conflict and competing conceptual models within a single individual because it offers a sense of tension as he debated the merits of each. The ongoing tension is not conveyed when examining model revision among multiple individuals.

Trystane explicitly considers three different mathematical framings of the task that eventually lead to model revision. The framings were identified because he returned to understanding and simplifying/structuring the problem, according to the MMC. He first considers using algebraic kinematics equations, which wouldn't account for air resistance. He then considers algebraic energy equations, until he realizes he doesn't have information about the man's weight. He then tries to derive a differential equation in order to account for air resistance. These distinct framings account for different variables, require different sets of assumptions, and rely on different mathematical concepts and structures.

Within-model mismatch

Trystane began by reading the task silently to himself and then stated that he would need to know how big the man was in order to account for air resistance. He indicated that mass was not an important factor and that his approach would be to use a "bunch of kinematics formulas." The kinematics formulas would not account for air resistance. He then indicated that he would like to use energy equations but that he'd need to know the man's weight. Since that information was unavailable, he'd have to "do it a different way and [he'd] have to know how high he fell from." Trystane's conflict arose due to a between-model mismatch. He was considering the merits of adopting different mathematical framings. Note that this is distinct from adopting different mathematical representations because both framings would be expressed in terms of an algebraic formula. There was a mismatch between the information perceived as available (note that he could have used m for mass) and his preferred mathematical framing (energy equations) leading him to adopt the kinematics equations. This framing subsequently influenced the variables and assumptions that could be made, prompting him to seek the height the body fell from.

Model-reality mismatch

Next, Trystane estimated that the window was 16 feet above ground. He attempted to use kinematics equations to calculate the impact velocity but encountered difficulty since "it's [the equation] got time in it and I don't know how long he fell." He resolved this issue and obtained an impact velocity of 32 ft/sec. He validated his answer: "At 32 feet, is actually the height of my house, is how I would think of it. He would fall, he would go that distance in 1 second. I'd say that's reasonably fast for a human to go." The interviewer then asked whether the size of the man mattered. Trystane responded, "Technically, yeah. If he's a really big guy, he's gonna have more wind resistance falling down. Other than that it wouldn't." Trystane indicated that he wanted to take wind resistance into account, but didn't know how to. This suggests a model-reality mismatch because he knows that the extended body will experience air resistance but his model does not account for it.. The competing conceptual systems were acknowledged (desire to include wind resistance and the fact that the kinematics equations did not incorporate it) and led to Trystane rejecting the kinematics equations and seeking to use a differential equation to describe the body's velocity throughout the fall.

Trystane began speaking about changing rate-of-descent and the interviewer responded by challenging his choice of model.

> Trystane: Air resistance is more the faster you go, rather than the slower you go. So in [the man's] case, the higher up he fell from, the more wind resistance until he reached his terminal velocity. Which is when wind resistance is pushing up as much as gravity is pulling down.

> Interviewer: What I'm hearing you say is that in very few cases wind resistance is actually negligible. You have to have a very short fall or be in a vacuum. Does it bother you at all that you don't learn how to take that into account or that the models you use all assume no air resistance?

Trystane responded that he felt that including air resistance in the models early in his physics studies would have been "needlessly complicated." He explained that wind resistance is negligible for most applications and offered an empirical demonstration by dropping a pencil and stated that "the pencil falling this high [placed pencil lower, near the desk surface] and this high [placed pencil higher, further from the desk surface] is so close to the same that it's not worth taking the effort to figure out what it is." This explanation demonstrates that negligibility of wind resistance has to do with height rather than other factors he had mentioned previously such as mass or the value or change in value of the force due to wind, or size of the object. In general, negligibility of a variable is an assumption that is related to the estimated sensitivity of predictions to that variable and also to available information.

Between-model mismatch

Trystane's treatment of wind resistance demonstrates that he weighed expended effort against improvement of results when determining which variables to include and therefore which mathematical model to adopt. Though not in competition, Trystane entertained two conceptual models of the problem situation, one with wind resistance and one without. The source of the dispute was a between-model mismatch. The resolution was to choose the less-complex model because the extra effort necessary to build a more accurate model was not worthwhile.

Later, the interviewer suggested that Trystane use a differential equation to model the velocity of the falling body. He concurred and then began by indentifying the man's movement rate as a function of time as an important variable. He then wrote the equation (in Figure 1), which is a first-order, linear, homogeneous equation in standard form. He then wrote the generic solution where Q represents position and dQ/dt represents velocity. He wrote the solution with the intention of determining the value of . Assuming the initial positions and velocity were both zero, he substituted the general solution into the equation and obtained the expression and thus the result

Within-model mismatch

Trystane pondered the correctness of the model:

> I'm not sure that that's right because I'm not sure if there should be some sort of constant increase

$$Q(t) = Ce^{\lambda t} \qquad Q(t) = 0$$
$$Q(0) = 0$$
$$\frac{dQ(t)}{dt} + \beta Q(t) = 0 \qquad \lambda Ce^{\lambda t} + Ce^{\lambda t} = 0$$
$$\lambda Ce^{0} + \beta Ce^{0} = 0$$
$$Q(t) = Ce^{-\beta t} \qquad \lambda = -\beta$$

Figure 1: First order linear differential equation with initial conditions

> as you get faster, um, I guess that just stems from fluid mechanics. For instance, I don't know if it's a linear graph [draws left graph in Figure 2] or if as you're going faster it gets [traces figure on the right].

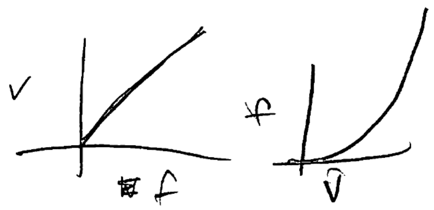

Figure 2: Exponential and linear relationships between air resistance and velocity

The graphs represented force due to wind vs velocity. Trystane knew that there should be an increasing relationship between the two quantities, but didn't know what the relationship would look like and he was debating which representation best matched the situation. He concluded that the model was incorrect because velocity should not increase exponentially with position as but that it was the best he could do without a fluid dynamics book.

Here, Trystane struggled to validate the model he selected because his conceptual model of the rate of change of the body's velocity was not well defined. The within-model mismatch surfaced as he tried to use the two distinct graphical representations of force versus velocity. The consequence was that he rejected the differential equations model because it was incompatible with his conceptual interpretation of ever-increasing velocity described by an exponential solution.

Next, Trystane noted that wind resistance was

> sort of like friction where it's an opposing force. Actually, that kinda sparks an idea. If the wind is always just an opposing force, it could be treated like the force of friction. I dunno. Maybe in a certain density wind you would multiply the coefficient of wind friction or air friction times the surface area of the body moving.

Here Trystane reintroduced the size of the man as an important variable and spent the remainder of the session attempting to include it in his differential equations representation. His final mathematical model (after he "waive[d] the white flag") is the nonlinear differential equation shown in Figure 3.

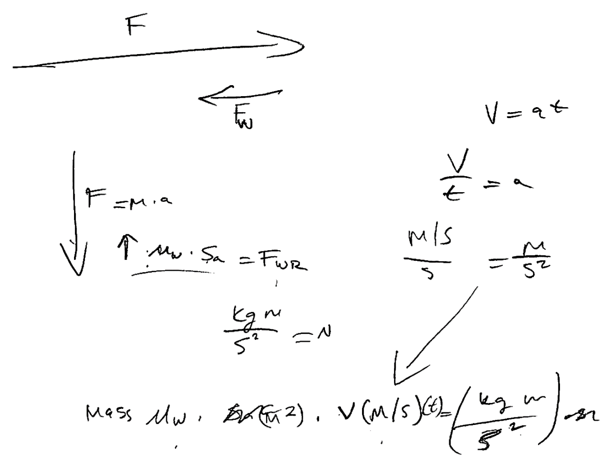

Figure 3: Nonlinear differential equation including wind resistance and surface area

Between-model mismatch

The transition from a first order linear equation with an incorrect solution to the final version of Trystane's model was provoked by competing conceptual systems that had been acknowledged at the start of the modelling task. One conceptualization of the physical system included wind resistance, which depended on the man's size, as a variable. The other conceptualizations ignored wind resistance in favour of algebraic equations with either known or easily-estimated parameters. This ongoing conflict, which was set aside while Trystane explored other possible models, was a between-model mismatch. Trystane's insistence on including wind resistance is a testament to how competing conceptual systems can provoke model development. Moreover, this look at competing conceptual systems suggests that the conceptual system determined the framing of the problem. That is, his selection of which mathematics to use was interactively determined by the data he had available.

DISCUSSION AND CONCLUSIONS

Analysis illustrated three kinds of mismatches that arise as cognitive conflict during mathematical model construction. The mismatches were symptoms of competing conceptual systems each of which came with their own sets of assumptions, variables, and expectations. The conflicts were identified during validating activity when the modeller noted mismatches between expectations of how the mathematical model should be constructed and its congruence with his idealized conceptualization of the situation to be modelled. Thus, expectations were based on both real world experiences, such as Trystane's claim that mass does not affect the velocity of a falling body, and the mathematical structure selected underlying the model, such as when Trystane rejected an exponential growth model for velocity.

Both mathematical and nonmathematical knowledge have been identified as important to model construction. Indeed, modelling is often characterized as bringing together both bodies of knowledge. In the vignettes above, Trystane generated and referred to ideas not explicitly in the problem statement. He relied on additional resources beyond mathematical and contextual knowledge. His spontaneous demonstration with the falling pencils and his resolution of the within-model mismatch between the two graphs of force against velocity suggest that he appeals to common sense and to thought experiments as resources for generating missing information. At times, Trystane had to abandon a particular framing because of a conflict where he did not have the relevant information to use the framing. In another case he was able to generate the missing information via a thought experiment. For example, the task did not give the height the body fell from but he was able to validate his answer (32 feet per second) by using the corresponding length of time it would take to fall 32 feet. In such a case, Trystane used a thought experiment to obtain and validate an appropriate estimate of the height.

Conflict resolution may depend certainly depends on student knowledge about the context and student characteristics (such as persistence). It may also depend on the modeler's values in model construction, and these values may be discipline specific. For example, Trystane demonstrated that he valued economy of effort. He valued avoiding "needlessly complicated" problem idealizations. He also valued avoiding revisions that made the model overly complex for only a marginal gain in accuracy or predictive power.

Trystane used knowledge both of plausible alternative mathematical representations and structures as well as sufficient knowledge of the problem context in order to resolve the conflict product productively. Trystane demonstrated reasoning based on a blend of mathematical and nonmathematical knowledge: his value judgments about effort and worthwhileness and his ability to willingness missing information. This suggests that mathematical and contextual knowledge alone do not account for how the conceptual model or subsequent mathematical model are revised. Future research should investigate criteria or factors students use to decide how to resolve competing conceptual systems.

Two related aspects that merit further examination are *how* the conflict resolution is executed and how conflict recognition can be promoted. When cognitive conflicts arose, at times Trystane rejected the initial model (such as his rejection of the energy equations) or to the revision of a model (such as his decision to include air resistance in the differential equation). One resolution required an additional assumption (neglect air resistance) while another required a change in mathematical structure (regarding net force as a function of time).

Mismatches between the individual's expectations and the model produce the cognitive conflict and the mismatches point to the presence of competing conceptual systems. When conflict is absent, the task may be too easy or familiar or the student may not recognize a particular kind of incongruence that is noticeable to the teacher or researcher. It should be noted that Trystane, as an engineering student was trained to look for such mismatches.

The conflicts arose when Trystane tried to fit available data to selected framings, such as setting initial velocity to zero in the differential equation. He changed representation and structure as information became available rather than undertaking derivations. He made progress when he had a mathematical frame that fit his personal (or scientific) experiences. This suggests that at least in some cases, framing precedes (or even determines) the relevant assumptions and variables sets.

Aside from theoretical consequences to perceptions of mathematical modelling as a cyclic, linear process, these observations have practical consequences. First, tasks should be selected that are amenable to multiple possible framings. Second, if a goal of using modelling tasks is to help students learn to make simplifying assumptions, it may be beneficial to use modelling tasks where potential framings are not obvious.

Mathematical modelling cycles have been long offered as descriptions of the mathematical modelling process, but the community still does not have an adequate explanation for how mathematical and non-mathematical knowledge are blended to render a real world problem as a mathematical one. The models and modelling perspective adopted here, along with the cognitive conflict framework (Lesh et al., 2003) revealed that competing conceptual systems play an important role in the selection of appropriate mathematical structure, mathematical representation and the subsequent fitting of available data into the selected model. The cognitive conflicts framework is a promising avenue to reveal how validation of the mathematical model leads to resolution (or lack of resolution) of the competing conceptual systems.

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Exploring grade 9 students' assumption making when mathematizing

Brikena Djepaxhija, Pauline Vos and Anne Berit Fuglestad

University of Agder, Kristiansand, Norway, brikena.djepaxhija@uia.no

Making assumptions is a key activity in modelling. The present study aims to explore the variety of assumptions that lower secondary students make in this process. As theoretical basis for the data analysis, we used the modelling cycle by Blum and Leiss (2007) and framed a definition of assumptions. The study was carried out with grade 9 students. The results show three categories of assumptions: (1) parameter assumptions, (2) assumptions for the choice of the mathematical model, and (3) assumptions about task expectations. Assumptions from the first two categories assist students to use extra-mathematical knowledge to construct a mathematical model, while the third category of assumptions can assist but also hinder them. Our study shows, that students were well aware of their assumption making.

Keywords: Assumption making, task expectations, modelling, mathematizing, lower secondary students.

INTRODUCTION

In mathematics education there is an increasing emphasis on applications and mathematical modelling (Vorhölter, Kaiser, & Borromeo Ferri, 2014). This international development is also reflected in the intended curriculum of Albania, the country in which the current study is based. In the curriculum framework for primary and lower secondary schools (grades 1–9), the Ministry of Education and Sports highlights that students need to understand the role of mathematics in everyday life and use mathematics adequately to solve problems from everyday life (IED, 2013).

However, Albanian students have displayed a rather poor performance on modelling problems as used in the Programme for International Student Assessment (PISA), in which Albania now has participated three times (Harizaj, 2011; OECD, 2014). The Albanian results on PISA urge for a closer investigation of the process when students deal with modelling problems. Therefore, we have started a study on the first phases of this process, when students are facing a problem situation and have to make a translation to a mathematical model in order to reach a solution of the problem. In particular, we aim to explore the various assumptions that students make in this start-up phase of the modelling process.

THEORETICAL FRAMEWORK

Blum and Leiss (2007) have framed mathematical modelling as a process, which consists of subsequent activities. See Figure 1. The process begins with understanding 'the real situation' given in the task, which leads to the construction of a 'situation model' (e.g., a rough drawing of the problem situation). Then, this 'situation model' is idealised into a 'real model' through relevant structuring, and by making assumptions and simplifications (e.g., the drawing is made more specific).

In the next step this 'real model' is translated into a 'mathematical model' (e.g., an algebraic formula). The 'mathematical model' is then used to obtain 'mathematical results'. These results are interpreted into 'real results' and then validated in light of the given problem. In the case the results are considered inadequate for the real situation the entire modelling process is run through again.

The above description of the mathematical modelling process is an idealized one, because in practice the modelling process is more complex and often non-linear (Borromeo Ferri, 2006; Galbraith & Stillman, 2001). Moreover, Borromeo Ferri (2006) empirically showed that some phases in the modelling process can overlap and that the problem structure affects the process.

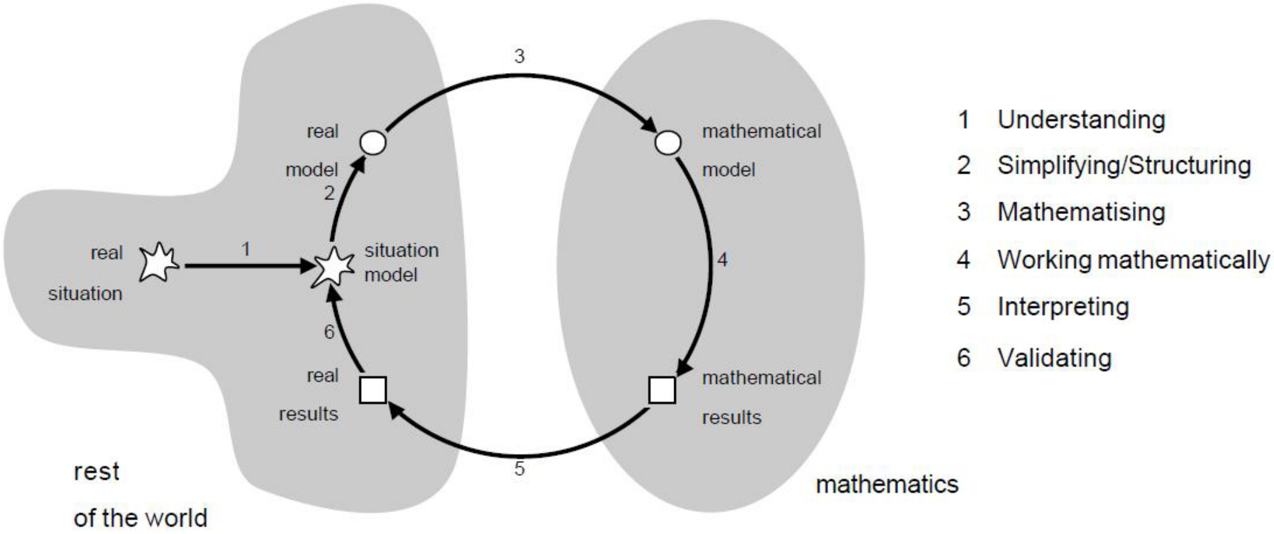

Figure 1: Modelling cycle from Blum and Leiss (2007)

An important term in the modelling process is *mathematizing*. Blum and Leiss (2007) use this term to describe the translation activity from the 'real model' to the 'mathematical model'. However, Borromeo Ferri (2006) has argued that it can be difficult to distinguish between 'situation model', 'real model', and 'mathematical model', because these differ between students and tasks. For example, many tasks with problems from the real world are already structured and de-authenticated by the task authors, to communicate the problem situation unambiguously to the students (Vos, 2011). Therefore, in the present paper the term mathematizing will be used to describe holistically all activities from 'problem situation' to 'mathematical model'. In this frame, mathematizing comprises all activities before a student starts on the purely mathematical work, such as: understanding; simplifying; scouting the problem (Schaap, Vos, & Goedhart, 2011); assuming; structuring; idealising; and finally translating into mathematics. Results of several studies in the field of mathematical modelling (Schaap et al., 2011; Stillman & Brown, 2012; Stillman, Brown, & Galbraith, 2010) display that the mathematization process is complex and that students face various obstacles within this process.

Making assumptions is one of the activities in the mathematization process (Edwards, 1989; Galbraith & Stillman, 2001). Maaß (2006) empirically showed that inadequate assumptions (unrealistic or over-simplified) lead to an inadequate real model, which further leads to an inadequate mathematical model for the problem situation.

Goldin (2002) defines assumptions as being:

> [...] often propositionally encoded, taken as a basis for exploration or discussion but (at least temporarily or provisionally) without attribution of truth, validity or applicability. (p. 65)

This definition converges with the definition in the Oxford English dictionary (www.oed.com), which says that an assumption is:

> The taking of anything for granted as the basis of argument or action.

An assumption can be formulated as *"let's take <statement A>"*, proposing statement A to temporally be true without further argumentation, but statement A offers a productive opening towards subsequent activities. Assumptions made during the mathematization process "provide building materials from the real world to bridge the divide between a descriptive problem statement and its representation in mathematical terms" (Galbraith & Stillman, 2001, p. 304). Assumptions are practical, enabling students to continue their work; assumptions may even assist in overcoming cognitive blockages. The present study aims to explore the variation of assumptions during the mathematization process. The research question is: what assumptions do students at lower secondary school level make while mathematizing?

METHODS

The study was operationalized by studying Albanian students while working on modelling items. The participants in the study were four grade 9 students from a lower secondary school in Albania. The four students were average achievers in mathematics and they were selected by their teacher. They had no particular experience in modelling. The students participated voluntarily and they will be identified by pseudonyms.

A three step design was used to capture a holistic view about students' mathematization process, consisting of (1) observation, (2) stimulated recall and (3) interview (Busse & Borromeo Ferri, 2003). The combination of these three methods draws on strengths of each and makes it possible to capture the connection between internal and external processes which occurred when mathematizing. The first author was the interviewer, being able to communicate with the students in their language. The present paper will report only on the data associated with the assumptions that students made while mathematizing.

In the first phase (observation), students worked in pairs on solving three modelling items collaboratively. To enhance their thinking aloud, just one pen was made available to both students. In this way they could not write both at the same time. The second and third phases (stimulated recall and interview) took place in another session (later on the same day or on the next day), whereby each student was individually invited to watch excerpts of the video recordings from the first session. The students were asked to comment and reflect on their activities in the video. In this way stimulated recall helped students to be "close to the process of working on the task without interfering [with] the process itself" (Busse & Borromeo Ferri, 2003, p. 257). In the third phase (interview), the interviewer cited some statements made during the solution process and asked them to comment on these statements. By using statements of the students, the risk of making students speak with the interviewer's words was avoided.

All the three phases of data collection were video recorded and then transcribed. First, transcriptions were used to identify all assumptions students made in the complete modelling process. Then, based on Blum and Leiss (2007) modelling cycle, we identified assumptions that the students made in the mathematization process. Assumptions associated with the mathematization process were investigated for their purpose (why), their emergence (when), and the awareness that the students displayed on the making of these assumptions.

The modelling items for the present study were taken from PISA. The PISA items for mathematics are designed as problems with a real world origin and they have been designed in a careful process (OECD, 2013). Moreover, their globally widespread use should enable us to compare, validate and generalize research results internationally. We selected PISA items on several criteria: (1) they should require students to integrate the offered task information with extra-mathematical knowledge in order to construct a mathematical model; (2) the cognitive demand should match with students' abilities and therefore the mathematical content was taken from the curriculum of at least one year below the students' level; (3) the real-life situations presented in the tasks should align with Albanian students preferences on the relevance of mathematics, as studied by Kacerja (2012).

FINDINGS

Because of the page limitations of this paper, we report only on the data from one pair of students, the girls Joni and Megi, on two of the PISA items used in the study. The first item is named Rock Concert.

> For a rock concert, a rectangular field of size 100 m by 50 m was reserved for the audience. The concert was completely sold out and the field was full with all the fans standing. Which one of the following is likely to be the best estimate of the total number of people attending the concert?
>
> a) 2000 c) 20000 e) 100000
> b) 5000 d) 50000

Joni and Megi started to calculate the area of the field (100x50) and then started a discussion on how to connect this to the other information in the problem statement.

Megi: Since the concert was completely sold out then there will be 5000 fans because the field was full.
Joni: But one fan per one square meter will be...here it says that all fans are stand-

ing, therefore they are more than 5000... there will not stand one fan per square meter because it has no meaning for a concert...how many fans can stand in one square meter?

Megi: But since nothing else is given, it will be 5000 fans... Just option b) has to do with it. Because we have the numbers 50 and 100 in the problem statement, we do not have any other numbers... The total number of fans according to me is 5000 because we cannot solve a problem by supposing.

In the above episode Megi equals the area with the number of fans, and then Joni translates this into the assumption: *one fan per* m^2. However, she thinks that this is not realistic for a rock concert, considering the information that all the fans are standing, so she implies the assumption that *there are more than one fan per* m^2, and she asks explicitly the clarifying question on the density of fans standing in a concert: *"how many fans can stand in one square meter?"*

Her partner, Megi hesitates to connect the problem statement to the real world. She bases her argumentation on two other assumptions that we reformulate as: *only given numbers should be used to build a mathematical model* and *one cannot make assumptions in these kinds of tasks*. However, these assumptions are not accepted and they continue the discussion on the density of fans. This leads to another assumption:

Megi: The rock concert is attended only by adults, because rock concerts cannot be attended by children... It is attended only by adults and adults use more space.

Here, Megi has changed her perception of what the task is asking her, namely that she can use extra-mathematical knowledge. So, she makes the assumption *only adults attend rock concerts*, which is relevant to define the density of fans.

Thereafter, Joni and Megi are carried away by the context of the task, talking about whether the rock players are popular, or not and there may not be too many fans coming. They also talk about whether there can be chairs in the field. In these discussions we did not identify assumptions, until the next utterance:

Megi: It depends on how fans stand...but we can divide the field in rows and columns... it depends on how fans stand because in a rock concert the audience tries to be as near as possible to the stage, but we can consider the same everywhere.

This utterance shows that Megi recognizes the complexity of defining the density of fans, and that one needs the assumption: *fans are uniformly distributed in the field*.

The above findings from the students' work on the Rock Concert item show that they made different assumptions in the mathematization process. Some of these assumptions can help students to incorporate new information into the mathematical model that is missing in the problem statement, while other assumptions are about their perception of task expectations. Also, we observe that students display an awareness of their assumption making.

The other PISA item, on which we report here, is the Pizza item:

> A pizzeria serves two round pizzas of the same thickness in different sizes. The small one has a diameter of 30 cm and costs 300 ALL. The larger one has a diameter of 40 cm and costs 400 ALL. Which pizza is better value for money? Show your reasoning.

After reading the problem, Joni and Megi make a straightforward mathematization of the situation by taking the ratio between the diameter of the circle and the price. Implicitly, they take for granted that *the size of the pizza is linear to its diameter,* which fits the definition of an assumption. After calculating these ratios they conclude: *"both pizzas have the same value for money"*. However, they do not find this result satisfying – they validate the obtained answer. To do this, they return to the mathematization process and start to ask clarifying questions on the meaning of "better value for money":

Megi: In this case, it means which one is bigger, which one makes you full.

Joni: But it also can be: which one is cheaper, which makes you save money...or which one is smaller, which helps to keep your body in a nice shape.

In this short episode the students make both a dietary assumption ("*makes you full*" and "*keep your body in a nice shape*") and a financial assumption ("*makes you save money*"), which both give a basis for further calculations. After some discussion, the students select one assumption: *better value for money means it makes you full,* because, according to Megi, *"this is what we do in our everyday life"*. Then she uses this assumption to carry on:

Megi: Since they have same ratio, the same thickness, but different size, it is the big pizza because it is bigger…therefore it has more calories …offers more opportunities to make you full.

So, the mathematical model consists of selecting the maximum. However, Joni is uncertain about this approach/choice:

Joni: It is a mathematical problem and we cannot solve it supposing what we do in our everyday life… We involved our individual opinions from our life into the solution and the solution was not a fixed one, we got it by supposing, while in the problems of our mathematical textbook…there are given more numbers and information and the solutions are fixed.

In this utterance Joni expresses uncertainty about the use of familiar, everyday assumptions into the mathematization process. This 'assumption on the making of assumptions' was also expressed in the Rock Concert item: *one cannot make assumptions in these kinds of tasks*. Interestingly, in that case it was expressed by the other student, Megi.

The above findings from the students' work on the Pizzas item show again that they purposefully made different kinds of assumptions. With this Pizza item the assumptions are made either during the mathematizing or the validating process. Some assumptions are made by using extra-mathematical knowledge to choose a mathematical model. Other assumptions limit the students to focus only on the given information in the task.

To summarize, we have categorised the assumptions identified in the above episodes into three broad categories:

Parameter assumptions

Parameter assumptions are assumptions that students make to refine a mathematical model, such as refining the formula in the Rock Concert item: *density of fans x area*. These assumptions incorporate extra-mathematical knowledge into a parameter of the mathematical model, such as in *density of fans*. These assumptions emerge when students understand that the given information in the problem statement is not enough for mathematizing.

Assumption for the choice of the mathematical model

Assumptions for the choice of the mathematical model are assumptions made to select a mathematical model, such as taking either the ratio of diameter and price, or taking the maximum of the size. These assumptions are made in the mathematizing process (for creating a model) or validating process (for critiquing the model and refining it).

Assumptions about task expectations

Assumptions about task expectations emerge from the learning environment of mathematics education, for example when students have been trained on so-called *word problems*. These word problems consist of short texts, generally describing inauthentic situations and irrelevant questions, and students have to deduct some numbers and an operation from the text to find a number answer. Word problems are part of a school culture with unwritten rules, also known as 'didactical contract' (Brousseau, 1997). Assumptions such as *only given numbers should be used* and *making assumptions is not allowed* make students hesitate to consider the situation context and to generate assumptions based on their extra mathematical knowledge. When students' perceptions do not align with the task, their assumptions about task expectations may hinder them. However, not all assumptions about task expectations will be counter-productive. For example, an assumption such as *we should use our extra-mathematical knowledge* is also about task expectations, but it is a productive opening in modelling.

Assumptions about task expectations can be statements on whether or not extra-mathematical knowledge can be used, and these assumptions can assist or hinder students to make modelling assumptions. Assumptions about task expectations emerge in the mathematizing or validating process.

CONCLUSION AND DISCUSSION

In the present study, we explored the various assumptions that grade 9 Albanian students make while mathematizing, that is while constructing a mathematical model from a problem situation. From the data analysis we identified three categories of assumptions:

- *parameter assumptions,*
- *assumptions for the choice of the mathematical model*, and
- *assumptions about task expectations.*

The first category, *parameter assumptions* are assumptions made to refine a mathematical model by using an additional parameter. The second category, *assumptions for the choice of the mathematical model* are assumptions made to choose, or even justify the mathematical model. These two types of assumptions emerged in different phases of the modelling process, such as during mathematizing or validating activities. Their need arose, when the students were constructing or criticising the mathematical model and they used extra-mathematical knowledge for choosing operations, and as such they used these assumptions for creating or refining the mathematical model.

We observed that the students were well aware of their assumptions. This is evident in the words they used: "we can consider", "we can suppose". Moreover, students were explicitly commenting on whether it was allowed to make assumptions based on extra-mathematical knowledge. This lead to the third category of assumptions: *assumptions about task expectations*, such as *only given numbers should be used* and *making assumptions is not allowed*. These are assumptions made by students to deal with the norms and values of a mathematical culture. These assumptions derive from students' beliefs on how they should work, in particular in relation to work on inauthentic word problems. Beliefs "are abstracted from one's experiences and from the culture in which one is embedded" (Schoenfeld, 1992, p. 74). Other researchers have termed the educational culture, in which students behave according to certain norms as a 'didactical contract' (Brousseau, 1997).

The above three categories also were observed in other episodes of our study, but space does not allow us to report this here. We do not claim that the three above named assumption categories are the only assumptions that students make while mathematizing. Our study was a pilot study done with only a few participants (sample of convenience) and a limited number of problems. Empirical findings from other studies (e.g., see Borromeo Ferri, 2006) show that mathematization activities differ between tasks. Therefore, further research with the same focus on assumptions is recommended. If items are used that require students to define variables or constants, it is possible that they will make assumptions on those variables and constants.

Our study revealed that *assumptions about task expectations* increased students' insecurity in the mathematization process. Therefore, we suggest as teaching implication, that assumption making should receive emphasis in mathematics lessons, as also said by Seino (2005). By discussing assumptions on the parameters or on the choice of the model, both teachers and students can learn about their *assumptions about task expectations*. They can then better understand the role of extra-mathematical knowledge in the mathematization process and in the entire modelling process. Moreover, such a focus can advance their awareness on validating their assumptions.

Our study displays that students make assumptions, which can assist in the mathematizing process. Some authors have described assumptions as "building materials" (Galbraith & Stillman, 2001, p. 304) or "the cement" (Edwards, 1989, p. 95). We support these metaphors, but at the same time we observed some assumption being made, which held back students from the situation context. Thus, some assumptions can be metaphorically described as "*blocked doors*". Our study shows that if students can open these doors, they can make meaningful assumptions in mathematizing, which will then lead to successful modelling.

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Fostering students' independence in modelling activities

Helen M. Doerr[1] and Jonas B. Ärlebäck[2]

1 Syracuse University, Syracuse, USA, hmdoerr@syr.edu

2 Linköping University, Linköping, Sweden

Over the last twenty years, research on the teaching of mathematical modelling has recognized the difficulties that students encounter in becoming independent modellers. In this paper, we put forward the notion that supporting and preserving student independence should be a central principle in guiding teaching practices that support students' modelling activities. This potentially provides new ways to address the tensions, dilemmas and in the moment decision making that occurs when teaching mathematical modelling. We provide empirical evidence of teaching practices that encourage students' self-evaluation of their modelling activities in ways that foster their independence as learners and modellers.

Keywords: Modelling, teaching practices, student independence, self-evaluation.

If we want students to be productive independent individuals and problem solvers with the abilities to apply mathematical reasoning in various situations, they need to have a range of possibilities for acting mathematically readily available when faced with a problem. Mason and Davis (2013) argue that this readiness is not fostered by pedagogies that keep students dependent on their teachers. Rather, teachers need to support students to act in ways that can become part of their own repertoire of mathematical ways of thinking. Students' mathematical learning needs include developing productive dispositions, flexible strategies, persistence and independent thinking (National Research Council, 2001). Mathematical modelling problems would appear to be a particularly rich site for developing these dispositions while learning mathematics in realistic problem situations. However, this poses many challenges for the teacher, especially how to: tackle classroom discussions, structure group interactions, and provide effective feedback to students (Brodie, 2011; Lobato, Clarke, & Ellis, 2005; Magiera & Zawojewski, 2011). In this paper, we provide empirical evidence of teaching practices that appear to support and encourage students' independence in their modelling activities and we identify dilemmas that present challenges for teachers in their pedagogical decision making.

THEORETICAL BACKGROUND

Adequately describing and theorizing about what teachers actually do in and around their classrooms is a complex task – one in which research has been slow and sometimes elusive in providing a holistic and comprehensive picture of teaching practices (Even & Ball, 2009). One important strand of research has emphasized and seriously acknowledged the important role of giving voice to and using students' own work and ideas in the teaching and learning of mathematics. Recently, Stein and colleagues (2008) proposed a model of five practices (*anticipating*, *monitoring*, *selecting*, *sequencing*, and *making connections between student responses)* that can be taken up by novice K-12 teachers, as they learn to orchestrate productive mathematical discussions by simultaneously building on students' ideas and important mathematics. Stein et al. argue that this model gives guidance to teachers so that "the teacher remains in control of which students will present their strategies, and therefore what the mathematical content of the discussion will likely be" (p. 328). However, by centering the control of the mathematical discussion with the teacher, these novice teaching strategies may not help students become more independent learners and, as such, offer little guidance for experienced teachers in managing more complex learning situations, such as modelling tasks. Such situations occur when the teacher has to respond to unanticipated student ideas and manage the emergence of student interactions that cannot be fully anticipated ahead of time. Mason and Davis (2013) refer to these situations as requiring teachers to make "in the moment" pedagogical decisions, something that presents a dilemma and tension for experienced teachers.

A common response to this dilemma is for the teacher to engage in telling or explaining to the students the intended content. Lobato, Clark and Ellis (2005) suggest a reconceptualization of the telling or not-telling dilemma by distinguishing between the teacher action of *telling as initiating* and the action of *eliciting*. According to Lobato et al., *telling as initiating* refers to teacher actions "that serve the function of stimulating students' mathematical constructions via the introduction of new mathematical ideas into a classroom conversation" (p. 110). The teacher's intention is to promote student sense making and is often followed by eliciting students' ideas. Brodie (2010; 2011) elaborates the dilemmas faced by experienced teachers when, having elicited students' ideas, they must make in the moment decisions about when and how long to press individual students for making meaning and giving justifications. As teachers engage with model eliciting activities (Lesh & Doerr, 2003), they are faced with pedagogical dilemmas and in the moment decision making about how to move students' mathematical learning forward. In this paper, we want to put forward the notion that student independence should be one of the most central principles in teaching practices, guiding both planning and in the moment pedagogical decisions and actions. The focus of this paper is on elaborating teaching practices that foster student independence.

Our research is situated in the models and modelling perspective on teaching and learning mathematics (Lesh & Doerr, 2003). Within this framework the notions of eliciting student thinking and developing their emerging models are central to learning mathematics. In this work, we take models to be externally represented conceptual systems that consist of objects, operations, relations, and interaction-governing rules used to predict, explain, describe, or understand some other system (Lesh & Doerr, 2003). By engaging in a sequence of model development activities, students' models are repeatedly developed, modified, extended and revised through "multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and reconstructed by the learner" (Doerr & English, 2003, p. 112). From this perspective, learning is equated with model development. The ability to develop, apply, and adapt a generalized model to be used in a range of contexts is the essence of what it means for students to be independent learners and problem solvers.

METHODOLOGY AND SETTING

This study took place in a six week summer mathematics course for beginning engineering students, designed around a model development sequence centred on the concept of average rate of change (Ärlebäck, Doerr, & O'Neil, 2013). Model development sequences are sequences of structurally related activities that are intended to engage students in multiple opportunities to describe, interpret, make conjectures, explain, develop and iteratively refine their models while interacting with other students (Lesh, Cramer, Doerr, Post, & Zawojewski, 2003). A model development sequence begins with a model eliciting activity where students' ideas about a problem situation are elicited and made explicit in forms that can be tested, revised and refined. Following the model eliciting activity are one or more model exploration activities and model application activities. In the model exploration activities, the underlying mathematical structure and representations (such as tables, interactive graphics, diagrams, or animations) of the elicited model are further investigated and developed. In the model application activities, the students apply and adapt their previously elicited and explored models in new contexts and situations.

The model development sequence in this study consisted of a model eliciting activity (MEA) using the context of bodily motion along a straight path; a model exploration activity (MXA) using a computer simulated walking-world (cf., Kaput & Roschelle, 1996); and, two model application activities (MAA). In the first model application activity the students developed a model for how light intensity varies with the distance from a light source, and in the second model application activity, the students modelled how the voltage changes over a fully charged discharging capacitor in a simple circuit (Ärlebäck et al., 2013). In this paper, we report on our analysis of the video recordings of the lessons focusing on the teacher moves (or actions) identified in the data as either *independence preserving* or *neutral* or *non-independence preserving* moves. We also analysed debriefing interviews with the teachers to understand her intentions. The data presented in this paper are from the MEA and the first MAA.

An *independence preserving move*, is a teacher move whose intention and consequence is that it preserves students' independent on-going work. A teacher move that provides students with a tool or strategy to facilitate the students' independence in a future activity/situation would be considered independence preserving. Encouraging student persistence ("keep thinking", "keep talking", "keep working") and collaborative work ("talk to your partner") is considered to be independence preserving. As well as being independence preserving, a teacher move can be *neutral* or *non-independence preserving*. A non-independence preserving move typically implicitly suggests that meeting external expectations ("label your columns") is valued rather than independent thinking and work. A neutral teacher move would be the teacher's observations as the students are working on a task.

We identified one particular sub-category of independence preserving moves that occurred regularly within our data set: *encouraging self-evaluation moves*. Teacher moves that either implicitly or explicitly encouraged students to be evaluators of the correctness, appropriateness, usefulness or goodness of their work are called *encouraging self-evaluation moves*. Examples of encouraging self-evaluation include moves that support students in having confidence in their solutions, in validating solutions with a partner, and in giving reasons or justifications about why their solution has to be correct. Encouraging self-evaluation also occurs when the teacher simply responds to the student "think more" about a particular result, representation, relationship, object or idea. We take encouraging self-evaluation to be a particular category within the more general category of independence preserving moves.

RESULTS

We give examples that occurred in two parts of the model development sequence: during the model eliciting activity and during a model application activity. In these examples, the teacher encouraged students' self-evaluation within the larger goal of fostering students' independence in their modelling activities. In the first example, over three days of the MEA, the teacher engaged in a range of moves (described below) that fostered aspects of student independence at the individual, small group and whole class level. In the second example, the teacher engaged students in an extended argument among the students and led by the students about the rate at which light intensity changes with respect to distance. Rather than resolve the argument for the students, the teacher preserved student independence by engaging them in collecting empirical data that they could use to resolve the question.

Examples from a model eliciting activity

The MEA was designed to elicit students' ideas about constant rates, about the distinction between velocity and speed, and about position graphs for which the motion was not physically possible. The students began the task by working in small groups to create a line with negative slope, using their graphing calculators, motions detectors and graphing software on a computer. The design of the task required them to transfer data from their calculators to other group members' calculators and to a computer. Thus, this MEA was comprised of a mathematical task (using bodily motion to create a line with negative slope) and the learning of technology skills. The teacher assumed that many of the students had limited background with the technology from high school and knew that their fluency with the technology would be needed throughout the model development sequence. Throughout the MEA, the teacher made numerous moves that were intended to set expectations for student independent work at both the individual and the group level and expectations for students to self-evaluate the goodness of their work and that of their peers. We will briefly illustrate five teacher moves: establishing the utility of a reference tool; turning back questions; gathering data; encouraging persistence; establishing criteria for self-evaluation.

At the beginning of the MEA, the teacher distributed a "data management reference sheet." Since one goal of this particular MEA included developing fluency with technology that would be needed throughout the model development sequence, the teacher had created a reference sheet that contained technical details for using the technology. Nearly all of the teacher's interactions with the students were brief as she answered their technology related questions with "do you have your data management reference sheet?" or "look at the data management reference sheet." She consistently referred to the need for each individual to acquire the necessary technology skills. In referring to the computer data transfer, she commented, "your whole group should go because everyone should know how to do that." We take this as the teacher's intentional moves to support the independence of each individual in gaining fluency with the technology that would be needed throughout the modelling activities.

In a segment that occurred after the students had worked on finding and interpreting equations for a linear position versus time graph with negative slope, the students initially incorrectly referred to speed as the slope of the line. As a discussion of this was about to conclude, a student asked, "what does the motion detector measure?" Rather than answer this question directly, the teacher turned this question back to the students. We take this as a move to engage students in self-evaluation, that is, in answering other students' questions. Eventually, there was an answer from the students that what the detector measured was distance in feet, recorded every tenth of a second. The segment ended with the teacher inserting the units for speed as feet/second.

One part of this modelling task focused students' attention on the possibility of creating a U-shaped graph with the motion detector by walking at the same speed the whole time. The students were sharply divided on this issue. The segment ended as one student suggested "why don't we just try it?" We take this as an example of students suggesting self-evaluation and of the students taking up the teacher's encouragement of their independence by gathering data and evidence to support their claims.

One aspect of student independence would seem to be persistence in working on tasks and solving problems. An example of this occurred at the end of the first lesson where one student had worked on finding an equation for the line with negative slope that the group had created. The student asked the teacher if his solution was correct and the teacher responded by asking him how he could evaluate the correctness of his answer for himself ("how do you check it?" "figure it out"). Later, when he determined that there was an error in his work, the student again approached the teacher, who responded with "find your mistake. I'm sure you can find it." Finally, at the end of class, when the student came to her again, she again reassured him that he can find his mistake. We take this as both encouraging student persistence and as a form of encouraging student self evaluation.

In the third lesson, the teacher displayed examples of students' descriptions of motion from work done in an earlier lesson. The teacher asked the students to choose a good description and provide a reason why the description was good. The students worked on this individually. We take this as an example of students evaluating descriptions created by other students. However, it was unclear what criteria the students had for deciding what constitutes a "good" or "best" description? This dilemma became clearer as the teacher asked the students to discuss in groups which description was the best. The students had difficulty in coming to consensus in their groups. The teacher had a whole class discussion where students gave their reasons for choosing various alternatives. The teacher shifted the goal of the task when she suggested that they consider how to improve a description as some students claimed "none of them are really good." As this task ended, the teacher asked the students, "could you create the graph if you were given this description?" This final statement by the teacher appears to be a form of establishing the criteria by which one could decide if a description were "good" enough. However, this criteria was not clear to the students from the beginning of the task.

Examples from a model application activity

This example is from a model application activity, where students investigate how the rate at which light intensity changes with the distance to a light source. At the beginning of the activity, the students are asked the following question as part of their homework prior to the lesson to elicit their thinking about changing light intensity:

> Imagine the tail light of a car moving at a constant speed away from you. Is the light intensity:
>
> 1) fading at a constant rate
> 2) fading slowly at first then quickly
> 3) **fading quickly at first and then slowly**
> 4) unsure

Although all of the students had taken a prior course in physics in secondary school, where the relationship between light intensity and distance is studied, only one of the students correctly identified the rate at which the light intensity fades: quickly at first and then slowly. The majority of students concluded that either the light faded at a constant rate (60% of the responses) or slowly at first and then quickly (27% of the responses). One student was unsure.

These responses were displayed for the students via a student response system. The teacher commented that there was "lots and lots to talk about" and that she wanted to know from them "why did you choose the answer you chose?" To accomplish this, she arranged the students in groups and asked them to discuss their answers. After a few minutes, while the teacher was walking around listening to students' reasoning, the teacher pulled the class together for discussion. Each of these teacher moves – eliciting their ideas with the initial question, asking them to engage in peer discussion, and listening to their reasoning – served to encourage the students to self-evaluate their responses to the question on changing light intensity.

The ensuing whole class discussion began as the teacher asked S1, a member of one group, to start the discussion:

S1: My other group members, they voted for number two, but I voted for number one because I explained that the car is going away at constant speed, so I thought like the light would go away at a constant speed too. But then I don't know if that's the same thing. It would be like how you see the light and the intensity of the light, if they drop in the same way.

The teacher re-stated S1's comment and invited students to consider S1's argument or offer their own:

Teacher: Are you guys thinking similarly to S1 or why did you think it was at a constant rate? [several students mumble] S2, what did you say?
S2: The car is moving away at a constant speed so I think the intensity decreases at a constant speed.
Teacher: S3
S3: The light travels at a different speed...
S4: ...than the car...
S3: ...than the car...
S4: So, it would actually be different.
S5: Isn't the speed of light constant?
S6: If the speed of light is constant, why...
S4: ...why is the car moving? – because it creates a variable!
All: [Laughter and many students talking at the same time]
S7: But yeah [inaudible] the speed of light and speed of...
S3: ...S7, what do you think the answer is?

The students were actively engaged in arguing whether or not the light was fading at a constant rate. Many of those who thought the rate was constant were arguing that it had to be constant because the speed of light was constant. However, the intensity of light with respect to distance is not related to the speed of light with respect to time, as S1 had suggested in her initial argument. What is striking about the conversation above is that the teacher is not mediating, restating, or directing the discussion. She is listening, off to the side. The argument takes a turn in a new direction as S3 asked S7 what she thought:

S7: Me?
S3: Yeah!
S7: I put two [slowly then quickly], but I'm not sure that, but I don't know what the ratio is between light intensity and [S7 starts gesturing]...
S2: I mean for instance, if the speed of light's constant, and the cars' constant...
S7: ...Yes, but the speed of light is the travelling speed of light. We're talking light intensity which is what you see
S2: Right...

In this segment, S7 has made the critical distinction (which was foreshadowed in S1's initial argument) between the speed of light and the intensity of the light. S2 seemed to acknowledge this, with his comment "right." But it is not entirely clear what he meant by this. But the next response to S7 is from S3, who had asked S7 for her answer in the first place. S3 claimed to have an example, which was then immediately followed by another student's (S4) example. From the teacher's perspective, it was not possible to fully anticipate what these examples might be, what they would mean, and how they would relate to the central question about the rate of change of intensity of light.

S3: Yes, so in 7th grade we were studying the change of motion and stuff and whatever. And so, we found out, like if you're in, okay, I'm from New York City, we're doing that, like, if you're walking in a train, like running inside of a train, at, you know let's say your running like, 20 feet per second or whatever, whatever, that's unrealistic, but just say you're doing that, right. And the train is going at, like 100 feet per second, then the total would be 120 feet per second. That's how fast it would seem you're running, because the train is like moving, and then you're moving...

S4: ...I think I have a better example, of how you know that light is not travelling at a constant rate. Why do they put the eenie-teenie-tiny [very tiny] lights on the top of buildings that planes aren't gonna be able to see from 50 yards away? [inaudible] explain that to me – why are runway lights so small, you see the guy [starts gesturing as if he was taxiing a plane to its gate]

All: [many students talking at the same time]

The discussion was ended by the teacher, but not by drawing a conclusion for the students. Instead, the teacher continued to engage the students in evaluating their emerging models of light intensity by initiating the next task of collecting real data that would enable them to resolve the question based on empirical evidence.

The results elicited from the initial question showed that 60% of the students thought that the relationship between light intensity and distance from the car was linear. To address this, the teacher encouraged the students to self-evaluate their emerging models of rates of change and of light intensity. As the episode unfolded, the students unpacked their models by themselves, first in small group discussion and then in a student-led and student-driven whole class discussion. S1 raised the core question about the relationship between the constant speed of the car and the rate of change of the light intensity. The discussion then revolved around the speed of light and its constancy, prompting many students to express their ideas on this matter. When S7 tried to focus the discussion back on S1's core question, S3 and S4 both drew on personal experience in trying to understand the situation. S3 remained focused on the role of the constancy of the speed of light, whereas S4 argued about the rate of change of the light intensity. The teacher functioned as a listener throughout the discussion. The teacher ended the discussion by initiating the next activity where students were to collect real data to resolve the issue. This shift was a key move in encouraging further student self-evaluation of their emerging models for how the light intensity varies with the distance from the light source and in preserving student independence.

DISCUSSION AND CONCLUSION

In this paper, we wanted to further the research on using students' thinking in mathematics lessons and the dilemmas connected with this practice, as reported in the literature, to forefront an argument and empirical evidence of teaching practices that encourage students' self-evaluation of their modelling activities in ways that preserve their independence as learners and problem solvers. This focus presents a shift in emphasis on students' and teachers' roles and responsibilities. For example, the essence of *sequencing* in the Stein and colleagues' (2008) model is for the teacher to be in control of the form and content in a whole class discussion. An independence preserving stance would instead advocate sharing with students the responsibility for the sequencing of contributions in the discussion in order to engage them in the self-evaluation of their ideas and their emerging models. The first example illustrates the role of the teacher in setting expectations for students to use tools such as a data management reference sheet to solve technology related problems, to answer questions from peers, to gather data to support arguments, to persist in finding mistakes, and to collectively establish criteria for evaluating the goodness of written descriptions. Taken together, these moves by the teacher appear to support the students in becoming more independent learners and problem solvers. The second example suggests the benefits of engaging students in peer-discussion and encouraging students' self-evaluation by having them collect data that can be used to self-evaluate the goodness of their emerging models. This practice seemingly resolved the teaching dilemma that would otherwise have confronted the teacher as to how to resolve students' conflicting ideas about changes in light intensity, while at the same time providing the students with peer-discussion as a self-evaluation tool for them to use in modelling activities. We offer these examples as ways of thinking about the notions of *students' independence* and *stu-*

dents' self-evaluation, with the hopes of contributing to the on going discussion of effective teaching practices.

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One-sided limits of a function at a point in a drug metabolism context as explained by non-compulsory secondary students

José Antonio Fernández-Plaza, Luis Rico and Juan Francisco Ruiz-Hidalgo

University of Granada, Granada, Spain, joseanfplaza@ugr.es, lrico@ugr.es, jfruiz@ugr.es

We present the results of an exploratory and descriptive study performed with Spanish students in Non-Compulsory Secondary Education focusing on how they explain the meaning of both one-sided limits within a temporal phenomenon (drug metabolism) given by a graphical non-authentic model. We organised the given explanations according to the following options: Only calculation; interpretation in a neighbourhood (locally); meaning of the value (pointwise); direction of approaching in time with regard to right-sided limit. We also highlight a particular attention to other elements of the model apart from limiting notions and some difficulty to give sense to right-sided limit, possibly because the direction of approximation is contrary to the natural progression of time.

Keywords: Partial modelling activity, one-sided limits, non-authentic drug metabolism model, spontaneous extension of a model, graphics.

PROBLEM

Mathematical modelling has been integrated in international programmes of students' assessment (e.g., PISA) and it has been a fundamental part of the mathematics education curricula for students aged 6–15 years old for several years (OECD, 2013). In some countries, such as Spain, mathematics teaching in Non-Compulsory Secondary Education (16–17 years old) is intended as a preparation for tertiary studies. However it seems not to keep this trend but rather emphasizes abstract concepts and procedures from advanced mathematical activity (Crouch & Haines, 2004). Successful teachers' training programmes in design and assessment of modelling classroom proposals (Ortiz, Rico, & Castro, 2006) make possible to transfer such proposals to pre-university education. Mathematical modelling is a field of mathematics education widely explored (Fischbein, 1987; De Lange, 1987; Niss, Blum, & Galbraith, 2007; Swetz, 1991).

This study aimed to explore how students perform a pre-modelled task related to the concept of limit of a function at a point. Concretely, we choose a task contextualised in how a human body removes a drug in two days period, focusing on both one-sided limits of the amount of drug at the moment at which a new dose of drug is introduced.

Since we had previously characterised a misconception about the limit of a function related to the continuity (Fernández-Plaza, Rico, & Ruiz-Hidalgo, 2013a, 2013b), we considered for this study a function with a discontinuity at the limit point.

The outcomes of this study provide a better understanding of students' learning of the concept of finite limit of a function at a point, specifically jump discontinuities where there is an instantaneous change to the function at a point in the domain. We examined students' learning of advanced mathematical content, but also their understanding of continuity as a tool to model real phenomena.

The specific aim we propose for this study is:

"To describe student meanings of one-sided limits of a function at a point as they explore a given graphical model describing a temporal phenomena, and the possible influence of variable time on left-sided and right-sided limits interpretations."

THEORETICAL FRAMEWORK

We provide a brief description of what we understand by a model and how to model. We then establish a relevant distinction between modelling and application, with particular reference to the concept of limit of a function at a point.

Notion of model, applications and modelling procedure

We consider a mathematical model as a mathematical structure that approaches or describes certain relationships within a phenomenon in order to explore, understand, explain and eventually control it (Swetz, 1989). As Fischbein (1987, p. 21) notes, not only physical facts can be modelled but also concepts can be associated with a model and properties of the abstract concept may be better understood from the corresponding model.

Niss and colleagues (2007, pp. 10–11) stress a significant distinction between an application and a modelization. Modelling focuses on finding mathematical knowledge from a certain part of the real world, for example, the cycloid is the model related to the motion of a point in a wheel as it rolls along a straight line without slippage. In contrast, application focuses on the opposite direction. Given a model the problem is finding what parts of the real world are susceptible of being modelled by such a model. For example, the inverted cycloid provides a solution to the Brachistochrone and Tautochrone Problems.

The modelling ability as considered by PISA 2015 draft framework (OECD, 2013) (called *mathematising*) involves capabilities such as:

> To structure the field or situation to be modelled,
>
> To make assumptions
>
> To translate the reality into a mathematical structure
>
> To work on the mathematical model to obtain findings
>
> To reinterpret these findings in terms of the real situation, and
>
> To establish limited or generalized conditions to validate or modify the model.

Greefrath and Riess (2013) summarise the modelling procedure into five steps, Understanding of the problem; approach selection; performing; explanation of results; checking results, calculations and approach. They developed and implemented a solution plan (it consists of these five steps with questions and clarifying points) with 6th grade students. In spite of some students engaged appropriately with this aid, other of them had some difficulties.

We pay special attention to the last two aforementioned capabilities: Interpreting, criticizing and modifying a given model. To sum up, modelling is the process to find a mathematical structure which approaches relationships within a phenomenon, which consists of an understanding of the problem, approach selection (assumptions), performing (translation of the reality to the mathematical structure and work on it to obtain findings), explanation of results (to reinterpret the findings in terms of the real situation) and checking results (to establish limited or generalized conditions to validate or modify the model).

Applications and Modelling related to the concept of limit of a function at a point

Classical problems, which were modelled using the concept of the limit of a function at a point, dealt with movement of an object and variations in magnitudes with respect to time.

According to the distinction between application and modelling, other phenomena may involve relationships between variables and time and the limit concept could be applied. The basic question to which the concept of limit of a function at a point tries to give an answer is the following:

> Given a flow of amount of a magnitude along an interval of time, $y = f(t)$ and an instant $t = t0$, obtain the best approximation L of the amount of magnitude near $t = t0$, providing that $t = t0$ is an accumulation point of the interval of time. If it does not exist, explore the reasons why not. L is the best approximation of $f(t)$ near $t = t0$, if for any K approximation, there exists an instant tK, such that $|t - t0| < |t0 - tK|$ implies $|f(t) - L| < |f(t) - K|$.

This question leads us to four different models related to the concept of finite limit of a function at a point depending on the properties of the function:

Instantaneous invariance model, related to the existence of limit. If the image of the point is the same as the limit, we referred to a *continuous model*, otherwise, it is a *hole model*.

The non-existence of the limit leads to the other models:

Jump or instantaneous change, when there are both one-sided best approximations, but they are different, so there is not a best approximation of the function at any neighbourhood centered at $t = t_0$.

Asymptotic change model, when one or both of the one-sided best approximations do not exist but there is a tendency to plus or minus infinity.

Oscillating change model, when none of one-sided best approximations exist, either finite or infinite.

For this study we are going to consider students' work on a jump model, because the asymptotic one involves infinity and oscillating one is not usually taught at their educational level.

METHOD

This is an exploratory and descriptive study based on a survey method. We designed and implemented a questionnaire including open-ended and closed-ended questions. This paper is focused on the following one:

> A patient is given a 0.05 mg injection of a drug daily, and each day 40% of the drug in the body is eliminated. The following graph (Figure 1) corresponds to the function $y = f(t)$ that relates time to the amount of the drug in the body during the first two days of treatment. Interpret $\lim_{t\to 1^-} f(t)$ (from now on *Lim_Left*) and $\lim_{t\to 1^+} f(t)$ (*Lim_Right*)

The sample was composed of 36 Spanish students in the first year of Non-Compulsory Secondary Education (grade 11th), 16–17 years of age, who were taking Mathematics for the Science and Technology track. The students were chosen deliberately based on their availability.

The survey was administered to the sample described above in the middle of the academic year 2010/2011 during a regular session of the math class (1 hour) counting on the collaboration of the teacher. Subjects had received prior instruction on the concept of limit by their teacher.

The analysis of students' answers (interpretations of left-sided limit and right-sided limit) is based on a *content analysis methodology*. Firstly, characterization of students' interpretation of left and right-sided limits. Secondly, detection of different approaches used to the interpretations of one-sided limits. Finally, detection of spontaneous attempts related to a further analysis of the model.

According to limit models framework, the task describes a phenomenon with a jump instantaneous change. However, the real phenomenon is continuous. According to Andresen (2007, p. 2044) is a non-authentic model. We discarded the time taken to inject the drug which is very small in comparison with the unit of the variable t (days), otherwise the function would seem to have a vertical line at $t = 1$.

It is important to note that the task does not consider the whole modelling procedure, because a model is given beforehand and students only have to interpret the value of the one-sided limits according to the provided model. However, our results show that some students spontaneously focused on other aspects of the model and tried to develop or modify it. So we argue that in part they were doing modelling activity and therefore bringing into play the two last capabilities according to PISA 2015 framework.

RESULTS

We describe the interpretations provided by the students of left-sided and right-sided limits, also different

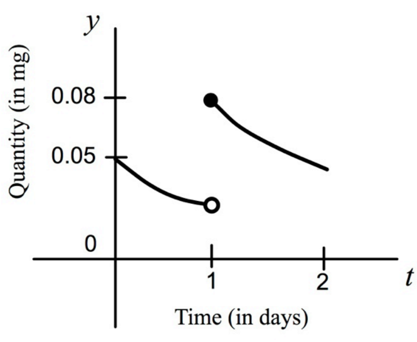

Figure 1: Graphical model of the situation described on the text

approaches to conceive them, as well as the modelling actions of students apart from the requirements of the task.

Students' interpretations of left-sided limit from the model

The different interpretations of the left-sided limit can be organised into these main categories, developed from answers, considering two dimensions: Information included from the graph and Contextualization. Answers in "Other Category" related to each dimension are incomplete or vague.

Dimension 1. *How much information from the graph is included*

Only the value of the left-sided limit. Some answers only provide the specific value of the left-sided limit (Example 1).

Example 1. " = 60% of 0.05 which is 0.03"

Local / pointwise interpretation. Some answers describe the behaviour of the function around the point t=1 (locally) (Example 2), particularly using specific terms such as, "to approach," "to tend," or "to get closer and closer," or focusing on the possibility of the limit to be reached, while other ones focus exclusively on the "left side" of the point t=1 making explicit the meaning of the left-sided limit value (pointwise) (Example 3).

Example 2. "The limit as t tends to 1 from the left represents how the amount of drug is decreasing along the day and at the end of the day increases by 0.05 mg due to the injection"

Example 3. "To know the remaining amount of drug in the body after the first dose"

Dimension 2. *Contextualization*

Contextualized. Students interpret the meaning of left-sided limit in terms of the real situation. The clearest interpretations used infinitesimal expressions such as "just before" or "before" (Example 4), or "at the moment drug has been removed" (Example 5).

Example 4. "To know amount of drug that patient has before 40% is removed"

Example 5. "As time goes on, body is removing drug until the moment in which a 40% has already been removed"

Decontextualized. Students interpret the meaning of left-sided limit in a purely mathematical context, not in the real situation (Example 1).

Table 1 shows the frequencies of each interpretation category. These categories are mutually exclusive.

Dimension 1	Frequencies (N=36)
Only value	2
Local	19
Pointwise	10
No answer/Other	5

Table 1: Frequencies of interpretations of Lim_left related to Dimension 1

Table 2 shows the frequencies of contextualized and decontextualized interpretations of left-sided limit.

Dimension 2	Frequencies (N=36)
Contextualized	29
Decontextualized	4
No answer/Other	3

Table 2: Frequencies of interpretations of Lim_left related to Dimension 2

Students' interpretations of right-sided limit from the model

The different interpretations of the right-sided limit can be organised into the aforementioned categories, but there is a new singular category:

To relate the tendency of t to 1 from the right to stepping back in time. Some students become aware that tending to 1 from the right implies "counting" backwards in time (Example 4). However, we note the contrary fact in the example 5.

Example 4. "As we get close to 1 from the right, we see that the drug has just been injected and the patient has not eliminated any amount of drug"

Example 5. "If we take limit from the right, the approximate amount of drug will tend to 0 mg"

Table 3 shows the frequencies of each interpretation category. These categories also are mutually exclusive:

Dimension 1	Frequencies (N=36)
Only value	1
Local / Tendency as stepping back in time	7
Pointwise	11
Tendency as stepping forward in time	13
No answer/Other	4

Table 3: Frequencies of interpretations of Lim_Right related to Dimension 1

An example of contextualized interpretation of right-sided limit is Example 6.

> Example 6. "As t approaches 1 from the right we see drug has just been injected and the patient has not removed any amount of drug"

Table 4 shows the frequencies of contextualized and decontextualized interpretations of right-sided limit.

Dimension 2	Frequencies (N=36)
Contextualized	28
Decontextualized	6
No answer/Other	2

Table 4: Frequencies of interpretations of Lim_Right related to Dimension 2

Different approaches of left-sided and right-sided limit interpretations

Even though students clearly identified the value of the left-sided limit, we consider that there are two different approaches when the right-sided limit is interpreted: (a) *formal*, according to the formal notion of right-sided limit, and (b) *contextual*, according to the natural development of the phenomenon along the time, as expressed in Figure 2 and Example 5.

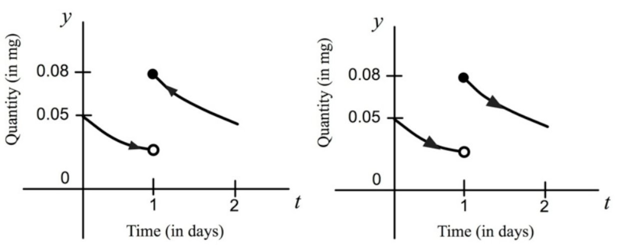

Figure 2: Formal (left) and contextual (right) direction of approximation

Table 5 shows the frequencies of formal and contextual approaches related to students' pairs of interpretations (N= number of formal left-sided limit direction of approximation).

Approach	Frequencies (N=30)
Formal	19
Contextual	11

Table 5: Frequencies of formal or contextual pairs of interpretations

Unexpected students' attempts of further analysis of the model

Students spontaneously focused on other aspects of the provided model (8 out of 36), such as:

To set an absolute start-end of the day. 2 out 8 students arbitrarily do not consider the day as a measure of time between two instants of time, but like a day in the calendar (from 00:00 a.m. today to 00:00 a.m. next day) (Example 8).

To consider that the total amount eliminated during a day is constant for every day. 2 out of 8 students do not consider that the 40% is taken out of the current amount of drug in the body, but of 0.05 mg dose, so the eliminated amount of drug is constant just like the length of the jump (Example 9 and Figure 3).

To extend the model to other days. 7 out of 8 students generalised the model to next days (inductive reasoning, Example 9). Some of them (3 out of 7) consider as well that the velocity of elimination is increasing day to day, because the amount of drug to remove in the same interval of time (1 day) is higher (Example 10), but in fact the phenomenon reaches a stationary behaviour around 0.05 mg of drug eliminated per day.

Discussion on the arbitrary setting of the hole in the graph. Only one student discussed the arbitrary place of the hole in the graph, i.e., the image of the function at t =1, because the limit is independent of what happens at the point t =1. He stressed the finite jump (Example 11).

> Example 8. " [Lim_Left] For example, let us suppose that the injection is administered at 0:00 a.m. As the time goes the amount of drug in the

body is decreasing. 24 hours later the amount has reached 0.03 mg...”

Example 9. “[Lim_Left] The patient eliminates 0.02 mg per day. [Figure 3 provided]”

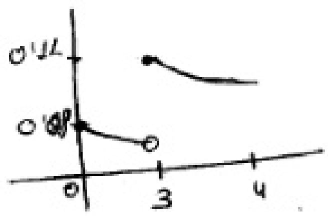

Figure 3: Inductive reasoning from the model (Example 9)

Example 10. “[Lim_Right] After 24 hours on the next day amount has increased by 0.05 from the remaining 0.03, that is 0.08. So, each day 1, 2, 3... there would be a jump and step by step the patient will have a higher amount of drug in his body”

Example 11. “[Lim_Right]...it has been set the hole at the end of the first day and the image at the beginning of the second one, but it could be done anyway. There is a finite jump because the one-sided limits are different”

Finally, given the similarity between right-sided limit at $t = 2$ (0.048 mg) and the amount of drug at $t = 0$ (0.05 mg), Example 12 could be interpreted as establishing by the student the equality between both values graphically rather than by calculation.

Example 12. “We observe that the patient takes the drug and so the amount in the body increases, along the day, the amount of drug is reducing until the same level when the first dose was administered.”

DISCUSSION AND CONCLUSIONS

According to the aim we draw the following conclusions:

The pointwise interpretations of both one-sided limits have been slightly frequent (10 and 11 out of 36), so we can tell that some students have a routine procedural conception of both one-sided limits. On the other hand, the local interpretations are more frequent for the limit from the left (19 out of 36) than are those from the right (7 out of 36), possibly due to the natural progression of time that produced some conflicts of interpretations as is shown by the answers (11 out of 36). Such a kind of conflicts was reported by other studies, such as Blázquez (2000). For further research, new examples with independent variable different from time could be chosen.

It is important to mention the spontaneous references to other aspects of the model (8 out of 36), such as the invariance of the daily-eliminated amount, the gradual increment of the amount of drug in the body in the future as well as the velocity of elimination. Only one student suggested that the image of $t = 1$ could be either 0.03 or 0.08. These actions are characteristic of modelling activity (OECD, 2013).

Surprisingly, no student discussed about the necessity of taking into account the time employed to inject the drug in order to check the continuity of the real phenomenon. For further research, no model would be provided in order to elicit students’ own proposals and specific classroom proposals could be planned according to PISA recommendations.

ACKNOWLEDGMENT

This study was performed with aid and financing from Fellowship FPU AP2010-0906 (MEC-FEDER), project EDU2012-33030 of the National Plan for R&D&R (MICIN), Subprogram EDUC, and group FQM-193 of the 3rd Andalusian Research Plan (PAIDI).

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A comparison between strategies applied by mathematicians and mathematics teachers to solve a problem

Carolina Guerrero-Ortiz and Jaime Mena-Lorca

Pontificia Universidad Católica Valparaíso, Instituto de Matemáticas, Valparaíso, Chile, c_cguerrero@yahoo.com.mx

This study analyses the results obtained from comparing the paths shown by expert mathematicians on the one hand and mathematics teachers on the other, when addressing a hypothetical problem that requires the construction of a mathematical model. The research was conducted with a qualitative approach, applying a case study which involved a group of mathematics teachers and three experts from different mathematical areas. The results show that the process of constructing a mathematical model differs between these two groups mainly by the type of cognitive processes developed. It was observed that the modelling routes depended on the graphical representations used by the individuals to address the problem. The mental model related to the situation plays an important role in the externalisations of the participants.

Keywords: Modelling, representations, mathematics teachers, paths of modelling.

INTRODUCTION

In the context of mathematical modelling (MM), there are two approaches: applications and modelling for mathematics learning, and learning mathematics to develop skills in the construction of mathematical models. The first considers the use of modelling activities as a vehicle for the construction of mathematical concepts; the second involves the application of mathematics to build models (Niss, Blum, & Galbraith, 2007). These two approaches may occur simultaneously or in isolation, and involve a difference in the emphasis put in the development of teaching strategies.

Speaking of teaching activities necessarily brings to mind the teacher-student-task triad, where it is unarguable that the mathematics teacher has great importance, both fulfilling the role of modeller and participating in the preparation and/or selection of tasks in order to support the students to understand the mathematical concepts involved in the activity. Hence, this study focuses on the observation of modelling strategies developed by the teachers when they model a hypothetical situation. Our initial hypothesis is that mathematicians and mathematics teachers develop different ways of modelling. Therefore, we discuss some modelling strategies implemented by our two study groups: mathematicians responsible for the training of mathematics teachers, and in-service mathematics teachers receiving training.

In the literature related to strategies activated by expert and novice modellers when solving a problem, it has been observed that novice modellers do not take enough time to understand the situation in question, thus experiencing difficulties in selecting the relevant information and using it in the construction of a suitable mathematical model (Crouch & Haines, 2004; Crouch & Haines, 2007, Bransford et al., 2000). Therefore, it becomes clear that some of the obstacles in the construction and interpretation of mathematical models are related to difficulties in accessing appropriate mathematical concepts and procedures in order to find a solution.

The modelling process of novices shows a trend toward the use of linear rather than cyclical modelling strategies, while the validation of the obtained models is not a relevant element to them. As a result, they have difficulties recognising the model associated to a given situation and fail to relate their results to the situation that gives rise to it. On the other hand, expert modellers access relevant knowledge more efficiently. Moreover, some authors estimate that it is likely that many years of practical experience are needed to become to be an expert modeller (Haines & Crouch, 2010; Crouch & Haines, 2007). However, there is little research that focuses on the processes and strategies that novices and experts develop when involved in modelling activities. Therefore, we aim to contribute to previous research by showing some aspects of the construction of a model that both groups (experts and novices) use. We identify some elements that permit them to move more effectively through the various stages or nodes of modelling.

In the context of modelling as a vehicle for learning mathematics, a general question arises: Are teachers able to convey to their students the strategies that they themselves develop to model a situation? This is under the assumption that some of the steps that comprise a modelling path take place in the mind. Therefore, if the teachers are not aware of what they are doing mentally when modelling a situation, they will be incapable of conveying their thought process to the students. Therefore, the objective of this research is to identify some of the processes that mathematicians and mathematics teachers in training develop in order to help them to reflect on their own process.

Our research was conducted with two groups. One was a group of expert mathematicians who conduct research in pure mathematics and also participate in in-service teacher training. They are not necessarily involved in modelling activities as part of their research. The other group consisted of mathematics teachers who work at secondary or college level, they received a solid background in mathematics as part of their undergraduate training, but have little experience in the development of modelling tasks in the classroom. The intention of the study is not to make comparisons of the modelling competence between the two groups; rather, we aimed to identify the tools and representations that the members of each group use. This knowledge may serve as a guide to identifying those aspects that would be appropriate to develop both in in-service teachers and in pedagogy students. We argue that one way to address the problems related to the transition of students through the modelling cycle is to identify the strategies used by mathematicians (experts) and try to have students (novices) develop the same tools and skills. This becomes especially relevant since so many mathematicians are involved in training future mathematics teachers. It is clear that knowledge generated in the interaction between these two groups is what is ultimately passed on to students; so it is desirable to identify and analyse the strategies developed in each of our groups. Therefore, the question that drives our research is: What differences are there between the strategies mobilized by mathematicians to model a situation and those mobilized by mathematics teachers?

We present the results of analysing the modelling paths (Borromeo-Ferri, 2007) developed by the participant groups, taking into account the strategies and mathematical tools they rely on.

CONCEPTUAL FRAMEWORK

The main challenges that arise when modelling a situation are related to the transition from reality to the world of mathematics and to the reinterpretation of a Mathematical Model (MM) in terms of reality. For an expert, reinterpretation may be trivial, but not so for novices (Crouch & Haines, 2004). In connection with the transition from reality to the mathematical model, Borromeo-Ferri (2006) identified some stages in a modelling process, focusing mainly on the analysis of the cognitive processes involved, where she distinguishes four phases: Real Situation (RS), Mental Representation of the Situation (MRS), Real Model (RM) and Mathematical Model (MM).

The Real Situation (RS) represents the situation described in the problem; it can be an image or a text. When going from the RS to the Mental Representation of the Situation (MRS) the individual somehow understands the problem more, and she or he mentally reconstructs the situation; even if they do not fully understand the problem, they can start working on it.

MRS may be different in each individual, depending on his or her mathematical thinking style[1]. It can be visual in relation to the experience, or attention can focus on numerical data and relationships given in the problem, depending on the associations that the individual chooses while understanding the task. In addition, the MRS can differ depending on the role

1 Mathematical thinking style is defined as: "The way through which an individual prefers to present, understand and think mathematical facts and make connections between certain internal imaginations and/or outsourced representations" (Borromeo-Ferri, 2012). In individuals aged between 15 and 16 years three different thinking styles have been identified: visual, analytical and integrated thinking.

mathematical activity has for the individual professionally speaking. Borromeo-Ferri (2006) identifies two aspects that mark the difference between RS and MRS: 1) unconscious simplifications of the task, and 2) personal choice on how to deal with the problem.

In the passage from the MRS to the real model (RM), more conscious simplifications and idealisations take place in the individual, since in MRS phase the individual has already made decisions that influence the filtering of information. The transition process may require extra-mathematical knowledge, depending on the type of task.

The RM phase is related to MRS, as RM is practically built internally and external representations are the Real Model depending on the statements that the individual makes when externalising the model. When in transit from RM to the Mathematical Model (MM), an individual's mathematisation progress appears, in which, according to the task in hand, it may be necessary to also use extra-mathematical knowledge.

The MM phase consists of external representations in mathematical expressions or drawings. The expressions of the individual are more related to mathematical facts and to a lesser degree to reality.

The transitions that occur between these phases are crucial since the externalisation that the individual expresses through images, mathematical language and statements are representations of mental activities, which also depend on previous experience and knowledge.

In the transition from MM to the mathematical results (labelled 4 in Figure 1), an individual's mathematics skills are put into play, such as mathematical resources and strategies to analyse and explore the model and to obtain results or conclusions. The mathematical results consist of writing the results of the model. The transition (labelled 5) from mathematical results to real results is given by the reinterpretation of the solution in terms of the problem. Borromeo-Ferri (2006) notes that individuals often make this transition unconsciously.

In the Real Results phase, the mathematical results are discussed concerning their correspondence to the situation. During the validation of the results, the individual seeks relationships between his or her results and MRS, depending on what kind of validation he or she chooses, either intuitive validation or knowledge-based validation.

The items described play a fundamental role in the constitution of our conceptual framework. Mathematicians and mathematics teachers do not follow the same modelling route, since this is determined by their experience and expertise in the subject area and their thinking style, among others things. Blum and Borromeo-Ferri (2009) have shown how the individuals pass through different phases focusing on some phases and ignoring others. We add that visual imagery (Aspinwall, Kenneth, & Presmeg, 1997) also influences the way in which the individual moves through the modelling cycle, this is because some images may persist in the mind limiting other ways of thinking. Our interest is to document the mathemat-

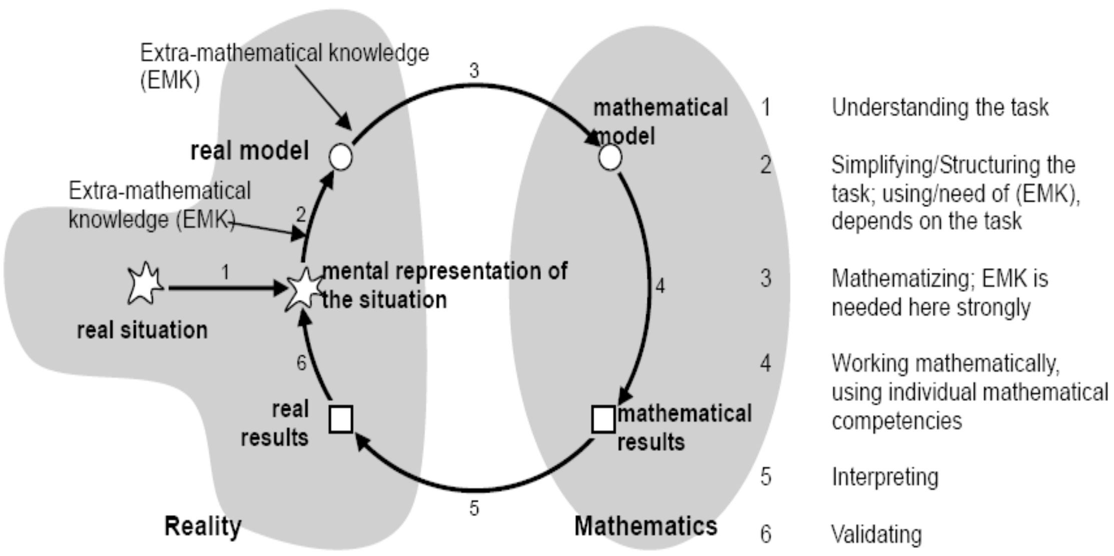

Figure 1: Modelling circle (Borromeo-Ferri, 2006)

ical processes that take place when our observation groups address the modelling of a situation.

METHODOLOGY

This is a qualitative research study (Miles & Huberman, 1994), aimed to analyse the behaviour of participants when they model a hypothetical situation. The data analysis is based on observation, interviews and written material of the individual work of the participants and their arguments while solving the proposed task.

Due to fact that cognitive processes are generally not directly expressed by the individuals, efforts were made to document the mental processes of the participants by analysing individual modelling routes (Borromeo-Ferri, 2010, p. 112). We analysed verbal expressions and representations externalised by the participants when building a mathematical model by asking questions such as: What are you thinking? Why did you do this? How can the situation be related to the representation?

Participants: Three expert mathematicians (Roger, Hugo and Evan) and 20 teachers in training on a Masters' program in Mathematics Education. All were given the same mathematical task. The mathematicians solved the task individually. They were asked to verbally express their thought processes and were interviewed regarding procedures that were unclear to the researchers. The teachers solved the task individually and expressed their reasoning and solving processes to their classmates.

The task was an adapted version of a Vasilyev and Gutenmájer's (1980) task, which can be solved by using knowledge of synthetic geometry and/or analytic Cartesian geometry:

A ladder rests against a wall; on the ladder, there is a cat. The ladder begins to slide on the ground, always touching the wall. What is the path that the cat describes? What would the path be if the cat were not sitting in the middle of the ladder?

DISCUSSION OF RESULTS

The analysis is organised into 4 sections: 1) Understanding the problem, which includes first ideas and impressions, 2) Searching for strategies to address the problem, 3) Model building, 4) Obtaining results and conclusions. The structure of the analysis is consistent with the phases of the cycle of modelling. The first block groups the transition from RS to RM. Block two analyses the externalisations made by the individuals, derived from the transition from MRS to RM. The third block considers the process of mathematisation. The fourth block includes transitions 4, 5 and 6 (Figure 1).

1. Understanding the problem – first ideas and impressions

This task conditions the solution process through the use of pictorial diagrams to explain the situation or to represent the problem, the pictures drawn by individuals bring to life their visual imagery related to the locus plotted by the dynamics of the situation. The mathematicians almost immediately moved the problem to a mathematical representation (Figure 2a) and focused on finding the mathematical formalism. All the teachers supported their solution processes with the use of diagrams to get an idea of the movement described in the situation and remained in this phase for an extended period of time (Figure 2b).

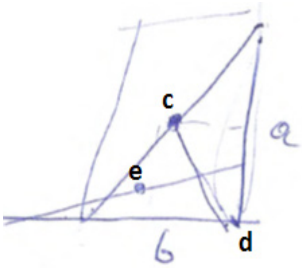

Figure 2a: Diagram made by Roger (mathematician)

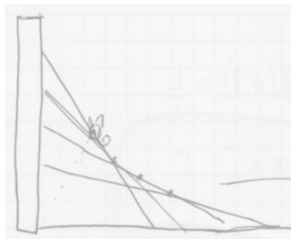

Figure 2b: Diagram made by one of the teachers

The transcriptions provide evidence of the simplification and idealisation that the teachers carried out. For example, they indicate that the cat is considered as a point, which has no weight, that there is no friction between the ladder and the floor, etc. These aspects show the transition between MRS and MR, highlighting the simplifications that the individual consciously performs. The mathematicians do not offer explanations of the characteristics of the phenomena or the elements that are idealised to build the model. They only allow us to observe the transition between the phases of the Real Model (RM) and the Mathematical Model (MM); this may be because the mathematicians immediately look at this situation as a mathematical problem. It appears that the teachers make more connections with reality while mathematicians quickly dissociate themselves from it.

2. Searching for strategies to address the problem

It is difficult to identify the exact moment when the participants begin to search for solution strategies because in many cases this process is determined by the understanding of the situation and the simplifications applied to it. These strategies can be mental operations that cannot be observed until the individual externalises them verbally or in written form. For example, Roger represented the problem geometrically (Figure 2a) and showed some aspects of his strategy only in the interview:

Researcher: What did you think first?

Roger: Of looking at whether the distance of this point [d] to the point that I drew [d and e] remained constant; for example, to see if that was a constant [segment cd],... then I tried to complete this figure like this, to see if the symmetry helped.

On the other hand, after the teachers analysed the problem and got a sense of the trajectory of the midpoint, they expressed different strategies for solving the situation. For example, one of them addressed the problem by placing it in a coordinate system and solved it with the use of tools of analytic geometry (Figure 3a). Another teacher turned to synthetic geometry by observing congruence between triangles (Figure 3b).

In this block teachers already knew the path of the cat so they focused on finding different strategies to determine the algebraic model. Many of their strategies did not help them build the algebraic model and they had to redefine how they achieved the solution. Conversely, we note that two mathematicians apparently reconstructed the locus while they were reading the problem.

3. Model building

The search for solution strategies and model building are closely related. When an individual follows a solution strategy and does not find a model that she/he considers relevant, he/ she tends to propose a new strategy, thus creating a cyclical process between understanding the problem, finding solution strategies and building the algebraic model. This transition was easier to observe in teachers, as the mathematicians went directly to the mathematical representation of the problem.

The teachers initially identified qualitative properties of the situation, and then they used geometric tools and located the problem in a coordinate system, seeking to obtain algebraic expressions that are traditionally associated with a locus (equation of the ellipse and circle); an example of this is seen in Figure 3a.

Two of the mathematicians offered a solution closer to the representation built in grasping the situation. Later they mobilized their mathematical resources for the construction of a mathematical model, showing preference for the use of concepts related to synthetic geometry. Only as a last resource they used analytic geometry tools.

Two moments in the construction of the mathematical model were identified: one related to the visual identification of the locus, and the other after the visualization and related to the construction of the algebraic model. Both moments are manifested in the transition from the mental representation of the situation (MRS) to the Mathematical Model (MM).

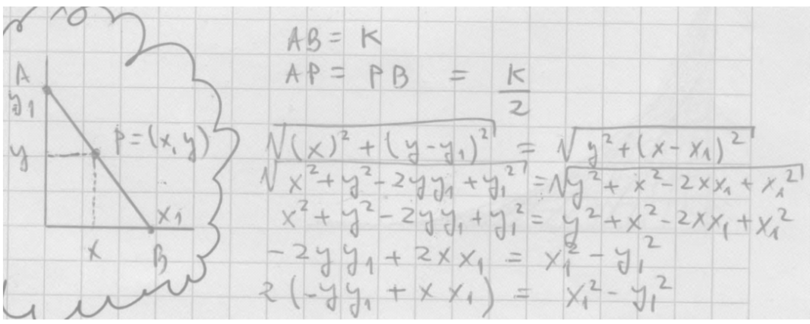

Figure 3a: Solution strategy show by a teacher using analytic geometry

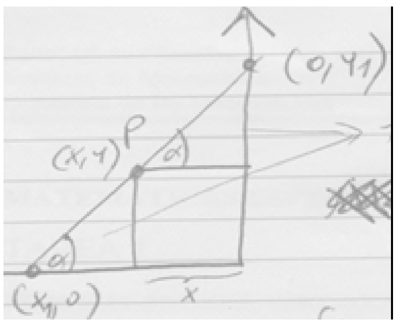

Figure 3b: Solution strategy using congruence between triangles

4. Getting results and conclusions

Finally all participants identified the path of the object. Different strategies were observed. Roger, although he knew the locus, did not refer to it, until he translated the situation to a geometric representation. Hugo did this during the construction of the geometric model. The teachers were aware of the fact that it was the locus of a semi-circle or an ellipse only after plotting different graphic representations and obtaining a mathematical model to represent it.

We summarize our main results in the next table.

CONCLUSIONS

The mathematicians and teachers use diagrams as a strategy to solve the problem. While Hugo and Roger mentally built the path generated by the movement based on its geometric properties and Evan found the locus by using additional geometric and dynamic construction, the teachers were not able to imagine the locus without the support of pictorial representation. For teachers the idea of a locus was also constituted by writing an analytical expression.

We found evidence of some differences in the mental representation of the situation (MRS) and the real model (RM) constructed by each individual. In the case of some of our participants, it can be wrong and largely mediated by their intuition of the behaviour of the situation, sometimes the erroneous visual imagery is persistent which leads to difficulties in obtaining the algebraic model. The fact that the mathematicians and teachers show different approaches to the modelling process can be associated with their experience and their domain of mathematical knowledge, but it can also be related to the way in which everyone is able to abstract the mathematics relationships in a problem. These aspects are links that determine the transitions of the participants in the modelling cycle in an efficient way.

Different strategies were identified when observing how the participants validate their results. For Hugo and Roger validation is associated with their prior knowledge of the locus and to a lesser degree with the verification of the mathematical procedure. For the teachers validation was based on the association of the algebraic model with its graphical representation.

We have shown some differences in the routes followed by mathematicians and mathematics teachers when solving a simple hypothetical problem, of course given a more complicated problem the differences would become more pronounced. In the educational context, we believe it is necessary to make teachers aware of the ability of abstraction possessed by some individuals (as characterised by the corresponding transition between the RS and MM phases), in order that they can make a rational effort to show their students the internal thought processes that occur when modelling a particular situation.

	Mathematicians	Math teachers
1. Understanding the problem -first ideas and impressions	They immediately visualise and construct a mathematical representation. They do not express the simplifications of the situation. It is only possible to observe the transition from RM and MM.	They use pictorial representations to help their understanding of the situation. They externalise the simplifications made to build the model. It is possible to observe the transition between MRS to RM.
2. Searching for strategies to address the problem	The strategies are related to the analysis of the mathematical properties of the situation.	The strategies are directed toward the algebraic solution of the problem.
3. Model building	Use of the Pythagoras and Thales theorem. They showed a preference for concepts associated to synthetic geometry and, as a last resort they used tools from analytical geometry.	The majority of the models were constructed using tools from analytical geometry.
4. Getting results and conclusions	Two of the participants reconstructed the geometric locus parallel to the construction of the MM.	They were aware of the geometric locus after drawing various scenarios. In some occasions they drew it until they determined the corresponding equation.

Table 1: Summary of results

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Modelling: From theory to practice

Britta Eyrich Jessen, Tinne Hoff Kjeldsen and Carl Winsløw

University of Copenhagen, Department of Science Education, Copenhagen, Denmark, britta.jessen@ind.ku.dk

Mathematical Competence Theory and the Anthropological Theory of the Didactic each offer different frameworks for the analysis and design of "modelling" as a central component of mathematics teaching. Based on two comparable cases from each research programme, we investigate how these differences appear in concrete design work, and what their practical consequences may be.

Keywords: Mathematical modelling, mathematical competence theory, anthropological theory of didactics, bidisciplinarity.

WHAT IS MODELLING AND DOES IT MATTER?

The fact that primary and secondary school students all over the world study a subject called "mathematics", with relatively similar contents and methods, is intrinsically linked to certain assumptions about the *relevance* if not *necessity* of this subject for every citizen in modern society. The formulation of these assumptions change over time and they are of course the object of constant debates, but an invariant common contention seems to be the *utility* of what is taught *in the actual or future lives of students*, or at least *its roles outside of school mathematics*.

One formulation, which has gained importance in the mathematics education community over the past 30 years, is based on the notion of *mathematical modelling*, defined roughly as "translations between reality and mathematics" (Blum & Borromeo-Ferri, 2009, p. 45). More complex descriptions of the modelling process, usually in the form of a *modelling cycle* (e.g., ibid, p. 46) have become commonly known and used in research into the ways in which these translations can appear in the school subject. It is a common assumption among researchers within this line of research that students' experience with all steps of the modelling cycle is essential to the justification of school mathematics in society (ibid, p. 46). In particular, Niss and colleagues (2002) proposed to consider *modelling competence* – the students' capacity to carry out mathematical modelling – as one of eight universal competence goals for the teaching of mathematics, linked to other goals equally defined in terms of competences. Their mathematical competence theory (MCT) thus integrates and develops earlier work on mathematical modelling, as an educational activity and goal, in a comprehensive framework for the analysis and design of school mathematics in a broad sense.

Another perspective on modelling stems from inquiries into the nature of mathematics as a school subject: how it is related to the science called mathematics, and more generally to "mathematical practices" appearing in society outside school? The anthropological theory of the didactical (ATD) emerged from the notion of *didactic transposition* (Chevallard, 1985) according to which school mathematics is a cultural set of practices and knowings which are inseparable from the institutions (schools) in which they are taught and learnt. In this theoretical framework, "mathematics" and "reality" are not *a priori* defined or distinguished; all human activity and knowledge is described in terms of *praxeologies* (Chevallard, 1999). Modelling has a wider meaning in this framework, as the elaboration of praxeologies in one domain in view of studying one or more *questions* in another domain. The school institution refers to this as "intra-mathematical modelling" when both domains are recognized as belonging to school mathematics, e.g. if school algebraic praxeologies are elaborated to study a question from school geometry (García, Gascón, Ruíz-Higueras, & Bosch, 2006). In ATD, modelling thus serves to create meaningful links between otherwise separate praxeological domains, whether or not these are considered as belonging to school mathematics or not.

The two theories are related to specific *design formats* which are often used for the design of teaching that involves modelling (cf. Miyakawa & Winsløw, 2009, for the distinction of theory and design format). In

MCT, it is *problem oriented project work* (PPW), in which students are to develop their competences while experiencing some or all steps in a modelling cycle (Blomhøj & Kjeldsen, 2006). In ATD, it is *study and research paths* (SRP), departing from one or more questions; the further development is sometimes represented with a tree like "map" of derived questions and praxeologies which students did construct while working with the questions (Barquero, Bosch, & Gascón, 2008; Jessen, 2014).

At this point, we have only hinted at some of the differences between two perspectives on modelling. The research question which interests us is a theoretical, but also quite practical, one: *What differences, if any, does it make for the design of new teaching practices, whether the theoretical control apparatus comes from MCT or ATD? In particular, are there differences between uses of the design formats PPW and SRP which can be related directly to the different theoretical notions of modelling found within MCT and ATD?*

We shall take an inductive approach to this question: we first present two cases of design of modelling activities for students in Danish upper secondary school, constructed from each of the two perspectives but otherwise similar in contents. Then we analyse the differences in view of providing tentative answers to the research question. To prepare that analysis, the presentations of the cases focus on the following variable features of modelling activities:

(V1) *Practical meaning of "modelling"* in the activity, as described by the authors

(V2) *Goals for the activity* (e.g. for student learning) *and their assessment*

(V3) *Organisation in time* of the activity

(V4) *Distribution of roles among students and teacher(s)*, in particular the way in which student autonomy is controlled (limited, furthered, differentiated, etc.)

(V5) *Adaptation to local conditions and constraints* (features of the activity which result from these adaptations, including choices made for (2)–(4)).

The case presentations given below are based on more extensive studies (Jessen, 2014; Blomhøj & Kjeldsen, 2006). V5 is further treated in these papers.

CASE 1: A STUDY AND RESEARCH PATH

The first case we will present comes from an experiment with study and research paths (SRP) in the context of Danish high school students' study line reports written in the second year of high school (a study line report is a bidisciplinary report students write in the second year as a preparation for the bidisciplinary "study line project", which is a high stake final exam in upper secondary school in Denmark, cf. Jessen, 2014, p. 2). The reports are about 15 pages long accounts of an autonomous work done by one or two students, within 6 weeks and with very limited access to help from teachers (V3). The study line of the students determines what disciplines are to be involved in the report. Before the 6 week period, the teachers formulate a set of questions for the students to work on (V4). For the study line of the experiment, the theme should combine the disciplines *mathematics* and *biology* with equal weight. These circumstances are constraints (V5) which affect the concrete design and in particular the variables V1-V4.

The aim for the study and research path (V2) was for students to develop new praxeologies in the domains of nervous physiology and differential equations by working with a certain generating question, given by the teacher together with some supplementary questions to ensure the involvement of both disciplines:

Q_0: How can a patient be relieved from his pain by painkillers like paracetamol – how does deposit medication work and how can we model this mathematically? Q_1: Explain the biological functioning and consequences of taking paracetamol orally versus taking it intravenously. Q_2: Create a mathematical model using differential equations that illustrates the two processes and solve the equations in the general case. Q_3: Give a concrete example, where the patient is relieved from pain and estimate from your own model how often paracetamol has to be dosed – which parameters (absorption, elimination factor, bioavailability) are important to be aware of? $Q_{3,1}$: Does it make any difference whether the dose is given oral or intravenously? Use your models while giving your answer. (*Translation of the original questions in Danish*)

Notice that in ATD, *modelling* means the elaboration of praxeologies in the two domains – done by students in view of answering the generating question (V1). However, in the assignment, "mathematical model" refers to a more restricted sense, which is closer to the notion of model found in MCT and, at least in outline, is the one found in official documents and text books for Danish high school.

The above assignment is based on a generating question Q_0 which the students can immediately understand, but not answer. In general, a generating question should be so strong, that it is necessary for the student to formulate derived questions Q_i, each representing a branch of inquiry, in order to answer Q_0. The answers R_i to the derived questions adds up to a final answer of Q_0 (Chevallard, 2012, p. 6). At the same time it is purposed that the generating question must be "alive" in the sense that students should be able to relate the question to things they perceive as interesting and real. These aims were deliberately pursued by the teaching design, knowing that several students in the class wanted to study medicine or similar after graduating.

The derived questions formulated by the teachers serve as supports for the students' study process (V4). In general, it is crucial that the students are not left with "big questions" that are unrelated to their praxeological equipment (Chevallard, 2012, p. 11); the relation to praxeologies from specific disciplines must be ensured. This was even more crucial in our context since no teaching activity was accompanying the SRP work of the students. Some students met after classes and formed their own working groups discussing strategies for answering the questions. The teachers were allowed to answer questions during the six weeks, and in order to keep track of the students working progress, the exchange of questions and answers was only permitted in writing (V4). For the same reason, students were asked to provide their immediate answer to the generating question Q_0 when it was handed out (without the derived questions Q_1–$Q_{3,1}$). After that, the entire assignment was given to them. After two weeks, and again two weeks later, the students were asked to answer the following questions in writing:

> What is your answer to the generating question right now? What have you done to answer the question? What are you planning to do next in order to come up with more fulfilled answers?

We cannot go into all the details of the analysis of this SRP, neither before nor after the experience (the latter being analyses of students' reports, cf. Jessen, 2014). However we notice that to construct the "mathematical model" asked for in Q_0, students must somehow examine the relationship between the amount of drug given, and the distribution of the drug in the body. How the pain is cured and how the drug is eliminated must be answered by praxeologies from the domain of physiology. The latter leads to consider that the pain is relieved in relation to how often the drug is given, the size of the body and the pain perception. Thus, the progressive development of a mathematical praxeology (involving tasks, which can be solved using techniques available to the students, e.g., CAS-based solution methods for differential equations) is closely articulated to the development of a biological praxeology. The modelling process in terms of ATD is not a question of following certain steps, it is an individual process where the students uses their praxeological equipment to investigate domains, form new questions, answer them with existing or new praxeologies unfolding the disciplinary organisation at stake (V1).

The intermediate answers from the students showed a variety in their working progress, which reflected different praxeological equipment among the students. Some students responded the first time, that they needed to know the half-life of the painkillers this indicates, that the students suspect, that there is a time dependence in the model, and that the model includes an exponential function. During mathematic classes they have seen that exponential equations are part of the solution to many differential equations. This implied, that they were trying to relate the generating question to the newly developed praxeologies in mathematics. Also they studied relevant medias since they were able to formulate relevant search topics. The students formulated derived questions such as the following: Q_1: How is pain registered? Q_2: How does paracetamol relieve pain (pharmaco dynamic)? Q_4: How can the dosing be modelled mathematically based on the biological knowledge? (Jessen, 2014, p. 11). The entire analysis shows that the students are constantly narrowing down their inquiry, by alternately studying the questions through physiology and differential equations.

The teacher involved was sure that for some students the generating question would not suffice to develop a reasonable model. It was for this reason that a part of the derived questions was handed out before the independent work of the students. Some of the students would otherwise not have been able to develop new praxeologies in the intended domains. With these more precise questions, they were able to identify relevant media (web-pages etc.) and although some of them uncritically adopted models constructed by others, they were all able to make use of them for simple calculations (e.g. of the amount of drug in the vein of a patient) (Jessen, 2014). Thus their modelling of the intended praxeologies was not as richly developed as in the previous case.

CASE 2: A PROBLEM ORIENTED PROJECT ON ASTHMA MEDICINE

Our second case presents a PPW on mathematical modelling related to the administration of asthma medicine. In MCT modelling competency is defined (V1) as

> A person's insightful readiness to autonomously carry through all aspects of a mathematical modelling process in a certain context and to reflect on the modelling process and the use of the model (Blomhøj & Jensen, 2003, p. 127).

The key words are *autonomy*, *modelling process*, *reflections*. PPW is particularly well suited to foster students' autonomous participation in the modelling process (Blomhøj & Kjeldsen, 2011). The goal for students' learning (V2) in MCT is to develop and/or enhance their competency.

A mathematical modelling process can be depicted analytically as a cycle consisting of six sub-processes (ibid., p. 387). Concrete modelling activities, like the case presented here, may have a variety of more specific goals for students' learning (V2) in order to adapt to local conditions and constraints (V5).

In a PPW, students work in teams with a problem for a longer period of time to produce a product representing the team's solution (V2+V3). The central idea is that the problem should function as the "guiding star" for all decisions made by the students in the sense, that all decisions should be justified by their contribution to the solution of the problem. This provides the students with (parts of) the responsibility of directing the project. It is crucial that the students are involved in (most of) the decisions taken in the modelling process and become involved in reflections upon the different steps in the modelling cycle. PPW opens for a distribution of roles among students and teacher(s) that makes it possible to direct the students' autonomy e.g. through specific requirements to the product of the project (V2+V4). PPW has the potential to foster in the students all the key elements in developing modelling competency which makes this format an obvious pedagogical choice in MCT.

The asthma project was designed by two teachers for first year students in mathematics in high school. The students were to: 1) work more independently than usually over a longer period (ten mathematics lessons of 1.5 hour each and a similar amount of homework); 2) develop new theory by working with modelling within a subject area (exponential growth) they hadn't worked with before; 3) work with a more complex and authentic problem for which they did not possess a standard method or technique such that the modelling, the mathematization, the interpretation of the results and the reflections about the modelling process and the use of the model became part of the project; 4) analyse a set of data in order to build a mathematical model; 5) use familiar concepts such as graphs and equations for functions in a concrete context; 6) develop their mathematical communication skills; 7) use ICT throughout the project. (V2)

These aims were achieved through a strict organization of time (V3) and a setup that allowed for and supported the students' autonomy (V4). The teachers divided the project into four phases (Figure 1). The teachers controlled phase 1–3, and the students controlled phase 4. The aim of the first three phases was to prepare the students for their independent work in phase 4. In phase 4, the teachers took on the role of

1. (1.5 module) Presentation of the problem (see Fig 2). Excel course, IT for project management, social contract. Crash course in the modeling process.
2. (1 module) Problem formulation Phase and decision.
3. (3 modules) Work with a set of four exercises related to the project in phase 4.
4. (4 modules) Working with the actual project.

Figure 1: The four phases of the design. 1 module corresponds to a 90-minute lesson (Blomhøj & Kjeldsen, 2006)

consultants (V4) that the students could ask for advice on specific problems.

In phase 1, the teachers introduced the students to a cyclic representation of the modelling process. The teachers used the process to inform the students about the various elements in mathematical modelling within MCT, and they asked the students to be aware of and to explain where in the modelling process they were at any given stage in their work. Hereby, the teachers made sure that the students became engaged in posing the modelling problem, constructing the model, solving the mathematical system and suggesting solutions to the problem (V2, V3 & V4/V5). In phase 2, the teachers trained the students' competence in posing mathematical modelling problems through discussions in the class room guided by the teachers (V1/V5).

The problem from phase 2 was given to all students with some data (Figure 2). The exercises in phase 3 were not included in the students' independent work. They served as inspiration and illustrated the level of mathematics, communication and documentation expected in phase 4. The product of the project work was a report, handed in by each group after phase 4 (V4/V5). The teachers formulated a set of requirements for the report to direct the students' autonomy in phase 4.

ANALYTIC COMPARISON OF THE CASES

A synthetic presentation and comparison of the two cases can be achieved using the five variables identified in the first section and indicated as they are "filled" by the above presentations (see Figure 3).

Despite evident similarities between Q0 in case 1, and the problem (Figure 2) underlying case 2, the contexts and constraints are quite different: in case 1, the students must work independently most of the time, and have to combine the two major disciplines (mathematics and biology) of their study line; while in case 2, the work is done as part of the regular teaching of one discipline (mathematics). In the Danish regulations for high school, *mathematical modelling* more or less understood as in MCT forms part of the competency goals for mathematics as a discipline (Niss et al., 2002); the bidisciplinarity required for study line projects is a more diffuse and general principle for the study line projects while in the case of mathematics, it is also often associated with the same notion of mathematical modelling. Despite these differences coming from the contexts, some more principal differences arising from the theoretical background of the two cases can also be identified.

Differences coming from the design formats

The variables V2–V4 are clearly shaped by the design formats. In PPW, everything begins with a *problem* defined in more or less commonly accessible terms, which should then be sharpened and translated into mathematical terms, in order to allow for applications of relevant mathematical machinery, either known in advance or developed through the project work. The PPW in itself does not suggest explicit structuring

Asthma patients' problems with exhalations may be alleviated medically by increasing the concentration of the drug theophylline in the blood of the patient. If the concentration of theophylline is below 5 mg / L it has hardly any positive effect. If the concentration is above 20 mg / L it has toxic effects. The problem is to administer medication such that the concentration of theophylline stays within a certain range in which the medicine is effective, say a concentration between 5 and 15 mg / L. The substance is excreted from the body through the kidneys; hence the amount of the substance in the blood will drop with time, which means that the patient will suffer, unless you "fill up" periodically. At the hospital where the patient is hospitalized you try, for the sake of the daily organization of work and to reduce the risk of errors, to schedule the medication so the patient is supplemented with an equal dose, D mg, with equal intervals of time, T hours. A doctor is examining how to choose D and T so that the concentration of theophylline remains within the range of 5- 15mg / L. On a patient, he has measured how the concentration of the substance decreases with time following an injection of 60 mg of the drug.

h	0	2	4	6	8	10	12	14	16	18
mg/L	10.0	7	5.0	3.5	2.5	1.9	1.3	0.9	0.6	0.5

Figure 2: The problem and the data (Blomhøj & Kjeldsen, 2006)

and requirements regarding the students' work besides the fact that the problem should be formulated in such a way that it can function as a guide. The formulation of the problem is part of PPW. Hence, it is left to the teacher to set the "scene" for the students' work within the given context, depending on his or her learning goals. A SRP begins with a *question* which, like the problem in PPW, is too open to allow for immediate, complete answers. In order to proceed, students need to work with subquestions arising from supplementary assumptions, suggested by the original question or by some first, intuitive hypotheses or answers. Both design formats leave the teachers with tools for *directing* the students work: in PPW, the structuring can allow students more or less autonomy depending on how the teacher choose to structure the work, and through specific requirements for the product – in this case a report - the students should deliver (Blomhøj & Kjeldsen, 2006, p. 168), while in SRP, the teacher may supply students with some derived questions to start with, some specific media to study, etc. (Winsløw, Matheron, & Mercier, 2013, pp. 271–282). In both cases, an initial planning may be adjusted to the work of the students, with the tree diagram of the SRP and the learning goals and (parts of) the modelling cycle as the main tools for control of these adjustments of the initial design.

Differences coming from the theories

MCT assumes a clear and evident boundary between mathematical and extra-mathematical phenomena, which implies (through the processes of problem formulation, demarcation of a domain of inquiry, and systematization), the construction of an object to be modelled. This object is then translated into a mathematical representation, which in daily work is also often referred to as *the model*. The preparation and conduct of the PPW can thus be structured according to the movements from the problem to the mathematical domain, and back – with an explicit notion of being "outside" and "inside" mathematics. ATD, on the other hand, is based on a general theory of human practice and knowledge, in which the organisation of praxeologies into disciplines is merely an institutional construction; the boundaries of what is called "mathematical" are not universal but contingent.

In MCT, it is part and parcel of mathematics teaching to develop students' explicit knowledge and experience of how mathematics (as a universal entity) *applies* to problems outside of that domain. In ATD, praxeologies are simply answers to questions which have been developed sufficiently to allow students to find culturally established answers through media or through research based on praxeologies they are familiar with; the main feature of modelling to experience is the development of praxeologies through this dynamics of study and research, independently of institutional classifications into disciplines of the praxeologies.

These theoretical differences have an impact on practice. In PPW based on MCT the disciplinary contents are in principle subordinate to the problem. The chief purpose is to reach a satisfactory solution to the problem through realisation of (specific features of) the

	Case 1: study and research path	Case 2: problem oriented project work
V1	Starting from a big question Q_0, develop derived questions and praxeologies which can answer these and in the end, at least partially, Q_0. Didactic theory is not taught.	Starting from a problem *P outside* mathematics, reformulate it as a mathematical problem, treat this, and evaluate solution relative to *P*. The modelling process is explicitly taught.
V2	Develop specific bidisciplinary praxeologies as answers to Q_0.	Modelling competency through phases of modelling of data and problem.
V3	Six weeks of independent work (individually or in pairs) based on Q_0 and some derived questions, with encouragement to search for media.	Project team work for ten 90-minute modules and similar amount of homework, structured by phases of modelling as shown in Figure 1.
V4	Teachers deliver Q_0 and some derived questions; students do study and research on these, with very limited access to teachers, to prepare their study line reports.	Teachers structure the work of teams according to the phases, with most autonomy required in the last phase (once mathematical formulation and expectations are established).
V5	Regulations of study line reports (combining math and biology)	Aims for regular mathematics lessons, which include mathematical modelling.

Figure 3: Syntheses of didactic variables as set by the two cases

mathematical modelling process including choosing disciplinary theory relevant for solving the problem. The mathematical content brought into play will depend on the mathematical competencies and knowledge of the modellers and their abilities to expand these. In the ATD approach to modelling, a more or less strongly directed SRP can be planned based on a priori analysis of its potential to realise certain institutionally defined disciplinary praxeologies as answers to the initial question. This could make the ATD approach to modelling implemented through SRP more attractive in institutional contexts where the disciplinary focus is strongly constrained. On the other hand, as we have argued and illustrated, the choice of design has theoretically determined consequences for the kinds and qualities of mathematical modelling activity, which students get to engage in. For further investigation one might analyse the activity students carry out in the classroom (how are answered produced and validated, etc.) and to what extend are the students able to solve other modelling problems in the future.

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Evaluating the effectiveness of a framework for measuring students' engagement with problem solving episodes

Patrick Johnson and Seán Moylett

University of Limerick, Limerick, Ireland, patrick.johnson@ul.ie

The purpose of this study is to investigate the problem solving episodes that a group of post-primary students in Ireland engaged with during the problem solving process and to determine an effective assessment technique for measuring their problem solving abilities. A framework developed by Artzt and Armour-Thomas was implemented and evaluated within the classroom over a nine week period to assist students in developing their problem solving skills. Testing was conducted on the students to be able to differentiate between their abilities prior to the introduction of the framework. It was found that when students utilised the framework the number of episodes engaged with by the students increased and the number of correct answers to the problems also increased.

Keywords: Problem solving, episodes, implementation, assessment.

INTRODUCTION

With the introduction of a new post-primary mathematics syllabus in Ireland in 2008, named Project Maths, increased emphasis has been placed on developing students' problem solving abilities. Around this time worrying findings from reports such as the *Statement on Raising National Mathematical Achievement* (Expert Group on Future Skills Needs [EGFSN], 2008) highlighted concerns about students' capacity to engage with problem solving. When designing Project Maths the National Council for Curriculum and Assessment (NCCA, 2012) identified five key skills that they saw as being central to effective teaching and learning across the new curriculum. These are *information processing, being personally effective, communicating, critical and creative thinking* and *working with others*. The new syllabus has placed increased emphasis on teaching "through" problem solving as it affords students the opportunities to develop these key skills although there have been concerns raised about what exactly teaching through problem solving means (Lubienski, 2011). Regardless of this confusion an aim of the new syllabus is to allow students more time to explore mathematics and to move away from an over-reliance on drill and practice techniques as had been evident in the old syllabus (Lyons, Lynch, Close, Sheerin, & Boland, 2003).

Even though problem solving is seen as a means by which "students deepen their understanding of mathematical concepts by analyzing and synthesizing their knowledge" (Erbas & Okur, 2012, p. 89) this is not always the case. Posamentier and Krulik (1998, p. 15) pointed out that "a substantial portion of problem solving is done by rote. Students struggle through one problem in the section, the teacher reveals a model solution and the remainder of the problems in the section are solved in the same manner". Mimicking a teacher's solution strategy will work for some problems but when presented with unfamiliar or non-standard problems students still struggle (Harskamp & Suhre, 2007). Having a solid knowledge base, good past experience and knowledge of strategies are considered to be important in successful problem solving (Erbas & Okur, 2012). Now that problem solving has been officially cited as a learning outcome on the post-primary mathematics syllabus in Ireland it is imperative that teachers are aware of effective frameworks for teaching and assessing problem solving to guarantee the successful integration of problem solving into the classroom.

The purpose of this study was to implement an existing problem solving framework in a mathematics classroom and to then attempt to use this framework as a guide for assessing students' problem solving competence. The following research questions guided this study:

— Do teachers feel that this framework could assist in the implementation and assessment of problem solving in the classroom?

— Which problem solving episodes can be observed while students solve mathematical problems?

SELECTION OF A SUITABLE FRAMEWORK

With the increased emphasis placed on problem solving in the new syllabus the Project Maths Development Team [1] set about designing a modular course with the aim of providing teachers with practical advice on how to approach problem solving in the classroom as well as providing them with a holistic rubric for the marking of solutions. The rubric presented by the Project Maths Development Team primarily focuses on three activities of problem solving – choosing a correct strategy; explaining your choice of strategy and getting the correct answer. Unfortunately this rubric makes no attempt to measure all the remaining, and often important, tasks that individuals engage with during the problem-solving process. To this end a more detailed framework for implementing problem solving was sought by the authors with the additional requirement that a rubric for the assessment of problem solving activities could be easily developed around this implementation framework.

Schoenfeld (1985) developed a framework that separated the problem-solving process into a number of "episodes". He defined an episode to be "a period of time during which an individual or a problem-solving group is engaged in one large task" (p. 292). The initial episodes according to Schoenfeld (1985) were *read, analyse, explore, plan, implement,* and *verify*. Using Schoenfeld's framework as a foundation, Artzt and Armour-Thomas (1992) adjusted the framework to "delineate explicitly the type and level of cognitive processes individuals use" (p. 141). To this end modifications of the original episodes within Schoenfeld's framework were needed. For example, the original episode of *read* was separated into the episodes of *read* and *understand*. Artzt and Armour-Thomas (1992) finally settled on eight episodes when looking at problem solving in small groups – *read, understand, analyse, plan, explore, implement, verify,* and *watch and listen*. This framework, with the exception of the 8th episode as this is specific to small groups and our study focused on students working individually, was adopted by the authors for the purpose of this study as it was deemed to address all the major "stages" of the problem-solving process and it was felt that it could be easily implemented in a classroom scenario.

Additionally each of the episodes within the Artzt and Armour-Thomas (1992) framework is sub-divided depending on whether it involves predominantly cognitive or metacognitive processes. This is important as several researchers (e.g., Goos, Galbraith, & Renshaw, 2000; Teong, 2003) have confirmed the importance of the relationship between cognitive and metacognitive processes where in the words of Artzt and Armour-Thomas (1992) "an appropriate interplay between the two is necessary for successful problem solving to occur" (p. 162).

Due to the structure of this framework it was relatively straightforward to develop an assessment rubric where the focus is placed on each of the individual episodes. In this way no single episode would be overlooked and additionally no individual episode would be deemed more "important" within the problem-solving process than any other. Erbas and Okur (2012, p. 97) have correctly noted that "all episodes don't need to occur to find a correct answer" but since this study is more focused on the process of problem-solving rather than the answer the focus remained on the inclusion or omission of the problem-solving episodes from the student solutions. Additionally it should be noted that the students being assessed in this study could be termed as "novice" in terms of their problem solving abilities and so the authors felt it would be more beneficial to the students if they fully engaged with all the episodes of the framework at this stage of their mathematical development.

METHODOLOGY

Participants for this study were selected from a school in the mid-west region of Ireland. A second year (typical student aged 14 years) higher level [2] mathematics class was selected as the study group for the duration of the nine week study. The selected class was a mixed class of 21 students consisting of 12 boys and 9 girls. Seven of the students volunteered to participate in the study (4 boys and 3 girls). Four teachers also agreed to participate in the study by reviewing the frameworks presented by the authors and taking part in a focus group to garner their opinions on the suitability of the frameworks for implementing and assessing problem solving in the classroom.

Quantitative data was gathered from the study via testing. Students were tested on their problem solving ability prior to being introduced to the framework within their class structure. During the nine-week intervention one of the authors replaced the regular teacher in the classroom and taught the topics outlined by the regular teacher. Additionally the problem solving framework was introduced during this timeframe and then regularly revisited to consolidate this new problem solving approach among the students. Testing was conducted at intermittent stages throughout the nine-week intervention to gain a more comprehensive view on whether or not the students were integrating the framework into their daily problem solving routine. A total of five tests were carried out during the nine-week intervention – one pre-test and four tests during the actual intervention.

Each test consisted of one mathematical problem that students had to solve. The new Project Maths syllabus in Ireland places increased emphasis on the development of students' literacy skills and to that end the questions selected were not just purely mathematical but instead were what has typically in Irish circles been described as "word problems". Word problems, according to Verschaffel, Greer, and De Corte (2000), is the term often used to refer to any mathematical task where significant background information on the problem is presented as text rather than in mathematical notation.

The question utilised in each test was carefully selected in an attempt to maintain the validity and reliability of the study. In the end it was decided that the best way to select questions of a similar standard and difficult, that were relevant to the new syllabus, would be to utilise past examination papers and sample papers from the Junior Cycle [3] examinations. Each question was selected to be slightly more challenging than the questions that the students encountered during class. This was the case as it was hoped that students would need to employ the problem solving framework shown to them during the lessons rather than being able to solve the question immediately upon reading it.

To assist with the assessment of the test in terms of being able to measure students' success at each of the different problem-solving episodes a template was designed around which each question was structured. When designing the template it was necessary to ensure that each episode of the Artzt and Armour-Thomas (1992) framework was addressed. Unfortunately it was not possible to assess each of the episodes in the framework e.g. *Read*. Since certain episodes could not be assessed independently they were instead combined with other episodes, e.g., *Read* was combined with *Understand* as if students demonstrated an understanding of the problem then we assumed that they had successfully read the problem. For this reason some of the tasks in the assessment template are deemed to assess two of the problem solving episodes as outlined in the Artzt and Armour-Thomas (1992) framework.

The six tasks, along with the episode(s) (in brackets) which each task measured, that students were asked to engage with as part of the assessment template were:

1) Underline or highlight the important information/facts given in the question (Read and Understand)

2) What is the question asking you to do? (Understand and Analyse)

3) Is there a process or method involved in solving this question? (i.e., What methods have you used to solve similar questions like this before?) (Plan)

4) What is the first step in attempting to solve this question? (Plan and Explore)

5) Complete the question and display all your workings in the space provided. (Implement)

6) Does your answer satisfy what is being asked in the question? (Verify)

The scoring of the assessment rubric is rather straightforward. If a student displays evident of completing a particular episode then they are given a score of 1. If no evidence is found of a certain episode then a score of 0 is awarded. Therefore in each test a student could score a total of 7 marks depending on whether evidence of all 7 problem solving episodes was present or not. As previously stated it is not always necessary to carry out every episode when problem solving but since we are dealing with novice problem solvers an absence of evidence of an episode will be viewed as an omission on the part of the student.

To ascertain whether or not teachers of the new Project Maths syllabus felt that this was a suitable framework for implementing and assessing problem solving in the classroom a focus group was conducted with four teachers towards the end of the nine week intervention. A total of twelve questions were put to the teachers such as "Do you think this framework is relevant to the aims of the Project Maths syllabus?", "What strategies do you currently use to implement problem solving?" and "Do you think this framework would be easy to implement in your classroom?" and their responses and feedback were recorded and then analysed.

FINDINGS

Episode engagement

From the analysis of the results it is clear that overall there was an increase in the total number of episodes that each student engaged with as part of their problem solving process. Table 1 presents a breakdown of the episodes that students successfully completed in the test that they completed prior to being introduced to the Artzt and Armour-Thomas (1992) framework.

From Table 1 we can see that all students showed evidence of reading the problem with a similarly high number showing evidence of understanding the problem. Unfortunately these positive results do not continue as none of the students showed evidence of analysing the problem (episode 3) or even verifying their solution (episode 7). The lack of analysis by the students is worrying as the analysis episode is important as it is at this stage that students examine the relationships between the information provided in the question and what they are required to show. Similarly to the *Analyse* episode zero students showed evidence of completing the *Verify* episode. An additional goal of Project Maths is that students develop the skills to justify/explain/verify their answers. It is clear from these results that these students have had very little exposure to this methodology so far in their mathematical careers.

The results from the four tests conducted after the introduction of the Artzt and Armour-Thomas (1992) framework are more promising. Immediately upon introduction to the framework all students showed evidence of engaging with the *Analyse* episode of the problem solving process. The number of students completing the *Plan*, *Explore* and *Implement* episode all show improvement from the pre-intervention test although there is some minor fluctuation between tests regarding the number of students engaging with each episode. Regarding the *Verify* episode it can be seen in Figure 1 that it took longer for student to habituate this episode into their problem solving process but promisingly by the end of the nine week intervention all seven students were displaying evidence of evaluating the outcome of their work. Overall the trend appears to be positive with regards to student engagement with the problem solving episodes.

Correct solutions to problems

Irrespective of the number of episodes that students engage with during their problem solving process it is still necessary to consider the number of correct answers in the tests. Even if a student completed all the episodes it is still possible for them to arrive at an incorrect answer. You would hope that upon reviewing their answer at the *Verify* episode and realising they are incorrect students would persevere and return to earlier episodes and attempt to determine where

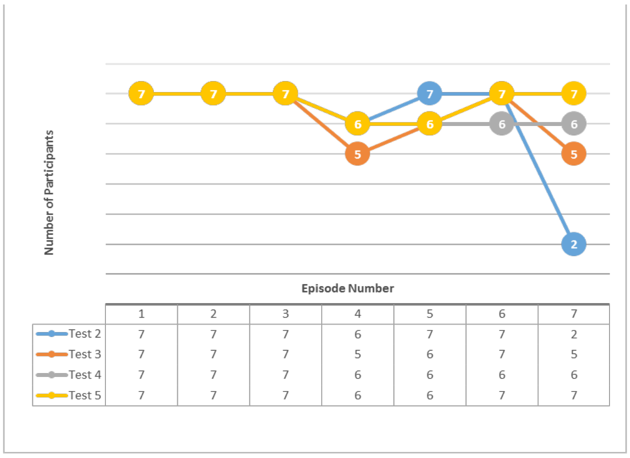

Figure 1: Results of episode inclusion during the intervention

	Read	Understand	Analyse	Plan	Explore	Implement	Verify
No. of students who completed each episode	7	6	0	4	3	3	0

Table 1: Results of episode inclusion in pre-intervention test

they have made a mistake along the way e.g. misunderstood the question in the *Understand* episode or overlooked a vital piece of information at the *Analyse* episode stage.

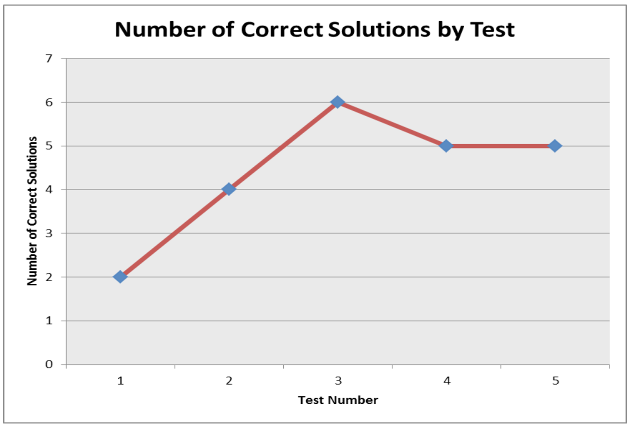

Figure 2: Results of Correct Solutions from Tests

Figure 2 highlights the trend regarding the number of correct answers across all 5 tests. The trend is positive, in general, showing that more students answered the questions correctly once the intervention commenced. There is a minor dip in the number of correct answers in the final two tests but even with this dip the number of correct answers is still higher than the results from test 1. The results regarding the total number of correct answers across the 5 tests offer additional justification regarding the effectiveness of the problem solving framework implemented but also highlight the need, in general, to introduce novice problem solvers to a structured approach to problem solving.

Feedback from teachers

Overall the teachers expressed satisfaction with the framework utilised by the authors to implement problem solving in the classroom. The teachers expressed their views that a framework like this would prove beneficial when attempting to familiarise students with the problem solving process. When asked if they had previously received or utilised any frameworks designed to assist in the teaching of problem solving the teachers stated:

> Teacher2: Well I would say I haven't. Have you?
>
> Teacher4: Yes I have, but I ignored it! Well we were given something like that (referring to Project Maths problem solving strategies poster)
>
> Teacher2: Yes that's the only thing, but I wouldn't say there was any time given in terms of training for teachers with the new Project Maths. In my opinion anyway, maybe it went over my head.

One teacher did go on to state that although she didn't utilise a "structured" framework she felt that within her class, and the classes of her colleagues, they embedded some of the problem solving episodes outlined in the framework.

> Teacher3: But I think to some extent that we do a lot of this. I mean I don't feel like this is something I've never seen before. We do say, well how would you solve this? What do we do now? What are you being asked? What are we looking for? Let's see, what are we missing? Do we have this? I think I do this, I'm sure everyone else does too. But I think that a structure where the kids know to put their information down, I think that this would help me.

This final comment about the students having a structure where they know to put their information down is referring to the assessment template structure outlined by the authors. This structure was also commented on by a fellow teacher who felt that it would be beneficial to students to have a structure, especially in an exam situation.

> Teacher1: But I think it would be useful maybe in an exam situation where all they see is a jumble of words and they just don't know where to start. At least they have a framework.

On a negative note one teacher commented that she felt it would take significant time to implement a framework of this type. Additionally she commented that trying to get students to focus on a problem for a long period of time might be an issue and that this could lead to discipline problems within the classroom.

> Interviewer: Just in terms of what you see there (Framework and Sample Question) do you think it would be worthwhile maybe trying to implement or trying to use this in a lesson?

Teacher4: I do think it would be, but what I do see this as is very time consuming. And in a classroom situation having each student reading and going through all of this a discipline issue probably would arise.

CONCLUSIONS

The results of this study showed that when teaching novice problem solvers it is important to offer appropriate instruction on problem solving so that these students can properly develop their problem solving abilities. This finding confirms Hembree (1992) and Higgins (1997) who observed similar results in their classroom studies. Once students engaged with the episodes of the problem solving framework the overall number of correct solutions to the attempted problems increased.

The result from the first test conducted as part of this study highlighted that very few students engaged with the *Verify* episode once they have reached a solution although almost every problem solving framework highlights the importance of this episode (Polya, 1973). This finding is also consistent with what other researchers such as Erbas and Okur (2012) have found. Encouragingly as the students became more familiar with the framework their results show that more and more of them conducted this episode and made some attempt to check their solutions with regards to the original problem statement.

Overall the teachers interviewed were positive regarding the structure and layout of the framework. Some issues around the time needed to problem solve within a class scenario were mentioned and other issues regarding students going off task when completing certain episodes were raised but neither of these concerns detracted from the overall positive comments from the teachers. The benefit of the framework as a resource within examinations was highlighted by the teachers as once students are familiar with the framework it should help them by scaffolding their problem solving efforts.

Finally it is worth mentioning as a discussion point the "age" of the original Artzt and Armour-Thomas framework upon which this work, and the work of Erbas and Okur (2012), was based. Does a more modern framework exist which would be more suitable for this purpose or is there a more up-to-date/adapted version of this Artzt and Armour-Thomas framework which might better aid teachers to implement problem solving in the classroom?

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ENDNOTES

1. The Project Maths Development Team is a team of experienced teachers of mathematics who have been recruited to provide professional development support to post-primary teachers of mathematics.

2. All subjects studied at post-primary level in Ireland can be studied at either ordinary or higher level with higher level being the more challenging.

3. The Junior Cycle is the first three years of post-primary education in Ireland. A state-wide examination takes place at the end of the Junior Cycle. Students normally sit for the examination at the age of 14 or 15.

Student's interpretations of visual models

Thomas Lingefjärd and Djamshid Farahani

University of Gothenburg, Department of Mathematics Education, Gothenburg, Sweden,
thomas.lingefjard@gu.se, djamshid.farahani@gu.se

Humans perceive and interpret graphical representations of models daily. Student's ability to do accurate interpretations of graphical and symbolical represented information is an important goal for the educational system. A study on 17 student's interpretations of graphical and symbolical representations of linear motion is reported. The collected material consists of about 10 hours video recording divided into 20 sequences about 15–50 minutes each. The tasks used to approach student's interpretation of graphical representation of linear motion where related to distance-time graph and analytical representations of distance function. The theoretical framework used is the theory of conceptual change for learning developed by Chi together with Tall and Vinner's framework on concept image and concept definition.

Keywords: Representations, iconic, every-day and scientific discourse, lateral categories of conceptual allocations.

The research reported here is not aimed at only identifying or describing students' alternative conceptions but instead provide possible explanation of why some students interpret graphical representation differently than other students. Mathematical representations such as diagrams, histograms, functions, graphs, tables and symbols normally makes it easier for us to understand abstract mathematical concepts or phenomenon described in mathematical terms. Representations are, according to Wittmann (2005), usually structured systems with strong connections to theories and may be seen as constructions that links abstract and concrete mathematics together.

Humans of today are facing a world that is shaped by increasingly complex, dynamic, and powerful systems of information that we meet through various different media. Being able to interpret, understand, and work with complex systems involves important mathematical processes that humans need to understand and be able to address when facing interpretation challenges.

Mathematical representations, structures and constructions are also usual when you study other subjects, such as physics. We wanted to investigate upper secondary school student's understanding and interpretations of mathematical representations of linear motion. Our research question is therefore:

- How do upper secondary school student interpret graphical and symbolic representations of linear motion?

THEORETICAL FRAMEWORK

We humans develop alternative reference systems in order to explain and understand the physical world we meet. We do this through daily life experiences. Several thousands of studies has described students alternative and resistant opinions about mathematics and natural science (Duit, 2004; Pfundt & Duit, 1993), together with reports on how teaching could be altered and developed in order to counter act on students alternative opinions (Slotta & Chi, 2006, p. 262).

Research has also shown that alternative ideas sometimes can be incompatible with scientific models and theories (Hammer, 2000; Shaffer & McDermott, 2005). There are several different labels on individually alternative opinions concerning daily life phenomena such as children's science, alternative ideas, intuitive ideas, and common sense beliefs.

The source of our daily life conceptions is shaped, according to diSessa (1993), by a row of well-developed intuitive ideas which diSessa defines as "Phenomenological primitives" or p-primes. P-primes are generalizable and different than our memories from events we have experienced. P-primes are also different than learning of scientific theories which generally are studied with a specific purpose, while p-prime are constructed and built rather unconsciously, less formal, intuitively and without a specific aim.

A p-prime can be either right or wrong, it may be compatible with scientific theories or not. To run a bike in hard upwind is more strenuous (p-prime: one has to work harder when the resistance is greater), motion with constant speed demands constant force (p-prime: there is always a force in an objects direction) or gravitation only applies to objects in movement (p-prime: no movement no force). A p-prime is generalizable since it may be used by the individual in different contexts. Even if we never had been closed to an earthquake, we still could determine we would run in a different direction (p-prime: the closer we come to a source, the stronger impact). A p-prime could be correct but used in the wrong context. The fact that the weather (on the northern hemisphere) becomes warmer in the summer is not because we are getting closer to the sun.

A collection of p-primes form the first and most elementary educational system, something diSessa calls the intuitive physical sense of mechanism. The way we humans interpret our surrounding world is the foundation for our learning of basic physics.

> The intuitive sense of mechanism contributes substantially to understanding school physics. (diSessa, 1993, p. 105).

diSessa claims that all p-primes are valuable and necessary and not misconceptions that we need to erase. diSessa also uses knowledge in pieces, intuitive conceptions or persistent false intuitions when he talks about p-primes. It seems that a p-prime sometimes needs to be modified, enhanced, or to be generalized.

GRAPHICAL REPRESENTATIONS

Representations, according to Wittmann (2005), are normally from a structured system with clear relations to theories and may be seen as constructions that connects abstract and concrete mathematics. A representation is something that stands for something else (Duval, 2006, p. 103). An object's speed or acceleration described through a speed-time-graph may be determined from the graphs slope, something not explicit obvious from the graphical representation.

Elby (2000) argues for a specific form of p-prim which activates with interaction with graphical artefacts. Elby calls this "What-you-see-is-what-you-get", or WYSIWYG:

> In my view, that's because the hill mistake and similar iconic interpretations spring, in part, from the activation of a cognitive structure, specifically, an intuitive knowledge element I call What-you-see-is-what-you-get. (Elby, 2000, p. 483)

Elby (2000) refers to a study where the students are asked to interpret a speed – time graph describing a bicycle path. See Figure 1.

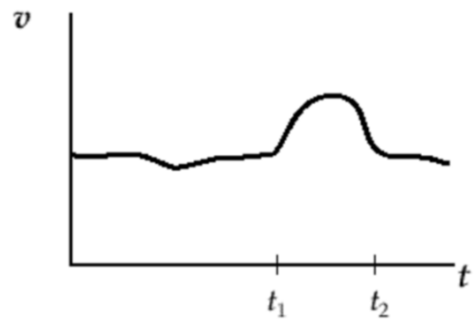

Figure 1: A bicycle path (Elby, 2000, p. 487)

Some students interpret this as evidence that the bicycle path goes over a hill. Some representations include special attributes which quickly catch our attention. Elby call this Compelling visual attributes. It may be all visual qualities and shades that are imbedded in the graphical representation, such as vertex, edges, contours, etcetera.

> My claim is that, even though what-you-see-is-what-you-get is not cued strongly in all contexts, it is cued strongly with respect to the compelling visual attribute of a representation. (Elby, 2000, p. 484)

Which one of these visual properties we notice first is strongly context dependent. We need to know what a bicycle is before we can connect its path to a graph.

> The over-literal readings call to mind the "what you see is what you get" (WYSIWYG) knowledge element proposed by Elby (2000), where students interpret a representation in the simplest, most literal way possible (a bump on a graph corre-

> sponds to a hill). This WYSIWYG element is a representational analog of the phenomenological primitives (or p-prim) described by diSessa (1993) which include such basic reasoning elements. (Kohl, 2001, p. 107)

CONCEPTUAL CHANGE

In order to understand information presented in a graph, we need to understand the underlying concepts of the phenomena presented. Conceptual understanding springs from the idea that knowledge is constructed actively by the individual for instance by testing alternative ideas. All concepts relate to other concepts and most concepts have both definition and image in our mind. Tall and Vinner (1981) said:

> We shall use the term concept image to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes. It is built up over the years through experiences of all kinds, changing as the individual meets new stimuli and matures.

On the other hand:

> The concept definition [is] a form of words used to specify that concept. (p. 152)

Gray and Tall suggest three mathematical worlds to explain development of mathematical understanding. This involves a parallel development of conceptual embodiment, as the complementary use of human perception and action and proceptual symbolism, involving manipulation of symbols that arise from operations. The term 'procept' address something that can be used flexibly as process or concept, including numbers, fractions, algebraic expressions, derivatives, integrals, and so on (Gray & Tall, 1994).

A key challenge to research in conceptual development and learning is to understand how individual constitute aspects of a scientific understanding of a concept. Knowledge acquisition is viewed as a process that involves actively generating and testing alternative propositions. Conceptual change refers to any change in conceptual understanding.

Three main perspectives on conceptual development are: the epistemological, the metaphysical and the cognitive perspective. The epistemological perspective on conceptions is based on criteria for difference. Our understanding regarding the world around us and how it is constructed relates to three ontologically "lateral" categories; mental state, entities and processes (Chi et. al., 2012). See Figure 2.

When we humans learn concepts, we also learn where the concept belongs in an ontological sense. What is the weight of the world series in soccer sounds like a strange question since the weight of something belongs to "Entities", while the world series in soccer is categorized as an "Event". Since the two concepts belong to different categories they cannot create an understandable context.

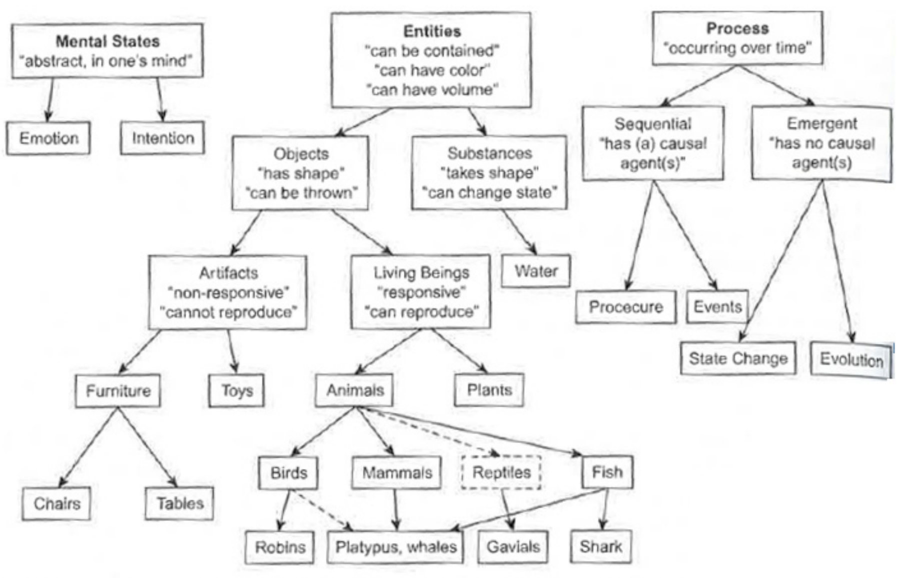

Figure 2: Conceptual change in an ontological structure (Chi et al., 2012)

Even close sub categories under the same branch might be ontological different lateral categories. Most young children know the difference between artefacts and living beings. Conceptual change occurs when we address concepts to different ontological categories. This constantly ongoing process happens inside us and strives for interpretations of new situations and to make them understandable.

Sometimes our interpretation of the world place concepts in the wrong place. One such well known phenomenon is the fact that students might view light or electrical current as material substances. Scientifically theories describe electrical current and light as sequential processes (Chi et. al., 2012, p. 57).

Conceptual understanding from the metaphysical standpoint is described as a restructuring of concepts over ontological lateral (not hierarchical) categories. Sometimes learning of scientific concepts is difficult since the process requires concept has to be re-assigned to an ontologically distinct category (across trees). The cognitive perspective on conceptual understanding is based on nature of misconceptions. Sometimes there are clear qualitative differences between the individual's (alternative interpretations, p-primes) and scientific models or expert's descriptions of a phenomenon.

Conceptual understanding based on a cognitivist's perspective can be seen as a process that tends to obtain a gradual increase in the correlation between alternative ideas (knowledge into pieces) and scientific theories (diSessa, 1993; Elby, 2002, & Hammer, 2000). The cognitive perspective on conceptual change starts in identifiable qualitative differences between an individual's (alternative interpretations, p-prime, and knowledge system) and scientific models or expert's description of systems.

METHODOLOGY

Our cohort consisted of 17 upper secondary school students (9 girls and 8 boys) at the natural science program, all enrolled in year 2. All students' volunteered to take part in the study. The students were grouped in small groups with 2 or 3 students in each group; a video camera was mounted to record the discussion. No observer was present to help the students to be natural in their comments and attitudes. Students written solutions and notes were collected.

Our intention was to record the individual students responses but we considered that this was easiest done by encouraging group discussions. The students responded to 13 questions all together, this article will present the dialogue in relation to a distance – time graph. The students took note and did some calculations on paper, which we collected. We will only present some short results here, depending on page limitations.

Problem

The graph in Figure 4 illustrates a train that moves during 4 hours. The distance s is in kilometer and the time t is in hours

1a) When do the train run at the highest speed?
1b) How do you know that?
1c) Are there more than one occasion when the train has this high speed?
2a) When do the train run at the lowest speed?
2b) How do you know that?
2c) Are there more than one occasion when the train has this low speed?
3) What could be said about the train's direction by interpretation of the graph?
4) Sketch a graph to describe the trains speed during these 4 hours.

RESULTS

Student group Arman, Alexander and Gideon – written answers

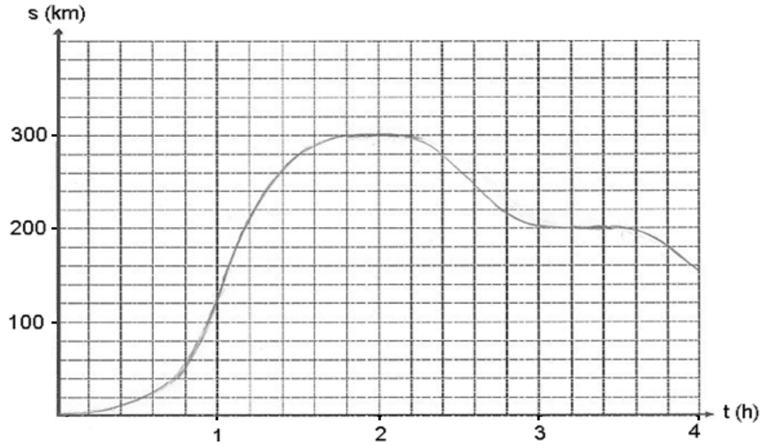

Figure 3: A moving train

Student: 1a. The train has the largest speed at 2 hours.
Student: 1b. Because that is the highest point in the graph.
Student: 1c. Yes, at 1.48 and 2.12. That speed last for 24 minutes.

Student group Arman, Alexander and Gideon – video recording

Student1: The train is fastest up here or ...
Student2: Write when it has the largest speed at 2 hours.
Student1: Here it goes down a hill; here it goes up a hill, up and down...
Student2: I think we should be careful writing about hills...
Student3: I do not think I never have seen a train go up a hill like that
Student1: Well, maybe when it is going up or around a mountain or something...
Student3: Like a rollercoaster? In an amusement park?
Student2: In San Francisco they have cable cars where you can hang outside...

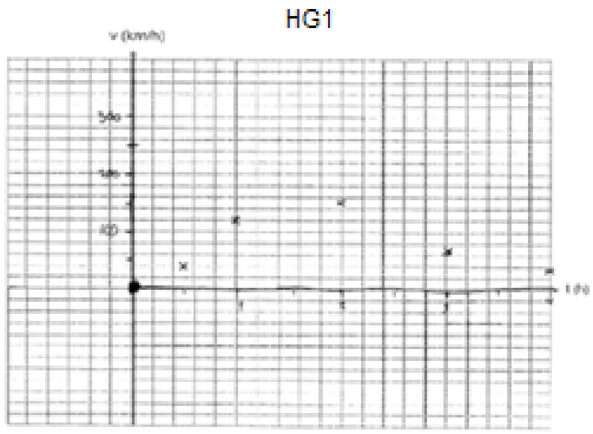

Figure 4: Alexander's response to task 4

Arman claims that the largest speed occur after 2 hours. His interpretation of the graph is built on the opinion that the graph is illustrating a train going "up a hill" and "down a hill". Arman interpret the graph as a "true image" which is called an iconic interpretation.

Alexander Shares the opinion that the speed is highest "at the top point up here" and that the train is going up and down. Alexander also connects the graph to the streets of San Francisco. Gideon, after some doubts, associates the graph to a rollercoaster in an amusement park.

Students' alternative iconic interpretations of graphical representations are well known from research and also obvious in this study. Arman's, Alexander's and Gideon's interpretation of the distance – time graph could be seen as an alternative descriptive model. The students talk about different places, refer to different events, and gives examples in different ways. Nevertheless, one thing they do have in common. Their interpretations seem to be iconic and they are all focusing on the wrong attribute.

Alexander's description that "In San Francisco they have cable cars where you can hang outside ...", or as Arman express it "Here it goes down a hill; here it goes up a hill, up and down...", are example of intuitive ideas and daily life concepts That seems to be deeply imbedded in the imagination world of these students. In reality it is probably impossible for a train to go up such a steep hill. The interpretation process together with the need to create a context for the train seems to be more important for the students than to think about the models validity and reliability. This in turn keeps their alternative ideas safe and resistant towards change.

Student group Jakob, Maria and Linnea – written answers

Student: 1a. When the curve has the steepest slope.
Student: 1b. Longest length on shortest time.
Student3: No, once.

Student group Jakob, Maria and Linnea – video recording

Student1: One box up in the graph paper is a fifth equal to 20 km, so the train is moving 20 km in short time while over here the train is going 20 km in a long time.
Student2: Yes, it is faster there.
Student3: It is fastest where the slope is steepest since the graph is showing us the real speed, the moment speed, so we have the answer to 1a.

One qualitative aspect of students understanding of a *distance – time graph* is found in Jakob's reasoning.

Jakob illustrates a good technique for determination of the slope of the graph. The discussion also reveals that the group manages to separate the meaning of distance and speed.

Written answers to question 2a-2c

Student: 2a. When the curve's slope is zero.
Student: 2b. Distance is not changing although the time is running.
Student3: Yes, at three occasions.

Quotations from transcribed video material related to 2a-2c

Student1: When the slope is zero, the train is standing still.
Student2: The train goes to one point and the back again and then it is standing still for some time.
Student3: Positive direction upwards and negative direction downwards means that the train is going back but not the whole way.

Through this conversation, the students are validating and develop their reasoning concerning the meaning of the graphs slope. The student's responses show that they are not confused by the graph, instead they are arguing for the relation between the slope of the graph and the movement of the train. The fact that they are not doing any iconic interpretation of the graph, allows them to give a full description of the meaning of the graph. As a result, they arrive to a quite different graph of the speed – time graph asked for in task 4.

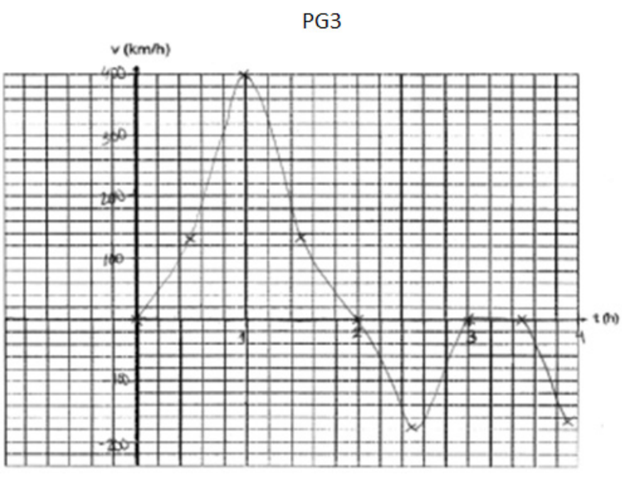

Figure 5: Maria's response to question 4

CONCLUSIONS

Sometimes may daily life experiences be too strong and to overwhelming when students try to make interpretations of graphical representations. This is obviously the case of Arman, Alexander and Gideon. We have shown you the results of two groups here, results that may be seen as the two extremes in our collection of results.

When students relate to iconic interpretations as cable cars in San Francisco, they do this because such interpretations have some advantages. They are easy to remember, seemingly realistic, easy to identify in a daily life language, easy to compare, and sounds convincing.

Our results indicate that the more a student build on iconic ideas as model for explanation, the more demand he or she will also need for daily life concepts in the arguments.

It is important for us to underline that we do not see the results as misconception, more as misplaced concepts. When Arman, Alexander and Gideon and probably many other students with similar iconic interpretations, are trying to explain the distance – time graph, they seem to relate the meaning to an event. This is important, since a distance – time graph should be seen and interpret as a representation of a sequential process. Event and sequential process are two close sub categories in Figure 2 (Chi, 2013). The fact that the students we studied are so called specialists in mathematics and science indicate that it is perhaps harder to make this restructuring than researcher have thought so far.

As researchers we might marvel over how much material that was derived from these students discussions regarding a situation with a train. But this article is also something teachers in general could read and discuss. The fact that humans in general may misinterpret some of all the graphic information we meet on a daily basis is nothing strange. How could we do otherwise? This article might help teachers to support student's learning and understanding of both mathematical entities and mathematical processes in a more successful way.

In the discussions between students regarding the train's movement we could identify occasions when students learned from each other. Even though this was a research study, it is beneficial when students learn from the organization of the study.

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Differences in the situation model construction for a textbook problem: The broken tree or the broken bamboo?

José Antonio Juárez López[1], Josip Slisko Ignjatov[1], Lidia Aurora Hernández Rebollar[1] and Mónica Monroy Kuhn[2]

1 Benemérita Universidad Autónoma de Puebla, Puebla, México, jajul@fcfm.buap.mx, lhernan@fcfm.buap.mx, jslisko@fcfm.buap.mx

2 Universidad Popular Autónoma del Estado de Puebla, Puebla, México, monica.monroy@upaep.mx

The present study addresses the differences in the construction of ninth graders' situation models, when they were confronted with an application problem that is found in middle-school mathematics textbooks in Mexico. This study focuses on the reading-comprehension phase, better known as the situation model, which is considered a prerequisite for the mathematical model construction of word problems. Four versions of the problem "The broken tree by the wind" were given to a total of 192 ninth- grade students. Students' drawings were analyzed in order to identify any possible causes by which few of the students were able to build an adequate situation model. The results suggest a potential influence of problem wording on students' visualizations and ways of interpreting mathematical problems during the situation model construction.

Keywords: Situation model, word problems, mathematics textbook.

INTRODUCTION

Text comprehension in mathematical word problems has been investigated in order to improve problem-solving processes (Cummins, Kintsch, Reusser, & Weimer, 1988; Mayer & Heagarty, 1996). Researchers have been trying for some time to get an insight into the causes that affect such performance in order to improve it (Kintsch & Greeno, 1985). Students' difficulties increase when problems imply a mathematical modelling process (Galbraith & Stillman, 2006).

It is already well established, both theoretically and experimentally, that the modelling process comprises several phases (Borromeo-Ferri, 2006). The first one consists of the situation model (SM) construction that the problem refers to, and it is necessary for the understanding of the mathematical problemand its further resolution, which is considered as "...a mental representation of the situation described by the text..." (Kintsch, 1986, p. 88).

Some results in a recent exploratory investigation (Juárez & Slisko, 2013) show that middle-school students have great difficulties to build the situation model of a problem named "The fallen tree". The mental images or models of the situation were studied through the requested drawings. The statement of the problem is the following:

> The wind has broken a tree trunk in such a way that two of its parts form a right triangle with the ground. The upper part forms a 35° angle with the ground, and the distance, measured on the floor, from the trunk to the fallen top of the tree is 5m. Find the height of the tree (Mancera, 2008, p. 333).

Juárez and Slisko (2013) found that after thirty middle-school students had been given "The broken tree" problem, only ten of them were able to depict the actual situation described in the text. However, these ten students interpreted it in different ways.

This problem and its various formulations appear in several middle-school textbooks published in Mexico authorized by the Ministry of Education. In some of them, it is just an application problem (Briseño, Carrasco, Martínez, Palmas, Struck, & Verdugo, 2007, p. 223; Waldegg, Villaseñor, García, & Montes, 2008,

p. 205; Farfán, Cantoral, Montiel, Lezama, Cabañas, Castañeda, Martínez, & Ferrari, 2008, p. 202). In certain books, its historical origin is mentioned (Pérez & Pérez, 2008, pp. 227–228; Arteaga & Sánchez, 2008, p. 75; Sánchez, 2008, p. 216).

The results mentioned above motivated us to conduct a more comprehensive study, where the factors and processes involved in the situation model construction as a prior step leading to a resolution of the problem could be investigated. In particular, we were interested in analyzing the possible effects of changing certain elements from the text of the problem, i.e. the title and the object of the problem, on students' text-comprehension process and their corresponding situation models.

Results of previous studies based on students' drawings have shown that students find it difficult to imagine the situation. What makes it hard for most of them, is the mental image of a "tree" (the trunk's thickness and the treetop formed by branches and leaves), because it is hard for them to "get rid" of the treetop. This study central assumption is that if students get a version of the problem, where the tree is changed for a simpler "tree" plant, like a "bamboo" (it has a straight shape and it does not have any branches that hamper the floor support), they will perform better in the development of the situation model than the students who get "the broken tree" version.

So, our research question is the following: Do changes in the problem's title, as well as in the object used in the text have an effect on middle-grade students' situation model construction?

THE CONSTRUCTION OF THE SITUATION MODEL AND MATHEMATICAL WORD PROBLEM SOLVING

Polya's model for problem solving has played a significant role in Mathematics Education since it has clarified many of the cognitive processes involved in it (Polya, 1976). In the first stage, the understanding of the problem, strategies can be found that help with the representation of the situation and the comprehension of the problem's conditions. The importance of using some heuristics is also emphasized here, like 'draw a picture or a diagram'.

Nathan, Kintsch and Young (1992) propose a theoretical model, where students have to read and understand the statement first, and then the information obtained is "organized into a (qualitative) situation model and mapped into a (quantitative) problem model that captures the algebraic structure" (p. 332).

According to Diezmann (2000), to make a drawing of the problem situation may be crucial for a person trying to solve it. In this way, many word problems describing a real situation are presented to the student as if its mental representation was an immediate process. However, it is a translation process that involves linguistic information decoding and visual information encoding. In this sense, Heagarty and Kozhevnikov (1999) investigated the relationship between mathematical visualization and mathematical problem-solving, being the first one understood as an individual's ability to use images or diagrams. They classified visual-spatial representations as schematic or pictorial and, in their study, schematic representations, which encode spatial relations described in the problem, were positively correlated with success in mathematical problem solving. On the other hand, pictorial representations, which are related to the visual appearance of the objects described in the problem, were negatively correlated.

In the same line, Edens and Potter (2007) examined the relationship between student performance on particular drawing tasks and their achievement in mathematical problem solving. They provided evidence that teaching strategies based on drawings can help teachers in obtaining useful information about their students' level of spatial understanding.

From a cognitive perspective, several studies have focused on the complex process of reading comprehension during word problem-solving, both in algebra and realistic problems, as well as in arithmetic problems (Vicente & Orrantia, 2007). These researchers acknowledge the need to create a model of the problem situation, by applying the real-world knowledge possessed by the student. Meanwhile, Reusser (1988) suggests that it is necessary to produce what he calls an "episodic model of the situation" between the text base and the problem's mathematical model.

One of the consequences that the lack of an appropriate situation model construction would have, for example, is the student's behavior strongly linked to practices, as mentioned by Reusser and Stebler (1997):

> As a result of schooling, students' behavior is pragmatically functional if they take into account any information they can draw from both problem texts and contexts. That is, their mathematical sense-making is functional if they actively and continuously construct a mental representation not only of the specific task (problem model…), but also of the socio-contextual situation which they are in (construction of a social context model)… (pp. 325–326).

Despite the amount of research indicating the importance of the situation model for understanding the text of the problem, some researchers such as Voyer (2010) argue that the question of the influence of the model constructionon student performance in solving word problems remains open. This conclusion might be correct for some arithmetical problems used in Voyer's research. Nevertheless, it seems that in trigonometric problems, the situation model construction influences significantly posterior students' problem-solving performance. It is easy to agree with van Dijk and Kintsch (1983), who emphasize that "… we know very little about the conditions that promote or inhibit the construction of situation models from texts …" (p. 346).

METHODOLOGY

The participants were 192 students in 9th grade at a public middle school in central Mexico, all divided into four different groups. In terms of the curriculum, the ninth graders had already been exposed to the Pythagoras Theorem and trigonometric ratios.

Each student was given a worksheet containing one of the four versions of "the broken tree" problem and they were asked, after reading the text of the problem, to draw the described situation. There was no time limit, and they were not asked to solve the problem.

Each version of the problem was applied to a different group. Two central objects were used, i.e. the tree and the bamboo, resulting in two distinct texts. In addition, two titles for each one of them were drafted, yielding four versions of the problem that are shown in Table 1.

The data collected for this study included only students' drawings, since the focus of the study was on the situation models built by the students. In order to describe them, the collected drawings were categorized via our definition of situation model mentioned above. Students' responses were grouped by identifying those drawings that had common characteristics in their situation models, in the sense that they were leading to the mathematical model construction.

RESULTS AND ANALYSIS

From the analysis of the drawings, four categories were obtained: Situation Model with a Right Triangle (SM-RT), Situation Model with an Arbitrary Triangle Related to the Situation (SM-AT-RS), Situation Model with an Arbitrary Triangle Not Related to the Situation (SM-AT-NRS) and Situation Model Without Triangle (SM-WT).

	Version 1	Version 2	Version 3	Version 4
Problem's title	The broken tree that forms a triangle	The broken tree	The broken bamboo that forms a triangle	The broken bamboo
Problem's statement	The wind has broken a tree in such a way that two of its parts form a right triangle with the ground. The upper part forms a 35° angle with the ground, and the distance, measured on the floor, from the trunk to the fallen top of the tree is 5m. Find the height of the tree.		The wind has broken a bamboo in such a way that two of its parts form a right triangle with the ground. The upper part forms a 35° angle with the ground, and the distance, measured on the floor, from the bamboo's base to the fallen top is 5m. Find the height of the bamboo.	

Table 1: Four different versions of the "the fallen tree" problem

Different SM for the version "The broken tree"			
SM-RT	SM-AT-RS	SM-AT-NRS	SM-WT
29.5% 13/44	29.5% 13/44	9% 4/44	32% 14/44

Table 2: SM classification for the first version of the problem

Version 1: The broken tree

The number of students that constructed a particular type of SM for the first version of the problem "The broken tree" is shown in Table 2.

The above table shows that almost 30 percent of the students produced the SM considering a right triangle, but only one of them drew up the situation model correctly. This finding is consistent with what Diezmann (2000) reported with primary school children, that although 'drawing a diagram' is advocated as a useful strategy for solving problems, to generate an appropriate diagram is problematic for many students.

On the other hand, the same number of students built their SM with an arbitrary triangle related to the situation described in the problem. Figure 1 shows the SM of one of these students, Victor, where its construction could have been affected by the interference of his real-life knowledge evoked while imagining the situation.

Version 2: The broken tree that forms a triangle

The number of students that constructed a particular type of SM for the second version of the problem "The broken tree that forms a triangle" is shown in Table 3.

In most of the situation models, a high number of realistic elements related to the scenario were included, such as wind, tree branches, and leaves, as well as clouds and rain. When compared to the first version of the problem, the same percentage of students did not include any triangle in their situation models, even though the expression "...that forms a triangle" was present in the title of the second activity. It seems not to have influenced the understanding of the situation or the corresponding situation model. These results suggest that the change in the problem's title had no effect on the construction of the SM.

However, the percentage of students that included an arbitrary triangle not related to the situation increased when compared to the first version. Figure 2 shows a situation model, for example, where Daniel drew an arbitrary triangle that has nothing to do with the situation described in the text. It seems that the explicit reference to a triangle in the activity's title forced the students to draw a triangle, without taking into account the rest of the information included in the text.

This behavior can be explained according to what Borromeo-Ferri (2006) claims:

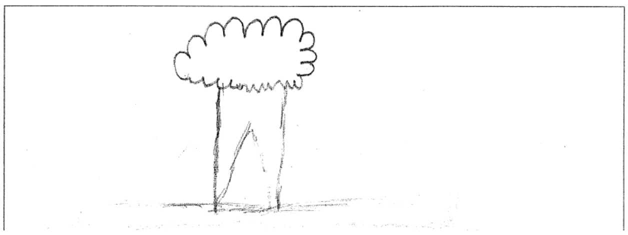

Figure 1: SM drawn by Victor (arbitrary triangle related to the situation)

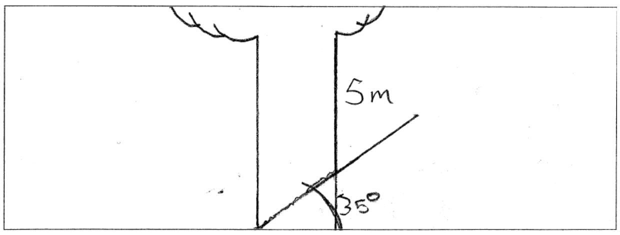

Figure 2: SM drawn by Daniel (arbitrary triangle not related to the situation)

Different SM for the version: "The broken tree that forms a triangle"			
SM-RT	SM-AT-RS	SM-AT-NRS	SM-WT
20% (10/50)	32% (16/50)	18% (9/50)	30% (15/50)

Table 3: SM classification for the second version of the problem

> The individual has a mental representation of the situation, which is given in the problem. This MRS can be very different, for example depending on the mathematical thinking style of the individual: visual imaginations in connection with strong associations to own experiences; or the focus lies more in the numbers and facts given in the problem, which the individual wants to combine or relate. (p. 92)

Version 3: The broken bamboo

The "broken bamboo" version was given to 48 students. Table 4 shows the percentages for each SM. Only 12 of them (25%) included a right triangle in their model. In the second column it can be observed that 40% of the students constructed a SM with an arbitrary triangle related to the situation. This could mean that, changing the tree for the bamboo could have been the cause for more students to include a triangle in their situation model. In Figure 3 one example of this type of situation model is shown.

Version 4: The broken bamboo that forms a triangle

For the problem's version "The broken bamboo that forms a triangle", 26 students out of the 50 (52%) that were given this version of the problem, were able to draw the right triangle required in the statement. This percentage is significantly higher when compared with versions 1, 2 and 3, thus confirming our hypothesis about the positive effect of including the "bamboo" instead of the "tree" in the statement of theproblemand "...that forms a triangle" in the title. Figure 4 shows an example of such drawings. Also, 26% of the students included an arbitrary triangle related to the situation in their SM.

The number and percentage of students' types of situation models for the problem's version "The broken bamboo that forms a triangle" is shown in Table 5.

FINAL COMMENTS

After analyzing the drawings made by students when confronted with a mathematical word problem, we note that the construct 'situation model' as conceptualized in the literature, was effective to account for the various productions and elements that hamper reading comprehension, and therefore the construction of mental representations corresponding to the situation described in the problem.

One of the first interesting findings in this study is the fact that a very low percentage of students created the situation model so as to enable them to solve

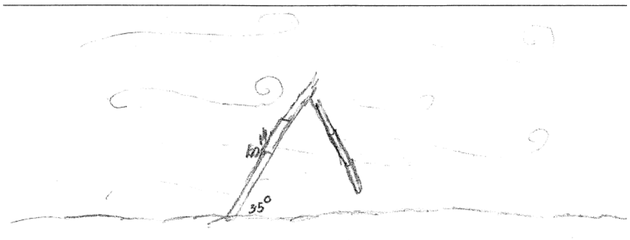

Figure 3: SM drawn by Monica (arbitrary triangle related to the situation)

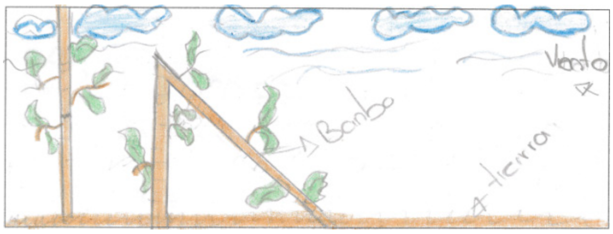

Figure 4: SM drawn by Giselle (right triangle)

Different SM for the version "The broken bamboo"			
SM-RT	SM-AT-RS	SM-AT-NRS	SM-WT
25% 12/48	39.6% 19/48	4.2% 2/48	31.2% 15/48

Table 4: SM classification for the third version of the problem

Different SM for the version "The broken bamboo that forms a triangle"			
SM-RT	SM-AT-RS	SM-AT-NRS	SM-WT
52% 26/50	26% 13/50	8% 4/50	14% 7/50

Table 5: SM classification for the fourth version of the problem

the problem in any of the four versions presented to them. This shows, once again, the importance of the proper construction of the situation model as part of the text comprehension process and as a step towards the development of the mathematical model.

Students' productions in this study included a variety of triangles, and although they do not represent the real situation, we realized that in the problem's version "The broken bamboo that forms a triangle", it was more natural and easier for them to imagine a situation where a triangle is formed than in "The broken tree that forms a triangle" version. One possible explanation is that for students it is more likely to imagine how the wind can break the vertical and thin bamboo structure than the one of the tree, possibly because in the latter case it might be thought that its root is stronger and its trunk is thicker, and therefore more difficult to bend.

It has been observed that in many mathematics textbooks, the text comprehension phase in word problems is presented as if the coherent situation model construction was a trivial process. This phenomenon could be one of the reasons students do not distinguish and are not able to sense its main characteristics. We believe that the findings of this study can be used to conduct further research to help clarify the entire understanding process and to serve as a reference in textbooks design and development, specifically related to word-problem solving.

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Mathematical models for chemistry and biochemistry service courses

Victor Martinez-Luaces

University of the Republic of Uruguay, I.I.Q.-Fing., Montevideo, Uruguay, victorml@fing.edu.uy

In this paper, applications and modelling problems for chemistry and or biochemistry courses are analysed. They obtain better results than others, in order to motivate students of chemical careers towards mathematical problem solving, since these problems are specially tailored for their needs. A teaching experience, carried out during two decades is analysed and a particular example is exposed with further details. The results of this experience led us to several conclusions that are included in the last section of the article.

Keywords: Nonlinear models, chemical careers, students' motivation.

INTRODUCTION

In chemical careers mathematical courses like numerical methods, differential equations and statistics are widely used to solve problems concerned with modelling real problems relevant for chemistry and/or biochemistry.

Nevertheless, most mathematical texts include only physical problems more relevant for other disciplines, like mechanics or electromagnetism. The need for relevance was highlighted by many writers as being important in motivating students when learning mathematics. For example, Bajpai, Mustoe and Walker (1975) suggested a range of improvements including a modelling approach and providing more relevant examples. In the same direction, according to McAlevey and Sullivan (2001), there is a need for using real-life problems since 'Students are best motivated by exposure to real applications, problems, cases and projects'.

For this reason, several previous articles (Martinez-Luaces, 2003, 2006, 2009a) have focused on how modelling may be used to motivate students in those careers. In particular, ordinary differential equations (O.D.E.) linear systems appear regularly in Chemical Engineering, Food Technology Engineering and Environmental Engineering courses. This is due to the usefulness in modelling chemical kinetics (Martinez-Luaces, 2012a), or water solutions, mixtures and reactors problems (Martinez-Luaces, 2005).

The introduction of chemical and/or biochemical problems help to motivate students and can be widely used for modelling and applications examples, particularly in differential equations (Martinez-Luaces, 2006), probability and statistics (Martinez-Luaces, Velazquez, & Dee, 2009) and numerical methods (Guineo Cobs & Martinez-Luaces, 2003).

In previous papers and in a pair of books (Martinez-Luaces, 2009b, 2012b), problems involving chemical kinetics, mixing problems, reactors, etc. have been exposed and analysed from its educative potential viewpoint. In this paper, more simple models will be considered and a concrete example will be discussed in the third section. Despite the simplicity of these models, they appear regularly in several branches of chemistry and biochemistry; they allow teachers to pose interesting questions and even project work to be carried out by the students in small groups.

Conclusions based on the results of the teaching methods used, will be drawn for differential equations, statistics and numerical methods courses for chemical careers. Several of these conclusions can be easily extrapolated to other mathematical service courses.

A PARTICULAR EXPERIENCE IN URUGUAY

In the late 90's, a small group of teachers and researchers was formed in the Chemistry Faculty at the University of the Republic of Uruguay (UdelaR), having their members an applied profile. For several years, this team included chemists, engineers and applied mathematicians. This group – in addition to other tasks as mathematical and engineering consultants – was appointed to be in charge of a course called Mathematics III, created in 1996 especially for Food Engineering students. The original syllabus included Ordinary Differential Equations (ODE), Partial Differential Equations (PDE) and an introduction to Laplace Transform, with emphasis towards applications and other disciplines. After the change of plans in year 2000, this course was replaced by another two – Mathematics 005 for Food Technology Engineering and Mathematics 105 for Chemical Engineering – with more class hours per week and a greater presence of modelling activities and applications.

Since the beginning of this course, direct and inverse modelling problems were proposed to the students (Martinez-Luaces, 2009a). Three thematic areas were especially suitable for this purpose: Chemical Kinetics (Martinez-Luaces, 2012a), mixing problems (Martinez-Luaces, 2009b) and Mass Transfer (Martinez-Luaces, 2003).

Chemical Kinetics is an important source for interesting problems. In our research work, the most refined versions of those problems were published in books, papers and proceedings, such as: Guineo Cobs and Martinez-Luaces (2002), Martinez-Luaces (2005, 2009b, 2012a, 2012b). Examples of these problems are: mutarotation of glucose, mechanisms with two or three reactions in series, carbon dioxide adsorption on platinum surfaces, etc.

Mixing problems have been another important source for modelling problems, such as: interconnected tanks system, tank divided into several compartments, stirred tank with recirculation, etc. These problems and many others have been included in papers published in international journals (Martinez-Luaces, 2005, 2009a), in books (Martinez-Luaces, 2009b, 2012b) and conference proceedings (Martinez-Luaces & Alfonso, 2000; Martinez-Luaces, 2007).

Finally, with regard to the problems involving parabolic PDE applied to Mass Transfer real situations, there are many examples that can be mentioned. For instance: drying a vegetable through its faces, sugar diffusion in a cherry, chives drying process, dissolved oxygen electrode, packed bed chemical reactor, diffusion of pollutants in the Chernobyl accident and pollution in the Rio Uruguay. These problems were presented briefly – with its analytical and/or numerical solution discussed – in an article published in New Zealand Journal of Mathematics (Martinez-Luaces, 2003), but there are also full versions of the modelling process with further details of the resolution in a book published a few years later (Martinez-Luaces, 2009b). Partial preliminary versions of those problems were presented in mathematical education conferences like Martinez-Luaces and colleagues (2000, 2001), Guineo Cobs and Martinez-Luaces (2003) and Martinez-Luaces and Guineo Cobs (2005), among others.

A NONLINEAR MODEL USEFUL FOR CHEMICAL CAREERS

The nonlinear mathematical formula $y = \frac{ax}{x+b}$ (Eq. 1) is widely used in chemistry and biochemistry for different purposes. For instance, the Michaelis-Menten kinetics is a well-known model in biochemistry of the form $v_0 = \frac{v_{\max}[S]}{K_m + [S]}$ (Eq. 2) where v_0 and $v_{\max}$ are the initial and the maximum velocity of the enzymatic reaction, $[S]$ is the substrate concentration and K_m is a constant (called Michaelis constant), which depends on the enzymatic reaction considered (Nelson & Cox, 2008).

Irving Langmuir, a Nobel Prize winner in chemistry, developed an equation that relates the coverage or adsorption of molecules on a solid surface to gas pressure or concentration of a medium above the solid surface at fixed temperature (Masel, 1996). The equation is $\Theta = \frac{\alpha P}{1 + \alpha P}$ (Eq. 3) where Θ is the fractional coverage of the surface, P is the gas pressure (or concentration in the case of liquids) and α is a constant. A very simple algebraic manipulation gives $\Theta = \frac{P}{1/\alpha + P}$ (Eq. 4) which is just a particular case of (Eq. 1).

The last example is a mathematical model for the growth of microorganisms proposed by Jacques Monod (1949). The mathematical formula is $\mu = \frac{\mu_{\max} S}{K_S + S}$ (Eq. 5), where μ is the specific growth rate of microorganisms and $\mu_{\max}$ represents its maximum value, S is the concentration of limiting substrate for growth and K_S is called the "half-velocity constant" (Martinez-Luaces, 2009b), since it corresponds to the value of S when $\frac{\mu}{\mu_{\max}} = \frac{1}{2}$ as well as the constant K_m in (Eq. 2).

The Monod equation has the same form as the Michaelis-Menten equation, but it was developed empirically whereas the Michaelis-Menten model is based on theoretical considerations.

A typical problem that arises in the treatment of data corresponding to these equations is the parameters determination since all of them are nonlinear models. In order to solve this problem, several methods were proposed to linearize these equations, being Lineweaver-Burk, Hanes-Woolf, Eadie-Hofstee, Scatchard, and Eisenthal-Cornish-Bowden the most important ones (Nelson & Cox, 2008).

For instance, the Lineaweaver-Burk (or double reciprocal plot method), proposes a graphical representation of $\frac{1}{V_0}$ vs $\frac{1}{S}$. It is easy to observe that the reciprocal of (Eq. 2) gives $\frac{1}{V_0} = \frac{K_M}{V_M}\frac{1}{S} + \frac{1}{V_M}$ (Eq. 6), so the x−intercept of the graph represents $-\frac{1}{K_M}$ and the y−intercept is equivalent to the inverse of V_M. An alternative way is to obtain the coefficients of a linear regression (i.e. $\frac{K_M}{V_M}$ and $\frac{1}{V_M}$) and finally get K_M and V_M from these coefficients.

Other methods propose a different linearization or another mathematical procedure for recovering K_M and V_M from experimental data.

All these methods can be compared in terms of exactitude and precision (Martinez-Luaces & Silva, 2014), using simulated data perturbed with Gaussian noise with different amplitudes. This comparison is particularly significant when the relation between trend and noise tends to increase.

Since equations (2), (3) and (4) represent the same mathematical model (Eq. 1), one of them (the Michaelis-Menten equation) may be chosen as an example in order to show the methodology to be followed.

In the selected equation, the parameters are K_M and V_{max} and variables are [S] and v_0. In a previous paper (Op. cit., 2014), typical values of these parameters and variables were chosen and theoretical curves were obtained. A Gaussian noise with different amplitudes was superimposed to the theoretical data obtained from (Eq. 2) with typical values of V_M, K_M and [S].

The graphics in Figure 1 show the simulated curves for the Michaelis-Menten equation (initial velocity vs substrate concentration) with the Gaussian noise multiplied by coefficients 1, 2, and 3, respectively.

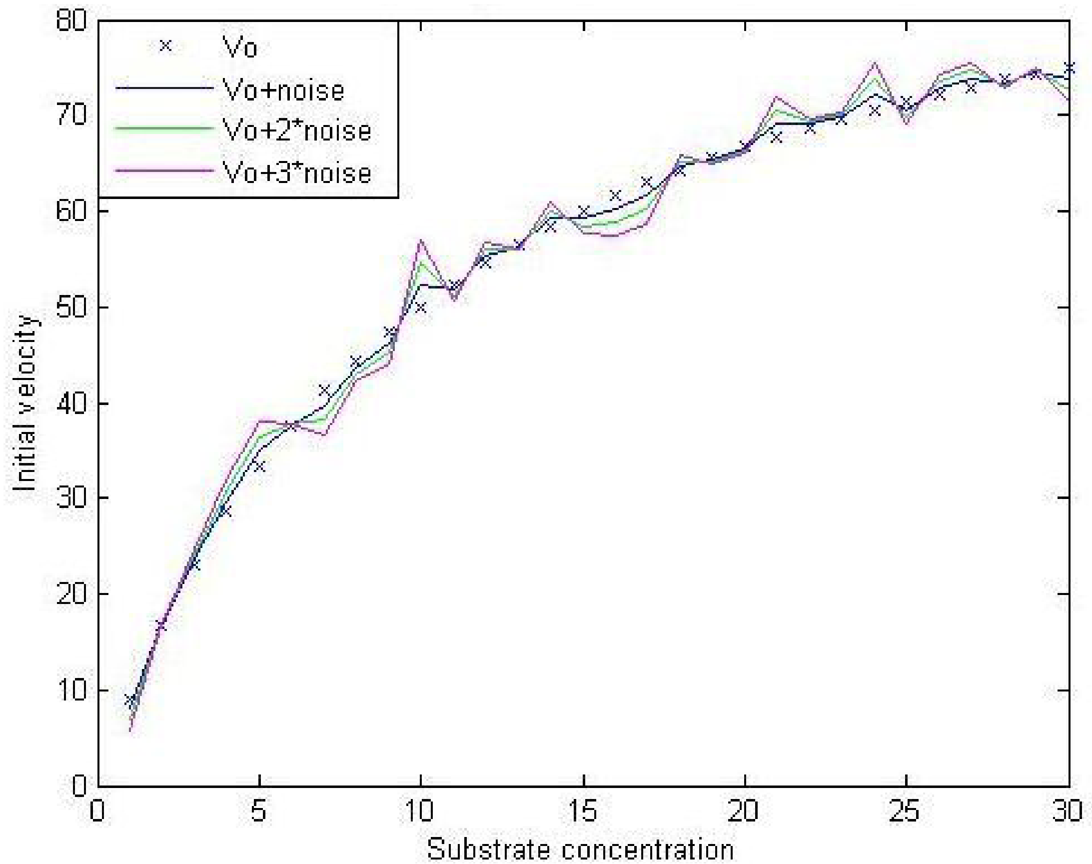

Figure 1: Simulated data for the Michaelis-Menten equation

Noise	Min. abs. error Km	Min. abs. error Vm	Min. rel. error Km	Min. rel. error Vm	Min. abs. Error Method Km	Min. abs. Error Method Vm	Min. rel. Error Method Km	Min. rel. Error Method Vm
Amplitud Noise 1	0,0167	0,3243	0,0017	0,0032	CBE	CBE	CBE	CBE
Amplitud Noise 2	0,0883	0,2823	0,0088	0,0028	H	CBE	H	CBE
Amplitud Noise 3	0,3941	1,4027	0,0394	0,0140	CBE	CBE	CBE	CBE

Table 1: Comparison of the results for the different methods

These simulated data took the place of the real experimental data and were used to determine the parameters K_M and V_M, which real values were known, so the different methods were easily compared in terms of exactitude and precision. Table 1 shows the minimum absolute and relative errors in K_M and V_M and which methodology was the best in each case, depending on the noise amplitude.

The method due to Eisenthal and Cornish-Bowden (CBE in Table 1) was the best one in all the cases, except when the Gaussian noise had double amplitude and K_M is the considered parameter. In this last case, Eadie and Hofstee's method (H in Table 1) obtained the minimum absolute and relative error in the parameter K_M, but not in V_M where once again Eisenthal and Cornish-Bowden gave the best estimate.

A similar methodology was followed in another article devoted to Electrochemical Noise studies (M. Martinez-Luaces, V. Martinez-Luaces, & Ohanian, 2006).

RESULTS

Modelling was introduced in UdelaR Statistics and Differential Equations courses for Chemical Engineering and related careers in 1996. Since then, modelling and application problems regularly appeared in final examinations and other forms of assessment. A similar situation took place in other subjects like Design of Experiments and Numerical Methods, as it was analysed in previous papers (Martinez-Luaces, 2005, 2009a; Martinez-Luaces et al., 2009).

Throughout this teaching experience, as it was already remarked, modelling problems and applications were not just discussed in class, but they also played an important role in the assessment. This is a very important issue, for example Smith and Wood said that "...appropriate assessment methods are of major importance in encouraging students to adopt successful approaches to their learning. Changing teaching without due attention to assessment is not sufficient" (Smith & Wood, 2000).

In this experience with students of Chemical Engineering and Food Technology Engineering, several surveys of student opinions were conducted through anonymous questionnaires. The transcribed below are some of the students´ responses to those semi-open questionnaires; they reflect a very positive attitude towards the courses themselves and the problems proposed during those courses:

> Now I find that mathematics can be useful.

> A really super course, I got a great deal out of it.

> An interesting course, with quite a lot of applications in real life.

Specifically, about tasks related to modelling and applications, they had this to say:

> If these topics were omitted, the course would just be another standard maths course, a 'hard' subject filled only with methods, calculations and numbers.

Once the new study plan was started, the course known as Mathematics III was replaced with Mathematics 005 and 105. After these changes, the students were again consulted by means of semi-open questionnaires. These are some of their views:

> Very directly applicable to my undergraduate professional career; it renewed a taste for mathematics and it was well taught, guiding the solutions to the exercises and not working them all out.

> I thought the course was very useful and dynamic, and I think it will have very useful applications in coming years.

As for the modelling, applications, and the problems set, these were some of their comments:

> The problems were motivational because you can see the usefulness of mathematics in daily life, and they clearly show the interaction that exists with other subjects.

> I have taken courses in which the applications dealt with here were relevant.

> Very useful examples for future years of undergraduate study.

As can be seen, the students reacted very positively not only to the course they were taught (before and after the initiation of a new study plan), but also in particular, within the course, they appreciated everything to do with the problems, involving modelling and applications.

The statistical results of these surveys – published in Martinez-Luaces (2009a) – confirm those views corresponding to several selected opinions.

CONCLUSIONS

The main goals of the activities described in the previous sections are both technical and educative. From the technical viewpoint the results obtained with simulated data showed that the method most widely used (Lineweaver-Burk) it is not the most accurate and this fact constitutes a surprising result for the majority of the students. From the mathematical education viewpoint students' positive reactions can be explained in terms of the following characteristics of the studied problems:

a) relevance
b) applicability
c) specificity
d) authenticity
e) low pre-requisites

The need for relevance was specially remarked in the discussion at CERME9, TWG06. The example considered above is especially relevant for students of several careers such as Chemistry, Biochemistry, Chemical Engineering, Food Technology Engineering and Environmental Engineering. The problem relevance – at least in this case – is linked with the problem applicability and its specificity. The results showed that students usually react more positively to these problems than to other application problems which are not so specific (like problems about circuits or mechanics, etc.).

Other issue that was widely discussed in the TWG06 was the need of authenticity for the modelling activities. In this case not only the models are real models, even more, the methods for recovering parameters are the "real ones", in the sense that these are the methods that the students will use in other subjects like Biochemistry, Microbiology and Physical Chemistry.

Last but not least, pre-requisites are an important constraint when choosing modelling and application activities. The problem described above and the corresponding linearization methods have very low pre-requisites and they only need some algebraic manipulations to be understood. For the proposed activities (i.e., the data simulation and the recovery of the parameters) the students only need a previous course in Probability and Statistics and some basic skills when using MATLAB® or any other software.

It is important to remark that very simple mathematical models, like the one discussed above, are excellent sources for this kind of problems. Moreover, there exists an important set of real-life problems from these areas, which remain almost unexplored from the point of view of their mathematical education richness.

Searching for new real-life problems to be used for project-work in chemical and biochemical careers represents an interesting challenge for mathematical education researchers. At the same time, it provides a good opportunity for an interdisciplinary work with teachers and researchers from other disciplines with the resulting benefits.

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School mathematical modelling: Developing mathematics or developing modelling?

Chris Olley

King's College London, London, UK, chris.olley@kcl.ac.uk

In this paper, I will examine mathematical modelling as a pedagogic activity. I will contrast this activity with mathematical modelling in other professional settings and with mathematics in a school and professional setting. Mathematical modelling is commonly drawn on in the literature as a resource for supporting the learning of mathematics in schools, often as a motivating or contextualising mechanism. I will argue, however, that the activity of modelling is mythologised in these instances, since the recognition rules for the practice of modelling per se are frequently not deployed. Instead, I will argue, illustrated by a school activity, that school mathematical modelling may be distinguished from its professional counterpart as a distinct pedagogic activity.

Keywords: Modelling, school, activity.

MATHEMATICAL MODELLING: WHAT IT IS AND IS NOT

Mathematical modelling is a tool used by engineers, financiers, in medicine and so on. From a brief review of papers describing mathematical models (e.g., Mendoza-Arriaga, Carr, & Linetsky, 2010; Gunzelmann, Gross, Gluck, & Dinges, 2009; Stolz, 2002), it is clear that when modelling is deployed to engage with issues where the issue itself is the purpose of the deployment, the level of complexity is far beyond the scope of a school course in mathematics. In each case an issue is raised in a setting which is not itself mathematical, for example; sleep deprivation and performance (Gunzelmann et al., 2009), regrowth of tissue cells (Whittaker et al., 2009), equity derivatives and pricing derivatives (Mendoza-Arriaga et al., 2010). A mathematical model is created in which key indicators are specified and measures constructed to generate the variables in the model. A mathematical structure is then presented which generates an interrelationship between the measures similar to that which exists with the indicators. This structure is the model itself. The model is validated by measuring the extent to which the output it produces is sufficiently accurate for the specification of the issue. The cyclical nature of the modelling process in which a model is suggested, critiqued and refined is a central feature which will necessarily have been a key component of the work in reaching the stage of publication, although in the examples cited, this is not made explicit in the final work.

Blum, Galbraith, Henn and Niss (2007) provide a comprehensive overview to the modelling process, viewed from the perspective of educational practices:

> The modelling perspective begins with the conceptualisation of some problem situation … Through a process of mathematization, the relevant objects, data, relations, conditions and assumptions from the extra-mathematical domain are then translated into mathematics … mathematical methods are used to derive mathematical results … [which] must then be translated back into the extra-mathematical domain …The problem solver then validates the model … the model is evaluated … when one of these 'tests' is deemed unsatisfactory, the whole process needs to be repeated …(Blum et al., 2007, p. 9).

In modelling there is little interest in a specific solution, but instead a solution which is good enough. This aspect is particularly alien to school mathematics. In the educational setting mathematical modelling is presented in different ways. Most commonly it is represented by the contexts that authors of school materials use for their examples and exercises. Burke (2013) presents examples from leading English school teaching text books. In a text designed for middle achieving students, the path of a rugby ball being kicked over the posts is to be matched against graphs of its speed against time. Firstly, this is an example of what Paul Dowling (1998) refers to as the myth of reference. Clearly, the kicker cannot be credibly perceived as constructing the graph as a purposeful aid to the kick and moreover none of the graphs given is a particularly reasonable representation of the change of speed and time of the ball. As Burke suggests; "the activity is reading a graph and the apparent non-mathematical context is a way in, a 'selling' point, for the problem" Burke (2013). This is not an example of mathematical modelling, in that it does not present any of the features of the modelling process suggested above. The problem is posed such that validation would need to reside with the rugby player, whereas the solution can only be validated as school mathematics. Nonetheless, it has the appearance of an example of modelling with a 'real-world' problem solving context with a mathematical model, yet is of a very common type in school text books. The learner is apprenticed into an apparent version of mathematical modelling which does not contain its essential features.

Sometimes the credibility of the problem leads mathematics educators to validate solutions without reference to the setting being used. Galbraith (2011) reports a question from the PISA project which provides a multiple choice set of answers to the question of how many fans could stand in a field at a concert 100m by 50m. The correct answer is given as C. 20,000. However, there is also an answer B. 5,000 which must be taken as incorrect. Galbraith says; "This sample item involves spatial insight, as students need to decide on a suitable model to quantify the amount of space occupied by a human, then perform an appropriate calculation to estimate how many people would fit into a given space. Only about 26% of the multinational sample of students answered the item correctly (C), illustrative of the depressed performance associated with contextualised problems" Galbraith (2011, p. 9). The field is 5000 square metres and the correct answer suggests that 4 people per square metre could reasonably fit. Naturally, this would need to be an average, since it would be hard to conceive of every square metre being completely full especially around the edges, hence some of the square metres would need to fit rather more than 4 people. The UK Health and Safety Executive publishes it's Purple Guide (HSE, 1999, p. 17) which says; "Generally, 0.5 m^2 of available floor space per person is used for outdoor music events." Here, the mathematics educator engages with a particular mathematisation of a problem from a given (non-mathematical) practice, which at least is open to critique within that practice, but nonetheless is taken as the 'correct' model, where, at least in the UK, a different answer would be more correct. One significant feature of mathematical modelling is that the model is critiqued by the owners of the problem who will be expert in their field, and not the owners of the mathematics.

MODELLING IN SCHOOL MATHEMATICS; MODELLING TASKS OR MATHEMATICS TASKS?

Dowling (2009; 2013) suggests a structuring of the domains of pedagogic action. This contrasts the esoteric domain of a practice with its public domain in terms of the strength of institutionalisation found in the expression and content of the text (Dowling, 2007). He represents this schematically:

	Content	
Expression	I+	I-
I+	*Esoteric domain*	*Descriptive domain*
I-	*Expressive domain*	*Public domain*

Figure 1: Domains of Action (after Dowling, 2009)

I have presented a characterisation of the esoteric domain of mathematical modelling with strongly institutionalised expression and content (the modelling cycle, the indicator/measure relationship, validation, etc.). This constructs a public domain of mathematical modelling characterised by low levels of institutionalised expression and content. Here, problems are solved by the deployment of mathematics without reference to the modelling process. So, rugby players can improve their kicking without a critical engagement with the relationship between the indicators and measures or indeed any requirement for validation of the outcomes.

Realistic Mathematics Education is a theoretical and curriculum development strand in mathematics education in which the term model is used in a mathematical context. It is presented here as a rich example within mathematics education where the idea of mathematical modelling is presented as an important feature. Gravemeijer (1999) gives a description of the central methodology thus; "what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality" (Gravemeijer, 1999, p. 156). This is seen as a modelling activity, in which students construct increasingly formal (esoteric domain) mathematical statements developed from their public domain discourse in mathematical settings. The modelling process starts from the practical setting; "... RME models are not derived from the intended mathematics. These models are seen as student generated ways of organising their mathematically grounded activity" (Gravemeijer, Bowers, & Stephan, 2003, p. 53). It is apparent that the purpose remains rooted within school mathematics practice. Issues of validation are entirely within mathematics practice and directed at an induction into the esoteric domain of mathematics. For example, describing a sequence of teaching for students on co-variance, the setting of T-cell counts in AIDS patients is used as the data. However, when the students' work is analysed, a series of issues are suggested as being the key development points for the next stage in this topic. All three of these are clearly framed in terms of esoteric domain mathematics: to better describe the shapes of the datasets, to structure the datasets in terms of patterns and multiplicative reasoning (Cobb, McClain, & Gravemeijer, 2003). Despite the potency of the data, there is no discussion of the engagement with it in the extra-mathematical setting. Again, this does not contain any of the esoteric domain of mathematical modelling and hence, can only be seen as a practice in school mathematics.

Cyril Julie (Julie, 2002) set a group of teachers a series of tasks rooted in issues of potential concern to them or their communities. Two contrasting examples were firstly, to find a model for pay scales in school, on the basis of equal pay for equal work. Secondly, a model of the accumulation of plastic shopping bags on school fences (a current environmental concern to that community). Julie notes that different teachers were more or less engaged with these tasks differentially according to the immediacy of the outcome to their professional position or political interest. It is clear here that where teachers engaged enthusiastically, the outcomes were potentially interesting models. The contrast I wish to make, is that the teachers were engaged in the practice of mathematical modelling. They used only the resources they already had, to engage with the central modelling issues of the construction of the model and the validation of the outcomes. Effectively, they looked for a formula which fitted well enough to appear workable. When prompted to improve their model they made minor changes to the formula rather than engage with what might count as an effective model.

SCHOOL MATHEMATICAL MODELLING AS A DISTINCT PRACTICE

I will describe as school mathematical modelling a practice where the explicit intention would be to apprentice learners into the esoteric domain of mathematical modelling. I have argued above that this includes a critical engagement with the relationship between problem and model, together with issues of the relationship between measure and indicator and issues of accuracy and validation.

In this sense, although Julie's teachers are engaging in a mathematical modelling practice, they are doing this in a 'common sense' way. So, there is no esoteric domain engagement in the practice of school mathematical modelling. That is not to say that the practice is neither valid, nor useful. It is clear that they have found models which they have found useful in their professional lives, beyond the pedagogic setting. Indeed the informal judgements of validity could be seen as esoteric domain practice in professional modelling, because here the realisation principle would include an effective solution, which there was. However, in school mathematical modelling practice, the esoteric domain would require an elaboration of the mechanisms for validation of the model, not just a statement of it. Here, the central modelling features would be engaged with directly, rather than being deployed as they would in mathematical modelling practices. For example, how the model is constructed, what counts as acceptable validation, how measures are constructed for the indicators.

For the teacher to be able to communicate the mathematics, they can constitute their conception as a 'map' of the practice (see Burke & Papadimitriou, 2002). This may be entirely implicit or as a teacher development strategy; explicit, external and structured in detail. Necessarily, the map will have elements which are personal and rooted in the subject and pedagogical knowledge of the teacher, nonetheless often with a closeness of fit to a presumed pre-existing definition of the practice.

This map can be constructed which structures school mathematical modelling as the cyclical creation of critique and refinement of a mathematical structure, finding good enough measures for the key indicators which generate outcomes, validated as sufficiently accurate with reference to the problem posed. The recontextualisation from originating practice to mathematical practice is the central phase, generally referred to as generating the 'model'. The validation of the acceptability of the model can only be tested ultimately by the project commissioner, who has expertise in the originating context. Hence, we construct a teaching narrative which will seek to make these issues explicit and engage overtly with the location and basis of judgements of validity. This is school mathematical modelling, a pedagogic practice, with the aim being to apprentice the modeller into the practice of modelling.

Gabriele Kaiser gives a detailed overview of the parameters proscribing the reported practice of mathematical modelling in a pedagogic setting (Kaiser, 2013). She suggests that the role of the teacher has been insufficiently researched. Indeed the centrality of the nature of the modelling process and of possible models themselves form the main part of the overview. Our aim here is to focus on the role of the teacher in creating and negotiating the apprenticing relationship, through explicit planning of school mathematical modelling practices. Presenting school mathematical modelling as distinct from school mathematics appears to be suggesting a *separation approach* (Kaiser, 2013, italics in original) where mathematical modelling is taught as a separate course. However, "The most advanced approach, the *interdisciplinary integrated approach*" would require each participant to generate their own map and narrative of the aspects of which they were the key expert; problem owner, mathematical modeller, mathematician. This potentially presents an opportunity for apprenticeship in the totality of the mathematical modelling, certainly a school mathematical modelling practice.

PIZZA DELIVERY: SCHOOL MATHEMATICAL MODELLING IN PRACTICE

I wish to contrast the preceding examples with an activity expressly designed to induct learners into the esoteric domain of mathematical modelling, hence operating in the esoteric domain of school mathematical modelling. Here, a project sets out a narrative based on a story problem about the delivery of pizzas. This is a pedagogic activity located in the practice of school mathematical modelling. It was originally developed for the Bowland Trust by Burke, Hodgen and Olley (Burke, Hodgen, & Olley, 2007). The activity is presented for a teacher audience in Olley (2011). The description is a compilation of experiences from two London schools; one selective and suburban, the other non-selective and inner city and from groups of mathematics and science teachers.

The initial phase of the narrative aims to imbue a sense of purpose. The problem is placed in the context of the owner of a new pizza outlet, who wishes to recruit a consultant to advise on the range of issues to consider in determining the profitability of the new enterprise. Mathematics educators are generally expert in neither running a pizza shop, nor business management, so it is important that their expertise is made clear. So, the activity begins with a public domain discourse on pizza shop management. This always throws up one central issue (amongst many others): you can reach more customers if you can keep your pizza hot for longer. The students have experience in pizza purchasing and hence are aware of the variety of packaging that pizza shops use to keep their delivery products fresh and the means by which they deliver them. Up to this point, there has been engagement in a marketing relationship. The educator has an activity to sell: mathematical modelling, and the marketing strategy is through the use of a compelling and apparently engaging setting. Pizza shop ownership has been mythologised; the educator has no means to credibly validate the outcomes in the practice of pizza shop ownership.

Measures for the key indicators need to be found. 'Sufficiently fresh' is indicated by a minimum topping temperature of 48°C (which was found initially with a 'taste' test in which a pizza cooled and was tasted until the taster considered it unacceptable). 'Reach more customers' is indicated by the time taken for the pizza to cool to 48° (given that we can find the average speed that the scooters delivering them travel at, and hence a circular route of that now calculable distance seems credible as a deliverable zone). The relationship between measure and indicator is a very important site for critique, although at this stage the first iteration of the modelling process progressed with these face value acceptable measures.

An experiment in which a pizza was heated, then allowed to cool, plotting its cooling against time, was then set up. The educator again must be clear to stress the limit of their expertise in issues of experimental design. Collaborating with the science department who can focus on this aspect would be better. The narrative then reached the point where participants needed to reflect on the rate of change. Hence, a data sheet was given out which asks participants to say what they thought the 'just cooked' temperature would be and give a reason. The experiment was then started and participants estimated the temperature after one minute, giving a reason. At the one minute point the temperature was announced and participants estimated for the end of the second minute and so on up to 10 minutes. As the experiment progressed, they were encouraged to refine their 'reason' and increasingly express it as a calculation, increasingly with more than one element. Finally, they were estimated the long term temperatures (30 mins, 2 hrs, 24 hrs).

As the pizza cooled, participants watched a graph of the cooling against time being generated by data logging apparatus connected to a temperature probe. Over the 10 minute period of the experiment, the cooling graph looked very linear indeed. Asked to describe the basis on which they estimated successive temperatures, a common response would be; "it's going down by roughly 2.4°C per minute". Some participants said that the rate of decrease was changing from around 2.6°C per minute to about 2.2°C per minute. Amongst groups of teachers discussion of first and second differentials frequently emerged at this point. Participants were asked to hold the thought of the changing rate and see the effect of the initial model. This led to the generation of a model; the starting temperature minus the rate of decrease times the number of minutes, i.e. something like $92 - t \times 2.4$. At this point participants could solve the pizza shop owner's problem. Here, pizza is still just acceptable at time t where $92 - (t/60)2.4 = 48$.

One complete iteration of the modelling process had been completed. So the results needed validation: where they good enough? The graph looked very linear over the range of the experiment and the solution was commonly only a small multiple of the experimental range. It quite probably would be good enough for the pizza shop owner. Again, the owner is a fiction, we are simply keeping up the marketing ploy. However, the narrative set up some unease, notably the expectation that in the medium term the temperature of the pizza would plateau at the temperature of the room. Using the functional model and values of t for 30 mins, 2 hrs and 24 hrs, generated absurd values for the temperature. When the graph was rescaled with the maximum value of t changed to these value s the gradient appeared extremely steep. However, the context provided a clear basis for critique. Participants knew that pizzas, left to themselves do not freeze of their own accord. That the rate of change was slowing was then incorporated to generate an improved model. This suggests a quadratic, but again the context provides a critique; unattended pizzas do not heat up again. This left two possible functions which meet both conditions; a reciprocal function and an exponential function. These provide extremely good fits to the data and meet both criteria (plateau at room temperature and a diminishing rate of change) and hence appear validated in the context of the problem. The next iteration of critique demands validation beyond the scope of the mathematician. Why Newton's law of cooling is exponential requires an explanation rooted in chemistry and physics. Again, the mathematician must be clear about the limits of their expertise.

MODELLING ACTIVITIES FOR SCHOOLS AND THEIR MATHEMATISATION STRATEGIES

The narrative is designed to preserve the focus of the learner on the construction of the model. In an earlier paper we refer to this as the mathematisation strategy (Burke, et al., 2014, p. 33). We set out an analytic framework describing the different strategies according to two axes: the first concerns the extent to which there is a rationale for deriving the relationships in the model (A Quantification rule) and the second concerns the rationale for the rule itself (A Mapping rule). The strength of these is determined by its discursive saturation (DS+/DS-), being the extent to which the principles of evaluation are contained within the practice (Dowling, 2013). This generates four mathematisation strategies (Burke et al., 2014, p. 33) (Figure 2).

The use of modelling in RME is as the construction of a relationship between the originating context and the model. This is beyond the reach of the student (and indeed the teacher) as it is generated at the level of author/researcher. So, it operates as a derived mathematisation. A mapping whose principles can be evaluated (DS+ Mapping), but with no access to

Quantification rule (external syntax)	Mapping rule (internal syntax)	
	DS+	DS-
DS+	*Definitive Mathematisation*	*Ad-hoc Mathematisation*
DS-	*Derived Mathematisation*	*Originative Mathematisation*

Figure 2

the means of construction (DS- Quantification). By contrast, Julie's teachers constructed their models with clear relationships between problem and model (DS+ Quantification) but do not provide a theoretical basis for their models (DS- Mapping), an ad-hoc mathematisation. In both cases, throughout the pedagogic practice, the mathematisation strategy is fixed. There is no possibility to engage with the practice of mathematical modelling per se and hence this is unrealised as a pedagogic practice. There is no school mathematical modelling.

For the pizza narrative, we describe "a move from originative through ad-hoc mathematisation that results in a definitive mathematisation". (Burke et al., 2014) This provides the apprenticeship, the mechanism by which the student is exposed to the processes of mathematical modelling. The relationship between teacher and student remains central as the narrative unfolds with the teacher having planned the inclusion and suppression of elements of the dialogue, to preserve the focus on the process of creating the model.

The detailed presentation of the pizza delivery narrative is intended to argue the case that the outcomes of the activity would be recognised as mathematical modelling by a professional, but the practice inducts the participant into the esoteric domain of mathematical modelling, by strategically emphasising key elements of the modelling process. It is clearly a pedagogic activity. The narrative is structured to selectively expose and suppress the elements of the modelling process in strategic ways, because we are in fact not solving a real problem, which allows us to extend the remit of the original problem to see the effects of requiring a higher level of validation. I make no claims for this as a school mathematics activity. It may be the case that students have learned some mathematics in doing this, but that would need to be tested empirically. What I do wish to claim is that the activity is constructed as a school mathematical modelling practice, created from a reflection on a map of that practice. This provides a structure from which a principled narrative is constructed, through which the participant comes to operate in the esoteric domain of school mathematical modelling.

IMPLICATIONS FOR RESEARCH AND FOR EDUCATION

This paper has set out an argument to clarify a distinction between the practice of mathematical modelling and pedagogic activity aimed at developing skills in mathematical modelling. I have used examples from the literature in which mathematical modelling is presented in an educational context but there is no intention in the activity to develop modelling skills. I would suggest further work within this research area to identify and potentially develop further instances where the activity can be identified as the pedagogic activity; school mathematical modelling. Here, the specific intention is to focus on the development of aspects of the modelling process and how this can be achieved as in the pizza delivery example I have given. Other elements of the modelling cycle would be amenable to this way of thinking and notably I would see the issue of validation as central to the modelling process and a potential next step for focussed pedagogic activity.

In educational policy, there is an increasing emphasis on solving problems in mathematics and framing these problems in a real world context. In the new UK National Curriculum (DfE, 2014), problem solving is one of the 3 foregrounded components (the others being reasoning and fluency). Students are required to; "begin to model situations mathematically". As I have argued, they will not be able to develop skills in this unless there is an explicit focus on mathematical modelling. Here, the elements of the modelling process need to be developed through explicit pedagogic activity. This is an issue for curriculum planning and materials design. The pizza delivery example provides one activity with a focus on one specific aspect of the mathematical modelling process. It could potentially fit in to a more systematic collection providing a more complete curriculum in school mathematical modelling.

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Thought structures as an instrument to determine the degree of difficulty of modelling tasks

Xenia-Rosemarie Reit and Matthias Ludwig

Goethe-University Frankfurt, Institute for Mathematics Education, Frankfurt, Germany, reit@math.uni-frankfurt.de

Although efforts have been made to integrate the concept of mathematical modelling in school, studies show that it has not arrived yet in everyday school classes. From a teacher's point of view, multiple solution approaches and a varying task difficulty complicate especially the development and assessment of modelling tasks. Taking up this issue, this study aims at developing a method to determine the degree of difficulty of solution approaches of modelling tasks based on so called thought structures of student solutions. Thought structures of student solutions provide information about the task difficulty and can be taken as a basis for a well-founded rating scheme. We want to consider the question of whether the method used to describe the degree of difficulty can be reproduced by empirical results.

Keywords: Mathematical modelling, assessment, degree of difficulty, task space.

INTRODUCTION

Within the community of mathematics education there is a broad consensus that the integration of mathematical modelling and applications must be promoted and increased. Of course this awareness is not new and efforts have been made during the last decade. However, several studies provide evidence that modelling is far away from playing an integral role in daily school teaching in Germany and also elsewhere (Blum, 2007). The proportion of modelling in daily school routine is rather low (Jordan et al., 2006). When researching into that problem it is worthwhile to have a closer look on the teachers' point of view. What prevents teachers from teaching modelling tasks? Schmidt (2010) found out that teachers often mention complexity and lack of predictability as motives for waving modelling tasks. A look in textbooks shows that modelling tasks are still rare, especially those which can be used in a normal school class setting in contrast to modelling tasks used within larger projects encompassing several lessons. The development of modelling tasks leads, more than ordinary task formats, to difficulties concerning the estimation of the task space, the task difficulty and finally also concerning the assessment.

The study presented here takes up the issues from a teacher's perspective and aims at a better structuring of modelling tasks in terms of a better manageability for teachers. In detail we developed a method to determine the degree of difficulty of modelling tasks and an assessment scheme, both, building on solution approaches with its particular thought structures. Empirical results will show whether the theoretically determined degree of difficulty is verifiable and certifies a good applicability.

THEORETICAL FRAMEWORK AND METHOD

A common instrument to determine the degree of difficulty is the solution rate by applying a dichotomous rating. Since the answer format of modelling tasks is open and compared to others rather extensive, this procedure seems not adequate to reflect the full scope which is provided by modelling tasks. Cohors-Fresenborg, Sjuts and Sommer (2004) applied a method analysing the text of the task. They identified task specific indicators for the difficulty of tasks by investigating PISA-2000 items. The task format of the investigated items was not restricted to modelling tasks. Since the task space of modelling tasks is, compared to other task formats, rather large, this aspect would get lost by focusing on the text of the task. In our study this specificity of modelling tasks is taken into account since the method for determining the degree of difficulty comes directly out of the solution approaches of the student solutions.

In a first step we analysed the student solutions of every modelling task to identify main solution approaches. On the one hand, this classification was based on the mathematical model used and, on the other hand, on the solution process, which can be different although the same mathematical model is used. Each solution approach has a specific structure which can be revealed by dismantling the solution approach into its single thought steps, which provide the opportunity to consider solution steps from cognitive aspects. This idea builds on structural considerations in the field of word problems. In this context Breidenbach implemented the term "Simplex" as a task consisting of three items and every item can be determined by the two others (Breidenbach, 1963, p. 200). By visualizing the logical structure of the mathematical operations to be done in terms of such Simplexes he built up a kind of arithmetic tree or flow chart. These flowcharts have been refined by Winter and Ziegler (1969) and serve as a basis for the so called thought structure analysis of the study described here. For the present study we refined Breidenbach's definition by identifying the single cognitive steps, here called thought operations, which have to be carried out in order to arrive at a solution. We define thought operations as follows:

A thought operation is a necessary (intermediate) result which is obtained directly (without intermediate calculations) from one or several (initial) data.

These thought operations can then be arranged as a kind of flow chart which illustrates the incremental proceeding and in addition, also the complexity of the solution process (see Figure 1).

A natural but empirically not yet validated conclusion is that the complexity of a mathematics task is dependent on the number of simplexes and the nesting of them (Graumann, 2002, p. 93). Cohors-Fresenborg and colleagues (2004) emphasize in this context the simultaneity and nesting of thought steps. With the help of theories within the field of cognitive psychology, we can operationalize the effects of nesting and simultaneity on complexity. Fletcher and Bloom (1998) found in their study about text comprehension, where they assumed text comprehension to be a kind of problem solving process, that information being the direct predecessor of another information must be kept actively in the working memory. Under the assumption of a working memory with limited capacity (Sweller, 1988) these findings indicate that several aspects in a task which are related to each other and have to be considered and understood at the same time, may load the working memory. Thought operations are considered to be parallel if they either originate from the same thought operation or determine the same thought operation of the subsequent solution level.

By applying the findings from above to the thought structure considerations, especially the amount of parallel thought operations (as it is the case in level two of Figure 1) of a solution approach appears to be a difficulty generating aspect. Thus, during the process of determining the degree of difficulty, parallel thought operations complicate the solution approach to a greater extent than those being processed consecutively. To describe this circumstance we developed a model which gives more weight to parallel than consecutive thought operations. Each level in the thought structure contributes to the overall degree of difficulty according to its number of parallel thought operations. From the number of parallel thought operations per solution level we calculate the factorial and then finally all levels are added up. In the following we want to reconstruct this procedure by taking modelling task "Potato" (Figure 2) as an example.

One solution approach, which we could identify, is called "Layer" (Figure 3, left). Based on the given length of the potato the student assumes a height and a diameter or depth of the potato. Together with an assumption about the measures of a potato stick the student is then able to calculate the number of sticks

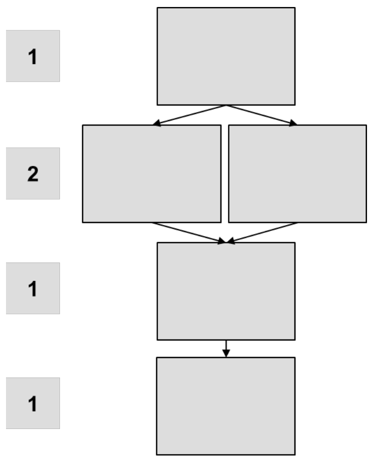
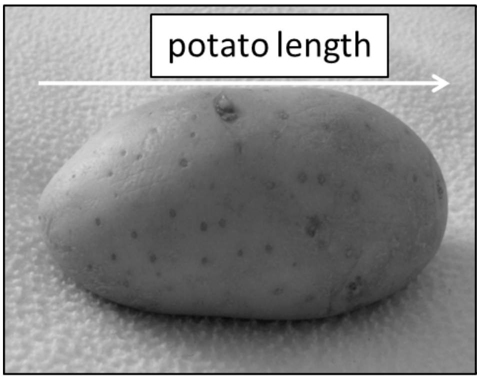


Figure 1: Exemplary thought structure of a solution approach together with the number of thought operations per level



Industrial manufactured french fries are supposed to be equal in size and the single sticks are cut out lengthwise. Therefore not the whole potato can be used. The potato tubers look similar to the picture above, are regularly formed and approximately 10 cm in length.
How many of these potato sticks can be obtained from one potato? Reason mathematically.

Figure 2: Modelling task "potato"

per layer in height and depth. Multiplication of these two numbers leads to the total number of potato sticks. The corresponding thought structure has been developed which arranges these thought operations as sort of arithmetic tree (Figure 3, right). Besides the chronological order of the calculations to be made, the thought structure also provides information about the difficulty of the solution approach. The numbers right next to each solution level indicate the number of parallel thought operations to be completed per thought structure level. As explained above summing up the factorial per level yields the degree of difficulty of the solution approach.Thus, in case of solution approach "Layer", the degree of difficlty is 1!+2!+1!+1!=5 (see grey Figure 3, right).

Based on these considerations an assessment scheme has been developed to utilize the full scope of the promising method on the one hand and to verify the theoretically determined degree of difficulty on the other hand. When observing the assessment routine of mathematics teachers it becomes clear, that assessment of tasks is based on a sample solution and its important partial aspects. The teacher compares each solution with a sample solution and awards important intermediate results. Our assessment scheme is based on this everyday school routine procedure. Intermediate results are represented here by thought operations and are assessed according to whether they have been conducted correctly or wrong. For partly right thought operations half points can be awarded. That means that thought operations are scored 0, 0.5 or 1 dependent on their completion. Thus, the maximum score per solution approach is defined by the number of thought operations. In case of solution approach "layer" the maximum score is 5 (number of thought operations (grey boxes) in Figure 3, right).

Before being able to solve the task, one has to identify the relevant information of the presented linguistic context. According to Cohors-Fresenborg & Sjuts (2001) difficulties are especially evoked by linguistic constructions concerning the logical structure and formulations conditioned by the authenticity of the situation. The textual differences of the tasks are integrated in the process of determining the level of

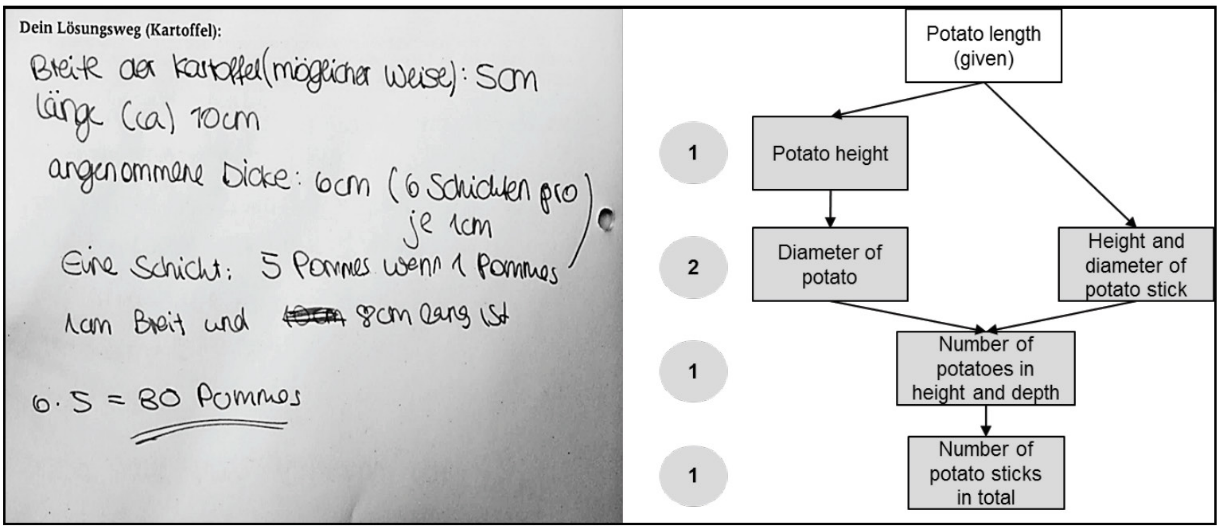


Figure 3: Solution of solution approach "Layer" and corresponding thought structure

difficulty by adding zero, one or two single thought operations according to the linguistic requirement of the task text.

0 Picture and/or text contain all relevant data. Only simple main clauses are used.

1 Missing data has to be estimated and there are several main and subordinate clauses due to a larger text, containing explanatory but mathematically irrelevant text passages.

2 Picture and/or text do not explicitly clarify the dimension of the object of the task, e.g. the two dimensional picture of a potato is out of keeping with the three dimensional reality leading to not include a third dimension in the calculations.

Modelling task potato has been assigned to linguistic level two according to its requirements coming from the text. Therefore we additionally add two single thought operations to the solution approach based degree of difficulty. Thus the total degree of difficulty of the solution approach “Layer” consists of linguistic requirements (2) and solution approach specific thought operations (5) and finally adds up to 2+5=7.

STUDY DESIGN

Five modelling tasks have been developed according to predefined criteria (see Reit & Ludwig, 2013):

— Authentic context (Maaß, 2007)

— Realistic numeric values (Müller et al., 2007)

— Problem solving character (Maaß, 2007)

— Naturalistic format for questions

— Openness relating to the task space

Authenticity and relation to reality are core elements of modelling tasks. We want to avoid ostensible relations to reality like they are often used in word problems in textbooks. There is no a priori known solution algorithm for the task which can be directly applied by the students. That means that the solution makes itself out to be a problem on students’ level. The questioning is supposed to either be close to the living environment of the students or take up a realistic question which could arises in reality. Openness of tasks is reflected by the task space. There has to be more than one solution approach which leads to a solution. The solution approaches distinguish themselves by their mathematical model. Thereby the students are able to have more options to arrive at a solution. Openness should rather be based on the alternatives of mathematical models to solve the tasks than on approximating sizes. Demanding this we do not deny that making assumptions is an important part of mathematical modelling but we want it to be limited to a degree which ensures an assessable solution interval.

The study splits up into pilot study and main study. The pilot study encompassed

— an a-priori definition of the task space of the tasks,

— the identification of the thought structure for each solution approach

— and the establishment of a rating scheme for each solution approach.

The implementation into a normal 45-minute lesson also required a time limitation of approximately 10–12 minutes per task. During the main study from December 2013 until April 2014 approximately 1800 students of grade 9 (15/16 years of age) of German grammar schools took part and solved three modelling tasks each.

To validate the thought structure method it must be investigated in how far the theoretically determined degree of difficulty is empirically reproducible. In other words, will tasks, which have been rated as difficult, be solved worse than those rated as rather simple? To do so the student solution together with its assessment is associated with the predetermined degree of difficulty of the respective solution approach.

VALIDITY OF THE THOUGHT STRUCTURE ANALYSIS

The modelling tasks, which have been developed especially for the purposes of the study, show a good variability of applied solution approaches. Although this was an important criteria during the process of development it could not be stated with certainty how the students solve the task. At least three target aimed solution approaches could be identified per modelling task (Figure 4). Hence, from the viewpoint of openness in the sense of an adequate task space the developed modelling tasks seem to meet the requirements.

The main objective of the study is the validation of the thought structure method to determine the degree of difficulty by comparing the student performance with the statement of the theoretical model. Hereafter we consider the question of whether the method makes a valid statement about the degree of difficulty of the modelling tasks.

Within a modelling task the student performance verifies the degree of difficulty in terms of scoring higher using solution approaches determined as easier and vice versa. This situation is illustrated in Figure 5. Taking three of the five modelling tasks into account (modelling task “Cola” and “Bridge” are not yet analysed completely) we can see a distinct decrease of the average score when using a solution approach being determined as more difficult. The value of the power of the regression function is in case of modelling task “Tennis” and “Taj Mahal” nearly similar what indicates that the coherence of score and degree of difficulty reacts almost in the same way. The degree of difficulty of modelling task “Potato” seems to have a firmer influence on the average score than it is the case with the two other modelling tasks. An increase in difficulty results in a steeper deterioration. In general a comparability of solution approaches within a modelling task, based on the theoretical degree of difficulty is reasonable.

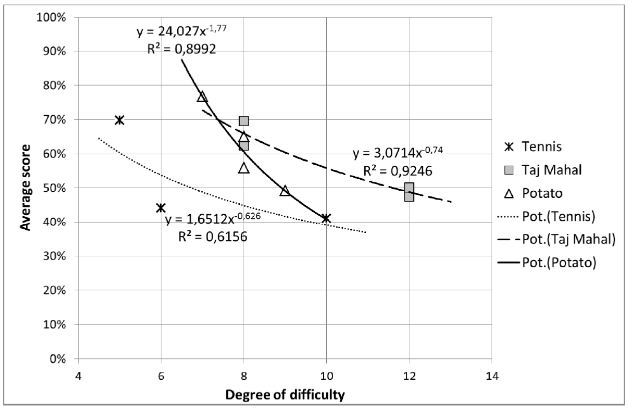


Figure 5: Coherence of score and degree of difficulty concerning the solution approaches per task

Another interesting question is to what extent the determination of the degree of difficulty can be used in a cross-task context. What statements can be made by comparing different modelling tasks? To answer this question we compared the average score of a modelling task dependent on its average degree of difficulty. The decreasing tendency of the average score with increasing degree of difficulty is in evidence (Figure 6). In detail the power of the regression function suggest that if the degree of difficulty doubles, then the score will half.

SUMMARY AND OUTLOOK

Especially when considering modelling tasks it is challenging to estimate the degree of difficulty and to assess their solutions reasonably and satisfying for students. On the part of teachers this brings along uncertainty and might contribute to a restrained attitude towards modelling tasks in everyday teaching. The presented study considers that problem and de-

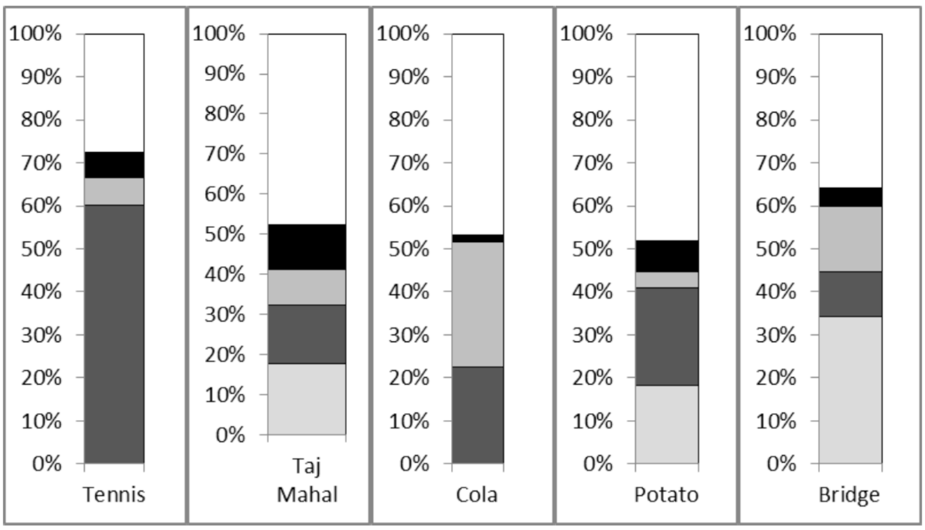


Figure 4: Distribution of solution approaches per modelling task (target aimed solutions in greyscale, others white)

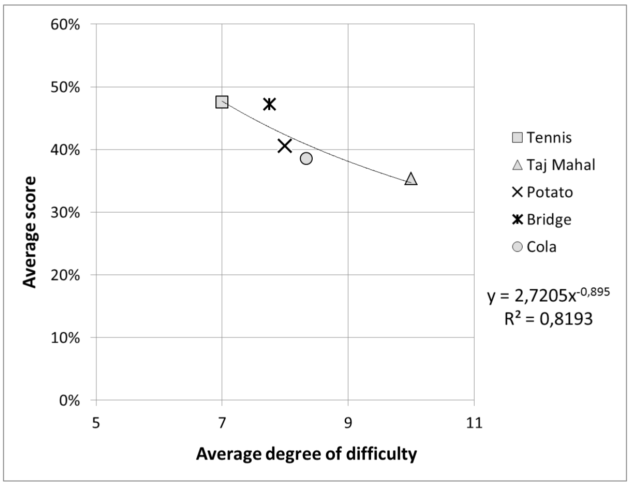


Figure 6: Average score of the modelling tasks subject to the degree of difficulty

veloped a promising method to firstly determine the degree of difficulty and secondly to form the basis for a reasonable and conclusively substantiated assessment scheme. The analysis so far encourages the assumption of a good applicability. The results up to now show that using the thought structure based method to determine the degree of difficulty is in line with the empirical results. Students show better results when using those solution approaches with a rather low degree of difficulty and vice versa. Similar results could be obtained considering the modelling tasks as a whole. The average student performance is better at easier tasks and worse at tasks being rated as more difficult. Additional analysis especially to modelling tasks "Potato" and "Bridge" will show how reliable and convincing the method is. Besides these affirmative outcomes modelling task "Potato" gives rise to questions concerning the limits of the method. The divergent results of this task may be due to the fact that their solution approaches are somehow intertwined in the sense of being difficult to distinguish. This sometimes leads to the problem of false solution approach classification, thus to a falsification of results. This may support the conclusion that a distinct discriminability is a necessary requirement for the thought structure method. Further reflection and the complete analysis of the two remaining modelling task may give deeper insights to that aspect.

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A multidisciplinary approach to model some aspects of historical events

Gemma Sala, Berta Barquero, Vicenç Font and Joaquim Giménez

Universitat de Barcelona, Faculty of Education, Barcelona, Spain, gsala@ub.edu, bbarquero@ub.edu, vfont@ub.edu, quimgimenez@ub.edu

This paper focuses on the problem of integrating a multidisciplinary approach to the study of extra-mathematical questions where mathematical modelling appears as a central tool in the teaching and learning processes. We present here the design, implementation and analysis of a sequence based upon a historical context, the war of Spanish Succession in Catalonia during 1714, which was implemented in two courses with 12–14-years-old students. We also show to what extent these implementations help to integrate an interdisciplinary study where history and mathematics will have to work together.

Keywords: Mathematical modelling, inquiry, multidisciplinary approach, historical and cultural context.

INTRODUCTION

The main aim of our research is to find complex situations of real life that can give rise to the design of a sequence of tasks that can promote inquiry and modelling students' competences. These real situations will be the origin of the design of a sequence of questions and tasks, integrating the necessary didactic devices enhancing students to mobilize the necessary cognitive processes to face them. More concretely: (1) to integrate diversity of tools coming from different disciplines and subjects and (2) to integrate mathematical modelling as an essential tool to facilitate a multidisciplinary approach to face such situations.

It is in this context of fostering the inquiry and modelling in which the selection of initial situations that will originate the process of design of tasks' sequences takes real importance. This situation leads to the formulation of the following specific research question that will be at the core of our research:

> Given the classical description of contents of different subjects (Mathematics, History, Languages, etc.) at Secondary school level, and given the traditional isolation between these disciplines, how can we design didactic devices which locate problematic questions at the starting point of the study process where: (1) mathematics (and mathematical modelling) appear as fundamental tools to provide answers to them, and (2) can the mathematical study inhabit and interact with other disciplines (or subjects) to validate and enlarge the study?

Here we consider the particular case of Secondary school level. The sequence described in this paper was named '*Understanding the incidents of 1714*' ('Valorant els fets de 1714'). The main purpose for students (and also the final task) was to participate in a literary contest about the consequences of the war of Spanish Succession in Catalonia, which happened 300 years ago. Students were asked to face quite an open starting historical problem about this war. They had to begin by looking for real data in some historic resources, and to use mathematical tools for the analysis, understanding and looking for answers to some initial historical problems. Finally, they were asked to individually write a short story as means to share and officialise their final responses and conclusions about the consequences of the war. This short story was sent to a literary contest announced by a Catalan publisher[1] opened to all secondary schools.

In this paper we describe some of the main traits of the didactic sequence's design that was finally implemented. We also analyse how it contributes to the development of students' inquiry and modelling competences by using the talks' registers, productions from students and their answers to a final questionnaire, but also by describing the a priori mathematical and didactic designs in which researchers and teachers work together.

THEORETICAL APPROACH

In the design of the sequence of tasks different approaches interact. On the one hand, the mathematical and didactic design quality is justified based on the three criteria of didactic 'suitability' proposed by the Ontosemiotic Approach, EOS (Godino, Batanero, & Font, 2007): 1) the emotional suitability, 2) the epistemic suitability and 3) the ecological suitability. The aim is to design a sequence of tasks in which different suitabilities were included. For instance, its 'emotional suitability' can be justified by the fact that students were asked to participate in a literary contest from which some winners were selected (as it finally happened); its 'mathematical quality' (the epistemic suitability) can be justified based on the view that implementation allows students to trigger relevant processes of mathematical activity, in particular of mathematical modelling processes. In its turn, the 'ecological suitability' is justified by the fact that the curricula of these secondary-school students have a competency-based approach, where the teaching and learning processes provided by the curricula should promote competences to deal with complex and varied real-life situations.

These curricula guidelines are on the same directions of recommendations from other countries and international organizations like the National Research Council (NRC, 1996). For instance, according to the NRC, teachers should support the development of abilities of inquiry. These inquiry abilities are hardly related to modelling perspectives, and sometimes it is difficult to find the differences between both processes (Artigue & Blomhoj, 2013). A recent discourse of inquiry in mathematics education focuses on the use of methods and mathematisation processes, promoting the construction of mathematical hypothesis and models, and the need for arguing, valuing and controlling in an appropriate way to solve the contextual problem (Elbers, 2003). From our viewpoint, placing mathematical modelling processes at the core of activities involves promoting other kind of processes important in a rich and functional and mathematical activity (understood as mathematical richness of quality processes).

On the other hand, a way to get a high epistemic suitability is to design the sequence of tasks using the notion of study and research paths (SRP) (Chevallard, 2006; 2012) as a didactic device to facilitate the inclusion of mathematical modelling in educational systems, and more importantly, to explicitly situate mathematical modelling problems in the centre of teaching and learning processes (Barquero, Bosch, & Gascón, 2008). We assume and use the structure of SRP (Barquero, Serrano, & Serrano, 2014) as the main theoretical construct to design the didactic sequence we will present in this paper: (a) The starting point of a SRP will be a 'lively' generating question with real interest for the community of study (students, teachers and researchers); (b) During a SRP, the study of the generating question will evolve and open many others 'derived questions'. The study of all these questions will lead to successive temporary responses, which would be tracing out the possible 'routes' to be followed in the effective experimentation of the SRP; (c) The teacher will thus have to assume a new role of acting as the leader of the study process, instead of lecturing the students; (d) An important dialectic that will be integrated in the SRP is the task of posing questions and that of the continuous search for answers; (e) Against the temptation of imposing some answers that are acceptable within the educational institution, the group of students needs to be invited to defend the successive answers they provide; and finally (f) The dialectics between the media and 'milieu' will be also essential to control what exiting resources and answers are available 'outside' the classroom (in the media), but also what tools will help us to validate and integrate them in our study.

The proposal presented in this paper uses some historical contexts to develop inquiry and modelling competences in the same line as some previous works, for instance: Vilatzara Group (2003) used historical questions context questions to introduce mathematical modelling processes or Sala, Giménez, & Font (2013) proposed a selection of tasks in historical and cultural context to promote inquiry.

CONDITIONS OF THE DESIGN AND IMPLEMENTATION

The starting question that opened the teaching sequence was placed in the context of Barcelona's siege during the war of the Spanish Succession.

> S_i: Which were the consequences of the incidents of 1714 for the society, culture and political organization of Catalonia?

Students were asked from the beginning to write a short story about the war' consequences for Catalonia with their own conclusions from their study developed in the classroom work. It was important this story was rigorous enough, based on contrastable historical data and involving a robust study about some related phenomena, like: Barcelona's demography evolution, economical changes for the society or for some guilds of the city, etc.

The teaching sequence was implemented during the course 2013/14 with two experimental groups at Secondary school level (12–14 years old) of 29 and 30 students respectively, in a high school in Badalona (Catalonia, Spain). Its experimentation was carried out during two weeks for each group, with a total of 7–8 weekly hours. One exceptional condition was that the subjects of History, Mathematics and Catalan literature were linked along this experimentation. They acted together in what was called the 'Mathematical-Historical workshop'. Along the workshop, the teachers of these subjects were all involved in the experimentation, with the help of the first author of the paper, who acted as observer and guide of the implementation. For their part, the students were organized in 'inquiry teams' that changed along the different stages of the workshop.

The study began describing the general initial situation S_i that was broken into some more concrete questions (some of them explained below) to better distribute and coordinate the inquiry-teams' work. They all began with the analysis of some real data (concerning each topic) in order to quantify and value the consequences of the incidents occurred in 1714 in Barcelona. Through the study of the initial question, together with the need of getting historical knowledge, evolved towards some other derived questions. The students had to search information outside the classroom in all available media (webpages, historical resources with real data, etc.). These teams had to build up and use different mathematical models to study some of these derived questions and to contrast the validity of their proposals by using tools from mathematics and also from history.

Most of the information related to the mathematical-historical workshop (initial worksheets, topics and questions to study, phases followed along the workshop, etc.), and also some of the results from its implementation, are available in the blog called 'Understanding the incidents of 1714': http://valorar1714.blogspot.com.es/. This blog was basically structured in four parts: a central part, where students could read teacher's posts with the aim of encouraging students to keep on working and temporizing each of the phases and tasks; on the right part, there were some materials and resources that students could use to do the first tasks of the workshop (historical frame contextualization). The left part contained the blog archive, the search bar, some links of interest and other resources 2.0 related to S_i; on the top, four different boxes contained the worksheets grouped by topics and other documents concerning each of the topics. The didactic sequence was organised in the following stages:

First stage: The historical frame contextualization. Cooperative work teams (the so-called 'inquiry teams') were formed and each student assumed different roles and some concrete tasks. Students were also asked to individually sign a contract about the compliance of certain working rules. Each team prepared, exposed and defended a poster board) where they had to explain a part of the historical frame since the beginning of the war. The rest of the students had to listen to, question and assess the rest of the poster boards and presentations. After that, new couple-teams were responsible for carrying out a time line with the free software 'Timetoast' to ensure they had a clear idea of the chronology of the historical events.

Second stage: Inquiry teams facing different topic related to S_i. The students, organized in the inquiry teams of two or three members, focused on the study of some phenomena (proposed by the teacher) related to S_i. Some real information was also provided to allow students to initiate an accurate study. To begin, all the teams had a general introduction about data's origin, historical resources and some other information to contextualize the inquiry. S_i was broken into different lines of research that were grouped in three main topics: A, B and C. After the general introduction, the teachers, who acted as supervisors of the inquiry, assigned each topic to each of the inquiry team.

Topic A focused on the demographic dynamics of Barcelona' population before and after 1714. It was composed by four worksheets (one for each inquiry team):

> $Q_{A.1}$: Evolution of the population between 1600 and 1720; $Q_{A.2}$: Evolution of the births in Catalonia between 1600 and 1715; $Q_{A.3}$: Evolution of the illegitimate births between 1600 and 1715; and $Q_{A.4}$: Comparative study of the salaries and prices between 1680 and 1800.

Topic B focused on the study of Barcelona's guilds before and after 1714, composed by seven dossiers that introduced the question of describing and fostering the evolution of the following guilds before and after 1714:

> $Q_{B.1}$: The fishermen and sailors' guild; $Q_{B.2}$: The weavers' guild; $Q_{B.3}$: The wool crafters' guild; $Q_{B.4}$: The shoemakers' guild; $Q_{B.5}$: The tanners' guild; $Q_{B.6}$: The tailors' guild; $Q_{B.6}$: The 'blanquers' (another type of tanners) guild.

Topic C focused on the study about the changes in the configuration of the city before and after 1714, composed by two worksheets:

> $Q_{C.1}$: The construction of 'Ciutadella' (a citadel) and its consequences for the city; QC.2: Changes on the configuration of the city (Barcelona).

Each of these inquiry lines, starts with the contextualization of the particular topic included in the worksheet, which was structured in different parts (two at least) including questions from less to more complexity (all of these questions are available in the worksheets published in the blog). Students were always asked to add new questions to follow with their study, but some feasible questions were included in the worksheets to help teachers to guide the study. For instance, if we focus on the topic A – $Q_{A.2}$ about the births' evolution in Catalonia between 1601 and 1700, it was introduced with the analysis of real data collected by Sant Just parish church (Barcelona) (Figure 1 – data provided by Ferrer, 2007). We began with the following general question:

> Given the data of the number of births in Catalonia during some years before 1714, how can we describe the evolution of the number of births? Can we identify any change? How can we make forecasts about the future number of births? How can we contrast prediction and reality?

Franges d'anys	Nombre de fills il·legítims
1601-1605	49
1606-1610	40
1611-1615	53
1616-1620	32
1621-1625	57
1626-1630	47
1631-1635	40
1636-1640	45
1641-1645	40
1646-1650	43
1651-1655	34
1656-1660	58
1661-1665	41
1666-1670	43
1671-1675	29
1676-1680	40
1681-1685	36
1686-1690	40
1691-1695	43
1696-1700	48

Figure 1a: Number of births per every period of five years

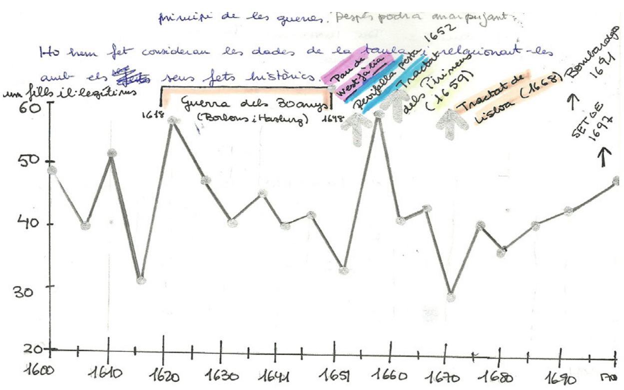

Figure 1b: Part of the partial report of an inquiry team working with $Q_{A.2}$

The first part focused on facing questions concerning the description of real data about the number of births between 1600 and 1714, with some other questions about how to predict the births' evolution around to 1714. Most of the students began using the graphical representation of real data (see Figure 1b) to complement the numerical analysis and to describe, for instance: the year intervals when births increase or decrease, linking the growth description to some historical events, etc. Despite all the rich description most of the teams provided, mathematical tools were not rigorous enough to provide forecasts about the short- and long-term births' evolution. At the end of the first part, students began to propose the use of discrete models (recurrent sequences models) or some basic continuous functions (constant or linear) to fit real data and also to predict birth's evolution. The second part expanded the previous answers and results. It was focused on creating tools to compare and contrast forecasts provided by the mathematical models with real data (more data about real births from 1714 were given to the inquiry teams). Furthermore, between the first and the second phase, each group had to prepare the 'interaction report' with which different teams working with similar topics (A, in this case) had to interact. From this interaction some common questions arise that could need some help from the teacher to follow with the activity (comparing proposals, introducing some unclear mathematical concept, formalising mathematical tools, etc.). At the end, each inquiry team was asked to complete, describe and report their work by writing their 'final report' about this line of research.

Third phase: Sharing the conclusions from the inquiry teams. In this final phase, each inquiry team had to prepare a presentation to share their study and conclusions. Some more 'interaction' moments were planned, depending on the conclusions they had found. For instance, it was necessary that some teams dealing with $Q_{C.1.}$ about the construction of 'Ciutadella' interacted with teams on $Q_{B.i.}$ about the changes on the guilds in the city. To better understand and explain why some changes in the city cause some movements in where guilds were placed.

Another example of the interlinked and necessary interaction between different lines of inquiry was shown by the work within $Q_{C.1}$ (see Figure 2) and $Q_{C.2}$ (topic C, Figure 3) about the creative proposals about the construction of the trenches (Torras & Sobrequés, 2005). In the first phase of the study, each team focused on its own topic:

> $Q_{C.1}$: The 'Ciutadella' was built a few years after the defeat of September 11th, 1714 with a special architectural structure (Figure 2). What role do you think it had? What benefits could its architectural structure provide to the city (in military terms, protection of inhabitants, etc.)?
>
> $Q_{C.2}$: It's been documented that the structure of the trenches (Figure 3) during the War of Succession introduced some innovative changes. What kind of novelties did this system include? What elements allow us to compare the shape of the trenches with the advantages in number of deaths?

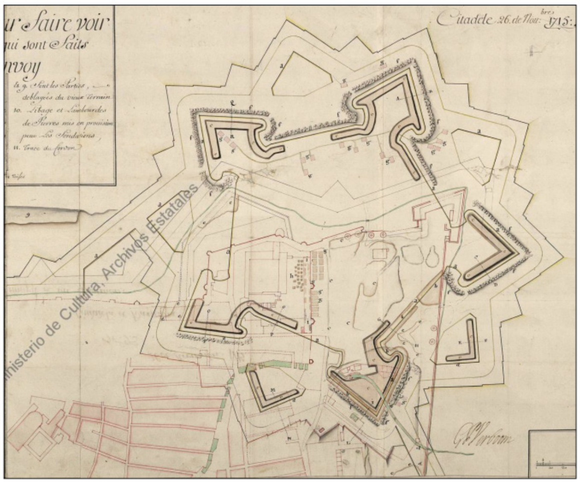

Figure 2: Part of the plan of the citadel (1715). Available in http://ves.cat/l9_P

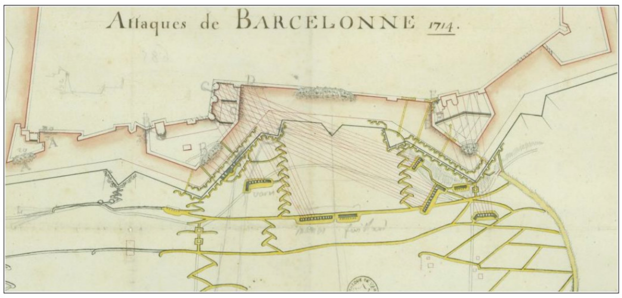

Figure 3: Trenches to the walls of Barcelona during the siege of the War of Succession in 1714. Available from http://gallica.bnf.fr/ark:/12148/btv1b7100421p/f1.zoom

After each team developed its own study of their particular lines, some new questions were asked to link both lines. They mainly focused on studying the relationship between the structure of the 'Ciutadella' and the defensive strategy and the resistance capacity of the trenches structure to bear the continuous attacks during the siege.

At the end of this phase (as forth stage), student had to individually draft a short story, based on the work done by all the groups working in the class, following the rules of the literature contest. After a process of co-evaluation of their writings, their writings were sent to the contest. The good news were that one of our students won the second prize of the contest (named 'Catalan memories'!)

RESULTS AND DISCUSSION

In all the work developed around the initial situation S_i (what were the consequences of the war of Spanish Succession in 1714 to Catalonia) and derivative questions that characterize each topic, it was essential that the students assumed many responsibilities, traditionally assigned exclusively to the teacher. For example: work in a inquiry team; schedule their own study; discuss the proposals and results in the classroom; question the historical validity according to their answers; analyse real data and look for patterns to build models and make predictions; contrast their conclusions with the most appropriate experimental milieu (looking for data in external resources, trying with other models, etc.); summarize results and defend them orally and in writing; check the validity of the mathematical tools that are being used; raise new questions to continue with the study, etc. Moreover, students had to compare, search agreement with team members and also with other teams based on their own productions and results.

This way of working helped students to place themselves in a critical perspective towards the reality around them and gave them tools to come up with their own decisions and ways to analyse situations in our society (Vanegas & Giménez, 2010), like the initial situation we have presented in this didactic sequence. That is, despite the path of tasks planned and followed by the students in this implementation, the teaching sequence promoted autonomy and a critical thinking thanks to the openness of the questions.

Among other devices that were available to students and teachers, we emphasize the importance of the 'inquiry guide', and some documents that each inquiry group had to write so that the teachers could follow the progress of the study. These documents are: two partial reports (Inquiry teams had to fill them at the end of the first and second research phase respectively, presenting a synthesis of the issues treated, and explaining their responses and the mathematical tools used); an 'interaction report' between groups (it had to be completed to guide the interaction between inquiry teams); and a 'final report' that they had to write at the end of the study. This final report helped the students to individually write the short story to send to the literature contest.

However, we want to emphasize the importance of planning some sessions to share all teams' work and approaches with the rest of the teams. Students answered a questionnaire that allowed us to know their perception on various aspects of their learning. They said that they had had a lot of difficulties on: understanding in the beginning what it was being asked and why, integrating their contributions and results in their research, generalizing their results, synthesising their work and writing the team's reports, etc. However, in general, their opinions were positive. It was in these 'sharing sessions' where the teacher had to act as guide or as mediator, helping students with all these new tasks that they had assigned.

Finally, we think that the final task of submitting an essay to a literary contest was a key element to engage students in the research project. Moreover, the integration and use of new didactic devices, as the ones described, that are not usually integrated in school reality help students to assume many responsibilities that are traditionally assigned only to teachers.

FINAL REFLECTIONS

The research we have presented in this paper describes some of the characteristics of the teaching sequence of 'Understanding the incidents of 1714', which was designed giving an especial role to inquiry, modelling and the interdisciplinary approach. The design gives special importance to this interdisciplinary approach in order to allow students to consider mathematics, and specifically mathematical modelling, as a tool for finding answers to questions (in this case, from History). Besides the first resistances of students to use and mix tools and information from different subjects (history and mathematics, in our case), mainly doubting that teachers were 'asking' for this combined used, they could finally go into the 'dialogue' and interaction of such traditionally isolated disciplines. History was not only providing the initial context where the initial questions where posed, its tools were constantly integrated along the study to validate their responses. And in the same way, models and mathematical tools were at the disposal of the research activity, they had to help us to follow with our inquiry activity and all proposals had to be persistently checked with the appropriate historical-mathematical milieu.

Along the SRP students were asked to build up their own 'piece of history' through: the formulation of hypotheses based on the analysis of real data and historical information, looking for mathematical tools to enlarge the answers to historical problematic questions, contrasting the validity of mathematical tools with historical resources, posing new questions to follow with, etc. That is, as Dean (1995) describes, learning by 'doing history', approaching History as a scientific discipline but also, learning by 'using mathematics' as a fundamental tool to provide answers.

Finally, considering all resistances manifested by the students and also by the teachers facing this implementation, we plan to continue searching on what kind of new design and didactic devices are necessary to integrate this kind of studies in which a 'real study' of questions will require to combine and 'make' interact tools emerging from several disciplines, where, in particular, mathematics and mathematical modelling will have an essential role.

ACKNOWLEDGMENT

This work was carried out in the framework of R & D: And EDU2012-32644 EDU2012-39312-C03-01 from Spanish Government.

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ENDNOTE

The rules to participate in the literature contest were available in the Bromera Publishers web: http://www.bromera.com/tl_files/pdfs/altres/basesconcurs_cicatrius1714_doblepag.pdf

The link between the cognitive structure and modelling to improve mathematics education

Laura van de Weerd and Nellie Verhoef

University of Twente, Institute for Teacher Training and Professional Development, Enschede, Netherlands, lauravandeweerd@gmail.com

This study focuses on the aspects in the cognitive structure that should be trained to develop Dutch students modelling without actual modelling lessons. The research used 16 fifth-grade beta coursed students to study the development of the cognitive structure and its relation to modelling. The methodology is based on the Conceptual Content Cognitive Map Method of Kearney and Kaplan (1997), while Tall's (2013) theory is the theoretical framework. Data are collected by a cognitive test and a modelling test. The results show that students with a rich cognitive structure develop compression. Thus, teachers should focus on the connections between mathematical concepts as a solution for this problem.

Keywords: Modelling, cognitive structure, student learning, cognitive map.

INTRODUCTION

Modelling is an important discipline in beta sciences. However, it does not belong to the curriculum of Dutch secondary schools, which results in a gap between secondary school and university (Renkl, 1997; Savelsbergh, 2008). This problem has been studied from different points of view. Barquero, Bosch, and Gascón (2008) investigated a new didactic device to teach mathematical modelling at university. Verhoef, Zwarteveen-Roosenbrand, Joolingen, and Pieters (2013) studied the themes for mathematical modelling that interest Dutch students in secondary school. Lots of studies searched the cognitive structure, but only a small amount used quantitative analyses for measurement. Therefore this research focuses on the relation between the cognitive structure and modelling in a quantitative way, by use of cognitive maps. To quantify this, the study was well-delineated to trigonometry. This leads to the following main question:

> *Which aspects of the cognitive structure of secondary school students are determining factors in their development of modelling?*

To answer this question the theoretical framework contains theory of the cognitive units, the cognitive structure and modelling. The method is formulated by means of the Conceptual Content Cognitive Map method of Kearney and Kaplan (1997), because of the quantitative analyses.

THEORETICAL FRAMEWORK

Cognitive units and structure

The human brain is a complex biological structure that forms a multi-processing system. It is able to make decisions and focuses on important information as a result of the electrical communication among nerve cells, also called neurons. Signals are sent from a neuron's axon and will be received by another neuron's dendrite. The signal frequency of the neurons produces neurological activity. Increasing activity results in stronger connections, as decreasing activity results in weaker connections. The more often a connection is used, the thicker the connection will be. This is a result of long-term potentiation, which is a long-term increase of the spiking frequency of the neurons. This is very important in the formation of brain structures. It is directly related to the memory (Purves, 2008) and makes subconscious actions possible (Barnard & Tall, 1997; Starzyk, Li, & Vogel, 2005). The plasticity of neuron connections makes change of thoughts possible by opening and closing the connections. The working memory, that can be used to solve problems, is a result of this plasticity in closely connected neurons (Barnard & Tall, 1997; Crowley & Tall, 1999; Starzyk et al., 2005; Vogel, 2005; Widrow & Aragon, 2013). Such densely packed neurons form interneurons. Activation of a neuron in such interneuron group, activates all connected neurons if the connections are strong, called compression (Barnard & Tall, 1997; Purves, 2008).

In this study a schematic representation of the brain is used to simplify this theory. Terms as cognitive unit and cognitive structure are used. A cognitive unit is a small piece of information that the brain can focus on. As Tall (2001) says: "A cognitive unit consists of a cognitive item that can be held in the focus of attention of an individual at one time, together with other ideas that can immediately be linked to it." (Tall & Barnard, 2001, p. 2). We define a rich cognitive unit, when the cognitive unit contains a great amount of connections between small pieces of information (Vogel, 2005). The cognitive structure can be described in different ways. According to Hiebert and Carpenter (1992) it can be described by the two metaphors; vertical hierarchies or webs. Tall and Barnard (2001) combine both and define the cognitive units as the nodes of the cognitive structure. The related cognitive units are connected by the threads of the web. In a rich cognitive structure there are many connections.

Compression is also an important schematic term in mathematical thinking. It describes the way in which small, rich cognitive units are formed within a cognitive structure. The junctions in the spider's web are so close that they touch each other (Tall & Barnard, 2001). This process is important because all cognitive units will be activated if one part of the information have been evoked, which is important in the thinking process. According to Tuminaro and Redish (2007), secondary schools focus too much on the students' results, while the connections and the learning process are more important.

Modelling

Modelling activates learning processes and confronts the scientist with the effects of his theories. Blum studied the difficulties with modelling and tried to explain these difficulties by the students' cognitive demands of these tasks. He emphasised that mathematical modelling has to be learnt specifically by students, and that modelling can indeed be learned if teaching obeys certain quality criteria, in particular maintaining a permanent balance between teacher's guidance and students' independence (Blum & Ferri, 2009).

In our study we focus on the cognitive aspects that are required for the development of modelling. By understanding the differences in the cognitive structure between students that are bad in modelling and good in modelling, we try to find the key aspect that is necessary in teaching modelling. For this study a mathematical model will be used which should comply with the conditions of modelling. According to Blum (2002), a modelling process contains five steps: a) Simplifying the real problem into a real model; b) Mathematizing the real model into a mathematical model; c) Searching for a solution for the mathematical model; d) Interpreting the solution of the mathematical model and; e) Validating the solution within the context of the real-life problem the real model into a mathematical model.

The Conceptual Content Cognitive Map Method

Lots of studies investigated the cognitive structure, but only some used quantitative analyses for measurement. Because this study also requests a quantitative measurement of the cognitive structure, the methodology is mainly based on the Conceptual Content Cognitive Map (3CM) Method[1] of Kearney and Kaplan (1997). Not only does their research show to be reliable and valid, their method is also a good basis for this study (Kearney & Kaplan, 1997; Somers, Passerini, Parhankangas, & Casal, 2014).

RESEARCH METHOD

Participants

Sixteen 16–17 aged students were subjected to the tests. They all had a beta programmed curriculum (physics and chemistry) with mathematics education.

Research instruments

This study used four instruments: a) the cognitive test; b) its evaluation, c) the modelling test and d) the grade list. Cognitive structures and modelling are respectively measured by the cognitive test and the modelling test. The study started with a cognitive test as a benchmark. One week later the modelling test followed, testing the students' modelling and their increasing insight in trigonometry and the ability to make connections between concepts. A week later, the second cognitive test studied the development in modelling concerning trigonometry. This second test was the same as the first cognitive test, so the differences in the results could be used to study the

1 For a detailed description of the 3CM Method, read http://eab.sagepub.com/content/29/5/579.full.pdf+html

cognitive development. After each cognitive test the students were asked to fill in the evaluation form and to note their grades for mathematics, physics, chemistry and if such is the case, informatics. Figure 1 shows an overview of the research method.

a) For the construction of the cognitive tests (used for the first and second cognitive test) mathematics concepts were collected. Therefore new participants were selected by their expertise, experiences, field of study and/or age. This results in a list with the following participants: a professional mathematician (f), ten master students (m/f) and one secondary school student (f). The participants made mind maps as a spider's web, which resulted in 68 concepts.

To test the cognitive test two of above-mentioned participants were used. They were chosen by their result of the mind map, because they had respectively the widest and deepest order. On basis of this pilot study the instructions were improved. This was tested by two secondary school students, which showed no further additions were necessary.

For the implementation of the cognitive test all participants (fifth-grade students) got an envelope filled with 68 concepts, one instruction form and one A_2 paper with in the middle the term trigonometry. The instruction form explained the participants to make a mind map around the word trigonometry in 15 minutes time. They could only use the concepts on the cards and a pen to connect them. Not all concepts had to be used. After 15 minutes the result was photographed and later analysed. The cognitive test measures the cognitive structure by constructing a mind map, called a cognitive map.

b) After each cognitive test the participants filled in an evaluation form. The answers to these questions were an underpinning of the results of the cognitive tests. The form contained the questions:

- *What was the assignment according to you?*
- *Did you know all used concepts?*
- *Would your cognitive map be different if you had more time? If so, what would be the differences?*

c) The modelling test was based on the theoretical framework of model conditions. The participants for this test's pilot were five master students (m/f) and two secondary school students (f). The pilot study showed that an illustration of a schematic representation had to be added to the test instruction. The modelling test was a trigonometry based problem, so the students should find connections between the right triangle, unit circle and sine as signal as function in time. The problem was a riddle that had to be solved to crack a safe. This riddle had to be decoded in goniometric steps. The students were asked to answer this problem by constructing a mathematical model. The problem mainly focused on the first two steps of a mathematical model (Blum, 2002), but could be solved with all modelling steps. The results were analysed by the following scale: insufficient, sufficient, good and very good. If the results didn't satisfy the first steps of a mathematical model it was graded insufficient. Any further step in the model resulted in a better grade.

d) The student's grades were collected during the cognitive tests. The students wrote their grades of mathematics, physics, chemistry and informatics (only if such is the case) on the envelopes, which were collected at the end of the tests and the grades were listed.

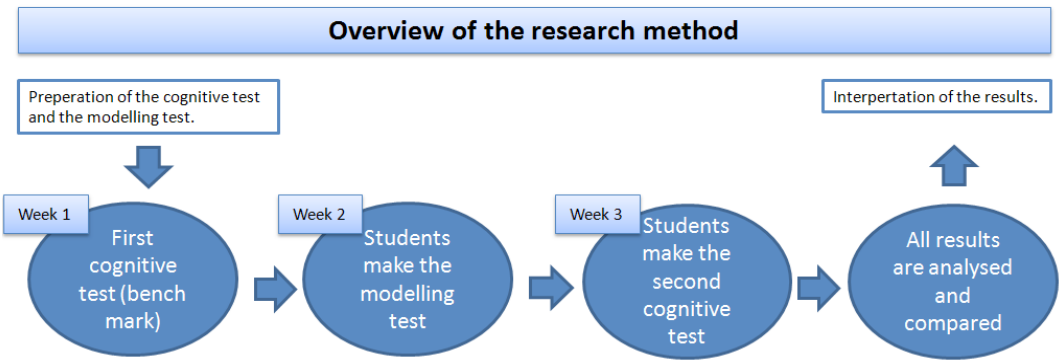

Figure 1: Overview of the research method

Data analysis

The four research instruments were each individually analysed and gathered in the overview table, which relates the data.

a) Cognitive test

The results of each participant were categorised by the depth of the cognitive map. As Figure 2 shows, the depth was determined by the amount of concepts counted from the main concept 'trigonometry'. For instance, if the maximum depth of a cognitive map was two concepts, the participant was placed in class 1. The class division was counted up to 5, as the pilot showed that a maximum of class 5 was enough. The pilot also showed that students who have a depth further than five can be placed in class 5.

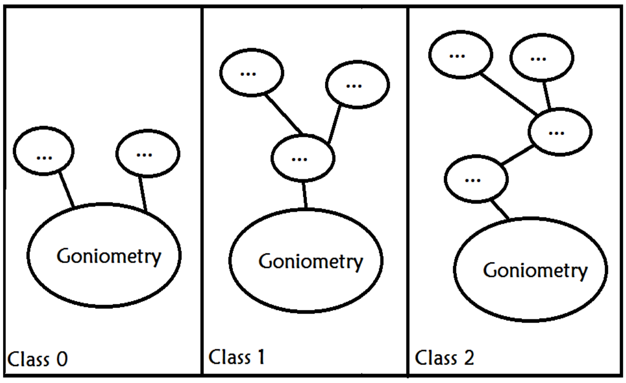

Figure 2: Examples cognitive map classification

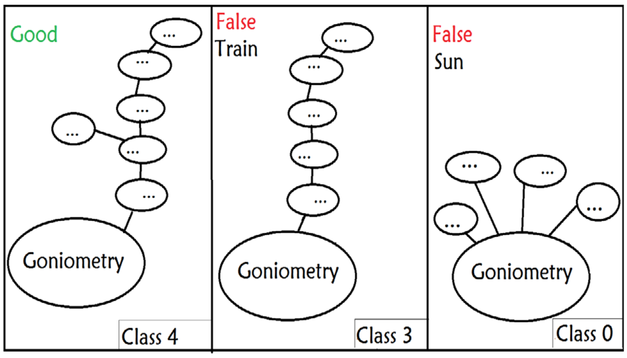

Figure 3: Cognitive maps containing 'trains' or 'suns'

There are some conditions in the classification of the cognitive maps. If students create 'trains' or 'suns' (Figure 3) in their maps it shows lack of depth. As 'sun' has no depth and a 'train' can be divided in more branches. That's why 'trains' longer than three concepts will be counted as three. Every deeper concept won't be counted, because it is always possible to further connect a concept this deep. This could mean that a cognitive map with depth 4 will be classified in class 3. The results of each participant were translated to a matrix, irrespective of classes.

The results were processed in three different kinds of matrices. Each a 68x68 matrix, constructed of *concept* x *concept*. Matrix type I was formed for each participant. The matrix cells were filled with '1' if the concerning concepts were connected. If the concepts were not connected the cell was filled with '0'. In matrix type II each [*ij*] element states the percentage of participants that connected the concept *i* with concept *j*. So, if 80% of the participants connected concept *i* with concept *j*, element *ij* was filled with 0.8. Matrix type III is a deviate of matrix type II. Each percentage smaller than 0.5 was equalled to 0. Percentages of 0.5 or larger than 0.5 were equalled to 1.

The correlation between matrix type I and matrix type III was calculated. This was done per participant per class. That means that the results of each participant was compared with the results of the other participants in the same class. The results were listed in the overview table (Table 3).

b) Modelling test

The results of this test were graded by the scale shown in Table 2.

c) Grade list

The school grades of mathematics, physics, chemistry and informatics (if such is the case) of each participant were listed. They are scaled from 1 to the upper val-

Result	Grade	Abbreviation
No form of a model and clearly no attempt to	Insufficient	I
Attempt to model, contains some elements of a model	Sufficient	S
Model, but result misses some model elements	Good	G
Model, clear and strong structure	Very good	VG

Table 2: Explanation for the evaluation of the modelling test

ue 10. The results were collected by forms that were handed out during the cognitive test.

d) Overview table

The results of each participant were collected in Table 3. The results of the cognitive test were expressed in the correlation between the students' test and the class average. This correlation represents the degree of connected concepts in the student's cognitive test compared to the class average. Students with a correlation smaller than 0.25 were selected. Based on the 3CM method, the results of these students deviate from the average results. This means that a student makes more or less connections than the average student.

RESULTS

Table 3 shows an overview of the collected results, sorted per student. It is noticeable that most students rose in classes and that more than half passed the modelling test. Also most students show sufficient results for the school grades.

Students 3, 4, 6, 7, 9, 12, 14 and 16 show correlations smaller than 0.25, but Table 3 shows no mutual connections. That means every cognitive map had to be analysed individually. Starting with the students that have a correlation smaller than 0.25, three students are insufficient graded for their modelling test. As students 3, 6 and 14 have different cognitive maps, all show little connections between the used concepts. The students with a correlation smaller than 0.25 and with a "very good" for the modelling test are students 12 and 16. Both seem to use many connections between their concepts. The results of the three other students that are graded with a "very good", 5, 10 and 13, show the same. In general, most striking cognitive maps are those of student 5 and 12. Both students use many concepts that are linked in many ways. Figure 4 shows the cognitive map of student 12. The student makes many connections, that even form loops which is not by all students. In fact, the opposite can be found in the results of student 2, 8 and 11.

CONCLUSION AND DISCUSSION

The results show some noticeable students that should be discussed, like students 3, 4 and 16. As the grades of student 3 were very high, the modelling test was graded insufficient. So it seemed that this student is highly talented. Student 4 showed a huge increase in classes. According to his cognitive maps, this seems to result from the amount of used concepts and the development in the used connections. This student seemed to give a clear result regarding the influence of the modelling test and the repetition of the cognitive test. The same applies for student 16. This student showed no progress in classes and used a small amount of concepts. Though he increased the used

	Class test 1	Correlation test 1	Class test 2	Correlation test 2	Modelling test	Grade 'wiskunde B'	Grade 'wiskunde D'	Grade physics	Grade chemistry	Knew all used concepts test 1	Knew all used concepts test 2	More time, better test 1	More time, better test 2
Student 1	4	0,55	5	0,40	S	6	-	7	7		x	x	
Student 2	3	0,37	5	0,63	I	6	-	7	7		x	x	x
Student 3	3	0,00	4	0,59	I	7	9	9	9			x	x
Student 4	2	1,00	5	0,13	G	5	-	6	7				x
Student 5	5	0,29	5	0,43	VG	8	6	7,5	6,5		x	x	
Student 6	4	0,12	5	0,46	I	6,3	-	6,4	7		x		x
Student 7	5	0,23	-	-	S	6	-	6	6			x	
Student 8	4	0,40	5	0,34	I	6	-	6	8		x	x	x
Student 9	3	0,25	-	-	S	5	-	6	6	x		x	
Student 10	3	0,37	-	-	VG	7	-	8	9			x	
Student 11	3	0,37	4	0,80	I	5	-	7	7		x	x	
Student 12	5	0,15	5	0,16	VG	9,5	9,5	8	8	x	x		x
Student 13	5	0,35	5	0,38	VG	6	-	6	6	x	x	x	x
Student 14	5	0,30	5	0,09	I	8	-	7	7	x		x	x
Student 15	3	0,28	5	0,52	G	5	-	6	7			x	
Student 16	3	0,00	3	1,00	VG	8	-	7	7	x	x	x	x

Table 3: All collected data listed for each student*

* None of the participants had informatics, so this is not present. The evaluation results of the cognitive test are represented as an 'x' for positive answers.

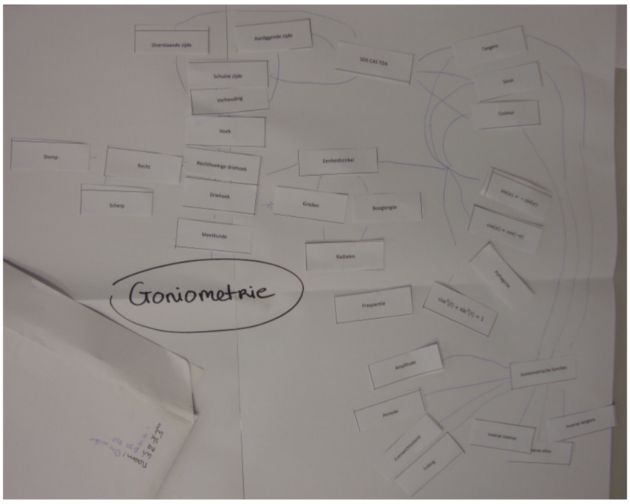

Figure 4: Cognitive map of student 12

connections in the last cognitive test. Only, it is not clear what influenced this development, was it the modelling test or the repetition of the cognitive test? This effect should be studied in further research. The results of this study prove to be a good indication for a large scale study. Although, it would be better for the analysis of the modelling test to use a more detailed checklist for evaluation. This would make it more reliable and valid.

It follows from the results that Dutch students are better in modelling if they have many connections between the concepts in their cognitive maps. This can be interpreted as the connections in the students' cognitive structure. It also follows from the results that the amount of the used concepts defines the category of the cognitive map. This represents the cognitive unit. Evident examples for this conclusion are the result of student 3, 4, 8, 12, 13, 14 and 16. So it can be stated that:

Students that are good in modelling have a rich cognitive structure. Students that are placed in a high class have a rich cognitive unit. When a student is good in modelling and is placed in a high class, there is compression. These students have a rich cognitive unit as well as a rich cognitive structure.

The cognitive structure can be developed to a rich cognitive structure by repetition. This seems to emerge from the connection between the results of the first and the second cognitive test. Repetition is definitely of indirect significance in the development of modelling

RECOMMENDATIONS

As this study shows good and useful results, it can be recommend to repeat this study with a larger group of participants. Further research should be focussed on the influences of the repetition, regarding the modelling test on the development of the cognitive structure. This study focussed on the mathematical subject trigonometry, but it can be useful to focus on different mathematical items.

ACKNOWLEDGEMENT

The authors wish to thank Michael Hekker for his contribution in this research.

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Understanding issues in teaching mathematical modelling: Lessons from lesson study

Geoff Wake, Colin Foster and Malcolm Swan

School of Education, University of Nottingham, Nottingham, United Kingdom,
geoffrey.wake@nottingham.ac.uk, colin.foster@nottingham.ac.uk, malcolm.swan@nottingham.ac.uk

This paper challenges the conceptualisation of the OECD's PISA model for assessment of mathematical processes and questions common approaches to modelling in the classroom. Drawing on evidence from research using a lesson study model, we argue that the crucial formulation phase of modelling, in which bridging between context and mathematics takes place, is undervalued. Consequently, we conclude that teaching towards assessment such as, or modelled on, PISA items could provide students with an impoverished experience of modelling and leave them inadequately prepared to engage in this important mathematical practice.

Keywords: Modelling, problem solving processes, lesson study, PISA.

INTRODUCTION

> *"The formulation of a problem is often more essential than its solution"*
> (Einstein & Infeld, 1938, p. 92)

Cai and Howson (2013) argue that there is evidence of some convergence of mathematics curricula around the world due to the TIMSS and PISA series of international comparative studies. In particular, as a result of the PISA series, there has been a noticeable increase in interest in mathematical modelling and problem solving. In this article, therefore, we focus on these important aspects of mathematics in schools. We consider this a particularly pertinent time to raise issues in relation to the PISA conceptualisation of problem solving and the validity of the framework that is used to define the domain of mathematics and the test items that result.

At the heart of our concern, and research, has been developing greater understanding of how, in classroom learning, students might develop a mathematical literacy that will better prepare them to be able to apply mathematics effectively in modelling problems so that they are able to make sense of situations that arise from a range of different contexts. There has been considerable theorising and research in areas that might inform our concerns. For example, in relation to mathematical literacy see Steen (2001), for mathematical modelling see the 14th ICMI Study (Blum, Galbraith, Henn, & Niss, 2007) and for problem solving see Schoenfeld (1992). However, as a mathematics education community, our detailed understanding of teachers' classroom practices and students' actions in problem solving and mathematical modelling is much less well developed than our understanding of conceptual development. We report here results from the first year of our ongoing research in relation to the teaching of problem solving and modelling.

CONCEPTUALISATIONS OF PROBLEM SOLVING, MODELLING AND MATHEMATICAL PROCESSES

Acknowledging the important influence that the PISA series of international assessments play in informing the development of curricula, and by implication teaching and learning, around the world, we consider the PISA definition of mathematical literacy as an important starting point in understanding acknowledged conceptualisations of the field:

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments

and decisions needed by constructive, engaged and reflective citizens. (OECD, 2013, p. 25)

Fundamental to the goal of PISA is a quest to measure the ability of students to be able to use mathematics to make sense of different contexts that have relevance and authenticity. This has implications for the age appropriateness of problems/tasks used. In developing a framework or vision of mathematical literacy in practice PISA has at its core a modelling cycle (Figure 1). Such cycles are well known (for example, see Blum & Leiß, 2007; Kaiser & Sriraman, 2006; Maaß, 2006; etc.). Although there are many variations both in the detail of the conceptualisation of the practice and its diagrammatic representation, the main the PISA diagram captures the essence of all.

Here the important processes as one moves from a contextual to a mathematical world and back again are:

- formulating – in which relevant mathematics that can lead to a solution, or sense-making, of the problem is identified. An appropriate mathematical structure and representation(s) are developed by making simplifying assumptions and identifying variables.
- employing – involves mathematical reasoning that draws on a range of concepts, procedures, facts and tools to provide a mathematical solution.
- interpreting and evaluating – involves making sense, and considering the validity, of the mathematical results/solution obtained in terms of the context in which the problem situation arises.

The cyclical representation of this overall process provides for the expectation that a refinement, or complete rethink, of the mathematical structure representing the real world situation may be desired, or even necessary. It is also important to bear in mind that progression around the cycle is not necessarily entirely one way, as there may be the need to refine thinking at any stage as the potential effects of decisions being taken become apparent and need modification as one proceeds.

The PISA framework, as in Figure 1, draws our attention to how problems may arise in a range of different contexts that can be classified as being *personal*, *societal*, *occupational* or *scientific*. Also of major importance are the mathematical content domains that in problem solving and modelling interact in symbiotic relationship with the problem-solving/modelling processes: PISA identifies these as being *quantity*, *uncertainty and data*, *change and relationships* and *shape and space*.

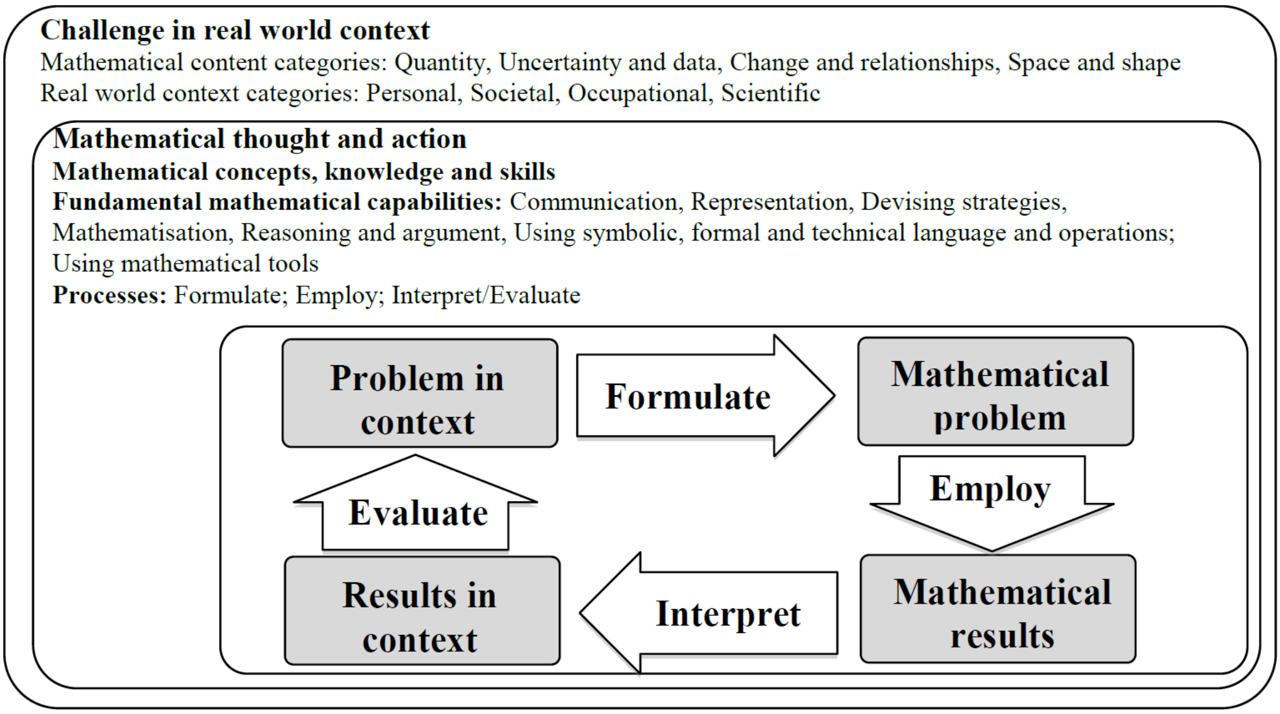

Figure 1: "A model of mathematical literacy in practice" (OECD, 2013)

In developing assessment items, it is acknowledged that it is not necessary that students engage with the whole modelling cycle: particular items may focus on only parts of modelling as a mathematical practice. This reflects the way in which adults engage with mathematics in practice, for example, in the workplace (Wake, 2014), where it is more usual to work with, or develop further, the mathematical models of others rather than start from scratch.

METHODOLOGY

Fundamental to the research project that informs this paper is a concept of professional learning that is focused on practitioner enquiry into teaching, learning and classroom practice. The project aims, therefore, to develop and research professional learning communities in which teachers work together and learn from each other, informed by 'knowledgeable others', who have a role of stimulating the community by drawing on a range of expertise that is research-informed. An adaptation of the Japanese lesson-study model (Fernandez & Yoshida, 2004) has been used and continues to evolve. This involves a community of teachers and a 'knowledgeable other' collaborating in a cyclical process that involves planning a 'research lesson', joint observation of the lesson and critical reflection in a detailed post-lesson discussion (Wake, Foster, & Swan, 2013). Lesson Study is perhaps particularly attractive, as it has the potential to meet the requirements that we know facilitate effective professional learning (Joubert & Sutherland, 2009); namely, that it is:

- sustained over substantial periods of time;
- collaborative within mathematics departments/teams;
- informed by outside expertise;
- evidence-based/research-informed;
- attentive to the development of the mathematics itself.

Here we draw on our research which has involved 3–4 teachers at each of nine schools organised in two geographically located clusters (of 5 and 4 schools), with teachers collaborating and involved in research lessons across their cluster. We adopted a case-study methodology in order to obtain rich, contextual data. This data consists of video recordings of the planning meetings, research lessons and post-lesson discussions, researcher records of students' working in research lessons, and audio recordings of interviews with teachers and other participants.

In this paper we present, as indicative of the outcomes we observed across the 30 research lessons that have been carried out within the project to date, examples from the work of two students that encapsulate their mathematical activity and learning in relation to mathematical modelling.

CASE STUDY

The task and lesson

The research lesson was with a class of 13 to 14 year-olds with little, but some, experience of working on problem-solving/modelling tasks. The students had worked on the task in the lesson prior to the research lesson, providing the teacher with insight into the

110 years on
This photograph was taken about 110 years ago.
The girl on the left was about the same age as you. As she got older, she had children, grandchildren, great grandchildren and so on.
Now, 110 years later, all this girl's descendants are meeting for a family party.
How many descendants would you expect there to be altogether?

Twentieth Century facts

At the beginning of the 20th century the average number of children per family was 3.5. By the end of the century this number had fallen to 1.7	In 1900, life expectancy of new born children was 45 years for boys and 49 years for girls. By the end of the century it was 75 years for boys and 80 years for girls.

Figure 2: Task '110 years on'. (Source: Bowland Maths Assessment tasks: http://www.bowlandmaths.org.uk/)

different ways in which they were beginning to understand the context and problem and the ways in which they were formulating a mathematical model of the problem. The inquiry focus of the lesson-study group in this particular research lesson was to better understand how mathematical representations may assist structuring and supporting mathematical thinking.

The chosen task is one from a collection of assessment tasks to be found at http://www.bowlandmaths.org.uk/. These differ from PISA tasks in that they are open-ended modelling tasks; albeit with guidance for teachers about how they might observe progression in each of the process skills such as formulating, employing and so on. In this way they are much less structured than PISA tasks leaving students with considerably more scope to explore the context. Due to restrictions of space here, we illustrate student outcomes by reference to the work of only two students, these being chosen to provide some evidence of the diversity in student thinking in this particular lesson.

Student A

Student A presented a list of key information that he considered relevant to the problem and also a list of assumptions which included the quantified factors presented in the formulation of the task. He also listed other factors that are not quantified in any way but are factors that could affect his solution. None of this is illustrated here due to space limitations. Student A's visualisation (Figure 3) of the situation effectively includes a timeline showing key years following the taking of the photograph. For example, 1903 is taken as the year in which the girl in the photograph is 13 years old. He assumed that after 3 years, in 1906, the girl married and after another 4 years, at age 20 she gave birth to 4 children. Throughout the period Student A assumed that people marry at age 16 and have children at age 20. He attempted to take account of the information that the average number of children per family was initially 3.5 by rounding to give 4 children in the first generation and then allocating a total of 14 children in the next generation (that is, that each of the girl's four children have 3.5 children). A similar logic underpinned his calculations to give the number of children in the next generation, although it is unclear how subsequent values in his diagram were calculated. Finally the student made some decisions about who from the early generations would have died in advance of the party.

Student B

Student B's representation of the situation (Figure 4) appears to show *fewer* descendants in each subsequent generation, which is contrary to what we would expect. On closer inspection of the diagram other features appear equally strange: for example, Student B distinguishes between males and females in the diagram, using a different marker for each. This leads to brothers and sisters, as well as sisters and sisters or brothers and brothers, being the parents of offspring. The calculations presented by Student B (not illus-

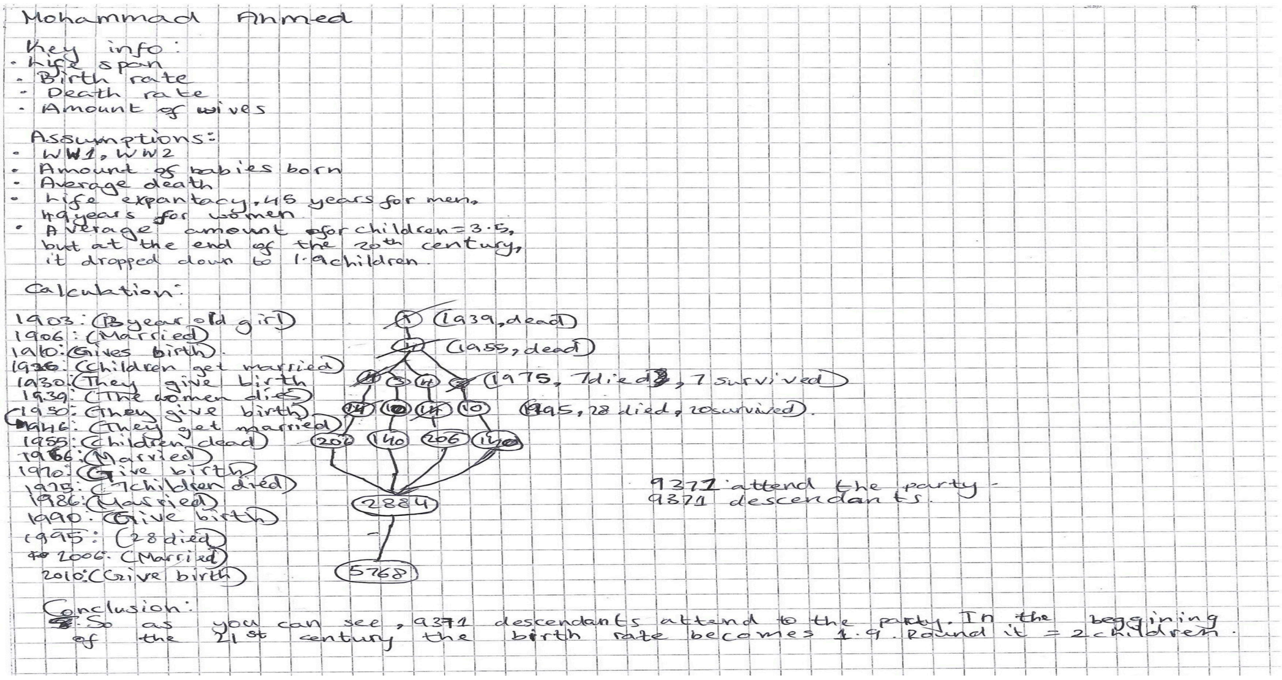

Figure 3: Extract from the response to the task '110 years on' by Student A

trated) provide insight into some of the thinking that underpins his diagram. It appears that Student B fell into the trap of selecting values given in the posing of the task and operating with these to calculate values he believed he needed to make progress. For example, he divided the information that there is a total time of 110 years between the photograph and the party by the average number of children per family (3.5) to calculate that altogether there should be 32 children. He also calculated that the girl in the photograph had 9 children by dividing 110 by 13 (the girl's age at the time of the photograph).

ANALYSIS AND DISCUSSION

As can be seen from the examples of the work of the two students illustrated here, the students not only arrived at very different solutions, they have very different understandings of the context/situation. Indeed, it appears that student B did not have an appropriate understanding of the situation at all. This was also true of some of the other students in the class, as evidenced by their representations and calculations. Of course we would expect varied approaches by students working on a modelling task, but here it is clear that some of the models being *formulated* were invalid.

Our experience across, and analysis of data from, lessons confirms that this is a common occurrence and leads us to contend that there are essential aspects of both problem solving and modelling that are under-emphasised in school practices when compared to similar practices in out-of-school settings. Crucially, in the process of *formulating*, the difficulty associated with understanding a complex context, and how this needs to be simplified so that it can be represented mathematically, is underestimated. This process requires an understanding of how a range of mathematics (concepts, procedures, facts and tools) might best be marshalled to provide a mathematical structure to represent the problem context/situation. Such understanding is important and needs to inform the simplification of the context/situation. Pollack (1969), Borromeo Ferri (2006) and Treilibs (1979), among others, draw attention to this important aspect of mathematical modelling. Across our case studies we note that students have particular difficulties with this crucial first step in modelling.

This leads us to emphasise that the mathematical model being developed to provide a mathematical structure that maps to a *simplified* structure of the reality it represents. In the initial phase of developing a mathematical model, this simplification of reality to provide what is in effect a model of reality or 'real model' is not necessarily a simple matter. The structure that is proposed has to be a realistic representation, capturing essential elements of what might in fact be a complex situation, *and* it has to be such that the modeller can use mathematics that they know and are comfortable to work with. Borromeo Ferri (2006) discusses how

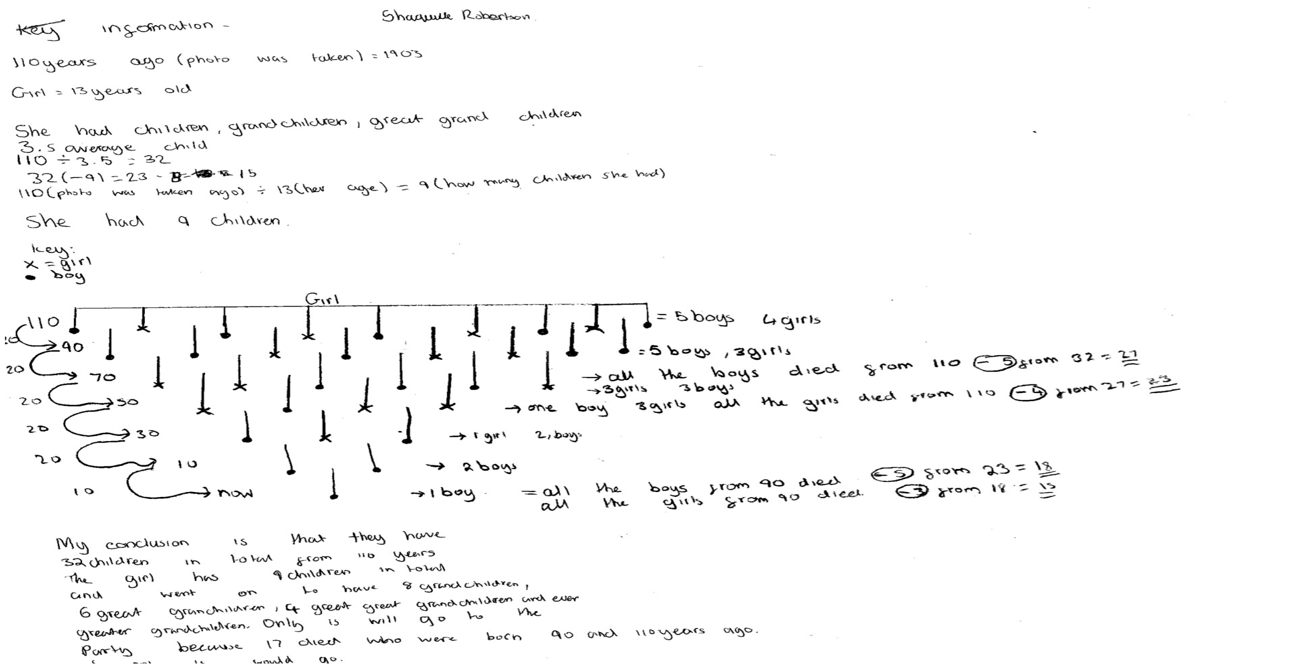

Figure 4: Response to the task '110 years on' by Student B

different researchers have attempted to characterise this early stage in the modelling cycle and identifies the terms 'situation model', 'mental representation of the situation' and 'real model' as being of significance here. Integral to this stage of mathematical modelling is the development of an understanding of the task: this should not necessarily be assumed as unproblematic. Blomhøj and Højgaard Jensen (2003) in their description of the modelling process and associated schema identify early processes as involving 'formulation of task' and 'systematisation', with the latter relating to making sense of the reality of the problem situation. Treilibs (1979) breaks down the formulation phase into subtasks, such as: modelling the situation by making simplifying assumptions, identifying relevant variables, generating relationships, and so on. Whatever terms we use to describe these initial steps towards being able to work mathematically, we have found that for many students the formulating stage can prove problematic and generate a lot of questions and discussions before a mathematical model can be developed.

Our consideration of this issue and how we might present tasks that involve students in effective mathematical modelling practices has raised the question of distinguishing between problem solving and modelling. We consider that mathematical models are mathematical structures that map to, or represent, a simplification of a real context, and as such they are useful when they have repeated use so as to consider variation of (a) factor(s) in the reality. For example, in the students' response to the task illustrated in this article assumptions made about the age at which people have children could be varied and the impact on results considered. In relation to this aspect of a mathematical model, we recognise that there are canonical models, such as exponential functions, inverse square laws, etc. that can, with adaptation, be used across numerous contexts and situations. In such cases, features of the mathematical model relate to factors and structure of the reality. For example, the growth of a population can be considered exponential when the rate of growth is proportional to the size, P, of the population, with the value of k in the equation $P=P_0e^{kt}$ being related to the time it takes the population to double in size. In mathematics teaching we are concerned with such adaptations as well as with models that are bespoke to particular contexts and situations, such as in the task presented in this article.

In either a canonical model or a bespoke model, change of an identified factor in the reality results in change of a variable in the mathematical structure (model), and vice versa: that is, varying factors in the mathematical model has implications for the reality it represents. This, we use to distinguish between solving a problem using mathematics (which also has a mathematical structure that maps to a simplified reality) and modelling. In the case of problem solving, there is a single solution, albeit dependent on decisions taken to simplify the reality (which may eventually be modified/refined); on the other hand, in the case of modelling, the expectation is that there will be variation of important factors and repeated use of the model. We consider drawing attention to this characteristic of a mathematical model as having importance as a potential pedagogical vehicle that can be used to focus students' attention on the aspects of modelling that we identify as being under-emphasised in school mathematics. These aspects are (i) the simplification of reality and (ii) the development of a mathematical structure that represents, or maps onto, the simplified reality, with each of these needing to be informed by detailed understanding of the implications and potential of each other.

In the examples of student working that we present here we note that it is clear that the students' representations, and by implication their simplification of reality, do not adequately capture the essence of the situation in order to allow them to successfully arrive at a valid solution. In this particular lesson the variation in student approach to the problem and consequently their understanding was perhaps the most significant and immediate observation made by the observers of the lesson. Although developing a family tree was the most typical approach each student's manifestation of this suggested at least minor differences in their understanding of the context leading to significant differences in the assumptions being made, and consequently variables and fixed values decided upon. Some students took very different approaches with at least one student providing an entirely textual solution with no apparent diagrammatic visualisation or calculations evident. This variation in understanding resulted in solutions that were also very different with the number of descendants varying from values that were less than fifty to values greater than 1000. The focus for the students appears to have been on arriving at a single solution rather than developing an appropriate mathematical structure for the situation. This is particularly starkly visible in the work of Student B. It seems likely that a pedagogical approach that focused on developing a model that could have repeated application, allowing for variation of a key factor, such as the time between generations, has the potential to force this issue in the classroom. In considering the design of tasks that might be appropriate to bring to the surface the important aspects of modelling that we have identified, we have found that introducing the requirement for students to work towards a product, such as an explanation to a particular audience about the effects of varying a particular factor in the reality, is potentially helpful in generating awareness of this aspect of modelling/using models. For example, in the case of '110 years on' we suggest that students might be required to write advice and explain to a caterer possible maximum and minimum numbers of guests at the party.

We suggest that greater emphasis should be given to model development in mathematics lessons; this seems crucial if we are to support student learning of effective modelling. We note that this appears under-emphasised in the PISA organising framework for the mathematics domain and, as a consequence, and importantly, in their assessment items (OECD, 2013). We therefore urge that there needs to be careful thought about how best to support students' learning in curricula based on PISA's conceptualisation of the mathematics domain. As it currently stands, we view that teaching towards a PISA notion of problem-solving/modelling may not support the mathematically literate students we seek. We also recommend further research in this area of modelling as a classroom practice in mathematics so that we are better informed about student learning, and what it means to make progress in learning, in this important area.

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Identifying ways to improve student performance on context-based mathematics tasks

Ariyadi Wijaya[1], Marja van den Heuvel-Panhuizen[2,3] and Michiel Doorman[2]

1 Yogyakarta State University, Mathematics Education Department, Yogyakarta, Indonesia, a.wijaya@staff.uny.ac.id

2 Utrecht University, Freudenthal Institute for Science and Mathematics Education (FIsme), Utrecht, Netherlands, m.vandenheuvel-panhuizen@uu.nl, m.doorman@uu.nl

3 Faculty of Social and Behavioural Sciences, Utrecht, Netherlands

This paper reports the Context-based Mathematics Tasks Indonesia (CoMTI) project that was aimed at getting a better insight into Indonesian students' low performance on context-based tasks and identifying ways to improve it. The project addressed three main issues: (1) Indonesian students' difficulties when solving context-based tasks; (2) possible reasons for students' difficulties; and (3) offering students opportunity-to-learn and testing its effect on student performance. These issues were investigated in four consecutive studies. The studies revealed that the students' difficulties are related to students' opportunity-to-learn.

Keywords: Context-based mathematics tasks, modelling, low achievement, Indonesian students, opportunity-to-learn.

BACKGROUND OF THE STUDY

The ability to apply mathematics is considered as a core goal of mathematics education around the world (see, e.g., Eurydice, 2011; NCTM, 2000). This goal is similar to what in the Programme for International Student Assessment (PISA) is called mathematical literacy, which refers to students' ability "to identify, and understand, the role that mathematics plays in the world, to make well-founded judgments and to use and engage with mathematics in ways that meet the needs of that individual's life as a constructive, concerned, and reflective citizen" (OECD, 2003, p. 24). To develop students' ability to apply mathematics, it is recommended to offer students mathematics problems situated in real-world contexts (De Lange, 2003; NCTM, 2000). In PISA study problems with real-world contexts are used to assess mathematical literacy (OECD, 2003).In this paper such problems are called context-based tasks and defined as tasks that are situated in real-world settings and provide elements or information that need to be organized and modelled mathematically.

Similar to many other countries, Indonesia also places a premium on applying mathematics as a core goal of mathematics education and pays attention to the use of context-based tasks (Pusat Kurikulum, 2003). Nevertheless, there is an apparent discrepancy between this goal and student achievement. The PISA results showed that Indonesian students perform low on context-based tasks. More than three quarters of Indonesian students did not reach the baseline Level 2, which means they could only answer tasks that have familiar contexts and present all relevant information (OECD, 2010). The low performance of Indonesian students on context-based tasks prompted an establishment of a project called "Context-based Mathematics Tasks Indonesia" (CoMTI), which was aimed at getting a better insight into Indonesian students' low performance on context-based tasks and identifying ways to improve student performance.

KEY IDEAS FROM LITERATURE

Solving context-based mathematics tasks

Solving mathematics problem situated in real-world contexts, which in this paper are called context-based tasks, requires interplay between the real world and mathematics, which is often described as a modelling process. According to Blum and Leiss (2007) the process of modelling is considered to be carried out in seven steps. The first step is establishing a 'situation model' to understand the real-world problem. Second, developing the situation model into a 'real model' through the process of simplifying and structuring. Third, constructing a 'mathematical model' by mathematizing the real model. The fourth step is carrying out mathematical procedure to get a mathematical solution. In the fifth and sixth steps, the mathematical solution is interpreted and, then, validated in terms of the real-world problem. The final step is communicating the real-world solution. This modelling process is similar to what is called 'mathematization' in PISA studies (OECD, 2003). Mathematization involves: understanding the problem situated in reality; organizing the real-world problem according to mathematical concepts and identifying the relevant mathematics; transforming the real-world problem into a mathematical problem which represents the situation; solving the mathematical problem; and interpreting the mathematical solution in terms of the real situation.

Opportunity-to-learn

A so called 'opportunity-to-learn' is often used to find an explanation for students' mathematics performance. In the First International Mathematics Study opportunity-to-learn was defined as "whether or not [...] students have had the opportunity to study a particular topic or learn how to solve a particular type of problem" (Husén, 1967, pp. 162–163). This definition was further specified by Brewer and Stasz (1996) who distinguished three aspects for measuring opportunity-to-learn. First, *curriculum content*, which refers to the scope of the topics offered to students. Second, *teaching strategies* that are used by teachers to present the topics and to engage students. Third, *learning materials*, such as textbooks, which are used to teach the students.

THE COMTI PROJECT

Although there are different ways used in different countries to improve educational achievement, improving Indonesian students' performance cannot be simply done by applying an educational practice that is used in other countries because, according to Pearson (2014), what works in one particular country will not necessarily give the same result in other countries. Careful thought about what is missing in current educational practices and what might be needed by students is necessary. Therefore, in order to identify possible ways to improve Indonesian students' performance the CoMTI project focused on three interrelated issues regarding context-based tasks in Indonesia. First, *what* difficulties are experienced by students when solving context-based tasks. Second, *why* students have difficulties, for which we investigated opportunity-to-learn to solve context-based tasks offered in Indonesian textbooks and in teachers' teaching practices. Lastly, to study *how* student performance can be improved, we offered students opportunity-to-learn to solve context-based tasks and test its effects on students' performance. These three issues were investigated in four studies, which are described in the following sub-sections.

Indonesian students' difficulties in solving context-based tasks [Study 1]

The first study of the CoMTI project was aimed at getting a better insight into the low performance of Indonesian students on context-based tasks. In this study the difficulties experienced by Indonesian students when solving context-based tasks were examined through an analysis of students' errors. This approach was chosen because students' errors provide access to students' reasoning and are considered as a powerful source to diagnose learning difficulties (Borasi, 1987). With respect to analysing students' difficulties in solving mathematical word problems, Newman (1977) developed a model that is known as Newman Error Analysis. Newman proposed five categories of errors, i.e. reading (error in simple recognition of words), comprehension (error in understanding the meaning of a problem), transformation (error in transforming a word problem into an appropriate mathematical problem), process skills (error in performing mathematical procedures), and encoding (error in representing the mathematical solution into written form).

Method

A total of 362 students from 11 schools in the Province of Yogyakarta, Indonesia participated in a so called CoMTI test. The test items were selected from the released PISA mathematics tasks. After the CoMTI test, an error analysis was carried out on the basis of students' incorrect responses to investigate the difficulties experienced by students. For this purpose, an analysis framework was developed based on Newman's error categories that were associated with the stages of modelling process and PISA mathematization. The analysis framework comprised four types of errors: comprehension, transformation, mathematical processing, and encoding. Newman's reading error was not used in our framework because this error category refers to a technical aspect and does not match to modelling process or PISA's mathematization.

Results and discussion
The error analysis revealed that of 1718 errors made by the students 38% were comprehension errors, 42% were transformation errors, 17% were mathematical processing errors, and 3% were encoding errors [1]. A closer examination of the comprehension and transformation errors disclosed that a half of the comprehension errors were errors in selecting relevant information. We also found that two thirds of the transformation errors were errors in selecting mathematical procedures required to solve the tasks.

The results of the error analysis indicate that Indonesian students mostly experienced difficulties in comprehending a context-based task and in transforming it into a mathematical problem. In addition to these specific results, this study showed how analysing students' difficulties can be a crucial preliminary step in the process of improving student performance because it sheds light on key aspects of solving context-based tasks that need to be developed. The findings of this study suggest that improving the task comprehension of Indonesian students requires a focus not only on students' language competence, but also on the ability to select relevant information. Furthermore, the ability to identify the required procedure or concept was found to be another key competence that needs to be improved.

Opportunity-to-learn to solve context-based tasks offered in Indonesian mathematics textbooks [Study 2]
The next step in the CoMTI project was identifying possible explanations for students' difficulties. Several studies have shown that student performance is often influenced by textbooks. Tornroos (2005) found a relation between student achievement on a test and the amount of textbook content related to the test items. The method used in a textbook to help students understand the content is also an important aspect influencing student performance. As found by Xin (2007), students tend to solve word problems by using the solution strategies suggested in the textbooks. Another aspect of a textbook that has an influence on student performance is the cognitive demands of the tasks. What competences students will master depends on the cognitive demands of mathematics tasks they are engage in.

Considering the important influence of textbooks on student performance, in the second study of the CoMTI project we investigated the opportunity-to-learn context-based tasks offered in Indonesian textbooks. Three issues were addressed in this study: (1) the amount of exposure to context-based tasks in Indonesian textbooks, (2) the characteristics of the context-based tasks in the textbooks, and (3) the relation between the characteristics of textbook tasks and students' errors.

Method
Three mathematics textbooks that were used in the schools participating in the first study of the CoMTI project were analysed. For this purpose, we developed an analysis framework that focused on three task characteristics. First, the type of context for which we used three types of context: relevant and essential context, camouflage context, and no context. Second, the type of information provided in a task: matching information, missing information, and superfluous information. Third, the cognitive demands of a task: reproduction, connection, and reflection tasks.

Results and discussion
The textbook analysis revealed insufficient number of context-based tasks in Indonesian mathematics textbooks [2]. Only 10% of tasks in the textbooks were tasks that used either camouflage or relevant and essential context. Of these tasks, three quarters used camouflage context, i.e. the context can be ignored in the solving process, and explicitly implied the required mathematical procedures. This finding indicates that Indonesian textbooks do not offer students enough opportunity-to-learn to identify mathematical procedure that is required to solve a context-based task, which might explain the high number of transformation errors made by students. An in-depth analysis of the task characteristics revealed that of 276 context-based tasks in the three textbooks 88% provided only the information that is needed to solve the tasks (matching information). This result signifies a lack of opportunities for students to learn to select relevant information, which might contribute to students' comprehension errors, in particular errors in selecting information. Lastly, regarding the cognitive demands, of all context-based tasks in the textbooks almost no

reflection tasks, i.e. tasks that require complex reasoning and a construction of original mathematical approaches.

Opportunity-to-learn to solve context-based tasks offered by Indonesian teachers' teaching practices [Study 3]

Several studies (e.g., Eurydice, 2011; Grouws & Cebulla, 2000) showed that student performance is affected by the teaching strategies used by teachers. How teachers teach mathematics and engage their students influences how well students learn. With respect to the teaching of context-based tasks, Antonius, Haines, Jensen, Niss, and Burkhardt (2007) argued that it requires more than an 'explanation-example-exercise' ritual because such directive approach does not offer students opportunity to develop strategic competences that are necessary to solve context-based tasks. Instead of using direct teaching, teachers should use a teaching approach in which they take a consultative role and give students opportunities to actively build new knowledge and reflect on their learning process (Antonius et al., 2007; Blum, 2011).

The purpose of this third study was to investigate the opportunity-to-learn (OTL) to solve context-based tasks offered in teachers' teaching practices. For this purpose, we investigated the teaching approach that was used by teachers to help students learn to solve context-based tasks. Teachers' beliefs were also investigated because they often influence teachers' teaching practices (see, e.g. Wilkins, 2008). Lastly, we investigate whether there was a relationship between the OTL to solve context-based tasks offered by teachers and the errors made by students when solving such tasks.

Method

A teacher survey and a series of classroom observations were used in this study. The survey was aimed at investigating teachers' beliefs and teachers' reported practices regarding the characteristics of context-based tasks offered to students. The classroom observations were carried out to investigate the teaching approaches used by teachers to help their students learn to solve context-based tasks.

Twenty-seven teachers from the schools involved in the first study of the CoMTI project participated in the teacher survey and four of them participated in the classroom observations.

Results and discussion

The survey data showed that the teachers tended to perceive context-based tasks merely as plain word problems. They believed that the mathematical procedure required to solve a context-based task should be given explicitly. Furthermore, the teachers also did not consider missing and superfluous information as an important characteristic of a context-based task. In agreement with these beliefs, the teachers reported that they frequently offered their students context-based tasks that have explicit procedures, but rarely gave context-based tasks that provide superfluous or missing information. Such practice might explain students' difficulties in identifying the required procedures and in selecting relevant information.

The classroom observations revealed that the teachers mainly used directive teaching approach in which they tell the students what a context-based task is about, translate the task into a mathematical problem, and explain what mathematical procedure to be carried out. In such teaching students were not encouraged to actively carry out and reflect on the stages of solving context-based tasks. This directive teaching approach was mostly used in the comprehension and transformation stages. Consultative teaching in which students were actively engaged in the process of solving context-based tasks was barely used by the teachers. Moreover, this teaching approach was mostly observed in the mathematical processing stage; a stage in which students do not have to deal with the context of a task.

Correspondences were indicated between teachers' teaching practices and students' difficulties. A lack of opportunities for students to paraphrase a context-based task might be related to students' difficulty in comprehending the task because, as pointed out by Kletzien (2009), paraphrasing helps students understand the text of a task. Moreover, teachers' direct advice regarding the procedures to be carried out might correspond to students' transformation errors because it might discourage students from thinking about the mathematics concepts addressed in the task.

Effects of opportunity-to-learn on Indonesian students' performance in context-based tasks [Study 4]

For teaching context-based tasks it is recommended to use teaching practice that emphasizes on guiding students to construct new knowledge actively and independently by using their prior knowledge and experiences (Antonius et al., 2007; Blum, 2011), which in the CoMTI project is called 'consultative teaching'. Blum (2011) found that students who learned through such teaching approach made a better progress regarding their modeling competence in comparison to students who were taught with directive teaching. In addition to teaching practices, it is also important to give students tasks that have superfluous and missing information and do not provide explicit suggestions about the required procedures (Maass, 2007).

The second and the third studies of the CoMTI project revealed a lack of opportunity-to-learn offered in Indonesian textbooks and in teachers' teaching practices. Therefore, in the final study of the project an intervention that offers students opportunity-to-learn to solve context-based tasks was developed. The effects of opportunity-to-learn on students' performance in solving context-based tasks were examined from the perspectives of students' score gains and students' errors.

Method

A field experiment with a pretest-posttest control-group design was used in this study, which involved a total of 299 students (144 students were in the experimental group and 155 students in the control group) from six schools.

An intervention program comprising a set of context-based tasks and a consultative teaching approach was used in the experimental group. The context-based tasks used in the intervention had three important characteristics: relevant and essential context, superfluous or missing information, and not explicit suggestion about the required mathematical procedures. The consultative teaching used metacognitive prompts, which included self-addressed questions and verbal prompts or instructions to help students focus attention on particular aspects of the solving process such as asking students to paraphrase a task and to underline relevant information.

Results [3] and discussion

A univariate ANOVA with the gain score (posttest score minus pretest score) as dependent variable and intervention as a fixed factor was carried out to investigate the effect of the intervention. Contrary to our expectations, the difference in gain scores between the students in the experimental group ($M_{experimental}$ = 0.11, $SD_{experimental}$ = 0.99) and the students in the control group ($M_{control}$ = -0.09, $SD_{control}$ = 0.95) was only marginally significant and the effect of the intervention was small ($p = .068$; $\eta_p^2 = .011$). Nevertheless, a closer examination of the effect of the intervention on students' errors revealed a significant difference between the experimental group and the control group for the decrease in the total number of errors (χ^2 (1, n = 4127) = 4.149, $p = .042$). This finding reflects a positive influence of the opportunity-to-learn on reducing students' errors. Students who received the opportunity-to-learn could better understand the instruction for a context-based task and had improved performance in selecting relevant information. With respect to transforming a real-world problem into a mathematical problem in general no influence of the opportunity-to-learn was found. However, a positive influence was found for context-based tasks addressing graphs – i.e. the topic taught during the intervention period – in which students who got the opportunity-to-learn were better able to give a mathematical interpretation of a graph. Reflecting upon this finding and referring to Howson (2010), it can be learned that to improve students' ability to identify the required procedure it is essential to provide not only context-based tasks that are related to the topic being taught, but also context-based tasks that address other topics.

FINAL REMARKS

In general, the results of the CoMTI study suggest two important ways to identify and to improve student performance; i.e. diagnosing students' difficulties and identifying opportunity-to-learn offered to students. By connecting students' difficulties with opportunity-to-learn, we could identify what was missing in the educational process. Our results show that textbooks and teaching practices are key aspects to improve students' performance on context-based tasks.

In the appendix we provide the summary of the findings of the four studies in the CoMTI project and show how the results of the four studies are interrelated.

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ENDNOTES

1. Examples of students' errors can be found in (Wijaya et al., 2014) on pages 569–573.

2. Examples of tasks in Indonesian mathematics textbooks can be found in (Wijaya et al., 2015) on pages 14–17.

3. We would like to thank Michiel Veldhuis for his contribution to the statistical analysis of the data.

APPENDIX: THE RESULTS OF THE COMTI PROJECT

STUDENTS' DIFFICULTIES (*Study 1*)

Analysis of Indonesian students' errors when solving context-based tasks

The most dominant error types:
- **comprehension** errors; in particular, errors in selecting relevant information.
- **transformation** errors; in particular, errors identifying the required mathematical procedures.

POSSIBLE REASONS FOR STUDENTS' DIFFICULTIES: OTLS AS A KEY CONCEPT (*Study 2* and *Study 3*)

Analysis of Indonesian mathematics textbooks

Exposure of the context-based tasks:
- Only about 10% of all tasks were context-based.

Characteristics of the context-based tasks:
- most of the tasks used *camouflage contexts* and provide explicit indications about the required mathematical procedures.
- most of the tasks provide *matching information*, i.e. only the information that is needed to solve the tasks.
- almost no *reflection tasks*, i.e. tasks with highest cognitive demands which require constructing original mathematical approaches and communicating complex arguments and complex reasoning.

An investigation into Indonesian mathematics teachers' teaching practices

Teachers' report about the **characteristics of context-based tasks** offered to students:
- most of the teachers frequently give tasks with *explicit procedures*
- most of the teachers frequently give tasks with *matching information*
- a half of the teachers never or rarely give tasks with *superfluous information*
- a half of the teachers never or rarely tasks with *missing information*.

Teachers' **teaching approach**:
Over all stages of solving context-based tasks:
- *No instruction* was given in 42% of all questions discussed in the lessons.
- *Directive teaching* was applied in 47% of all questions discussed in the lessons.
- *Consultative teaching* was applied in only 12% of all questions discussed in the lessons.

Specified for the stages of solving context-based tasks:
- Directive teaching was most frequently applied in the comprehension and the transformation stages.
- Consultative teaching was mostly applied in the mathematical processing stage.
- Almost no attention was paid to the encoding stage.

OTL AND STUDENT PERFORMANCE (*Study 4*)

Offering students opportunity-to-learn (OTL)		**Effects of the OTL on students' performance**
Context-based tasks:	**Consultative teaching approach with metacognitive prompts:**	
- Context-based tasks with missing or superfluous information.	- Paraphrasing: asking students to formulate a task in their own words. - Underlining all information and circling only the relevant information - Self- questioning; e.g. *"Do we have enough information to solve the task?"*	A positive effect of the OTL on students' task comprehension was found: - Students could understand better the instruction of the task - Students' ability to select relevant information improved
- Context-based tasks with a relevant context that requires modeling - Context-based tasks with non-explicit procedure	- Self-questioning; e.g. *"What are possible strategies to solve the task?"*	- In general no effect of the OTL on students' ability to transform a real-world problem into a mathematical problem. However, a positive effect was found for tasks addressing an interpretation of a graph, which in fact was related to the topic taught during the intervention. - This finding leads to a recommendation to offer students 'mixed exercises', i.e. a set of context-based tasks that address various topics.

'Literature' on mathematical modelling from a teacher perspective: A textbook's portrayal

Anders Wolfsberg

University of Copenhagen, Faculty of Science, Copenhagen, Denmark, rnp123@alumni.ku.dk

In this paper, it is proposed to reformulate the question of whether modelling as depicted in academic literature is insufficiently implemented in school. Rather, the question proposed is: what form of modelling is actually portrayed in textbooks and curricula – understood as teachers' 'literature' on mathematical modelling?

For illustration, a case is offered wherein perspectives on modelling are traced from academic literature, through curriculum, and an upper secondary textbook, all taken from a Danish context. It is discussed how the textbook's portrayal of mathematical modelling deviates from that in academic literature. Further, it is suggested that this deviation – this gap – may be closed by fitting the version of modelling to be implemented better to the institutional structure of upper secondary school.

Keywords: Mathematical modelling, textbook portrayal, connecting research to practice.

INTRODUCTION AND DISPOSITION

Large scale implementation of mathematical modelling (hereinafter modelling) activities in mathematics education is a visionary project that has been under way for the past several decades. According to many researchers it is still present in mathematics classrooms to a degree far from satisfactory (Barquero, Bosch, & Gascón, 2010).

The aim of this paper is to understand better the causes and consequences of the apparent gap between research on modelling, and modelling as taught in practice.

One may take the position that researchers have a particular 'vision' of a subject to implement in practice – their version of modelling. Observing that it is not satisfactorily implemented, one is led to the question: 'what are the obstacles hindering its implementation?' This paper proposes a change of perspective.

Instead, the viewpoint taken is that a subject known as mathematical modelling exists in the upper secondary school institution, one that is different from the subject conceived of in academic research. Hence, it is integral to find out what this subject is, how it relates to academic perspectives, and why it takes the form it does. This might guide the way in conceiving of a version of modelling that is better fit to be implemented in teaching-in-practice in school classrooms.

A case is presented wherein a particular notion of mathematical modelling is traced from academic literature, throughout curriculum documents to a textbook's portrayal. All three samples constituting the case are taken from a Danish context.

The case is taken as point of departure for a discussion of the differences between two versions of modelling: that which is portrayed in academic literature, and that portrayed in textbooks and curricula – 'literature' implemented in secondary school.

DIDACTIC TRANSPOSITION

Theory of Didactic Transposition (TDT) is a theoretical framework developed by Yves Chevallard. TDT describes the transition of scholastic knowledge produced in universities throughout the educational system, e.g., curriculum and textbooks, to teaching situations. The mathematics being taught is not the same as that being produced in research institutions. Indeed, academic knowledge naturally undergoes changes when transposed through curriculum into textbooks (Winsløw, 2011).

The *external* transposition process takes place in what is called the 'noosphere' where 'scholarly knowledge' is transformed into 'knowledge to be taught', to be found in, e.g., curriculum and textbooks.

> The first step [the external transposition] corresponds to the study of the *formation* of the 'teaching text' (...) and highlights the conditions and constraints under which the 'knowledge to be taught' is constituted (...) (Bosch & Gascón, 2006, p. 56)

The *internal* transposition describes how teachers adapt and implement the knowledge into the very teaching situation (Winsløw, 2011).

Didactic knowledge such as that on *teaching mathematical modelling and promoting modelling competency* is produced in communities of mathematics and educational science researchers. However, the knowledge emerges in curriculum and textbooks (often by the work of others) in a transposed form more or less dissimilar to that envisioned by educational researchers.

Thus, in a TDT view, incoherencies between knowledge of (some) researchers, and that portrayed in textbooks, and applied by teachers, is a naturally occurring phenomenon. Indeed, constructing a so-called *epistemological reference model* (hereinafter reference model) of knowledge on a given topic in all levels of the didactic transposition allows for analysing these incoherencies (Winsløw, 2011).

METHOD OF ANALYSIS

The case study consists of three samples of 'knowledge on mathematical modelling': one from academic literature, one from curriculum, and one from a textbook. The contents of these samples are analysed qualitatively. In view of TDT this method corresponds to constructing a three-level reference model for 'knowledge on modelling' – a view of modelling in three different 'spheres' of the educational system. Hence, only the external transposition is analysed in this case.

All three samples are taken from a Danish context but are of international relevance. Thus, the scope on literature focuses on Danish researchers who work with mathematical modelling and the concept of modelling competency as used and developed by Mogens Niss. Niss has had significant influence on the wording of Danish mathematics curriculum via the KOM project on mathematical competencies (Blomhøj & Kjeldsen, 2010). Furthermore, interpretations of the modelling subject such as that of Niss are widely used in international literature on modelling (see, e.g., Niss, Blum, & Galbraith, 2007). The textbook is one of the most widely used in Danish upper secondary mathematics teaching.

In the following three sections, views on mathematical modelling as portrayed in academic literature, curriculum, and textbook are extracted.

Inspiration has been drawn from Julie and Mudaly (2007), who identify two categories for tendencies in didactical research on teaching modelling: 'modelling as vehicle' and 'modelling as content'. In a modelling as vehicle perspective, modelling is conceived of as providing motivation and support for learning of other mathematical subjects. As content, modelling is conceived of as a topic in itself.

In the following analysis and extraction of academic views, and curriculum and textbook descriptions, the focus is set on the category of modelling as content. Thus, content is sampled that describes modelling as educational content, not as a way of teaching. What is excluded are views on how modelling could or should be taught.

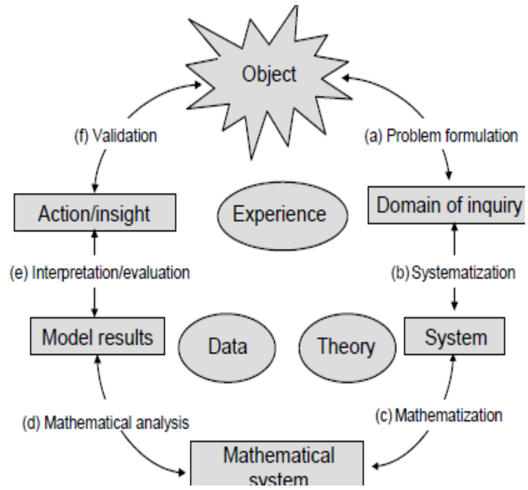
ACADEMIC VIEW ON MODELLING

Models and modelling

Niss (1989) defines a mathematical model as a triplet (A,f,M) of three domains denoting an extra-mathematical domain, A, a mathematical domain, M, and then a mapping, f, between the two. The domains must be understood abstractly as "collections of relationships, phenomena, questions (and possible answers) and such-like" (Niss, 1989, p. 28).

Niss identifies modelling with model construction, and in defining the concept, he characterises the process that signifies general model construction processes. We find an equivalent characterisation of the modelling process in Blomhøj & Kjeldsen (2010, pp. 3–5), illustrated in Figure 1 – the modelling cycle.

The process is illustrated cyclically since the object to be modelled often is redefined progressively. Between the sub-processes are double-headed arrows to indi-



a) Formulation of a task (more or less explicit) that is related to a perceived reality and influenced by the modeller's interests. Through this process the object of the modelling process is constructed. The object can be reconstructed as a result of the modelling process. However, it is the object and the formulated task that guides the identification and construction of a domain of inquiry.
b) Selection and construction of the relevant objects, relations etc. from the domain of inquiry, and idealisation of these, in order to make a mathematical representation possible.
c) Transformation and translation of selected objects and relations from their initial mode of appearance to mathematics by further abstraction and idealisation.
d) Using mathematical methods to achieve mathematical results and conclusions.
e) Interpretation of these as results and conclusions regarding the system or th initiating domain of inquiry.
f) Evaluating the validity of the model by comparison with data (observed or predicted) and/or with already established knowledge (theoretically based c shared/personal experience based)

Figure 1: Left: the modelling cycle/process. Right: the six sub-processes (Blomhøj & Kjeldsen, 2010, p. 4)

cate that activity in any one sub-process influences the entire model in either direction simultaneously. The sources of information for the model are shown in the three ellipses.

Niss (1989) points out, on the nature of models, that "Models are designed to model *something* – not to be confused with something *unique*." (p. 31).

Modelling competency

The modelling competency is one of eight mathematical competencies identified by Niss in the KOM Project which are characteristic to mathematical activity. Mathematical competency is described as "someone's insightful readiness to act in response to the challenges of a given situation" (Blomhøj & Jensen, 2007, p. 47). Specifically, the modelling competency is characterised as:

> A person's insightful readiness to autonomously carrying through all aspects of a mathematical modelling process in a certain context and to reflect on the modelling process and the use of the model (Blomhøj & Kjeldsen, 2010, p. 3)

Thus, "insightful" and "reflect on" indicates that, for teaching purposes, abilities beyond mere mechanical skills are sought in students. The student must be able to construct models, and "autonomously" at that. Furthermore, it is emphasised that the student in this regard be able to consider a model and the process of constructing it *holistically*, not simply as separate algorithmic steps. Finally, the student must be capable of reflecting on "the use of the model", which can be understood as having capabilities for both *internal* and *external* reflections:

> Internal reflections add meaning and quality to the sub-processes involved in a mathematical modelling process, (...) the external reflections address the role and function of the model in actual or potential applications. (Blomhøj & Kjeldsen, 2011, p. 1)

Purposes

In (Blomhøj, 2009, p. 6) the main purposes for teaching and learning mathematical modelling, as given by the above definitions, are divided into three categories. Below these are described in a shortened, partly reformulated, but content preserving form:

- *Proficiency:* To learn modelling, it being a subject of the mathematical discipline and to provide motivation and support for learning of other mathematical topics.

- *Applications:* To apply mathematics in dealing with the challenges of private and professional life.

- *Cultivation:* To strengthen students' competences as critical and insightful participants of a highly technologically developed democratic society.

MODELLING AS PORTRAYED IN CURRICULUM

In this section references to modelling and models in Danish curriculum for mathematics Stx A, B and C levels are explained (Danish Ministry of Education [DME], 2013, Bilag 35–37). The levels A, B, and C refer to upper secondary mathematics education. A level is the highest.

Under "disciplinary goals" (DME, 2013, Bilag 35, 2.1) models and modelling are mentioned several times. Exact quotes are not given here, but are presented in synthesised form in Table 1 illustrating the results of the analysis.

The curriculum subsection "Supplementary material" proposes teaching modelling on B level, and differential equation models on A level. This supplementary material must "provide perspective and depth to the central material" (DME, 2013, Bilag 35, 2.3).

Under "Interaction with other disciplines" it is pointed out that material must be included so that the student can understand "the significance of considering and discussing the preconditions for any mathematical description of reality, and the validity of results obtained hereby" (DME, 2013, Bilag 35, 3.4).

MODELLING IN THE TEXTBOOK

The B level mathematics textbook "Gyldendals Gymnasiematematik Grundbog 2B" (Clausen, Schomacker, & Tolnø, 2007b) contains the chapter "Models" (Danish: "Modeller") of 37 pages, wherein modelling and models are treated.

The entire textbook series, this book included, introduces models when treating a diversity of other subjects (variable correlations, functions, etc.). Many of the examples given there are revisited in the analysed section, focusing now on their specific qualities as models and the construction of these by modelling. In the following, Clausen and colleagues (2007b) is referenced whenever only page numbers are stated in parenthesis.

The textbook chapter commences with a motivational text giving a number of examples of models (nine pages). Thereafter follows the theory on modelling (two pages). It is emphasised that "The mathematical model describes a real-world situation" (p. 91) that "may have limited range" (p. 91), and that it can "give insight into, and overview of, the real-world situation that is described" (p. 91).

A simplified version of the modelling cycle is shown illustrating the sub-processes "Translation to mathematics", "mathematical treatment" and "translation to reality". These steps are presented in a non-cyclic, algorithmic manner, connected by single-headed arrows (p. 91).

Throughout the succeeding two pages, examples and exercises – by references to the exercise book (Clausen, Schomacker, & Tolnø, 2007a, pp. 27–33) – are given on model analysis. In particular, emphasis is put on identifying variables and parameters, respectively, and on their range: "Consequently, (...), x must take values between 200 and 1200" (p. 92).

The succeeding 21 pages contain more examples and exercises, now with emphasis on constructing models – modelling. All six sub-processes from Figure 1 are visited in linear succession in nearly every example. With several examples it is clarified that a model can be generalised: "Considering an alternative thickness for the paper sheet (...)" (p. 92), and "It is easy to generalise the model to considering alternative widths of the canal (...)" (p. 98). However, no examples or exercises are given wherein a point is made of progressively redefining real-world situations to be modelled. The approach is algorithmic rather than holistic.

The exercises focus on the student activity of extracting and interpreting information from a model, as well as constructing models. As with the examples, no exercises are given wherein the student must assess or reflect on the quality or validity of a model.

The remaining three pages of the chapter consist of a text considering existing "professional" models, and how their construction and use depend on applicants' interests. For example: "An oil company and an environmental organisation will hardly construct and interpret a model in the same way." (p. 116).

WHAT IS MODELLING? THREE POINTS OF REFERENCE

In this section the content extracted in the above three sections is organised and illustrated in Table 1. The table is organised so that distinct aspects of modelling may be viewed comparatively in terms of their depiction in each of the three levels of the educational system analysed.

Firstly, extracts from the analysis have been sorted under either purposes of *applications* and *proficiency*, or, under the purpose *cultivation* (see "Academic view on modelling" above). Secondly, four colours; purple, red, green, and blue highlight the four main comparable aspects of modelling that I wish to emphasise. These four points focus on portrayals of the modelling cycle (purple), the modelling process (blue), internal critical reflections (red), and external critical reflections (green).

DISCUSSION

This discussion sets out first to explore what versions of modelling are portrayed in the specific case elaborated above. This is done relative to the scope on academic literature on modelling that the case presents. Subsequently, possible causes for the gap between modelling in academic literature and textbooks are discussed. Lastly, it is suggested that a version of modelling better fit for implementation might be conceived of by considering the institutional structure of secondary school.

Lack of holistic and critical perspectives

In the textbook only examples and exercises are given where models are considered as results of following certain mathematical calculations in linear order (see blue and purple coloured text in Table 1). This approach is predominantly algorithmic as it never invites students (and teacher) to consider the model *holistically*: perspective is in every example and exercise confined to sub-processes of the modelling process.

Note that a holistic understanding of the modelling process was emphasised in literature (as presented in the case) for being essential in developing *autonomous* competencies. This is so since autonomy is understood as deriving from seeing both what is to be done to solve a problem, and why. An algorithmic viewpoint hence confines students and teacher to considering what step to do next, not what it means for the model in its entirety. As a consequence, the form of modelling portrayed in the textbook does not promote well autonomous competences.

Also, *critical* competences are underemphasised (see red and green coloured text in Table 1). In the textbook, assessment of a model's validity and range amounts merely to examining ranges of variables and parameters. This is a devaluing interpretation of the curriculum statement on understanding the "range of models". Indeed, consider the notion of *critical reflections* in the scope of the case on academic literature. In this view, "range of models" is understood as what the model in its entirety explains, its descriptive power and its limitations, not simply the range of variables used in it.

A text example is offered in the textbook describing how an oil company and an environmental organisation might interpret a model differently. Assessing others' use of models, which academic literature emphasises as part of modelling competency, is therefore somewhat an element of the modelling subject por-

	Literature	Curriculum	Textbook
Purpose declarations	-Applications - Proficiency	- Knowledge of math's usability in formulating and treating problems from other disciplines	No indication
Specific points of note	- Complex modelling cycle with double-headed arrows - Holistic perspective on modelling process - Internal critical reflections on models and model construction	- Skills in constructing and using models (geometrical, statistical, simulations) - Knowledge of basic properties of models and modelling - Reflection on idealisations and range of models	- Linear modelling cycle with single-headed arrows - Iterative modelling process. No examples or exercises displaying a holistic perspective - Examples and exercises on range of variables in models
Purpose declarations	- Cultivation	- Knowledge of mathematics' interaction with culture, science, technology	No indication
Specific points of note	- External critical reflections	- Reflection on assumptions and validity of mathematical descriptions (under "interaction with other disciplines")	- Text example of an oil company and an environmental organisations' different interpretations of a model

Table 1: A case of 'mathematical modelling' in three levels of the educational system

trayed in the textbook. Yet, no techniques to making these critical model assessments are offered.

Hereby, model criticism seems but a matter of examining ranges of functions and variables, not assessing its significance for the real-world situation it is supposed to describe. Thus, the *cultivation* purposes of academic literature, and the corresponding purpose declarations of the curriculum are underemphasised, if present at all, in the textbook. That is, the use and role of models in decision making is hardly part of the modelling subject portrayed in the textbook.

A fragmented version of modelling

The lack of holistic and critical perspectives in the textbooks indicate an overall conception of modelling as a *mathematical* technique to solve mathematically stated problems, not real-world problems. This may seem natural as it is a topic presented in a mathematics textbook. Yet, it is worthwhile to ponder the consequences of portraying this mathematics-centred version of modelling.

In the scope on academic literature presented in the case, holistic and critical perspectives are integral aspects of the modelling process and competency. These aspects, however, necessitate an approach to modelling that goes beyond mathematical algorithms and concepts. When transposing the subject into teaching material for mathematics class proper, we might suspect that the subject that results becomes centred on the mathematics. That is, aspects that go beyond that discipline become underrepresented. We saw in the above that this suspicion seems justified in the case studied.

Under which subject, then, do the holistic and critical aspects of modelling sort, if not under the subject known as modelling in mathematics textbooks?

The curriculum phrasing indicates a hint. Note that it is stated in Table 1 (in green) that reflections on assumptions and validity of mathematical descriptions sort under 'interaction with other disciplines'. Thus, it seems reasonable to assume that these aspects of modelling are considered content to be taught in interdisciplinary work.

Hence, it is reasonable to suggest that modelling in upper secondary are in fact two subjects: a mainly mathematical one found in mathematics textbooks, and one pertaining to extra-mathematical aspects found in interdisciplinary teaching material. This might seem reasonable as the teacher can use different material – different literature – when teaching modelling.

Yet, assuming that modelling should be a subject that in its very nature aims to *connect* the mathematical and the extra-mathematical, this fragmentation is problematic. Indeed, reflecting on the extra-mathematical situation that one aims to model, constructing a suitable model, and then assessing its validity is a process of *connecting* mathematical and the extra-mathematical domains.

However, this situation is only problematic when seen relative to academic perspectives on modelling. Modelling as a subject seems to find its own form in practice, one that fits the institutional structure, e.g., with subjects divided in disciplines and activities considered to be either mathematical or interdisciplinary. Understanding better this version of modelling-in-practice, and the processes that determine the form it takes is what this paper has aimed at.

CONCLUSION AND PERSPECTIVES

This study set out to investigate the gap between research on modelling and its degree of implementation in school. Rather than viewing the gap as an issue, it was considered a *condition* of connecting research to practice.

From this viewpoint, the question was not which obstacles lay in the way of implementing modelling in practice, and how they may be overcome. Rather, it was to understand how, and why, the subject of modelling takes the form it does in textbooks and curriculum.

Departing from a specific case, it was discussed how modelling as portrayed in textbooks and curricula differs from modelling as envisioned by researchers. In particular, it was indicated that this subject, modelling-in-practice, is fragmented into two subjects. One to be taught in mathematics class proper focusing on the mathematical work in modelling activities, and one aimed at interdisciplinary work focusing on critical reflections and holistic aspects of the modelling process.

It was argued that this fragmentation is a natural consequence of the division of activities common in secondary school, between mathematical and interdisciplinary activities. Taking this situation to be a condition rather than an issue (a viewpoint that must also be taken!), it is reasonable to ask whether the version of modelling to be implemented must be reshaped into a form more fit to the common secondary school structure. Indeed, could research provide a version of modelling that in distinct ways targets mathematics class proper, interdisciplinary work, as well as activities that bridge the two?

To be sure, researchers must attempt to implement the version of modelling that they see fit, and ensure that the teaching institution is changed accordingly to accommodate this subject. Yet, the point made in this paper is that investigating the version of modelling that actually occurs in teaching materials, one might see that the particular vision of modelling one has could be fitted better to the institutional structure of school.

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TWG06

Posters

The Fraunhofer MINT-EC Math Talents Programme

Martin Bracke, Patrick Capraro, Jana Kreckler and Andreas Roth

University of Kaiserslautern, KOMMS, Kaiserslautern, Germany, roth@mathematik.uni-kl.de

Fraunhofer MINT-EC Talents supports talented, selected students from ages 16 to 18 and encourages them to develop their own scientific research interests and projects. Those projects are suggested and designed by the students as part of their independent work. It is important, that each project contains theoretical mathematical work alongside technical aspects like the control and construction of electric devices. Members of KOMMS support them by providing mathematical and technical expert knowledge and soft skill training during several workshops and by offering infrastructure for communication and computing.

Keywords: STEM, talent, secondary education, workshops.

SETUP

The programme is a cooperation between KOMMS (Competence Centre for Mathematical Modelling in STEM Projects at Schools, http://komms.uni-kl.de), the ITWM (Fraunhofer Institute for Industrial Mathematics, http://www.itwm.fraunhofer.de), and MINT-EC (Association of Mathematical and Technical Excellence Centres at Schools, http://www.mint-ec.de). MINT-EC certifies secondary schools (Gymnasia), which exhibit a distinctive profile in STEM disciplines. Talented students at ages 16 are selected from these schools, and get the opportunity to work on mathematical-technical projects together during several workshops, parallel to their standard secondary education. The project does not have to end with their final exam; given the quality of the developments made in the two-year course, participation at competitions like “Jugend forscht” (“Youth researches” – a German youth science competition, http://www.jugend-forscht.de) is possible and desired. The number of project groups and their participants is determined in an initial workshop. We make sure, that the challenges of each topic include the necessity of mathematical modelling as well as the actual implementation of software and construction of devices. It is important, that the students themselves decide how to approach their projects and which resources are needed to proceed. The supervisors, who are KOMMS personnel, can then offer support from their experience with modelling and solution of real-world problems. Fraunhofer ITWM provides financial support to purchase necessary hardware components, offers infrastructure for communication via the myTalent portal and also pays for accommodation and transport during workshops. Due to a sponsoring agreement with MathWorks, the students in the current programme have the opportunity to work with the Matlab and Simulink Student Suite. This tool provides enough abstraction from the hardware layer, so the students can focus more on mathematical challenges and have to deal with less implementation issues. During the workshops, the supervisors or external experts offer compact courses for skills in several programming languages or for different types of hardware, if needed. The poster includes a graphical representation of the activities belonging to the MINT Talents programme and displays the involved parties and their tasks.

PROJECT EXAMPLES

Some project examples are explained with concise statements and pictures.

Playing billiards with electronic assistance

A computer software shall help a billiard player to decide which ball is the easiest to play next. A camera is used to capture the development of the game, hence it is necessary to get acquainted with basic methods of image processing. A mathematical model has to be established to rank different possible moves. As an extension, the software should be able to screen the players' skills in order to generate a profile and to give personalised suggestions.

Automatic steering of model quadcopters
A quadcopter model aircraft is supposed to navigate autonomously in a specified region and to recognise certain objects. Mathematical methods for image processing and strategies for autonomous navigation have to be developed. A swarm model can also include more than one aircraft.

GOALS

Students should emerge from the programme as self-confident young individuals, who will be able to do excellent research in the future. Acquiring expertise in project-oriented work is also the basis for putting their own ideas into market some day. But most importantly, the students learn to work with applied mathematics intuitively and produce visible results from a science, which is often considered to be of an abstract and not application-oriented type. This experience is in general not achieved by means of classic school teaching. The workshop character of the programme additionally teaches a variety of soft skills, which can come in handy in the future academic or job-related lives of the students.

A didactic problem around the elementary differential calculus and functional modelling

Catarina Lucas[1], Josep Gascón[2] and Cecilio Fonseca[1]

1 University of Vigo, Department of Applied Mathematics I, Pontevedra, Spain, catarinalucas.mail@gmail.com, cfonseca@uvigo.es

2 Autonomous University of Barcelona, Faculty of Sciences, Department of Mathematics, Barcelona, Spain, gascon@mat.uab.cat

This work is part of a PhD thesis in the field of the Anthropological Theory of the Didactic: "Se propone un modelo epistemológico de referencia (MER) del cálculo diferencial elemental (CDE) alternativo al modelo dominante en el sistema escolar." It proposes an alternative reference epistemological model to the dominant model of elementary differential calculus (EDC) for Secondary school systems. We assume that its rationale arises in the context of functional modelling (FM).

Keywords: Calculus, modeling, anthropological theory of the didactic, dimensions of a didactical problem.

The didactic problem approached will be here characterized by three fundamental dimensions: *epistemological*, *economic* and *ecological* (Gascón, 2011):

EPISTEMOLOGICAL DIMENSION

What is the rationale of EDC in the transition secondary school-university?

How to interpret and describe EDC linked to FM?

The *Activity Diagram* of the FM constitutes the basis for answering these questions (Figure 1).

ECONOMIC DIMENSION

How is EDC organized and managed in the Portuguese educational system? What is the rationale currently assigned to it? Which were the transpositives changes (Chevallard, 1985) suffered by its role in the FM activity to pass from the scientific community to the education system?

Some answers to these questions emerged from an historical evolution analysis of the EDC role in the Portuguese curriculum and the potential relationship FM-EDC. So, we characterized the EDC 'official' rationale in secondary-university as a way to study isolated functions with a weak connection to FM.

ECOLOGICAL DIMENSION

What conditions are needed to develop functional modelling in the transition secondary-university? *Dimensión ecológica:*What role could play the EDC to establish these conditions? What restrictions could hinder its development?

To partially answer these issues a *study and research path* (Chevallard, 2009) was designed and experimented, on the basis of a part of the proposed reference epistemological model.

ACKNOWLEDGEMENT

1. Funded by the FCT PhD fellowship (SFRH/BD/77335/2011) and the Project *"La modelización matemática para la formación del profesorado de secundaria: del algebra al cálculo diferencial"* (EDU2012-39312-C03-03).

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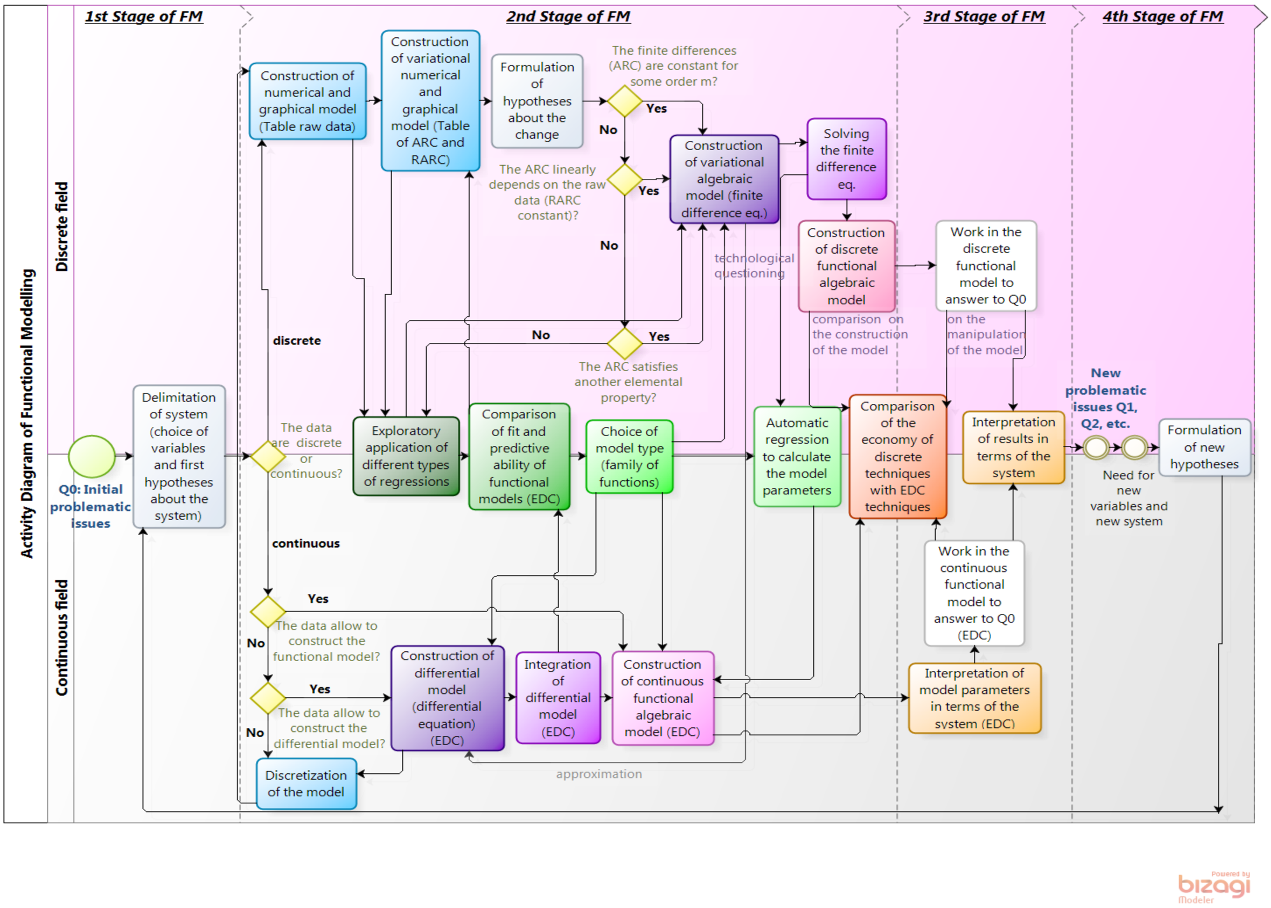

Figure 1: Activity diagram of functional modelling

Educativa, *14*(2), 203–231. Retrieved from http://www.redalyc.org/articulo.oa?id=33519238004

Learning the concept of family of functions through the modelling process using tablets

Miriam Ortega and Luis Puig

University of Valencia, Faculty of Education, Valencia, Spain, miriam.ortega@uv.es, luis.puig@uv.es

In this study, we present a teaching situation to work on the modelling process using real data taken in classroom with tablets. The situation allowed us to work on the concept of the quadratic function with a group of 11th grade students and take into account the hypothesis that the qualitative analysis of the phenomenon, the knowledge of the qualitative properties of the families of functions and the meaning of the parameters are crucial elements for the management and control of the modelling process. We observed they are key elements to choose the function used as a model and to check its adequacy throughout the modelling process. Some student's conceptions on height and time were found and analysed.

Keywords: Modelling process, functions, real data, tablet, problem solving.

INTRODUCTION

We present the design and some results of a pilot study on the teaching of the concept of family of functions through a modelling process. In the teaching situation, real data are taken in classroom using the possibilities given by apps for *iPads*®, and data are processed using free apps. In particular, the situation studied is the bounce of a ball dropped from a certain height, restricting the model to a single bounce, i. e., since the moment it touches the ground for the first time until it touches it again.

The teaching situation takes also into account the hypothesis that the qualitative analysis of the phenomenon, the knowledge of the qualitative properties of the families of functions and the meaning of the parameters are crucial elements for the management and control of the modelling process, and includes this metacognitive element in the design of the materials, as we have done in a previous study (Puig & Monzó, 2013).

This teaching model was explored with a group of 11th grade students with the following aims: (a) Check the influence of the qualitative analysis in the management and control of the process and (b) Explore the behaviour and the previous ideas of the students.

MATERIALS AND METHODS

Our study is globally organised by the theoretical and methodological framework of the Local Theoretical Models (Filloy, Rojano, & Puig, 2008).

The pilot study was made in a natural group of 11th grade science students in València, Spain. They hadn't worked on the modelling process before, but they have been taught with a problem solving methodology.

The teaching experiment was carried on in two sessions and a set of interviews. In the first session, the students were given a worksheet with a set of questions that begins by asking to draw a sketch of the graph they expected to find when making the experiment, in order to explicitly include in the teaching the control of the modelling process by the qualitative analysis. After that, they represented and recorded the phenomenon studied using *Video Physics*®. In the second session, they introduced the data obtained from this app in *Data Analysis*® to choose the function which fits better, and they answered some other questions related to the phenomenon represented using *Free GraCalc*®, which is a graphing calculator. In the interviews they were asked about their answers recorded in the worksheets, and were given some hints to get over the difficulties observed.

The results of the research were obtained from two sources. On the one hand, the data from the two worksheets and from the tablets were analysed by doing a rational reconstruction of the problem solving process. On the other hand, the data from the interviews were analysed in detail to get more information about the answers' origin and the students' conceptions.

SOME RESULTS AND CONCLUSIONS

Our study supports the hypothesis that the qualitative analysis of the phenomenon, the knowledge about the meaning of the parameters and the qualitative properties of the families are decisive elements in the control and management of the modelling process. Specially, they are key elements to choose the function used as a model and to check its adequacy throughout the modelling process.

Regarding students' conceptions, we have found a deeply rooted idea that the height can't take negative values, and the idea that time is absolute.

ACKNOWLEDGEMENT

This is a result of a research program (EDU2012-35638) and a PhD grant (BES-2013-063826) both of them funded by the Ministerio de Economía y Competitividad (Spain).

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TWG07

Mathematical potential, creativity and talent

Introduction to the papers of TWG07: Mathematical potential, creativity and talent

Roza Leikin[1], Alexander Karp[2], Demetra Pitta-Pantazi[3] and Florence Mihaela Singer[4]

1 University of Haifa, Haifa, Israel, rozal@construct.haifa.ac.il
2 USA/Russia, apk16@columbia.edu
3 Cyprus, dpitta@ucy.ac.cy
4 Romania, mikisinger@gmail.com

This was the third CERME at which Thematic Working Group (TWG) "Mathematical potential, creativity and talent" took place. One of the central goals of this TWG was to raise attention of the mathematics education community to the field of mathematical potential, creativity and talent, and to promote empirical and theoretical research that will contribute to the development of our understanding in the field. In 2011, 25 participants from 14 countries discussed 15 contributions. This time (2015), 35 participants took part in the TWG discussions about 24 contributions by researchers and practitioners from 15 countries from different parts of the world. The TWG facilitated communication between educational researchers, mathematics educators and research mathematicians focusing on the nature and nurture of mathematical creativity in all students and high mathematical ability in particular individuals.

Following the debates at TWG07 at CERME7 and CERME8, we continued an international exchange of ideas related to the research into the identification of mathematical talent, the didactics of teaching highly able students as well as the promotion of creativity in all students.

Four main topics framed the TWG discussion: Topic 1: Creativity – definitions, identification, development; Topic 2: Mathematical tasks for different levels of abilities and expertise; Topic 3: High mathematical abilities: Characteristics and components; Topic 4: Teacher preparation: teaching for creativity, teaching the gifted. In order to disseminate the ideas of different participants, each session started with small group work followed by a whole group discussion. Thus, the TWG concentrated on the exchange of the ideas rather than on individual presentations.

Topic 1: Creativity – definitions, identification, development

During two sessions devoted to Topic 1, the discussion focused on the relationship between general creativity and mathematical creativity following the study of Kattou, Christou and Pitta-Pantazi. The researchers demonstrated that mathematical creativity exists as a distinct type of creativity and concluded that no predictions can be made on general creativity based on mathematical creativity and vice versa. Bronislaw Czarnocha suggested that "bisociation and simultaneity of attention" could serve a useful theoretical framework (based on Arthur Koestler's ideas expressed in his monograph "Act of creation") for the analysis of creativity and argued that mental flexibility is not related at all to creative processing. In contrast to this study, Rott analysed heuristics and mental flexibility in the problem solving processes of regular and gifted fifth and sixth graders. The study of Desli and Zioga directed the attention of the group's participants to the creativity embedded in primary school mathematical tasks which are useful for educating teachers in the management of creativity in their classes. The authors argued that prospective teachers mostly connect mathematical creativity to arousing student interest whereas in-service teachers connect it to book-oriented problem posing indicating their narrow and blurred perceptions.

While Winkler and Brandl suggested process-based analysis of mathematically gifted pupils in a regular class at the primary school level, Karakok, Savic, Tang and El Turkey suggested a model for the analysis of mathematicians' views on undergraduate students' creativity that addresses flexibility as one of the central characteristics of creativity. They suggested the following rubric for mathematical creativity: Creative thinking, risk-taking, innovating thinking, connecting and transforming, originality and aesthetics of the solutions. Based on the work of Monstad, participants discussed mathematical practices involved in mathematical creativity. Émin, Essonnier, Filho, Mercat and Trgalova analysed didactical contract negotiation and technology assigned to creativity. They suggested additional ways of fostering mathematical creativity through the implementation of electronic books.

Obviously the studies varied in their definitions of creativity, research methodologies, target populations and the ways in which creativity was evaluated and the proportion between cognitive and social characteristics addressed. No agreement among the researchers was achieved on the definitions of creativity and the relationships between creativity and ability grouping. This variety of views stressed the importance of further research in the field.

Topic 2: Mathematical tasks for different levels of abilities and expertise

Task design and use is a core issue in research in mathematics education. In the context of the TWG "Mathematical potential, creativity and talent" the design of mathematical tasks for the identification and development of mathematical potential and creativity attracted the attention of all participants. Leikin and Elgrably analysed integrative investigation geometry tasks as a tool for the development and evaluation of creativity for pre-service mathematics teachers. They argue that problem solving expertise influences mental flexibility which is reflected in the richness of discovered properties as well as investigation strategies. Pitta-Pantazi, Christou, Kattou, Sophocleous and Pittalis suggest a system of mathematical competences for the evaluation of the mathematical challenge embedded in the mathematical tasks: Digital, social, communication in mother tongue, learning to learn and sense of initiative. Study participants found "digital competence" and "learning to learn the most difficult aspects to incorporate in their mathematical tasks". Palha, Schuitema, van Boxtel and Peetsma discussed the effect of high versus low guidance structured tasks on mathematical creativity. The study extended the previous research on mathematical creativity by accounting for the relationship between the learning environment and creativity and by providing a way to operationalize fluency and flexibility in conceptual mathematical terms. Safuanov, Atanasyan and Ovsyannikova argued that open-ended exploratory learning in the mathematics classroom is an effective tool for the development of knowledge and creativity, while Schindler and Joklitschke suggested criteria for designing tasks according to students' capabilities with special attention to mathematically talented students. Singer, Pelczer and Voica explored behaviours of 10 to 16 year old high achievers during a problem modification process. The researchers studied the relationship between students' creative approaches and the quality of the mathematics problems they generated, and concluded that these children either show low amplitude driven creativity, or, when they try to be more creative, they fail to pose consistent problems that make use of deep structures of mathematical concepts and strategies. Mellroth presented the Kangaroo Mathematical Competition as a very effective tool for the popularisation of mathematics among students with a wide range of mathematical abilities. In turn, Bureš and Nováková explored developing students' culture of problem solving via heuristic solving strategies. The participants discussed the construct of heuristic strategies.

Mathematical investigations, challenging problem solving and problem posing were the most common tasks considered by the TWG07 participants. The tasks were considered in different contexts: Development of creativity in students and teachers, assessing creativity, and teacher education. The participants raised the following questions for further investigation: What is done in teacher education internationally in order to assist prospective teachers to develop creativity? What are textbooks doing to develop creativity in different countries? Can "problems from the scientific studies" be used in regular classes?

Topic 3: High mathematical abilities: Characteristics and components

High abilities in mathematics is an ill-defined concept. There was no agreement between the group participants on who are the students with high ability in mathematics. Baruch-Paz, Leikin and Leikin analysed cognitive characteristics (with a focus on Speed of Information Processing) among students with superior mathematical performance. This research involved 190 students from four groups of 16–18 years old participants varying in levels of general giftedness and excellence in mathematics. The researchers demonstrated that the differences between the groups are task dependent; that is, between-group differences vary among different cognitive skills. They argued that excellence in school mathematics differs from mathematical giftedness. In contrast to the previous study that focused on the basic cognitive traits as related to mathematical abilities, Benölken's study explored emotional factors, namely, he investigated the impact of interest in Mathematics on the identification of girls' mathematical talent. In this study, the impact of mathematical interest and attitudes were used in order to identify girls' mathematical talent. Boys showed a stronger interest in mathematics compared to girls among those who were not identified as mathematically talented. Motivational factors were shown to be critical for the identification of mathematical talent.

Nordheimer and Brandl presented challenges facing the identification of mathematical giftedness in students with hearing impairment. The researchers claimed that for teachers, it is vital to recognise their pupils' diverging abilities effectively in order to meet their respective learning requirements. They compared empirically the possibilities of identifying mathematical giftedness of 3rd/4th graders by combining written tests and by process-based analyses of lessons. Szabo examined the interaction of mathematical abilities and the role of mathematical memory in students with high achievements in mathematics. The study demonstrated that mathematical memory has a critical role in the choice of problem-solving methods. The study showed that if the initially selected methods do not lead to the desired outcome, the students find it very difficult to modify them. The study confirms some qualitative differences in problem solving between high-achievers who are not essentially mathematically gifted and mathematically gifted students. However, the inflexibility of the participants could also be explained by two main functions of the cerebral cortex, where working memory operates. Winkler and Brandl argued that with heterogeneity at schools growing, individualisation of education has become more important than ever. For teachers, it is vital to recognise their pupils' diverging abilities effectively in order to meet the pupils' respective learning requirements. The researchers presented a project aiming at showing ways to identify and characterise mathematically gifted pupils during regular lessons.

An interesting discussion developed during this session. Just as at the first two sessions devoted to the studying nature and nurture of mathematical creativity, the researchers did not agree on who the mathematically gifted are, or what are the indicators of high mathematical abilities. Moreover, one of the participants in the group argued that "such a thing as giftedness does not exist". After the discussion, it was clear that more attention should be paid, through systematic research to the following questions: Can giftedness be developed or is it an innate trait? How can mathematical giftedness be determined? Are school achievements predictors of mathematical giftedness? Is mathematical creativity a part of mathematical giftedness?

Topic 4: Teacher preparation: Teaching for creativity, teaching the gifted

It is clear to every mathematics educator that teachers' expertise, knowledge and skills determine the quality of mathematics teaching. This is equally true when teaching the gifted or when goals of teaching mathematics include the development of creativity in all students. Karp and Busev devoted their study to people who taught highly gifted schoolchildren and developed educational materials for them. They claimed that teaching the highly gifted requires the teachers' creativity the development of which can be provoked and supported by the social environment. To exemplify their ideas, researchers discussed two prominent Russian figures in the advanced course of mathematics. In contrast, Birkeland analysed mathematical reasoning of pre-service teachers through the distinction between imitative and creative reasoning. This study reveals variety of the types of students' mathematical reasoning and indicates that if their reasoning is not imitative it is perhaps not creative either. Safuanov's work is devoted to different ways of teaching students in pedagogical universities to solve various types of non-routine mathematical problems, while Sinitsky's work aimed to highlight the different aspects of promoting creativity from the point of view of an educator involved in the professional development of pre-service teachers. Challenging the prospective teachers with open-ended mathematical problems provided data on their beliefs and behaviour concerning creativity and creativity encouragement in the classroom.

Overall, participants agreed that the development of teachers' understanding of the special characteristics of gifted students is essential for teaching the gifted, and that the development of teachers' creativity in mathematics teaching is essential for the development of teachers' overall creativity.

TWG07

Research papers

The impact of mathematics interest and attitudes as determinants in order to identify girls' mathematical talent

Ralf Benölken

University of Münster, Münster, Germany, rben@wwu.de

In proportion, girls often are decidedly underrepresented in support programs that aim at mathematically talented primary school children. Thus, it is of interest to ascertain aspects that might make possible a more differentiated identification and support. In the following article, a questionnaire study will be presented which can clarify the significance of mathematics interest and attitudes as determinants for the identification of mathematical talent: Boys and girls who were identified to be mathematically talented, and boys who were not showed a stronger mathematics interest (in and beyond the classroom) and more advantageous mathematics attitudes compared to girls who were not identified to be mathematically talented.

Keywords: Mathematical talent, mathematical giftedness, interest, attitudes, gender.

INTRODUCTION AND RATIONALE

In Germany just like in other western European countries, girls are in proportion decidedly underrepresented in programs that foster mathematical talent (Benölken, 2011). This phenomenon contradicts the consensus on the fact that both sexes have equal potentials across all academic domains (Endepohls-Ulpe, 2012). When it comes to primary school children, aspects such as gender stereotyping of mathematical occupational fields cannot really act as possible explanations, especially because there cannot be found any gender-specific differences in mathematical competencies at this age (Lindberg, Hyde, Petersen, & Linn, 2010). In addition, studies have indicated a decline of such differences at subsequent ages for many years (Hyde, Lindberg, Linn, Ellis, & Williams, 2008). This is why it is of interest to look for aspects that improve the identification of girls' mathematical talent (see [1]). With a holistic approach, diagnostics should be organized as a process considering both cognitive and co-cognitive, e.g. motivational, parameters as determinants in order to identify talents. For instance, girls and boys who were identified to be mathematically talented ("imt") as well as boys who were not ("n-imt"), often show more advantageous self-concepts and attributions in mathematics than n-imt girls (Benölken, 2014). Findings like these raise the question how other motivational factors can be characterized in view of these groups. In this article, the significance of both mathematics interest and attitudes as determinants for the identification of mathematical talent at primary school age will be examined by a questionnaire study. Its aim is to investigate boys' and girls' frequent characteristics as to these factors by a comparison of the four groups mentioned above. Based on literature reviews, hypotheses on the characteristics in question will be deduced that correspond to the questions of the study. Afterwards, the design and the results of the study will be reported.

BACKGROUND

Theoretical frameworks

Mathematics interest: The conception of interest applied in the study refers to Prenzel, Krapp and Schiefele (1986): Interest is seen as a result of an interaction between a person and an object that – along with adjuvant conditions – might cause to focus on a long-term preoccupation with this specific object. This relation is characterized by (1) value-related, (2) affective and (3) cognitive aspects. Additionally, in accordance with current approaches on a multidimensional structure of interests, a distinction between subject-, context- and topic-related interest was considered (Krapp, 2010). The first two dimensions were summarized in the term of "mathematics interest in the classroom" because it cannot be expected that primary school children differ between activities and contexts applied in classrooms (Hellmich, 2006). The third one is referred to by the term "mathematics interest beyond the classroom".

Mathematics attitudes: The construct of "attitudes" focuses on an evaluation of objects which an individual imagines or perceives in his or her environment. Attitudes can be explicitly and consciously accessed, or they can emerge implicitly and spontaneously – in both cases influencing an individual's behavior (Bohner, 2003). The conception of attitudes applied in the study refers to the classical operationalization consisting of (1) cognitive, (2) affective and value-related as well as (3) behavior-related components (Aronson, Wilson, & Akert, 2004).

Brief literature reviews

Preliminary notes: Research on both mathematics interest and attitudes mostly focuses either on children at middle school age or on gender-specific differences without regarding specific aspects of giftedness or talent. Studies which investigated mathematics interest or attitudes in the context of exceeding abilities mostly refer to "giftedness" as a "g-factor-concept" implying standardized diagnostics. Thus, their results cannot be transferred automatically to "mathematical talents" regarding domain-specific criteria and implying long-term process diagnostics (see [1]). The findings collectively show, however, the significance of both factors as determinants for the identification of girls' mathematical potentials. Therefore, they are suited to provide a basis for the intended deduction of hypotheses.

Mathematics interest: Primary school children often have a lot of interests like sports, TV, computer games or reading (Pruisken, 2005). Furthermore, gender-specific differences can already be found at this early age (Hoberg & Rost, 2000): horseback riding, animals or reading seem to be "typical" interests of girls; football, technics or computer "typical" interests of boys (Fölling-Albers, 1995). Boys more often show stronger mathematics interest – even at primary school age and both in and beyond the classroom; girls interest in language or literature (Hellmich, 2006; Pruisken, 2005). Though gifted children show the same differences, they do not have any extraordinary interests compared to non-gifted children. However, gifted children generally seem to be more interested in both mathematics and languages or literature (Pruisken, 2005). In contrast to non-gifted girls, gifted girls have more interests which are supposed to be "typical" interests of boys, and they have a larger spectrum of interests than gifted boys (Kerr, 2000). Regarding specific mathematical talents (in the sense of [1]), girls (irrespective of the identification of talent) more often show a larger spectrum of interests than boys (Benölken, 2014). The majority of primary school children does not differ between mathematics interest in the classroom and beyond the classroom (Hellmich, 2006). However, current studies do not focus on gender- or giftedness- respectively talent-specific aspects in this context. Furthermore, there are only very few studies with a focus on ability-related mathematics interest. Their findings indicate, that the mathematics interest of students with lower achievements exceeds that one of higher achievers (Frenzel, Goetz, Pekrun, & Watt, 2010), but these studies do not focus on gifted or talented students. Finally, an often reported phenomenon is a decline in mathematics interest in the years of adolescence (Fredricks & Eccles, 2002), which is of little importance when conducting studies with primary school children.

Mathematics attitudes: Boys show advantageous mathematics attitudes more often than girls (Hyde, Fennema, Ryan, Frost, & Hopp, 1990). As to the cognitive aspect, studies primarily focus on individuals' assessments of usefulness and difficulty of mathematics. There seem to be no gender- or talent-specific differences between imt and n-imt children regarding usefulness (Benölken, 2011), but some studies indicate that mathematically gifted boys and girls as well as non-gifted boys ascribe mathematics a lower level of difficulty compared to non-gifted girls (Wieczerkowski & Jansen, 1990). Finally, there are findings on gender stereotypes: The older girls are the more they ascribe mathematics to males (Newton & Newton, 1998), which seems to be less important at primary school age, since such differences mostly appear from an age of ten onwards. Concerning the affective aspect, results on gender- or giftedness-specific differences of individuals' intrinsic values (such as enjoying mathematical task solving) seem to play the most important role: Similar to characteristics of the assessment of mathematics' difficulty, some studies show that mathematically gifted boys and girls as well as non-gifted boys show a higher intrinsic value doing mathematics compared to non-gifted girls (Wieczerkowski & Jansen, 1990). On the other hand, studies indicate that boys in general ascribe a higher intrinsic value to mathematics than girls (Bos, Wendt, Köller, & Selter, 2012). As to the behavior-related aspect, boys seem to engage in mathematics beyond mathematical school lessons more often than girls (Schiepe-Tiska & Schmidtner, 2013).

THE STUDY

Questions

The study was designed to answer the question how mathematics interest and attitudes can be characterized with the regarded groups. The following hypotheses were deduced from the theoretical findings: (1a) Imt girls and boys as well as n-imt boys show a stronger mathematics interest in the classroom than n-imt girls. (1b) Imt girls and boys as well as n-imt boys show a stronger mathematics interest beyond the classroom than n-imt girls. (2) Imt girls and boys as well as n-imt boys show more advantageous mathematics attitudes than n-imt girls.

Design

The study adds to previous research on the significance of motivational factors as determinants for the identification of mathematical talent using questionnaires that are appropriate to primary school children, and that can be completed within a short time (e.g., Benölken, 2011; 2014). Operationalizations of mathematics interest in and beyond the classroom as well as of attitudes were tested within pilot studies.

Sample and procedure

The sample contains N=162 children of the third and fourth grade (71 girls, 91 boys). The subsample of imt children is n=83 (32 girls, 51 boys). Children who are assessed as "imt" take part in a project that fosters mathematical talent at the University of Münster called "math for small pundits". They were chosen by long-term process-diagnostics that are a synthesis of standardized and non-standardized tools (see [1]; Benölken, 2014). The sample contains n=79 n-imt primary school children (39 girls, 40 boys) from common classes. The probands were questioned during the school year of 2014/2015. All procedures of questioning were consistent: The children were told how to fill in the questionnaire. In this context, possible differences between mathematics interest in and beyond the classroom were emphasized (see [2]). The children completed the questionnaire on their own without any time limit (no one took more than ten minutes and no one refused to fill in the questionnaire).

Method

Apart from declaring sex, the questionnaire was anonymized. The phrasing of all items (following styles of common operationalizations in each case) was formulated in German. In order to measure mathematics interest in the classroom by a value-related, an affective and a cognitive aspect, the following instruction was given: "This is about mathematics in the classroom. Mark with a cross a statement that you think fits best to you: (1) Mathematics in the classroom is really important to me. (2) I always look forward to mathematics in the classroom. (3) I am interested in mathematics in the classroom." An analog instruction was composed to collect data about mathematics interest beyond the classroom: "This is about mathematics beyond the classroom. Mark with a cross a statement that you think fits best to you: (1) Mathematics is really important to me. (2) I always look forward to doing mathematics. (3) I am interested in mathematics." In order to measure attitudes by cognitive, affective and behavior-related aspects, the following instruction was given: "Mark with a cross a statement that you think fits best to you: (1) Mathematical tasks are sometimes too difficult. (2) I enjoy doing mathematics. (3) I engage in mathematics beyond mathematical school lessons." To evaluate the items, in each case a four-step Likert-scale was offered ("that's not correct", "that's almost not correct", "that's almost correct", "that's correct"; instead, the children could choose "I don't know").

Evaluation

Statements about all items except the one relating to cognitive attitudes were translated into numbers from 1 ("that's not correct") to 4 ("that's correct"). As to the cognitive attitude-item, the assignment was turned around: "that's not correct", e.g., was translated into 4 and "that's correct" into 1, because statements that focus on a low level of difficulty reflect advantageous characteristics of attitudes. Regarding the mathematics-interest-in-the-classroom-scale, the coefficient of correlation as defined by Pearson between the included items moves in a range from .366 to .475 (with $p<.01$ in each case) and the internal consistency is only just acceptable (Cronbachs α=.680). As to the mathematics-interest-beyond-the-classroom-scale the coefficient of correlation as defined by Pearson between the included items is in a range from .378 to .576

(with $p<.01$ in each case) and the internal consistency is between acceptable and good (Cronbachs α=.731). Finally, the coefficient of correlation as defined by Pearson between the included attitudes items moves in a range from .334 to .617 (with $p<.01$ in each case) and the internal consistency is between acceptable and good, too (Cronbachs α=.710). In all cases, the items have been combined to one scale with mean values. Data have been evaluated by an analysis of variance with two factors ("talent" and "sex") to find significant differences between the four groups. In addition to that, η^2-values have been calculated to see the possible importance of both the factors and their interaction by their effect size. The requirements of the statistical procedure need the independence of subsamples and a normal distribution of the regarded trait within the groups amongst homogeneity of variance: The subsamples are obviously independent because of the distinction between sex and talent-identification. As a consequence of a graphical analysis of the distributions-histograms and the corresponding quantile-quantile-plots, the data are leptokurtic, but sufficiently similar to normal distributions (Hatzinger & Nagel, 2009). The requirement of homogeneity of variance is statistically firm as a result of Levene-testings.

RESULTS

Mathematics interest

Regarding *mathematics interest in the classroom*, the averages of imt boys and girls are relatively similar, while the value of n-imt boys is slightly larger and the value of n-imt girls is slightly lower (Table 1). There is no significant main effect on talent ($F(1,158)<.001$, $p=.990$, $\eta^2<.001$), but there can be found a significant main effect on sex ($F(1,158)=12.795$, $p<.001$, $\eta^2=.075$) as well as a significant effect of interaction ($F(1,158)=4.139$, $p=.044$, $\eta^2=.026$). As indicated by η^2-values, sex (medium effect of 7,5%) plays a bigger part to explain variance than the interaction (medium effect of 2.6%). Thus, the boys' groups, especially the n-imt boys, show a stronger mathematics interest in the classroom compared to the girls' groups, but as indicated by the significant effect of interaction, imt girls are more similar to the boys' groups than to the n-imt girls, who show a lower mathematics interest in the classroom on average compared to all other groups. Therefore, the statistical evaluation confirms hypothesis 1a in principle.

As to *mathematics interest beyond the classroom*, the averages of imt children and n-imt boys are very similar and exceed the value of n-imt girls (Table 1). There are significant main effects on talent ($F(1,157)=10.579$, $p=.001$, $\eta^2=.063$) and sex ($F(1,157)=8.435$, $p=.004$, $\eta^2=.051$) just as there is a significant effect of interaction ($F(1,157)=7.579$, $p=.007$, $\eta^2=.046$). As indicated by η^2-values, talent (medium effect of 6,3%) and sex (medium effect of 5.1%) play a similar role to explain variance. Thus, imt children and n-imt boys show similar characteristics of mathematics interest beyond the classroom which is stronger compared to n-imt girls, i.e. hypothesis 1b is confirmed. In addition, a descriptive data analysis of all groups' mean values regarding both mathematics interest in and beyond the classroom indicates that only imt children seem to differ between these dimensions, since the values of n-imt boys and girls are quite similar in each case, while imt children's mathematics interest beyond the classroom is stronger than in the classroom.

Mathematics attitudes

As to mathematics attitudes, the mean values of imt boys, imt girls and n-imt boys are relatively similar, even though the value of imt boys is slightly larger than the values of imt girls and n-imt boys. The value of n-imt girls is clearly lower in comparison to all other groups (Table 2). There are significant main effects on both talent ($F(1,158)=29.023$, $p<.001$, $\eta^2=.155$) and sex ($F(1,158)=21.550$, $p<.001$, $\eta^2=.120$). Finally, there is a

	mathematics interest in the classroom		mathematics interest beyond the classroom	
	boys	girls	boys	girls
imt children	3.07 (.83)	2.89 (.51)	3.39 (.70)	3.37 (.48)
	n=51	n=32	n=51	n=32
n-imt children	3.30 (.76)	2.65 (.72)	3.33 (.79)	2.74 (.55)
	n=40	n=39	n=40	n=38

Table 1: Averages (standard deviations) of mathematics interest-statements

significant effect of interaction ($F(1,158)=7.597$, $p=.007$, $\eta^2=.046$). As indicated by η^2-values, talent (strong effect of 15,5%) and sex (medium effect of 12.0%) play a similar role to explain variance, even though the talent effect is stronger. Thus, attitudes of imt children are more advantageous compared to n-imt children, but n-imt boys merely differ a little from the imt children. The statistical evaluation confirms hypothesis 2.

DISCUSSION

Synopsis: In this article, the significance of both mathematics interest – by a distinction between in and beyond the classroom – and attitudes as determinants for the identification of mathematical talent at primary school age was investigated by a comparison of frequent characteristics with boys and girls who were identified to be mathematically talented (imt) as well as with boys and girls who were not (n-imt). Based on a review of existing empirical evidence, hypotheses on the mentioned characteristics were deduced: It has to be expected that imt children and n-imt boys show a stronger mathematics interest and more advantageous attitudes than n-imt girls. The hypotheses were investigated by a questionnaire study. The statistical results confirm the assumptions in principle: First, the boys' groups, especially n-imt boys, show a stronger mathematics interest in the classroom compared to the girls' groups, while imt girls are more similar to the boys' groups than to n-imt girls, who show a lower mathematics interest regarding this aspect than all other groups (similar to Pruisken, 2005). Second, imt children and n-imt boys show a stronger mathematics interest beyond the classroom than n-imt girls. In addition, only imt children seem to differ between mathematics interest in and beyond the classroom showing stronger interest beyond the classroom, while n-imt children on average took similar stances in both cases (which could explain the results of Hellmich, 2006). Finally, attitudes of imt children are more advantageous compared to n-imt children, but n-imt boys merely differ a little (similar to Wieczerkowski & Jansen, 1990).

A deeper interpretation of the results indicates that all groups show a relatively strong mathematics interest and advantageous attitudes, even though the values of n-imt girls are lower compared to the other groups in each case. Regarding the significance of mathematics interest and attitudes for the identification of talent, the results insinuate that both a particularly strong mathematics interest (in and beyond the classroom) and advantageous attitudes can be found – independent of the identification of talent – more often with boys, while girls who have been identified to be mathematically talented are very similar to these groups. An observable stronger mathematics interest, especially in the classroom, and more advantageous attitudes might cause more efficient diagnostics of boys' talents, because they might tend to a strong preoccupation with mathematics, or teachers might perceive their potentials primarily. By contrast, both lower mathematics interest, especially in the classroom, and disadvantageous attitudes might lead to the fact that children do not develop a stronger preoccupation with mathematics and, e.g., turn to different interests. This might also apply to children who have a high potential that might be more difficult to identify. Though the findings are not suitable to predict how mathematics interest and attitudes can be characterized with mathematically talented but not identified girls, they imply the following thesis: Disadvantageous characteristics of mathematics interest and attitudes are important aspects effecting a more infrequent identification of high potentials with girls. In addition, interest and attitudes have to be seen in a strong interdependence with other motivational factors as well as with influences of socialization or gender-specific preferences in solving tasks (Benölken, 2011, 2014).

As to *limitations of the study and directions for future research*, within the imt group the underrepresentation of girls has to be discussed: Because of the rare

	boys	girls
imt children	3.32 (.58)	3.11 (.60)
	n=51	n=32
n-imt children	3.03 (.78)	2.23 (.76)
	n=40	n=39

Table 2: Averages (standard deviations) of mathematics attitudes-statements

identification of mathematical talent with girls, it takes a long time to compose suitable subsamples (in particular, by process diagnostics). Nevertheless, despite the relative imbalance within the imt group, the size of all subsamples is sufficient in principle, even though follow-up studies should enlarge all subsamples and ensure a balance. The diagnostics procedures of talent identification that are used to compose the imt subsample have been established for many years. Thus, "imt" children most probably are rightly assessed in that way. In addition, there might be motivational effects caused by their participation in "math for small pundits" that cannot be found with children who have high potentials, but who are not taking part in such a program. Finally, the subsample of n-imt children is nothing more than an insufficient image of population. Thus, the sample's representativeness has to be seen as limited. The questionnaire was adequate to the aims of the study in principle. It is suited for a pragmatic use in classrooms because its design is appropriate to children, and it can be completed in a short time. However, mathematics interest and attitudes are strongly reduced in their conceptions, and the evaluation depends on very simple measurements. The external validity of the findings cannot be judged because tools that evidentially regard criteria of quality were not applied (in favor of the appropriateness to young children) and because the imt sample is very specifically composed. In sum, the study has obvious limitations, and it rather has an explorative character. Despite the significant results, subsequent studies might focus on a deeper clarification using established tools.

As to a survey of exemplary *practical consequences*, first, any gender stereotyping of mathematics should be avoided. Second, the development of mathematics interest and advantageous attitudes seems to play an important role for girls in order to support their potentials to emerge. In this context, e.g., task-fields that are composed to foster girls especially – without stereotyping – might be useful (Benölken, 2013). The distinction between mathematics interest in and beyond the classroom that was observed with imt children indicates the significance of a challenging education, e.g., by using enrichment tasks in common classes (Fuchs & Käpnick, 2009).

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ENDNOTES

1. According to Fuchs and Käpnick (2009), "mathematical talent" is seen as an above-average potential regarding the criteria of Käpnick (1998), i.e., remembering mathematical facts, sensitivity and fantasy, structuring and transferring structures or reversing thoughts. This potential is characterized by individual determinants and a dynamic development depending on inter- and intrapersonal influences in interdependence with personality traits supporting the talent.

2. As to the distinction between the interest dimensions, the questionnaire instructions contained the following elucidation (translated from German): "I would like to know how you like mathematics in and beyond the classroom. 'Mathematics in the classroom' focuses on everything you do in mathematical school lessons. 'Mathematics beyond the classroom' focuses on, e.g., mathematical activities or themes in your life beyond mathematical school lessons or even outside the school."

Pre-service teachers' mathematical reasoning

Alv Birkeland

UiT The Arctic University of Norway, Tromsø, Norway, alv.birkeland@uit.no

The focus of this study is the mathematical reasoning of pre-service teachers. One class of pre-service teachers preparing to teach from grade 5 to 10 was organized in small groups where they worked on certain mathematical exercises. While working on these exercises the students were video and audio recorded. The dialogues of each group constitute the unit of analysis. The research framework used in this study distinguishes between imitative and creative reasoning. This distinction is based on the idea that rote learning is imitative, while the opposite kind of reasoning is creative. However, this study reveals some of the variety of the students' mathematical reasoning and indicates that if their reasoning is not imitative it is perhaps not creative either.

Keywords: Pre-service teachers, imitative reasoning, non-imitative reasoning, creative mathematical reasoning, creativity.

INTRODUCTION

To foster creativity among students in general, it would probably be useful to have teachers that could engage students in creative mathematical work. Therefore it might be of interest to study the mathematical work of pre-service teachers. Thus knowledge about pre-service teachers' mathematical reasoning would be useful. To know if some of their reasoning could be characterized as creative or not would be valuable. Such knowledge would perhaps make it possible to prepare pre-service teachers better for their future work. This is what motivates this study.

Skovsmose (2001) distinguishes between the exercise paradigm and landscapes of investigations. Within the exercise paradigm the textbook is central for classroom practice. The relevance of the exercises is not part of the mathematics lesson, and there is one and only one correct answer to each exercise. In contrast to the exercise paradigm, landscapes of investigations is an investigative approach where students are involved in processes of exploration and explanation. Skovsmose makes the point that traditional mathematics education often falls within the exercise paradigm. If students' work with mathematical exercises falls within the exercise paradigm and essentially involves copying the solutions they find in the textbook, their reasoning can hardly be characterized as creative. To copy or imitate solutions from the textbook would be what Lithner (2008) labels as imitative reasoning. He denotes the opposite kind of reasoning as creative reasoning. Haylock (1987) is concerned with creativity in school mathematics and makes the point that overcoming certain kinds of fixations is essential. He calls overcoming fixations "flexibility". Sriraman (2009) has investigated the work of research mathematicians and defines creativity to be the ability to produce novel or original work. The novelty of students would normally be at a personal level only, which is called relative creativity by Leikin and Pitta-Pantazi (2013). To copy the solutions in the textbook might be a normal procedure for students of mathematics except perhaps for students at an advanced or graduate level (Lithner, 2004). An undergraduate textbook giving several examples of solutions to the mathematical exercises in the book, is perhaps asking the students to reason imitatively (Lithner, 2008) rather than to engage them in an investigative approach (Skovsmose, 2001). However if the students cannot find a solution in the textbook to copy, the situation is different. To find a solution to an exercise would then probably require relative creativity (Leikin & Pitta-Pantazi, 2013).

One class of pre-service teachers, preparing to teach students from grade 5 to 10, participated in a study of creative mathematical reasoning. The participating students were not selected for any kind of mathematical giftedness or excellence. The class was organized in small groups and given some mathematical exercises to work on. The number of students in each group varied from two to four. The topic was basically number theory and the exercises were part of a course. The students were recorded on video and audio, and transcripts of the students' dialogues were prepared and analyzed. It was hoped that the dialogues of the students would reflect the actual mathematical reasoning of the students, and perhaps reveal more than what their written works only would have done. This led to the following research question:

Is Lithner's (2008) distinction between imitative and creative reasoning sufficient to analyze pre-service teachers' mathematical reasoning, or can some of their reasoning be neither imitative nor creative?

REVIEW OF THE LITERATURE

In a review of the literature mainly from English speaking countries, Haylock (1987) is concerned with creative thinking in school mathematics. The review indicates that both overcoming fixations and the ability for divergent production are essential components in any assessment of mathematical creativity. One aspect of creative reasoning that would be relevant for mathematical work would thus be to overcome fixations or rigidity. Haylock suggests two key aspects of fixations or rigidity in mathematical reasoning. One is called content-universe fixation where the reasoning is unnecessarily restricted to an insufficient range of elements that may be used or related to the mathematical situation. The other kind of fixation is called algorithmic fixation, where the reasoning continually adheres to an initially successful algorithm even when this becomes less than optimal. The counterpart of fixation or rigidity is called flexibility. In divergent production tests the common element is that the subject is given a mathematical situation with many responses. The creativity of the responses in such tests is conventionally assessed by evaluating them in terms of the number of responses (fluency), the number of categories of responses (flexibility) and originality (the statistical infrequency of the responses). The opposite of divergent thinking is convergent thinking where the subject is supposed to find a single solution to a given problem.

Sriraman (2009) investigated the work of five creative mathematicians. The study indicated that in general, the mathematicians' creative process followed the four stage Gestalt model of preparation-incubation-illumination-verification (Wallas, 1926) as indicated by Hadamard (1954). According to this model the creative process starts with a period of preparation where in spite of hard work over a period of time apparently no results are achieved. In a period of incubation the problem is left aside and partly forgotten. However, even though the problem is put aside for some time, it is not completely forgotten and it is thought that the mind is occupied with the problem, but in a subconscious way. Later in a moment of illumination an idea comes up which possibly solves the problem. This idea may seem to come more or less out of the blue. Therefore the moment of illumination is also seen as a result of the work of the subconscious mind. Finally, it is necessary to verify the solution. Incubation and illumination may be the work of the subconscious mind. However, both preparation and verification obviously take place in a fully conscious way.

Leikin and Pitta-Pantazi (2013) reveal that the relationship between creativity and giftedness is complex. Some researchers claim that creativity is one form of giftedness, whereas others feel that creativity is an essential part of giftedness, and still other researchers suggest that creativity and giftedness are two independent characteristics of human beings. A distinction is made between relative and absolute creativity. Creativity is relative if the creativity is at a personal level only, as opposed to absolute creativity where creativity is regarded as novel to the professional community. Students' ability to produce solutions to mathematical exercises that are new to the students only would typically be relative creativity, whereas new mathematical discoveries such as those awarded the Abel Prize would be seen as the result of absolute creativity. Researchers have different focuses on where the creativity lies. The focus is either on the creative person, the creative process, the creative product or the creative environment. Research studies that focus on the creative person deal with individuals' cognitive and personality traits. Other research studies focus on the way creative work is produced such as the four stage Gestalt model of Wallas (1926). Research studies that focus on product concentrate on ideas translated into tangible forms. Researchers that focus on environment concentrate on where the creative person acts. In educational settings this could be the educational environment where the creative activity takes place and where the creativity is studied.

Lithner (2004) gives a detailed description of how exercises in undergraduate calculus textbooks may be solved by mathematically superficial strategies. A distinction is made between intrinsic and surface mathematical properties of the components involved in the reasoning. An intrinsic mathematical property is central to the problem as opposed to a surface property which has little or no relevance to the problem (Haavold, 2013). When considering surface mathematical properties it is not necessary to understand the central mathematical ideas and analyze the consequences of their properties. Bergqvist (2007) has classified tasks and task solutions from all introductory calculus courses at four Swedish universities during the academic year of 2003/2004. The analysis shows that about 70% of the tasks do not require creative reasoning. All exams except one were possible to pass without the use of creative reasoning of any kind. In one quarter of the cases it was possible to pass exams with distinction without using creative reasoning of any kind. Lithner (2008) has introduced a research framework for creative and imitative reasoning. The basic idea behind this framework is that rote learning reasoning is imitative while the opposite type of reasoning is creative. The characteristic for imitative reasoning is that the reasoning individual is copying solutions e.g. by looking at a textbook example or remembering a textbook algorithm. The opposite kind of reasoning is called creative mathematically-founded reasoning. This kind of reasoning is characterized by novelty, plausibility and that the reasoning is mathematically founded.

The research framework introduced by Lithner based on empirical data is applied in this study. This framework identifies two types of mathematical reasoning called imitative reasoning (IR) and creative mathematically-founded reasoning (CMR). More precisely imitative reasoning is based on imitating or copying a line of reasoning laid out step by step for the student. On the other hand, the characteristics of creative mathematically-founded reasoning are (Lithner, 2008, p. 266):

1) Novelty. A new (to the reasoner) reasoning sequence is created, or a forgotten one is re-created.

2) Plausibility. There are arguments supporting the strategy choice and/or strategy implementation motivating why the conclusions are true or plausible.

3) Mathematical foundation. The arguments are anchored in intrinsic mathematical properties of the components involved in the reasoning.

The components involved in the reasoning could be objects such as numbers, functions and matrices, (Haavold, 2013). According to Haavold the notion of plausibility in this framework is inspired by Polya (1954). Polya makes the point that there are two kinds of mathematical reasoning. There is demonstrative reasoning and there is plausible reasoning. Demonstrative reasoning is what mathematical proofs are made of, whereas the reasoning used to solve a mathematical problem or find a proof is plausible. According to Polya, the result of the mathematician's creative work is demonstrative reasoning. However, to find a solution to a mathematical problem plausible reasoning is used.

METHODOLOGY

The design of this study was to divide one class of pre-service teachers into small groups and give them some mathematical exercises. The video and audio recordings of the work of each group constitute the data of the study. The exercises were selected from the course. Thus the mathematical work of the teaching experiment would also be relevant for the students. The students were in fact preparing for their exam doing the exercises. It was not expected that the students would solve the exercises using imitative reasoning only. The reason for this was that although the students were given an idea on how to get started, they were not given a complete solution. The episode from the video recording was chosen because it indicates that when pre-service teachers' reasoning is not imitative, it is perhaps not creative either. Similar mathematical reasoning was found in many of the groups involved in the study. However, one example was chosen for this paper to show what was found.

The term commognition has been coined by Sfard (2008) meaning a combination of communication and cognition. This means that interpersonal communication and individual thinking are two facets of the same phenomenon. Thinking is defined by Sfard as the individualized version of interpersonal communication. In this paper we view reasoning as a form of thinking. Thus we study the mathematical reasoning of each individual by studying the interpersonal communication of each group. Therefore the unit of analysis is the dialogue of each group. The idea behind this approach is that the dialogue of each group would perhaps reveal more mathematical reasoning than the written works only would have done.

The analysis was based on the research framework of Lithner (2008), which makes the distinction between imitative and creative reasoning. However, the analysis of the dialogues indicated that some of the students' reasoning was not compatible with Lithner's distinction. The analysis indicated that if the students' reasoning was not imitative it was perhaps not creative either. Therefore it became interesting to analyze the students' reasoning if it was neither imitative nor creative and find a way to characterize it. Hence, in order to analyze pre-service teachers' mathematical reasoning a new distinction of reasoning was introduced. In addition, new categories were introduced.

ANALYSIS

One of the problems the students worked on was the sequence (a_n) starting with the terms 0,4,10,18,28,40... The students were asked to find an expression for a_n. Let us start by looking at the dialogue of one of the groups with three students. The instructor made the students familiar with the idea that they could write down the differences between consecutive terms of the sequence to get a set of equations that could be added. The video shows that when the dialogue begins the students have written down the following equations:

$$a_2 - a_1 = 4 = 2 \times 2$$
$$a_3 - a_2 = 6 = 2 \times 6$$
$$a_4 - a_3 = 8 = 2 \times 4$$
$$\ldots$$
$$a_n - a_{n-1} = 2n$$

Adding these equations, the students arrived at the equation:

$$a_n - a_1 = 2 \times 2 + 2 \times 3 + 2 \times 4 + \ldots + 2n$$

Thus the students did what one would expect if they were to follow the line of reasoning given to them by the instructor. Therefore their reasoning was probably imitative so far (Lithner, 2008). We enter the dialogue with the following episode from the video. The numbered transcription of the episode is given with some comments.

Episode:

1. Katherine: And then we can write 2 outside,
2. Elizabeth: And a_1 is zero, we don't need it, we can simply skip it.

Following this dialogue the students have written down the equation:

$$a_n = 2(2 + 3 + 4 + \ldots + n)$$

In lines 1 and 2 Katherine and Elizabeth use formulations such as "we can write 2 outside" or "we can simply skip it" as opposed to formulations such as "what do we have to do here?" or "what are we supposed to do here?" This could mean that the students make their own choices about what to do, rather than asking themselves what they are supposed to do which would be characteristic for imitative reasoning. If the students make their own choices then their reasoning would not be imitative. The dialogue of the episode continues as follows:

3. Jennifer: We are now looking for the triangular numbers.
4. Elizabeth: Hm.
5. Katherine: No, plus n, the sum of the n first positive integers, so it is really the triangular numbers we are looking for, but can we...?
6. Jennifer: We are missing 1.
7. Katherine: We are missing 1, yes if we add...
8. Elizabeth: Add 1 to each side.
9. Katherine: We have to add 2...2...2 times 1... to both sides, because we have the number 2...Yes, if we try that, add 2 times 1, then you get a_n plus 2 times 1 equals 2, and then we get 1 plus 2 plus 3 plus 4...plus n.
10. Elizabeth: Yes, we do.
11. Katherine: Yes.
12. Jennifer: Can we just add like that?
13. Katherine: Yes we may add to both sides.
14. Jennifer: Yes, and then we just have to...

Finally, the video recording shows that the students arrive at the following equation:

$$a_n + 2 \times 1 = 2(1 + 2 + 3 + \ldots + n)$$

As the students are familiar with triangular numbers, the exercise is now resolved.

Jennifer continues the dialogue in line 3 by making the point that they are looking for the triangular numbers. After having agreed that they should look for the triangular numbers, Jennifer begins a line of reasoning in line 6 by observing that they are missing the number 1. She is obviously referring to the fact that the sum within the brackets should have started with the number 1. Katherine suggests in line 7 that they should add something. This is followed up in line 8 by Elizabeth saying that they should add 1 to each side of the equation. However, in line 9 Katherine introduces a different idea saying that they should add 2 times 1 to each side of the equation. Thus the students change their point of view while reasoning, which would not be typical for imitative reasoning. Instead, changing point of view characterizes flexible reasoning (Haylock, 1997). The formulation "Yes, if we try that" used by Katherine in line 9 further indicates that the students are trying out their own ideas rather than following step by step a given line of reasoning, and thus that their reasoning is not imitative (Lithner, 2008). Hence, their reasoning should not be characterized as imitative but, rather, as flexible.

When Katherine introduces her idea in line 9 she shows little uncertainty. In fact she says: "Yes, if we try that, add 2 times 1, then (...)" thus perhaps showing some confidence. Elizabeth easily accepts the idea of Katherine in line 10. Only Jennifer hesitates a little in line 12 (Birkeland, 2013) but accepts the idea in line 14. Thus the idea they use is introduced rather smoothly. This would hardly be the case if the idea was new to the group. Therefore nothing indicates that the idea has novelty to the group. If the students' reasoning is based on a relational understanding (Skemp, 1978) of the components involved, then it is reasonable to assume that it is mathematically founded (Lithner, 2008). This would indicate that the reasoning of the group is mathematically founded, has flexibility (Haylock, 1987) but no novelty or originality.

DISCUSSION

The first part of the students' reasoning was to write down the differences between consecutive terms of the given sequence to get a set of equations that could be added. The students were made familiar with this idea by the instructor. Therefore the first part of their reasoning was probably imitative (Lithner, 2008).

The second part of the students' reasoning was not laid out step by step for them. The instructor did not give them any hints. The students tried out certain ideas they had and finally chose their own line of reasoning. If their reasoning was based on their own choices then it should not be characterized as imitative. Assuming that their reasoning was based on relational understanding (Skemp, 1978) it was probably both plausible and mathematically founded. Further, nothing indicated that their reasoning had novelty. However, their reasoning was found to have flexibility. Probably the students were working flexibly with familiar lines of reasoning.

One may argue that flexible reasoning is part of creative reasoning. However, if creativity is the ability to produce novel or original work (Sriraman, 2009), then flexibility alone is not sufficient for the reasoning to be creative. Therefore the students' reasoning should perhaps not be characterized as novel or creative. Consequently the first part of the students' reasoning could be said to be imitative, however the last part of their reasoning would be neither imitative nor creative as defined by Lithner (2008).

The episode chosen for this paper was quite typical for the mathematical reasoning of most of the groups involved in the study. It appears that they all followed the line of reasoning given to them by the instructor on how to get started. Therefore, for all groups the first part should be characterized as imitative reasoning. However, having started each group followed their own line of mathematical reasoning, varying slightly from one group to another. The reasoning of all the groups except for one should not be characterized as imitative. Their reasoning had flexibility but nothing indicated novelty. Only one group continued with imitative reasoning. The video shows that this group found an earlier example to compare with. The group appears to have found similarities between the two examples. However, the similarities should be characterized as surface similarities. Therefore their mathematical reasoning should be characterized as superficial reasoning.

Hence, according to this study, an analysis of the mathematical reasoning of pre-service teachers should be based on a distinction between imitative reasoning (IR) and non-imitative reasoning (NIR). Imitative reasoning is defined by Lithner (2008) as the kind of reasoning where each element of the reasoning is laid out step by step for the reasoning subject. Non-imitative reasoning is introduced in this paper as the kind of mathematical reasoning which is not imitative.

It is quite possible that the imitative reasoning of the first part was based on a relational understanding (Skemp, 1978). However, if the reasoning is based on surface property considerations (Lithner, 2008) only, it should be characterized as superficial reasoning (SR).

This study indicates that imitative reasoning may be mathematically founded or plausible (IMR). However, imitative reasoning may also be based on surface property considerations only (SR). Hence, imitative mathematical reasoning may or may not be mathematically founded or plausible.

Non-imitative reasoning (NIR) can hardly be based on surface property considerations only because that kind of reasoning would always involve some elements of copying. If plausible reasoning (Polya, 1954) is about supporting conjectures, then plausible reasoning would not be imitative reasoning. Therefore plausible reasoning could be an example of non-imitative reasoning (NIR). The kind of non-imitative reasoning found in this study is characterized by the fact that it is plausible and mathematically founded (Lithner, 2008), has flexibility (Haylock, 1987) but no novelty. This kind of non-imitative reasoning is called flexible mathematically founded reasoning (FMR) in this study.

FINAL REMARKS

This study shows that to analyze the mathematical reasoning of pre-service teachers, it might be useful to base the analysis on a distinction between imitative and non-imitative mathematical reasoning (NIR) rather than imitative and creative reasoning (Lithner, 2008). Imitative reasoning includes both superficial reasoning and imitative mathematically-founded reasoning (IMR). Non-imitative reasoning (NIR) includes both flexible reasoning (FMR) and creative mathematically-founded reasoning (CMR). Thus non-imitative reasoning may or may not involve novelty. If novelty is essential to the concept of creativity as defined by Sriraman (2009) then non-imitative reasoning would include mathematical reasoning that is neither imitative nor creative. Hence, to analyze pre-service teachers' mathematical reasoning, Lithner's (2008) distinction between imitative and creative reasoning might not be sufficient. Some of the students' reasoning might be neither imitative nor creative.

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Developing students' culture of problem solving via heuristic solving strategies

Jiří Bureš and Hana Nováková

Charles University in Prague, Faculty of Education, Prague, Czech Republic, jiri.bures@pedf.cuni.cz

The goal of our long-term project presented in this paper is to develop students' culture of problem solving based on using heuristic solving strategies. In this paper, we describe the method of implementation of the strategies in class preceded by a detailed a priori analysis and followed by an a posteriori analysis and a comparison between the teacher's expectations and the reality in the class. We focus on the variables of a problem to be treated in class that may influence the role and use of the problem in developing students' creativity and their culture of problem solving.

Keywords: Problem solving, word problems, heuristic solving strategies, creativity, a priori analysis.

INTRODUCTION

Problem solving is generally acknowledged as one of the key features of mathematical education. The role and the use of problem solving tasks vary with different learning situation. Our observations and some studies (Novotná, 2000) show that, in secondary schools in the Czech Republic, problem solving is generally used as a tool for applying basic mathematical concepts, such as linear equations or systems of linear equations. Teachers often show an algorithmic way to solve a class of problems and students are lead to identify and to use this method to solve other problems. This approach to problem solving in school mathematics can create an illusion of students' good abilities to solve various problems; however, it does not fulfil the potential of word problems to promote creativity and ability to solve non-standard or real-world problems.

The research presented in this paper is focused on the culture of word problem solving based on a different perspective and way of using word problems and solving strategies in school mathematics. One of the main goals of the research is to investigate heuristic solving strategies and problem types that support students' ability to apply the acquired knowledge and skills in various situations to work creatively in mathematics. According to Silver (1997) or Kopka (2010, Foreword), we believe that solving carefully selected problems may help to develop and cultivate students' creativity. By students' creativity in problem solving, we understand the ability of finding non-standard ways of solution as well as the ability to find more than one method of solving a problem.

The key concept of our research is the culture of solving mathematical problems, which is understood as a structure of internal factors that influence a pupil's performance and success in problem solving. Based on the four categories of skills needed to have success in mathematics (Schoenfeld, 1985), the culture consists of four components: intelligence, creativity, ability to use existing knowledge and reading comprehension skills (Eisenmann, Novotná, & Přibyl, 2014).

The main research activities were: designing problem solving situations based on problems encouraging the students to use non-standard solving strategies, a detailed a priori analysis of these problems and using these problems and strategies in different classes. The implementation of the situations in class was preceded and followed by psychological screening, by testing the ability to use existing knowledge and by evaluation by the teacher of mathematics (see, for example, Eisenmann, Novotná, & Přibyl, 2014). The psychological screening consisted of a test of intelligence for the category of age 12–18, the test of ability of reading with comprehension and the Christensen-Guilford test of creativity measuring fluency, flexibility, originality and elaboration.

In this paper, we focus on the method of designing a problem-solving situation based on a detailed a priori analysis used in the research. The realization of a problem-solving situation in class is preceded by the *a priori* analysis and followed by the *a posteriori* analysis. The objective of the a priori analysis is to give, in advance, a detailed description of a teaching unit and to predict as accurately as possible the course of this unit, students' and teacher's attitudes and reactions, solving strategies and knowledge prerequisite for the use of the solving strategies (Brousseau, 1998; Novotná, Nováková, 2014). In the a posteriori analysis, the a priori analysis is compared with the experience of the realization of the situation in the class. The comparison is based on the observation of a video sequence from the lesson and on the discussion with the teacher. Recommended changes arise from this comparison.

The problem solving situation described in the paper is one of many situations during the period where the teacher teaches the students to use various solving strategies. In order to promote creativity in problem solving, four heuristic solving strategies were treated in the class in this situation. Two strategies were presented by the students (Direct method – graphical representation – solution drawing and Direct method without graphical representation) for Problem 1 and the second strategy was generalized and used to solve Problem 2. After solving both problems, the use of the strategy of analogy for solving a general problem was discussed in the classroom.

We used the following problems in the situation:

Problem 1. We have to dispose 24 stakes around a square ground. How many stakes at most can we dispose on each side of the ground if there has to be the same number of stakes on all sides?

Problem 2. We have to dispose c stakes around a square ground (c is a number divisible by 4). How many stakes at most can we dispose on each side of the ground if there has to be the same number of stakes on all sides?

Problem 2 represents in fact a set of problems created from Problem 1 by changing the variable of the number of stakes. The set of problems issued from changing a variable of the given problem is called a cluster of problems and represents a possibility to motivate the students to use various solving strategies.

A PRIORI ANALYSIS OF PROBLEM 1

The main purpose of this paragraph is to describe in detail the a priori analysis of Problem 1. Most of the characteristics of Problem 2 are similar, with the exception of the solving strategies. Since the problems are given to the students together as a cluster of problems, we believe that the solution of Problem 1 will precede the solution of Problem 2. If Problem 2 is posed to the students separately, the solving strategies will probably start with a specification of the number of stakes if the student is not able to solve the general problem.

The a priori analysis of Problem 1 is described below.

Nature of the assignment

Knowledge prerequisite to grasping the problem: general expressions (stake, around a ground, side of a ground), mathematical expressions (square, at most, the same)

Potential problems in comprehension of the assignment: We think that younger students might have a problem with the expression "at most" in the formulation of the question. It would be better to use the word "maximally". We also think that the formulation "How many stakes at most can we dispose on each side of the ground" is not exact and we propose to reformulate it as "How many stakes can one see..."

Thematic unit

Word problems (possible resolution with a linear equation or without an equation)

Goal of the problem

Goal for the students: to find the maximal number of stakes that one can see on one side of the garden

Didactical goal for the teacher: to find out if the students are able to represent the situation in a picture, to identify their solving strategies and to make a generalization of the problem with the students

Time needed for the solution of the problem

We estimate 10 minutes for the solution of the problem.

Class management

Individual work or work in pairs or groups. Homework or work in class.

Aids
paper, pen, squared paper

Cognitive didactic variables
(squares) The ground might have another form (triangle, irregular ground etc.).

(24) The choice of the number is crucial for choosing a solving strategy. With a small number like 4, the problem can be presented to young students; however, with a big number like for example 444, students cannot use graphical representation and the difficulty of the problem rises significantly.

(same) We can modify the problem by giving a given difference of the number of stakes on each side of the ground.

Formulation variables
There are the following formulation variables: (dispose), (stakes), (on each side), (number), (how many).

Students' reactions and attitudes
We suppose that the problem may be interesting for the students because of its real-life context. We think that they might have some problems with understanding the formulation of the problem.

Teacher's reactions
We suppose that the problem may be interesting for teachers because of its real-life context and its potential of modification and generalization.

Correct solving strategies
1) Systematic experimentation

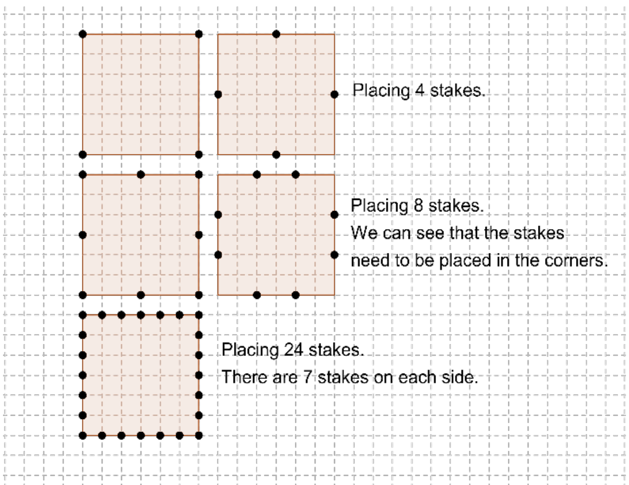

Figure 1: Systematic experimentation

Systematic experimentation leads us to a graphical representation of the situation. It leads to the fact that there must be a stake in each corner to get the maximum number of stakes on each side. Since we dispose of 24 stakes, there are 7 stakes on each side.

Knowledge needed: square and its properties, addition of natural numbers

Possible difficulties: lack of system in the experimentation, clarity of the drawing, idea of placement of stakes in the corners

2) Direct method – graphical representation – solution drawing

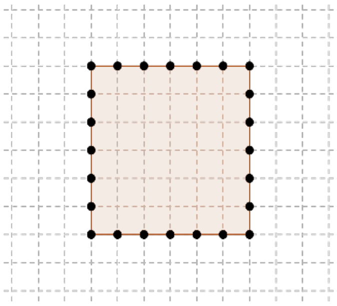

Figure 2: Direct method – graphical representation – solution drawing

There are 7 stakes on each side.

Knowledge needed: square and its properties, addition of natural numbers

Possible difficulties: lack of system in the experimentation, clarity of the drawing, idea of placement of the stakes in the corners

3) Analogy

We simplify the solution of the problem by choosing a smaller number of stakes (for example 4 or 8 stakes) and we solve the analogical problem. The choice of 4 stakes or 8 stakes leads to the idea of placing the stakes in the corners of the ground.
A stake placed in the corner counts for two sides. By increasing the number of stakes to 24, we find the solution – 7 stakes on each side.

Knowledge needed: square and its properties, addition of natural numbers

Possible difficulties: choice of numbers creating an insolvable problem, clarity of the drawing, idea of placement of stakes in the corners

4) Direct method without graphical representation

A direct way of solving the problem is based on the idea that placing a stake in the corner makes it being counted for two sides. When we place 4 stakes in four corners, we have 20 stakes left. Since 20 can be divided by 4, we can dispose 5 stakes on each side, which gives totally 7 stakes on each side.

Knowledge needed: square and its properties, addition of natural numbers

Possible difficulties: idea of placement of stakes in the corners, numerical error

Incorrect solving strategies

1) Arithmetical way

There are 24 stakes to place on 4 sides of a square, so there are 6 stakes on each side.

2) Solution drawing

The pupil places 6 stakes on each side of the ground and leaves the corners free.

The a priori analysis of Problem 2 is based on the analysis of Problem 1. In our opinion, students will first use the same solving strategies based on a concrete number of stakes and then they will generalize the results. We suppose that the time needed for solving the problem might be longer.

REALIZATION OF THE SITUATION IN THE CLASS

In this part we describe a problem-solving situation from June 2013 in the class of students aged 15 or 16 in a grammar school in Prague, Czech Republic. The class is composed of students selected during the entrance exams. There are some very gifted students, but there is also a big difference among the students in their performance in Mathematics. In the years 2012–2014, this class participated on the long-term research mentioned in the introduction. The students dealt regularly with problems using different heuristic problem solving strategies.

The situation was based on the problems described in the previous section and its goal was to learn how to grasp a general problem by solving a simplified problem with specified variables. The description of the situation is based on a video sequence and on the observations of the teacher. There were 18 students in the class. First, they were given 10 minutes to prepare the solution of the problems. They could work individually or in pairs and they could use only paper and pen or pencil to solve the problems. Then, some volunteers presented their way of solving the problems to the others. The presentation was always followed by a whole class discussion about the methods and results.

Student 1 presented her way of solving the problem based on a graphical representation of the square and the stakes. She found the correct solution and presented also the idea of dividing 24 by four as a wrong idea, without explanation. The correct argument (the expression "at most" from the assignment) appeared in the following whole class discussion. Most of the students used the same solving strategy based on a graphical representation.

Student 2 presented her solution based on the idea of placing a stake in each corner of the ground and the same number of stakes on each side. Since there are 2 stakes at the corners and 5 on the side, the total number of stakes on a side is 7. Only three students used the same solving strategy.

16 students found the correct answer, 2 students answered 6 stakes. Since there were not any other solving strategies used for the Problem 1, the next students presented their solution to Problem 2.

Student 3 showed the following method: The number of stakes is divisible by 4, so we divide it by 4 and then, we add 1 to the result because of the stake at the corner that counts two times. The general formula is $n : 4 + 1$.

Student 4 showed another formula: $(n - 4):4 + 2$ with a commentary that the result is the same as for the formula of Student 3.

There were 13 students who found the first or the second formula and 5 students who did not find any general formula. Student 5 showed that both formulas are equivalent by rewriting the second formula as the first one.

A POSTERIORI ANALYSIS AND COMPARISON

The a posteriori analysis and comparison will focus on some characteristics of the situation that appear to us as the most important.

Nature of the assignment

The students did not have problems understanding the assignment. Nevertheless, there appeared a difference among students in realizing the importance of the expression "at most" at the moment when they had to explain why 6 is not a correct solution. During the discussion, they used both expressions "at most" and "maximally". In our opinion, younger students might have more problems with these expressions.

Goal of the problems

Goal for the students: The students were mostly able to find the maximal number of stakes in Problem 1; however, the generalization of the formula presented a difficulty for some students.

Didactical goal for the teacher: The students were able to represent the situation by a picture. For some of them, the graphical representation was not necessary to find the solution and it was probably replaced by a mental representation. The teacher could also identify students' solving strategies. The generalization of the problem appeared only in the number of stakes. Due to lack of time, the teacher did not deal with the other possible generalizations of the problem, as for example a change of the form of the ground or a given difference between the number of stakes on each side.

Time needed for the solution, class management, aids

We supposed 10 minutes for the Problem 1 would be needed. Most students were able to solve both problems in less than 10 minutes. Some of them worked individually but most of them worked in pairs using a paper and a pen.

Cognitive didactical and formulation variables

(24) The choice of the number 24 for the stakes permitted the students to use graphical representation as a solving strategy. As most of the students found the solution very quickly and then considered the problem an easy one, we believe that the choice of a higher number (72, for example) could provoke more reflection than the given number, in particular for students aged 15 or 16. Problem 1 was suitable for showing different solving strategies; however, it was not a challenge for the more gifted students or for all the students having the idea of placing the stakes at the corners. Problem 2 was more challenging for the students.

As the assignment was understood by all students, we did not identify any importance of the formulation variables in the solution of Problem 1. However, we believe that omitting the information „*c* is a number divisible by 4" in the assignment of Problem 2 could provoke more reflection and also more errors in solution of the problem.

Students' reactions and attitudes

Problem 1 was interesting for students with average results in Mathematics because it was easy for them. Problem 2 was more interesting for the more gifted students. Nevertheless, some students with average results were satisfied with finding a solution to both problems, which could improve their self-confidence and attitude to problem solving.

Correct solving strategies

15 students used the Direct method – graphical representation – solution drawing. Some students drew the complete solution, some of them only the corners and one side of the ground. 3 students used the Direct method without graphical representation. We did not notice any solution using systematic experimentation, but a non-systematic experimentation appeared in most of the solutions using graphical representation.

Incorrect solving strategies

Incorrect solving strategies appeared only in two cases. These students did not realize the fact that a stake placed at the corner of the ground counts for both sides.

The comparison between the a priori and the a posteriori analysis shows that most of characteristics of the situation described in the a priori analysis appeared in the class. In our opinion, there could be more reflection on the choice of didactical variable (number of stakes) with respect to the age of the students. The a posteriori analysis shows that the number 24 was suitable for promoting the use of both principal solving strategies, the reflection upon the generalization of the problem might be provoked by giving a number of stakes that did not allow a graphical representation. For this reason, we suggest replacing the number 24 by another number, for example 72, and we believe that the students will be lead to use the graphical representation for a smaller number of stakes by the constraints of the milieu and not by the fact that it is easy to represent.

One of the goals of the situation was to make a generalization of the problem with the students. Even though there was enough time to solve both problems, there was not any time and enough support for the teacher. In our opinion, the cluster of problems suitable for generalization should contain also problems dealing with other variables, for example:

Problem 3. We have to dispose c stakes around a *pentagonal* ground. How many stakes at most can we dispose on each side of the ground if there has to be the same number of stakes on all sides?

CONCLUSION

The methodology described in the paper was used for various word problems and solving strategies (see, for example, Břehovský et al., 2013; Eisenmann, Novotná, & Přibyl, 2014). The results of the research show that number of using of two solving strategies, *Systematic experimentation* and *Adding of an auxiliary element*, increased significantly. The analysis of the psychological screening show that the indicators of creativity increased in an important way, in particular the fluency and the flexibility. The didactical situation based on the use of various solving strategies seems to be a suitable environment to enhance the development of these key features of creativity. A question about the way how the problems-solving activities can promote also the originality and the elaboration arises immediately from these results. The role of the key features of a problem solving situation in development of fluency and flexibility gives also an implication for future research.

The most important changes appeared in the attitudes of the students towards problem solving. We could observe half of them make an increasing effort to solve a problem despite the fact that they didn't know how to solve it at the beginning. The participating teachers became more tolerant to different students' solving strategies. Furthermore, there is a list of problems and solving strategies prepared to use by teachers promoting students' creativity in problem solving.

ACKNOWLEDGEMENT

This research was supported by the grant GACR P407/12/1939.

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Looking for creativity in primary school mathematical tasks

Despina Desli and Marianthi Zioga

Aristotle University of Thessaloniki, Faculty of Education, School of Primary Education, Thessaloniki, Greece, ddesli@eled.auth.gr

The purpose of this study is to explore the features that primary school teachers consider appropriate when choosing tasks intended to promote mathematical creativity. 26 prospective and 48 in-service teachers completed a questionnaire whereby they were asked to choose a mathematical task and explain why it fosters mathematical creativity. Interviews were also conducted with 2 prospective and 2 in-service teachers to further investigate their perceptions of creativity in mathematics as well as issues related to teaching mathematics using creative problems. Disparate views were revealed between the two groups: prospective teachers mostly connect mathematical creativity to arousing student interest whereas in-service teachers connect it to book-oriented problem posing indicating their narrow and blurred perceptions. Outcomes highlight the need for educating teachers about creativity in mathematics.

Keywords: Mathematical creativity, teachers, primary school.

INTRODUCTION

Creativity, traditionally linked to art and literature, has been characterized as an individual activity intended to produce something new (Bolden, Harries, & Newton, 2010). Creativity has also recently been associated with mathematics, it being the ability to create new mathematical insights and ideas (Sriraman, 2009). Similar to this idea, the ability to combine previously known concepts or discover unknown relations between mathematical facts and employ non-algorithmic decision-making can be considered as a creative act of doing mathematics (Ervynck, 1991). Divergent thinking in problem solving is also associated with mathematical creativity (Haylock, 1997; Chamberlin & Moon, 2005). It allows one to analyze a problem from different perspectives without applying one fixed answer, identify patterns, differences and similarities and choose an appropriate method for tackling it. Although there are many definitions of mathematical creativity, in general, two common trends describe it - the generation of new mathematical knowledge and flexible problem solving abilities (Kwon, Park, & Park, 2006).

Even though the importance of facilitating mathematical creativity in educational settings has been well established, there is a clear need to study the choices teachers make in terms of tasks that could develop children's creativity in mathematics. The present study focuses on the features they consider necessary for mathematical creativity and the reasons for these choices.

RESEARCH BACKGROUND

Mathematical creativity is often assessed on the basis of the four indices of creativity proposed by Torrance (as cited in Silver, 1997) and Guilford (as cited in Klavir & Hershkovitz, 2008). These indices include: a) fluency, referring to the number of correct responses that the student produces, b) flexibility, referring to the number of different mathematical concepts and ideas that the student discovers, usually breaking away from stereotypes, c) elaboration, indicating the complexity of mathematical thinking, as the student integrates different pieces of mathematical knowledge, and d) originality, illuminating the extent that the student's ideas are insightful, new and lead to unexpected and unconventional solutions.

Studies of mathematical creativity have revealed that students' creative mathematical thinking and indices of mathematical creativity could be encouraged by providing divergent product tasks. Klavir and Hershkovitz (2008), for example, suggested tools for teachers to analyze and evaluate the work of fifth-grade students when dealing with an open-ended problem. These tools referred to indices of mathematical creativity as well as levels of complexity in mathematical knowledge. Their results led them to suggest that open-ended problems tend to distance us from the stereotype that there is only one solution to any given problem and recognized their value as an assessment tool for both teachers and students. Kwon and colleagues (2006) also found that divergent thinking in mathematics could be cultivated through an open-ended approach: in their study with seventh-grade students, open-ended problems were found more cognitively challenging, because they allowed for multiple interpretations and solutions and offered students the opportunity to solve problems using their actual skills. As Mann (2006) suggests, solving these types of problems let the students take the first steps towards mathematical creativity.

The encouragement for promoting children's mathematical creativity in the classroom is advocated in mathematics curricula worldwide that regard it as a desirable outcome of mathematical education. Given the fact that mathematical creativity is also considered as a dynamic faculty that can be improved and enriched or, conversely, decline (Leikin, 2009), great attention has recently been paid to how teachers perceive creativity in mathematics. Bolden and colleagues (2010) found that pre-service teachers in UK hold narrow conceptions regarding creativity in mathematics: these conceptions are mainly associated with the use of resources and technology and, while attempting to create a 'fun' environment, they 'teach creatively' rather than 'teach for creativity'. Their great difficulty in recognizing creativity in teaching mathematics and also in identifying ways of encouraging it in the classroom was also observed. More positive results were revealed in Chiu's study (2009), which found that in-service teachers profess a greater preference for creative problems, compared to non-creative problems for teaching fractions: a liberal approach was proposed as the most appropriate method for teaching creative problems with emphasis on the children's imaginative and diverse solutions. Secondary classroom teachers, however, identified both opportunities and constraints in posing more challenging mathematical tasks, especially those related to changes to their pedagogies and assessment of student work (Sullivan & Mornane, 2014).

However, a key component of mathematical creativity is how teachers select and use appropriate tasks which enhance children's creativity in terms of school mathematics. In her study, Levenson (2013) investigated general trends that prospective and in-service teachers attribute to tasks that may occasion mathematical creativity. Of these trends, the most common one was the implication that creativity pertains to being different and unusual. The responses of participants also focused on the cognitive demands of the chosen tasks, as well as their affective aspect. Interestingly, with regards to the latter, teachers took into consideration possible feelings which a task may elicit from students.

These studies suggest that excluding the teachers as important factors in determining the use of creative mathematics problems in the classroom could lead to an incomplete understanding of the situation. Thus, investigating teachers' perspectives of creativity in primary mathematics, by asking them to choose such tasks is important, in order to understand the knowledge they hold that could influence their interpretation of creativity in the curriculum and what they do in their teaching. Specifically, this study attempted to examine the following questions: a) Which task characteristics do prospective and in-service primary school teachers associate with the promotion of mathematical creativity?, b) Are there any discrepancies between the task characteristics identified by prospective teachers and in-service teachers?, and c) How they envision creativity being taught in the mathematics classroom?

METHOD

Participants. Twenty-six prospective teachers and forty-eight in-service teachers participated in the study. At the time of the study, prospective teachers, who were in their last year of an elementary teaching education programme at a large-sized university in Thessaloniki, participated in a course on teaching mathematics that was mainly focused on observing, planning and teaching mathematics in local primary schools. They were predominantly women (92%) with a mean age of 22 years and 2 months. In-service teachers, most of which were women (83%) with a mean teaching experience of 18 years, worked at several grades of state primary schools from various geographical regions of Northern Greece and their ages ranged from 26 to 58 years old. A small percentage of in-service teachers (15%) who took part in the study had completed postgraduate studies, but did not mention that they have attended specific training courses on creativity. All participants' selection was random and their participation was voluntary.

Instrument. Data were collected via a questionnaire and semi-structured interviews. The questionnaire used was based on Levenson's (2013) research tool and asked participants to: a) choose a task that they consider appropriate for promoting mathematical creativity. This task could be from either the state mathematics textbooks or any mathematics book or even one proposed by themselves., b) indicate their source, c) state the grade to which the task is targeted and whether it is intended for individual or group work, and d) describe why they consider the task they had chosen suitable for promoting mathematical creativity. All questions in the questionnaire were open-ended and no possible answers were provided. Additionally, no instruction about the term 'task' was given to participants who were free in their interpretation. However, the present study relies on Stein and Smith (1998), who clarified the multiple roles of tasks: these may be set up by teachers during their instruction as learning tasks, review tasks, practice and assessment tasks with the main purpose of developing a particular mathematical idea.

Follow-up interviews were conducted on the basis of the questions developed from the completed questionnaires. Thus, supplementary data were gathered, in order to verify that questionnaire responses were being interpreted as they were initially intended. Interview questions explored the participants' envision of creativity in mathematics and the use of creative problems in teaching mathematics in primary schools. In the present study the results of only two prospective and two in-service -randomly chosen- teachers are presented as these were the first interviewed and became part of a larger sample for the purposes of a later relevant study.

Procedure. The questionnaires were administered to prospective and in-service teachers during their classes, at the University and the schools, respectively, and they were collected after two weeks. The majority of those contacted returned the completed questionnaires, reaching 93% and 89% participation response rates for prospective and in-service teachers, respectively.

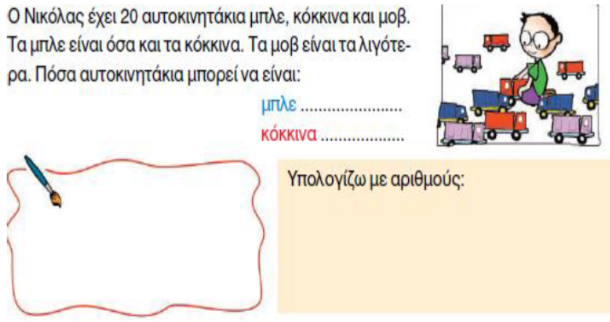
Interviews were conducted individually at a place convenient to interviewees, two weeks after they had completed the questionnaires, and lasted approximately 20 min.

RESULTS

The results are presented in two main sections. It is not possible to present all the tasks proposed by the participants in this paper, therefore only three are presented in the first section. In the second section we describe common trends in the responses of participants, drawn from answers to the questionnaire and the interviews, regarding the teaching of mathematics and learning with creative problems.

a. Tasks proposed by the participants. The majority of the tasks the participants chose were taken from the mathematics textbooks that are given to all students in the nation (85% and 83% for prospective and in-service teachers, respectively). Only one in-service teacher seized the opportunity to propose a task herself and handed in a task that was her own devising. Prospective teachers proposed about 40% of the tasks for individual work, whereas the rest of the chosen tasks were divided equally for either group work or both individual and group work. Great preference for tasks carried out individually (42%) – rather than for all students (38%) – was also observed by in-service teachers.

Katerina, a prospective teacher, chose the task shown in Figure 1, taken from a Grade 2 mathematics student book, and proposed that children could work on it individually. The task in question is special because it does not provide students with a familiar methodology for solving it. Katerina mentioned it in her response and explained why this task promoted mathematical creativity: '*it is a task that cannot be easily solved in the usual way. It doesn't use typical formulas ... children will experience the struggles of finding explanations and solutions... it leads to creative thinking*'. Even the opportunity to use a trial-and-error strategy when solving the problem was seen as conducive to mathematical creativity. For her, mathematical creativity was associated with breaking away from well-trodden paths. However, she did not mention the fact that there are several ways of solving this problem, a characteristic that raises flexibility as involved in creativity. Last, she put great emphasis on the story involved (toy cars) and believed that anything that motivates students to work – indicating mainly the problem story – can promote creativity.



Nikolas has a total of 20 blue, red and purple toy cars. Blue cars are as many as red ones, purple cars are less. How many are the blue and red cars? Compute with numbers.

Figure 1: Katerina's task

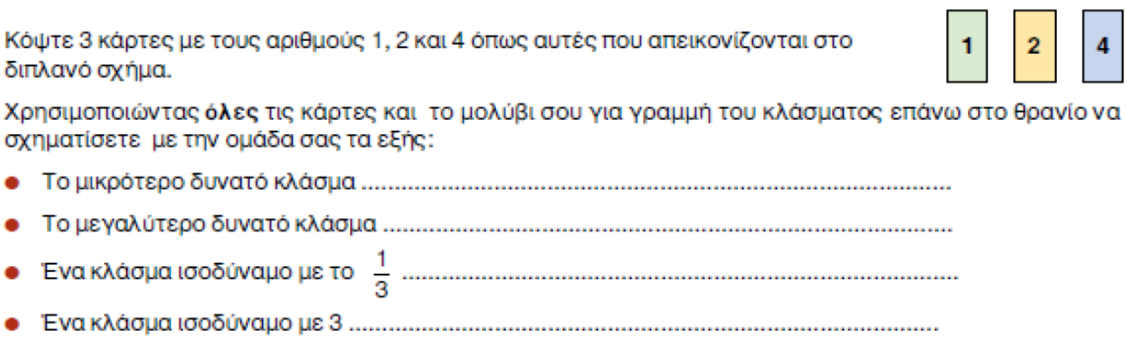
Make your own problems that suit the pictures.

Figure 2: Anna's task

The task in Figure 2 was proposed by Anna, an in-service teacher. It was taken from a Grade 1 mathematics workbook and it was meant for all students to solve. Anna mentioned three components she believes are promoted by the task. First, every child may phrase and create her own problem and, thus, may take an active part in solving it. In this sense, '*every child may bring her interests into mathematics and feel involved*'. Although this feature calls mostly on the emotional impact that a task may have on children, Anna raises the issue of the mathematical content involved. According to her, '*children may see addition or subtraction involved, insert numbers and perform calculations*'. This feature, though, is more reminiscent of a conventional problem that requires certain ordered steps to reach algorithmic answers. Third, she acknowledges that the fact that children can pose different problems leads them to different solutions that may encourage debates and contribute to a productive classroom discussion.

Figure 3 shows a prospective teacher's choice of a creative mathematical task, taken from a Grade 6 mathematics workbook. Maria's activity includes four mini questions -each of which is meant to have one and only one algorithmic solution- that do not require new ideas. It may be conducted in the classroom using pen and paper and by following steps in a specific order. However, this fact did not deter Maria from considering this task as one promoting creativity. Instead, she considered the fact that children are asked to act (e.g. by moving the cards), are presented with the mathematical knowledge and communicate their findings as important. The fact that some features are overestimated or others are ignored is often the case, as it happens with Maria's choice.

b. Participants' reasons for choosing tasks. Participants listed various reasons for choosing their task. Because they provided more than one reasons, these are analyzed on the basis of the number of participants (see Table 1). Attractive and entertaining stories (30,8%) involved in the task that may arouse children's interest and inquiry was one of the two most important elements for fostering mathematical creativity, according to prospective teachers. Creative mathemat-



Cut three cards and place numbers 1, 2 and 4 on them, as shown in the figure. Using all cards and marking a fraction line with your pencil, work in groups and make: the smallest possible fraction, the biggest possible fraction, a fraction equal to 1/3, a fraction equal to 3.

Figure 3: Maria's task

ical tasks were also those that allow children to make their own constructions (30,8%), placing emphasis on the children's active engagement in mathematical activities, as well as those that invite work between students and encourage cooperative learning (23%).

In-service teachers seem to relate creativity to the activity of problem-posing on the part of the children (37,5%) and the placement of the task in a real-life context (16,7%) that may offer opportunities for the flexible use and application of mathematics in everyday life. However, many in-service teachers insisted on more 'traditional' ideas: these tied creativity to solution methods provided from the start and presented in multiple steps following an hierarchical order (29,1%) and focused on algorithmic thinking and techniques (20,8%).

Both groups of participants mentioned creativity in terms of searching for new ways to cope with mathematical problems on the basis of their previous mathematical knowledge (23% and 8,4% for prospective and in-service teachers, respectively). Presenting children with unusual questions (quizzes were the most frequently mentioned) was connected to motivation for imagination and creativity (23% and 20,8% for the two groups). Interestingly, prospective teachers only seem to regard that difficult technical calculations are not necessarily required; they view mathematical creativity as not being constrained by precise methods or anticipated outcomes and support non-algorithmic thinking and computational estimation (8,5%). Last, seeing the same problem offering several different solutions as well as connections with other, less (or non-) mathematical, school subjects were also considered to elicit mathematical creativity.

DISCUSSION

The characteristics participants associated with mathematical creativity, as shown in the aspects they considered in choosing tasks, demonstrated predominantly that both prospective and in-service teachers identify creativity in primary mathematics classrooms. Based on the mathematical tasks they chose, however, it was revealed that identifying mathematical creativity was not an easy thing to do. Interviews further confirmed the participants' difficulties in being clear about planning and encouraging creativity in the classroom. Although they seem to believe that mathematics can be a subject that is offered for the promotion of creativity in children, however, they acknowledge constraints in the classroom, mainly related to the role of the teacher and the mathematical content.

There were some common ideas between the two participant groups, but the issue of mathematical creativity and how this can be fostered drew disparate views, which were probably influenced by their academic training. Prospective teachers mostly related mathematical creativity to arousing children's interest and

Elements of tasks that promote mathematical creativity	Prospective	In-service
	Teachers	
Story that arouses children's interest and inquiry	30,8%	8,4%
Problem-posing by children		37,5%
Children construct (e.g., draw, combine shapes)	30,8%	8,4%
The solution method is provided in multiple steps	15,4%	29,1%
Presenting with unusual questions	23%	20,8%
Connections between mathematical topics and everyday life	23%	16,7%
Use of manipulatives	23%	12,6%
Use and extension of previous knowledge	23%	8,4%
Children work together – Cooperative learning	23%	
Focus on algorithmic thinking and techniques	7,7%	20,8%
Large number of possible solutions to a problem	15,4%	8,4%
Focus on computational estimation	8,5%	
Connections with other school subjects	7,7%	4,2%
Other	3%	4,2%

Table 1: Characteristics identified for tasks that promote mathematical creativity by prospective and in-service teachers

motivation, mainly with the use of attractive stories or puzzles that keep them engaged in mathematics. This finding is in agreement with Bolden and colleagues' (2010) results that, for prospective teachers, creativity was bound up with the use of resources in order to create a 'fun' environment for children. Although this might be a desirable outcome, care is needed to avoid underestimating children's access to mathematical ideas. Prospective teachers also gave emphasis on children's own constructions when doing mathematics, use of materials and non-algorithmic thinking. It is encouraging to see prospective teachers moving away from a conservative view of mathematics with algorithmic calculations. This orientation that brings flexibility to the front was also observed in Bolden and colleagues' (2010) study. In general, prospective teachers recognize some of the aspects of creativity as do researchers (Klavir & Hershkovitz, 2008; Levenson, 2013; Kwon et al., 2006). Last, cooperative work was often considered necessary for the promotion of mathematical creativity; children working on a task may contribute insights, experiences and ideas building eventually to a solution brought forth by more than one student. The issue of collective mathematical creativity was originally raised by Levenson (2011) who deems it equally important as individual mathematical creativity.

In-service teachers, on the other hand, related mathematical creativity to children's problem posing. They explain that asking children to make up a problem provides them with the opportunity to feel involved and take an active role in solving their own problem. Other elements they considered as important referred to applying some algorithm learned in class and following strictly-defined steps towards a solution method. These were limited to rule-based applications and were relatively different from what researchers consider as mathematical creativity. Their choices of tasks, in general, were less imaginative without recognizing the essence of the problem to be solved, and indicated their narrow and blurred perceptions. Thus we wonder how easily a teacher may overestimate the more conservative aspects of teaching and discard the opportunities for mathematical creativity that a task might provide. Searching for the new in mathematics and expressing the unusual have also been considered by a few participants who highlighted novelty and placed emphasis on creativity in mathematics. These participants raised issues such as real-life connections to mathematics, connections with other school subjects and extension of previous knowledge. Unusualness as an element for creativity was strongly supported by the use of non-routine problems (e.g., quizzes), non-standard problems that involved unexpected and unfamiliar solutions (Yeo, 2009).

An additional finding of the study showed that although participants identified characteristics of mathematical creativity in general, these characteristics were not always revealed in the tasks they had chosen or did not fit well with their choices of tasks. For example, a few participants regarded multiple solution methods as a key element to creativity. Although searching for different solutions can definitely be seen as an aspect of both flexible thinking and fluency, its value is mediated, if an explicit direction to consider several different ways of solving the problem is given at the end of the task. The demand to solve a task using different methods may have the opposite effect on creativity. Similarly, in the search for a task that may foster children's mathematical creativity, the vast majority of the participants searched for tasks taken from standard classroom textbooks. This may be interpreted in two ways: participants are unwilling to look for tasks other than those provided, or, less pessimistically, participants may transform any task to a creative one, depending on how they themselves implement it in the classroom. Although attention to the nature of mathematical tasks that foster mathematical creativity is important, an equal level of it in the classroom processes associated with mathematical creativity is needed. We are aware that analyzing only the tasks chosen by the participants is an inadequate method for predicting if the lessons would run for the purpose of teaching for creativity. Further investigation of this topic is needed.

The findings from this study suggested that the awareness of prospective and in-service teachers in terms of choosing tasks that promote mathematical creativity was apparent, at least at its manifested level; however, encouraging creativity in primary mathematics classrooms can generally be an elusive accomplishment. There is a need for clarification on how the idea of mathematical creativity is implemented and referred to in a mathematics lesson. This leads to the need for providing teachers with opportunities to be educated in mathematics teaching for creativity.

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Assigned to creativity: Didactical contract negotiation and technology

Valérie Émin, Nataly Essonnier, Pedro Lealdino Filho, Christian Mercat and Jana Trgalová

Université de Lyon, Université Claude Bernard Lyon 1, S2HEP EA4148, École Normale Supérieure de Lyon, Lyon, France, pedrolealdino@gmail.com

The mcSquared European project aims at studying Social Creativity among pedagogical resources designers and Creative Mathematical Thinking in their users, through technology, namely a creative e-book software infrastructure and resources called "c-book units". This article focuses on a study carried out in the framework of this project in France experimenting such a c-book unit and highlights a particular obstacle created by a didactical contract effect related to creativity in the French mathematics classroom: high achieving students perform well on content related to an official assignment but have difficulties engaging in unusual creative problems. The article concludes on possible ways to circumvent it in order to foster the unleashing of mathematical creativity in all students.

Keywords: C-book technology, creative mathematical thinking, didactical contract.

INTRODUCTION

Mathematicians profess that performing mathematics is a creative activity (Hadamard, 1954). While "capital C" creativity is clearly of the essence, can "small c" creativity (Csikszentmihalyi, 1996) be implemented in the classroom in a way to transpose professional activity as a learning tool? Technology supported inquiry based learning is a possible way to put students in situations where their creativity is needed and can be expressed (Blumenfeld et al., 1991). Yet, many obstacles pave its way (Edelson et al., 1999). Despite these obstacles, inquiry based learning is rather put in practice and somehow familiar to students in sciences; but it might not be so in mathematics. This article focuses on didactical contract effects that may deter its adoption, in a manner similar to (Brandl, 2011) in the realm of giftedness.

In this article, we first introduce the mcSquared European project [2] and its structure in Communities of Interest (CoI) (Fischer, 2001) aiming at designing electronic books for teaching mathematics enhancing Creative Mathematical Thinking (CMT). We then present the educational resource under consideration in the reported experiment, the "Velocity" c-book, designed by the French CoI. Then we describe and analyze the experimentation carried out in two Grade 9 classrooms, where 14–15 years old students were invited to work on the Velocity c-book math activities. This lead to a didactical contract clash that we describe, followed by an outline of possible remediation. We then conclude on possible ways to improve acceptability and devolution of activities aiming at promoting creativity in mathematics classroom.

THE MC SQUARED EUROPEAN PROJECT

The mcSquared project aims at designing and developing an intelligent computational environment, a new genre of authorable e-book, which we call 'the c-book' (c for creative), extending e-book technologies to include diverse dynamic widgets, an authorable data analytics engine and a tool supporting asynchronous collaborative design of pedagogical resources, which we call 'c-book units'. The c-book environment aims at stimulating and enhancing creative designs for fostering mathematical creativity in mathematics classes.

Creativity is studied in two complementary ways: Creative Mathematical Thinking (CMT) in students using technology and Social Creativity (SC) in the design of c-book units intended to enhance CMT in the users. The c-book units are produced by four different Communities of Interest (CoI), organised by consortium partners' countries (France, Greece, Spain and UK), bringing together stakeholders from different professional domains, such as publishers, game developers, math education researchers, school educators... The French CoI is composed of representatives of several Communities of Practice (CoP) (Wenger, 1998), mainly gathered around the IREM [3] in Lyon and in Grenoble, and a few individuals (Figure 1).

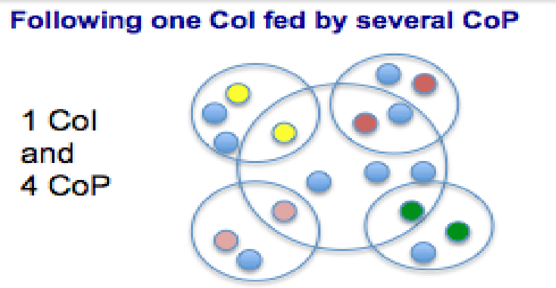

Figure 1: Several CoPs around a CoI

In this paper we only consider the creative mathematical thinking part of the project.

Creative mathematical thinking

Based on the existing literature review on creativity (Guilford, 1950; Kaufman & Sternberg, 2010), mathematical creativity (Sriraman, 2004, 2005; Leikin & Lev, 2007) and mathematical thinking (Tall, 2002; Blinder, 2013), we understand CMT as the combination of divergent and convergent thinking in mathematics.

Divergent thinking in mathematics is characterized by:

- fluency: number of solutions;
- flexibility: number of categories (representations and settings) of solutions;
- originality: statistical frequency of solutions;
- elaboration: depth and detail of solutions.

Convergent thinking is characterized by:

- mathematical correctness or conventional answers,
- use of cognitive processes to produce one or very few possible solutions.

Besides, CMT can be fostered by adequate feedback regarding its different dimensions: fluency, flexibility, originality/novelty, appropriateness and usefulness, provided by the c-book unit, teachers, fellow students... CMT is associated with an individual and relative to a given community, a given context, in which the process is envisioned to be used.

We can also integrate the social aspect of creativity in CMT dealing with motivation of participants, the issue of informal norms that promote cooperation and assistance, the social recognition of one's work value. The present article treats specifically this point and some obstacles that hinder creativity, namely didactical contract effects.

THE VELOCITY C-BOOK UNIT

First, a c-book unit is a digital pedagogical resource developed in a specific environment of the project, the c-book technology, viewable and editable in any modern Internet browser, organised in a set of pages which bundle together texts and communicating widgets from different origins enabling a great variety of affordances. They can be movie or sound players, 2D and 3D object viewers, but also constructionist bricks of software such as dynamic geometry software (Geogebra, Cinderella...), dynamic algebra software (epsilonwriter), programming environments (eSlate Logo TurtleWorlds, javaScript, cindyScript, GeogebraScript...), specialized visualization constructs ("widget factories", spreadsheets, graph of a function, algebraic expression editor, calculator...). These widgets can be saved and shared in a particular state, ranging from an empty canvas to a finished full-fledged "press and play" interactive resource, as well as half-baked micro-worlds (Kynigos, 2007) to be appropriated and worked upon by the students.

The idea of the Velocity c-book unit stems from the Community of Practice called TraAM (Mutualised Academic Works [4]) group in IREM Lyon, mainly composed of secondary mathematics teachers, focusing on the design of open-ended problems and problem-solving with technology. Their aim is to develop a shared repertoire of resources based on interesting use of ICT in the tackling of interdisciplinary open-ended problems in everyday life situations. They collaboratively design resources that they cross-experiment in their own classes. This CoP production takes place into a national framework coordinated by the ministry of education.

After a presentation and a quick *a priori* analysis of the c-book unit, we present the results of the experimentation carried out in two classes, with different average achievement levels in mathematics.

Learning goal

The Velocity c-book unit, aimed for Grade 9 students, has as the main learning goal the notion of speed as distance divided by time leading to the notion of derivation as a faraway objective.

The main targeted competency is modelling real world situations, but a series of sub-competences useful in this context and promoting creativity are also at stake: Understand a problem, engage in a research, show initiatives, be original; suggest answers, propose hypotheses or conjectures, formulate questions; prove that something is true or that it is false; communicate, orally or in a written form.

The idea behind this c-book unit is to let students gather information from the real world in order to analyse what they see with a "scientific eye", including the need to "be true to the data", reflecting the fact that real life does not provide you with polished data that makes sense at once: real data is full of glitches and does not follow exactly the model that you want to force on it. Specifically for this c-book unit, speed is the topical notion at work. Therefore an important goal is to make the students realize that position and time can be defined exactly only in theory but nevertheless, that a crude approximation is enough to be able to make science and take decisions based on it. We know from the start that this is a real change in didactical contract (Brousseau, 1988) for most of the students, used to be fed artificial exercises with a unique well defined answer to a question they know they can answer with readily available tools. This experiment focuses on the study of the devolution phase.

Description of the learning situation

The c-book unit comprises a series of four activities organized in 11 c-book pages (Table 1).

Each activity presents raw data of some sort and a very simple question that should engage the students in making sense of the data in order to answer these questions (see Table 1). Initiatives, which are manifold, have to be taken in order to overcome the limitations inherent to real world phenomena and reach definitive conclusions despite uncertainty. Being able to validate hypotheses such as "the car was driving too fast", or "the truck drove for more than 500 km", without knowing everything precisely is a goal that is attainable but requires mathematical creativity from the students. In what follows, we present in more details the Tunnel activity and its brief *a priori* analysis.

In this activity, students have to analyse videos taken from within a car while driving through a tunnel and try to figure out whether the speed limit is reached or not. They have to look for clues, such as the trails of lights or chevrons in order to estimate the car speed, by tabulating the timing and positions of these events, computing the average speed, dealing with imprecision and confidence intervals. Three different videos are provided (Figure 2), with computer adjusted speed profiles falling in three different scenarios showing two types of speed camera: a fixed speed camera and an average speed measuring camera, requiring the notions of instantaneous and average speed. In the first scenario, the car is not caught by any of the two speed cameras, in the second one it is caught only by the fixed speed camera, and in the third scenario it is caught only by the average speed measuring camera.

The total length of the tunnel is 1757 m. The students can identify chevrons and lights that are evenly spaced,

Activities	"widgets" used	Questions asked to students
Tunnel (3 pages)	Three videos taken from a car driving through a tunnel, Cinderella chronometer, GeoGebra	According to the video, will the car driver receive a fine for exceeding the speed limit in the tunnel?
Particles (4 pages)	Simulation of a particle in Cinderella, Graph2, GeoGebra, Microsoft Kinect	Dance your way as a function: graph the movement of a particle, and move to replicate a given graph.
Control (3 pages)	Picture of a paper disk used in trucks to monitor their speed along the day, Geogebra	According to the tachograph, what is the total distance driven by the truck driver that day?
Average speed (1 p)	GeoGebra	Give the average speed of a car in a given condition.

Table 1: Velocity c-book unit activities

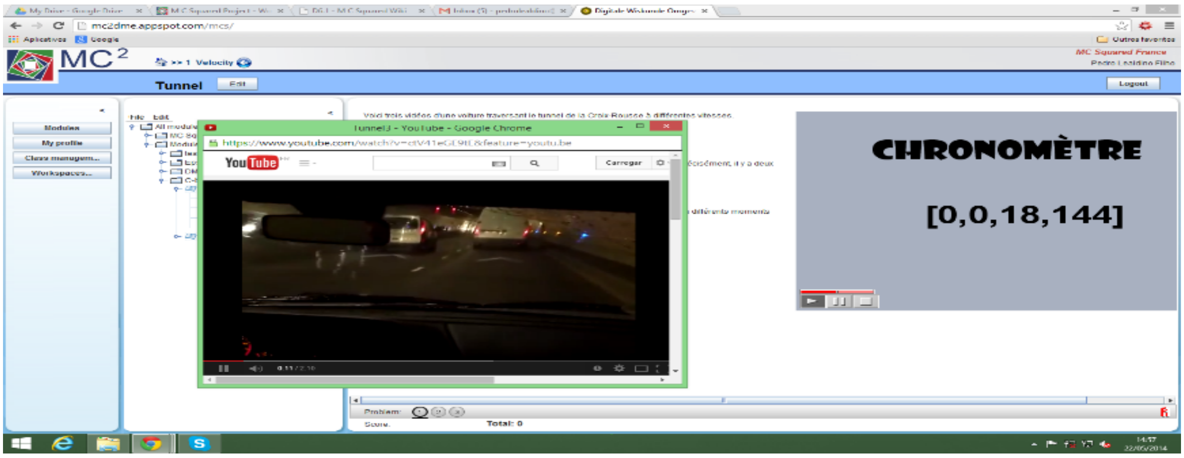

Figure 2: The tunnel activity

with a space prescribed in some official documents that can be found on the internet, or estimated with the number of such events and the total length. The position of the fixed speed camera should be found as well. Then a chronometer available on the c-book page can be used to record some events like the moment a chevron or a light disappear from the screen.

The expected students' behaviour is to search for the total length of the tunnel, to view the video and to measure with the embedded chronometer the total time from the beginning to the end and finally to infer the average speed for the first question.

Several issues are expected to be encountered. What about the format Hours : Minutes : Seconds . Milliseconds, how to work with this data in order to make some math? The students are supposed to paste the minutes and seconds in two columns and elaborate a formula that gives the total amount in seconds. Another issue is the accuracy of the data. When two students record the same events, the data can vary. How can they account for it? Shall they take the mean of the different values between students? Before that they have to agree on a starting time and a translation in time might be in order. Once this is done, they can begin to do some calculations, but they were already involved in mathematics: modelling a phenomenon, transforming it to observations which can be quantified, is part of mathematics. A more difficult problem is the comparison of data recording different events: especially the ones that record timings of chevrons and timings of lights: it is only after the plotting of both position graphs overtime that the comparison can be performed, because the discrete set of data for both events do not mingle that easily on the tables of numerical values. Only the graphical plot of position with respect to time (and not the rank of item!), which are both discrete sequences of the same integer rank, can show similarities. This is a difficult issue because plotting position against the item rank (the line number in the spreadsheet) is the obvious way to plot a sequence. And the item ranks of the chevrons and lights are related to the associated positions through a linear transformation (the respective distances between two occurrences). This requires a deep understanding of the notion of function of one variable: the position is a function of time but given as values in a sequence with given ranks and time itself is a function of the same rank. Therefore we can infer values of the position as a function of time and forget about the rank as an intermediate variable, artefact of our modelling.

Starting with the raw data, we produced different elaborate constructions and each page in the c-book unit unravels some possible new features, including widgets of many different kinds. We count on the orchestration of a teacher, knowing a good portion of what is feasible given the available technology and examples of implementations that are proposed in the c-book, to help the students leapfrog from one instrumented situation to another. But it is our hope that some students will eventually go beyond the proposed implementations, or in totally new unexpected directions to answer the first question.

THE EXPERIMENTATION

Although our objective was to design learning situations in which the use of the c-book unit could be autonomous by groups of students, the pilot study reported in this section is regulated by a teacher, in order not to "spoil the fun" and yet see progress. Moreover, mastering all the aspects of the powerful tools afforded by the c-book technology is not obvious. For the time being the transition from one page to a next one is not automatic and should be controlled by a teacher orchestrating the activity. The pedagogical context in which we conducted the experimentation was provided by a teacher with her students. Two Grade 9 classes (secondary 3rd in French school system, called therefore 3C and 3D in what follows) with 14–15 years old students participated in the study, working in groups of 3–4, equipped with computers and a beamer that could project the work of a given group. The two classes are very different, the first one is composed of students which perform well in mathematics and the second one of students that have difficulties coping with the core of the curriculum and who circumvent frontal confrontation with mathematics.

We report only on the "Tunnel" activity. Investigation on the average speed, different speed controls and some research on the tunnel was given as homework before the classroom session during which the students worked together on the activity.

3D class – lower achieving students

The students were relating the scenario to their own experience and discussed about the issue, it really meant something for them. After a few viewings of the video, they understood how to compute the average speed by measuring the time and the distance, having learnt how to do it themselves in similar situations. They addressed, discussed and solved the issues, difficult for them, of converting the ratio of a distance by a time from m/s to km/h, using proportionality and dealing with decimal representation of time instead of the usual hexadecimal HHMMSS representation. They showed progress in the direction of defining a protocol suitable to estimate the instantaneous speed (lights, marks on the road) but nobody came to a definite answer, only the average speed was computed and related to their own estimates based on their experience. The session was nevertheless felt as successful by all parties.

3C class – higher achieving students

When comparing the assigned videos between groups, the class, in a very homogeneous fashion, realized that it was the same one, tuned to fit special purposes. They answered at once without doing computations, sorting correctly the three videos into three different scenarios regarding the fine. They gave the answer they thought the teacher was expecting, minimizing their effort and failing to engage into the activity. When pressed, they correctly explained the measurements and computations that had to be done, both for the average and instantaneous speeds but nobody actually did them. What refrained them from doing anything is the obvious fact that only crude estimates were possible and that the answer had to be somehow unique, giving that an approximation would have been wrong and was not in order. The issue of giving error margins was debated. Incited once again to make measurements, they were forced to reluctantly take decisions, measure events, and give answers. They had to admit that, whereas their actual numbers were indeed different, the final conclusions were the same for each group on a given video. The expressed feeling was that of an abuse of power and a lousy work ethic on the part of the teacher who went beyond her right in asking such questions, that this was not mathematics and that nothing of that sort was ever asked at the junior high school final national exam "brevet des collèges". This students' behaviour can be interpreted as a reaction to the didactical contract break-up (Brousseau, 1988).

Remediation

A possible way out of this didactical contract clash was found by making explicit a list of competencies taken from the national curriculum. This list of competencies, one of the results of the EvaCoDICE project [5], is now introducing the c-book, in order to be self-evaluated throughout the activity. For example, students have to choose, for the competency labelled "Understand the problem, do some research, take initiatives, be original" an item in the following list:

— "I don't understand what we are looking for, I cannot begin",

— "I understand what we are looking for but I don't know how to begin, I don't have any good idea",

— "I understand what we are looking for, I am trying but I make mistakes in my research, I have some ideas",

— "I understand what we are looking for, I make some experiments, I have ideas".

Showing the students a table with the list of competencies reassured them in the fact that, whereas these are indeed never assessed at a national exam, they are nevertheless officially expected from them. But on the other hand, this remediation leads to a reinforcement of the didactical contract that only what is explicitly and officially required is to be used in the classroom. In a system where national assessments explicit the expectations for all partners of the educational system (students, teachers, parents...), and which sticks to technical and standardized tasks, promoting CMT is a real challenge when the achievement measured by these assessments does not correlate easily with CMT.

CONCLUSION

Fostering Creative Mathematical Thinking in the classroom needs a tailored "ecology" (Barquero, Serrano, & Serrano, 2010), a trained teacher, a rich *milieu* and a special didactical contract (Chevallard, 2012; Wozniak, 2012) to be negotiated: *it's alright to think! Make some guess! Explore!* Such competencies are seldom valued in the curriculum.

In the agenda for progress in math education, Schoenfeld (2011) states that "assessments that are consistent with [mathematically rich content and sense making activities]" is one of the conditions to achieve the goal of a "meaningful engagement with powerful mathematics for all children". This condition might be the most taxing in the didactical contract effect that was observed here: high achieving students tend to minimize their effort and see no direct interest in engaging into what they see as exceeding their job as a student. Teacher training has as well to address the evaluation of competencies.

It is all the more true with technology enhanced learning especially in unsupervised situations: in order to earn student's interest, we might have to put our activities in the cyber-space perspective in which they live and which is so engaging for them. Adding social-networking and timing might turn a dull set of marks and assessments into a friendly competition; the 21st century didactic engineer should identify, alongside the didactical variables, the playful appealing ones, which can turn a mathematical task into a game where devolution of the task means not trying to please the teacher but to have fun, where fellow students cooperate online, turn upside down their assignments and boast their achievements on social networks. Recent works (Pelay, 2011) and political stands (Vallaud-Belkacem, 2014) tend to point in the right direction!

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ENDNOTES

1. Supported as well by the CAPES - Proc . N° 0791/14-8, Ministry of Education of Brazil.

2. MC SQUARED European project ICT STREP 100712. http://mc2-project.eu. The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n° 610467 - project "M C Squared". This publication reflects only the author's views and Union is not liable for any use that may be made of the information contained therein.

3. IREM: Institute for Research in Mathematics Education, a network of 28 such institutes in France is devoted to studying math education and math teachers training. http://www.univ-irem.fr

4. TraAM (Travaux Académiques Mutualisés) are the IREM groups under regional educational authorities that develop and share resources aiming at supporting the use of technology in classrooms.

5. EvaCoDICE project (Évaluation par compétences dans les démarches d'investigation au collège et à l'école) http://ife.ens-lyon.fr/lea/le-reseau/les-differents-lea/evacodice

Mathematicians' views on undergraduate students' creativity

Gulden Karakok[1], Milos Savic[2], Gail Tang[3] and Houssein El Turkey[4]

1 University of Northern Colorado, Greeley, USA, gulden.karakok@unco.edu
2 University of Oklahoma, Norman, USA
3 University of La Verne, La Verne, USA
4 University of New Haven, New Haven, USA

There are studies investigating mathematical creativity in the primary and secondary levels, but there is still a need to explore creativity in the tertiary level. Our effort of expanding research to this level started with investigating mathematicians' views on creativity and its role in teaching and student learning of mathematics. One 60-minute interview was conducted with six mathematicians who teach courses at the tertiary level and are active in research. Two themes, Originality and Aesthetics, were observed capturing participants' views of creativity in their work, aligning with existing process and product views. In addition, all participants believed creativity could be encouraged in undergraduate courses and provided suggestions on how to cultivate and value creativity in courses focusing on proving and problem solving.

Keywords: Creativity rubric, proof process, undergraduate mathematics education.

INTRODUCTION

Creativity is one of the important aspects of professional mathematicians' work. It has been documented that many mathematicians describe creativity in their work as an enlightenment that is somewhat unexpected (Hadamard, 1945; Poincare, 1958). Furthermore, creativity helps in the development of mathematics as a whole (Sriraman, 2009). However, creativity is an intricate research construct to explore, made apparent by the myriad of definitions (over 100 as reported by Mann, 2005). In fact, Borwein, Liljedahl, and Zhai (2014) demonstrated that many brilliant mathematicians had differing views about mathematical creativity.

Some conceptualizations of creativity focus on emphasizing whether the *end-product* is original and useful (Runco & Jaeger, 2012), while others describe it as a *process* that involves different modes of thinking, some of an unusual nature (Balka, 1974). Liljedahl and Sriraman (2006) suggested a definition in which creativity was viewed as a personal trait of a mathematician; "the ability to produce original work that significantly extends the body of knowledge (which could include significant syntheses and extensions of known ideas)" (p. 18). In addition, Sriraman (2005) argued that creativity in K-12 classrooms is different than the kind employed by mathematicians and that students' creativity needs to be evaluated according to their prior experiences. This particular point highlights the difference between *absolute* and *relative* creativity; the former one refers to historical inventions or discoveries at a global level and the latter one is defined as, "the discoveries by a specific person within a specific reference group, to human imagination that creates something new" (Vygotsky, 1982, 1984; as cited by Leikin, 2009, p. 131). Using a relativistic perspective, Sriraman and Liljedahl (2006) define mathematical creativity at the school level as a process of offering new solutions or insights that are unexpected for the student, with respect to his/her mathematics background or the problems s/he has seen before. This particular definition acknowledges that students "have moments of creativity that may, or may not, result in the creation of a product that may, or may not, be either useful or novel" (Liljedahl, 2013, p. 256).

Despite the acknowledgment of differences in creativity between professional mathematicians and K-12 students and studies in K-12 level, the mathematical creativity research in undergraduate mathematics education has been sparse. The purpose of our research project is to explore creativity in undergraduate level teaching and learning by first focusing on mathematicians' views on creativity. Researchers have investigated mathematicians' views on creativity (e.g., Hadamard, 1945; Sriraman, 2005), but we contribute to the existing literature by focusing on mathematicians' views on the role of creativity in teaching and students' learning, especially in the case of proving and problem solving. More precisely, our research addresses the following research questions: (1) How do mathematicians define creativity? (2) How do mathematicians view creativity in undergraduate mathematics courses, especially the ones that focus on proving and/or problem solving? (3) Can we (or should we) value and/or assess undergraduate students' creativity in proving (and/or problem solving)?

Prior to sharing findings of our analysis, we briefly summarize some studies that explored creativity from mathematicians' views and in K-12 levels.

LITERATURE REVIEW

Exploring mathematical creativity is an ongoing quest of researchers, with the earliest known attempt by two psychologists Claparède and Flournoy in 1902 (as cited in Borwein et al., 2014, and Sriraman, 2009). Following this survey-focused attempt, which elicited voluntary responses from mathematicians of that time, Hadamard (1945) resumed the exploration of mathematical creativity with surveys of his own sent to prominent mathematicians across the world. Using a psychological framework created by Wallas (1926), Hadamard theorized four stages in the process of creativity: preparation (thoroughly understanding the problem), incubation (when the mind goes about solving a problem subconsciously and automatically), illumination (internally generating an idea after the incubation process), and verification (determining whether that idea is correct). Sriraman (2005) found that Hadamard's four stages are still applicable to modern day mathematicians by interviewing five research mathematicians. Furthermore, his study provided more detail of the stages by considering the roles of personal and social attributes such as imagery, intuition, and interaction with others. Guilford (1950), however, found Hadamard's stages "superficial from the psychological point of view" (p. 451). His concerns were that these stages were not informing us on the mental processes that occur and the stages were not testable. He suggested some testable factors such as fluency, flexibility, production of novel ideas, synthesizing and analysing ability, and evaluation ability. His list was refined to *fluency*, *flexibility*, *originality*, and *elaboration*, which were expanded and used in forthcoming creativity research by others (e.g., Balka, 1974; Leikin, 2009; Torrance, 1966; Silver, 1997).

Fluency in general refers to the quantity of outputs. Silver (1997) defined it in the problem-solving setting as the "number of ideas generated in response to a prompt" (p. 76). *Flexibility* is "shifts of approaches taken when generating responses to a prompt" (Silver, 1997, p. 76). This could mean that a student approached a certain task, was not successful with finding a solution or did not feel the approach was going to be fruitful, and changes to a new approach. *Originality* (or novelty) is described as a unique production or an unusual thinking (Torrance, 1966). *Elaboration* refers to the ability of producing detailed plan and generalizing ideas (Torrance, ibid). These factors of creativity have been used in K-12 levels to determine students' creativity. For example, Leikin (2009) focused on fluency, flexibility and originality to create a creativity rubric (using a point system) that evaluated how creative a student was when s/he produced solutions of certain tasks. Similarly, Yuan and Sriraman (2011) integrated the same three aspects to measure participants' creativity in mathematical problem posing. Recently, Chamberlin and Mann (2014) proposed a fifth aspect of creativity, *iconoclasm,* which "entails the penchant of mathematically creative individuals to dissent from commonly accepted principles and solutions" (p. 35). They suggest the possibility of observing iconoclastic behaviour in individuals who are considered to have a high degree of creativity.

Even though exploring students' creativity in K-12 level is a common practice, such efforts are sparsely expanded to undergraduate mathematics level. There have been on-going efforts of implementing new pedagogical strategies (such as inquiry-based learning or problem-based learning) to improve undergraduate students' skills that are related to creativity (such as investigating ideas, providing multiple solutions, analysing others' strategies). However, we know little about how to explicitly value or assess undergraduate students' creativity in courses involving proving, or in more traditional teaching settings. To expand our understanding of how creativity can be cultivated while learning mathematics, we conducted a qualitative research study investigating mathematicians' views of creativity and their perspectives on its role in teaching and learning of mathematics.

METHODOLOGY

Participants

Participants of this study were six mathematicians who are active researchers in their mathematical areas and teach undergraduate and graduate level mathematics courses. Two of the participants are from a mid-size Ph.D. granting, but predominantly teaching mid-western university, and four participants are from a large Ph.D. granting and research dominant mid-western university. Two are tenured associate professors and four are tenured full professors, and one of whom is female. Participants had 8 to 30 years of teaching and research experience. Research fields vary from algebraic geometry, nonstandard analysis, geometry and topology, representation theory, signal analysis, and number theory.

Data collection

We conducted one 1-hour interview with each participant in his/her office. Interviews were audio- and video-taped and transcribed. The semi-structured interview had three parts. Participants were asked to talk about their views of creativity in their mathematics work in part one. In the second part, participants were asked to comment on a given a set of creativity definitions from different theoretical perspectives. The third part of the interview focused on participants' views on teaching and learning. To elicit their initial thoughts, participants were first asked to talk about their perspectives of teaching creativity and its potential role in students' learning. After, we gave three proofs constructed by three students (Birky, Campbell, Raman, Sandefur, & Somers, 2011), one at a time, and asked them to comment on creativeness of these proofs (See Appendix). All participants were then given a Creative Thinking Value Rubric (Rhodes, 2010) to evaluate the same proofs. This rubric was chosen by the authors because it claims to assess undergraduate students' creative thinking across disciplines.

Analysis

We employed grounded-theory methodology (Strauss & Corbin, 1998). Researchers independently read the transcripts several times to select passages "that express a distinct idea related to [our] research ideas" (Auerbach & Sliverstein, 2003, p. 46). After identifying these relevant texts, we searched for repeating ideas that could be combined into themes. The emergent themes describing mathematicians' views of creativity were *Originality* and *Aesthetics*. These two themes match with the *process* view and the *end-product* view in the creativity literature, respectively. Quotes that highlight the characteristic of each theme are shared with brief discussions in the next section. We also present our analyses of the mathematicians' views on teaching and student learning in relation to creativity. More precisely, we describe our participants' views on how to cultivate students' creativity in proof and problem solving courses, and how to identify, evaluate and value student creativity in such courses.

RESULTS

Views on creativity

Though participants were asked for their definitions during the first part of the interview, they stated different aspects of creativity as they responded to questions in other parts. For example, some participants would refer back to their definitions of creativity as they explored given students' proofs or as they discussed given definitions. We also noticed that some participants would contradict their definitions when discussing the students' proofs, so we asked, "How does this particular idea you mentioned align with your previously mentioned view of creativity?" For these reasons, we analysed each participant's entire interview to uncover his/her views of creativity, thereby generating a more holistic view of his/her perspective.

Overall, we observed two main themes in our participants' views of creativity: *Originality* and *Aesthetics*. Since the first theme shared some similar aspect with the originality (novelty) described in the previous literature, we used the same word. All of our participants mentioned creating a "new way", "new approach/strategy" or "new trick" when they described creativity in mathematics, generally or in their own work.

Dr. B — So for me the creative aspect is you introduce a new way to look at the problem.

Dr. C — [While talking about his creative moment] So, it wasn't that creative in the sense that there was already stuff out there that I didn't have to think about it myself, but in the process of applying it I think I created something new.

Dr. E I think the definition of creativity is approaching a problem from a different perspective or with different tools.

Within the theme of *Originality,* we noticed some sub-themes that the participants referred to as they described the process of creativity: (i) *Making Connections,* (ii) *Attempts,* and (iii) *Insight.*

All of our participants highlighted the importance of making some sort of connection. For example, Dr. A mentioned seeing connections between the task at hand and other theorems, whereas Dr. D mentioned making connections between various topics of mathematics to approach a task from a new way. Similarly, Dr. F stated,

Dr. F For example, notic[ing] that some equations result in geometry, with some geometry connects to some algebra, doing something unusual.

Participants also emphasized that in the process of creating something original, having several attempts, even incorrect ones played an important role.

Dr. A If it did happen [an incorrect attempt], then it is creative in some sense because exploring a wrong answer helps.

Dr. D Well, let's see the most recent example I can think of is when my creativity took us in a wrong or took us in a negative direction.

Further in the interview when Dr. D was asked about the role of creativity it was observed that s/he again mentioned the importance of making attempts by saying, "You have to be willing to try something."

We observed that some of our participants mentioned the role of intuition or insight in the process of creativity. This particular sub-theme, *Insight*, was not shared by all of our participants. The quotes below demonstrate ideas from these participants.

Dr. A But having the idea [of a proof in his research] was the spark...It's that initial moment that is the creative part, not the actual carrying out the thing.

Dr. D I think [...] there are people that have more and less or different types of creativity. There are mathematicians who are very intuitive, [...] who can see a broad result.

We also noticed that some of our participants stated the importance in making conjectures. For example,

Dr. A So, I was writing a paper with a colleague just last week. So I was working on that and there was an implication that the colleague had proved, A⇒B. And it occurred to me...is the converse true? Does B also imply A?

Even though this idea was not repeated by other participants, this quote from Dr. A provides a valuable insight for encouraging students' to make conjectures in courses.

The first theme of *Originality* and its sub-themes thus far speak to the *process* of proving rather than the *end-product* proof. The latter view is encompassed in the second theme: *Aesthetics*. Almost all of our participants mentioned some aspect of creativity that relates to the look of the final proof. In the following quotes, we underlined some code words that helped us create this particular theme.

Dr. B [when talking about a "creative proof"] But there is a notion...there is a <u>notion of economy</u>, of something surprising that you would not expect in that proof, and something <u>lovely</u>.

Dr. C There's kind of an element of <u>aesthetics</u> involved, or <u>beauty</u>, and so when I think of creativity in mathematics I think of people that are able to pull that out of themselves and come up with nice problems to solve that are <u>attractive</u>, somehow. Or follow a line of thought that is <u>attractive</u>.

Dr. E [when evaluating a student proof] Wow! Yeah, that does seem more creative, that is <u>cute</u>. That is really <u>cute</u>!

Dr. F [when discussing creativity in teaching] When I'm teaching classes for example, sometimes I find a <u>cute</u> way of doing the proof, or I find an <u>elegant</u> way of doing computation.

Role of creativity in teaching and learning

To understand our participants' perspectives on the role of creativity in teaching and learning undergraduate mathematics, we focused on the responses that were given during the third part of the interview. In this part, participants were first asked to talk about their ideas on how to cultivate students' creativity when teaching and then were given three student-constructed proofs to assess the creativity. The analysis of summaries from each participant yielded similar ideas.

All of our participants believed that creativity could be encouraged in undergraduate courses. They provided similar teaching ideas, which could foster creativity. For example, providing problems and letting students "play" with them and discuss different solution techniques would be one way to cultivate creativity. Another example was to show students different proofs of the same theorem and discussing the ones that are more creative and why they would be considered more creative.

Participants also discussed possible ways of evaluating creativity in proofs. They were given three student-constructed proofs (See Appendix) to the theorem "If n is an integer such that $n \geq 3$, then $n^3 \geq (n+1)^2$." All of our participants determined that the first student's proof was not creative due to the fact that the induction proof technique was an expected method to implement in this particular question. They thought the second and third proofs were more creative, and some participants discussed the importance of using prior knowledge and making connections between the tasks and the student's existing knowledge as they talked about these two proofs.

Dr. E [referring to the second proof] It is possible that they had a course where this kind of trick was really looked at.

Furthermore, participants acknowledged the mistake in the third proof, which started a conversation about the role of correctness in creativity. All participants thought incorrect attempts could play an important role during the process of creativity. However, some of our participants thought that such incorrect ideas should be fixed in the final product.

Dr. F I will risk it and say that [a proof] doesn't have to be correct to be creative. But at least it should be fixable. It can happen that you have an original idea and you mess up details, which is not surprising because if it is an original idea then it means that you haven't practiced that, you would make mistakes.

Some participants stated that they do give "higher" points to students' proofs or solutions to problems if they thought that the approach was original or unexpected. Other participants were hesitant to give extra points to "creative" proofs or solutions but said they would provide written encouraging comments to students' work.

When participants viewed and tried to implement the Creative Thinking Value Rubric (Rhodes, 2010) to evaluate three proofs, they designated applicable and inapplicable categories to mathematics. For example, all participants agreed that *Taking Risks, Innovative Thinking,* and *Connecting, Synthesizing, Transforming* categories would be applicable. They believed that the *Solving Problems* category by itself was what they expect their students to do in mathematics so it would not be applicable. Some participants thought that there were too many levels provided in this rubric (*Capstone, Milestones,* and *Benchmarks*).

DISCUSSION AND CONCLUSION

The purpose of this particular study was to explore participating mathematicians' views of creativity and its role in teaching and student learning at tertiary level. The first question relating to participating mathematicians' definitions of creativity is described by two observed themes: *Originality* and *Aesthetics*. All six mathematicians' views of creativity highlighted the notion of *Originality*. That is, we noticed that our participants discussed the process of creating ideas and mentioned the importance of making connections, trying or attempting different solutions, and having an insight or "spark," which Wallas (1926) called the "illumination" stage of creativity. In the process of creating ideas, connections between different mathematical knowledge require the individual to understand and absorb many previous definitions and theorems, which Wallas (1926) called the "preparation" stage of creativity.

With our second research question, we investigated the actions and thoughts of mathematicians with regard to the teaching and learning of mathematical creativity. We found an emphasis on the process of proving when participants were asked to evaluate students' proofs based on creativity. They thought in order to evaluate a final product or final proof, they either judged them against one another, or, similar to Sriraman and Liljedahl (2006), they needed to know the student's prior knowledge and thought process. Some mathematicians thought that the proving process might reveal mathematical creativity aspects of a student or a mathematician not revealed in the final proof. For example, the student that created proof 1 may have tried proofs that resemble proofs 2 or 3 and realized that they may not be as fruitful as the technique for proof 1. Also, having courage to take a risk and create an attempt or multiple attempts was determined to be an aspect of mathematical creativity. Therefore, valuing multiple attempts might encourage students to be creative in their proving process (Leikin, in press). But, as Dr. E stated, there is some caution in how to value those attempts; a student might make multiple attempts that have "zero chance of working."

Finally, we attempted to answer the third research question by utilizing the interviews and previous creativity rubrics (Leikin, 2009; Rhodes, 2010) to create the Creativity-in-Progress Rubric (CPR) on proving (see Savic et al., 2015, and Tang et al., 2015, for more details.) This rubric is a formative assessment to explicitly value undergraduate students' creativity during the proving process in proof-based courses at the tertiary level. Given the heavy emphasis the mathematicians placed on the process of construction, the CPR focuses on assessing the process of proving (*Originality)* rather than the final product of the proof itself (*Aesthetics*). As a formative assessment tool, the rubric has three major categories *Making Connections, Taking Risks* and *Creating Ideas*. These categories align with our participants' views of creativity and its role in undergraduate mathematics. In addition, the use of the rubric as a formative tool has merged from participants' suggestions to encourage creativity in the classroom. The greatest use of the CPR on proving, we believe, is that it can start the discussion of creativity and the proving process in the classroom. Mann (2005) states that avoiding the acknowledgment of creativity could "drive the creatively talented underground or, worse yet, cause them to give up the study of mathematics altogether" (p. 239). Since there is an increased need for students to have research-like experiences (e.g., Research Experiences for Undergraduates [REUs] (Garcia & Wyels, 2014)), valuing mathematical creativity may bridge the gap between undergraduate mathematics and research mathematics. In particular, using formative assessment tools, such as the *CPR* on proving would help in such efforts.

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APPENDIX - THREE STUDENTS' PROOFS FROM (BIRKY ET AL., 2011)

Proof 1: If $n=3$, then $n^3=27$ and $(n+1)^2=16$, so $n^3>(n+1)^2=16$. Now assume that $k^3>(k+1)^2$, for some integer $k\geq 3$. On the left-hand side, we add $(3k^2+3k+1)$ and get $k^3+3k^2+3k+1=(k+1)^3$. On the right-hand side, we add the same thing to get $(k+1)^2+(3k^22+3k+1)=(4k^2+5k+2)$. We see that $(4k^2+5k+2)>(k^2+4k+4)$ because $4k^2>k^2$ and, since $k>2$, $5k+2>4k+2+2=4k+4$. Thus we see that $(4k^2+5k+2)>(k+2)^2$ and so we have $(k+1)^3>(4k^2+5k+2)>(k+2)^2$. Therefore, by the principle of mathematical induction, $n^3>(n+1)^2$ for all integers $n\geq 3$. QED

Proof 2: Assume $n\geq 3$. Then $n>2$, so $1>2/n$, and $n^2>1$, so $1>1/n^2$ also. This means that $n\geq 3=1+1+1>1+2/n+1/n^2$. So $n>1+2/n+1/n^2$. If we multiply each side of this last inequality by n^2, we get $n^3>n^2+2n+1$. Thus $n^3>(n+1)^2$. QED.

Proof 3: Assume $n\geq 3$. Then, since $n-2$ is a positive integer, $(n+1)^2<(n+1)^2(n-2)$. Thus, $(n+1)^2<n^3-3n-2=n^3-(3n+2)<n^3$. Therefore, $n^3>(n+1)^2$. QED.

Teachers of the mathematically gifted: Two case studies

Alexander Karp[1] and Vasily Busev[2]

1 Columbia University, Teachers College, New York, USA, apk16@columbia.edu

2 Ushinsky State Pedagogical Library, Moscow, Russia

This article is devoted to people who taught highly gifted schoolchildren and developed educational materials for them. Teaching the highly gifted involves changing the contents of education and, possibly more importantly, changing its form and style. Consequently, mathematics educators who are involved in such teaching face new problems, and in fact such teaching often means the appearance of a special new type of teacher at a school. In this paper, we will discuss two prominent Russian figures in the advanced course of study in mathematics. An analysis of their biographies will facilitate a better understanding of both the methodological and the pedagogical history of the advanced course of study in mathematics, as well as the social-political circumstances under which such study took place in the Soviet Union.

Keywords: Advanced course of study in mathematics, schools with an advanced course of study in mathematics, teachers of the mathematically gifted.

INTRODUCTION

Highly gifted children need highly gifted teachers. This fact was recognized quite long ago and quite widely (Stanley, 1987; Thornton and Peel, 1997), yet the study of the biographies and preparation of teachers of the highly gifted is only just beginning (see Even, Karsenty, & Friedlander, 2009; Evered & Karp, 2000; Karp, 2010; Leikin, 2011). Below, we will discuss two Russian figures in mathematics education, Vladimir Ashkinuze and Semyon Shvartsburd, who played important roles in the formation of the advanced course of study in mathematics and, first and foremost, in the formation of Russian schools with an advanced course of study in mathematics. Each of them is the author of dozens of important works. However, far less is known about their own biographies (see the obituary of S. I. Shvartsburd in the journal *Matematika v shkole*), and these will be the main focus of what follows, since the analysis of their biographies, we would argue, is useful for understanding where teachers for the advanced course of study in mathematics come from—an understanding that this study severely needs. The present article is based first and foremost on archival materials related to these two figures, which have been discovered by Vasily Busev.

ON SCHOOLS WITH AN ADVANCED COURSE OF STUDY IN MATHEMATICS

Russian schools with an advanced course of study in mathematics began attracting international attention practically from the moment of their inception (Vogeli, 1968); nonetheless, it will be useful to review certain key moments in their development.

The first Russian classes with an advanced course of study in mathematics appeared in 1959 under the supervision of Semyon Shvartsburd in Moscow under the banner of the "polytechnization" campaign that was taking place at that time, which sought to combine school-based education with production-based education (Karp, 2011). Subsequently, a number of leading Soviet mathematicians became involved in the creation and development of a system of schools with an advanced course of study in mathematics, including A. N. Kolmogorov, I. M. Gelfand, M. A. Lavrentyev, V. I. Smirnov, D. K. Faddeev, and many others. As a result, fundamentally new educational institutions were formed, in which gifted schoolchildren were offered the opportunity not only to learn mathematics more and better than usual, but also to become involved in genuine mathematical creativity. The unfolding methodological formation of the new system was evidenced by publications in the series *Problems of Mathematics Schools*, the first of which, *Teaching in*

Mathematics Schools, was edited by S.I. Shvartsburd, V.M. Monakhov and V.G. Ashkinuze (1965).

The first mathematics schools were established during the period of the so-called thaw, after the death of Stalin, when there were hopes that the Soviet regime would become more liberal. When after the resignation of Khrushchev and especially after the invasion of Czechoslovakia by Soviet troops in 1968, these hopes had to be abandoned, the position of these schools began to change. The authorities did not dare to shut them down completely, since they saw in them a most valuable source of trained workers for the military-technological struggle with the West. However, every effort was made to restrain the spirit of freedom that inevitably sprung up in schools which awakened and nurtured the striving for independent research. Naturally, the position of teachers in these schools changed as well (Karp, 2011).

At the same time, a certain development of the mathematics schools continued, despite the rigidification of the political regime. In particular, textbooks for such schools became popular during this period, for example, *Algebra* by N. Ya. Vilenkin, R. S. Guter, S. I. Shvartsburd, B. L. Ovchinsky, & V. G. Ashkinuze (1968), and *Mathematical Analysis* by N. Ya. Vilenkin & S. I. Shvartsburd (1969).

Gorbachev's *Perestroika* made mathematics schools popular and brought them government support. Their numbers grew rapidly (which was probably not especially good for their quality—Karp, 2011). One of the textbooks that enjoyed greatest popularity at this time was written by M. L. Galitsky, M. M. Moshkovich, & S. I. Shvartsburd (1986). S. I. Shvartsburd died in 1996 in Haifa, and V. G. Ashkinuze yet in 1970s, so we will not address mathematics schools during later periods in any detail. We will merely state that, despite the difficulties that they have encountered, they continue to exist and successfully to prepare future mathematicians, physicists, and engineers.

ON THE ARCHIVAL FILES OF V. G. ASHKINUZE AND S. I. SHVARTSBURD

All three files—two relating to V. G. Ashkinuze (Ashkinuze, 1963, 1967) and one relating to S. I. Shvartsburd (Shvartsburd, 1974) —were discovered in the archives of the Russian Academy of Education (RAO). The RAO is a continuation of the Academy of the Pedagogical Sciences (APN), which was established in 1943 in order to oversee and coordinate the work of scientific organizations of a psychological-pedagogical profile. Such organizations (scientific institutes and laboratories) were many, and they employed many famous scientists, for example, most of the Russian scholarly works on psychology that are known in the West were written by employees of the institutes of the Academy of the Pedagogical Sciences.

One can only speculate about the nature of the circumstances under which the files analyzed below began to be kept. Vladimir Ashkinuze worked for a certain time at the Academic Institute of General and Polytechnic Education (which until 1960 was called the Institute of Methods of Instruction), and therefore the existence of a file on him during this period was to be expected. And indeed this file mainly contains materials documenting his hiring and discharge from the institute, including his personal history statement, official forms (the so-called personal form for personnel records), character references, and his reports (this file was ended in 1963). When at a later date V. G. Ashkinuze, now employed elsewhere, returned to the APN system as a part-time employee, a new file on him was begun (and ended in 1967).

S. I. Shvartsburd likewise worked in the APN system from 1962 on, initially at the Institute of General and Polytechnic Education, and then at the Institute of the Contents and Methods of Instruction. By all appearances, however, the file on Shvartsburd began to be kept for a special reason, namely, because he was nominated as a candidate for full membership in the APN (he had been elected as a corresponding member in 1968). The cover of the file bears the following heading: "application for full member status in the APN." The file contains biographical details, including Shvartsburd's personal history statement, reviews of his work, lists of his publications, and so on (this file was ended in 1974).

TOWARD A BIOGRAPHY OF V. G. ASHKINUZE

V. G. Ashkinuze was born in Moscow in 1927, as he writes in his 1958 personal history statement, into a "family of Communist public servants." He goes on to state that "at the present time my parents are special pensioners" (Ashkinuze, 1963, sheet 5). A "special pension" was a pension assigned by special warrant, rather than in accordance with general rules; in a later personal history statement from 1966, he writes more clearly that his parents had been Communist Party functionaries (Ashkinuze, 1967, sheet 5), who had, however, already lost their positions by 1958, when he wrote his first personal history statement. In 1944 Ashkinuze enrolled in the Mathematics-Mechanics Department and in 1949 he graduated from it. As can be seen from the list of published works in his file (Ashkinuze, 1963, sheet 8), in 1951 he published an article in the *Ukrainian Mathematics Journal*: "Theorem on the Splitability of J-Algebras." Taking into account the fact that publication in a journal does not happen overnight, one can assume that the article contained Ashkinuze's scientific findings from his last years of study at the university (possibly his thesis). The promising young mathematician, however, began his professional career as a mere teacher, and the institutions at which he taught included the so-called School for Working Youth, by no means the most prestigious educational establishment. It should be borne in mind that the end of the 1940s and beginning of the 1950s were the years of the anti-Semitic, anti-cosmopolitan campaign, so it was probably difficult for Ashkinuze to find any better employment. In January 1954 (in other words, less than a year after Stalin's death), Ashkinuze was accepted to the graduate school of the Institute of Methods of Instruction, and moreover he enrolled at once in the second year of the program. In August 1955, Ashkinuze finished his graduate education and began to work at the Shakhty Pedagogical Institute (in the city of Shakhty) as teacher and department chairman. After working there for the required three years, Ashkinuze returned to Moscow, becoming in June 1958 a junior research employee of the Institute of Methods of Instruction (later called the Institute of General and Polytechnic Education). The papers that he published after his 1951 article were devoted to methodological issues, with the exception of an article published in the anthology *Matematicheskoye prosvescheniye* in 1957: "On the Number of Semiregular Polyhedra." Ashkinuze himself, listing his achievements for 1958–1959 (Ashkinuze, 1963, sheet 16), noted his participation in developing programs in mathematics for eight-year and three-year courses (in other words, for the upper grades); articles discussing these programs; testing out lessons on "The Derivative and Its Application in the Investigation of Functions" in classroom conditions, a topic that had not been studied in schools previously; delivering lectures on the methodology of teaching elementary mathematical analysis in secondary schools for teachers; working to develop a textbook for a course in algebra; and others.

Subsequently, these studies were continued, and new ones were added. A character reference from 1963 indicates that Ashkinuze was supervising two planned projects—the writing of a textbook in algebra and the writing of a study on the methodology of teaching algebra—in addition to heading a laboratory devoted to schools specializing in mathematics, which had been formed on his suggestion, and in connection with which he spent four months studying mathematical machines at another institute (Ashkinuze, 1963, sheets 48–49).

This character reference was written in connection with Ashkinuze's transfer to a different place of employment (the Moscow State Pedagogical Institute—MGPI). It is noteworthy that the purpose of the character reference was to smear Ashkinuze, to the extent possible, by describing him as an individual who was leaving a job and thus interrupting his planned projects, which "have already cost the Soviet state over 10 thousand rubles" (Ashkinuze, 1963, sheet 49). The director of the institute felt so strongly about the need to besmirch Ashkinuze's reputation that did not hesitate to add a personal attack, pointing out that Ashkinuze did not exhibit sufficient moral purity, taking work-related trips with his female laboratory assistant.

Nonetheless, in 1966 Ashkinuze returned to the APN system, while continuing to conduct his main work as a docent (associate professor) of the MGPI. He was accepted as a part-time employee at the newly founded Institute of School Equipment and Technological Means of Instructions (Levitas, 2010). However, he did not work there for long—only until 1967 (Ashkinuze, 1967, sheet 7). At this point, his files end.

TOWARD A BIOGRAPHY OF S. I. SHVARTSBURD

S. I. Shvartsburd was born in 1918 in a small town in Moldavia. His father, as he states in his personal history statement from 1974 (Shvartsburd, 1974, sheet 4), worked as a grain receiver before the Revolution and was killed in 1941. After graduating from a ten-year school in 1935, Shvartsburd enrolled in the Physics-Mathematics Department of Odessa University, from which he graduated in 1940, staying on at the same university as an assistant at the Department of Theoretical Mechanics. Having contracted polio in his childhood, he remained handicapped, and therefore was not drafted into the army. After the war began, in July 1941, together with the university, he relocated to the city of Jalalabad, where from 1941 until 1944 he worked as a mathematics teacher. In 1942, he joined the Communist Party (having become a candidate for membership already in 1939) and in 1943–44 he was bureau head and deputy head of the department of agitation and propaganda of the Jalalabad Municipal Committee of the CPSU. In August 1944, Shvartsburd returned to Moscow, where he worked as a teacher of mathematics until 1962, when he became a senior research fellow at the Institute of General and Polytechnic Education. He defended his candidate's dissertation in 1961 and his doctoral dissertation in 1973 (Russia has two advanced degrees: the degree of a candidate of sciences corresponds to a Ph.D., while the so-called doctoral degree is higher—roughly speaking, one can say that this degree corresponds to the rank of full professor). In 1968, he was elected as a corresponding member of the APN. Shvartsburd received various awards and honorary titles; he served as head of a number of influential organization: as chairman of the mathematics section of Academic Council of the State Committee on Professional Education of the USSR, as deputy head of the Academic Council on the Problem of the Advanced Study of Individual Subjects; as head of a seminar for the mathematics teachers of Moscow, etc. Shvartsburd developed numerous curricula, including—in collaboration of V. G. Ashkinuze, in 1963—curricula for secondary schools with on-the-job education for the "computer operator" track, which were important for the subsequent development of mathematics schools.

From the late 1950s on, Shvartsburd began closely collaborating with Professor Naum Vilenkin, a well-known mathematician and mathematics educator (their first joint publication, "On Certain Applications of Exponential and Logarithmic Functions," appeared in the journal *Matematika v shkole* in 1959). This collaboration continued in the field of the advanced study of mathematics, and in the development of curricula for grades 4–5 (the general supervisor of this work was A. I. Markushevich).

Shvartsburd's file contains reviews of his work by a number of leading figures in mathematics and mathematics education. Thus, Andrey Kolmogorov wrote:

> In the person of S. I. Shvartsburd, we have an outstanding representative of our mathematics education. In his field of organizing student preparation in applied mathematics in secondary schools and secondary educational institutions, he is a central figure. (Shvartsburd, 1974, sheet 24)

Elsewhere, A. N. Kolmogorov emphasizes Shvartsburd's contribution to the construction of "a system of advanced mathematical preparation with a focus on numerical analysis in the upper grades of secondary schools" (sheet 23), which he distinguishes from working with schools with an advanced course of study in mathematics (which he also credits Shvartsburd with doing, listing the textbooks that Shvartsburd wrote for such schools).

A. A. Lyapunov, a corresponding member of the USSR Academy of Sciences, mentions the fact that S. I. Shvartsburd has done a great deal

> for various types of intensive, specialized secondary mathematics education. These include different forms of elective work (such as mathematics circles, Olympiads, etc.), specialized physics-mathematics schools, classes in computer programming, and also mathematical technical colleges preparing mid-ranking professional mathematicians. (Shvartsburd, 1974, sheet 25)

The file also contains a long letter signed by a number of figures in mathematics, pedagogy, and technology, supporting S. I. Shvartsburd's promotion to the rank of doctor. The signatories of this letter include such well-known mathematicians as V. A. Uspensky.

I. Shvartsburd was not elected to full membership in the APN. His file, however, contains no information about this.

DISCUSSION AND CONCLUSION

The files examined above are interesting for several reasons. First, they offer examples of Soviet clerical practices in the scientific and scientific-methodological sphere. Today, millions of people are still alive who took part in that life, but even so, although a great deal has remained unchanged since Soviet times, many details have disappeared and are becoming forgotten, for which reason it is important to record them (for example, the fact that in order for Ashkinuze to be hired as a part-time worker, permission had to be obtained from the Deputy Minister of Higher and Secondary Education of the USSR).

Second, the files pertain to individuals whose roles in the creation of mathematics schools and of intensive, specialized education in general—to use A. A. Lyapunov's terminology—was extraordinarily important. A. N. Kolmogorov does not name Shvartsburd as the founder of the advanced study of mathematics in Russia, drawing a subtle distinction between applied advanced study, in which he clearly acknowledges Shvartsburd's preeminence, and advanced study in general, in which Shvartsburd is allotted an important, but not a leading role. In reality, the advanced study of mathematics as a whole, as has already been noted, grew out of applied advanced study, although, of course, in this process the role of, say, Kolmogorov himself was undoubtedly unparalleled. The names of the creators of a new system that exerted an influence over the entire world should not be forgotten, and consequently, the collection of details about their biographies is valuable in itself.

But perhaps even more important is the question posed at the beginning of this article: where do leaders in the education of the mathematically gifted come from? In an earlier work (Karp, 2010), we discussed teachers, famous ones, but nonetheless specifically teachers, who continued working in schools. Both Ashkinuze and Shvartsburd left schools and became involved in secondary mathematics education at a different level. Nonetheless, in this case, too, it may be said that the biographies of future leaders in the advanced study of mathematics can be very varied. It would be inaccurate, for example, based on the fact that mathematics schools were often perceived as hotbeds of liberalism (which they indeed were), to draw the conclusion that the individuals involved in organizing the advanced study of mathematics were inevitably opposed to the Soviet regime. Such individuals could well wish to serve the regime in good faith, occupying high-ranking positions if this was possible. To be sure, the regime did not always agree to offer them such positions, and could even refuse their service altogether (in the still-liberal year of 1968, S. I. Shvartsburd got elected to the APN as a corresponding member, while in 1974, as a Jew who moreover did not occupy any high administrative position, he was not elected to the same organization as a full member). Both Ashkinuze and Shvartsburd evidently had excellent preparation in mathematics, and this is probably the only characteristic shared by all prominent figures in the advanced study of mathematics. Another characteristic that is typical of the biographies of many of them, although naturally far from all, is a relative lack of access to the usual career paths, at least at the early stages of their careers. There is little use in speculating what might have happened if V. G. Ashkinuze had been admitted to graduate school in mathematics immediately upon graduating from university. We have no information concerning the circumstances that caused S. I. Shvartsburd to break off his party-member career and to remain a school teacher for almost twenty years (even while maintaining ties with academic circles, see Sector of mathematics, 1948).

Analyzing the life of her father, Aron Maizelis, who was an outstanding teacher in mathematics schools, Yelena Platonova (2007) objects to the explanation that Maizelis went to work in schools because he had been unable to find employment elsewhere, showing that even as a student Maizelis had been passionately interested in questions of teaching and instruction (which, however, does not seem to us to rule out the possibility that, if he had been given the opportunity to teach at a higher educational institution, he would not have done so). In any case, given the absence of precise biographical information, one can only speculate. It may be argued, however, that both Ashkinuze and Shvartsburd were in situations in which, in order to achieve success, they had to turn their lives around somehow, to find something new, which had not been envisioned previously. Thornton and Peel (1999) wrote about the importance of the teacher's creativity for cultivating creativity in the students. In certain cases, life itself impelled the teachers to be creative.

In conclusion, we should say that the aforementioned shortage of information poses new problems for researchers. It would be extremely interesting to continue gathering information both about those figures in mathematics education who are the focus of this article, and about others. One would like to hope that personal and state archives still exist in which documents have survived that are capable of shedding a light on the history of mathematics schools and their creators. Such research must be continued.

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Mathematical creativity or general creativity?

Maria Kattou, Constantinos Christou and Demetra Pitta-Pantazi

University of Cyprus, Department of Education, Nicosia, Cyprus, kattou.maria@ucy.ac.cy

The present study aims to investigate whether creativity is domain general or domain specific, by relating students' performance on two tests: the Creative Thinking Test and the Mathematical Creativity Test. Four hundred and seventy six students (Grades 4–6) participated in the study. Through confirmatory factor analysis we purported to compare the fitness of two a-priori theoretical models, representing creativity either as domain general or domain specific. Correlation and crosstabs analysis were also conducted in order to examine whether the data obtained from the two creativity instruments were related and/or were in agreement, respectively. Data analysis converged that creativity is domain specific.

Keywords: Domain general, domain specific creativity.

INTRODUCTION

The generality-specificity issue of creativity "goes to the heart of the question of what it means to be creative" (Baer & Kaufman, 2005, p. 158). Indeed, theoretical and practical issues are related to the way that creativity is approached. For instance, some of the proposed definitions embrace context, assuming that the creative ability is specialized in one or more cognitive fields, whereas others do not make any reference to specific areas (Baer & Kaufman, 2005). Moreover, the adopted identification procedures are either based on a general creativity test, in order to predict individual's creative potential in every field, or use domain specific instruments. Similarly, the educational practices and material for the enhancement of students' creative ability either focus on the development of general abilities that are applicable across domains or they are differentiated amongst various domains.

Taking into consideration the previous discussion, the necessity for conducting research that will shed light on this controversial issue of generality-specificity seems warranted (Silvia, Kaufman, & Pretz, 2009). Indeed, such research is prerequisite of any other research in the domain due to its fundamental position to direct the appropriate adopted theoretical definition as well as the methodology employed and data analysis conducted. Hence, the present study purports to contribute in this direction by examining whether creativity is domain general or domain specific.

What is following is a review on the existing literature about defining and approaching creativity as domain general or as domain specific. Afterwards the methodology used in the study will be presented. Finally we will present the results obtained from the data analysis and discuss them.

THEORETICAL CONSIDERATIONS

Creativity as domain general

Domain general perspective views creativity as a universal ability which contributes to all creative achievements (Plucker, 1999), assuming that this ability is transferable and applicable in any cognitive area (Plucker & Zabelina, 2009). According to Beghetto and Kaufman (2009), the creative expression is similar in all cognitive areas and therefore, a person who has demonstrated a high level of creativity in a field is anticipated to show correspondingly high level of creativity in various fields.

Domain general view of creativity has been supported by empirical findings which either verified the predictive power of domain general instruments on creative achievements in several fields or concluded with similarities by comparing the creative ability in different fields (Beghetto & Kaufman, 2009). In particular, Charyton and Merill's study (2009) found high consistency between a general creativity instrument and a domain specific instrument, concluding that the creative behavior is independent of the field under investigation. Kaufman and Baer (2005) described the creative process and the creative product in various disciplines (e.g., poetry, psychology, mathematics), concluding that creative individuals exhibit similar characteristics and skills in a wide range of disciplines. Similarly, Root-Bernstein and Root-Bernstein (2004) examined the background of creative scientists and artists and found that many scientists were artistic and vice versa. Thus, they proposed that the distinction between different types of creativity might be artificial.

Creativity as domain specific

Domain-specific creativity emerged as a major theoretical position in the 1980s, assuming that creativity cannot be understood if there is no reference in the domain in which it takes place (Plucker & Zabelina, 2009). In this context, the skills, the traits and knowledge that underlie creative ability are differentiated between cognitive areas (Beghetto & Kaufman, 2009). Several studies presented evidence to support domain specificity of creativity (e.g., Baer & Kaufman, 2005; Plucker, 1999).

In these studies researchers concluded with low correlations between creative results in different fields of knowledge indicating that there is not a general creative ability that contributes to creative performance in different disciplines (e.g., Plucker & Zabelina, 2009). In particular, Hocevar (1976, in Baer, 1998) found that the correlations among various self-report indices of creativity in different domains were low to moderate. In the same line, Baer (1991) found low or non-statistically significant correlations between verbal and mathematical creative tasks. In a systematic work conducted by Baer (1993) with students from childhood to adulthood, he concluded that independently of age groups and content the correlations were consistently low.

Furthermore, using more advanced statistical analysis researchers reached multiple-factors models to describe the structure of creativity, verifying that there are distinct domains that differentiate creative ability (Silvia, Kaufman, & Pretz, 2009). For instance, Kaufman and colleagues (e.g., Kaufman, Cole, & Baer, 2009) suggested a multi-factors model to describe creativity.

PURPOSE OF THE STUDY

By considering the abovementioned discussion, it is obvious that both the domain general and domain specific perspectives of creativity are supported by strong arguments leading to a polarization of the debate (Silvia, Kaufman, & Pretz, 2009). The field is still blurred and the question whether creativity is domain general or specific remains unanswered. Hence, this study aims to examine this issue.

Moreover, Silvia, Kaufman and Pretz (2009) claimed that the methodology and statistical analyses followed by researchers may affect their conclusion regarding generality and specificity. For instance, the majority of research studies which used correlation analyses concluded to creative generality while research studies which used advanced statistical methods found evidences for the domain specificity of creativity. In order to eliminate these limitations, in the present study we will use a combination of statistical analyses, in order to investigate whether different statistical approaches offer affirmative evidences for one direction.

This study has two purposes: (a) to investigate whether creativity is domain general or domain specific; (b) to examine whether the exploitation of different statistical approaches leads to similar conclusions concerning the generality-specificity issue.

METHODOLOGY

Four hundred and seventy six students attending Grades 4 (N=202), 5 (N=165) and 6 (N=109) in public schools in Cyprus participated in the present study. Aiming to investigate whether creativity is domain general or domain specific, two tests were administered: the Mathematical Creativity Test (MCT) and the Creative Thinking Test (CTT). The tests were administered in paper and pencil form and one hour was allocated to students for completing them (MCT: 40 minutes, CTT: 20 minutes).

Mathematical Creativity Test (MCT)

The MCT consisted of four multiple-solutions tasks with problem solving and problem posing situations (see Fig1), taking into consideration relevant research studies on mathematical creativity (e.g., Kattou, Kontoyianni, Pitta-Pantazi, & Christou, 2013). The selection of the four tasks included in the MCT was based on the results of a task analysis that took place at a previous phase of the study. For each task students were asked to provide multiple solutions, solutions that were distinct from each other and solutions that none of their peers could provide. This was done in an effort to capture students' fluency, flexibility and originality. These three abilities constituted the assessment criteria. In particular, the assessment was based on the number of correct mathematical solutions students' proposed (fluency), the number of different mathematical ideas included in students' answers (flexibility) and the scarcity of answers (originality) (Kattou, et al., 2013).

Specifically, we employed the assessment method proposed by Kattou and colleagues (2013): (a) Fluency score: we calculated the ratio between the number of correct mathematical solutions that the student provided, to the maximum number of correct mathematical solutions provided by a student in the population under investigation. (b) Flexibility score: we calculated the ratio between the number of different types of correct solutions that the student provided, to the maximum number of different types of solutions provided by a student in the population under investigation. (c) Originality score: it was calculated according to the frequency of a student's solutions in relation to the solutions provided by all the students (score 1 was given to students whom one or more of their solutions appeared in less than 2% of the sample's solutions, score 0.8 was given to students whom one or more of their solutions appeared between 2% and 5%, score 0.6 was given to students whom one or more of their solutions appeared between 6% and 10%, score 0.4 was given to students whom one or more of their solutions appeared between 11% and 20%, score 0.2 was given to students whom one or more of their solutions appeared in more than 20% of the sample's answers). Therefore, three different scores yielded for each student in each task. The final score of the test was obtained by adding the respective scores of fluency, flexibility and originality in the four tasks and then by converging them to a scale ranging from 0 to 1.

Creative Thinking Test (CTT)

The CTT included two tasks, one verbal and one figural, taken from the respective subtests of the Torrance Tests of Creative Thinking (Torrance, 1974). Concretely, the first task required students to provide unusual uses of a common everyday object, while the second one asked students to complete simple repeated figures to make a picture. Students were asked to provide as many answers as they could in a specific time interval. Their answers were assessed according to their fluency, flexibility and originality as described above.

Data analysis

The data were quantitatively analysed with the modeling program Mplus (Muthén & Muthén, 1998) in combination with the statistical package SPSS. Specifically, correlation analyses took place aiming to examine the correlation between participants' performances in the two tests. Moreover, crosstabs analysis allowed us to examine whether the two tests provided similar or different results regarding the identification of creative individuals. Confirmatory Factor Analysis (CFA) was employed, to test the validity of alternative theoretical models that present creativity as domain specific and domain general. The alternative models that were compared are discussed below.

Model 1 regards creativity as domain general. In this model creativity is defined across fluency, flexibility and originality, independently of the instruments used. Model 2 regards creativity as domain specific, implying that distinct creative abilities exist. Thus domain specific creativity in mathematics is differentiated from the creative ability that was measured with a test not targeted in mathematics. Each of the domain specific creativities is comprised by the abilities of fluency, flexibility and originality.

RESULTS

Correlation analysis

Aiming to investigate the existence of correlations between the creative abilities (fluency, flexibility, originality) of the same test or/and between the same crea-

Make as many groups of numbers as you can, using the numbers given below. Label each group with its characteristic.

2, 3, 7, 9, 13, 15, 17, 25, 36, 39, 49, 51, 60, 64, 91, 119, 121, 125, 136, 143, 150

Warnings: You can use each number in more than one group.
Each group should contain more than two numbers.

Figure 1: Example of tasks from the Mathematical Creativity Test

tive ability between the two tests, correlation analysis was conducted, as presented in Table 1. From the correlation analysis it can be deduced that all variables were significantly correlated with each other (p<.01). However, the correlations exist between the abilities of the same test were higher (ranging between r=.579 and r=.812), as they are compared to the correlations of the same ability across the two tests (ranging between r=.208 and r=.421).

To examine whether the two tests used in the present study identify the same participants as creative, crosstabs analysis was conducted. It is important to mention that the participants were primarily split in categories according to their performance in each test. Specifically, four groups emerged based on students' performance on the MCT and four other groups emerged based on their performance on the CTT as follow: Group 1 included students whose performance belonged to the lowest 15% of performances on the corresponding test, whereas Groups 2 and 3 included students whose performance belonged between 15%-50% and 50%-85% of performances, respectively. Group 4 included students whose performance belonged to the highest 15% of the performances (max score was 3 – the score was obtained by adding fluency, flexibility and originality scores). The descriptive statistics of each group of students are presented in Table 2.

Crosstabs analysis was also conducted aiming to investigate whether students who were classified as creative using the MCT were also creative according to the CTT and vice versa. The results of the analysis are presented in Table 3.

Data analysis showed that 4.20% of the participants were identified as creative with both instruments. However, among the students who were regarded as high mathematical creative (Group 4) only the 28.99% (20 out of 69) were also regarded as creative according to the CTT. Similarly, only 27.78% (20 out of 72) of the students who were highly creative using the CTT for their assessment (Group 4) were mathematically creative as well. Based on the abovementioned percentages, a student who is creative using as indicator his/her performance in one of the instruments is not necessarily identified as creative by the other instrument.

	CTT			MCT		
	Fluency	Flexibility	Originality	Fluency	Flexibility	Originality
Fluency CTT	1	.615	.579	.421	.237	.166
Flexibility CTT	.615	1	.595	.301	.208	.198
Originality CTT	.579	.595	1	.314	.225	.214
Fluency MCT	.421	.301	.314	1	.719	.621
Flexibility MCT	.237	.208	.225	.719	1	.812
Originality MCT	.166	.198	.214	.621	.812	1

Table 1: Correlations between fluency, flexibility, originality in MCT and CTT

	Groups according to CTT		Groups according to MCT	
	N (%)	Mean (SD)	N (%)	Mean (SD)
Group 1	72 (15.13)	.99 (.23)	71 (14.92)	.56 (.12)
Group 2	164 (34.45)	1.54 (.14)	168 (35.29)	.91 (.10)
Group 3	168 (35.29)	1.93 (.10)	168 (35.29)	1.24 (.11)
Group 4	72 (15.13)	2.26 (.14)	69 (14.50)	1.66 (.16)
Total	476 (100)	1.70 (.41)	476 (100)	1.08 (.35)

Table 2: Descriptive statistics of the groups of students

CTT	MCT				
	Group 1 N (%)	Group 2 N (%)	Group 3 N (%)	Group 4 N (%)	Total N (%)
Group 1	25 (5.25)	27 (5.67)	16 (3.36)	4 (0.84)	72 (15.13)
Group 2	19 (3.99)	69 (14.50)	60 (12.61)	16 (3.36)	164 (34.45)
Group 3	23 (4.83)	57 (11.97)	59 (12.40)	29 (6.09)	168 (35.29)
Group 4	4 (0.84)	15 (3.15)	33 (6.93)	20 (4.20)	72 (15.13)
Total	71 (14.92)	168 (35.29)	168 (35.29)	69 (14.50)	476 (100)

Table 3: Results of the crosstabs analysis

Confirmatory factor analysis

Confirmatory factor analysis allowed us to compare the validity of the structure of two alternative models for creativity. For the evaluation of model fitness, three indices were taken into consideration: The chi-square to its degree of freedom ratio (χ^2/df), the comparative fit index (CFI), and the root mean-square error of approximation (RMSEA) (Marcoulides & Schumacker, 1996). An acceptable model should have the value of CFI higher than .90, the value of χ^2/df lower than 2 and the value of RMSEA lower than .08 (Marcoulides & Schumacker, 1996). During the comparison of alternative models, apart from the appropriate values of indices, we choose the model with the highest CFI index and the lowest AIC and BIC indices (Marcoulides & Schumacker, 1996). What follows is a description of the structure of the two alternative models, however, only the model with the best “fitness” on the data of the present study will be presented diagrammatically, due to space limitation.

In Model 1 creativity is a second order factor which consists of the general abilities of fluency, flexibility and originality. Each of these three abilities constitute first order factors that are constructed by students corresponding performance in the two tests. This model implies that independently of the measure or the domain, fluency, flexibility and originality form a general factor, that of domain general creativity. Data analysis indicated that the Model 1 does not have good indicators of adjustment to the research data (CFI=.961, χ^2=276.614, df=115, χ^2/df=2.405, RMSEA=.063, AIC=-7515.508, BIC=-7282.245), due to the fact that the χ^2/df is higher than 2.

Model 2 (see Figure 2 in Appendix) suggests that different types of creativity exist which form different factors: creativity measured using the MCT and creativity measured with the CTT. Each domain specific type of creativity is comprised of fluency, flexibility and originality. Confirmatory factor analysis showed that Model 2 has very good “fitness” on the data of the present study (CFI=.990, χ^2=152.926, df=111, χ^2/df=1.378, RMSEA=.039, AIC=-7631.196, BIC=-7381.703). In particular, the analysis suggested that two independent second order factors exist. Each of these context-dependent creative abilities is formed by fluency (MCT: r=.96, p<.05, CTT: r=.87, p<.05), flexibility (MCT: r=.99, p<.05, CTT: r=.98, p<.05), and originality (MCT: r=.95, p<.05, CTT: r=.86, p<.05), implying that fluency, flexibility and originality in the two instruments do not constitute a common factor but they are distinguished according to the stimuli. Indeed, each of the three components of content-dependent creativity is comprised by the corresponding performances on the specific measurement. For instance, the second order factor “Fluency” in the MCT is comprised by “Fluency 1”- “Fluency 4”, that is the measured fluency ability in the four tasks of the MCT.

Comparing the two models, the second model has a better fit to the data; firstly the CFI index has higher value in Model 2 as compared to Model 1 (CFI_{Model1}=.961, CFI_{Model2}=.990,); secondly Model 2 has the lowest value of AIC and BIC indices (AIC_{Model1}=-7515.508, AIC_{Model2}=-7631.196, BIC_{Model1}=-7282.245, BIC_{Model2}=-7381.703,), thirdly Model 1 has an inappropriate value for one of the indices that are taking into account for the evaluation of model fitness (χ^2/df=2.405>2).

DISCUSSION

Is creativity domain specific or domain general? Although both opposing views have been examined and empirically supported by several researchers, is still one of the enduring controversies of the creativity research. According to Baer and colleagues (Baer, 1998; Baer & Kaufman, 2005) the importance in answering that question goes to the heart of the field and consequently influences the educational practices. Due to the importance of the domain generality-specificity issue both in educational and research domains (Baer, 1998; Kaufman & Baer, 2005), the present study attempted to investigate this controversial issue.

The result obtained through data analyses converged to the domain specificity of creativity. Specifically through the comparison of two alternative theoretical models which define creativity as specific or general, confirmatory factor analysis confirmed the appropriateness of the domain specific model. By employing similar statistical analysis other researchers found multiple-factors models to describe the structure of creativity, verifying that there are distinct domains of creative ability (Kaufman, Cole, & Baer, 2009). Therefore, psychologists and educators should no longer characterise individuals as creative, but instead, as creative in specific domains. Consequently, research work on creativity is anticipated to be specialized in different fields in order to develop an integrated picture of the concept, rather than considering creativity as a general aspect.

This result was also supported by the correlation analyses which aimed at investigating the relationships between creative abilities (fluency, flexibility, originality) on two different dimensions: correlations of the same creative ability between the two instruments and correlations of the three creative abilities within the same instrument. Negligible correlations were found between the same creative ability across the two measurements (e.g. MCT fluency- CTT fluency), whereas high correlations were observed between the three abilities (fluency, flexibility, originality) that were measured within the same instrument. Low correlations between creative results in different domains were identified by numerous researchers (Baer, 1991; Kaufman & Baer, 2005; Plucker & Zabelina, 2009). By interpreting this result, one may assume that there are not identical and systematic creative abilities which may arise to stimuli of different domains; hence the existence of "a universal creativity" which is transferrable from one cognitive field to another is rejected (Kaufman & Baer, 2005).

Finally, the results of the crosstabs analysis illustrated that students' performance on the two creative instruments were not in agreement. In particular, a high percentage of participants who were considered as creative using one of the instruments were not consistently found as creative using the other instrument, whereas only a low percentage of participants were identified as creative by both instruments. Based on these results we can conclude the inadequacy of general creativity instruments to identify creative thinking in specific domains. Hence, the necessity for developing domain specific instruments to measure creative ability is obvious (Hong & Milgram, 2010). Extending the above conclusion, by identifying a student who is creative in mathematics does not necessary imply that she/he is also creative in art or literature, and vice versa. Additionally, a student who has shown high creative ability in one field is not automatically excluded from being creative in the subject of mathematics.

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APPENDIX

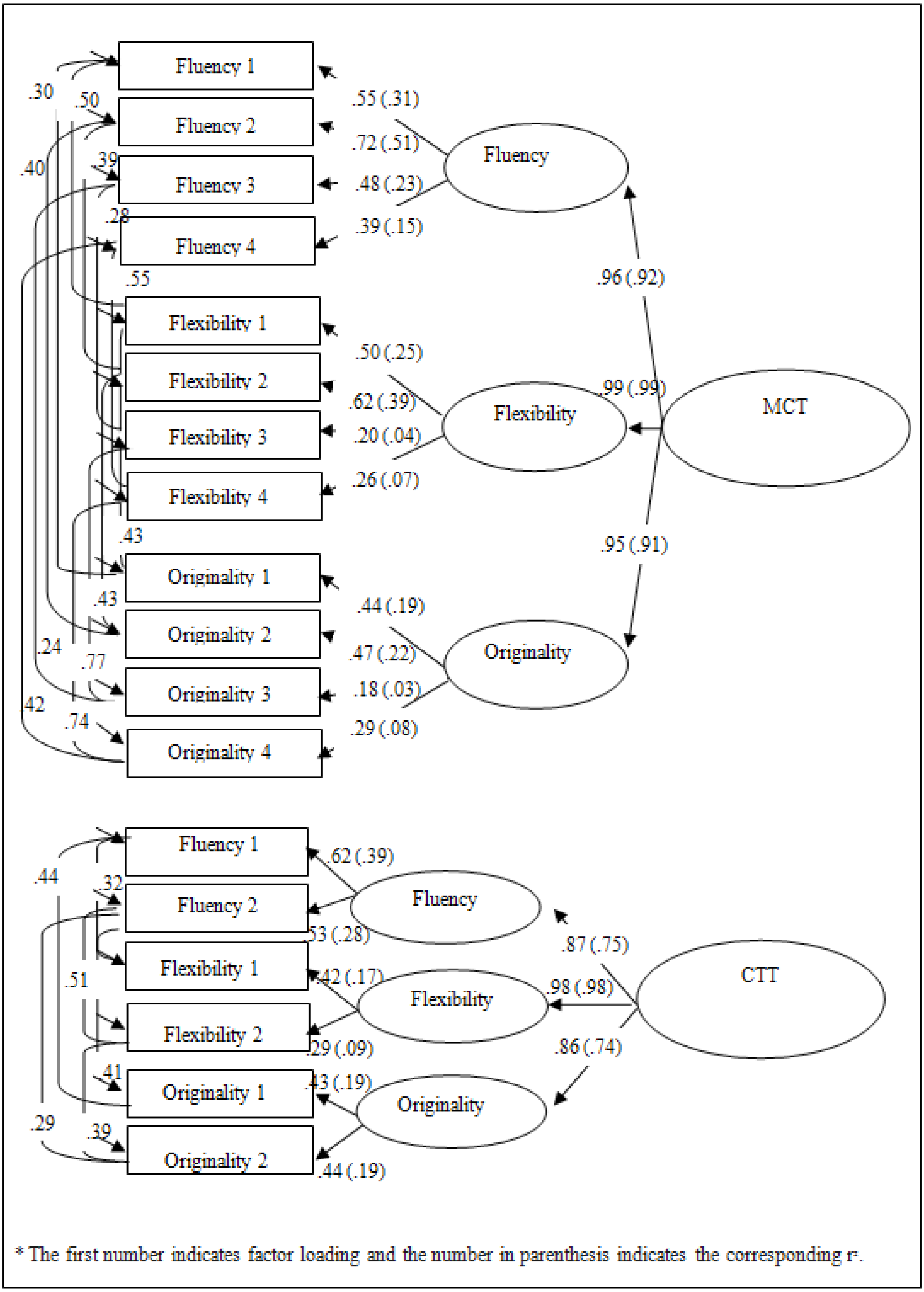


Figure 2: Model of domain specific creativity

Creativity and expertise: The chicken or the egg? Discovering properties of geometry figures in DGE

Roza Leikin and Haim Elgrabli

University of Haifa, Haifa, Israel, rozal@edu.haifa.ac.il

The relationship between mathematical creativity, knowledge and expertise is a phenomenon which can be seen as a "creativity-knowledge dilemma": Having knowledge is a necessary condition for a person to be creative; on the other hand, creativity is an important condition for knowledge construction. In this paper, we analyze mathematical activity that is directed at both the development of problem solving expertise and creativity in geometry. Creativity in this study is connected to the discovery of new properties of the given geometrical objects through investigation in DGE. We introduce a framework for the analysis of the complexity of a discovered property. Based on the analysis performed, we hypothesize that (1) discovery skills can be developed in people with different levels of problem solving expertise while the range of this development depends on the expertise; (2) the discovery process is rooted in the problem-solving expertise of a person.

Keywords: Mathematical creativity, problem-solving expertise, geometry investigations, complexity of discovery.

BACKGROUND

In his arguments about the importance of creativity for child development, Vygotsky ("Imagination and creativity in childhood," 1930, and "Imagination and creativity in Adolescents," 1931) maintains that imagination is goal-directed, culturally mediated, and emerges from the interweaving of fantasy and conceptual thinking. In his view, the development of creativity is one of the essential elements of the child's mental and social development. The educational system should pay attention to this construct.

Vygotsky (1982/1930) argues that imagination (creativity in our terms) is the central mechanism in the development of children's knowledge, since imagination allows them to construct connections between their existing knowledge and the new pieces that they study. Going a step further, we argue that creative activities in mathematics allow students to design mathematical connections and use their mathematical knowledge in an unlearned fashion. In this sense, creativity is a necessary condition for knowledge construction. To the contrary, creative processes in mathematics presume discovery of new mathematical constructs, properties and regularities to expand mathematical knowledge to new territory. This requires previous knowledge and the ability to critically evaluate that the discovered facts are new. In this sense, knowledge is a necessary condition for a creative process. Thus we consider the knowledge-creativity dilemma an intriguing phenomenon and try to explore it.

Relative and absolute creativity

The rationale for investigating creativity in school children lies in the shift from a static view on mathematical creativity to a dynamic characterization of personal development (Leikin, 2013). Rather than looking at creativity as a personal characteristic given at birth ("a gift"), we consider it a personal creative potential that can be developed if appropriate opportunities are provided for the learner. This position requires a distinction between absolute creativity as associated with discovery or invention at a universal level and relative creativity which is considered with respect to a specific person acting in a creative way within a specific reference group (cf., objective vs. subjective creativity in Lytton (1971), and that of Big C vs. Little C creativity in Csikszentmihalyi (1988)). For example, the distinction between absolute and relative creativity is obvious in Yerushalmy (2009) who provided analysis of curricular design, mainly based on mathematical investigations, which implies the development of mathematical creativity in all students.

Our study explores connections between relative creativity and expertise associated with geometry investigations. We compare their creative activity by prospective mathematics teachers (PMTs) with the creative activity of an expert in mathematical problem solving (Sharon – pseudonym).

Problem solving expertise

An expert in mathematics has been described as having a more robust mental imagery, more numerous images, the ability to switch efficiently and effectively between different images, the ability to focus attention on appropriate features of problems, and having more cognizance of their thought and of how others may think (Carlson & Bloom, 2005; Hiebert & Carpenter, 1992; Lester, 1994). Individuals with a coherent understanding of a particular mathematical topic have a complex system of internal and external representations that are joined together by numerous strong connections to form a network of knowledge. In contrast to experts, a student's system of representations of a mathematical concept may be deficient in number and deficient in connections to form an adequate network of knowledge (Hiebert & Carpenter, 1992; Lester 1994).

Whereas a novice uses a conventional means-ends analysis to solve problems, an expert categorizes problems according to solution principles and applies those principles in a forward-working manner to the givens of the problem. The expert's knowledge-based strategy is dominated by previous experience. He or she "knows into which category the problem should be placed and knows which moves are most appropriate, given that particular type of problem" (Sweller, Mawer, & Ward, 1983, p. 640).

Expert knowledge is also likely to be organised as hierarchical schemas (Chi, Feltovich, & Glaser, 1981). Problem-solving schemas are knowledge structures that consist of prototypical aspects of the problem type including declarative information about the features, facts, principles, and strategies associated with the problem. Experts have been shown to spend more time on features designated as critical to the problem (Morrow et al., 2009; Shanteau, 1992) and to rapidly encode features of problems based on goal-relevant representations. In the study on which this paper is based we were interested in examining how the investigation procedures performed by novice and expert problem solvers differ.

Mathematical investigations

We believe that mathematical investigations should become an core component of any mathematical course whether the participants in the course are teachers or students. While mathematical investigations are central to the activity of any research mathematician who is ultimately creative in the field of mathematics, mathematical discovery is always accompanied by enjoyment, it raises self-esteem and leads to curiosity and courage to make a new discovery. And as such it is necessary and should be in curricula for all learners.

Leikin (2014) analysed interplay between Mathematical Investigations (MIs) and Multiple Solution Tasks (MSTs). She argues that MSTs and MIs are effective instructional tools for balancing the level of mathematical challenge in the mathematics classroom and, thus, for realizing students' mathematical potential at different levels. Particular emphasis is placed on varying levels of mathematical challenge in school mathematical classroom by employing MSTs and MIs.

In this paper we provide an additional view on MIs with DGE used as a basic element of a teacher education course. In this context, MI is an integrative mathematical activity that includes experimenting, discov-

Given: AB – diameter in circle (O, R)
AD, BC, DC – tangent segments

Prove:

a. Prove the DOC = 90° in at least 2 different ways
b. Find at least 3 additional properties of the given object
c. Prove each discovered property in at least 2 different ways.

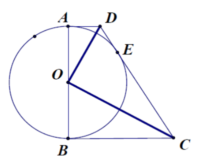


Figure 1: Problem 1

ering, conjecturing, verifying and proving. Problem 1 illustrates an investigation problem in our study.

THE PROPOSED FRAMEWORK FOR THE EVALUATION OF INVESTIGATIONS

The goal of the study presented in this paper was to design the criteria for the evaluation of MIs from the point of view of investigation processes and investigation products.

We analyse MIs using the construct of ***spaces of discovered properties*** as analogous to the notions of example spaces (Watson & Mason, 2001) and solution spaces (Leikin, 2007). We distinguish between *individual spaces of discovered properties* which are collections of properties discovered by an individual based on a particular problem and *collective spaces of discovered properties* which are a combination of the properties discovered by a group of individuals.

The analysis presented in this paper is based on two case studies (CS):

Case study-1 (CS-1) is focused on the collective space of properties discovered by Prospective Mathematics Teachers (PMTs) who are considered (in this study) as non-experts in geometry problem solving. The PMTs participated in the 56 hours courses directed at the development of their problem-solving expertise and the ability to create new geometry problems through investigations in DGE (see also Leikin, in press). The sessions with PMTs were videotaped and artefacts of their works were collected. Additionally the PMTs presented their investigations to the whole group of PMTs and these presentations were also video-recorded.

Case study-2 (CS-2) is focused on Sharon's (expert's) individual space of discovered properties. The investigation in this case was performed in the form of a thought experiment. Sharon was interviewed after the thought experiment to better understand the ways in which he arrived at the discoveries.

SPACES OF DISCOVERED PROPERTIES

In this section, Figure 2 depicts the non-expert collective space of discovered properties for Problem 1 which was achieved at the end of the 56 hours courses (CS-1). Note that the investigation activities were new for the PMTs at the beginning of the courses and the discovery skills were developed through the course. Figure 3 presents Sharon's expert space of discovered properties for Problem 1 (CS-2).

THE FRAMEWORK FOR THE ANALYSIS OF DISCOVERED PROPERTIES

The framework for the analysis of discovered properties is based on the analysis of the expert spaces of the discovered properties in CS-1. The distinctions between the *discovered properties* was defined as a complex function of

-- *the newness of the property* discovered in the courses of investigation,

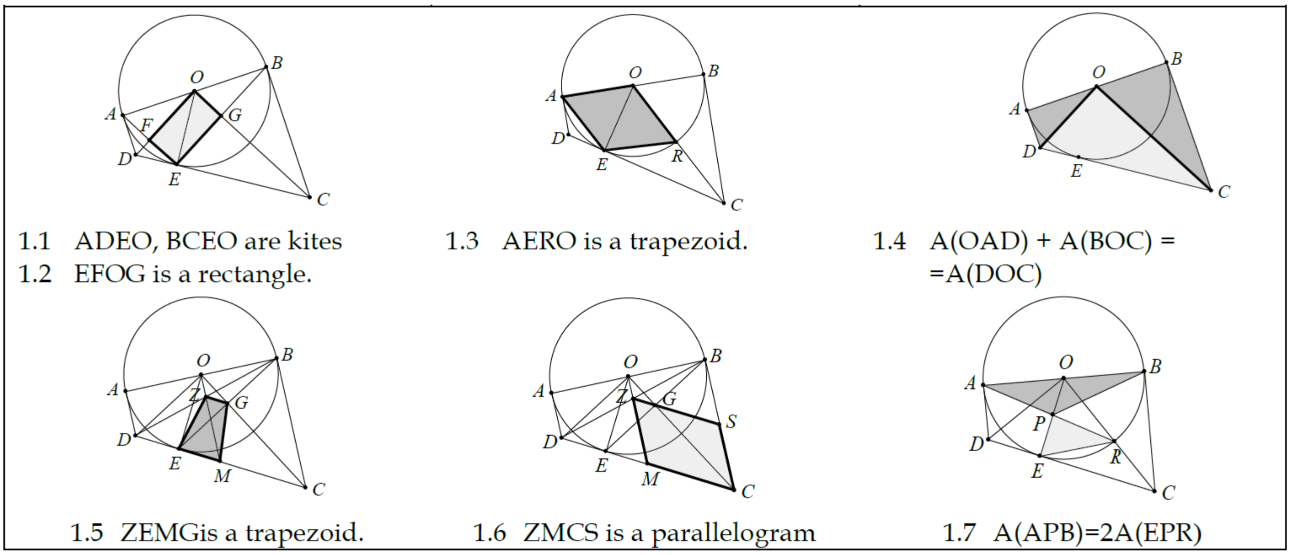
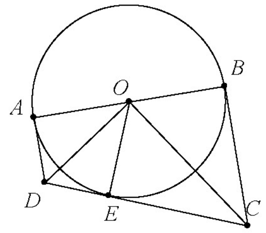
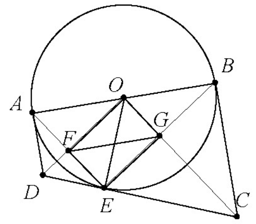


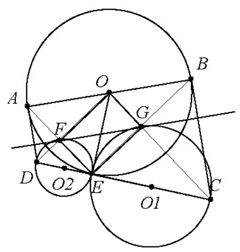
Figure 2: The PMTs non-expert collective space of discovered properties



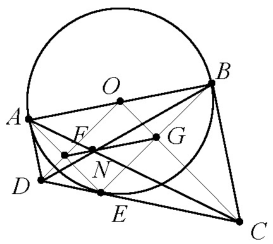
1a $OE \perp DC$
1b $BCEO$ – kite; $ADEO$ – kite
1c $ABCD$ trapezoid
1d $\sphericalangle COD = 90°$



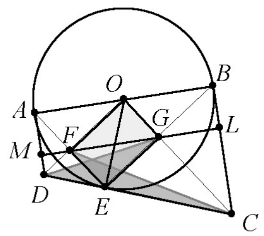
1e $EFOG$ rectangle
1f $AB \parallel FG$; $FG = \frac{1}{2}AB$
1g $BCEO$, $ADEO$ – cyclic quadrilaterals



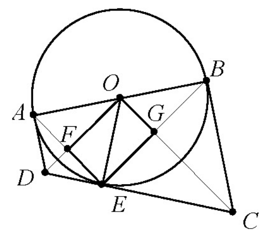
1h Circles $O1$ – diameter CE, $O2$ – diameter ED pass through F and G respectively
1i FG is tangent to circles $O1$ and $O2$
$GO1||BC$; $FO2||AD$; $FG||AB$



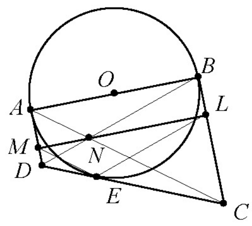
1* If $N = AC \cap BD$ then FG passes through N



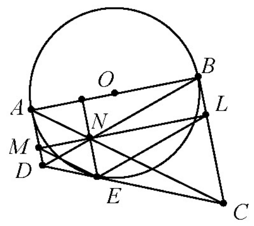
1j $A(CEF) = A(DEG) = \frac{1}{2}A(EFOG)$
1k $ABLM$ - rectangle
$A(ABE) = A(ABLM)$
$= 2 \cdot A(EFOG)$



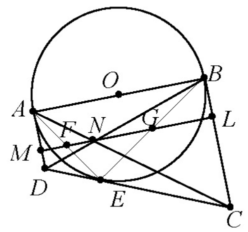
1l $\Delta ADE \sim \Delta BOE$, $\Delta AOE \sim \Delta BCE$
1m $\frac{A(BOE)}{A(ADE)} = \frac{A(BCE)}{A(AOE)}$
1n $\frac{A(ABCD)}{A(CDO)} = 2$



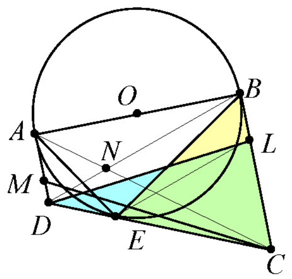
1o $N = AC \cap DB$;
$ML \perp BC \Rightarrow ABLM$ rectangle
1p $EL||BD$
1q $ME||AC$



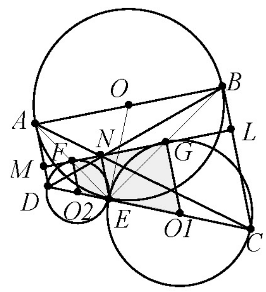
1r $NE||BC$
1s $BLEN$, $AMEN$ parallelograms
1t $JN = NE$



1u $F, G \in ML$
$NG = GL$
$MF = FN$



1v $A(CDL) = A(BCE)$
1w $A(CDM) = A(ADE)$



1x OE is tangent to circles O1 and O2
1y $FGO_1O_2 \sim ABCD$; $\frac{A(ABCD)}{A(FGO_1O_2)} = 4$
1z $FGO_1O_2 \sim ABCD \sim MNED \sim NLCE$

Figure 3: The expert individual space of discovered properties

-- *the complexity of the auxiliary construction* performed for the investigation and

-- *the complexity of the proof* of the new property.

Newness of the discovered property

The newness of the discovered property is relative to the educational history of the participant. This criterion reflects participants' critical reasoning and the ability to evaluate the property as either discovered in the process of investigation or one that was known previously. As such, we distinguish between three levels of newness of the discovered property: 0-trivial, 1-less trivial, 2-nontrivial.

For example, in Figure 3, properties 1a, b, c, d, g, are categorised as trivial, properties 1e, 1f, 1l, 1m are less trivial, whereas properties 1h, 1i, 1j, 1k, 1o. 1p, 1q, 1r, 1s, 1t, 1*, 1u, 1v, 1w, 1y, 1z are categorised as not trivial since they require complex proofs. Property 1* is a special one since through search for the proof of the property, Sharon discovered additionally several new properties

Complexity of the auxiliary construction

We distinguish between three levels of complexity of the auxiliary constructions (Table 1) according to two criteria:

Criterion 1: *The location of the auxiliary construction*. The property is discovered based on the auxiliary construction "within the given figure" vs. auxiliary construction "outside the given figure" (e.g., properties 1x, 1y, 1z Figure 3). Constructions "within/in the given figure" include (but are not restricted to) marking points on the border or in the interior part of the figure, construction of segments within the figure by connecting existing points or the new ones, construction of special lines (medians, bisectors, altitudes, inscribed circles) and more. Constructions "outside" the figure are more complex than those "within" the figure.

Criterion 2: *The number of the auxiliary constructions*. The property can be discovered without any auxiliary construction while the conjecture is raised on the basis of the measurement of segments, angles, areas and perimeters of the existing elements in the given figure and the comparison of these measurements. The number of the auxiliary constructions required for the discovery of a property determines the level of complexity of the property.

Through a combination of Criteria 1 and 2, we determine the complexity of the auxiliary construction as presented in Table 1.

For example, in Figure 3, the levels of complexity of the auxiliary constructions are 1 for properties 1l and 1m; 2 for properties 1e and 1f, 3 for properties 1o, 1r.

Complexity of the proof of a discovered property

Complexity of a proof of the discovered property is determined by the length of the logical chain of the proof and its conceptual density (Silver & Zawodjewsky 1997). Table 2 presents the levels of proof difficulty as determined in this study.

The complexity of a discovery is determined according to the combination of the complexity of auxiliary construction and the complexity of proof (see Table 3).

The number of constructions / Location	0	1	2	3 and more
Within	N/A	Easy (1)	Medium (2)	Difficult (3)
Outside	N/A	Medium (2)	Difficult (3)	Difficult (3)

Table 1: The complexity of the auxiliary construction

Proof length / Conceptual density (number of concepts and properties used during the proof)	1–3 steps easy	4–6 steps medium	7 and more difficult
1–2 concepts/properties – easy	Easy (1)	Medium (2)	Difficult (3)
3–4 concepts/properties – medium	Medium (2)	Medium (2)	Difficult (3)
5 and more concepts/properties – difficult	Difficult (3)	Difficult (3)	Difficult (3)

Table 2: The complexity of a proof

DISCOVERY STRATEGIES

Within the space constrains of this paper we present shortly the discovery strategies identified in this study (CS-2). We identified eight *types of discovery* as associated with the *process of discovery*:

1) Immediate discovery
2) Discovery by chance (through wondering dragging in DGE)
3) Discovery through association with another problem
4) Discovery in the search for a proof
5) Discovery based on the previous knowledge of related properties
6) Discovery in the course of proof
7) Discovery by symmetry considerations
8) Intuitive discovery

COMPARING EXPERT AND NON-EXPERT SPACES OF DISCOVERED PROPERTIES FOR

Table 3 shows the investigation process performed by the non-experts (properties 1.1.-1.7) and the expert (properties 1.a-1z) in problem solving. It also includes the analysis of the complexity of the discovered properties and describes the type of discovery.

	Property (in Tables 1 and 2)	Complexity of the discovery (A, B, C) A-newness of discovery B-complexity of auxiliary construction C-complexity of proof	*Type of discovery*
CS-1: Non-Expert space of discovered properties	1.1	(0, 0, 1)	*By chance (in DGE)*
	1.2	(1, 2, 1)	*By chance (in DGE)*
	1.3	(1, 2, 2)	*By chance (in DGE)*
	1.4	(0, 0, 2)	*By chance (in DGE)*
	1.5, 1.6, 1.7	(2, 3, 3)	*By chance (in DGE)*
CS-2: Expert space of discovered properties	1a, 1b, 1c, 1d	(0, 0, 1)	Immediate
	1n	(0, 0, 2)	Immediate
	1g	(0, 0, 1)	By knowledge of properties
	1e	(1, 2, 1)	*Through association with other problem*
	1f	(1, 2, 2)	*By knowledge of properties*
	1l	(1, 1, 2)	*By knowledge of properties*
	1m	(1, 1, 3)	By knowledge of properties
	1o	(2, 3, 1)	Through search of proof
	1r	(2, 3, 2)	By knowledge of properties
	1h, 1t	(2, 3, 2)	Through association with other problem
	1i, 1j, 1k	(2, 3, 3)	Through search of proof
	1*, 1p, 1v, 1y	(2, 3, 3)	*By chance (in DGE)*
	1s, 1u	(2, 3, 3)	Immediate
	1z	(2, 3, 3)	*Intuitively*
	1q, 1w	(2, 3, 3)	Symmetry considerations

Table 3: Complexity and the type of discovery

DISCUSSION

Based on the framework for the analysis of geometry investigations and the findings presented in this paper, we argue that creativity can be developed in people with different levels of problem solving expertise, but the range of the development depends on the expertise. The differences in such a range are reflected in the amount and the complexity of discoveries depicted in Figure 2 and Figure 3 and Table 3.

While surprisingly both expert and non-expert spaces of discovered properties include trivial and non-trivial discoveries, obviously, the expert space of the discovered properties is richer. Interestingly, non-experts discovered properties that were not discovered by the expert (properties1.5, 1.6, 1.7) and thus we do not consider any expert space of discovered properties as a complete one.

The major difference between expert and non-expert investigations is in the investigation strategies applied. Most of the discoveries by non-experts in problem solving are discovered "by chance" by observation of regularities which are immune to dragging. On the contrary, the discoveries by the expert were based on his mathematical knowledge and problem-solving expertise. Moreover, many of the discoveries "by chance" were based on the experts' attempt to search for a proof with the help of DGE.

Based on the findings presented in this short paper, we argue that problem solving expertise is a core element in the development of investigation skill in geometry that is a development of creativity. This development in PMTs is a rather "painful" process which requires them to overcome multiple difficulties related to geometry construction, grasping the meaning of dragging, proving and refuting multiple conjectures. Through the development of investigation skills, we also develop problem-solving expertise in PMTs and thus hope that creativity will be further developed. The process of this development is an objective of our current new study.

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Students with hearing impairment: Challenges facing the identification of mathematical giftedness

Swetlana Nordheimer[1] and Matthias Brandl[2]

1 Ellen-Key-Oberschule and Humboldt-University, Berlin, Germany, swetlana.nordheimer@gmail.com

2 University of Passau, Faculty of Computer Science and Mathematics, Passau, Germany, Matthias.Brandl@Uni-Passau.de

Usually a mixture of student's interest, qualitative teacher's observation and quantitative tests is used in order to identify mathematical giftedness. However, for children with hearing impairment, traditional ways of identification and diagnosis need to be adjusted, especially when it comes to quantitative aspects. The challenges of this process are illustrated by a case study and conclusions (like less time limits in tests, more visual helps) are drawn.

Keywords: Mathematical giftedness, hearing impairment, identification, inclusion.

INTRODUCTION

In 2008, the UN *Convention on the Rights of Persons with Disabilities* was ratified in consequence of *The Salamanca Statement and Framework for Action on Special Needs Education* in the context of the UNESCO *World Conference of Special Needs Education: Access and Quality* in 1994. The Convention aims at ensuring „an inclusive education system at all levels" (UNO, 2008, Article 24) which satisfies the needs of „*all children* regardless of their physical, intellectual, social, emotional, linguistic or other conditions. This should include disabled and gifted children" (UNESCO, 1994, 6). And therefore, the question of how to recognize and foster children that bring together two different specifics, i.e., *hearing impairment* and *mathematical giftedness*, is massively challenging and important.

The purpose of the study is to present a first exploration of the field and start the development of appropriate research questions with regard to the identification of mathematically gifted students with hearing disabilities. Further, we outline several complexities in the identification of gifted children with special needs that require future systematic research.

THEORETICAL BACKGROUND

The following paragraphs shortly summarise some facts from literature on the identification of mathematical giftedness, children with hearing impairment (hi) and the mathematical development of hi-children.

Identification of mathematical giftedness

According to Renzulli (1978), giftedness is characterised by *above average ability*, *high levels of task commitment*, and *high levels of creativity*. These aspects are acknowledged to be important for giftedness in mathematics education as well (Leikin, 2011). As already discussed in Brandl (2014), there are different ways of identifying and selecting promising students for reasons of fostering (in mathematics). One option is to rely on the students' own *interest* in mathematics as the main motivational force and most important factor for mathematical giftedness (see amongst many others Kruteskii, 1976). A second way is to trust in teachers' choices, adding a *qualitative* external selection component to the students' interest (see Linke & Steinhöfel, 1986, for example). A third option would focus on *quantitative* methods and result in testing the students (see Nolte, 2012; Kontoyianni et al., 2011, amongst many others). In general, identification processes often are designed as a combination of these three ways (see Wagner & Zimmermann, 1986, for example).

Children with hearing impairment (hi-children)

The term 'hearing impairment' comprises several cases: dysfunctions of the auditory system, hardness of hearing, deafness. It may be caused by genetic reasons, trauma or disease. However, the terms "deaf" (d) or "hard of hearing" (hh) which are accepted in their community of people, not only imply the degree, type or configuration of hearing impairment, but also the way of communication the person uses. Most of hh-people rely on residual hearing and communicate through speaking and lip-reading. In noisy environments, however, it can be very hard for them to communicate verbally. Thus, they are likely to face various difficulties at school and differ in their educational and psycho-social development from their hearing (h) classmates. An hh-person may use a hearing aid to perceive what is spoken around him or her. In case of severe impairment, the person may not be able to distinguish any sounds or spoken language, even with a hearing aid. The majority of d-people use sign language to communicate. The majority of the h-people neither use sign language, nor do they have an understanding of the special needs of hi-children. This is why hi-children often find themselves facing great challenges with regard to their language skills, potentially carrying away gaps in vocabulary and difficulties in articulation, for example. This is likely to have an impact on their cognitive and social-emotional development (Leonhard 2002, pp. 71ff.).

Mathematical development and giftedness of hi-children

Pagliaro and Kritzer (2012, p. 139) summarise the results of current studies and conclude: "Consistently, over decades and across grade levels, deaf/hard-of-hearing (d/hh) students in various countries have scored poorly on mathematics assessments." The authors try to explain this "math gap" of d- and hh-children by the limited experiences and lack of ability to learn mathematical concepts, e. g. numbers by incidence, because of the language barriers separating them from their natural environment. Gregory (1998, p. 122) names two other possible reasons for the difficulties especially of d-children to cope with mathematical tests and assessments: the nature of specialized mathematical language and the suppression of sign language in mathematical education. Because of this reduction, mathematical signs cannot be developed in an appropriate way.

Further reasons for the differences in the development of mathematical thinking can be found in the specifics and obstacles within communication processes between hi- and h-people. Solving mathematical word (and modelling) problems usually starts with reading and understanding written texts. These texts often contain short words (e.g., prepositions), which are underemphasized in spoken language and can easily be overheard (or overseen by hi-students, when relying on lip-reading or sign-language). Moreover, there are specific mathematical terms like "product" or "root" which have different meanings in the common spoken language. This proves to be difficult to express in sign-language or figure out by lip-reading.

Engel (2000, p. 17) reports that many hi-children develop strategies in order to solve word problems without entirely understanding the text. These children try to simplify and restructure the text. This process can be compared with the reading of fill-in-the-blank texts. In some cases, these strategies are successful, in many others they are not. Cohors-Fresenborg (1988, pp. 102ff.) claims that hi-children excelled in solving problems, which had been represented visually or haptically without using written or spoken language. These children achieved particularly high scores, when it came to problems which demanded high modelling competences (reconstruction and reorganization). Biographies of famous deaf mathematicians and scientists also show that hi-people can be brilliant problem solvers or gifted mathematicians (confer Lang & Meath-Lang, 1995, p. 407). Obviously, not every hi-person is a mathematical genius, and the same applies to h-people. There are, however, hi-students who are very interested and gifted in mathematics, and eager to solve mathematical problems. As for the discussed language difficulties of hi-students, it is not easy to recognise their mathematical giftedness, because most diagnostic instruments use spoken or written language to represent mathematical problems.

Giftedness of children with special needs can be masked by their disability. Gifted children with disabilities also often use their intelligence to compensate for their impairment. That is why it could be very difficult to recognise special needs of gifted children (Krochak, 2007; Nielson, 2002). Children whose hearing is impaired cannot respond to oral directions in the same way hearing children do, and they may also lack the vocabulary expressing the complexity of their ideas. They sometimes cannot respond to tests requiring verbal responses (Whitmore & Maker, 1985). Since the population of gifted students with special needs is difficult to locate, they are seldom represented in standardised test norming groups. This makes every comparison highly problematic. Pedagogical literature related to gifted hi-students lists criteria which should help to recognise giftedness, i.e., *Excellent memory, Rapid grasp of ideas, High reasoning ability, Superior performance in school, Wide range of interests, Nontraditional ways of getting information, Use of problem-solving skills in everyday situations, Delays in concept attainment, Self-motivation, Ingenuity in solving problems, Symbolic language abilities like different symbol system* (Cline & Schwarz, 1999; Whitmore & Maker, 1985).

METHOD

In order to illustrate aspects of the diagnostic challenge, a single case study was done.

Description of the student

The study deals with Leon[1], who is now visiting a 5th grade at the special school for hi-children in Berlin, where pupils are grouped in small classes of ten. The classrooms are equipped with technical aids and visual materials in order to support spatial thinking, and, compared to conventional schools for h-children, students get more time to work on their assessments. At the beginning of the study in August 2012, Leon was still a 3rd grader. According to the pupil's record, Leon suffers from bilateral[2] sensorineural[3] severe hearing loss of 60 to 70 dB. Leon is hard of hearing since he was born. Since his first year he has been using hearing aids, but it is still difficult for Leon to perceive spoken language without being able to watch the lips of his communication partners. Leon's parents and all his siblings are hearing and use oral language to communicate. It was not until he started school, that Leon was confronted with sign language. After one year, his parents decided to invest in one-to-one-language therapy. Because of Leon's special needs with regard to language, his parents finally sent him to the special school for hi-children.

At the beginning of 3rd grade, Leon had difficulties to express himself and to articulate his thoughts. Misunderstandings and conflicts with his classmates or even teachers were frequent and usually ended up in tears. Probably in order to avoid these situations, Leon very often waived explanations of his needs and feelings. Still, even more remarkable than Leon's communicational obstacles was and is his tremendous motivation to do mathematics. According to an interview done with his former mathematics teacher, he attracted her attention during the mathematics lessons through his eagerness to solve mathematical problems and his endurance when working on difficult and new tasks. According to his educational file, Leon discovered multiplication on his own by working with Montessori materials for preschoolers in kindergarten, when he was four years old.

At school, he works in mathematics classes with concentration, listens to the teacher and his classmates very carefully and examines what they said mathematically. He always manages to complete more mathematical tasks and problems than his classmates and sometimes poses his own questions and problems to the teacher as well. The tasks Leon tried to avoid in 3rd grade were word problems. Usually, his mood changed as soon as he was presented with a word problem. He often did not even start working on it, unless his teacher encouraged him strongly. Then he asked for the meaning of some words and tried to solve the problem, but he was not always successful, mainly because of textual misunderstandings and mismatches between the world of mathematics and the world of written German language.

Despite his difficulties, the mathematics teacher who taught him in 1st and 2nd grade recommended Leon for "Mathe-Treff" at Humboldt University in Berlin. This is a program for gifted students from primary schools from all over the city. The "little mathematicians" meet once a week for one and a half hour to solve mathematical puzzles and problems. They are assisted and observed by students of the educational department and the leader of the project, Prof. Marianne Grassmann. The biggest part of mathematical puzzles and problems is verbalized in written German language, so this represented a real big challenge for Leon. For this reason, two students from the department for special education were supporting Leon. They adapted mathematical word problems to his communication level and offered their help. Not all children managed to participate till the very end of the course, because it was not easy to stay motivated to do mathematics after a long school day. However, Leon was among those who completed the whole one-semester course successfully.

Identification method for mathematical giftedness

In Leon's case the decisive factor for the identification of mathematical giftedness was the teacher's recommendation. As mentioned earlier, the careful observation of the child during class can be a successful qualitative method. A very important part of this observation method was the study of the child's written products. We examined number tasks and word problems, both done within regular mathematics lessons. These observations were accompanied by a quantitative extracurricular assessment, intelligence and development test: In order to compare Leon with his hearing peers and to see his mathematical giftedness within the context of his general development, the IDS (Intelligence and Development Scales) for five- to ten-year-old children was used as a further diagnostic instrument (Grob et al., 2009). IDS was invented to provide differentiated insight into intelligence and general development in areas like Cognition, Psycho-Motorics, Social-emotional Competence, Mathematics, Language and Achievement Motivation at the beginning of the school career. It focuses on the dynamics of so-called individual "strengths" and "weaknesses" and puts them in relation to the child's development profile and to his corresponding peer-group. The test consists of different modules[4] and can be used as a whole or in parts. Except for the modules Phonological Memory and Auditive Long-Term Memory, it is possible to pose and solve the test tasks without spoken language. More language instruction is necessary when it comes to the modules describing General Development.

IDS was chosen as a diagnostic instrument, because it allows for comparison of Leon's intellectual abilities with the average population. It provides for precise and distinctive insights into different areas of the intellectual and social development and shows the interaction of talents and special needs. It could help to avoid that intellectual abilities and language needs collide. In this sense it could provide insights into the ability to compensate for hearing and language problems. Besides, the part *Achievement Motivation* of the IDS allows for conclusions about task commitment, which according to Renzulli (1978) can serve as an important characteristic of mathematical giftedness. It is difficult to make conclusions about mathematical creativity using IDS as a quantitative instrument, since the answers are standardised. That is why Leon was asked to talk about IDS-items and to explain his solutions, which were written down and then used as questions for qualitative interviews.

Since time was very limited, Leon was not exposed to the whole test. Instead, a selection of *Intelligence*, *Social-emotional Competence*, *Mathematical Competence*, *Expressive Language Competence* and *Achievement Motivation* was chosen. When taking the test Leon was 9 years and 7 months old. The test was completed in a one-on-one situation.

RESULTS

We illustrate how written products can indicate mathematical giftedness by describing and analysing some chosen cuttings from Leon's booklet and items of the IDS test.

Number tasks (regular lessons)

Leon's answer to the question: "Draw your favourite number." was the number 201031 which has six digits. Leon also knows that 200000 + 1000 + 31 = 201031. At this moment most children in the classroom were expected to deal with numbers under 1000. Some of them painted numbers above 2000 but not above 3000.

When it came to the figurative representation of numbers with three digits, Leon invented his own notation. The task for "Zahlenbild" was to find a symbolizing picture for numbers. (Example: the number is 300. Intended and trained solution: □□□). In contrast to other children he did not draw three squares in order to represent for example three hundred as a sum, but he used a multiplication sign to shorten the notation (□x3). So, Leon's solution does not confine itself to non-mathematical (purely additive) pictograms but extends it by bringing in abbreviating mathematical symbols.

Some months later the students learned how to add numbers under 1000 via an algorithm. Leon did not only use this algorithm to solve several given problems without a single mistake, but invented his own problems with bigger numbers and looked for new regularities.

Word problems (regular lessons)

Compared to a regular school, at the special school for hi-children mathematical word-problems are usually verbalized clearer and the numbers in it are kept small in order to make it easier for the students to solve them. Further, text problems have been translated into visual representations or supported by other special tools like small blocks and Montessori-materials. Students could use stamps to solve the problems or to control their results. The problems were offered at three difficulty levels. Leon chose the easiest problem first and controlled his result with stamps. In a second step he applied his solution to a more difficult problem dealing with spiders. Since he had understood the structure of the first problem, he was now able to transfer it onto the text structure of the spider-problem without using stamps as visual aids. Remarkably, Leon found it important to notice the commutativity of the multiplication.

Assessment, Intelligence and Development test IDS 5–10 (extracurricular)

Leon's intellectual performance with IQ = 109 lies according to the IDS in the upper average area in relation to his hearing peers. In the sub-test Selective Attention Leon achieved the highest possible score [Value Points (VP) = 19]. The scores in the sub-tests Auditive Long-Term Memory and Conceptual Thinking are above the average [VP = 14]. The other scores lay within the standard norm of his peers [VP = 9, 11, 12, 12]. It is remarkable, that Leon could manage all tasks of the sub-tests in Figural Thinking almost within the standardized time limits.[5] Leon's answers in the sub-test Social-emotional Development show that he can perceive and recognize other people's emotions within average norm [VP = 9]. Likewise, Leon is able to regulate his emotions [VP = 14] and understand social situations [VP = 13] better than most of his peers. However, he showed heavy deficits with regard to the aspect Social Acting [VP = 6]. In the sub-test Achievement Motivation Leon again reached the highest possible score. Leon's score in the sub-test Expressive Language lay way below the average norm [VP = 2]. He was able to express his thoughts, but he could hardly do it in the grammatically correct way as it was expected in the standard test. Some of his sentences are difficult to understand. However, if the sentences containing semantic and grammatical errors counted as correct, Leon would have reached VP = 12, which corresponds to the average. Within limited time he attained VP = 14 in the sub-test Mathematics, which is above the average. Without time-limitation Leon managed to solve all mathematical problems except one word problem and reached VP = 19, which is far above the average compared to children of his age. In order to gain deeper insight into Leon's mathematical thinking we will discuss some items of the test closer.

There were two very similar word problems which were read out loud to the children. The respective tested child was given toys (small doll, cat, dog and cups) to illustrate the solution and was expected to explain it. Leon managed to solve one problem and was not successful with the other. The reason was that the chosen vocabulary and the text structure of the unsolved problem were more complicated than in the problem Leon could solve. Other problems were verbalized as well, but they could be solved almost without understanding the text. The information needed was also represented by visuals or symbols. Leon soon ignored the text and solved them only by using mathematical symbols. In the second task he had to deal with numbers which, by that time, were no subject matter of the mathematics curriculum: „A counter is counting people, who are coming into the stadium. The counter shows this number. What would the counter show, if 201 more people would come to the stadium? Write the numbers into the nearby blanks" (Figure 1 left). Leon solved the problem and even put a point into the number. This demonstrates, that he understood the meaning of the written symbols. The next two problems referred to Geometry: "How many cubes are in this figure? Fill in" (Figure 1 bottom). In these tasks Leon showed his good abilities in figural and spatial thinking.

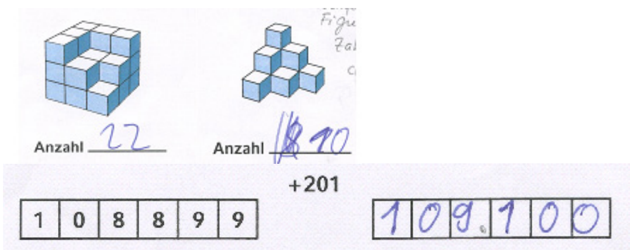

Figure 1: IDS/Items 14 to 16

The final problems checked his orientation in the range of numbers 0 to 100000. Since Leon had until then only learned numbers under 10000, he found himself working with new numbers and still managed to find the correct solutions in the given time. He could not solve the very last problem in the given time, but he found out the right numbers after the time was over. This appears even more remarkable, since he transferred his knowledge about numbers under 10000 to numbers over 10000 (Figure 2 bottom).

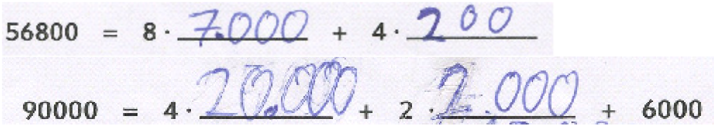

Figure 2: IDS/Item 17 & 18

CONCLUSION

Language barriers essentially complicate the identification of mathematically gifted hi-children. However, the documents presented in the results section clearly show that a hi-child like Leon can also develop the ability to analyse the structure of numbers, to play with them, to invent an own notation, to work with different modes of representation and to compensate for his language difficulties by visual help (for example stamps) or mathematical signs (multiplication sign). Although these results were achieved in the setting of a special school, the case study of Leon can inspire appropriate combinations of qualitative and quantitative methods that can work out successfully in regular (inclusive) schools, too: documents and products from student observations combined with standardised tests (perhaps including less time limits and more visual helps) may overcome language barriers and lead to differentiated insights into the child's possibilities. In the end, this could be of great help not only in order to predict a hi-student's success in programs for gifted children, but also in order to provide him with the individual support needed.

The results of the study correspond with criteria for mathematically gifted hi-children and the study leads to the following future research questions: What kind of information about mathematical giftedness of children with hearing difficulties allows for a quantitative comparison with children without hearing difficulties? Does it mean that hi-students with the same score as hearing students have greater competence, given the knowledge that they compensate for their disadvantages? In what way need criteria for mathematically giftedness to be altered in order to address students with hearing problems? How can this information help to predict hi-students' success in programmes for fostering mathematical giftedness that are usually designed for hearing students?

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ENDNOTES

1. "Leon" is a pseudonym because of protection of privacy. The child's real name is known to the authors.

2. in both ears

3. caused because of dysfunction of the inner ear

4. Visual Perception, Selective Attention, Phonological Working Memory, Spatial Working Memory, Auditive Long-Term Memory, Conceptual Thinking and Figural Thinking build General Intelligence.

5. Results which were scored without time limitation were not considered by the calculation of the IQ, but they still show that Leon could achieve more if he has more time and that he is able to concentrate as long as it takes him to get the correct solution.

The effect of high versus low guidance structured tasks on mathematical creativity

Sonia Palha, Jaap Schuitema, Carla van Boxtel and Thea Peetsma

University of Amsterdam, Research Institute of Child Development and Education, Amsterdam, The Netherlands, S.AbrantesGarcezPalha@uva.nl, J.A.Schuitema@uva.nl, C.A.M.vanBoxtel@uva.nl, T.T.D.Peetsma@uva.nl

To engage in challenging tasks, students need to feel some autonomy and competence. Providing structure within the task can help to meet these needs. This study investigates the influence of structure within a modelling task on mathematical creativity among 79 eleventh-grade groups of students. Two versions of the task were developed and the groups were randomly assigned within their classroom to one of these. The analysis explored: (i) the level of mathematical creativity in groups solutions and (ii) if they were dependent on the amount of structure. The results were not statistically significant and, therefore, the question remains open. Additional results and implication of this study to mathematics education are further discussed.

Keywords: Integral calculus, creativity, modelling, collaborative learning, structure.

INTRODUCTION

Researchers express different views with regard to creativity and its connection with the learning environment. Some claim that creativity can be seen as a disposition towards mathematical activity and therefore it can be fostered through specific instruction, such as problem-solving (Silver, 1997). Others see creativity as characteristic of extraordinary individuals (Weisberg, 1988) and thus, not likely to be strongly influenced by the learning environment. Also, several researchers connect creativity to self-regulated learning (Feldhusen & Goh, 1995) and psychological characteristics such as task commitment and motivation (Renzulli, 1978). In our research we share the view that mathematical creativity can be fostered by adequate instruction and we study the relationship between aspects of the learning environment (e.g., task characteristics) and mathematical creativity.

This study is part of a longitudinal intervention research in which we investigate how aspects of the learning environment influences students' motivation, self-regulation and academic performance in mathematics. We developed a learning arrangement in which we used differentiated tasks with a deeper and broader content and method to create a more authentic and challenging learning context. The participants are 16/17 years old students in pre-university education in The Netherlands. Part of our research is to investigate which amount of structure is optimal for the students. We developed two versions of the same learning arrangement. One version consists of low-structured *(LS)* tasks and provides more open tasks, more choice and initiative for students. The other version contains more high-structured *(HS)* tasks, which still provide some choice but also hints, more sub-questions and guidance.

In this paper we discuss our findings with regard to a modelling-task: the parachute jump (Figure 1), which was used within the topic Introduction to Integral Calculus. Modelling-tasks as problem-posing tasks have been seen by several researchers as excellent opportunities for mathematical creativity (Kim & Kim, 2010; Chamberlin & Moon, 2005). The research questions that guided our study were:

- What can we say about the mathematical creativity of students' productions with regard to the parachute jump task?
- In which way does variation in the amount of structure in the parachute jump task influences students' mathematical creativity?

THEORETICAL FRAMEWORK

Mathematical reasoning and creativity

Mathematical creativity can be seen as the ability of students to create useful and original solutions in authentic problem-solving situations (Chamberlin & Moon, 2005). The core activity of the parachute task is to build a model that can be applied in the particular example and other situations. The students' products can then be evaluated in terms of mathematical creativity. In the literature, mathematical creativity is often defined in terms of three components: *flexibility*, *fluency* and *originality* (Silver, 1997; Yuan & Sriraman, 2001). Flexibility can be seen as the ability to generate multiple solutions to a given problem. Fluency can be seen as the ability to use several relevant ideas to solve the task and, in problem-situation tasks it is connected to many interpretations, methods, or answers Silver (1997). Originality concerns different solutions or innovative ways to approach a problem.

Measurement of mathematical creativity remains critical. One reason is the absence of a universal definition applicable in different academic domains (Leikin & Lev, 2013; Kattou, Christou, & Pitta-Pantazi, 2015). Another reason is that one person's creativity can only be assessed indirectly (Piffer, 2012). The ability of posing problems given one mathematical scenario have been linked by several researchers to mathematical creativity (Silver, 1997; Yuan & Sriraman, 2001). Also, over the past years, researchers (Leikin & Lev, 2013) developed an analytical framework that can be used to evaluate creativity in students' productions using the components fluency, flexibility and originality. Mathematical creativity with regard to modelling activities often includes a fourth component: *usefulness*, which concerns the degree of relevance, adaptability and generality of solutions with regard to real world situations (Chamberlin & Moon, 2005). The criterion of usefulness has been contested by some authors. Sriraman (as cited in Yuan & Sriraman, 2001) argues that mathematics creative work might not be useful in terms of its applicability in the real world. Chamberlin and Moon (2005) propose the *Quality Assurance Guide* as a reliable instrument to evaluate creativity in students' products on modelling tasks. Each solution is scored within one of five levels. *Level 1*- requires redirection- the product is on the wrong track and working harder or longer will not improve it. At *level 2*, the product requires major extensions or refinements, the product is a good start towards meeting the goal of the task. At *level 3*, the product is nearly ready to be used; it is useful for the specific data or sharable or reusable. At *level 4*, no changes are needed and at *level 5*, others can use it as tool in similar situations.

High- and Low-structured tasks (HS and LS- tasks)

According to Silver (1997) problem-oriented instruction can assist students to develop more creative approaches to mathematics by increasing their capacity with respect to the core dimensions of creativity: fluency, flexibility, and originality. For instance, ill-structured problems require problem posing and conjecturing, which can foster the generation of novel conjectures. Silver (1997) stated: "It is in this interplay of formulating, attempting to solve, reformulating, and eventually solving a problem that one sees creative activity" (p. 76). However, engaging in problem-solving activity also requires certain ability and disposition to deal with uncertainty and challenge. Aspects of the learning environment that have been found to support the development of such disposition are autonomy support and structure provision (Deci & Ryan, 2000). According to these authors, in autonomy supportive environments students are allowed to make own decisions and are encouraged to solve problems. This can be achieved by providing authentic tasks and opportunities for taking initiative and minimize the use of controlling behaviour. Also, the provisions of structure contributes for students' feeling of competence and therefore is important for motivation. Providing structure involves communicating clear expectations, set limits to students' behaviour and provide help.

Task arrangement

We investigate the relationship between structure provision and mathematical creativity in a problem-oriented arrangement that consisted of the 'parachute jump' task (Figure 1) and small group work. Working together may enhance feelings of relatedness and a sense of autonomy (Schuitema, Peetsma, & Van der Veen, 2011). And, during students' collaboration there is an unpredictable flow of ideas and actions that emerge from the elements of the group while responding to each other. Levenson (2011) states: "Together, the group tries out various strategies and possibly produces solutions based on different mathematical properties or different representations" (p. 230). This is tied to mathematical creativity in the sense that participants must be flexible, establish

(both versions A and B)

Task 25 parachute jump

Dynamical processes, like a train ride, a traveling car and other speed-time processes can be described using a *mathematical model*. A mathematical model may include tables, graphs, formulas or any combination of these representations. These mathematical models can then be used to investigate (and solve) problems through calculations and reasoning or to invent better models to attack the stated problem. In a group of three students, you will create a mathematical model in which the distance travelled against time for a parachutist is described. You also prepare a demonstration (Powerpoint, poster or video clip) of your group's work as a homework task.

But first an example of a parachute jump is presented.

Example.
Imagine the following situation: A parachutist jumps from an airplane. The first five seconds she makes a free fall. Then she opens the parachute and because of that her fall velocity decreases linearly down until after 6 seconds she achieves a fall velocity of 4 meters per second. From this moment on the velocity remains constant during 70 seconds and she lands on the ground at this velocity.

(only version A)

In the example, time is called t (in seconds), with $t = 0$ at the jump from the plane. For the free fall the velocity is given by $v(t) = 9{,}8\,t$ with v in meter per second. The total jump, until reaching the ground, was 561.5 meters. A mathematical model for this example could be a formula (or some collection of formulas), a graph or table in which the falling process is described and that may help to solve the stated problem.

The process for another parachutist will be comparable although different in the three phases of the process.

(only version B)

The process for another parachutist will be comparable, although different in the three phases of the process. That process may be described using a mathematical model. It is usual to start such a model with a concrete example and after that you try to design a more general model or representation. In this class period you will develop a mathematical model that describes the distance traveled against elapsed time for a parachutist. There are guiding questions describing an example that will help you to understand what is going on (questions a-d) and after that you are asked to design your own model (question e).

In the example, time is called t (in seconds), with $t = 0$ at the jump from the plane. For the free fall the velocity is given by $v(t) = 9{,}8\,t$ with v in meter per second.

a. What distance does the parachutist cover during the free fall?

b. What is the total distance covered from start to landing??

a. (version A) c. (version B)

Watch the Youtube video of a parachutist jump: http://www.youtube.com/watch?v=STDlEFhIPrw. Which similarities and differences do you notice, compared to the situation of the example?

d. Re-watch the video and try to collect data to design the model that describes the parachute jump (describe the data with use of tables/graphs or both).

e. Can you find a relationship between distance covered and time, based on the data you collected fom the video?

b. (version A) e. (version B)

Create a mathematical model that describes the distance covered during the total parachute jump against time.
After that, you preare a group presentation (powerpoint, poster or video for a 2-5 minutes presentation) in which:

- the mathematical model is presented;
- you give a justification of the choices made;
- show some examples of situations in which the chosen model will work;
- a critical reflection on the model.

Figure 1: Parachute jump task

mathematical relations and approach the task in distinct or novel ways.

The 'parachute jump' task was entailed to provide challenge and authentic experiences, as these are important elements of autonomy supportive tasks. It was designed according to the following four criteria.

Appealing and accessible to all students. The context of a parachute jump and the YouTube video make the task interesting to the students. And, the task becomes more accessible by providing an initial example with concrete values and asking to compare it with the one in the video. The pre-knowledge needed to start working on the task was known from previous year (functions, graphs and derivatives).

Authentic. By providing students with an authentic task, and enough freedom of choice we expect that students will be willing to spend thinking effort on it.

Foster mathematical reasoning and creativity. The accomplishment of the task requires the use of mathematical understanding and high-level reasoning. The students must produce at least one representation of the integral function (table, formula, graph, words) and describe its variation at the different instances of the jump. This involves high-level reasoning, as the students must imagine the total accumulating distance varying over time (Thompson & Silverman, 2008).

Suitable for collaborative learning. The task is complex and it can be approached at several levels of understanding. Moreover, the students were encouraged to discuss their ideas and communicate their findings within the group.

Solving the task takes about two lessons of 50 minutes each and some homework time. We agreed with the teachers that the students would work in small groups during one lesson on the task and that they should finish it in their own time (not more than one week). The final product would have the format of a Power Point or a short video-film and would be delivered to the teacher, who would send it to us.

METHOD

Participants and data collection

Seventy-nine groups of 3 students (16/17 years old) from 10 classrooms in 5 schools participated in the study. The data was collected in the spring 2014 and consists of delivered groups products and lesson observations. The groups were formed based on a cognitive ability test. The 40 groups in the LS condition and the 39 groups at the HS condition were, in each classroom, random assigned to one of the conditions.

Instrument used for the evaluation of mathematical creativity

The instrument that we used to evaluate the students' solutions to the parachute jump is based on three of the four components discussed in the theoretical section (we excluded originality because of the difficulty on assessing it in our data).

Usefulness regards the creation of a model that is useful to describe a parachute jump. For each written solution, we decided whether the model was incorrect (level 1), was in the good way but needed major improvements (level 2) or it was ready to be used but needed editing (level 3). Levels 4 and 5 were not observed in our data.

Fluency was seen as the ability to use several mathematical relevant ideas to solve the task. In the context of the parachute task it should be connected to the mathematical concept of the integral function, which is here treated as the total accumulating distance. Based on our theoretical framework, we define mathematical fluency as the ability to (i) link integration and differentiation as inverse processes; (ii) represent the total accumulated distance as a process (operational concept) and as an object (object oriented concept) within at least one functional representation (analytical, graphical, by words or numerical in a table); (iii) Indicate parameters that influence the model and to explain choices made.

Flexibility refers to the ability to set up a model and to use values that go beyond the information provided in the examples.

Analysis

To investigate the first research question we operationalized mathematical creativity in terms of the three components and explored the frequencies found in the students solutions. To investigate the second research question we gave scores to the 3 components and sub-components. Each student solution was then scored within 1–3 for usefulness, 0–2 for each subcategories of fluency, 0–2 for flexibility. We used the Mann-Whitney test, which is indicated for data at ordinal level of measurement, to explore whether the products of the two conditions differed from each other.

RESULTS

Fifty-two of the 79 groups that worked on the task in classroom handed in their final product to the teacher. In the following of this section we report on these products.

Students' creativity in terms of usefulness, fluency and flexibility

The first research question concerned the mathematical creativity of student productions. Table 1 shows that the majority of the groups solutions (36) were at level 1 and therefore, not useful to model the parachute jump. Only 16 groups produced models that could be used.

Usefulness	Groups solutions (N=52)
Level 1	36 (45,6%)
Level 2	15 (19%)
Level 3	1 (1,3%)

Table 1: Results on usefulness

The results on fluency are shown in Table 2. Almost half of the groups (22) explicitly established the link between integration and differentiation. For instance, one group draw both graphs, with the text differentiation and integration and two arrows pointing opposite directions. Most of the solutions (37) presented traces of an operational- oriented conception of total distance. This means that students can draw a total distance graph, use formulas to calculate single values but have difficulty to conceptualize the total distance as a mathematical object on which operations can be performed (Sfard, 1991). Very few groups (7) showed to have an object-oriented conception of total distance. An example of a student explanation that we consider exemplary of object-oriented conception is: "The distance increases at the beginning very fast, during the free fall. After 36 second, when the parachute opens the velocity becomes more or less constant and the distance increases linearly (...)". In contrast, students who would have no functional concept would not refer to distance in their explanations but describe the changes along the jump in terms of velocity, slope of line graphs (the line goes up or down) or in phenomenological terms. Most of the groups (34) did not consider parameters or provided choices.

The results on flexibility are summarized in Table 3. The majority of the groups (35) used only the values from the example. Few groups (14) refer to the values of the video and only 3 groups went beyond the information given in the task setting. Figure 2 contrast one of these solutions (right column) with a solution of the major group.

Influence of HS and LS task on mathematical creativity

The second research question investigates whether the amount of structure in the task has effect on

Fluency	Criteria	Groups solutions (N=52)
link between integration and differentiation	Not visible Unclear Explicit	24 (30,4%) 6 (7,6%) 22 (27,8%)
Conceptions of accumulating distance function	No functional concept Operational concept Object oriented concept	8 (10,1%) 37 (46,8%) 7 (8,9%)
Parameters and choices	No parameters nor choices Parameters or choices Both	34 (43%) 11 (13,9%) 7 (8,9%)

Table 2: Results on fluency

Flexibility	Groups solutions (N=52)
Confined to example or undefined	35 (44,3%)
Beyond example and confined to film	14 (17,7%)
Beyond video and example	3 (3,8%)

Table 3: Results on flexibility

Confined to the example

interval	afstandsgrafiek	invullen
(0-5)	s(x)=4,9x^2	s(5)=122,5
(5-11)	s(x)=-3,75x^2 + 86,5x – 338,75	s(11)=159
(11-81)	s(x)=4x-44	s(81)=280
		561,5m

Beyond the example and the film

"Imagine that you want to make a parachute jump. You want to make a free fall of 7 seconds. After opening the parachute you have a constant velocity of 3 m/s. Opening the parachute takes 4 seconds. After opening it you want to stay 3 minutes in the sky . How high must be the jump?"

Figure 2: Examples of two levels of flexibility

student's mathematical creativity. Table 4 shows the results on usefulness, fluency, and flexibility in both conditions. A Mann-Whitney test indicated that there was no statistically significant difference between the two conditions for all components and sub-components of mathematical creativity.

DISCUSSION

In this paper we explored the influence of task structure on the mathematical creativity in students' productions in the context of a modelling task. Next we discuss our results in the light of the two research questions.

What can we say about the mathematical creativity of students' productions with regard to the parachute jump task? Overall the student solutions attained low scores with regard to the three components of mathematical creativity. Only 52 out 79 groups delivered their final product, none of the groups created a general and reusable solution (levels 4 and 5) and only 16 out of 52 groups have created a model with level 2 or 3. Most students' use of mathematical functions involved thinking in operational views rather than object-oriented. Also, most groups failed in considering relevant side conditions (wind, gravity, etc.) and parameters that are necessary to present a realistic model for the parachute jump. These difficulties suggest that the task as we presented to the students was too challenging for most of them. Several researchers (Silver, 1997; Lithner, 2008) suggest that relationships between creativity and problem solving might be the product of previous instructional patterns. Therefore it is possible that students' previous experiences with mathematical tasks (note that the students are not used to problem-oriented instruction) may have limited their searching process. For instance, only few students tried to go beyond the given examples, as it can be seen by the low levels of usefulness and flexibility. Or, they have tried to explain their choices and present different parameters, as most of the students scored very low on these subcomponents of fluency. Therefore, one suggestion to improve the task is to provide additional information on side conditions that are not part of the mandatory curriculum or provide explicitly directions to look for them. Other suggestion involve the improvement of students' problem-solving activity. The teacher should encourage more the students during the solving process, e.g., to explore different paths, to look for other examples and not to give up too easily. Other aspects that we did not discuss here but also should be taken into consideration are the amount of time available to solve the task in the classroom, the specific directions to be provided by the teachers and assessment practices.

In which way does variation in the amount of structure in the parachute jump task influences students' mathematical creativity? The products created by the groups of students in the two conditions are not statistically significant different with regard to mathematical creativity. Therefore, the question whether providing more/less guidance in the mathematical

	HS-task Median	(N=27) Range	LS-task Median	(N=25) Range	Mann-Whitney U test (two tailed)
Usefulness (scores 1–3)	1	2	1	1	U=257.000, p=.066
Fluency (scores 0–2)					
integration-differentiation	1	2	1	2	U=308.500, p=.559
concept accumulating distance	1	2	1	2	U=304.000, p=.441
parameters and choices	0	2	0	2	U=333.000, p=.992
Flexibility (scores 0–2)	0	2	0	1	U=272.000, p=.144

Table 4: Results on mathematical creativity within high- and low-structured tasks

tasks have impact on students' mathematical creativity remains open. In this paper we studied the effect of task structure on the groups products without refer to the solution process. However the way students approach the tasks and reasoning processes might reveal mathematical creativity aspects of the students not revealed in the final product (Karakok, Milos, Tang, & El Turkey, 2015). This is one question that deserves further investigation. Another interesting question to be further investigated regards the collective creativity process. In our research the students work in small groups, thus the intrapersonal creativity of one student produces a creative product which is then appropriated by others. In this case it is difficult to determine to what extend the final creative ideas and solutions are the product of particular students or from the collective (Levenson, 2011). An interesting question therefore is: in what extend this collective process is mediated by the amount of structure provision in the task?

Concluding, although our study could not provide a conclusive answer to the question whether the amount of structure in the task influences students' mathematical creativity, it contributes to the field of research and teacher education in two ways. It extends previous research on mathematical creativity by accounting the relationship between the learning environment and creativity and, by providing a way to operationalize fluency and flexibility in conceptual mathematical terms. And it provides a practical example (the parachute task) with potential to engage students in problem-solving and concrete suggestions for its implementation. The use of this kind of tasks in the classroom and in teacher education can help teachers to recognize mathematical creativity in their lessons and therefore to better support it.

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Visual processing and attention abilities of general gifted and excelling in mathematics students

Nurit Paz-Baruch, Mark Leikin and Roza Leikin

University of Haifa, Haifa, Israel, rozal@edu.haifa.ac.il

The present study examined the visual perception and attention abilities associated with general giftedness (G) and excellence in mathematics (EM). The research involved four groups of 16–18 years old participants varying in levels of G and EM. 190 participants were tested on a battery of visual processing tasks: visual-spatial memory (VSM), visual speed of information processing (SVIP),Visual-perception (VP) and Visual attention (VA). The results support the notion that the differences between the groups are task depended. On the VSM (Backward visual-spatial memory span) test, differences in performance were associated only with EM factor, while on the visual-perception (Pattern-recognition test) and attention (D2-CP score) tests only the G factor had a main performance effect. SVIP was associated with both G and EM factors.

Keywords: Visual processing, attention, visual speed of information processing, visual-spatial memory, giftedness, excellence in mathematics.

INTRODUCTION

The literature regarding visual processing and attention, in relation to both general G and EM, is limited. Up to now, most studies have examined G and EM factors separately, while leaving out EM students who are not gifted (Benbow & Minor, 1990) or G students who are not EM (e.g., Johnson et al., 2003; Zhang, et al., 2006). This study is part of a larger investigation aimed at defining mathematical-giftedness in mathematics. In a previous study (Leikin, Paz-Baruch, & Leikin, 2013; Paz-Baruch, Leikin, Aharon-Peretz, & Leikin, 2014) we compared relations between G and EM factors and other cognitive abilities such as: memory and speed of processing abilities. In this study, we examine these same relations with regard to visual processing and visual attention abilities.

BACKGROUND

Visual processing, giftedness and excellence in mathematics

Generally, visual processing ability is defined as the ability to generate, store, retrieve and transform visual images and sensations; visual processing is also related with the ability to recall the location of stimuli or to recall, identify or reproduce a design (McGrew, 2009).

Several studies suggest that visual speed of information processing (SVIP) abilities are related to intellectual giftedness (Dark & Benbow, 1991; Jensen, Cohn, & Cohn, 1989; Kranzler et al., 1994). Research has also demonstrated that visual-spatial ability is associated with general intelligence and academic achievement (Johnson & Bouchard, 2005). Gifted children have been found to respond more quickly than those with average IQ on a variety of SVIP (Deary, 2000; Duan, Dan, & Shi, 2013; Johnson et al., 2003) and visual-spatial tasks (Rizza, McIntosh, & McCunn, 2001).

Studies also showed a connection between SVIP and mathematical performing. Taub and colleagues (2008) demonstrated that visual processing speed is significantly related to quantitative knowledge for children. Moreover, Fuchs and colleagues (2006) found that in a group of third grade children, processing speed was a predictor of arithmetic ability when assessed by crossing-out tasks, and perceptual motor speed tasks. Geary (2011) revealed that processing speed, predicted achievement in mathematics, especially in numerical operations.

The ability to understand visual representations is considered by researchers as an important tool for mathematical learning and problem solving (Deliyianni, Monoyiou, Elia, Georgiou, & Zannettou, 2009). Excelling in mathematics students understand the problem by constructing and employing a diagram or a picture to help obtain a solution (Bishop, 1989).

Visual attention, giftedness and excellence in mathematics

The relationship between measures of attention and intelligence has been investigated repeatedly (e.g., Crawford, 1991; Rockstroh & Schweizer 2001; Schweizer, Zimmermann, & Koch, 2000; Schweizer & Moosbrugger, 2004). Few studies examined connection between sustained attention or divided attention and intelligence and showed that they are correlated with intelligence (Scheweizer et al., 2000; Schweizer & Moosbrugger, 2004). Being able to maintain attention for a long time at a high level is important whenever complex mental activities are to be performed, like problem solving and reasoning, which are closely associated with intelligence (Schweizer & Moosbrugger, 2004). Gifted individuals have swifter access to relevant knowledge due to faster automation of thought processes. As a result of which, they retain available attention capacity to tackle additional tasks (Memmert, 2008). Correlation between intelligence and divided attention depends on the tasks to be performed. Higher demanding tasks seem to yield higher correlations between measures of attention and intelligence than less demanding tasks (Scheweizer et al., 2000).

Most of the literature about mathematical ability and attention focus on children with learning disabilities and on the inhibition of irrelevant stimuli. Children who are less proficient in math have difficulties to suppress irrelevant information under high processing demand conditions (e.g., De Beni et al., 1998; Swanson, 2006). Anobile, Stievano, and Burr (2013) showed that attention and numerosity perception predict math scores. Individuals with higher math ability have less difficulty than average achievers in reducing accessibility of less relevant information that could overload and interfere during processing (Agostino, Johnson, & Pascual-Leone, 2010).

Accordingly, the goal of this study was to examine the connection between visual processing, attention and G and EM factors. We examined the hypothesis that G and EM factors are related differently to different visual processing abilities.

METHOD

Participants

We report herein our findings on 186 10th–12th grade students (16–18 years old) right-handed male and female students who were recruited for the study (see Table 1). The participants were subdivided in four experimental groups, determining the research population by a combination of EM and G factors: G-EM group: students who are identified as generally gifted and excelling in mathematics; G-NEM group: students who are identified as generally gifted but do not excel in mathematics; NG-EM group: students excelling in mathematics who are not identified as generally gifted; NG-NEM group: students who are neither identified as generally gifted nor excelling in mathematics.

Tasks and materials

Visio-Spatial Working Memory test (Corsi, 1972)

This block recall task consists of ten blocks arranged randomly on a wooden board. The test involves two parts: during the first part the researcher points at a sequence of blocks at a rate of one per second. After the researcher completes indicating the sequence, the participant is asked to replicate the sequence. If the

	Gifted (G) Raven > 27	Non-Gifted (NG) Raven < 26	Total
Excelling in mathematics (EM) SAT-M >26 or HL in mathematics with math score > 90	41	40	81
Non-excelling in mathematics (NEM) SAT-M <22 and RL in mathematics with math score > 90 or HL in mathematics with math score < 80.	53	56	109
Total	94	96	190

Table 1: Description of study groups

participant recalls the sequence of blocks correctly, another trial is administered. Successive trials are administered adding one more block each time and so forth until the participant fails two successive attempts. The maximum possible span is ten blocks.

During the second part, the researcher points at a sequence of blocks at a rate of one per second. After the researcher completes indicating the sequence, the participant is asked to replicate the sequence backwards. If the participant recalls the sequence of blocks correctly, another trial is administered. Successive trials are administered adding one more block each time and so forth until the participant fails two successive attempts. The maximum possible span is ten blocks. The measure of both test parts was a standard score according to the accepted Israeli scale (from Hebrew version of Visio-Spatial Working Memory test).

Visual- matching test (Woodcock-Johnson Tests of Cognitive Ability, 2001)

The test consists of rows that include one target symbol and 19 additional symbols. The participant has to circle all the symbols that are identical to the target symbol. The time limit for the assignment is 120 seconds.

Digit-symbol test (WISC III, 1997)

The test consists of a code table displaying pairs of digits and symbols, and rows of double boxes with a digit on the top box and nothing on the bottom box. The participant has to use the code table to determine the symbol associated with each digit (the test consists 133 digits), and to write as many symbols as possible in the empty boxes below each digit. The time limit for the assignment is 120 seconds.

Symbol-search (WISC III, 1997)

The test consists of rows marked by one target symbol and five additional symbols. The participant has to decide if the target symbols appear in the row of symbols and to mark YES or NO accordingly. The test consists of 60 items and the participant has to mark as many items as possible within 120 second.

Pattern recognition test (Thorndike, Hagen & Sattler, 1986)

The test consists of two columns of cross patterns: Pattern A is hidden in the larger pattern B. The participant has to draw a line around the crosses in B which make the same pattern as those in A. The test consists of 18 patterns and the time limit is nine minutes. The measure was accuracy (in %) of correct answers.

D2 Test of attention (Brickenkamp, 1994)

The D2 is a timed test for selective attention. The items are composed of the letters "d" and "p" with one, two, three or four dashes arranged either individually or in pairs above and below the letter. The participant is given 20 seconds to scan each line and mark all "d's" with two dashes. There are 14 lines of 47 characters each for a total of 658 items. Measures of performance include total number of items processed (TN), Total number of items correctly processed (TN-E) number of errors (E), an index of concentration performance (CP), and fluctuation rate (FR) across trials.

Data analysis

To investigate the questions addressed in this study, multivariate analysis of variance tests (MANOVA) were used to compare the scores of participants in each test. The between-subjects factors were: G and EM factors and the within-subjects factors were the scores on each visual processing and attention tests.

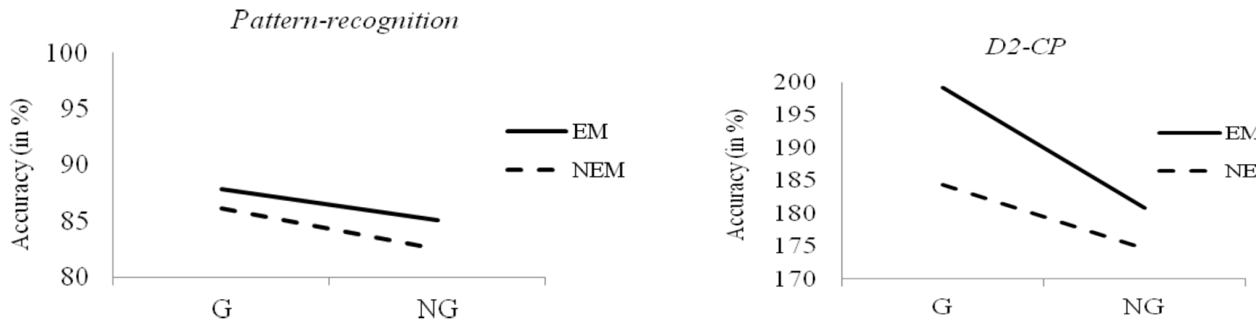

Figure 1: Significant main effect of G factor on VP and VA tests

RESULTS

Between groups differences on visual processing tests

MANOVA revealed an overall significant main effect for G factor (F(12,167)= 2.31, p< .01). Following univariate ANOVA tests showed that the sources of differences between the groups are the Pattern-recognition test (F(1,178)= 5.15, p<.05), D2-CP (F(1,178)= 9.63, p<.05). G student's accuracy on Pattern-recognition test (M= 87.04, SD= 9.33) and their D2-CP scores (M= 191.80, SD= 35.80) were significantly higher than NG students (Pattern-recognition M= 83.74, SD=11.62; D2-CP M=177.64, SD= 21.10) (Figure 1).

In addition, Univariate ANOVA tests revealed a significant main effect for EM factor in Symbol-search (F(1,178)= 4.64, p<.05) and Backward Corsi-span (F(1,178)= 3.96, p<.05) tasks. As shown in Figure 2, EM students outperformed NEM students on Symbol-search (EM: M= 73.62, SD= 10.22; NEM: M= 70.64, SD= 11.22) and Backward Corsi-span (EM: M= 6.29, SD= 1.04; NEM: M= 6.04, SD= 0.98).

CONCLUSIONS

The present study evaluated visual processing abilities linked to G (general giftedness), EM (excelling in mathematics). Between-group differences in visual processing were found to be task-dependent. On the VSM (Backward visual-span) test, differences in performance were associated only with EM factor, while on the visual-perception (Pattern-recognition test) and visual attention (D2-CP scores) tests only the G factor had a main performance effect. Visual SIP tasks were associated with both G and EM factors.

The results regarding visual-perception revealed that G students performed significantly better on this task regardless of their abilities in mathematics. These findings are in line with results of other studies which suggested that a superior visualizing ability characterizes highly gifted individuals (Silverman, 1995). The results regarding visual SIP revealed that G-EM students outperformed on two of the visual SIP tests (Symbol-search and Digit-symbol) compared to the other three groups. These findings are in line with previous studies which reported that processing speed is significantly related both to quantitative knowledge (Berg, 2008; Johnson et al., 2003; Swanson & Beebe-Frankenberger, 2004) and general giftedness (Dark & Benbow, 1991; Johnson et al., 2003; Kranzler et al., 1994), and are also partly reported in our previous study (Paz-Baruch et al., 2014).

Our study also demonstrates that the performance of G students on visual attention task, as regards concentration performance (D2-CP score) was better than

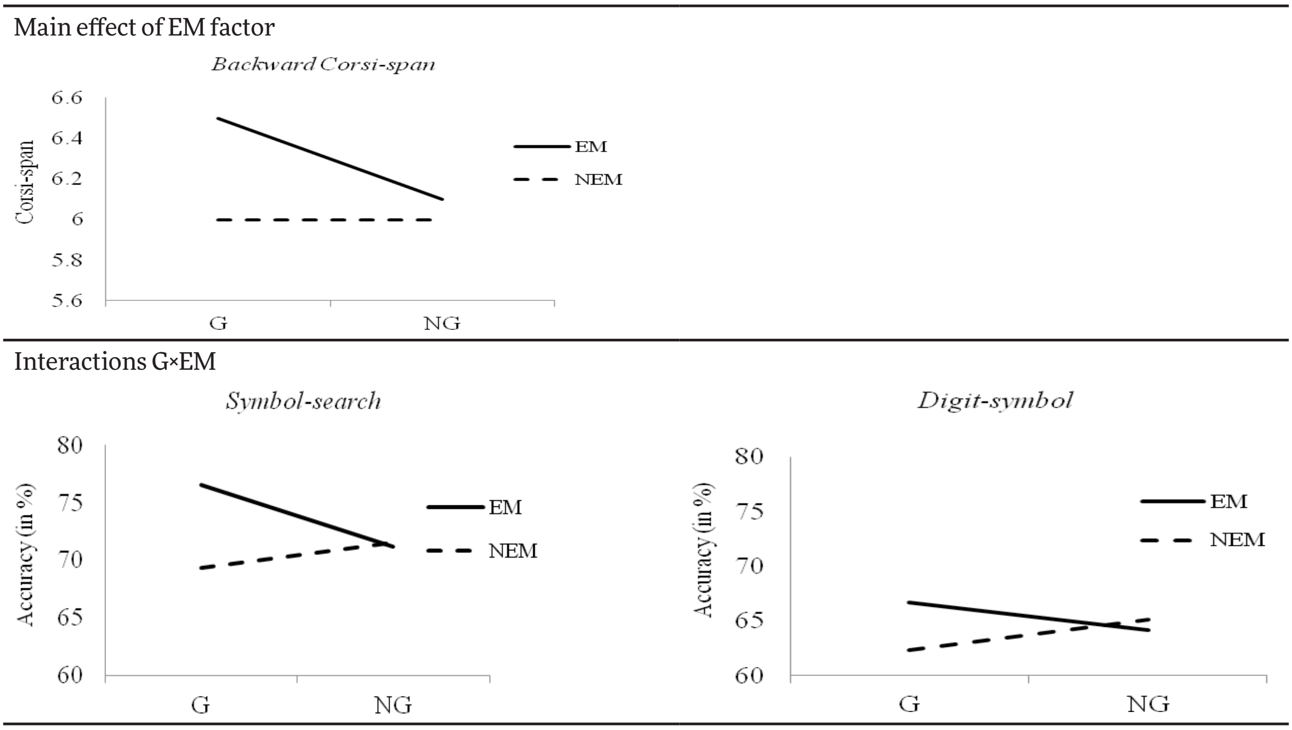

Figure 2: Significant main effects and interactions of G & EM factors on VP tests

that of NG students. It appears that gifted students are able to stay focussed on an assignment for a long time (elements of sustained attention) and are able to selectively attend to relevant stimuli while filtering out irrelevant stimuli in a rapid manner.

In summary, the present study generated data on the visual-processing abilities of adolescents, divided into four groups according to giftedness and excelling in mathematics. The study reveals that G and EM factors are different yet related mechanisms.

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Assessing mathematically challenging problems

Demetra Pitta-Pantazi, Constantinos Christou, Maria Kattou, Paraskevi Sophocleous and Marios Pittalis

University of Cyprus, Department of Education, Nicosia, Cyprus, dpitta@ucy.ac.cy

The aim of this article was twofold. First, to propose a model for assessing mathematical challenging problems and second, to investigate the abilities of a group of in-service teachers to propose mathematically challenging problems based on the model suggested. The results indicated that mathematical challenging tasks may be characterised as those that are cognitively demanding and also require from students to develop other key competences: digital, social, communication in mother tongue, learning to learn and sense of initiative. About half of the participants of this study were able to provide mathematically cognitively demanding tasks which encompassed at least three of the other key competences. Participants appeared to find most difficult to incorporate in their mathematical tasks the "digital competence" and "learning to learn".

Keywords: Challenging tasks, key competences.

INTRODUCTION

Mathematical tasks are considered to be in the core of student mathematics learning because they "convey messages about what mathematics is and what doing mathematics entails" (NCTM, 1991, p. 24). Different types of tasks may potentially influence students learning, thinking and understanding of mathematics in a different way (Henningsen & Stein, 1997; Kilpatrick, Swafford, & Findell, 2001). For low cognitive demand mathematics tasks, the emphasis is on practicing and repetition of known facts and procedures. In contrast, high cognitive/ challenging mathematics tasks require understanding and extending concepts (Hsu, 2013). In recent years the emphasis of mathematics education has turned towards teaching challenging cognitive demanding tasks, given that "Challenge is not only an important component of the learning process but also a vital skill for life" (Taylor, 2006, p. 2). International organizations stressed the importance of empowering students with key competences in order to be able to confront future challenges (e.g., European Parliament and Council, 2006).

The Professional Standards for Teaching Mathematics (NCTM, 1991) claimed that students' learning depends to a great extent on the way that teachers develop and implement mathematical tasks in their instruction. Therefore, it is important for teachers to gain a coherent understanding of the importance of mathematical challenge in teaching and learning mathematics, and moreover to be able to choose, design and implement such tasks in their teaching (Applebaum & Leikin, 2014). The purpose of the present study is twofold; first, to propose a mathematically challenging problem assessment tool, and second, to examine in-service teachers' ability to design challenging mathematical problems.

THEORETICAL CONSIDERATIONS

Defining mathematical challenging tasks

The main characteristic of mathematical challenging tasks is the fact that the solver is not immediately aware of the procedures or algorithms that are critical for its solution (Applebaum & Leikin, 2014; Powell, Borge, Fioriti, Kondratieva, Koublanova, & Sukthankar, 2009). Therefore, solvers are required to attempt to find a solution based on their knowledge and understanding (Applebaum & Leikin, 2014; Powell et al., 2009). Moreover, Guberman and Leikin (2013) added other characteristics in the definition of mathematical challenge: the tasks should be neither too easy nor too difficult, and should engage students in meaningful scenarios that develop mathematical curiosity and motivate students to persevere with task completion.

The characteristics of challenging mathematical tasks make them suitable to cover a range of audiences and didactical situations (Powell et al., 2009). In particular, challenging mathematical tasks can be attempted successfully by students of various mathematical backgrounds for diagnostic purposes, for learning new concepts and procedures, for developing mathematical understanding, for formative and final assessment (Powell et al., 2009). Recent studies verified the importance of challenging mathematics tasks in teaching and learning mathematics. A number of researchers (Hiebert et al., 2005; Hsu, 2013; Powell et al., 2009; Silver, Mesa, Morris, Star, & Benken, 2009; Stein & Lane, 1996) underlined their potential to maintain curiosity, stimulate creativity, promote flexible thinking, encourage collaboration and exploration, allow communication, increase students' understanding, promote conceptual understanding of mathematics, develop problem solving and reasoning abilities.

Mathematical challenging tasks and key competences

The abovementioned characteristics are essential in a society that requires citizens to be flexible at workplace, to adapt quickly to constant changes in an increasingly interconnected world and to be innovative, productive and competitive (European Parliament and Council, 2006; Halász & Michel, 2011). Taking into consideration the declaration of the European Parliament and Council (2006) for adapting the educational systems to the demands of todays' society by empowering citizens with lifelong learning abilities, it could be supported that the design of challenging problems that develop these abilities are of great importance.

Contemporary documents made use of the term "key competences" to determine the cognitive elements, the functional aspects (involving technical skills) as well as the interpersonal attributes (social or organizational skills) and ethical values that are important for personal fulfillment, active citizenship and employability (Halász & Michel, 2011). In this framework the European Parliament and Council proposed six key competences (KC), which are of equally importance and also are interrelated, as they are defined below (European Parliament and Council, 2006; Halász & Michel, 2011). (a) *Communication in the mother tongue and communication in foreign languages*: The ability to understand, express and interpret procedures, concepts, ideas, thoughts, and feelings, in both written and oral form is fundamental to human interaction, (b) *Mathematical competences*: The ability to develop and apply mathematical thinking in an attempt to understand situations, to find explanations and to solve a range of problems in everyday situations, (c) *Digital competences:* The confident and critical use of information and communication technologies for the execution of educational, vocational and everyday work as well as for leisure and communication, (d) *Learning to learn:* The ability to pursue and organize the learning procedure of an individual or group, taking into account their needs and difficulties, the available time and information and the given opportunities and restrictions, (e) *Social, civic and cultural competences:* These competences embrace personal, interpersonal and intercultural aspects which equip individuals to engage in active and democratic participation in social and working life, (f) *Sense of initiative and entrepreneurship:* Creativity, innovation and risk-taking are among the characteristics that assist individuals to materialize their objectives.

These competences are anticipated to be acquired both by students at the end of compulsory education, as well as by adults through a process of developing and updating their skills (European Parliament and Council, 2006). Thus initial education should offer all students the opportunities to develop key competences in a sufficient level that will equip them for adult and working life (European Parliament and Council, 2006).

Designing mathematical challenging tasks

Despite the importance of mathematical challenging tasks, research has shown that it is not easy for teachers to design and implement such tasks in mathematics classrooms (Henningsen & Stein, 1997; Silver et al., 2009). The first barrier lies on teachers' pedagogical and content knowledge to design challenging tasks (Applebaum & Leikin, 2014). Teachers' content knowledge determines their understanding of the essence of mathematical challenge, their knowledge of challenging mathematics and their ability to approach challenging tasks (Applebaum & Leikin, 2014). As for the pedagogical knowledge, teachers' knowledge of the way that students cope with challenging mathematics, as well as different approaches and learning setting to teaching challenging mathematics are included (Applebaum & Leikin, 2014). Secondly, teachers find it difficult to design tasks that have a rich mathematical content, either by incorporating different mathematics topics and ideas or by demanding high cognitive effort (Silver et al., 2009; Stigler & Hiebert, 1999). In particular, in a study conducted by Silver and colleagues (2009) 84% of the activities designed by teachers focused on a single mathematics topic area rather than on multiple topics. Only 1 out of 3 activities was classified as a high-demand task, since teachers had difficulties to incorporate requirements for inquiry or explanations.

An important dimension of teachers' work is to find and/or adapt tasks. Additionally, it is extremely important to evaluate whether a task is appropriate for a particular student, from various perspectives (level of difficulty, interest, prior knowledge) (Guberman & Leikin, 2013). Moreover during employing the tasks in their instruction, teachers have to decide what they want students to achieve, what they need to emphasize, how to sequence the various activities and in what way to support students without reducing the challenge (NCTM, 1991; Vale & Pimentel, 2011). Hence, it is apparent that by providing teachers with ready-to-use challenging mathematical tasks is not sufficient for their implementation. Teachers need to be convinced about the importance of mathematical challenge and develop abilities to deal with such kind of mathematics (Applebaum & Leikin, 2014).

Another issue that appears to be surfacing is the definition of a "mathematical challenging task" in today's society. Is it simply a task that offers a certain level of mathematical, cognitive challenge to students or should it encompass other competences? Thus, the aim of the study was twofold: first, to suggest a model for assessing mathematically challenging problems, and secondly, to investigate whether in-service teachers who participated in a post-graduate problem solving course could develop mathematical challenging problems that promote key competences.

METHODOLOGY

Participants, procedure and data collection

The research was conducted at the University of Cyprus with 29 post-graduate students (PGs), who were studying for an MA in Mathematics Education, during their Problem Solving Course. In this course, ideas arising from the Research Project KeyCoMath were utilized. Twenty three of the PGs had a BA in Primary Education, and six had a BSc in Mathematics.

The postgraduate course on Problem Solving was organized in 13 three-hour seminars (a seminar per week). During the course students worked on the following topics: (a) Mathematical problems and problem solving: Definitions, stages and strategies in problem solving, different types of problems, factors that affect problem solving abilities, (b) Modeling problems: description, modeling principles, modeling perspective vs problem solving perspective, assessment of modeling, (c) Problems in international competitions (PISA and TIMSS), (d) Problem posing, (e) Teaching approaches for the development of problem solving skills, (f) Inquiry-based learning and problem-based learning, (g) Teachers role during problem solving, and (h) Key competences in mathematics education.

One of the assignments that the students had to do, for the fulfillment of the requirements of the course, was to work either individually or in groups (of 2 or 3 people) and develop four challenging mathematical problems which would also promote key competences. Moreover students had to identify how key competences can be developed through these challenging problems.

Data analysis

In the present study we propose a methodological tool to examine the extent to which a problem could be considered as a challenging one. To do so, we synthesize different theoretical approaches. For instance, Silver and his colleagues (2009) adopted frameworks that were used to distinguish levels of demands in mathematical tasks (see Kilpatrick, Swafford, & Findel, 2001) and proposed criteria for coding activities as high or low demand. In addition, a number of organizations (see European Parliament and Council, 2006) assert that students should develop key competences to meet contemporary society needs. Table 1, presents thoroughly the set of criteria that compose our proposed assessment tool. Based on the adopted frameworks, we propose that a problem could be classified as a challenging one if it required high mathematical cognitive demand and involved at least three out of the five examined key competences.

The first assessment criterion is concerned with the problem's potential to develop student's mathematical competence. Taking into consideration Silver's framework (2009), it was decided that a problem could be characterized as a "mathematical high cognitive demand", if it explicitly required students to explain, describe, justify, compare, make decisions, plan, formulate questions or be creative in some way (Silver et al., 2009). On the contrary, a "mathematical low cognitive demand" problem requires merely routine applications of known procedures, extremely guided structure or challenging on non-mathematical issues.

The second criterion of our framework relates to the extent to which a problem provides opportunities to develop student's digital competence. A problem would be classified as a high digital competence task if it explicitly required the use of a digital device to search, collect or analyze data, support critical thinking, creativity or innovation. No-use, unclear mention to the use of digital media or use of technology solely for executing computations were classified as low digital competence. The third key competence criterion involves social competence. A problem would be classified as high social competence task if required students' involvement in communicating with others, to work collaboratively, to understand, share and reinforce other's ideas. Problems that did not require group working or peer interaction were classified as low social competence tasks. Communication in mother tongue was interpreted as the ability to express and interpret concepts and present with clarity and accuracy their mathematical ideas. Thus, a problem would be classified as high in communicating in mother/mathematical tongue, if it asked students to present, justify or convince regarding their solution or explain the way in which they used mathematical language to interpret the problem. The key competence learning to learn was defined as student's awareness of his/her learning process and the ability to build on previous learning experiences to transfer knowledge in a new context. Thus, a high demand learning to learn task required the use of reflective tools, such as explicit description of the solution plan, or the extension of the proposed solution to a different context. Finally, the key competence initiative was conceptualized as student's initiative to take decisions, propose creative ideas, risk-taking in planning and managing steps in the solving procedure. In this sense, the assessment criterion for high initiative key competence included the engagement of students in taking decision, planning or evaluating situation, the existence of multiple solutions or solving plans. This would require students to turn their ideas into actions by judging the risk of each solution plan or by applying creative ideas. On the contrary, an ill-define problem or extremely guided one would be classified as low initiative.

Two researchers were independently assigned to rate each task using the abovementioned criteria. There was near unanimity in this coding, and wherever there was any disagreement this was discussed until consensus was reached.

RESULTS

Table 2, presents the classification of the problems proposed by the students based on the assessment model described above. In particular, 37 out of 57 problems were classified as high demand mathematical cognitive competence. They required students

Mathematical competence	Digital competence	Social competence	Communication in mother tongue	Learning to learn	Initiative
Low					
Routine applications, extremely guided, imposed solution	No-use, unclear mention	Absence of effective interaction, no tolerance to other ideas	No use of mathematical language, poor expression of ideas, insufficient data or without different forms of data (verbal, graphical, symbolic)	No use of reflective questions or potential of transferring knowledge	Ill-structure problems or extremely guided
High					
Explain, describe, justify, make decisions, plan, analysis, investigate, explore	Explicit and effective use of digital device to search or analyze data	Constructive communication, group working, respect to other ideas	Express and interpret concepts, thoughts and facts in oral and written form, proper use of mathematical language, flexible use of different representations	Reflective tools and questions, transfer of knowledge to new contexts	Initiative to take decisions, creativity, risk-taking, plan and manage solution

Table 1: Description of assessment criteria

to investigate real life situations, use and connect different mathematical concepts, processes and relationships. The majority of them were decision-making problems. Four problems required from students to decide which the best monthly payment plan was, based on the advertisement of four communication companies. To do so, it was required to mathematize the problem by (i) suggesting a mathematical model for evaluating the monthly cost based on the offers of the companies, (ii) proposing the cheaper package that meets specific needs or (iii) proposing the best package for a person based on the statements of his account. Another problem required from students to propose the cheaper heating option, by taking into consideration the dimensions of a flat. Another interesting problem involved finding the best wine list for a cellar shop based on real costs, promotion costs, sale costs, delivery costs and people preferences. Two groups of teachers used football scenarios. They asked students to predict the winner of World Cup 2014 in Brazil, by proposing a mathematical model that takes into consideration several parameters. A second group of high cognitive demand mathematical problems required the analysis of a situation and the design of a system for a specific goal. For instance, students were asked to design a camping place and a car park.

Twenty out of the 57 problems were classified as low cognitive demand because they simply asked students to apply known procedures. A number of low demand problems asked students to solve open problems, without providing adequate data or their questions did not involve mathematical procedures or calculation (e.g., to describe the recipe for a birthday cake).

The results of the study showed that only 11 mathematical problems gave opportunities to develop students' digital competence. In particular, eight of these problems asked students to analyse, find connections and evaluate data given in spreadsheets or in specific links. For example, a problem required students to use the data provided in a spreadsheet regarding the population of Cyprus and Japan and use the tools of the software to find which of the two countries has the greater aging population problem. Another problem asked students to search on internet about the data of runners participated in the Olympic Games in London and the World Championship Athletics 2014 and to propose the four best runners of the last years. The rest of the high digital competence problems required students to find the solution to the problem by using software. For instance, a problem asked students to use a dynamic geometry software to design the floor plan of a library, by taking into consideration the dimensions of furniture that were provided in a spreadsheet.

The majority of the proposed problems provided opportunities to develop students' social competence. In particular, 53 out of the 57 problems required students to work in groups, collaborate and communicate for the solution of the problems. Some problems required job assignment, so that all members could work constructively. Problems also included instructions regarding tolerance and respect to everybody's ideas. The four low social competence problems did not make any reference on group work or communication among group members.

Almost half of the problems (28) provided opportunities to develop communication in mother tongue. They required students to use accurate and concrete mathematical language, to express and explain their ideas both in written and oral form and to utilise different representations. For instance, a problem asked students to design a poster for presenting their solution to their peers. Other problems asked students to use graphs and convincing arguments to support their suggestion. The remaining 29 problems were classified as low social competence because they did not involve accurate use of mathematical language or utilisation of different representations.

Only 20 out of the 57 problems could promote "learning to learn". In these problems participants included reflection tools (see Figure 1) or they required to extend the solution method in a new situation (e.g., after

	Math. competence	Digital competence	Social competence	Communication in mother tongue	Learning to learn	Initiative	**Challenging Problems**
Low	20	46	4	29	37	22	30
High	37	11	53	28	20	35	27

Table 2: Classification of problems

developing a model to select the most effective lights for University lecture theatre students were asked to extend their model to adapt in any room's light). On the contrary, the rest of the problems did not include reflection tools or any extension questions.

About half of the problems (35 out of the 57) provided opportunities to develop students' sense of initiative. In particular, these problems required the construction of a mathematical model and the assignment of weights to the various criteria used in this model. For instance, a problem asked students to select the best three Universities for a prospective student based on specific criteria. To do so, it was required to assign weight to a set of criteria, such as the ranking of the university, the distance from home, the number of amenities, the distance from the airport and availability of free wi-fi. Twenty two problems were rated as low initiative, due to the fact that they were extremely structured or they were too open and ill-defined.

Work in groups to solve the following problem. It is important to listen carefully to the ideas of your peers and collaborate in a constructive manner. A family searches for available flats to rent. As Mr. Antoniou mentioned they can afford a monthly rent up to 620 euro. Mr. Antoniou works in an accountancy firm, while Mrs. Antoniou is unemployed. They want to live in Latsia (area in Nicosia, Cyprus), since the children's school is in this area. Use the data provided in the spreadsheet (dimensions, area, age, rent, etc.) and the map from Google earth to decide the best option for the family to rent. Explain your answer by using words, symbols, tables or graphs. Once you have solved the problem, answer the following questions: (1) Write the mathematical concepts and processes you used in solving the problem, (2) Draw a diagram to show the changes in group thinking during the solution process, (3) Can the proposed model be used by another family? Explain, and (4) How difficult did you find this task? Explain.

Figure 1: An example of challenging mathematical problem

Summing up, we concluded that only 27 out of the 57 problems could be classified as challenging (high mathematical cognitive demand and three out of the five key competences). It should be noted that ten problems met the criterion for high cognitive mathematical demand, but not the criterion of developing three out of the five key competences. Figure 1, presents a decision-making problem provided by one of the participants, which fulfilled all the criteria of our suggested model.

CONCLUSIONS

In this paper we proposed a model for assessing challenging mathematical problems based on mathematical features as well as other key competences. We suggested that a mathematical challenging problem should encompass high cognitive demand and at the same time enhance at least three out of the following key competences: digital, social, communication in mother tongue, learning to learn and sense of initiative. This proposed model is in line with the definitions and characteristics of challenging mathematical problems suggested by other researchers (e.g., Hiebert et al., 2005; Hsu, 2013; Powell et al., 2009; Silver et al., 2009; Stein & Lane, 1996).

The postgraduate problem solving course allowed a group of in-service teachers to develop abilities to design challenging tasks up to a certain extent. Almost half of the proposed problems appeared to be able to develop highly mathematically cognitive tasks which incorporated at least three other key competences. However, it seemed that the in-service teachers of the study had a greater difficulty to develop mathematical challenging problems that promote "digital competence" and "learning to learn". Therefore, future studies should aim to investigate the way in which we may empower teachers to develop challenging mathematical problems which enhance students' key competences.

ACKNOWLEDGEMENT

The project "KeyCoMath – Developing Key Competences by Mathematics Education" has been funded with support from the European Commission. This publication reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

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Heuristics and mental flexibility in the problem solving processes of regular and gifted fifth and sixth graders

Benjamin Rott

University of Duisburg-Essen, Faculty of Mathematics, Essen, Germany, benjamin.rott@uni-due.de

This study reports on a mixed-methods design analyzing the problem solving processes of gifted and regular fifth and sixth graders (age 10–12). The analysis focuses on heuristic techniques, indicating that the regular pupils need more heuristics to be equally successful as the gifted ones. This finding is explained by the theory that heuristics can be used by the regular students to compensate a lack of mental flexibility.

Keywords: Mathematical problem solving, heuristics, intellectual flexibility.

INTRODUCTION

Within problem solving, which is a very important part of mathematics, heuristics play an integral role. It is generally assumed that the use of heuristics is related to problem solving with successful problem solvers using a greater number and variety of heuristics than less successful ones. But there are still some issues unresolved as these results have mostly been obtained with regular students. The number and appearance of heuristics in the processes of gifted or creative problem solvers might differ from those of regular problem solvers. In this article, the problem solving processes of two groups of fifth and sixth graders are analyzed and compared: one group consists of pupils from regular lower secondary schools while the other group consists of pupils that have been very successful at mathematical competitions.

THEORETICAL BACKGROUND

The theoretical framework of this article focuses on an integral aspect of mathematics that is important for gifted as well as for regular students: on mathematical problem solving and particularly on the use of heuristics in problem solving. After justifying the classification of gifted students by their participation in competitions, different aspects of problem solving are discussed. This implies the use of heuristics, their function of compensating a lack of intellectual flexibility, as well as reports on their trainability.

Mathematical competitions provide the opportunity for students to actively engage in mathematical problem solving and to compete with other problem solvers. Successful participants of such competitions are mostly mathematically gifted and talented students (Bicknell & Riley, 2012); a fact which should be true especially for younger students as they have less opportunity (only regarding their age) to compensate missing talent by training. Classifying the successful participants as gifted also fits Renzulli's (1978) classical definition of giftedness as these students show above-average abilities as well as task commitment and creativity by solving the problems posed at mathematical competitions.

The term "problem solving" has different meanings ranging from solving routine tasks to working in perplexing or difficult situations (e.g., Schoenfeld, 1992). This article refers to the latter interpretation, problem solving as working on non-routine tasks. This implies that the attribute "problem" depends on the solver, not on the task. A task is a problem for a person that does not know any procedures or algorithms to solve it. Instead of algorithms that just have to be followed step by step, heuristics can help solving problems by ordering or reducing the search space and by helping to generate new ideas (Rott, 2014). In this article, heuristics are considered as methods, mental tools, or mental operations such as "working backwards" or "drawing a figure". The use of heuristics is thoroughly discussed by Schoenfeld (1992) in general and by Engel (1998) who focuses on mathematical competitions.

Bruder (2003) and Bruder and Collet (2011) – drawing on the work of the psychologists Lompscher and Hasdorf – describe the qualities of creative and intuitive problem solvers: One main characteristic of these problem solvers is their intellectual flexibility that allows them (amongst others) to easily consider different aspects and to focus on important parts of problems. Bruder then divides intellectual flexibility into five groups of actions, namely Reduction, Reversibility, Consideration of Aspects, Change of Aspects, and Transferring. For the actions of each group of intellectual flexibility, she presents heuristic actions that are able to help less flexible problem solvers overcome their lack of intuitive problem solving skills. For example, the intuitive skill of structuring facts can be compensated by creating a table; the intuitive skill of reversing relationships can be substituted by working backwards.

Research results regarding the trainability and use of heuristics can be summarized as follows: In the 1960ies and 70ies there have been several studies in America showing (often weak) positive correlations between the use of heuristic strategies and performance on ability tests as well as on specially constructed problem solving tests (Schoenfeld, 1992). Newer studies support this supposed relationship between the use of heuristics and success in problem solving (e.g., Komorek et al., 2007; Rott, 2012). Most of these studies did not only measure the number of heuristics used and the participants' success in problem solving, but also conducted some sort of training. The results show consensually that usage of heuristics can be accomplished by training. However, these trainings have often been limited to small groups of problems with unknown transferability of strategies to other problems (cf. Schoenfeld, 1992); additionally, these trainings have been limited to regular (school and university) students without specifically addressing gifted students. The findings of these studies seem to support the claim of Bruder that learning heuristics can (at least partly) compensate the abilities of intuitive problem solvers in regular students. But we do not know enough about the abilities of creative or gifted problem solvers to really draw such conclusions.

The *research intention* of this article is to further explore the problem solving abilities of gifted students and to compare them to those of regular students. Can these two groups be distinguished by the number and appearance of the heuristics in their processes? In the short run, such a comparison can help us to better understand the way in which heuristics work. In the long run, this research can help us to better teach problem solving in schools (for gifted and regular students).

DESIGN OF THE STUDY

The aim of the research presented in this paper is to explore the problem solving behavior of fifth and sixth graders (aged 10 to 12) by analyzing and comparing the processes of two groups of pupils: novices and experts[1].

"Novices": The so-called novices were regular pupils from secondary schools in Hanover, Germany that took part in the first four terms of the support and research program MALU[2] for fifth graders, which lasted from November 2008 to June 2010 (with 10 – 15 pupils each term). These pupils came to the University of Hanover once a week for 1.5 hours and worked on problems for about half of this time. Ability tests and a consideration of school grades as selection criteria ensured a mixture of pupils that can be classified as "non-gifted".

"Experts": The so-called pupil experts were successful participants of mathematical competitions, namely of the final round of the German Mathematical Olympiad[3] in 2009/10 (8 pupils of grade 5 and 6) and price winners of the Mathematical Kangaroo[4] in 2009 (2 pupils of grade 6). As very successful problem solvers at a young age, these pupils are considered to be

1 The use of the terms „non-gifted" and "gifted" is mostly avoided in this context because of possible negative connotations of "non-gifted" and because there was no official test to ensure the "gifted" status of the pupil experts – however, the second group meets the criterion of "reproducible superior performance" for expertise.

2 Mathematik AG an der Leibniz Universität which means Mathematics Working Group at Leibniz University

3 The German Mathematical Olympiad consists of four rounds: (1) tasks to be solved at home to qualify for (2) a written tests at schools (180 minutes for grades 5 and 6). The best 200 students of all grades qualify for (3) the final round of the federal state which takes place at a central place. And the 12 winners of those state finals are invited to the final round of all German states – but this round is only for students of grade 8 and higher.

4 An international competition which consists of 30 tasks (75 min); 5 to 6 % of the German participants receive prices.

"gifted". They were asked to take part in this study at the venue of the Olympiad and the school of the Kangaroo winners; the pupils did so voluntarily and have not worked with their video-partners beforehand.

To explore the pupils' problem solving behavior, their processes were videotaped. To ensure uninfluenced problem solving attempts, the pupils worked without interruptions or hints from the researchers. For the same reason the pupils were not trained to think-aloud or interrupted by interview questions; instead, they were asked to work in pairs to enable an insight into their thoughts through their natural communication. Three problems were selected for the comparison (see Figure 1).

METHODOLOGY

To analyze the problem solving processes as well as the products (everything that had been written down or sketched during those processes) of both groups of pupils, a mixed methods design has been chosen. This allows for detailed analyses of singular processes as well as an overview of all the data.

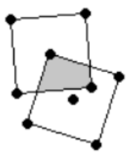
Product Coding: The pupils' products were graded in four categories of success: (1) *No access*, when they showed no signs of understanding the task properly or did not work on it meaningfully. (2) *Basic access*, when the pupils mainly understood the problem and showed basic approach. (3) *Advanced access*, when they understood the problem properly and solved it for the most part. And (4) *full access*, when the pupils solved the task properly and presented appropriate reasons, if necessary.

This grading system was customized for each task with examples for each category. Then, all the products were rated independently by the author and research assistants. After calculating Cohen's kappa ($k > 0.85$ for each task), the few products with differing ratings were discussed and recoded, reaching consensus every time.

Process Coding – Heuristics: Occurrences of heuristic techniques (like *drawing a figure* or *examining special cases*) were coded using a manual that was developed for analyzing videotaped problem solving processes (see Table 1); that development was based on empirical processes as well as on related research literature (Koichu et al., 2007, being the most noteworthy influence; see Rott, 2012, for details). The coding procedure is a sort of qualitative content analysis (cf. Mayring, 2000) which helps ensuring its reliability and objectivity and makes it suitable for qualitative as well as quantitative analyses.

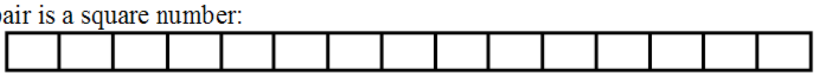
All videos were coded independently by several researchers, who first identified points of time in the processes where heuristics were used and then characterized the heuristics afterwards. In accordance with the TIMSS 1999 video study (cf. Jacobs et al., 2003, p. 103 f.), the "percentage of agreement" approach

Beverage Coasters
The two pictured squares depict coasters. They are placed so, that the corner of one coaster lies in the center of the other.



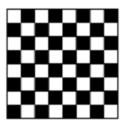
Examine the size of the area covered by **both** coasters.
[Idea: Schoenfeld (1985, p. 77): *Mathematical Problem Solving*. Orlando, Florida, Acad. Press.]

Marco's Number Series
Marco wants to arrange the numbers from 1 to 15 into the caskets so that the sum of every adjoining pair is a square number:



For instance, if there are the numbers 10, 6, 3 in three consecutive caskets, the 6 adds up to a square number with its left (10+6=16) and its right neighbor (6+3=9).
How could Marco fill-up his 15 caskets?
[Source: Fürther Mathematikolympiade, 2005/06, 1. round (www.fuemo.de)]

Squares on a Chessboard
Peter loves playing chess. He likes playing chess so much that he keeps thinking about it even when he isn't playing. Recently he asked himself how many squares there are on a chessboard. Try to answer Peter's question!



[Idea: Mason, Burton, & Stacey (2010, p. 17): *Thinking Mathematically*. Dorchester, Pearson, 2. ed.]

Figure 1: The three problems selected for the study

Code	Description	Examples
Drawing a figure	Drawing a figure, a graph, or a diagram.	**Coasters**: a drawing of possible positions of the two squares. **Number Series**: drawing a diagram of numbers with possible neighbors.
Auxiliary elements	Introducing auxiliary elements like auxiliary lines or additional variables.	**Coasters**: drawing auxiliary lines to indicate area. **Chessboard**: drawing borders of squares to illustrate their size or count them.
Special cases	Assigning special values (like 0 or 1) to algebraic problems or examining special positions in geometric problems.	**Coasters**: positions of the two squares which make it evident that the marked area amounts to one fourth of a square.
Mental flexibility	"Thinking outside the box"; special ideas and activities that are not captured by other heuristic categories	**Coasters**: (mental) rotation of the squares. **Number Series**: flexible way of adding numbers to both sides of the series. **Chessboard**: easily identifying the possible overlap of squares bigger than 1x1.

Table 1: An extract of the heuristics coding manual

(Bakeman & Gottman, 1997, p. 59)[5] was used to compute the interrater-reliability of randomly chosen videos (40 % of the processes). More than $P_A = 0.7$ for identifying points of time in the videos with heuristics and more than $P_A = 0.85$ for characterizing the heuristics was achieved. After calculating the reliability, all differing codes were analyzed and recoding consensually (100 % of the processes).

Pupils' products were coded individually with the result that in 6 of the 55 processes discussed here the two members of a pair obtained results with differing product ratings. To manage the data, for each pair the better result was chosen to further work with. The heuristics in the pupils' processes have also been coded individually. The numbers given here represent the number of different heuristics noticed for each pair; for example, when one member of a pair drew a figure while the other one didn't do so, this heuristic was counted for the pair.[6]

RESULTS

Quantitative results

The evaluation presented here starts quantitatively by comparing some statistical results regarding the product and process codings of both groups. Looking at the novice group (i.e., the regular, non-gifted pupils), the results are distributed widely among the four product categories in all three problems indicating no ground or ceiling effects. As expected, the pupil experts are significantly more successful ($c^2 = 14.54$; $p < 0.001$), scoring exclusively in categories 3 or 4.

For the novices, the number of coded heuristics is related to the success in solving the problems with mean scores of 1–3 heuristics in less and 3–5 heuristics in more successful processes (see Table 2 for details). There are significant Spearman rank-order correlations[7] for the Coasters ($r_s = 0.69$; $p < 0.01$), the Number Series ($r_s = 0.78$; $p < 0.001$), and the Chessboard ($r_s = 0.99$; $p < 0.05$) problems as well as for all three problems ($r_s = 0.72$; $p < 0.001$) combined. This finding is in accordance with research on the topic and the values mostly match the correlations reported by Komorek and colleagues (2007); this result meets the expectations as the use of heuristics should be helpful in solving problems.

However, there are successful processes with only one or two heuristics as there are unsuccessful ones with three or more. Some pairs picked a heuristic and used it to solve the problem outright; on the other hand, heuristics did not help every time. As expected, there is no straight "the-more-the-better" rule for the use of heuristics.

For the pupil experts, there is no such correlation between the number of heuristic techniques and suc-

5 Chance-corrected measures like Cohen's kappa are not suitable for this calculation, as there is no model to calculate the agreement by chance for a random number of heuristics distributed randomly over the course of the process.

6 Please note that in previous publications from the MALU data pool (e.g., Rott, 2012), the individually coded results have been reported. This time, the pair data is reported, therefore the numbers do not match those from previous articles.

7 As the product categories yield only ordinally scaled data, no Pearson correlation coefficient was calculated. The web-tool by R. Lowry (http://vassarstats.net/corr_rank.html) also provides a way to calculate the significance level for n < 10.

		Beverage Coasters		Marco's Number Series		Squares on a Chessboard	
		heu	count (%)	heu	count (%)	heu	count (%)
NOVICE	$\bar{x}_{cat1}$	2.00	6 (37.5)	0.33	3 (18.8)	0.00	5 (55.6)
	$\bar{x}_{cat2}$	3.80	5 (31.3)	1.00	3 (18.8)	2.00	2 (22.2)
	$\bar{x}_{cat3}$	5.33	3 (18.8)	3.33	6 (37.5)	3.50	2 (22.2)
	$\bar{x}_{cat4}$	5.00	2 (12.5)	4.00	4 (25.0)	---	0 (0.0)
EXPERT		heu	count (%)	heu	count (%)	heu	count (%)
	cat. 1/2	---	0 (0.0)	---	0 (0.0)	---	0 (0.0)
	$\bar{x}_{cat3}$	3.00	4 (80.0)	---	0 (0.0)	2.00	1 (25.0)
	$\bar{x}_{cat4}$	3.00	1 (20.0)	2.40	5 (100.0)	2.67	3 (75.0)

Table 2: Results of the heuristic coding for the pupil novices and experts

cess (r_s = -0.06; p = 0.84), but this can be explained by ceiling effects. Surprisingly, opposed to the vague "the-more-the-better" rule, the experts use significantly *less* heuristics compared to equally successful novices (i.e., pairs that reached product categories 3 or 4) (t = 2.73; p = 0.01).

Table 2 summarizes some statistical data of both groups. Column "count (%)" shows the number of pairs for each of the four product categories. Column "heu" shows the mean numbers of heuristics coded in the processes for each product category. The experts use less heuristics for each problem in the product categories 3 and 4.

Qualitative results

To further explore this surprising result – better problem solvers use less heuristics – the processes are analyzed qualitatively in the following paragraphs. Coded heuristic actions are indicated with *italics*.

The first (abbreviated and smoothed) example deals with the Coasters problem. After reading the problem formulation, the novices Hannelore and Lucy start to *measure* the length of the sides of the squares to somehow calculate the requested area; Hannelore also *introduces notations* to points in the given figure. They soon notice that their first approach does not work and start to question whether the squares could be arranged in another way. Lucy then *draws figures*, among them a *special case* in which the size of the area is easily identified (see Figure 2). They return to the given figure and start adding *auxiliary lines* to indicate a *decomposition* (see Figure 2). Lucy notices that she could "cut off" a triangle and add it at the other side to regain the special case. She concludes that the area is always as big as in the special case. They then do not write down "a fourth of a square" but calculate the size of the requested area. Overall, they worked nearly 12 minutes on this problem.

Bernd and Tobi, two of the pupil experts, worked on this problem for 2 minutes. After reading, Bernd says: "Wait, this is exactly one fourth. Because you can push it over there, so that you get exactly four parts." (*mental flexibility*) Tobi quickly agrees and they request the next problem without further justifying their solution. Bernd's recognition can be interpreted as Consideration of Aspects within the framework of intellectual flexibility, because he "recognize[s] the correlation of facts and [is] easily able to vary them." (Bruder, 2003, p. 17)

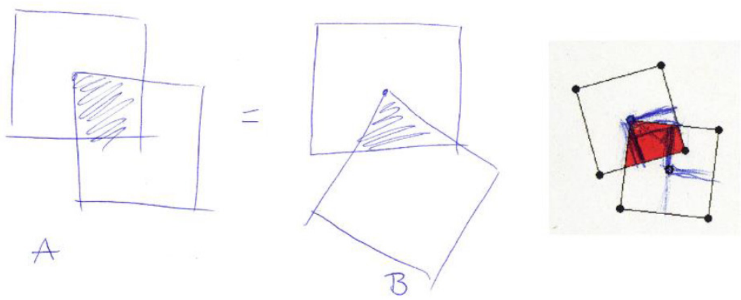


Figure 2: Lucy's figures working on the Coasters problem

The second comparison of novice and expert processes deals with Marco's Number Series. It takes the novices Birk and Janus more than 30 minutes to solve this problem. They start with "1, 3, 6, 10, 15" and are stuck, because there is no number left to add to 15 (they try

21 but reject this idea quickly). They start new rows with different numbers, always adding numbers to the right end until they get stuck and start anew. To keep track of the numbers they had already used, they keep a list of all numbers from 1 to 15 to cross out (*tool of systematization*). In this period, they occasionally use *backtracking*, i.e. deleting the last number(s) to continue with a different number instead of starting a new row. After more than 20 minutes, Birk introduces a new idea; he *creates a table* with all possible combinations of two numbers smaller than 16 adding up to a square number. This way, he realizes that the numbers 8 and 9 only have one possible neighbor each and have to start and end the row (*looking for patterns*). Shortly after this realization, they solve the problem.

It took the experts Robert and Lasse about 5 minutes to solve this problem. They immediately start with a list of numbers to cross out (*tool of systematization*) and begin their first row with the given example, "3, 6, 10". Instead of only adding numbers to the right, they work on both sides of the row (*mental flexibility*). To the right, they add "15, 1, 8". Finding no neighbor for the "8", they do not restart, but complete the row on the left side: "13", "12", "4", "5", "11", "14", "2", "7" and "9". Within the framework of intellectual flexibility, this idea can be interpreted as Change of Aspects, because "[b]y intuition they consider different aspects of the problem which avoids or overcomes getting stuck." (Bruder, 2003, p. 17)

Of course, these two examples are more obvious than the majority of the processes; they have been selected to illustrate an argument. However, the appearance of *mental flexibility* seems to be a distinguishing factor between novice and expert processes as this code appears significantly more often in the experts' processes: overall, mental flexibility appears in 15 of 43 novice processes compared to 10 of 14 expert processes ($c^2 = 5.73$; $p = 0.017$); this trend continues at the level of individual problems (Coasters: 8/16 compared to 3/5; Number Series: 6/16 compared to 4/5; Chessboard: 1/9 compared to 3/4).

The experts need less heuristics because of their mental flexibility. In other words, the novices that are not that flexible need more heuristics (and more time) to get similar ideas. This finding supports the claim of Bruder and Collet (2011) that a lack of intellectual flexibility can be compensated by the use of heuristics. For example, not being able to imaginary rotate the squares in the Coasters problem, most of the novices draw figures of squares in different positions.

DISCUSSION

The quantitative analysis of the regular pupils' problem solving processes showed a significant moderate to high correlation between the number of heuristics in those processes and the pupils' success. This result is in accordance with the literature as a vague "the-more-the-better" rule according to the number of heuristics has been reported many times. Surprisingly, this result does not apply to the group of gifted pupils (successful participants of mathematical competitions, i.e., "experts") whose processes have also been analyzed. These pupils have been more successful than the regular ones (as expected) but used significantly less heuristics than equally successful pupils of the first group. A qualitative comparison of the processes of both groups revealed an explanation for this finding: the experts' processes contained a significantly higher number of actions coded as "mental flexibility" than those of the novices (regular pupils). This finding can be explained by the theory of Bruder and Collet (2011). The lack of intellectual flexibility of the regular students (compared to the experts) can at least partly be compensated by the use of heuristics.

This has implications for practicing and teaching problem solving. Of course, intuitive problem solvers have an advantage working on problems as "the active intuition might be the most important device for discoveries, e.g. the sudden realization of analogies can only be thought of as an intuitive event." (Winter, 1989, p. 177, translated by the author) But not-so-intuitive problem solvers might overcome their disadvantage by learning and practicing heuristic techniques.

Of course, there are limitations to this study: Firstly, the group sizes are quite small; especially the group of the pupil experts consists of only ten pupils. A bigger number of both regular and gifted pupils would be desirable to see if the observed patterns can be confirmed. Secondly, the pupils have worked in pairs to enable access to their thoughts through their natural communication. This might have influenced their problem solving behavior and it might be problematic to generalize the findings of this study to individual problem solving behavior of regular and gifted pupils.

Further studies need to evaluate whether the actions of “mental flexibility” are a genuine part of gifted pupils’ problem solving behavior or whether being “mentally flexible is a learned (and thus learnable) ability these pupils picked up when participating in and training for mathematical competitions. It might be that the actions coded as “mental flexibility” are a combination of heuristics which are performed mostly cognitive and so elaborate that they are unidentifiable for the raters observing the problem solving processes of the pupil experts.

It would be interesting to see whether a follow up study with older students might provide a clearer distinction between regular and gifted students and similar results.

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Designing tasks for mathematically talented students

Maike Schindler[1] and Julia Joklitschke[2]

1 Örebro universitet, Institutionen för naturvetenskap och teknik, Örebro, Sweden, maike.schindler@oru.se
2 Ruhr-Universität Bochum, Bochum, Germany

For the Design Research project presented, a learning environment for mathematically talented and interested 7th-grade students was investigated. The results show that the subject matter of graph theory offers both opportunities and means for students to develop their abilities. The data analysis showed likewise how the tasks might be modified in order to impose on their potential and thereby foster students' abilities of a formalized perception and pervasion of mathematical information and of generalization.

Keywords: Talented students, Design Research, RME, task design, graph theory.

INTRODUCTION

Mathematical potential and talent is one of the topics in mathematics education research increasingly attracting attention (Leikin, Karp, Novotna, & Singe, 2013) – with good reason. Not only since PISA and TIMSS – which have impressively revealed the heterogeneity of students within countries and even within single school types – the huge variety of students' abilities is well-known. However, there is only a small number of investigations that show how high-achievers can be supported in order to develop their potential (cf., e.g., Kießwetter & Rehlich, 2005). In the context of mathematics education research, there are certain findings regarding the abilities and characteristics of mathematically talented students (e.g., Krutetskii, 1976). These constitute a stepping stone for the investigation at hand: On the basis of the existing results, a Design Research project (cf. Gravemeijer & Bakker, 2006, p. 1) was conducted in order to develop and refine a learning arrangement for a group of mathematically talented and interested students and to assess, for a concrete subject matter – namely graph coloring –, how mathematically talented students can be challenged and, at the same time, supported in their specific abilities. Therefore it was one main goal to get insights into the abilities and possible difficulties of the students, which are being focused in this paper.

THEORETICAL BACKGROUND

Mathematically talented students and their abilities

One of the most important investigations in the field of abilities of mathematically talented students was conducted by Krutetskii (1976). He found four components being constitutive for mathematical abilities during school age: *Obtaining mathematical information*, i.e. "the ability for formalized perception of mathematical material, for grasping the formal structure of a problem" (Krutetskii, 1976, p. 350); *Processing mathematical information*, which comprises, among others, "the ability for rapid and broad generalization of mathematical objects, relations, and operations" (Krutetskii, 1976, p. 350); *Retaining mathematical information*, meaning memorizing mathematical approaches etc.; and a *General synthetic component*, connecting and interrelating all other components and forming a mathematical cast of mind. Mathematically talented students can be characterized by possessing these abilities to a great extent. Therefore Krutetskii's categories should be considered when learning environments and tasks for these students are being designed.

In the investigation presented, the focus was – on the one hand – on the students' ability to *mathematize* situations and – on the other hand – on their ability to *generalize*, which means being able to (a) recognize similar situations and (b) handle the generalized solution in these situations (Krutetskii, 1976, p. 237). It was investigated in how far the students were able to mathematize and generalize when handling the tasks of the developed learning environment.

Task design and realistic mathematics education

For refining and (re-)designing our learning environment, we focused especially on students' abilities to *mathematize* and *generalize* (see above). In our project, research was not supposed to be separated from a practical perspective, on the contrary: research and design were closely interwoven (Gravemeijer & Bakker, 2006). The Design Research alignment meant a focus on *task design*, which is of significant importance in mathematics education (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013). However, the terms *task* and *task design* vary widely (Watson et al., 2013). The aim of our investigation was to develop and refine *tasks* in their meaning of being "what students are asked to do. Then 'activity' means the subsequent mathematical (and other) motives that emerge from interaction between student, teacher, resources, environment, and so on around the task" (Watson et al., 2013, p. 11). In our case, *task design* means the design of tasks for the above-mentioned purpose as well as possible oral impulses of the teacher and corresponding material.

The design of the learning environment within our investigation is being guided by the domain-specific instruction theory of *Realistic Mathematics Education (RME)*. RME "itself is the result of a long history of design research in the Netherlands" (Gravemeijer & Bakker, 2006, p. 2).

"According to RME, mathematics should be seen as an activity (Freudenthal, 1973), and students, rather than being receivers of ready-made mathematics, should be active participants in the educational process, in which they develop mathematical tools and insights by themselves" (Drijvers et al., 2013, p. 56).

Mathematical learning should – according to RME – originate from problem situations in realistic contexts, which do not necessarily need to be from the real world (Van den Heuvel-Panhuizen, 2005, p. 2). Based on their activity within these situations, students use their individual concepts in order to handle situations and therefore extend them. In our project, students were supposed to handle the presented tasks and hence the mathematical information (cf. Krutetskii, 1976, see above), and thereby apply their mathematical concepts, develop them and generalize them for being able to apply them in different situations (Krutetskii, 1976).

Graph theory as subject matter for mathematically talented students

The application of graph theory as a subject matter for talented students offers different advantages. Most of these are connected to the large applicability and therefore its huge potential for fostering students' creativity (cf. Leuders, 2007).

In graph theory, the notion of *graph* is fundamental: "A graph G consists of a finite set V, whose members are called vertices, and a set E of 2-subsets of V, whose members are called edges" (Biggs, 1985, p. 158; partially highlighted in the original). As graphs represent networking structures, they can be visualized by points and connecting lines between two points. Especially *graph coloring* is a central scope of application in the focus of our study. "A vertex-colouring of a graph $G = (V, E)$ is a function $c: V \to \mathbb{N}$ with the property that $c(x) \neq c(y)$ whenever $\{x, y\} \in E$. The chromatic number $\chi(G)$ is defined to be the least integer k for which there is a vertex-colouring of G using k colours" (Biggs, 1985, p. 172). Due to graph coloring, we can distinguish adjacent vertices and, thereby, create disjoint partitions of vertex-sets.

Research questions

In this Design Research project, the following questions were at the core of the research interests (based on Krutetskii, 1976). These are:

1) To what extent are the talented students able to *mathematize* the realistic problem situations in the given tasks?

2) In how far are they able to recognize the similarity of different situations and *generalize* their solutions?

On the basis of these questions, it is considered how the task design can be refined in order to optimize the learning processes.

DESIGN OF THE INVESTIGATION

Designing the tasks

We developed tasks, according to two kinds of problem situations (cf. Joklitschke, 2014). The first kind of problem situations is rather abstract. Here, a concrete problem situation may be the assignment of persons to different groups. Persons can be represented by vertices – and the do-not-like-relation by edges. If the vertices are being colored, the colors signify the emerging groups. The number of colors then represents the number of groups. A second field of application addresses problem situations with rather geometrical visualizations, for instance the coloring of maps. Here, the dual graph is needed, which includes the information about adjacent areas: every area is represented by a vertex, and adjacent areas are visualized by an edge (cf. Leuders, 2007, p. 141). By finding the chromatic number, we get to know how many colors are needed.

Our tasks comprised certain subtasks and were supposed to serve for a 90 minutes lesson. Two of them are being focused in this paper. As we assumed problem situations with a geometrical representation to be easier to grasp for the students (see Bronner, 2014), the first task was a map task. Students were supposed to find the minimum number of colors for coloring the 16 federal states of Germany (see Leuders, 2007, pp. 132ff, Figure 1). The second task, then, comprised a problem situation without geometrical visualization (cf. Leuders, 2007, pp. 133f). We chose the context of the soccer world championship with national supporters who are – due to rivalries – supposed to be accommodated in different places (Figure 1).

Implementation and data collection at school

The developed tasks were investigated in a group of eleven 7th-graders – aged twelve or 13 – at a German secondary school. This group had already been active for two years and was supposed to give mathematical talented and interested students the opportunity to enhance their abilities. The selection of the students for this course depended on their performance in mathematics (according to the teachers' assessment) and their motivation to participate. Since this course was already well-established, all students had experience in the field of graph theory: They had worked on realistic problem situations (shortest path problems, spanning trees and Euler graphs, developing algorithms) beforehand. The students were used to work in small groups on open problems and to generate the mathematical contents by themselves.

In this investigation, the students worked in groups of four (or three) as they were used to. The group work took 90 minutes. The group work of all groups was videotaped. The analysis focused on one "focus group" because here, the students communicated a lot so that the analysis could be undertaken on a profound empirical base. The videos were transcribed.

The four students of this focus group were additionally interviewed in semi-structured partner post interviews. Here, every group got the same tasks and questions. The students were interviewed in pairs of two as this was expected to foster their communication and give them safety. The interviews took place two weeks after the group work. The interview guide comprised questions on the approaches that the students had worked on before as well as two new, but analogous problem situations (one problem regarding map coloring and one problem regarding partitioning). The

Task 1: Color selection

GeoPaint Inc. wants to include a colored map of Germany with its federal states in their assortment.

a) What should the company consider? Note the necessary criteria and try to color the map.

b) As the company has to order each color separately, the map gets cheaper when they use as few colors as possible. How many colors are needed?

c) In order to develop a computer program that helps designing the colored map, GeoPaint has to create an illustration that represents the neighborhood of the federal states. How could such an illustration look like?

Task 2: A Peaceful World Championship

At the soccer world championship 2014, many European teams compete. But the national supporters partly do not stand well with each other. Therefore, fans with a special rivalry shall be accommodated in different locations. Now, the organizing committee has to find a suitable allocation of the national supporters. These are the conditions:

- German supporters do not stand well with fans from the Netherlands and England.
- ...

How many locations do the organizers need to ensure a peaceful meeting? How can you represent the problem adequately?

Figure 1: "Map coloring task" and "Soccer world championship task"

semi-structured interviews – which allowed not only to inquire their approaches to the same tasks, but also to go into detail to see certain differences between the pairs, especially in their inferential reasoning – took 30 minutes each. The students were supposed to comment on their approaches by thinking aloud. The interviews were videotaped and transcribed afterwards. For the data analysis, both, the transcripts of the group work and of the interviews, were taken into account.

ANALYTICAL FRAMEWORK

For investigating students' individual approaches, we used a framework that arises from a philosophical perspective, founding on ideas of Kant, Frege, Wittgenstein, Heidegger, and Brandom's (1994) theory of *semantic Inferentialism*. Based on these philosophical notions, a theoretical and analytical framework was developed for mathematics education (e.g., Schindler, 2014), whose applicability has been shown for different subject matters (e.g., Schindler & Hußmann, 2013). The philosophical background signifies that individual approaches and students' concepts can only be understood in their use, i.e. in reasoning processes. Therefore, the data analysis is being conducted on the basis of three crucial analytical elements, which are: individual commitments, inferences and focuses (see below).

The concept of *language game* plays an important part in the theoretical background. For our analysis it is important to analyze the use of concepts in these language games, since concepts and their meaning can – in this holistic perspective – only be understood by means of the role that they play here: "grasping a concept involves mastering the properties of inferential moves that connect it to many other concepts" (Brandom, 1994, p. 89). The basic elements in this framework are propositions that individuals hold to be true and explicate, like for example "Four colors are enough to color the map". These individual *commitments* constitute the building blocks in our data analysis, which is carried out turn-by-turn on the basis of the transcripts at hand.

Furthermore, the reasoning process itself is being analyzed, as it is crucial for reconstructing students' understanding. *Inferences* embody the reasoning process, as they constitute the relation between commitments if a student entitles a commitment with another commitment, like e.g. "Five colors are one too many, thus four colors are enough to color a planar graph". For our data analysis, it is important that these inferences, as well as the commitments, do not have to be formally correct but to be held as true by the individual student. Besides commitments and inferences, *focuses* – as individual categories that are used to pick up, select and handle the information at hand – are being reconstructed in our data analysis. Students can – consciously or not – utilize e.g. concepts, properties, or other entities as categories, such as the number of colors and partitions. Via the analysis of these three elements (students' focuses, the commitments and inferences), we analyzed how students mathematize the realistic situations and in how far they are able to generalize (see results).

RESULTS

An insight into students' mathematization process

In general, the analysis revealed that students showed an enormous performance handling the maps task and mathematize the information at hand. First, they focused on painting the map with colored pencils. While doing so, they did not yet gain access to the formal structure of the problem and some stuck to the real-world conditions of the problem situation. For example, they committed to "The company can blend colors, then they do not have to buy so many of them". During their work, their focus became more and more structure-oriented. It was only when they started with the task to support a computer program (see task 1c, Figure 1), that they focused on graphs for the first time.

Sasha:	Well, we have to do this with a graph, right?
Tim:	Yes. I already worked on two or three graphs, but they do not yet work out.
Klara:	(reading the task out loud) The neighbors of federal states...
Sasha:	Yes, let's see.
Klara:	So, simply a graph?
Sasha:	Yes.

In this excerpt, Sasha at once commits that they can focus on a graph. Tim agrees and therefore acknowledges Sasha's commitment. Klara seems to be in doubt if this focus is adequate, as she asks "Simply a graph?".

But after Sasha's conclusive confirmation, everyone of them starts drawing a graph (focus: graph).

In this dialogue, it is interesting that all students immediately focused on an *edge* representing a neighborhood, which is adequate from a mathematical perspective, as it implies constructing the dual graph (Figure 2). As Leuders (2007, p. 140) shows, it is also possible to choose the borders to represent the edges and the vertices to represent the "corners" of three countries bordering. But the students seemed to immediately realize that "it is not of interest to know the exact shape of country's frontiers [...], but only to know who is adjacent to whom" (Leuders, 2007, p. 138, translated by M.S./J.J.). During their prolonged activity, in which they constructed the dual graph, they focused on the numbering of the countries, i.e. vertices, in order to keep track of the neighborhoods and to communicate about them more easily. The focuses on (dual) graphs, on numbering, on edges representing the neighborhood reveals that the mathematization process was successful in the first task.

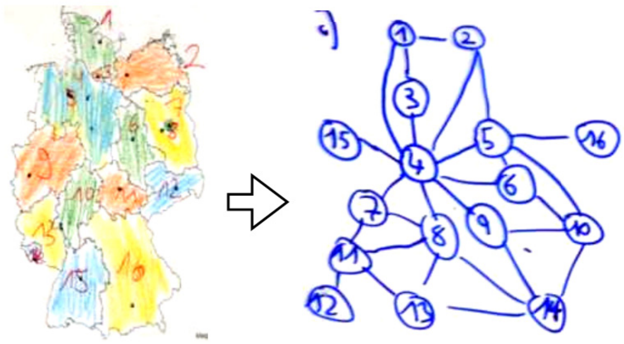

Figure 2: Students' drawings

Similarity recognition and generalization

The second task on the soccer world championship and national supporters represented more of a challenge for the students. Here, they had much more trouble to capture the formal structure of the problem situation and to recognize the similarity to the first task. When dealing with the task, students proceeded intuitively and tried splitting the supporters into distinct groups.

Alex: Well, we can definitely put Russia, Greece and Switzerland together.
Tim: I suggested the same.
Alex: Because they do not oppose each other.
Tim: With whom does Germany *not* get on well? Germany does not get on well with the Netherlands and England, but they get on well with Spain and France, don't they? Yes, Germany does not get along with the Netherlands. Therefore, I would put Germany together with Spain and France.

In this heuristic approach, the students' common focus is on dividing the sets of supporters into groups. In mathematical terms, it is the concept of partition, and the idea is to find an optimal (i.e. smallest possible) partition. The students justify their partitioning via harmonies or compatibilities.

In this excerpt of the group work, the students focused on splitting the sets of supporters (i.e. building partitions) and on the number of locations needed. They did not mention that the formal structure of this problem is similar to the first one; that the sets of supporters can be depicted – just like the countries – as vertices and the rivalries as edges – just like the borders. They did not seem to realize that both problem situations are about objects and their (binary) relations to each other and that the mathematization is obvious (cf. Leuders, 2007, p. 138). On the contrary, the focuses of the students in our investigation remained in the real-world situation in the second task. When the teacher tried to help students focusing on graphs, they disliked this focus and committed for instance to "But that would only be the same that we did with the other approach" or "That only brings the same result that we can find otherwise". They did not see a reason for focusing on graphs.

Then, in an additional, further task, students were confronted with a problem situation that dealt with four groups of national supporters, which each have a rivalry against each other. Here, students were again *explicitly* asked if it is possible to depict this with a graph and if there are edges, which necessarily have to cross over each other.

Tim: Well, I have drawn one without crossings. Where one can see that everyone hates everyone. (...)
Sasha: Huh? How do you want to draw that with a *graph*?
Tim: Four points and connect them one to each other. (...)

Sasha: But how do you want to draw *rivalry* with a graph?
Tim: Well, because if they are connected then they do not like each other. And here, everyone is connected to everyone.
Sasha: But this is actually the other way round!
Alex: Yes, I would also say that...
Tim: Yes, okay. So, simply four points. That also works. (...)
Sasha: (while writing, talking to himself) Mhm, but that's right.

Here, we see that the students acknowledge the focus on graphs. Tim directly seems to generalize – probably unconsciously – the idea of *dissociation represented as edges* from the first task (boundaries as edges) to the second task (rivalry as edges). Sasha and Alexander, however, are obviously not convinced: Sasha asks for reasons and explanations, but Tim's reasoning is not convincing for them. Instead, the group focuses on *compatibility as edges* – and not on the similarities between the two kinds of dissociation and of the two problem situations. When writing his findings down, Sasha then acknowledges Tim's focus: Afterwards, he continues drawing dissociation as edges – but this does not become subject of discussion anymore. Sasha's final commitment as well as his drawing might indicate a generalization process.

In the post interview, however, neither Tim nor Sasha saw the similarities between the geometrical and the abstract problem situation. The frequent changes of focuses as well as the low level of inferential reasoning during the lecture series go hand in hand with the lacking generalization. However, an impulse of the interviewer encouraged the similarity recognition in one case (Sasha). This was initiated by the interviewer's explicit focus on similarities between the maps problem and the problems with the football supporters.

Interviewer: Are there any similarities between the two problem situations?
Sasha: (Looking straight ahead, then looking out of the window, beginning to smile and looking back to the interviewer) Yes. Because here (pointing onto the map), adjacent countries were supposed to have different colors. And here, it is effectively the same. Because, then the colors are the locations, and the locations just differ from each other.

The interviewer's question seems to foster the student's similarity recognition: After thinking about the question, he affirms the similarity. The inferences that he makes underpin that he understands the reason: He is able to see the functional similarity of countries and locations and commits to "The colors are then the locations". This indicates that students *are* able to focus on graphs and on coloring in problem situations without geometrical visualization, but this needs specific support.

Consequences for the task design

Task 1: One of the prominent results revealing from the above-mentioned data analysis is that the mathematization process was fostered by the sub-task to think of an illustration for a computer program. Here, students were able to focus on graphs easily. This was very useful for systematic approaches in which the students focused more structurally and systematically and thus fostered the mathematization process. This indicated that this task does not need modifications in this regard.

Task 2: It was not easy for the talented students in our investigation to grasp the formal structure of a graph in problem situations, which do not have a geometrical visualization. On the one hand, this indicates that these tasks have a huge potential for talented students to exert themselves, because it was shown that these students can gather the mathematical structure when having the right focuses. On the other hand, the analysis reveals that the tasks and even the impulses of the teacher did not prompt the students to perceive the structure and to generalize their focuses and commitments from the geometrical problem situations. In order to foster these abilities in this context even more, the task has to be modified: There have to be more reflective aspects that make the students think of similarities. Possible impulses would be e.g. "Look for similarities of both problem situations. Is it possible to solve the second task with the same strategy that you used in task 1?". For generalization purposes, it seems to be important for the students to recognize the *focus on dissociation* in both kinds of problem situations. But without a profound inferential reasoning, even the right focus is not sufficient for students' similarity recognition and especially their generalization.

CONCLUSION

The overall aim of the empirical investigation at hand was to explore the students' approaches and abilities (esp. mathematization and generalization) for refining the design of a learning environment.

The results indicate that the students were able to perceive mathematical information and the structure of problem situations formally (cf. Krutetskii, 1976). But this ability strongly depended on the tasks presented. It was manifested as highly-developed in problem situations which had a geometrical visualization, whereas in problem situations without a direct geometrical visualization, students were less able to see the mathematical structure. In the consequence, it was not easy for the students to generalize their focuses from a problem situation with a geometrical visualization to one without the latter. Our findings indicate that Krutetskii's framework constitutes a profound basis for developing tasks for talented students: On the one hand, the students showed the abilities to a certain extent, on the other hand, it was shown how these abilities can be fostered even more.

The results of the investigation revealed many aspects which can be used for optimizing the tasks and support students in their specific abilities: for instance, giving adequate impulses that lead students to focus on graphs; and fostering a purposeful and conscious reflection about similarities of problem situations much more explicitly.

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Problem posing: Students between driven creativity and mathematical failure

Florence Mihaela Singer[1], Ildikó Pelczer[2] and Cristian Voica[3]

1 University of Ploiești, Ploiești, Romania, mikisinger@gmail.com

2 Concordia University, Montreal, Canada

3 University of Bucharest, Bucharest, Romania

We explore behaviors of high achievers 10 to 16 years old during a problem modification process by looking at the ways students vary the constraints of a given problem. We found that these children swing between low amplitude driven creativity and failure to pose mathematically consistent problems when it comes about understanding the deep structure of mathematical concepts and strategies.

Keywords: Problem posing, problem modification, constraints, mathematical creativity, endogenous and exogenous transformations.

INTRODUCTION

We exposed students from grades 4 to 10 to a problem posing context. We were interested in this case to see how mathematical creativity manifests in students and what is (if any) the relationship between their creative approaches and the quality of the mathematics problems they generate.

In general, in problem-posing contexts, students are stimulated to make observations, experiment through varying some parameters, and devise their own new problems (Singer, Ellerton, Cai, & Leung, 2013). In this paper, we accept that problem posing refers to the generation of (completely) new problems, and to the re-formulation/ modification of given problems (Silver, 1994). We specifically address here the context of problem modification.

Previous studies have shown that, in problem posing situations, mathematically able students tend to vary a single parameter in order to ensure the control over the relationships among all the elements of a posed problem, even when they make interesting generalizations. This tendency of small-step variations was interpreted as a student's need to keep as coherent and consistent as possible his/her proposals by controlling (sometimes unintentionally) the consequences of the proposed changes (Singer, 2012a; Voica & Singer, 2013). In a problem-modification study, Singer, Pelczer and Voica have shown that the capacity of a student to pose coherent and consistent changes of a problem and to understand the (mathematical) consequences of these changes proves student's capability for deep transfer creative approaches (Singer, Pelczer, & Voica, 2011). Therefore, this capability could predict mathematical creativity, which has been characterized as being different from general creativity (e.g., Piirto, 1999).

In describing mathematical creativity, we use a framework based on cognitive flexibility, which is characterized by: cognitive novelty, cognitive variety, and changes in cognitive framing (e.g., Voica & Singer, 2013). In a problem-modification context, we consider that a student proves cognitive flexibility when she or he poses different new problems starting from a given input (i.e. cognitive variety), generates new proposals that are far from the starting item (i.e. cognitive novelty), and is able to change his/her mental frame in solving problems or identifying/discovering new ones (i.e., change in cognitive framing).

In addition, these studies have shown that the more the student advances in the abstract dimension of a given problem, the more mathematically relevant are his/her newly obtained versions. It seems that the students who are cautious and minimally change the problem are in fact mathematically advanced students who show proper insights on some mathematical concepts that exceed their age level. Actually, they pay careful attention to controlling the values of the variables, as mathematicians usually do. These studies conclude that a feature of mathematical creativity, at least for the students involved in the research (10–16 years old), is a type of cognitive flexibility characterized not so much by novelty, but by incremental representational changes in cognitive frames (Singer, 2012b; Voica & Singer, 2013).

In the present study, we start from the assumption, suggested by previous experimental research (e.g., Voica & Singer, 2014), that problem posing (and its particular case – problem modification) is a useful context for identifying mathematical creativity, and we try to go further in determining correlations between students' mathematical creativity and the quality of their posed problems. Thus, we are trying to answer the question: What are the gains and the losses from the view of creativity, when students do mathematical problem posing?

METHODOLOGY

The sample of this study consists of students from grades 4 to 10 (10–16 years old), participants in a summer camp organized for the winners of a large-scale two-round competition (Kangaroo). The camp participants, representing the top 0.2% from a total of about 150 000 participants at the competition, are considered high achievers in mathematics. During this summer camp we launched a call for problems. The participants had to choose a problem from a list of three problems (given as reference points) and, after solving it, to pose three problems based on the starting reference problem: one simpler, one similar, and one more difficult. In addition, the students were supposed to write down the solutions of their posed problems and to explain how their proposals fit the requested criterion (simpler, similar, etc.). We used therefore a structured problem-posing situation (in the terminology of Stoyanova and Ellerton, 1996). Students had 3 days to formulate their answers. In all, 57 students submitted problems.

Each of the posed problems was evaluated by two independent evaluators who assessed it based on a set of criteria: correctness and completeness of the problem formulation and the proposed solution, originality of the text, and clarity of explanations. Based on this primary evaluation, we invited a few participants for interview. Among the selected students were those whose problems were highly ranked by the evaluators. We also invited for interview two students who posed problems considered with potential for further development. In all, 20 students were interviewed. The interviews were based on a general protocol concerning issues such as: What are the similarities/differences between the posed problem and the starting one? What other problems are possible to pose? What make you think to such approach?

The interviews were carried out with two students at a time, thus creating a context for interactions between the children. This helped us to reveal students' ideas through discussions with their peers (Vygotsky, 1978) and to get consistent feedback on their problem posing strategies and their level of mathematical understanding. The interviews were video recorded.

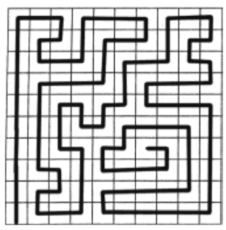
We focus the present paper on the problems generated by the students who have chosen the first problem in the given list (problem 1 below). Consequently, our sample consists of those 19 students (out of which 6 have participated in the interview sessions). The analysis that we do is a qualitative one, mainly based on students' written submissions. Because of the reduced dimension of this paper, we just present a holistic view on students' comments during the interviews to highlight our main conclusions.

CONTEXT

The starting problem

The following problem was initially given to students as a reference point: we will identify it as *the starting problem.*

Problem 1. The 10 × 10 grid is formed out of squares of 1dm. We glue a ribbon as shown in the image. What is the length of the ribbon?



A. 99 dm; B. 99.5 dm; C. 100 dm; D. 100.5 dm; E. 20 dm

We have chosen this problem from the Kangaroo contest to have statistical information on its degree of difficulty. In the competition, this problem was administered to a group of 5659 students from the 5th and the 6th grades. The Kangaroo contest takes place in two rounds and the participants have to answer 30 multiple-choice questions in 75 minutes; therefore,

	A	B	C	D	E	NA
Grade 5 (3278 participants)	10.2	18.5	37.2	8.9	7.7	17.5
Grade 6 (2381 participants)	10.4	25.1	33.9	7.7	7.0	15.8

Table 1: Frequencies of answers to the starting problem in the Kangaroo competition (Correct answer is B. NA stands for no-answer)

optimizing solution strategies is essential in obtaining a good score.

Statistical data about the performance of students participating in the Kangaroo contest show that they perceived the starting problem as difficult: this conclusion follows from the relatively low percentage of correct responses to this problem (see Table 1), the percentage of students that preferred not to answer this question, and the percentage of correct answers obtained by this cohort to other problems (around 75%). Even if we take together answers B and C (concluding that about half of the students had an insight concerning a strategy to optimize the solution to this problem), we can still conclude that the problem was perceived as difficult.

Components of a problem

To better understand the discussion that follows, we detail the components of the starting problem based on the methodology introduced by Singer and Voica (2013). In general, the text of a problem contains: a background theme, (numerical) data, operators (or operating schemes), constraints over the data and the operators, and constraints that involve at least one unknown value of a parameter (Singer & Voica, 2013). The background theme represents, briefly said, what is about in the problem. In our case, the background theme is represented by a grid and a trajectory marked on it. The background theme is characterized by one or more parameters: in our case, the parameter is the shape of the trajectory (of the ribbon). The data are (numerical or literal) values associated to these parameters: in the above problem, these are represented by the number of squares and their dimensions. The operating schemes are actions suggested by the text: here, these are the way of crossing, and the measurement of the trajectory. The constraints imposed on the data and the operators are restrictions that state the relations of the background theme with the data and the operating schemes. In the starting problem, some of the constraints are of *local* nature: for example, "the trajectory" unites the centers of the squares and crosses a square in one of the two modalities shown in Figure 1 (eventually rotated).

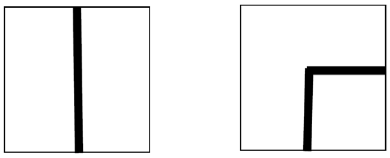


Figure 1: Local constraints for building the trajectory in problem 1

Other constraints are *global*, such as the filling condition – i.e. that the ribbon goes across each cell of the square. These constraints, in combination, allow a quick identification of the answer, since 99 out of 100 squares are crossed and the last one is only half marked (therefore, the correct result is 99.5). We believe that the difficulty of the problem comes from the fact that the solvers did not totally understand the local and the global constraints; usually, they try to perform sequential additions of the ribbon length, which can lead to miscalculations or to abandoning the problem because of the time limit crisis.

The starting problem contains two kinds of constraints: essential in-depth constraints that give the core-structure of the problem, and superficial constraints. The essential constraints manifest at both local and global levels: the global constraint refers to the grid coverage, which combined with the local algorithm (the trajectory crosses the squares in the midpoints of their sides following two different patterns, and there are no self-crossings) allows an optimal solution (as time is concerned). Superficial constraints refer to the dimension of the grid, the shape of the grid (including the base element of the grid), and to the particular path chosen within the grid.

Transformations

We classified the modifications made by the students when they developed a new problem in two categories: *exogenous* and *endogenous* transformations. The terms originate from the system theory, where endogenous transformations are transformations between models expressed in the same language, and exogenous ones are transformations between models expressed using different languages (Mens & Van Gorp, 2006). In problem modification (PM) situations, we consider that a student makes endogenous transformations when he/she uses the same types of operating

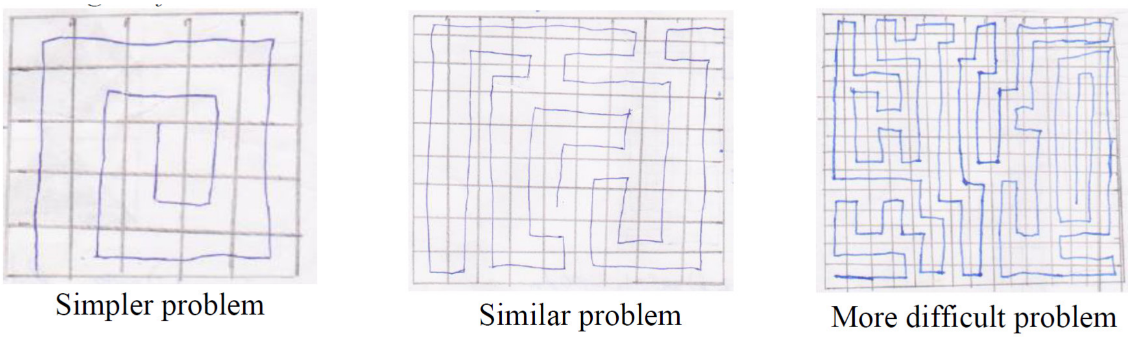


Figure 2: The drawings made by Victor (grade 4) in posing new problems

schemes and constraints as in the starting problem (i.e. makes a new mathematical problem using the same language - the same mathematical concepts as those of the starting problem); otherwise, we consider that the student makes an exogenous transformation. In other words: the endogenous transformation acts only on already present components in the starting problem, while an exogenous transformation will bring new components into the generated problem.

RESULTS

We present our analysis based on some illustrative examples.

Example 1. Victor (grade 4) proposed three problems based on the drawings contained in Figure 2. In all these problems, Victor formulated the same question: *What is the length of the ribbon?*

One can observe that Victor varies the dimensions and the shape of the trajectory. For the last problem (most difficult), Victor avoids two of the squares – therefore, he renounces to the complete coverage of the grid, but he keeps the same type of optimal strategy of solving. The transformations performed by Victor are endogenous: he poses a new problem using elements that exist in the starting problem.

Example 2. Doina (grade 8) based her posed problems on the patterns presented in Figure 3. She put the same requirements for all her problems: finding the length of the drawn broken line. Doina specifies in her problem texts the geometrical properties of the base cell (rhombus, in the first case; regular hexagon in the second; and equilateral triangle in the third). She is very careful in characterizing the local constraints; for example, in the last posed problem, she clarifies the fact that the ribbon crosses the centers of gravity of some of the triangles.

Overall, Doina kept both the local and global defining characteristics unchanged and modified only the layout and the basic element in the tessellation. The modifications took into account the unique association between each cell(s) and measure unit, the nature of the trajectory (no crossings) and the complete coverage of the grid, conditions that allow the optimal solution.

We can observe in this case cognitive flexibility in changing elements without changing the deep structure of the initial problem (solution). This case also illustrates endogenous transformations – since only the elements already present in the initial problem are manipulated.

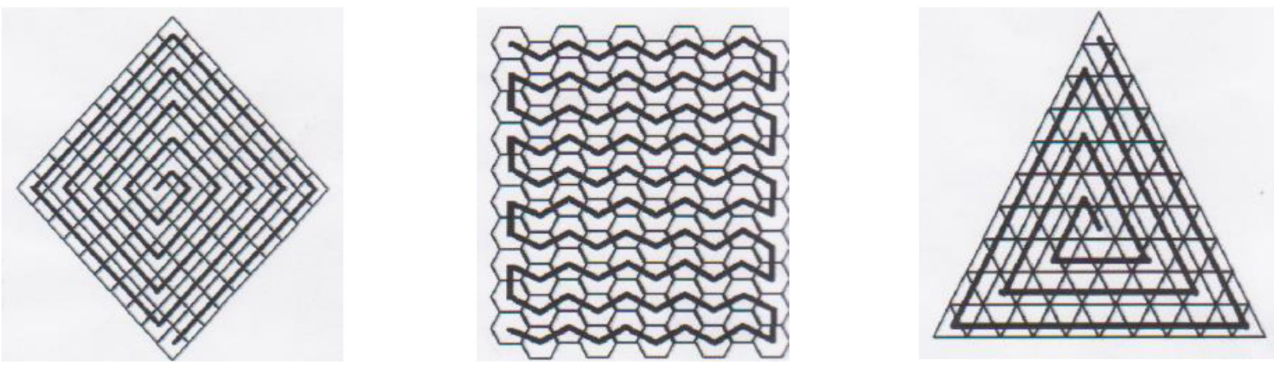


Figure 3: The drawings made by Doina (grade 8) in posing new problems

Example 3. The following problems were proposed by Silviu, a grade 9 student.

Simpler: Dan wants to cut out a shuttle from a textile covered by a square grid (Figure 4a). If the side of a square is 2, what is the remaining surface after the shuttle was cut out?

Similar: An ant crossed a piece of cake of an equilateral triangle shape with the side of 5cm. What is the length of the trajectory marked in the image (Figure 4b)?

More difficult (simplified phrasing): There are four animals competing on square-grid paths (Figure 4c); their speed is given in terms of relations between them. The question is on the order of arrival of the animals.

The "similar" problem has an exogenous modification on the global constraint. Although the student considered this proposal similar to the given problem, this is actually very different when its nature is considered: the only available strategy for solving is the counting of the line segments (two different kinds). Consequently, the student failed to use an optimized algorithm to solve this problem.

Silviu's simpler and more difficult problems suppose also exogenous transformations. By adding new elements, the requirements changed, and the difficulty remains to be judged based on the amount of calculations to perform. By an exogenous transformation, the new problems are much farther from the given one, but they become procedural, as global solving is not possible.

Example 4. The following problems were proposed by three students (Mircea, Andrei, and Nguyen, grade 5), who decided to work as a group.

Simpler: A monkey wants to reach bananas. In the 7 x 9 square below the route is indicated (Figure 5a). Knowing that the side of a square is 5m and the monkey crosses 19m in 5 minutes, find how long does it takes the monkey to reach the bananas.

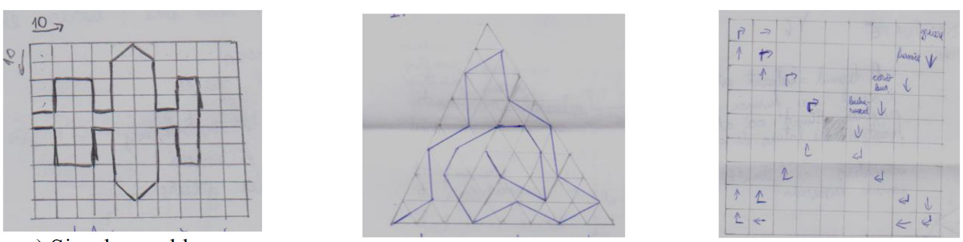
Similar: Ballerina lost her ribbon in the maze. The ribbon is represented in the square grid (Figure 5b). Knowing that the side of a small square is 3m, find the length of the ribbon.

More difficult: A maze is between the houses A and B. Jack wants to go from A to B, so he takes a string to indicate his way to go. On the 11 × 11 square grid below the Jack's travel is drawn (Figure 5c). Knowing that a square has a side of 1.2m, find the properties of the number that shows the length of the road.

In their proposals, the three students renounced to the local constraints (the similar and more difficult problems have cross-overs), but also to the global constraints (the trajectory doesn't fill in the grid). In addition, the problems contain a series of external elements meant to increase the degree of difficulty. For example, they requested to calculate the time needed for crossing the given path (simpler problem) or the fact that problem 3 contains a reference to an 11 × 11 grid, although the given image contains a 11 × 12 grid (consequently, the last column should be ignored).

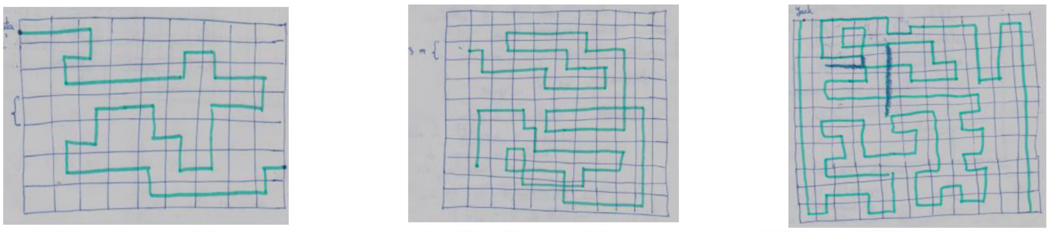
DISCUSSION

The data reported above highlight two situations. Some of the students vary superficial constraints by making endogenous transformations. For example, Victor (example 1) modifies the size of the grid and the shape of the trajectory, while Doina (example 2) modifies the generator element of the grid. In these cases, the students strictly follow the starting prob-



a) Simpler problem b) Similar problem c) More difficult problem

Figure 4: The drawings made by Silviu for his posed problems



a) Simpler problem b) Similar problem c) More difficult problem

Figure 5: The drawings made by Mircea, Andrei, and Nguyen for the posed problems

lem's structure and pose coherent and consistent problems that respect the nature of the given problem.

Some other students use their knowledge of advanced mathematical procedures and techniques to pose problems based on exogenous transformations. In these cases, the proposals are farther from the starting model, but at the expense of a lesser level of generalization. This is visible in the situations where students vary global constraints, as in the case of Silviu (example 3), or in the case of Mircea, Andrei, and Nguyen (example 4). In these cases, students introduce independent variables – that is, they perform exogenous transformations on the starting problem. In spite of their efforts to make their problems more difficult, they just increase the amount of procedural computations, but not the mathematical quality of the problem, measured by its mathematical consistency. Their interviews also revealed that they actually have limited understanding of the starting problem: this fact had an effect on their posed problems, which they were not able to solve using an optimal solution, but just making step-by-step procedural calculations.

Before getting to a final conclusion, we may consider that other factors could be at play in our analysis. The problems were posed for a problem-posing competition, and the students knew that the "value" of their problems would be judged. It might be that some students preferred to keep the structure of the starting problem, because it was already considered as being "valuable" (as it was a problem included into a real contest). Consequently, in appreciating the problems posed by students we need to consider their personal beliefs about what makes a good problem, as well as their ability to identify the problem structure and the mechanisms available for a controlled change of the problem elements. Thus, it is interesting to consider the nature of modifications at one-more finer detail.

Victor (example 1) has posed problems of identical nature to the given one, just by varying the size of the net. Even though in his solutions we saw that he was aware of the constraint allowing an optimal solution, attending only to the modification of the grid suggests that he has not yet developed efficient mechanisms for changing more elements. Such situation is understandable if we take into account that Victor is a 4 grader, therefore his experience with different shapes and grid structures may be very limited. In contrast, Doina (example 2) seems to master such mechanisms, while keeping close to the problem type might be a personal preference. Somehow, Doina simplifies the starting problem because the pattern she has chosen for the "broken line" in each of her three proposals allows immediate identification of the grid-filling property. This simplification allows variations in the cognitive frame generated by the initial problem while keeping intact the analogy with the deep features of the starting problem. These variations in cognitive framing give an indication of her mathematical creativity.

Silviu (example 3), and Mircea, Andrei, and Nguyen (example 4) change many more elements of the starting problem. Furthermore, we found during the interviews that these students participated in intensive training programs (with their school teachers and with their parents). Their experience from such training might suggest that good problems are those with many data, which require breaking the problem into several pieces, and, in overall, ask for more fluid procedural work.

These cases reveal an essential fact: the extent of which the new proposal reflects an understanding of the hard core of the given problem drives the quality of the newly posed issues. We further analyze the students' posed problems from this perspective. Beyond the students' beliefs about problems and competitions, a dilemma still stands. On the one hand, we have more creative approaches but at the expense of mathematical quality, on the other hand, the newly posed problems that reflect in-depth understanding are apparently less creative.

Some of the students who suggested changes of an exogenous type placed in their texts new mathematical concepts and usually provided traps to solvers. Although they seem to be more creative – when judged on the novelty dimension – they depart from a deep understanding of the starting problem structure. The problems they posed have a low degree of generality, and their difficulty is given by the solving procedure (overcoming traps intentionally included in the text and performing step-by-step calculations). The problems arising in our sample in this way proved weaker in terms of the mathematics involved. We interpret this result as a (mathematical) failure in posing problems that preserve significant aspects of the deep structure of the original problem, yet confirming a distinction between being creative and being *mathematically* creative.

What are the underlying factors that ensure for a new posed problem to be mathematically consistent? We found that, among the students who used endogenous transformations, those who kept the global way of solving of the starting problem made mathematically qualitative new problems. But, as they identified the optimal way of solving, the resulted problem remained close to the initial one, therefore in this case the student proved a low level of cognitive novelty, and the changes in cognitive framing were minimal. Their creativity was driven by the solution space, and inevitably, it had low amplitude.

Concluding, the evidence we got from problem modification tasks in our sample show that high achieving children swing between low amplitude driven creativity and failure to pose consistent problems when it comes about understanding the deep structure of mathematical concepts and strategies. Are these the only alternatives? Further research is needed to find a more definite answer.

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What can we learn from pre-service teachers' beliefs on and dealing with creativity stimulating activities?

Ilya Sinitsky

Gordon Pedagogical College, Haifa, Israel, sinitzsk@gordon.ac.il

This paper aims to highlight the different aspects of a discussion on creativity promotion from the point of view of an educator involved in professional development of pre-service teachers. Challenging the prospective teachers with open mathematical problems provides data on their beliefs and behaviour concerning creativity and creativity encouragement in the classroom. We emphasize a certain manner of revealing elements of relative creativity during students' activities. The final remarks suggest relevant agenda for further discussion and research.

Keywords: Mathematics creativity, teachers' education, problem solving, problem posing.

INTRODUCTION AND BACKGROUND

The promotion of creativity seems to be the central issue of mathematical education at all levels. As mathematics teacher educator, I believe in a crucial role of the teacher in promoting students' creativity, therefore, the development of creativity-inviting environment (Sinitsky, 2008) and analysis of students' activity in this environment are the issues of my primary attention.

During my long-term pedagogical practice, I have collected a vast amount of scattered empirical data on different approaches utilized by prospective and in-service teachers for treating open mathematical situations. The course called "Development of Mathematical Thinking" is a subject of my explicit interest, as its attendance consists of prospective elementary school teachers. This paper refers to various aspects of behaviour and the activities of future teachers themselves as the students of the course. With almost two hundred prospective elementary school mathematics teachers involved in different stages of a study, it may be regarded as fairly wide albeit not a very systematic one. Obviously, some aspects of the study have been related to creativity promotion, and provide an empirical basis for the discussion below.

Despite widespread declarations on cultivation of creativity as the core of mathematical education, there is no single accepted definition of creativity (Mann, 2006; Sriraman, Yaftian, & Lee, 2011). Since creativity is related to the process of problem solving, some research papers focus on in-depth investigations of several, mainly intermediate, stages of cognitive and mental processes - in the spirit of a four-component model of preparation, incubation, illumination and verification proposed by Wallas (Dodds, Ward, & Smith, 2003). Following this paradigm, researchers pay major attention to the structure of an 'Aha!' moment (Liljedahl, 2009; Prabhu & Czarnocha, 2014). At the same time, Leikin (2009) has enriched the model of creativity as a specific combination of fluency, flexibility and originality of Torrance, making it possible to measure various components of creativity. In the frame of this paper, we refer to this description of creativity that enables us to analyse the creative elements in mathematical behaviour of prospective mathematics teachers for elementary and secondary schools.

Recent researches on teachers' component of creativity have been carried out both on macro- (Leikin, Subotnik, Pitta-Pantazi, Singer, & Pelczer, 2013) and micro-levels (Pitta-Pantazi, Sophocleous, & Christou, 2013). Up to now, however, teachers' conceptions and practice in relation to creativity has not been studied systematically (Lev-Zamir & Leikin, 2013).

The aim of the paper is to propose additional issues for further research agenda in the field from the point of view of teachers' educator. The discussion refers to the empirical data concerning the following questions:

— What do prospective and in-service teachers think about the possibilities of promoting creativity through everyday learning of mathematics in elementary school?

— How would they deal with various creativity stimulating activities?

— Which features of creativity could be associated with the process of solving multi-step mathematical tasks by prospective mathematics teachers?

BELIEFS VS. DECLARATIONS: POSSIBLE REASONS OF THE GAP

Studying teachers' beliefs regarding the encouragement of students' creativity was not our primary goal. Yet, the issue has arisen when pre-service teachers were asked to list the reasons for the importance of the course "development of mathematical thinking". "We need to encourage creativity in framework of mathematics lesson in school; and the competence of mathematics teacher in aspect of creativity promotion is a key factor in this process," – they stated. This composite proposition has almost become an axiom in the last decade, and pre-service teachers are broadly familiar with both components of it. In the open-form questionnaire, almost 80% of them declared that teachers are required to develop mathematical thinking through learning mathematics rather than focusing solely on standard algorithms and procedures. Significant part of prospective teachers also delivered interesting thoughts on the necessity of involving all the students in the learning process by using suitable pedagogical tools. Nevertheless, it is known (Lev-Zamir & Leikin, 2013), that the likeness of declarative conceptions concerning creativity does not provide the similarity of pedagogical practice related to promoting creativity.

Let us examine the 'real value' of these claims. The same group of students discussed the well-known problem: divisibility of sum of consequent addends:

> Assignment 1. Construct different sums of three consequent addends. What is the common property of these sums concerning divisibility? Try to prove your assumption.
>
> Check the situation with another quantity of consequent addends. Try to generalize.

Following a multi-stage process of problem solving, discussing the results and summarizing the conclusions, students have been asked about possible ways of introducing this activity (partially or as a whole) in elementary school mathematics lessons. Only 20% of respondents have described more or less suitable arithmetic situations or problems.

The reasoning of the remaining 80% of students as to why they could not use such a task in a regular mathematics lesson in an elementary school classroom can be summarized as follows:

— This activity is a difficult one; it is suitable for advanced students only.

— I have no idea how to fit this activity to the needs and the abilities of elementary school students.

— The activity does not belong to any school curriculum.

We observed similar replies when a group of in-service teachers had been discussing analogous tasks during a professional training course. Moreover, many of them have added that "it seems to be a waste of time".

Let me put here two notes and to propose some related questions.

The first remark concerns the current elementary school curriculum. Is it possible that the extensive familiarity of teachers with textbooks and other teaching resources rules out the option of creativity-stimulating activities? In other words, *does the actual content of (elementary) school mathematics invite those activities and learning styles or at least provide a suitable environment for introducing them?* A survey by Sheffield (2013) contains some significant remarks on this topic, but the problem, indeed, requires a separate discussion concerning both evaluation standards and curricula issues.

Another comment is on the structure of pedagogical content knowledge (PCK). In terms of mathematical knowledge for teaching (Ball, Thames, & Phelps, 2008), we can easily conclude that pre-service teachers do not have enough PCK in the field of creativity promotion. However, *which specific component of pedagogical knowledge can be associated with tools that a prospective (or current) teacher might use in a classroom to encourage creativity?* It certainly does not fit into the 'knowledge of content and teaching' and cannot be included into the 'knowledge of content and students' category that deals with strategies of learning about specific issues. Since the ability of problem solving is both a principal component of mathematical reasoning and its main measure (Borasi, 1994; Silver, 1997), non-routine activities that stimulate creativity have to appear throughout all the content of subject matter. It seems that awareness on components of creative thinking lies beyond the topics of school mathematics and needs to be built as a core and cross-subject pedagogical mathematical knowledge. By analogy with horizon knowledge (HK) of subject matter content knowledge, it can be determined as HK component of PCK.

Let us turn now to the practice. How do prospective teachers deal with mathematically challenging situations? It is accepted that pre-service teachers need to construct their pedagogical knowledge about encouraging creativity by using their own experience of 'mathematical challenge and discovery' accompanied by further practice in the elaboration of creativity-stimulating activities for students (Shriki, 2010).

"WHAT DO I NEED TO DO?"

For years, students have started the course "Development of mathematical thinking" by filling in a closed-type questionnaire on the nature of problem solving in elementary mathematics. I have observed throughout the years that about 70% of prospective teachers have accepted the following description of mathematics problems and the ways of solving them:

— Each problem belongs to a specific mathematical topic, and there are explicit tools that are suitable for solving the problem within given mathematic areas only;
— To solve the problem, one needs to execute a series of operations (in the right order), similar to a sample, typically known to a student.

These data correlate with the findings of Zazkis and Liljedahl (2002) on perceiving school mathematics as a collection of isolated propositions and the tendency of forthcoming teachers to focus on formulas and algorithmic procedures.

All students had enough knowledge in elementary mathematics to progress with proposed assignments through the research. Nevertheless, the title of a section is, not surprisingly, a citation of central theme and a leitmotif observed in students' replies once they had faced a mathematical situation without an immediate solution algorithm. The pre-service teachers were real newcomers to the field: only 2% of respondents have acknowledged having practice with non-standard mathematical situations and/or open problems.

During a six-month course students were typically challenged with 7–9 assignments of open type (Silver, 1997). Certain assignments were multiple-solution tasks (Leikin, 2009) while others included the search of multiple solutions as an essential stage of the overall inquiry. Each assignment was presented as a multi-stage problem with auxiliary questions acting as a natural way to generalize results derived from the earlier stages. The set of assignments was designed according to a criteria elaborated by author (Sinitsky, 2008). I present here two of these assignments alongside the analysis of students' solving strategies. The first activity and a subsequent extensive discussion on the nature of open-problems has in fact served as a 'pedagogical preparation' for the following assignments.

> Assignment 2. Student has solved the following task: "For a given chain of natural numbers 1, 2, 3, insert signs of arithmetic operations between numbers in order to obtain arithmetic expression which equals zero." He has produced the solution as follows: $1 + 2 - 3 = 0$.
>
> Try to solve the similar problem for longer chains of natural numbers that start with 1, i.e. find suitable arithmetic operation signs to obtain the equality for

> 1 2 3 4 = 0; 1 2 3 4 5 = 0; 1 2 3 4 5 6 = 0; … etc. For each chain, find as many solutions as you can.
>
> Try to find solutions with addition and subtraction only. What can you say about the number of those solutions?
>
> Come back to the same chains with a number 1 as a target result. Explore each of the previously solved chains (for instance, 1 2 3 4 = 1).

When dealing with the first part of Assignment 2, the students have used two principal strategies, i.e. 'balancing' of added and subtracted operands (as $1 - 2 - 3 + 4 = 0$), and multiplying by an arbitrary factor of an algebraic sum that equals to 0 (as $(1 + 2 - 3) \times (4 + 5) = 0$). Each strategy provides a significant amount of relevant solutions, thus we suppose that the fluency in this problem is a function of the number of proposed solutions of a given type and of time the student spent on finding them. After a couple of preliminary stages, many students have worked more or less in an algorithmic manner in compliance to the scheme mentioned by Ervynck (1991). Yet, some students have 'rediscovered' the leading principle for each solution without using any routine procedures.

In contrast to fluency, the flexibility of solutions in this assignment is associated with a shift to another type of solution (for example, switch from expression $1 - (2 + 3) + 4 = 0$ to $(1 + 2 - 3) \times 4 = 0$). We also regard the ability to adjust previously constructed solution to another situation as a feature of flexibility. Thus, we interpret the transition from 'balanced' equality $[(1 + 2) \times 3] - (4 + 5) = 0$ to the equality with unity as a quotient of two equal numbers, $[(1 + 2) \times 3] : (4 + 5) = 1$ as flexibility as well.

Typically, for different groups of students, about 80% of future teachers have shown several degrees of fluency and almost 40% revealed some flexibility in their solutions for Assignment 2. Additionally, small portions of students (about 8–10%) have constructed a handful of surprising and non-trivial solutions that will no doubt belong to unconventional solution space (Leikin, 2007). I would like to present two notable expressions as an example: $((1 + 2) : 3 + 4) : 5$ and $(1 - 2) \times (3 - 4)$. Remarkably, these solutions have served as a starting point for further fluency, as in $(1 + 2) : 3 = ((1 + 2) : 3 + 4) : 5 = (((1 + 2) : 3 + 4) : 5 + 6) : 7 = \dots$, and flexibility as in $(1 - 2) \times (3 - 4) = 1 \times (2 - 3) \times (4 - 5)$, with a relevant search for limits of possible generalization. In terms of Koestler (1964), progress in understanding provides the basis for the exercise of understanding, and can even lead on to the "next level" of understanding.

According to the accumulated data, Assignment 2 had served as a reasonable tool to evaluate the components of creativity through constructed solutions. This assignment also invites a discussion in classroom about introducing some creativity-related concepts to the reference group. After reaching such a promising conclusion let us turn to the next assignment and analyse the student's activities within the framework of the following example:

> Assignment 3. We assigned the label 'exceptional' to number 11 because it can be expressed as a difference of two square numbers: $11 = 6^2 - 5^2$. Is this the only 'exceptional' number? How can we find other 'exceptional' numbers? For a given natural number, is there a way to write it as a difference of two perfect squares? Can we state that each natural number is an 'exceptional' one?

At the first stage of the solution, pre-service teachers have demonstrated a very limited repertoire of tools and ideas. Almost half of them claimed they 'can do nothing' with a problem and the following dialogue was a typical stimulating tool to start some progress towards a solution:

> Tutor: Do you really believe that 11 is the only "exceptional" number?
> Student: No, I think it is not the only example.
> Tutor: How can you *calculate* an additional "exceptional" number?
> Student: Aha! I can take any pair of perfect squares and subtract one from another. For example, $10^2 - 9^2$, therefore 19 is also an "exceptional" number.

Despite the fact that various pairs of squares give numerous "exceptional" numbers, the ability to record a series of technical results hardly seems to be associated with a fluency of thinking. Alternatively or additionally, some students have shown a fluency in their search for solutions, constructing chains of differences with some regularity. For example, students have constructed a series of differences of squares of consequent numbers or a series of differences of squares with a constant difference between bases, but in all their suggestions students did not *use algebraic expressions*. As a result, those students have derived a distinctive series of desirable representations of natural numbers as "exceptional" ones.

Since representation of any odd number as a difference of two (adjacent) square numbers was the ultimate outcome of the above scheme, a number of students have changed the pattern to produce solutions of diverse types. Following this route, they have discovered "multi-exceptional" numbers – those that have more than one representation as a difference of square numbers. Flexibility may certainly be attributed to this step of the solution.

Can we find any elements of originality among the routines explored by the students? Which mathematical tool did they use in order to complete the solution? Notably, the factorization of squares' differences alone almost immediately leads both to a list of possible wanted decompositions and a discrimination of criterion as an opportunity of such a representation for each natural number. Since a suitable formula has been applied, one may continue with a simple routine. In this context, the breakthrough is connected to a switch towards simple algebra, and not a single prospective elementary school mathematics teacher succeeded in making this switch.

Discussing this disappointing result of the last assignment is especially interesting given the fact that the situation was fairly similar to other assignments, but it is not the scope of this paper. Instead, let us come back to the main goal of our case: assignments as a room to explore creativity. Did we really construct the set of assignments in order to identify gifted students (those with extraordinary creativity) in a population of pre-service teachers? Note that I did not present any numerical data on measuring fluency and flexibility of students' solutions, and I did that for a reason. We want prospective teachers to deal with creativity-promoting assignments in order to make them familiar with the field and to equip them with principal notions and components of 'everyday' creativity (Kaufman & Beghetto, 2009). It is a matter of fact, that through such an experience prospective teachers have meet "global" mathematical ideas and concepts (Safuanov, 2015).

On the other hand, pre-service teachers' own experience may serve, to some extent, as a reasonable model of mathematical behaviour of students faced with challenging activities. Analysing the collected data, we can suggest some specific features for the characteristics of creativity-connected activity of non-experienced students in non-standard mathematical situations.

Most of these students *need* certain *guiding hint or solution as a starting point* to go further and explore the problem. This hint may simply be a minor reformulation of a problem in the terms of possible answers (as one can see from the presented Assignments). In the absence of this 'push', students typically come back to the above-mentioned question "What do I need to do?" in full compliance with the statement of Sternberg (2009) on the role of a supportive environment as a condition for demonstrating creativity.

Furthermore, we can attribute a number of particular features to the dimensions of creativity that have been developed by pre-service elementary school mathematics teachers when exploring multi-step problems. As we have seen, *fluency* is associated with a series of solutions produced on the basis of a discovered sample or pattern. Contrary to this, *flexibility* reflects the ability to switch to a completely different pattern or to take the search to another direction. It seems that the discussion on originality is not relevant in this case, since some straightforward algorithmic solutions have to be accepted as "original" ones: they do not belong to the set of solutions produced by similar groups.

INTEGRATION OF PROBLEM POSING AND PROBLEM SOLVING

The multi-stage and open character of proposed assignments invited and stimulated the students to pose specific and general questions on mathematical situations and also on the results of the preceding steps of their inquiry.

Since problem posing is a central component of creative processes (Silver, 1997), we have constructed and researched the situation of 'pure' problem posing in the frame of above-mentioned course for prospective teachers. A group of 21 pre-service teachers learned the course "Development of mathematical thinking" in self-regulated learning (SRL) format. Following exemplification with several assignments, students were requested to construct *their own open-type mathematical problem* and to explore it. With accordance to 'pure' SRL approach (Goodwin, 2010), students have been asked to design their study, including the choice of a relevant mathematical situation itself as well as the tools and rate to explore it. In this approach of 'absolute free problem posing', the mission proved to be impossible to complete: only 30% of students were able to even formulate a task, and all the tasks submitted had a form of enrichment questions concerning curriculum items of different levels. This result is not a surprise: prospective teachers needed to 'work like real mathematicians' (Shriki, 2010) through generating and solving new problems. Nonetheless, after the tutor's intervention and intensive group discussions, 75% of respondents were able to construct open mathematical situations of different grades of complexity and relevance.

Another item concerning mathematical creativity awareness of teachers is their ability to construct challenging activities for students. As a pilot test, we asked four prospective teachers to adjust their successfully constructed assignments for students of an elementary school. One of the results is presented below.

> Assignment 4 (original version). I have a set of 40 items that look identical but have different weights. The weight is a whole number in kilograms, from 1 to 40. In order to find the weight of each item, I can use a scale in one of the two manners: to balance an item on one side with one or more weights on another side; to balance an item and weights (if necessary) with a set of weights on another side. For each manner, what is the number and the value of required weights?

> Assignment 4 (adapted version). Represent arbitrary natural number from 1 to 40 with numbers 1, 3, 9, 27 and using operations of addition and subtraction (each number may be not be used more than once).

Unfortunately, the result is somewhat of a profanation of the initial assignment, and a total loss of the creativity component can be observed. This means that even when prospective teachers try to adapt their own problems to a "real-world" classroom, they are expected to meet some crucial difficulties. Recent research that was focused on the ways teachers posed problems for their students (Pitta-Pantazi, Christou, Kattou, Sophocleous, & Pittalis, 2015) proves that we still need to consider a wide range of cognitive, psychological and social factors.

FINAL REMARKS

Let me emphasize below the major questions dealing with encouraging creativity that, in my opinion, require further clarification.

— Part of pre-service teachers believe that the content of elementary school mathematics is not suitable for activities that promote creativity. Does (and to what extent) the current curricula allow, invite and encourage mathematically challenging activities?

— Pedagogical content knowledge emphasizes the importance of issue-dependant ways of teaching and learning. What is the place of awareness on creative thinking and which are the ways to promote it in the structure of pedagogical mathematical knowledge?

— According to our findings, external support at the initial stage has a crucial role for launching the process of creative thinking. Additionally, the elements of *fluency*, *flexibility* and *originality* appear: utilizing patterns and samples are two examples of that. Do those peculiarities depend on reference groups and/or on the nature of the proposed assignments?

— Posing of mathematically challenging problems must be a part of teacher's repertoire. How does this ability relate to the experience of problem posing through prospective teachers' own handling open mathematical situations?

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Mathematical problem-solving by high-achieving students: Interaction of mathematical abilities and the role of the mathematical memory

Attila Szabo

Stockholm University, Department of Mathematics and Science Education, Stockholm, Sweden, attila.szabo@stockholm.se

The present study deals with the interaction of mathematical abilities and the role of the mathematical memory in the problem-solving process. To examine those phenomena, the study investigates the behaviour of high-achieving students from secondary school when solving new and challenging problems. Although the mathematical memory accounts for a small part of the problem-solving process, it has a critical role in the choice of problem-solving methods. The study shows that if the initially selected methods do not lead to the desired outcome, the students find it very difficult to modify them. The study also shows that students who use algebraic problem-solving methods perform better than those who use numerical methods.

Keywords: High-achievers, mathematical memory, abilities, problem solving.

INTRODUCTION

Despite increasing emphasis on the education of gifted and high-achieving students, we still have limited empirical data about their mathematical abilities and use of memory functions during mathematical problem-solving. So far, much of the research on mathematical abilities has been conducted on low-achievers (e.g., Swanson & Jerman, 2006). Only a few studies are focusing the mathematical abilities of gifted and high-achieving students (e.g., Brandl, 2011; Vilkomir & O´Donoghue, 2009) or the connection between those students' memory functions and their mathematical performance (Leikin, Paz-Baruch, & Leikin, 2013; Raghubar, Barnes, & Hecht, 2010). Yet no study since Krutetskii (1976) has examined the role of the mathematical memory in the context of able students' problem-solving activities.

BACKGROUND

Mathematical abilities

We are not born with abilities that are explicitly mathematical, but an active contact with the subject may, under favourable circumstances, generate complex mathematical abilities (Krutetskii, 1976). When discussing the subject, we should remind ourselves that mathematics is not a topic defined by sufficient and necessary components and there is no uniform terminology for the abilities that we tend to define as mathematical (Csíkos & Dobi, 2001). Thus, it is not possible to define a structured system of mathematical thinking in which the units are satisfactory to understand the system. A historical review shows that Calkins (1894) concluded – based on replies from Harvard students – that mathematicians have concrete rather than verbal memories, that there are no differences in ease in memorising between mathematicians and other students and that, when doing mathematics, there is no significant difference between men and women. In the early 1900s, mainly because of the dominance of psychometric approaches, the research community's efforts to define mathematical abilities were unsatisfactory. Nevertheless, Binet, Piaget and Vygotsky made relevant contributions to the subject by replacing psychometric approaches with socio-cultural attitudes and thereby showing that abilities are nor static or innate, but qualities that can be assimilated and developed by the individual (Vilkomir & O´Donoghue, 2009).

An essential contribution to the subject was made by Krutetskii (1976) who observed around 200 pupils in a longitudinal study (1955–1966). Krutetskii's analysis of the pupils' problem-solving activities lead to a model of mathematical ability as a dynamic and complex phenomenon, consisting of: a) the ability to *obtain* and *formalize mathematical information* (e.g., formalized perception of mathematic material), b) the ability to *process mathematical information* (e.g., logical thought, flexibility in mental processes, striving for clarity and simplicity of solutions), c) the ability to *retain mathematical information* or *mathematical memory* (i.e., a generalized memory for mathematical relationships) and d) a *general synthetic component*, named a "mathematical cast of mind" (Krutetskii, 1976, pp. 350–351).

Although the above model is often used to identify mathematical giftedness – and studies (e.g., Brandl, 2011; Krutetskii, 1976; Öystein, 2011) show that high-achievers are not necessarily mathematically gifted – Krutetskii indicates that even students performing very well in the learning of the subject, e.g. high-achievers, manifest abilities that can be regarded as proper mathematical abilities (ibid, pp. 67–70).

Mathematical memory

Memory is thought to be critical to both learning and doing mathematics (e.g., Leikin et al., 2013; Raghubar et al., 2010). Research that deals with memory functions was conducted for more than 120 years, but during the first eight decades the topic was almost exclusively examined by quantitative measures (Byers & Erlwanger, 1985). However, in the 1940s, the research shifted focus toward more qualitative terms. Thus, Katona (1940) stated that information related to a method and based on understanding is easier to remember than arbitrary numbers. Later, Bruner (1962) noted that detailed knowledge can be recalled from memory with the use of simple interrelated representations. Although numbers are fundamental tools in mathematics, Krutetskii (1976) underlines that recalling numbers or multiplication tables cannot be equated with mathematical memory; highly able students memorise contextual information of a problem only during the problem-solving process and forget it mostly afterwards. Yet, they can still several months later recall the *general method* which solved the problem. In contrast, low-achievers often remember the context and exact figures related to a problem, but rarely the general problem-solving method. Thus, mathematical memory is a *generalized memory* for mathematical relationships, schemes of arguments and methods of problem-solving (ibid, p. 300).

Studies (e.g., Squire, 2004) show significant distinctions between different types of memory systems. Relating mathematical problem-solving to the cognitive model – by using a simplification – one can say that information is processed (e.g. the problem is solved) in the working memory and is stored (e.g. the problem-solving method) in the long term memory. Long term memory has two subcategories: *explicit* and *implicit* memory, depending on the type of information stored in the respective system. The implicit memory stores information about procedures, algorithms and patterns of movement that can be activated when certain events occur; in mathematical context, the *procedural* memory is a relevant part of this system (Olson et al., 2009; Squire, 2004). The explicit memory stores information about experiences and facts which can be consciously recalled and explained; thus, it is associated with the ability to create mental schemas for problem-solving (Davis, Hill, & Smith, 2000). Thus, we can assume that mathematical memory, as defined by Krutetskii, belongs to the explicit (hence not to the implicit) memory system.

Krutetskii (1976, p. 339) and Davis and colleagues (2000) suggest that proper manifestations of mathematical memory are not observable in the primary grades, because at that age able pupils usually remember relationships and concrete data equally well. Krutetskii indicates that mathematical memory is formed at later stages, most probably on the basis of the initial ability to generalize mathematical material (ibid, p. 341).

Accordingly, the present study (Szabo, 2013) examined the dynamics between mathematical abilities, as defined by Krutetskii, from the following perspectives:

1) The evidence and the interaction of mathematical abilities when high-achieving students are solving new and challenging mathematical problems.

2) The role of the mathematical memory in the process of solving new and challenging mathematical problems.

METHOD

Participants

According to Krutetskii: a) the mathematical memory cannot be observed properly in young pupils or in low-achievers and b) mathematical abilities are manifested by high-achievers. Consequently, the present study focused remarkably high-achieving, 16–17 years old students from Swedish secondary school. The participants attended an advanced mathematics programme and achieved the highest grade in mathematics. The participation was optional; after four months of classroom observations and consultations with their mathematics teacher, three boys and three girls were selected to attend the study.

Tasks

The analysis of a given problem, regardless of the mathematical field it belongs to, indicates the structure of the mathematical thinking needed to solve the problem (Halmos, 1980). Several studies confirm that the most effective way to discern mathematical abilities is to analyse the behaviour of individuals in the context of problem-solving activities (e.g., Gyarmathy, 2002; Krutetskii, 1976). Other results (e.g., Krutetskii, 1976; Öystein, 2011) indicate that individual experience influences students' ways of solving problems. The aim of the present study was to investigate the participants' mathematical abilities, not their knowledge of the subject; thus, to avoid as far as possible the influence of prior experiences, new problems were proposed that were not of a standard nature. After examining the participants' textbooks and consulting their math teacher, the following problems were selected:

Problem 1: In a semicircle we draw two additional semicircles, according to the figure. Is the length of the large semicircle longer, shorter or equal to the sum of the lengths of the two smaller semicircles? Justify your answer.

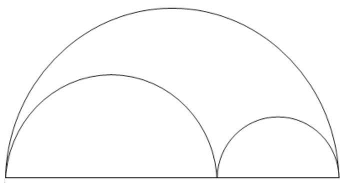

Figure 1

Problem 2: Mary and Peter want to buy a CD. At the store, they realise that Mary has 24 SEK less and Peter has 2 SEK less than the price of the CD. Even when they put their money together, they couldn't afford to buy the CD. What is the price of the CD and how much money has Mary and Peter respectively?

Observations and materials

Classroom interaction affects student's thought process and that interaction is not limited to verbal communication; even gestures or other minor events are affecting the process (Norris, 2002). Krutetskii also underlines that it is difficult to map individual mathematical abilities if pupils are observed in a classroom situation. To avoid these confounding factors, the students were observed individually and, in order to avoid stress, they had unlimited time for completing the tasks. They were asked to write down every step in the process and to "think out loud" whenever it was possible. If a student neither wrote, nor drew or spoke for a while, some of the following questions were posed: What is bothering you? Why do you do that? What do you want to do and why? What are you thinking about? All observations were carried out during a single day and recorded by using a technology which enables to digitalize handwritten notes and related verbal utterances (www.livescribe.com).

Pupils are not used to communicate their thoughts while solving problems (Krutetskii, 1976). To avoid the risk that significant parts of their cognitive actions would not be documented during the process, every problem-solving activity was followed by a contextual interview. The recordings of the interviews and problem-solving activities were transcribed verbatim. Although the unlimited time for solving the problems, no participant needed more than 14 minutes to solve a single problem and the average duration of the succeeding contextual interviews was four minutes.

Data analysis

The *general synthetic component* in Krutetskii's model, i.e., the "mathematical cast of mind", is difficult to observe during occasional problem-solving and is typical for mathematically gifted students (ibid, pp. 350–351). The participants were certainly high-achievers, but not tested for mathematical giftedness and besides that, only two tasks were proposed to them during the present study. Thus, the general synthetic component was not focused in the study. Conversely, the *ability to generalize* is frequently used when pupils establish mathematical memories (ibid, p. 341); thus, the study examined the presence of the ability to generalize mathematical information during the students' activities. A rigorous a-priori analysis of the proposed tasks led to an identification-model for the present study, which focused the following abilities from Krutetskii's framework: *obtaining* and *formalizing* mathematical information (O), *processing* mathematical information (P), *generalizing* mathematical relations and operations (G) and *mathematical memory* (M).

The digital recording of the problem-solving activities resulted in an exact linear reproduction of the pupils' written solutions, drawings and verbal utterances. This was very useful when performing *qualitative content analysis* of the empirical material, inspired by Graneheim and Lundman (2004) and van Leeuwen (2005). The students' solutions were analysed by identifying, coding and categorising the basic patterns in the empirical content. At first, the method highlighted those abilities that were directly expressed in the empirical material, i.e. the *manifest content*. After that, the *latent content* was analysed, by combining data from observations and contextual interviews. I exemplify this with data from Linda, who – when solving Problem 1 – looked at the task, drew some semicircles and whispered for herself:

Linda: Thus, eh... Oh, and here we are after all just using what radius they have and such. One would...

After this device, which occurred after 30 seconds from start, she solved the problem by not saying that much and it was not possible to decide if mathematical memory was present at the time or if she only used her ability to obtain and formalize mathematical information. Later, when analysing the latent content, the following sequence from the contextual interview referred to the above mentioned episode:

Linda: And then you express it simply as that, well, expressing their different diameters as something of each other.
Interviewer: Yes.
Linda: It is similar to another task that I like very much...
...
Linda: Like there, when solving that, the first thing to do... it is making formulas... How different triangles and squares... how the inside of it looks.
Interviewer: Yes... hum.
Linda: Just like there, if you express different sides through... and take one side minus the other, just like in that problem...

The statement "It is similar to another task that I like very much" and the following explanation, relating the actual problem to an apparently different task – with a context of triangles and squares – show the evidence of an (explicit) memory for a generalization, i.e. mathematical memory in the actual device. The combined analysis resulted in a matrix there every device which lasted at least one second during the observed activities, and the time period for its occurrence, was related to the mathematical abilities focused in the present study. The matrix displayed both the interaction between the focused abilities and the occurrence of the abilities, measured in seconds, during every particular problem-solving activity. The matrix also indicated that some devices were actually related to two interconnected abilities.

RESULTS

The interaction of the mathematical abilities

Every student confirmed that the problems were new and challenging, which was a key issue in the design of the study. The analysis displays that the students' problem-solving activities contain three main phases. All activities start with an *initial phase* which encloses both the ability to *obtain and formalize the mathematical information* and *mathematical memory*; these abilities are intimately connected and it is difficult to differentiate them. Directly after the initial phase, follows a phase where the ability to *process mathematical information* is prominent. Nevertheless, every activity ends with a different phase of *processing mathematical information*, where the students are checking their results. Beyond these three main phases, the observed mathematical abilities interact in irregular and unstructured configurations.

The analysis also shows that if the chosen method does not lead to a direct solution of the problem, students become stressful and discontinue processing the mathematical information; they return to the initial phase, which is once again followed by a phase of information processing. Some participants went through this shifting of phases three times. The stress was most evident at Problem 2, where three of those four students who used similar methods made the same error when solving the inequality $2x - 26 < x$. All three activities include the incorrect sequence "$2x - 26 < x$ gives $x - 13 < x$", before returning to the initial phase. All participants were familiar with inequalities; thus, one may naturally wonder why high-achievers make seemingly simple errors. The interviews reveal that the stress occurred when the formalization led to inequalities instead of the expected equations:

Earl: That´s I was a little surprised when it was... on the inequality you solved it.
Erin: It is always difficult to start thinking outside the box... It feels like your mind goes blank.
Linda: Because I get so... When I start with equations ... then I really want to solve it with equations.
Sebastian: This kind of tasks usually requires an equation.

Thus, it seems that the stress was due to the selected method, i.e. equation-solving, lacked those procedures that are necessary when solving inequalities.

The problem-solving methods used by the students

The problem-solving methods could be divided into two categories, as identified during the a-priori analysis of the proposed tasks: *algebraic* respectively *numerical* methods. Consequently, it was possible to distinguish 7 algebraic and 5 numerical methods among the 12 processes. All of the algebraic methods – despite different approaches – led to correct solutions. In contrast, when applying numerical methods, the problems were not solved in a proper way.

The general structure of the students' mathematical abilities

The analysis emerged in a matrix there every device in the problem-solving activities was related to at least one mathematical ability. On the other hand, the analysis revealed that at some devices there were two interrelated abilities present at the time – thus the methods of observation and analysis used in this study were not sufficient to differentiate those interrelated abilities. Accordingly, the ability to process mathematical information (P) is present at 52 % of the total time of the students' activities (see also Table 1). Obtaining and formalizing mathematical information (O) – solitary or in combination with other abilities – is present at 45 % and mathematical memory (M) at 17 % of the total time (Table 1).

According to the a-priori analysis of the tasks, the ability to generalize mathematical information (G) could be detected when numerical solutions were developed into general solutions. Consequently, when numerical solutions were presented by the students, they were asked if they were able to generalize the obtained results. The analysis shows that none of the participants has been able to generalize the obtained numerical solutions; thus, the ability to generalize mathematical information could not be observed in this study.

The role of the mathematical memory in the problem-solving process

The analysis demonstrates that the mathematical memory is present predominantly at the *initial phase* of the process, at a relatively small proportion. In isolated form – at 5 % of the process – the ability is present in the manifest content and it is mainly used for recalling mathematical relationships and problem-solving methods. In the latent content, the ability is present during 12 % of the process, mainly in combination with the ability to obtain and formalize mathematical information (O with M) (Table 1).

Despite of its minor proportion, the mathematical memory is essential to students' achievement in the problem-solving process, because: a) the students selected their methods in the initial phase of the process and b) the students found it very difficult to modify the selected methods. Although they started over the process by returning to the initial phase, none of them abandoned the initially selected method.

DISCUSSION

One of the study's main objectives was to map the interaction of high-achieving students' mathematical abilities during problem-solving. Three main phases of the problem-solving activities were identified: the *initial phase*, the *subsequent phase* of processing the

O	O with P	O with M	P	P with M	G	M
33 %	2 %	10 %	48 %	2 %	0 %	5 %

Note: O = the ability to obtain and formalize mathematical information; P = the ability to process information; G = the ability to generalize mathematical information; M = mathematical memory

Table 1: Average time for mathematical abilities, according to the total time of the problem-solving process

information and the *ending phase*, where results are checked by once again processing the information. Despite the limitations of the study, the chronological order of the mentioned phases emphasize to some extent Polya's (1957) model for problem-solving, which consist of four phases: a) *understanding* the problem, b) *devising a plan* in order to solve the problem, c) *carrying out the plan* and d) *looking back*. Thus, the study indicates that high-achievers solve new and challenging mathematical problems according to the ground stones in Polya's model.

According to the results, the role of the mathematical memory – despite its relatively small presence in the process – is critical, since the participants selected their methods at the start of the process and did not change them later, e.g. when the formalization led to inequalities instead of the expected equations, the participants returned to the initial phase but did not abandon the selected method. A selection of an improper method caused stress, time delay and errors during problem-solving. Thus, it seems that the participants experience a close and rigid interrelation between problem-solving methods and included procedures, i.e. they are not acting flexibly when solving new and challenging problems. By confirming the findings of other studies (e.g., Brandl, 2011) – where typical high-achievers are characterised by being dutiful, nonflexible and conformist – the results indicate that these participants were high-achievers but probably not mathematically gifted. In contrast, mathematically gifted students are described as flexible, high-level problem-solvers and out-of-the-box-thinkers (e.g., Brandl, 2011; Krutetskii, 1976; Leikin, 2014). Hence, the study confirms some qualitative differences in problem-solving between high-achievers who are not essentially mathematically gifted and mathematically gifted students.

However, the inflexibility of the participants can also be explained by two main functions of the cerebral cortex, where working memory operates. One function is to assemble all new information in relation to previous experiences (Olson et al., 2009). Thus, we can assume that at the initial phase, when obtaining and formalizing the information, the students are influenced by previous experiences (e.g. mathematical memory) and act as they are used to, e.g. by starting problem-solving with equations. Another main function of the cerebral cortex is to automate all knowledge (Olson et al., 2009). Yet, automated processes are rigid and extremely hard to modify during an on-going activity. Therefore, it seems that equation-solving is an automated process for typical high-achievers and that the interpretation of new information in the light of past experiences affects their possibility to think flexibly in unusual situations.

Finally, it has to be mentioned that the present study confirms Krutetskii's (1976) observation that during the initial phase it is extremely difficult to distinguish the ability to obtain mathematical information from the mathematical memory. Since information-units stored in the long term memory systems are retrieved at extremely high speed to the working memory (Olson et al., 2009), the methods used in this study were not sufficient to differentiate the information-units related to respective abilities. For a better understanding of the interaction of the mathematical abilities there is a need of further studies. One possible access is to design studies where the structure of the mathematical ability is examined with approaches from several research fields, e.g. by combining qualitative research methods with practices from cognitive neuroscience. In that way, we would possibly be able to answer the questions that were not possible to be answered in the present study.

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TWG07

Posters

Problem solving competency and the mathematical kangaroo

Elisabet Mellroth

Karlstad University, Karlstad, Sweden, elisabet.mellroth@karlstad.se

The goal of the study was to investigate differences in how two groups of students activated mathematical competencies in the mathematical kangaroo (MK). The two groups, group 1 and 2, were identified from a sample of 264 students (grade 7, age 13) through high achievement (top 20 %) in only one of the tests: the MK or a curriculum bounded test (CT). Analysis of mathematical competencies showed that the high achievers in the MK activated the problem solving competency to a greater extent than the high achievers in the CT, when doing the MK. The results indicate the importance of using non-traditional tests in the assessment process of students to be able to find students that might possess good mathematical competencies although they do not show it on curriculum bounded tests.

Keywords: Problem solving, mathematical competency, alternative assessment.

RESEARCH QUESTION

How can differences in students' strength in the problem solving competency be visualised through the mathematical kangaroo (MK)?

THEORETICAL BACKGROUND

Teachers in Sweden have observed that some students achieve highly in the international competition, the MK, although they have trouble with the national courses in mathematics. Some teachers suspect that those students are highly able in mathematics (Mattsson, 2013). To investigate if differences in achievement between groups of students on the MK can be explained by mathematical competencies, Mathematical Competencies: a Research Framework (MCRF) (Lithner et al., 2010) is used as framework. The framework describes six mathematical competencies: applying procedures, representation, connection, communication, reasoning and problem solving (Lithner et al., 2010).

METHOD

Empirical data is test results of same students' (n = 264) from the national test (CT) given in grade 6 and from the MK given in grade 7. Two groups of students were identified: *Group 1*, top 20 % achievers in the MK but not in the CT, i.e. among the bottom 80% in the CT. *Group2*, top 20% achievers in the CT but not in the MK. The two groups' activation of mathematical competencies on the MK was compared after eliminating the achievement factor. Competencies are therefore activated within an individual by showing the individuals' strengths and weaknesses. Each identified individual belonged to one of the two disjunctive groups, and differences between the groups were analysed.

RESULTS AND DISCUSSION

This study verifies teachers' observation, that there are students who achieve among the top in the MK but not in the CT. It is shown that on the MK, *Group 1* activates the problem solving competency to a greater extent than *Group 2*. Some tasks (6 out of 21), in the MK differ more than others in response rate between the two groups. Those tasks have in common that they all give opportunity to activate either the problem solving (n = 5) or the reasoning (n = 5) competencies. The reasoning competency is in the MCRF closely related to the problem solving competency, it is its juridical counterpart (Lithner et al., 2010). This study shows that the MK consists of a relatively high number of problem solving tasks, the MK also aims to offer interesting challenges. The use of challenging problems are important when working with mathematically highly able students (Nolte, 2012), The MK has inspired part of a model used to identify students highly able in mathematics (Pitta-Pantazi, Christou, Kontoyianni, & Kattou, 2011), maybe the students in *Group1* actually are mathematically highly able and given challenging mathematical problems makes them achieve. This study indicates that through the MK some students show good mathematical competencies that they not are able to show on the national test. It is therefore important to use both curriculum bounded test as well as non-curriculum bounded in assessment, so that students get more and varying possibilities to show their strength.

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Exploratory learning in the mathematical classroom (open-ended approach)

Ildar Safuanov, Sergey Atanasyan and Irina Ovsyannikova

Moscow City Pedagogical University, Moscow, Russia, safuanov@yahoo.com

This paper explores the possibilities of the organisation of research projects by pupils in the Secondary school when the open-ended approach is utilized.

Keywords: Exploratory learning, open-ended approach.

RATIONALE AND BACKGROUND

Exploratory learning has been often used in the mathematics teaching practice in Russia, typically, using the form of project (Shatskiy, 1989). As a result of the preliminary research, our interest focused on the open-ended approach to mathematics teaching that appeared in Japan in the 1970-s and was developed in the next decades (see, e.g., Becker & Shimada, 1997; Nohda, 1991). Essentially, this method was based on the use of problems with multiple solutions. After studying the philosophy of this approach, as well as its didactic opportunities and similar Russian practices, we developed a methodical system of using open-ended problem approach for lessons of mathematics in the secondary school. The goal of this study is to check the effectiveness of the exploratory learning based on the open-ended approach in the mathematical classroom.

SETTING AND METHODOLOGY

We created a toolkit consisting of two blocks: a methodical manual for teachers and a booklet for pupils. The methodical manual contains practical guidelines for using open-ended tasks in classes, a set of tasks for pupils of different age, examples of assessment systems, and instructions for composing open-ended tasks. The booklet for pupils is intended to assist in developing their metacognitive awareness.

Some of the important stages in the lesson preparation coming from the experience of schools using exploratory learning approach in their practice should be mentioned:, 1) solving open-ended tasks cannot be used more often than once a week, 2) at the same time such an activity will not be of any use if research assignments are not used at least once a month.

Also, it is necessary to decide if the exploration is to be done by the entire class acting as one group or the class is to be divided into groups.

The experiment has been carried out during three years. The first phase of research included the design of the format of classroom work, and the identification of the indicators to be controlled, as well as the development of teaching materials. The second (main) phase of the experiment began in the fall of 2012 in a secondary school in Moscow, in three grade 5 (12 years old pupils) classes, and was continued in those classes up to grade 7. One class (N=28) formed an experimental group, and two others (N=45) a control group. The objectives of the research project were:

1) to clarify possibilities and ways for using open-ended approach;

2) to examine how pupils' problem solving ability develops in response to the changes of teaching methods;

3) to reduce pupils' mathematical anxiety.

Before the beginning and at the end of the experiment, teachers' and pupils' opinions about methods of teaching mathematics have been collected using questionnaires and interviews (that are presented on the poster). At the second stage of the experiment, about 20% of lessons in the experimental group (a lesson once a week) were devoted to open-ended tasks.

FINDINGS AND DISCUSSION.

Summarizing the results of the study, we observed the following changes indicated by the responses to the questionnaires before and after the main stage of the study:

- increase of students' engagement in the classroom;
- progress in acquiring the experience of applying mathematics to real life;
- stable positive dynamics of improvement of research competences;
- improvement of the psychological climate in a classroom;
- changes in the role of a teacher as a participant of the educational process in a classroom;
- acquisition of skills of team work.

Some important questions still require further research. In particular, our further research will focus on the development of detailed recommendations for teachers for using open-ended tasks in their practice, and for preparing worksheets with opened-ended tasks for pupils of different ages. Changes in teachers' beliefs and practices seem to be crucial.

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Teaching prospective mathematics teachers to solve non-routine problems

Ildar Safuanov

Moscow City Pedagogical University, Moscow, Russia, safuanov@yahoo.com

This paper explores different ways of teaching pedagogical universities' students to solve various types of non-routine mathematical problems.

Keywords: Problem solving, teacher education.

RATIONALE AND BACKGROUND

The rationale of this paper is to describe a new experimental course in solving non-routine mathematical problems developed for prospective mathematics teachers. The course was based on the genetic approach combined with G. Polya's principle of consecutive phases and also with principles and rules developed by Soviet tradition of fostering problem-solving abilities. The creation of the course as well as the study of its results was guided by the following questions:

1) How new course will improve problem-solving skills of students?

2) How students' views of mathematical problem solving will change?

George Polya (1981, I, p. xii) emphasized that prospective mathematics teachers should be specially taught to solve mathematical problems. "...The solution of a non-routine mathematical problem is genuine creative work". Moreover, he indicated the importance of the discussing methods of solving problems in the classroom. However, as Abramovich & Brown (1996, p. 323) rightly mentioned, "traditional teacher training courses have offered little if any engagement in exploratory mathematics". Furthermore, non-routine mathematical problems now constitute the essential part of the Uniform State Examination in Mathematics for secondary school graduates (tasks C1-C6). The aim of this paper is to describe the experience of teaching prospective secondary mathematics teachers to solve non-routine mathematical problems including Olympiad problems and problems C1-C6 of the Unified State Examination in Mathematics. The new experimental course in solving non-routine mathematical problems for prospective mathematics teachers conducted with a group of 4-th year mathematics major students at the Moscow City Pedagogical University will be discussed.

SETTING AND METHODOLOGY

In our course, we used genetic approach (Safuanov, 2004). In particular, having solved a problem, we gradually establish its connection to some fundamental mathematical theories. For example, considering naturally arising problem of Konigsberg bridges, we arrive to the important mathematical theory of Eulerian graphs. G. Polya (1981, II, p. 133) wrote that "the genetic principle may suggest the principle of consecutive phases".

The principle of concentrated teaching (Safuanov, 1999) manifests itself in our course in several directions. Knowledge of some mathematical topics was deepened. Some simple problems serve for the anticipation of more complex problems and mathematical theories. Many problems serve not only for the raising of the interest to studies (due to their entertaining character) but also promote the acquisition of new theoretical knowledge because they are connected to modern mathematical theories – one can say that he combination of pedagogical functions was achieved. Finally, the "linkage" and "connections" were systematically used: one interesting problem led to other, in some way connected with the former; thus the chains of problems were considered. For example, we offer chains of problems on weighing coins, chains of problems on quad paper etc. It is important to tell student teachers about more general approaches, using the psychological characteristic of the process of problem solving. The new course was implemented within an undergraduate curriculum for a group (N=18) of 4-th year mathematics major students. Before the beginning and at the end of the course, students' skills of solving non-routine problems have been tested and their views of problem solving have been identified using questionnaires and interviews. The classroom work has been organized as a collective solving of some key problems and mostly as a work in small groups solving problems.

FINDINGS AND DISCUSSION

First outcomes of the implementation of our course demonstrated the positive changes in prospective mathematics teachers' skills in solving non-routine problems as well as in their beliefs about the problem solving.

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Process-based analysis of mathematically gifted pupils in a regular class at primary school

Stefanie Winkler and Matthias Brandl

University of Passau, Faculty of Computer Science and Mathematics, Passau, Germany,
winkler-steffie@arcor.de, Matthias.Brandl@Uni-Passau.de

With heterogeneity at schools growing, individualisation of education has become more important than ever. For teachers it is vital to recognise their pupils' diverging abilities effectively in order to meet the pupils' respective learning requirements. The following project, which is based on a regular education setting at primary school (3rd/4th graders, German Primary School), aims at showing ways to identify and characterise mathematically gifted pupils during regular lessons.

Keywords: Mathematical giftedness, mathematical thinking, inquiry-based learning, process-based analysis.

MATHEMATICAL GIFTEDNESS IN PRIMARY SCHOOL

With respect to the development of mathematical skills and interests at primary school, two relevant projects have been influencing effective teaching at German Primary Schools: F.W. Käpnick, who picked out children according to their own motivation or teachers' reference and let them take part in extra mathematics lessons where their methods of working mathematically could be monitored (Käpnick, 1998, pp. 36–38). M. Nolte initiated extra meetings for inter-

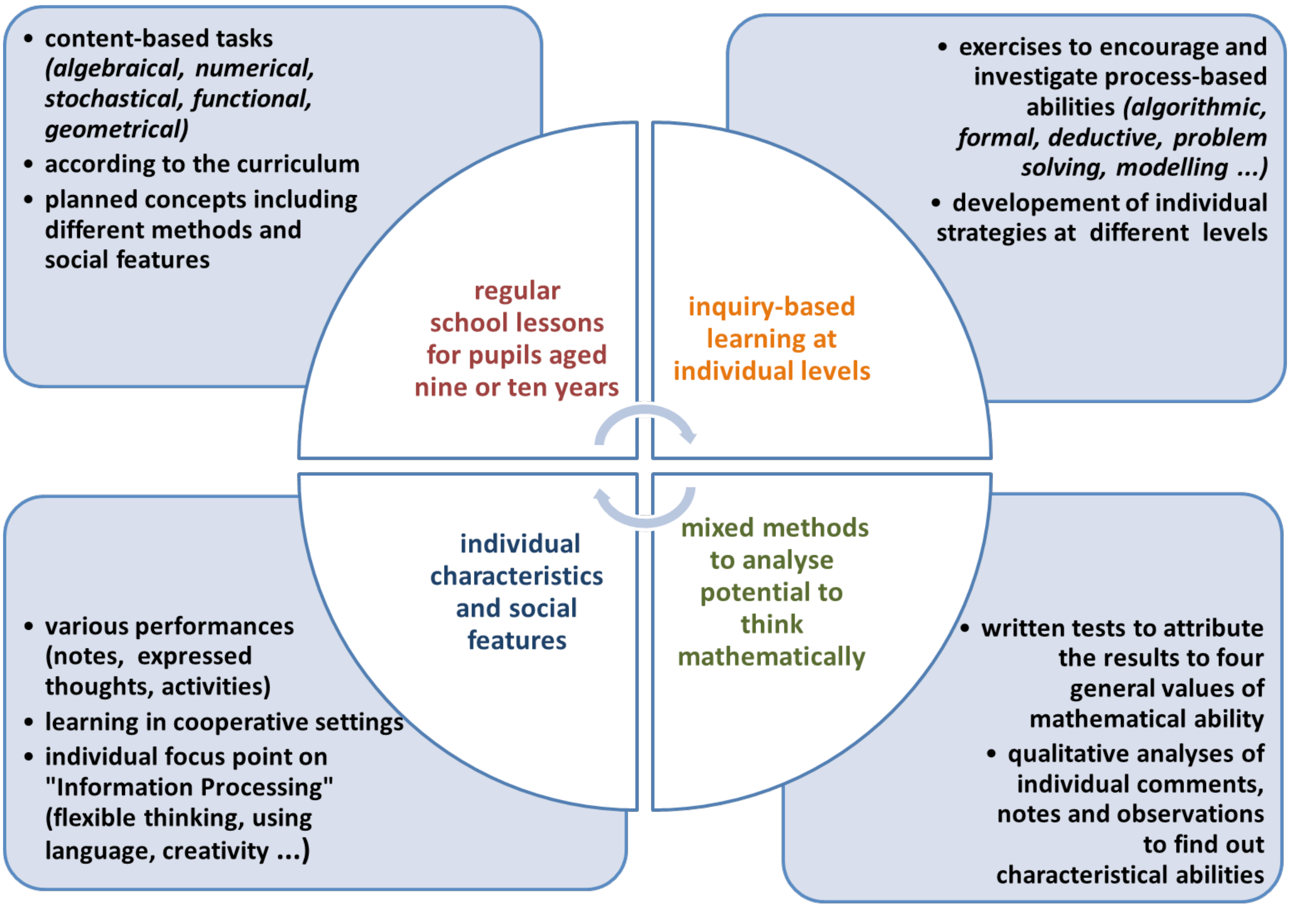

Figure 1: Empirical framework for a process-based analysis of mathematical gifted pupils in primary school

ested pupils, and implemented tests and challenging exercises (Nolte, 2004, pp. 17–20). Both stated that it is quite difficult to define or even categorise an exact level of mathematical giftedness, if pupils are chosen only according to teachers' references, parents' suggestion or their performance in standardised tests (Käpnick et al., 2005, pp. 18–28). We assume that "mathematical giftedness is the individual potential to think mathematically. (...) High mathematical giftedness is characterised by an above-average potential in several aspects of mathematically thinking." (Ulm, 2011, p. 7) Content-, process- and information-based thinking as well as environmental and individual features influence the pupils' ideas, results and performances (Ulm, 2011, pp. 6–9).

Empirical project and analysis

This project focusses on the research question whether there are significant differences between the mentioned ways of identifying mathematical giftedness, and what a simple identification tool for teachers' observation in everyday classroom situations must look like in order to allow for a successful identification of gifted pupils. With regard to research methodology, the development of different tasks which correlate with varying aspects of mathematical thinking builds the starting point when it comes to encouraging pupils during lesson to work on a topic independently and cooperatively (cf. upper half of Figure 1). In addition, various didactical settings and methods offer a detailed, qualitative view on the pupils' individual mathematical abilities (cf. Figure 1 below). Afterwards, observations are interpreted by a qualitative content analysis. The latter considers different aspects of mathematically thinking corresponding to four levels of mathematical ability. In a final step, the results are compared to the results of standardized tests.

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TWG08

Affect and mathematical thinking

Introduction to the papers of TWG08: Affect and mathematical thinking

Pietro Di Martino[1], Inés Ma Gómez-Chacón[2], Peter Liljedahl[3], Francesca Morselli[4], Marilena Pantziara[5] and Stanislaw Schukajlow[6]

1 University of Pisa, Pisa, Italy, dimartin@dm.unipi.it
2 Universidad Complutense de Madrid, Madrid, Spain
3 Simon Fraser University, Burnaby, Canada
4 University of Torino, Torino, Italy
5 Cyprus Pedagogical institute, Latsia, Cyprus
6 University of Münster, Münster, Germany

INTRODUCTION

The quantitative data about the participation to TWG08 at CERME9 highlights the growing interest toward affective issues in the field of Mathematics Education. 40 manuscripts were submitted to the group, 34 were accepted for the discussion, and finally in these proceedings 29 papers and 4 posters are included.

Although we have seen a general upward trend in the number of countries represented within this TWG, CERME9 set a new record in this regard with 16 countries present, representing four different continents. This meant that we had more papers both submitted and presented than ever before. 40 manuscripts were submitted to the group, 34 were accepted for the discussion, and finally in these proceedings 29 papers and 4 posters are included.

Due to the high number of papers and in order to have adequate time for a deep discussion of all the contributions, we organized five of the seven working group sessions splitting the group in two subgroups, maintaining the whole group for the first and the last session. Moreover, in order to avoid the split of the group into two fixed subgroups A and B, we collected participants' interests before the conference and organized different subgroups for each split session. During the five sessions, each participant had the opportunity to attain either group A or group B according to their preferences and in this way to meet all participants of the TWG08.

All submitted documents (paper or poster) were discussed: presenters had 10 minutes (5 minutes for posters) to introduce the key-ideas of their papers, then an assigned reactor had 5 minutes to underline issues or pose questions to the whole audience and finally there were 15 minutes for discussion (5 minutes for posters). In this way, discussions were generally centred and engaging. The final versions of the papers and posters have benefited and developed from the inspiring and motivating discussions conducted during the Conference, that have also involved 5 researchers without submitted papers.

THE STATE OF THE ART

Boero and Szendrei (1998) stress the cumulative and universal characters of the research in mathematics education: this universal character appears to be particularly important in the field of affect, characterized by several constructs derived by other domains of research. For this reason and due to participation of many newcomers in the TWG08, we used part of the first session to report the results emerged in the TWG08 in previous CERMEs conferences.

Marilena Pantziara (who chaired the previous two TWG08) developed and presented a very interesting overview, significantly titled "CERME TWG08: The past-The present-The future".

Starting with the past, i.e. the first studies in mathematics education where *affect* emerges as a field of research (McLeod, 1992), Pantziara retraced the reasons that induced many researchers to go beyond the cognitive for better understanding the complex process of learning and teaching mathematics. Then Pantziara introduced the main affective constructs – from the more "classic" (emotions, beliefs, motivation and attitudes) to other constructs introduced in the more recent editions of CERME (identity, mathematical security, uncertainty orientation). The presentation included the evolution of definitions, methods and questions in our field, as emerged by discussions in previous Affect TWGs, and the model for the structure of the affective domain introduced by Hannula in CERME7 (2011).

In the description of the progress of the research in the field of affect (corresponding largely to the evolution of the papers presented in the overview of Affect TWG), the presentation stressed:

— the growing attention to the clarification of concepts (despite that, the problem of different meanings given to the same words is even now not overcome) and to the mutual relationship between concepts;

— the trend towards the use of mixed methods (quantitative and qualitative) in the research on affect, overcoming the initial preponderance of quantitative methods. The interesting aspect is that this trend is related to the shift from the description of a phenomenon to the interpretation of the same phenomenon;

— the growing attention to the interpretation of the collected data (this aspect is clearly linked to the previous one, and in particular to the shift on affect from a normative approach to an interpretivist one).

Within this frame, the presentation highlighted some important possible directions for future research on affect. Some of these directions were exactly discussed during the sessions of TWG08 in CERME9.

THE PAPERS PRESENTED IN TWG08 AT CERME9

The analysis of research questions of the papers discussed reflects the current diversity of interests and approaches inherent in the field of affect research. The only commonality across these diverse perspectives is that the papers all deal, for the most part, with disaffection. We are still, as a field, working towards precision in terminology and the papers reflect this effort. Finally, as in previous meetings, the papers continue to take into consideration the cultural contexts (language, traditions, and history) within which the research is set.

However, these similarities aside, CERME9 also saw a significant evolution from past affect TWG's.

In this CERME9, in continuation with a long tradition in the field of affect, a spread topic concerns the study of (pre-service or in-service) teachers' beliefs: Arslan and Bulut study middle grades mathematics teachers' teaching efficacy beliefs and their sources; Schmitz and Eichler investigate teachers' beliefs about the roles of visualization; Yurekli and Isiksal discuss the origin of pre-service teachers' self efficacy beliefs; Haser, Arlsan and Kübra explore primary pre-service mathematics teachers' beliefs about mathematics teachers through asking them about their metaphors for mathematics teachers; Bräunling and Eichler exhibit a case study to reconstruct the whole belief system of a single teacher about the teaching and learning of arithmetic; Skilling and Stylianides investigate secondary teacher beliefs and practice that the teachers report using to promote cognitive engagement in their classes. Charalampous deals with students' beliefs and particularly with the question: "does mathematics pre-exist and hence is discovered or is it invented and owes its being to humans?".

Two papers are related to the development of instruments to analyse students' beliefs: Kibrislioglu and Haser develop mathematıcs-related belıefs questionnaires while Andrà, Brunetto, Parolini and Verani propose a codifying system for inferring the students' "I can" and "you can" during a groupwork activity.

Another issue of interest concerns the role of motivation/engagement in mathematics learning and the way to improve perseverance in students' mathematical activities: Lewis studies and describes patterns of motivation in mathematics classrooms; Pantziara and Philippou discuss the role of multiple goal in students' motivation and achievement; Barnes discusses how to improve children's perseverance in mathematical reasoning; Kazima investigates students' reasons for preferences of contexts in learning mathematics; Beumann analyses the impact of mathematical activities on motivation and interest (these last two papers are not included in the proceedings).

Several papers deal with the issue of affect in problem solving – Viitala discusses a case study of a grade 9 girl; Antognazza, Di Martino, Pellandini and Sbaragli and Daher, Swidan and Shahbari study the intertwining of affective and cognitive factors in problem solving in two different school levels (respectively kindergarten and grade 7 students); Müller-Hill and Spies analyse the role of aesthetic experiences in problem solving processes; Tuohilampi, Näveri and Laine present a three-year intervention designed to improve primary school pupils' problem solving skills, and mathematics-related affect; Morselli and Sabena present a study about primary pre-service teachers' affective pathways in problem solving – and, more in general, with emotions. Helmane describes basic emotions of primary students during mathematics lessons; Martínez-Sierra describes students' emotional experiences in high school mathematics, Schukajlow analyses a connection between boredom and students' performance; Fyhn deals with the original theme of the consideration of affective aspects of knowing mathematics in oral examinations in Norway.

De Simone and Lake discuss in their papers the emotional experience of teaching mathematics at the secondary school level.

Regarding the posters, Grothérus describes a method for teaching, evaluation and assessment in mathematics finalized to reduce students' math anxiety; Hansson investigates how students explain their selected failure in mathematics; Andrà, Brunetto, Parolini and Verani study teachers' interpretations of students' mathematical competencies; García González and Farfán Márquez analyse students' attitudes towards mathematics.

As usual in our group, there are papers that examine in depth theoretical aspects: Liljedahl uses the theory by Leont'ev to interpret pre-service teachers' changes after an intensely negative emotional experience and introduces the idea of hierarchy of teachers' motives; Moscucci and Bibbò describe relationship in the affect domain using theory by neuroscience; Pieronkiewicz introduces the notion of affective transgression in order to interpret students' negative emotions towards mathematics; Branchetti & Morselli study the relation between identity and rational behavior.

The discussion of theoretical aspects was particularly stimulating because it highlighted new trends. In particular, in CERME9 for the first time we had papers looking at affect from the participationist (social) perspective overcoming a pure aquisitionist (individual) perspective. Moreover, the discussion about identity (a construct that has a growing attention in the field of affect) has underlined the possible contributions for the study of this construct from a socio-psychological and interactionist approach emphasizing the construction of identity processes and conceiving identities as strategies.

The importance of considering the dynamic dimension of culture has been also underlined. In particular, it has been highlighted the need to unpack "cultures", considering how they have been enculturated into a set of pedagogical assumptions (that is, beliefs and orientation).

Summarizing this brief panning shot at a more qualitative level, the papers presented are focused much more on teacher affect (as opposed to student affect) than we have seen in the past.

Trait and *state* (Hannula, 2011) research has long been presented in the papers at past CERMEs. However, unlike previous years, the research at CERME9 was much more focused on the *state* side of this dichotomy. CERME9 also saw an increase in the number of different affective frameworks being used to analyse phenomena and more qualitative papers. This stands in stark contrast to, for example, CERME5, where almost all of the papers were quantitative.

We also saw several papers dealing with emotions, which is a significant change from past CERMEs. Likewise, there was an increased presence of research into meta-affective aspects, the role of interests, creativity, and self-regulation – topics that had previously received little attention at CERME.

Finally, for the first time we had two papers looking at affect from the participationist perspective.

Despite all this evolution the participants felt that more changes are still needed. In particular, there was a call for further work in improving our definitions. In addition, there was a feeling that more work was needed on the emergent topics of emotions, and meta-affect.

Although we continue to consider cultural contexts it may also be time to consider micro-cultures, such as the classroom or student-teacher relationships. More longitudinal research is needed and, with the shift from quantitative to qualitative research methods, it is now time to consider mixed methods. The introduction of participationist perspectives signals a need to pay more attention to theorizing and networking of theories as well as more comparative and cross-domain research. And, of course, more work on the implications of research on the constructs for curriculum development, teacher education, assessment, and intervention is warranted.

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TWG08

Research papers

‘I can – you can’: Cooperation in group activities

Chiara Andrà, Domenico Brunetto, Nicola Parolini and Marco Verani

Polytechnic of Milan, Department of Mathematics, MOX, Milan, Italy, chiara.andra@polimi.it

We stem from phenomenological stances that each thought originates from a perceived competence (“I can”), which has a strong affective nature. In groupwork activities the “you can”, namely the competence that a student recognizes in the classmates, is also important. We propose a codifying system for inferring the students’ “I can” and “you can” during a groupwork activity. The results are 2D-diagrams that capture the main moments of the activity and help us identifying the leader(s) of the activity and the different roles the students play.

Keywords: Group interaction, perceived competence, cooperation.

INTRODUCTION AND FRAMEWORK

Learning in mathematics occurs in and through interaction with others, the learner becomes conversant with new concepts emerging in the activity, as Sfard (2001) points out: “communication may be defined as a person’s attempt to make an interlocutor act, think or feel according to her intentions” (p.13). Activity allows the growth of mutual understanding and coordination between the individual and the rest of the community (Sfard, 2001). Cooperation is one of the main focuses. Different kinds of cooperation have been depicted (Gooding & Stacey, 1993), with respect to the objective of the activity. Also, conditions that foster cooperation have been highlighted: operational, computational requests of the activity enhance cooperation, whilst reflexive ones may impede it (Hertz-Lazarowitz, 1989). But, are there conditions that shape cooperation which can be traced back to the relationships among the students, rather than to the features of the task?

We stem from Merleau-Ponty (2002) claim that every thought originates from an “I can” rather than from an “I think of”. Di Martino and Zan (2011) have identified the students’ “I can” in the narrative accounts of their relationship with mathematics during the school years: they call it *perceived competence*. “I can” may prompt the student to intervene in a conversation, proposing a strategy, rephrasing another student’s utterance, adding a detail, fostering her thinking. “I can’t” may push a student to ask questions, seek for clarification, stay silent and listen to her mates (see Andrà & Liljedahl, 2014). Since each student may (or not) recognize a competence to each one of his classmates, there is another dimension that counts: “you can”. Andrà and Liljedahl (2014) have shown that, during a group activity, a student may care of a certain classmate’s attention and, at the same time, be annoyed by another student’s reaction to his proposal. “I can” and “you can” are seen as interior states, whilst utterances, postures, glances, gestures are the external expressions of such internal states that determine each student’s will to propose, to react, to stay silent.

METHOD OF ANALYSIS

We exploit an idea introduced by Sfard and Kieran (2001) to capture “two types of speaker’s meta-discursive intentions: the wish to react to a previous contribution of a partner or the wish to evoke a response in another interlocutor” (p.58): reactive and proactive utterances, respectively. Proactive and reactive statements are, in our view, indicators for inferring a sense of “I can”, a sense of likelihood of success that provides the basis for thought and communication. A sense of “I can’t”, conversely, can be inferred from a student’s silence, but also from his/her questioning the strategies/results proposed by his/her classmates. Postures, glances, tone of voice can sustain our inferences about a student’s sense of “I can” versus “I can’t”. A sense of “you can’t” is, as well, inferred from absence of reactions to other students’ utterances. A student who does not glance at a classmate, but looks at the paper, or elsewhere, may not recognize a competence to her (see also Andrà & Liljedahl, 2014). Conversely, a student who care about a classmate’s attention, a student who gives her way, who reacts to her, may feel a sense of “you can”.

We analyze two video excerpts. We codify the excerpts as follows. Each example is divided into *moves*, lasting few seconds each and being identified by the placement of the students into a 2D-diagram, representing the internal states: "I can" versus "I can't", "you can" versus "you can't". If a student change his/her placement in the 2D-diagram, a new move starts. Even if there can be various degrees of "I can" and "you can", in our first analysis we decided to depict only two possible, dichotomous states. The result of the analysis is a series of 2D-diagrams (4 possible "states") that capture the subsequent moves of the activity. We added arrows to denote changes (if any) for a student with respect to the previous move. The first episode has no arrows.

The diagrams are the result of: (i) extensive observation made by each author, (ii) intense discussion about possible interpretations of data. After hours of joint work in team, it is not possible to provide indices for inter-rater agreement within us, given that each interpretation is strictly intertwined with each other's one. This can be seen as a weakness of this work. Hence, in a poster (Andrà, Brunetto, Parolini, & Verani, 2015), we report on several secondary school teachers' comments, and we specifically seek for agreement/disagreement with respect to our interpretations.

DATA

We present two excerpts taken from a course in probability aimed at preventing gambling abuse. These excerpts come from 22 hours of video recording in 13 different schools of different types in Northern Italy. These two excerpts have been selected because they capture some interesting dynamics. True, the small sample is a limitation of this study. Another limitation is the focus only on the perceived competence: Di Martino and Zan (2011) identify another important dimension, 'I like'. Motivation is third dimension that plays a crucial role in framing students' interaction. As a first step, we decided to focus only on the 'I can'-'you can'.

The first excerpt comes from a group activity engaging 4 grade-12 (17 years old) students: Enrico (E), Federico (F), Giovanni (G), and Michele (M). They are attending a technical high school program and are asked to invent a fair game using two dices (possible outcomes are all the sums from 2 to 12). In a fair game, the expected winning equals the ticket price. They have computed the probability of each sum, reporting it on the paper (Figure 1A). They now have to assign the prize to each sum from 2 to 12: G wants to start from 7, and F from 12. The excerpt lasts almost 1 minute. Out-of-school knowledge emerge, as well as school mathematics.

The second excerpt comes from a group activity engaging 4 grade-10 (15 years old) students: Alice (A), Barbara (B), Carola (C), and Dora (D). They are attending a scientific high school program and are analyzing a slot machine, which has three rolls with 9 different symbols each. The number of different sequences is 9^3=729. There is only one winning symbol: the gold bar. The students have computed the probabilities of one, two and three gold bars, and reported them on the paper (Figure 1B). They now compute the expected

Sum	Probability	Prize
2	1/36	
3	2/36	
4	3/36	
5	4/36	
6	5/36	
7	6/36	
8	5/36	
9	4/36	
10	3/36	
11	2/36	
12	1/36	

(A)

Probability of 'gold bar-other-other': 64/729 → 9%

Probability of getting 1 gold bar: 192/729

Outcome	Prize	Probability
1 gold bar	1 euro	
2 gold bars	10 euros	
3 gold bars	100 euros	

(B)

Figure 1: Parts of tasks completed before the first (A) and second (B) excerpts begin

winning. To do so, they have to report the probabilities in the table, to multiply each of them by the corresponding prize, and to sum up the results (weighted prizes). The excerpt lasts 8 minutes.

DATA ANALYSIS

Excerpt 1—move 1. We now present the excerpts, inferring the students' "I can" and "you can" from their utterances, their postures, their gazes and the tone of their voice. Data analysis begins with the first 13 seconds of the first excerpt.

1 G: We should start with 7, which has the highest prize. ...No, the lowest prize.
2 F: No, let's do, let's bet 1 euro.
3 M: Easy.
4 F: If e.g. you bet on 12, it comes out, you win...
5 M: But bet 2 euros.
6 F: ...you win 36 euros. Let's do 36 to 1.

Both G and F make proactive statements. G looks at the paper but addresses F. F, in fact, reacts immediately to G ("No"), and makes a proactive statement ("let's do, let's bet 1 euro"), which involves everyone: in fact, M reacts to it in 3 ("Easy"), and in 5 ("bet 2 euros"), and E will react later, in 13 ("Else, bet 50 cents"). G opens the conversation, proposing to start with the prize to be assigned to 7. A proactive statement is an indicator for G to have a sense of "I can". We claim that such an "I can" is grounded on the intuition that 7 should be assigned the lowest prize, and this intuition comes from his experience with betting games: the higher the probability to win, the lower the prize. F makes three proactive statements, showing a clear sense of "I can": in 2, 4 and 6. In his proactive statement in 4 F proposes to start from the prize to assign to 12. At 6 he intuits that one should win 36 euros, and this intuition comes from his vivid experience with sport betting practice. The language he uses is the language of sport betters. In the first two proactive statements, F looks at the paper, in the last one he looks at G. To look at the paper is an indicator of "you can't", and in fact F speaks over his classmates both in 2 and 5. In 2 F contradicts G with a "No", without providing any justification to the group concerning why should them do differently from what G has proposed. In 5 M reacts to F, but F does not react to M. F discards M's proposal to bet 2 euros instead of 1.

In 1, indeed, G feels a sense of "you can", using the first plural person ("we"). He makes his proposal, but he remains silent to listen to the others, without trying to impose his standpoint. As well, also M shows a sense of both "I can" and "you can": the adjective "Easy" he uses at 3 to comment on F's proposal, clearly refers to his perceived competence, and his proposal in 5 to bet 2 euros is both involving and bearing a sense of competence with respect to the activity. E is silent in this move of the activity and we conjecture that E is listening ("you can"), but he is still making sense of the activity and hence his perceived competence is low ("I can't"). We infer this by looking also at his posture: from 1 to 5 he stares at F, but in 6 he takes the paper and brings it closer to him, starting reading the task silently.

Excerpt 1—move 2. Also this move starts with a statement from G, who reacts to F.

7 G: Wait: 7, how much is it? We should compute the average prize and...
8 F: It's enough to do this (points to 36, the denominator) divided by this (points to 6, the numerator of the probability to get a sum of 7). If you do 36 divided by 7, what do you get? (makes computations with the calculator) 5. If you bet 1 euro on 7, you win 5 euros.

In 7 G reacts to F, recalling his will to start with 7 and the mathematical activity on the mean prize during previous lessons at school. In this moment, G feels a sense of "I can't" and asks for F's help in computing such a mean. F reacts to G, showing a sense of "I can" and "you can't". The sense of "I can" can be inferred from his prompt, procedural reaction, and "you can't" can be inferred from his proposal that differs from the one recalled by G. In fact, F again refers back to sport betting practice rather than the mathematical activity. A conflict starts to emerge, the conflict between the experience at school (the average) personified by G, and the betting experience personified by F. Since E and M are silent, but they are listening to F and G, we conjecture that they feel a sense of "I can't" that hinders their will to speak, and a sense of "you can" since they give F and G their way. We further observe that F and G occupy two mirroring positions: F is the "I can"-"you can't" area, whilst G is in the "I can't"-"you can" area (Table 1, column 2). This area is occupied also by E and M and it seems that the group has reached a sort of frozen situation: there is F, who has a strong sense of "I can" and "you can't", whilst his mates feel that they "can't". How can the group sort this situation out? A possibility is that E, G or M start to question the sense of "you can" with respect to F. If also F "can't", the group activity can go on *as a group activity*. This is happening in *Excerpt 1—move 3*.

9 G: Hence, the minimum you can win is 5 euros.
10 E: (inaudible)
11 G: It's too much.
12 F: The highest prize is 36 euros.
13 E: Else, let's bet 50 cents.
14 F: It's the same, finally. If we bet 1 euro at least we have (inaudible)
15 M: (inaudible)

In 9 G reacts to F's result and tries to give sense to it. This confirms that, even if he has a low sense of perceived competence in the second move of the activity ("I can't"), he trusts F and he listens to him ("you can"). In 11 G adds: "it's too much". His emotions provide G with a sense of unlikelihood of success, a sense of both "I can't" (as in the previous phase of the group activity), and "you can't": F now can't solve the problem, the solution he is proposing is "too much". G looks at the paper, but E and F, conversely, look at him. F, specifically, recognizes his role ("you can"). Even if they are almost inaudible, also E and M contribute to the activity in its third move: their perceived competence has increased ("I can"), and—as we have already commented—we infer a sense of "you can" towards G and F.

In 12 F makes his fourth proactive statement to the whole group, and this is taken as an indicator of "I can". In 13 E reacts to F's proactive statement, and F in 14 makes his first (and only one) reactive statement, reacting to E. F looks at E, differently from his usual practice. Hence, F not only recognizes G's role ("you can"), but also sees E as an interlocutor (again, "you can").

G's move towards "you can't" is mirrored by F's move towards "you can": F has recognizes to G a role, F is recognizing G as an interlocutor (Table 1, column 3). Furthermore, E and M have moved from "I can't" to "I can", since they feel they can follow G's argument and contribute to the group activity. This is mirrored in the next move of the activity, where E nods at F and F stares at E. *Excerpt 1 – move 4*.

16 F: E.g. 12 is given ... like the SNAI[1] (all the students laugh)
17 F: 12 is given 36 to 1. If I bet 1 euro I win 36 euros.
18 E: (nods)
19 G: mmm it's too much, because, then, the 7...?

In 16–17 F makes another proactive statement: "12 is given 36 to 1. If I bet 1 euro, I win 36 euros". F looks at the paper in 16, and at E in 17. We read a sense of "I can" for F, but a sense of "you can't", since he looks at the paper and he does not involve his mates, he does not invite them to react to him. G reacts to F, and this reaction speaks to his sense of both "I can't" (in fact, he is asking again how to deal with the case of 7) and "you can't" (in fact, he feels that the result got by F "it's too much"). E has moved towards "I can", continuing to feel a sense of "you can" with respect to both F and G. Since M does not intervene, we infer that he moves again to "I can't".

We observe that the situation at this point is very balanced (Table 1, column 4). There is a leader, F, who feels a sense of "I can" as well as a sense of "you can't", but there is another student, E, who has been almost silent during the activity, who attracts F's gazes at this point, since he has a sense of "I can" and "you can". Also M feels a "you can", while G feels both an "I can't" and a "you can't". At this point, something really interesting happens (*Excerpt 1 – move 5*): F ignores G's doubt ("the 7?"), and he goes on with the prize to be assigned to 11. Then, he feels a sense of "I can't" saying

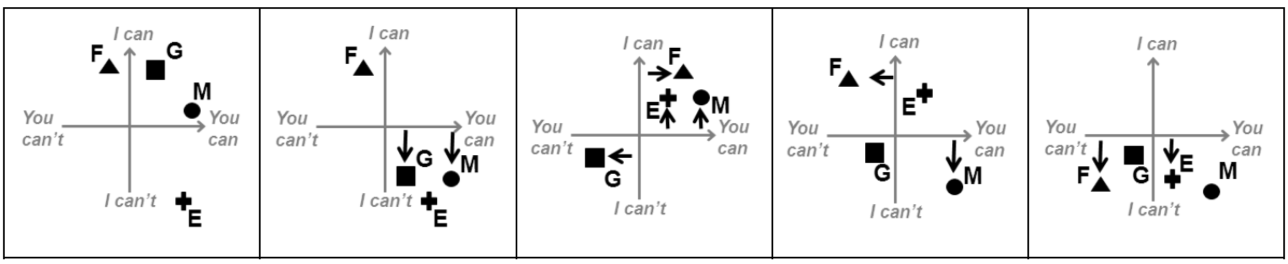

Table 1: The five subsequent moves of the activity involving E, F, G and M

"Eh, no, because…". His emotions provide F with a sense of unlikelihood of success, a sense of "I can't", which freezes him. Also M and E, despite their sense of "you can", feel a sense of "I can't". Table 1 summarizes the five subsequent moves that have been analyzed.

Excerpt 2 – move 1. In the first 15 seconds B reads the task, then

5 B: We should use combinatorial mathematics.
6 D: No, we have computed that the probability of finding one gold bar and two other symbols (she takes the sheet of paper to be able to read it) was…
7 B: …was it 192?
8 D: No. No. This one maybe (she points to the probability of getting the sequence 'gold bar–other–other')
9 B: (reads) 'probability of gold bar-other-other' (nods). 9%
10 D: 9%

The students look mostly at the paper, only B in 5 looks at D. B in 5 makes a proactive statement, but D reacts proposing to use the probabilities they have already computed. In 8 B reacts to D's proposal and she reads on the sheet of paper the correct answer: the probability of getting one gold bar is 192 (over 729), but D contradicts B and leads the group to use the probability of getting "gold bar–other–other", that is 9%. Both B and D feel a sense of "I can", whilst A and C––being silent—yields us thinking that they are feeling a sense of "I can't" and "you can", giving B and D their way. Also B feels a sense of "you can", but D feels a "you can't": in 6 and 8 she contradicts B ("no"), proposing something different from what B has proposed. There is a conflict that is emerging, a conflict between two strategies: B would like to resort to the formulas learned in previous lessons, while D would like to use the computations already made. The group follows D's proposal, as we will see in the next move, but both B and D are right: B (as well as the rest of the group) is not aware that they have already used combinatory. We add a local comment to this segment of data analysis: this lack of awareness is quite usual when the students deal with strategies and concepts that are "new" for them, since they are still not conversant with them. Learning is, in fact, a progressive becoming aware of mathematical meanings.

In the subsequent 2 minutes, the students make computations and report the results on the paper, then they ask the teacher if their work is correct, and the teacher confirms their doubt: it is wrong. *Excerpt 2 – move 2.* B recalls her first proposal:

42 B: That is why we should use combinatorial mathematics. Otherwise, they would not have given it to us.
43 C: Let's use our ingenuity!
44 B: For sure it's with combinatory, then let's invent something. Well, in combinatory one needs to multiply many numbers, so let's have a look…

D is silent, we infer that she is feeling a sense of "I can't", but C intervenes and say "let's use our ingenuity" looking at A, the classmate that has remained silent until now. We see a sense of "I can" in C, who looks at the student that has been silent like her, in a sense encouraging her to intervene. B is proactive ("I can"). A does not intervene for the present. The teacher provides a feedback about the mistake the group has made: the probability of getting one gold bar is not 9%. A takes the floor:

67 A: This one (she points to 192/729) is the probability to get one gold bar.
68 B: Hence, I have said it correctly at the beginning! It is 192 divided by 729.
69 C: (speaks over B) 192 divided by 729.
70 B: (addressing D, pleased) Ah! Ah!

B and D look at each other in this sequence, and B expresses a sense of revenge with respect to D: we read it in terms of "I can" and "you can't" on B's side. Moreover, the conflict between B and D is solved: D was right in having proposed to use their own computations, and B was right in using 192/729.

Excerpt 2 – move 3. The students copy the probabilities on the sheet, then they stop.

110 C: (looks at B) But what should we compute?
111 B: The total average prize. The… mm… namely…
112 D: All those prizes times the probability that you win them, namely that you win the prizes, divided by… all the cases?
113 C: Eh?

114 D: 729? Because there are many cases in which you win nothing.
115 A: Exactly. There are many cases where you get nothing, where you get 'other'.
116 B: Hence we should do the prize..? No, the cases in which you win, divided...
117 D: The cases in which you win, the total prize divided by all.
118 B: Why divided by all?

D makes two proactive statements in 112 and 117, and all her classmates look at her. A, B and C react to her proactive statements: C ("Eh?") at 113, A echoes D's comment at 114, and B at 116 makes sure she has understood, but at 118 asks for explanation. D feels a sense of "I can" and "you can": she is proactive, but she also reacts to her classmates. Also A intervenes, showing a sense of "I can" and "you can". B and C, instead, feel a sense of "you can" with respect to D, but they are not understanding D's proposal, and ask for clarifications: we infer a sense of "I can". *Excerpt 2 – move 4*. The group remains silent for a while, then A (the student that feels "I can"), tries to reformulate D's proposal:

119 A: You are saying: to sum all the average prizes. Is it what you're saying?
120 D: I do not know (smiles).
121 C: Eh, (echoing D) 'I don't know what I am doing'! (smiles)

We see a sense of "I can't" in D's utterance "I do not know", and of "you can't". *Excerpt 2 – move 5*. C, after having echoed D, addresses B and asks:

122 C: Wait. Is it asked (A pulls the paper close to her) the mean of the prizes?
123 B: But, but I would have done this: I would have multiplied the mean prizes for the probability, then summing the probabilities, divided for all the cases.
124 C: Like the last time!
125 B: Yes, it is the weighted average.

D remains silent and stares at the paper, but C in 122, even if she is addressing B, looks at D, and also B looks at D in 123. D's silence leads us inferring a feeling of "I can't" and "you can": she is listening to her mates. A shares the same state of D, being silent. B and C, indeed, share a sense of "you can", accompanied by a sense of "I can". They have found the correct strategy, and they are confirming it to each other trying to involve the rest of the group (looking at D, for example). They are recognizing the distinguishing features of a procedure they have already used, and their positive emotions provide them with a sense of likelihood of success. They are intuiting the analogy with respect to a procedure previously used, that can work.

DISCUSSION

Answering to the research question (are there conditions that shape cooperation which can be traced back to the relationships among the students?), we can see that: (a) proactive utterances and *vertical arrows* allow us to identify *the leader(s)*; (b) *horizontal arrows* allow us to identify *collaboration*. In excerpt 1, F provides the majority of proactive utterances, whilst G is proactive only at the beginning, then he reacts to F's proposals. In excerpt 2, B and D are proactive throughout the excerpt; B and D are the leaders, but also A and C make 2 and 3 proactive utterances, respectively. B and D are collaborative leaders.

E and M show only vertical movements: they occupy always the "you can" area, and they *oscillate* between the "I can" – "I can't" positions.

We may also notice that vertical arrows characterize the moves of students who are not leaders: E and M are clearly not leaders in excerpt 1, like A and C in excerpt 2. But, there are downward arrows also for F (Table 1, last move) and D (Table 2, move 2). These two downward arrows, indeed, depict two different situations. In F's case, it is F alone that stops speaking and starts to doubt about his strategy: his emotions

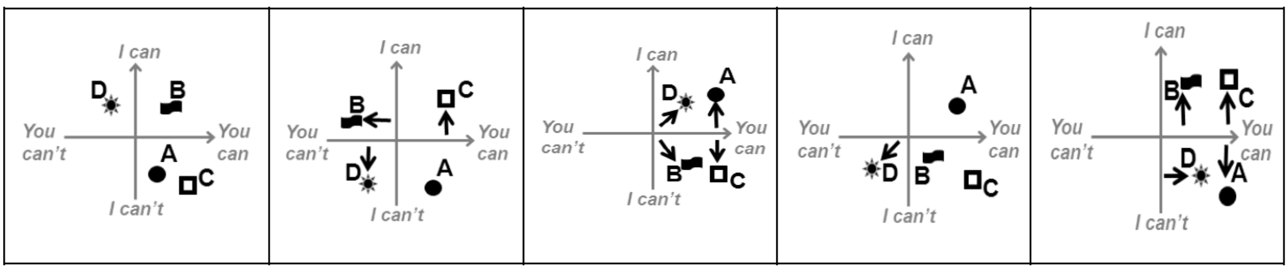

Table 2: The six subsequent moves of the activity involving A, B, C and D

provide him with a sense of unlikelihood of success. F, in fact, is a leader that is not collaborative. In D's case, D goes down because it is B that is leading in that move. Hence, the leaders' "I can" can be diminished either by his/her emotional sense of likelihood of success, or by the feedback of the classmates. This may open the possibility for the others to intervene (Andrà & Liljedahl, 2014). Negative feedback determine a decreasing of "I can". The other way around, to increase one's "I can" by providing positive feedbacks, does not work likewise: in excerpt 2, D receives the glances of her mates (that is to say "you can" to D), but her "I can" does not improve.

We have also seen that *conflicts* emerge: between G and F, between B and D. Conflicts can be read as contrasting views that can impede true dialogue between students. Each student's view of mathematics shape cooperation in groupwork activities, but cooperation is driven also by "you can", the competence recognized by the others. Horizontal arrows correspond to movements on the "you can"-"you can't" axis. In Table 1, we see that F shows only movements of this sort, and moreover he is quite stable in the "I can"-"you can't" position. F is a leader, which *does not recognize* an ability to his mates. In excerpt 2, also B and D occupy the "I can"-"you can't" area, but for few seconds. We sense that such a position is (or can be) necessary for the development of the activity, it is when a student occupies it *for the most of time* that cooperation between the students may be hindered.

One can question whether there are configurations, in the 2D-diagram, identifying situations where the group activity gets. We observe that each excerpt ends with either all the students in the "I can't" area (excerpt 1), or all of them in the "you can" one (excerpt 2). Interestingly, these two configurations are followed by an intervention from the teacher. This issue needs further reflection, however.

This paper reports a qualitative study aimed at defining a descriptive schema to analyze group activities, and the focus of this work is mainly methodological: stronger theoretical foundations are needed in future works.

ACKNOWLEDGEMENT

Project funded by Politecnico di Milano through the 5x1000 Polisocial Award 2013 in collaboration with Fondazione Politecnico di Milano.

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ENDNOTE

1. SNAI is an Italian acronym: "Sindacato Nazionale Agenzie Ippiche" [National Consortium of horse-race Agencies].

The flow of emotions in primary school problem solving

Davide Antognazza[1], Pietro Di Martino[2], Aline Pellandini[1] and Silvia Sbaragli[1]

1 SUPSI, Department of Education and Learning, Locarno, Switzerland

2 University of Pisa, Department of Mathematics, Italy, dimartin@dm.unipi.it

Research into if, and how, students' emotions change during problem-solving, the factors behind the change, and the potential impact of a certain emotional change on mathematical activity, may provide significant indications regarding students' problem-solving difficulties, and regarding the link between affective and cognitive factors "in context". In this paper, we describe the results of a pilot study conducted at primary school level, while also emphasising the methodological choices made in relation to the young ages of the students involved.

Keywords: Emotions, problem solving, affect, primary school, mathematics.

INTRODUCTION AND THEORETICAL FRAMEWORK

The role played by the affective factors in the study of the mathematics teaching and learning process has been little explored for a long time. The success of behaviourism in psychology, the clear division between emotions and rational thinking, and the conviction that this latter sphere is particularly predominant in mathematics marginalised research into affective factors in mathematical education until the 1980s. As emphasized by Zan, Brown, Evans and Hannula (2006, p. 113): "affect has generally been seen as 'other' than mathematical thinking, as just not part of it."

In the 1980s, the gradual affirmation of the interpretative paradigm in the social sciences, related to a greater attention to aspects regarding the complexity of human behavior, has led researchers to abandon the attempt to explain behavior through measurements or general rules based on a cause-effect scheme, and to search instead for interpretative tools. Within this approach, the scientific community in M.E. has recognized the need to go beyond purely cognitive interpretations of failure in mathematics, in particular to interpret the failure of individuals apparently possessing the necessary cognitive resources in mathematical problem solving (Silver, 1985).

In 1989, the book *Affect and mathematical problem solving by McLeod and Adams* opened up a new era in research into emotions in mathematics education. As a matter of fact, the book includes several contributions that, following the theoretical model for the cognitive origin of emotions proposed by the psychologist George Mandler, underline the role of emotion in one of the most important mathematical activities (problem solving). According to Mandler (1989), complex emotions are the result of the cognitive evaluation of a visceral arousal provoked by a discrepancy between the individual's expectations and the demands of ongoing activity.

In the meantime, evidence of the strong interaction between cognitive and affective factors – in particular, of the influence of emotions on decision-making processes – also emerges from studies from other fields of research such as neuroscience (Damasio, 1996) and psychology (Salovey & Mayer, 1990). The study by Salovey and Mayer, introducing the concept of *emotional intelligence,* completely changed the way of looking at the affective component of human personality and its relationship with the cognitive field. Emotions are no longer regarded as a disturbance to correct thinking, but as a potential for effective thinking, particularly when our (and those of other people) affective experiences are recognised and managed appropriately.

On the other hand, according to Mandler (1989, p. 16): "Affectless learning is not a possible goal for a theory or for the praxis of instruction. Common sense tells us that emotions and affective reactions are with us now and forever."

This all leads to the conviction that it is important to teach emotion management in learning, a conviction that is also gaining ground in the mathematics education research field, which confirms how emotions on the one hand affect cognitive processing, biasing attention and memory and activating action tendencies, and on the other have a key role in human coping and adaptation (Evans, 2000; Hannula, 2002). In particular, Op 'T Eynde, De Corte and Verschaffel (2006, p. 194) claim: "from a socio-constructivist perspective, students' emotions and other affective processes are conceived as an integral part of problem solving and learning."

An important objective of mathematics education should therefore be that of teaching how to manage emotional involvement, particularly that which involves negative emotions (Op 'T Eynde, De Corte, & Verschaffel, 2006, p. 204).

Teaching students how to solve mathematical problems then implies that we have to teach them also how to cope effectively with feelings of frustration or sometimes anger. In other words, when teaching and learning mathematical problem solving, the allowing of space for negative emotions might be an educational goal from a cognitive, as well as motivational, point of view. Indeed, only when experiencing negative emotions will students have the opportunity of learning how to deal with them.

The very features of problem solving mean that the process is typified by frequent *interruptions to pre-established plans* and by local failures, which, according to Mandler's model, initiate the emotional experience process. In his overview of the research on emotions in problem solving, Hannula (2012) concludes: "The main lesson to learn from the research on emotions in problem solving is that emotions are an essential part of problem solving."

Mathematical problem solving is therefore often marked by a strong emotional component, which can very rapidly evolve from positive to negative, pleasant to unpleasant, or vice versa, and emotion management profoundly affects students' performances. So, as emphasised by Op'T Eynde and colleagues (ibidem, p. 204) it is very important to conduct further study into these emotions, in order to "become aware of dynamics underlying the succession of several emotions over a short period of time".

The aim of our research was to study these dynamics at primary school level, investigating students' emotions when faced with a mathematical problem, how these emotions change, and the factors that the students recognise as being responsible for the emotions and for the changes.

METHODOLOGY

Population

The study involved five primary school classes in Canton Ticino (Switzerland): one 3rd grade class (18 students), three 4th grade classes (53 students) and one mixed 3rd- 4th grade class (20 students), making a total of 91 students.

Rationale

It is very difficult to study emotions: all instruments are limited in capturing emotional reactions that are not conscious (Schlögmann, 2002) and, as Ortony, Clore and Collins claim (1988, p. 9): "There is as yet no known objective measures that can conclusively establish that a person is experiencing some particular emotion." These same authors emphasise how emotions are nevertheless not linguistic things, but the most readily available non-phenomenal access we have to them is through language and we are willing to treat people's reports of their emotions as valid.

In the context of mathematics education it has been shown that it is important to adopt a multiple approach to data collection in the research on emotions (Evans, 2004), and how: "mathematical activity should be studied in context, and that researchers should take an actor's perspective [emphasis in the original] *that allows the meaning structure underlying students' behaviours and emotions to become explicit.*" (Evans, 2006, p. 241)

In order to "take an actor's perspective", we believe that it is important to choose instruments that contemplate open answers, where the respondent is free to express his emotions using his own words; as Cohen and colleagues (2007) underline:

> It is open-ended responses that might contain the 'germs' of information that otherwise might not have been caught in the questionnaire (...) An open-ended question can catch the authenticity, richness, depth of response, honesty and candor which are the hallmarks of qualitative data.

On the other hand, the low number of mathematics education studies on the emotions of "young" students (primary school) is probably due to the difficulties experienced by the students in terms of possessing and managing the "language of emotions". This is precisely why an important item of our research was an emotional literacy course conducted by Antognazza and Sciaroni (2010) in the classes participating in the research project, aimed at the allowing students to acquire an understanding of their emotions, and also to learn a specific vocabulary for clearly expressing which emotions they are experiencing at a specific time.

Procedure

There were three phases to our study:

1) **A pre-test** conducted on some pilot classes in order to assess, on the one hand, if the mathematics problems tested were calibrated, and to what extent (i.e. the average level of difficulty for the students), and, on the other hand, to identify a codification protocol for analysing the open-answer questions.

The questionnaire used in the research, and the codification protocol, were defined at the end of this phase, and six problems were identified as appropriate to the research objectives. The texts of the problems were shown to the class teachers, who were asked to choose one that they considered to have the *correct* level of difficulty for their students: neither too easy nor too difficult. One example was: "Giulio and Andrea play together with their toy cars. They have a total of 48 cars to play with. At the end of the game, they each take back their own cars. Andrea has three time as many cars as Giulio. How many cars does Andrea have at the end of the game? How many cars does Giulio have at the end of the game?"

2) **An activity phase**. The students were asked to read the selected problem on their own, to answer the questionnaire tested in phase 1, and, subsequently, to try to solve the problem. No time limit was given. This phase was conducted individually in school workplaces in which the students felt comfortable (class, creative activity laboratory or support class).

The questionnaire given to the students consisted of three questions investigating three aspects:

a) assessment of the difficulty of the problem. Closed-answer question "*how do you evaluate the problem?*" with the following possible alternative answers: easy, quite easy, medium, quite difficult, difficult;

b) expression of the emotion perceived when having to solve the problem. Open-answer question: "*how do you feel about having to solve this problem?*";

c) identification of the reasons underlying the perception of the emotion expressed: Open-answer question: "*Why do you think you feel like this?*".

3) **Semi-structured individual interview** (transcribed), focussing on the emotional states experienced when solving the problem, any changes from the emotion initially expressed ("*On the paper you wrote that you were* [emotion written down]. *Did you feel any other emotions when you were solving the problem?*"), and the reasons for these changes ("*Why did your emotion change? What happened?*").

The students were observed while they were solving the problem, in order to analyse their behaviour and their facial expressions. This observation took place under conditions of experimenter blindness (the experimenter did not know how the students had answered the questionnaire), and was also useful in terms of allowing specific questions to emerge in the course of the individual interviews.

RESULTS AND DISCUSSION

A priori assessment of the difficulty of the problem

An analysis of the questionnaire showed how, after having read the text, most students effectively classified the set problem as "quite easy" or "medium" (see Table 1 below).

A priori assessment of the difficulty of the problem	Easy	Quite easy	Medium	Quite Hard	Hard
Percentage	12%	43%	31%	12%	2%

Table 1: A priori assessment of the difficulty of the problem

Emotion perceived from reading the text

Analysis of the answers to the second question – open-ended and fluid responses – is more complex. On the one hand the positive/negative emotion dichotomy must be defined, while on the other hand it is necessary to identify the "representatives" in order to categorise various labels that appear to take inspiration from similar emotions.

According to Ortony and colleagues (1988), emotions are considered as "valenced reactions" to consequences of events, action of agents, or aspects of objects, and then it is possible to classify the reactions to events as being pleased or displeased, the reactions to agents as approving or disapproving, and those to objects as liking or disliking. These dichotomies permitted a first classification of emotions into positive and negative.

In terms of identifying the representatives, the codification protocol developed following the pre-test phase was utilised, making it possible to identify 6 categories consisting of 3 dichotomous pairs: *confident*, *worried*, *nervous*, *calm*, *happy*, *sad*, to which, a posteriori, was added the category *bored,* not found in the answers obtained in the pre-test. We emphasise that the answer was open, without any kind of restriction and with the possibility for the students to specify more than one emotion.

Table 2 summarises the quantitative results obtained.

Some interesting observations: the emotions initially stated, and supported by the subsequent interviews, are essentially perfectly divided into "positive" and "negative", but although this equilibrium is also found in the specific nervous/calm dichotomy (22% vs 20%), "worried" appears to predominate over "confident" in terms of the possibility of solving the problem, and "happy" appears to predominate over "sad" in terms of having to tackle the problem.

Another significant aspect consists of the fact that the balance between positive and negative emotions obtained from the sample of the five classes is less strong when analysing the individual classes involved in the study. For example, the 3rd grade class is typified by positive emotions selected by 17 out of 18 students (10 students express happiness, and 7 students express calmness), while in other classes negative emotions predominate. For example, in one of the 4th grade classes, consisting of 17 students, the following emotions were expressed: worry (6 students), nervousness (3 students) and sadness (2 students). This may depend partly on cognitive aspects such as superior competence, but also on affective aspects, such as conviction in one's own understanding regarding the specific problem, and on other aspects (emotions, convictions, attitudes and values) depending on the specific social context of the class: "students' interpretation and appraisals processes that initiate the emotional process are grounded in a specific context" (Op 'T Eynde & et al., 2006, p. 196).

As a qualitative note to the analysis of the second question of the questionnaire, we emphasise that in three cases the students' answers testify contrasting emotions. In one of these cases, Enea states testifies between the fear of not knowing how to solve the problem ("*a little scared*") and his confidence in his own abilities, or, in any case, the desire to muster his courage ("*but I'm sure that I can do it*"). In the other two cases, the contrasting emotions are associated with various aspects: *intrinsic*, in the sense of being related to the specific problem, and, *extrinsic*, in the sense of being extraneous to the specific problem. Martina appears worried about having to solve the problem of the toy cars ("*I am in a little bit of difficulty because I can't find the calculation*"), but at the same time she notes that she feels fine as she always does in class ("*I feel good, like in class*"). On the other hand, Mattia feels nervous because, as he later explains in the interview, he always feels nervous when he has to solve problems, but he feels relatively calm about the toy cars problem.

Correlation between the perception of the difficulty of the problem and the emotion stated

Most of the students who state that they think that the problem is easy/quite easy feel calm or happy; on the other hand, those who think that the problem is quite difficult/difficult feel worried or nervous. Out

Emotion	Confident	Worried	Nervous	Calm	Happy	Sad	Bored	Other
Percentage	7%	18%	22%	20%	22%	2%	8%	1%

Table 2: Emotions associated with the problem

of 91 students, only 13 do not assign positive emotions to presumed ease, or negative emotions to presumed difficulty.

Analysing the answers to the third question (the reasons for the emotions), these 13 students are seen to divide into two "categories". The first category consists of those who, in a certain sense, show that they appreciate the "intellectual challenge" of the mathematics problem (usually good-achievers) and for whom the easy problem is boring (Chiara: "*I feel bored because it's boring when problems are easy*"), and the difficult problem a real challenge (Irene, who thinks that the problem is difficult, writes: "*The emotion that I feel is joy because the problem is nice*"). The second category consists of those who, like Mattia cited above, say that they feel nervous when they have to solve mathematics problems, and although they think that the toy cars problem is easy, are overcome by what we might call a worry consistently associated with having to tackle mathematics. This is a very interesting phenomenon, in terms of both research on affect in mathematics education, and in terms of didactics. In fact, it is precisely when aversion, worry and fear are experienced *regardless* of the intrinsic aspects that a negative attitude of fatalism toward mathematics can develop, an attitude which is very difficult to modify (Zan & Di Martino, 2009).

Causes of the declared emotions

More generally, when analysing the causes of the declared emotions (question 3 of the questionnaire), the aspect we found most interesting was that of distinguishing between reasons referring to the specific problem (*intrinsic*) and reasons not referring to the specific problem (*extrinsic*).

If we consider only the explicit references to intrinsic or extrinsic aspects – not classifying in either sense statements difficult to interpret, such as "*I feel nervous because I am afraid of making a mistake*": the fear of making a mistake might be associated with doing mathematics (extrinsic) and unrelated to the specific problem, or, on the other hand, it might be associated with the assessment of the level of difficulty of the problem to tackle (intrinsic) – what emerges from the positive/negative emotions dichotomy appears to be rather significant. In fact, the positive emotions are motivated mainly by intrinsic aspects (for example "*because it's easy*"): 56% of those who said they felt happy, 62% of those who said they felt calm, and as much as 86% of those who said that they felt confident. The negative emotions are motivated mainly by extrinsic aspects (for example "*because I don't like doing problems*", "*because I don't like mathematics*", "*because I don't like school*"): 58% of those who said that they felt worried, 67% of those who said that they felt nervous, and more than 86% of those who said that they felt bored.

It should also be noted that the 54% of the students who said that they felt nervous, and the 45% of those who said that they felt worried, highlight the fear of making a mistake or the desire to do everything correctly (for example: *"because I worry about making mistakes"; "I don't like them. And if I get them wrong, how will I manage when I'm big"; "because I don't know if I do it all correctly"; "I think that it's because I don't know if the problem is right or wrong")*. This phenomenon is particularly significant because fear of making errors often becomes fear of math, a phenomenon that has serious consequences on mathematics learning (Di Martino & Zan, 2013).

Change in emotions perceived while solving the problem

One particularly interesting feature from what has been observed until now emerges from the interviews: most of the sample (52%) reports emotional changes from the *initial state*. Those who say that they did not experience any emotional changes are mainly those who started with a positive emotion that was confirmed in the course of the problem-solving process, or else those who started with a negative emotion, associated with extrinsic aspects, an emotion that affected their perseverance in trying to solve the problem. This confirms the link between negative emotions associated with extrinsic aspects and poor perseverance (Hannula, 2012).

The following observations emerge from an analysis of the interviews conducted with those who stated that they experienced emotional changes:

i) **changes in 'intensity'** (8%, only one case with positive emotions and 6 starting from negative emotions), like that reported by Chiara, who initially said that she felt worried because after having read the problem she thought she would not manage to finish it, but who in the interview explained that at a certain point she stopped feeling worried and became "*almost desperate, because already I didn't manage to finish it, and then,* *seeing all these calculations …*", or like Martino, who said that he felt happy during the solving process "*because I realised that the calculation wasn't so difficult*";

ii) **changes in 'direction'**: from positive to negative emotion (25%) (like Samuele, who initially said that he felt happy, but who in the interview explained that he felt worried "*because I was trying to do the calculation but I didn't manage*"), and from negative to positive emotion (14%) (like Selene, who said that she felt worried because she felt alone, but who at the end said that she felt happy because she hoped to have solved it correctly and because "*I feel I did it correctly*"). Moreover, 5% of the students expressed a double change in "emotional direction" while solving the problem: for example, Sofia said at first that she felt calm, then nervous *"because I was afraid of making a mistake"*, then calm again because *"it seemed easy to do"*. Sofia's account demonstrates a repeated assessment of the situation and of any progress made, and, in particular, this testifies the fact that her emotions are (also) associated with intrinsic aspects.

In both cases (changes in intensity and in direction) it is seen how, except in very rare cases, when explaining the emotional change reference is virtually always made to intrinsic aspects, to the perception of making, or not making, progress in solving the problem. This confirms the close link between affective and cognitive aspects: in fact, a fundamental role is played by convictions about making (or not making) progress toward solving the problem, about having solved the problem correctly or not, convictions that are profoundly linked to the mathematical knowledge of the students. In relation to this, and as an important comment, it is interesting to observe how many of those who changed from a negative to a positive emotion because they were sure that they had solved the problem correctly, in fact handed in a wrong solution to this problem. It would be interesting to investigate the emotion triggered off by discovering that the solution handed in, and believed to be correct, was in fact wrong.

CONCLUSION

In the analysis of the results we have focussed particularly on what we feel is the most interesting aspect of our study, in terms of both research and didactic practice. That is, the distinction between positive or negative emotions perceived in a specific mathematics problem solving activity and deriving from an assessment of the difficulty of the activity proposed (intrinsic aspects), or from more general aspects (for example, the stance of those who think "I don't like mathematics, you are asking me to do maths and I therefore have a negative emotion").

What clearly emerges is that most of those who express a positive emotion on finding out that they have to solve the problem in fact refer to intrinsic aspects, and the issue is therefore one of problem assessment. Conversely, most of those who express a negative emotion refer to aspects unrelated to the problem, or, in other words, to extrinsic aspects (except for the fact that it is recognised as a mathematics problem).

On the one hand this confirms how, already at primary school level, there is a generalised type of "a-priori" hostility towards all things mathematical, hostility that is very often typical of an unquestionable negative attitude toward mathematics (Di Martino & Zan, 2011). On the other hand it suggests the importance of working with students so that any negative attitudes of this type do not compromise the a-priori cognitive assessment of the mathematical activities to be tackled, and the resulting performance. In fact, the negative effects of this kind of phenomenon are clear: negative emotions associated with any mathematics-related activities, deciding against investing the cognitive resources required in order to tackle the activity.

However, the results of our study tell us something more: emotional changes (intensity and direction) during the problem solving activity occur almost exclusively as a result of considerations related to intrinsic aspects. These emotional changes appear to be fundamental in problem solving activities: both those "in the positive direction" (they play an important role in problem-solving perseverance), and those "in the negative direction" (they have an important function in terms of cognitive control stimulation).

Further investigation into the link between emotional change during problem solving, and aspects as perseverance and the activation of cognitive controls would certainly constitute an interesting research direction. It definitely appears that it may be important to promote emotional literacy and emotional awareness, also in order to develop problem solving skills already at primary school level. Our research also shows how, at the same time, it is fundamental to develop the "proper mathematical understanding": in our experiment we have shown how many of the positive direction emotional changes were related to the erroneous conviction of having solved the problem correctly.

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Turkish prospective middle grades mathematics teachers' teaching efficacy beliefs and sources of these beliefs

Okan Arslan and Aykut Bulut

Middle East Technical University, Faculty of Education, Ankara, Turkey, arokan@metu.edu.tr

In this study, 187 prospective mathematics teachers' teaching efficacy beliefs and sources of their beliefs were investigated through the administration of Teachers' Sense of Efficacy Scale (TSES) and The Sources of Self Efficacy Inventory (SOSI) scales. Furthermore, this study explored how well hypothesized sources (mastery experiences, vicarious experiences, verbal persuasions, physiological and affective states) predict participants' teaching efficacy beliefs. The calculated mean scores (out of 9) for dimensions of TSES were: 6,35 for Efficacy for Student Engagement, 6,57 for Efficacy for Instructional Strategies, 6,35 for Efficacy for Classroom Management. Multiple regression analysis results showed that combination of hypothesized sources significantly predicted overall teaching efficacy beliefs and all dimensions of TSES.

Keywords: Teaching efficacy beliefs and sources, prospective mathematics teachers.

INTRODUCTION

The term of self-efficacy was introduced first in Social Cognitive Theory and defined as one's personal judgments about doing a particular behavior and these beliefs have a great effect on the future behavior (Bandura, 1997). Its effect on the future behavior leads to an interest on this construct and there has been several research related with self-efficacy beliefs in the literature so far. It has also been an important topic in the field of education and much research has been done on the efficacy beliefs of students and teachers. In line with Bandura's definition of self-efficacy, Tschannen-Moran and Woolfolk Hoy (2001) introduced the term teaching efficacy as "…judgment of his or her capabilities to bring about desired outcomes of student engagement, and learning, even among those students who may be difficult or unmotivated" (p. 783).

Studies have revealed that teaching efficacy beliefs have effects on teachers' teaching related decisions and student learning outcomes (Işıksal & Çakıroğlu, 2006; Poulou, 2007). Teachers with high teaching efficacy beliefs keep their efforts in the classroom even if there are problems in the class (Tschannen-Moran & Woolfolk Hoy, 2001), behave more positively to students who fail in the class (Ashton & Webb, 1986), and have more effective teaching strategies when compared with the teachers who have a low teaching efficacy (Işıksal & Çakıroğlu, 2006). Such research may enable not only to predict prospective teachers' possible teaching behaviors but also to interpret the effectiveness of teacher education programs (Poulou, 2007). Therefore, the focus of the current study is to investigate prospective mathematics teachers' teaching efficacy beliefs.

The significance of teaching efficacy beliefs also lead to an interest on how these beliefs are constructed. Therefore, the sources of teaching efficacy beliefs began to take attention in the literature (Morris, 2010; Usher & Pajares, 2008). When the issue is the sources of teaching efficacy beliefs, the main interest in the literature is the sources of self-efficacy beliefs hypothesized by Bandura (1997) in Social Cognitive Theory. Bandura hypothesized four main sources of self-efficacy beliefs which are *mastery experience*, *vicarious experience*, *verbal persuasion*, *physiological and affective states*. *Mastery experiences* refer to the results of one's personal experiences in the past and based on these experiences, one knows in which conditions s/he would succeed or fail (Bandura, 1997). In other words, one feels more efficacious if s/he succeeded at doing that behavior before and feels less efficacious if s/he failed before. Bandura (1997) claimed that *mastery experiences* are the most powerful source of self-efficacy beliefs among the four sources. One not only gains self-efficacy beliefs through personal experiences, s/he also gains efficacy beliefs through observing the experiences of others, and this source is named as *vicarious experience* (Bandura, 1997). One might judge his/her capabilities on a particular behavior based on his/her model's accomplishment on that behavior, which functions as a standard for him/her (Tschannen-Moran & McMaster, 2009). Another source of self-efficacy beliefs is *verbal persuasions* received from others (Bandura, 1997). *Verbal persuasions* refer to judgments about one on his/her capability on doing a particular behavior and these judgments might come from different sources such as colleagues, parents, teachers or administrators (Tschannen-Moran & McMaster, 2009; Usher, 2009). As a last source, Bandura (1997) mentioned one's judgment of personal capabilities is affected from his/her physiological and affective states at that moment. For instance, one might feel less efficacious when s/he is under stress or anxiety (Tschannen-Moran & McMaster, 2009) or when s/he is exhausted (Usher, 2009).

When we consider these hypothesized sources in terms of teaching efficacy, theoretically it is possible to claim that prospective teachers' previous teaching experiences, their observations of peers' and previous teachers' practice, judgments about their own teaching performances, and their feelings such as excitement, tension, fear during teaching have an effect on the development of their teaching efficacy beliefs. However, the effect of these hypothesized sources on teaching efficacy beliefs are not consistent in all contexts and domains (Morris, 2010; Usher & Pajares, 2008). Therefore, investigating how the hypothesized sources contributed to teaching efficacy beliefs would provide insight on understanding the development of prospective teachers' teaching efficacy beliefs and teacher education programs might benefit from such studies.

In brief, the current study aims to investigate a group of Turkish prospective middle grades mathematics teachers' teaching efficacy beliefs and how well hypothesized sources predicts their teaching efficacy beliefs. In line with these purposes, the following research questions were sought in the current study:

1) How efficacious do Turkish prospective middle grades mathematics teachers feel themselves as a teacher? In which dimension(s) do they feel themselves mostly efficacious?

2) How well hypothesized sources of self-efficacy beliefs predict prospective middle grades mathematics teachers' teaching efficacy beliefs?

METHOD

Contexts and participants

The participants of the current study were conveniently selected 187 prospective middle grades mathematics teachers from two public universities in Ankara, Turkey. Middle grades Mathematics Education Programs are four-year programs designed to train prospective teachers for teaching middle grades (grades 5 to 8) mathematics in Turkish middle schools. The first two years of the program focus on mathematics courses and the last two years focus

Descriptive Information		Number of Prospective Teachers (N)	Percent (%)
Gender	Male	29	16
	Female	158	84
Methods 1 course	Yes	182	97
	No	5	3
Methods 2 course	Yes	85	46
	No	102	54
School Experience course	Yes	89	48
	No	98	52
Practice Teaching course	Yes	10	5
	No	177	95

Table 1: Information about participants

on pedagogical content knowledge and pedagogical knowledge courses. Two-semester methods (Methods 1 and Methods 2) of teaching mathematics courses are offered in the third year, and School Experience and Practice Teaching courses are offered in the fourth year of the program with very little actual teaching experience opportunities. Descriptive information about participants can be seen in Table 1.

Prospective teachers were also asked about their teaching experience. An important number of them (74%) indicated that they have a teaching experience mostly in the form of private tutoring. These teaching experiences are being a private tutor, being a math teacher in private teaching institutions, being a teacher in voluntary organizations/institutions or being an intern teacher in Practice Teaching course.

Data collection procedure and instruments

Data for the current study was collected in the final week of the fall semester of the 2013–2014 academic year. Prospective teachers were informed about the purpose of the study and only voluntary participants participated in the study. There were two data collection instruments called as "Teachers' Sense of Efficacy Scale" (TSES) and "The Sources of Self Efficacy Inventory" (SOSI). These two scales were widely used in the teaching efficacy related literature and both of the scales were translated into Turkish before and validated. Therefore, the cultural bias is eliminated.

The first scale was TSES which was developed by Tschanen-Moran and Woolfolk Hoy (2001) and adapted into Turkish by Çapa, Çakıroğlu and Sarıkaya (2005). The scale was in 9 point Likert format and consisted of three dimensions. Sample items for these dimensions are given in Table 2.

In order to get validity evidence for the obtained data through TSES, confirmatory factor analysis was applied. According to analysis results, Root Mean Square Error of Approximation (RMSEA) value was calculated as 0.080, Normed Fit Index (NFI) was found as 0.94, and Comparative Fit Index (CFI) was found as 0.97. Finding RMSEA value lower than 0.10, and CFI and NFI values close to 1 could be accepted as an indicator of good fit (Kelloway, 1998). Cronbach Alpha values were also calculated in order to interpret the reliability of the obtained data. For each TSES dimension and whole scale, it ranged from 0.74 to 0.93, which indicated satisfactory internal consistency (Pallant, 2007).

SOSI, which was developed in line with the sources of the self-efficacy in Social Cognitive Theory by Kieffer and Henson (2000), was used as the second data collection instrument. Therefore, these four hypothesized sources constituted the four dimensions of SOSI. This scale was adapted into Turkish by Çapa-Aydın, Uzuntiryaki-Kondakçı, Temli and Tarkın (2013). The scale is in 7 point Likert format and consisted of 27 items in four dimensions which are explained above. In order to have a better understanding of the scale, looking at the sample items from each dimension in Table 3 might be beneficial.

Confirmatory factor analysis was conducted in order to present a validity evidence for the data obtained through SOSI. Calculating RMSEA value lower than 0.10 (RMSEA = 0.085), NFI (0.86) and CFI (0.91) fit indices close to 1 was interpreted as satisfactory for the data fit (Kelloway, 1998). Furthermore, calculated Cronbach Alpha values for each SOSI dimension and whole scale ranged from 0.77 to 0.86, which indicated satisfactory reliability evidence (Pallant, 2007).

Data analysis procedure

In order to answer the first research question, data obtained through the administration of TSES were analyzed with descriptive statistics techniques and ANOVA. To investigate second research question, multiple regression analysis was conducted. Independent variables of the analysis were each dimension of SOSI. Dependent variable in multiple regression analysis

Dimension	Sample Item
Efficacy for Instructional Strategies	To what extent can you provide an alternative explanation or example when students are confused?
Efficacy for Classroom Management	How much can you do to control disruptive behavior in the classroom?
Efficacy for Student Engagement	How much can you do to get students to believe they can do well in school work?

Table 2: Sample of items of TSES

Dimension	Sample Item
Mastery Experiences	I became successful when trying to teach something to students.
Vicarious Experiences	I had chances to observe other teachers in class environment.
Verbal Persuasion	I often get feedback from experienced people about my teaching skills.
Physiological and Affective States	I get worried when I teach something wrong.

Table 3: Sample of items of SOSI

was all dimensions of TSES and overall teaching efficacy belief which was the total mean score obtained from TSES.

FINDINGS

Teaching efficacy beliefs

Descriptive analysis was conducted to explore the prospective mathematics teachers' teaching efficacy beliefs. Table 4 indicates mean values and standard deviations for participants' teaching efficacy beliefs scores for each dimension.

In TSES instrument, a rating of 7 out of 9 refers to the quite a bit efficiency on the given item, and 5 out of 9 refers to the some efficiency on the given item. Table 3 shows that mean values in the dimensions of teaching efficacy beliefs ranged from 6.3 to 6.6 out of 9. It might be inferred that participants felt efficacious themselves as a teacher. In order to understand whether there was a significant difference among dimensions of TSES, ANOVA was conducted. Before starting the analysis, all of the assumptions were checked and confirmed.

Findings of ANOVA indicated that there was a statistically significant difference among TSES dimensions (Wilks' Lambda = .88, F (2, 185) = 13.24, $p < .05$). As a follow up test, paired sample t tests were conducted and results were evaluated using Holm's Sequential Bonferroni Procedure.

According to Bonferroni Procedure, "Efficacy for Instructional Strategies" (M = 6.57, SD = .071) was significantly different from "Efficacy for Student Engagement" (M = 6.35, SD = .073) and "Efficacy for Classroom Management" (M = 6.35, SD = .074). The magnitude of the mean differences among dimensions were respectively .10 and .09, which indicated moderate effect size. It means that prospective mathematics teachers felt mostly efficacious in "Efficacy for Instructional Strategies" dimension.

Prediction of teaching efficacy beliefs by hypothesized sources

In order to explore the how well hypothesized sources of self-efficacy beliefs predict participants' teaching efficacy beliefs, multiple regression analysis was conducted. Before starting the analysis, eleven assumptions addressed by Tabachnick and Fidell (2007) had been checked, and one outlier was eliminated from the data since it highly exceeds the critical value of Mahalanobis Distance. After assuring the assumptions, the analysis was conducted with .05 alpha level and pairwise deletion method.

	Mean	SD	Skewness	Kurtosis
Efficacy for Student Engagement	6.35	.073	-.49	.61
Efficacy for Instructional Strategies	6.57	.071	-.23	-.04
Efficacy for Classroom Management	6.35	.074	-.47	.86

Table 4: Descriptive analysis for teaching efficacy beliefs dimensions

Pairs	t	df	Sig.
Student engagement – Instructional strategies	- 4.55	186	.00
Classroom management – Instructional strategies	- 4.40	186	.00
Student engagement – Classroom management	.05	186	.96

Table 5: Paired sample t-tests

	R	R Square	Std. Error of the Estimate	Durbin-Watson	F	Sig.
Student Eng.	.63	.395	.79	1.87	29.59	.00
Instructional Str.	.63	.392	.76	1.81	29.21	.00
Class. Management	.56	.318	.84	1.92	21.10	.00
Total TSES	.66	.433	.70	1.84	34.58	.00

Table 6: Regression analysis

The examination of multiple regression analysis values from Table 6 indicated that the combination of hypothesized sources of teaching efficacy beliefs significantly predicted all dimensions of TSES and overall teaching efficacy beliefs. Moreover, explained variance in dependent variables ranged from 31.8% to 43.3%.

As shown in Table 7, *mastery experiences* and *physiological and affective states* significantly predicted all dimensions of TSES and overall teaching efficacy beliefs. To consider the Beta scores, *mastery experiences* made the strongest contribution to explaining the overall teaching efficacy and TSES dimensions. In addition, *physiological and affective states* made more contribution than *verbal persuasion* and *vicarious experiences* to explaining them. On the other hand, *verbal persuasion* and *vicarious experiences* did not significantly predict teaching efficacy beliefs.

DISCUSSION

The present study aimed to examine one group of Turkish prospective middle grades mathematics teachers' teaching efficacy beliefs and how the hypothesized sources predict these teaching efficacy beliefs. The descriptive statistics results for TSES indicated that participants feel themselves efficacious in mathematics teaching. Relevant studies in the national context supported this finding. For instance, Işıksal and Çakıroğlu (2006) mentioned that prospective middle grade mathematics teachers' teaching efficacy level could be interpreted as high. In another study, Koç (2011) reported that prospective middle grades mathematics teachers had significantly higher teaching efficacy beliefs than prospective secondary mathematics teachers.

The current study also indicated that prospective mathematics teachers' teaching efficacy beliefs for instructional strategies were significantly higher than teaching efficacy for classroom management and student engagement. This might address that prospective mathematics teachers felt more competent and sophisticated in employing instructional strategies. Participants might have benefited from courses in teacher education program, especially from methods

		Student Eng.	Inst. Str.	Class. Management	Total TSES
Mastery Experiences (M = 5.09, SD = .78)	Beta	.60	.55	.47	.59
	T	6.53	6.04	4.80	6.62
	Sig.	.00	.00	.00	.00
Vicarious Experiences (M = 5.25, SD = .84)	Beta	.08	.08	.08	.09
	T	.77	.80	.79	.91
	Sig.	.44	.42	.43	.37
Verbal Persuasion (M = 5.39, SD = .94)	Beta	-.06	-.01	.03	-.02
	T	-.74	-.13	.31	-.21
	Sig.	.46	.89	.76	.84
Physiological and Affective States (M = 4.66, SD = 1.05)	Beta	-.14	-.18	-.16	-.17
	T	-2.42	-3.02	-2.54	-3.04
	Sig.	.02	.00	.01	.00

Table 7: Coefficients

of teaching mathematics courses, in which they learnt using different instructional strategies for teaching mathematics. On the other hand, mean scores of efficacy for classroom management and student engagement dimensions were relatively lower than efficacy for instructional strategies. It might be the case that the lack of participants' experience in real classroom environment resulted in relatively lower scores in these dimensions. Although a considerable part of participants (74%) had a teaching experience, most of these experiences were private tutoring in a one-to-one context, not in a classroom. Offering teaching experience courses not only in the last year, but also in the previous years might be beneficial for prospective teachers to improve their teaching efficacy for classroom management and student engagement.

Multiple regression analysis indicated that combination of hypothesized sources significantly predicted overall teaching efficacy belief and all dimensions of TSES. Therefore, it could be stated that the analysis results were in line with the theory. However, apart from the hypothesized sources, there might be other sources for teaching efficacy beliefs. To give an example, content knowledge (Can, 2015), invitations send and received by the individuals (Usher & Pajares, 2008), and personal characteristics and motivation for teaching (Poulou, 2007) might be additional sources which are mentioned in some of the related studies. Hypothesized sources might be supported with such additional sources in the further studies.

When the individual contribution of hypothesized sources are investigated, it was seen that mastery experiences and physiological and affective states significantly predicted teaching efficacy beliefs in all dimensions of TSES and in overall teaching efficacy beliefs. As hypothesized by Bandura (1997), mastery experiences are the most influential source for self-efficacy beliefs. Studies in the literature also consistently show that mastery experiences are the best predictor of teaching efficacy beliefs (Morris, 2010). Therefore, the results of the current study could be interpreted as in consistency with the theory and practical studies in the literature and highlights the importance of mastery experience for teaching efficacy beliefs. In line with this finding, it is possible to claim that there is a need to create environments which provide mastery experiences for prospective teachers during their training. The teaching experience courses can be enhanced for prospective teachers to include more actual teaching experience, and new courses in which they will have opportunities to improve their teaching experiences can be designed. However, when designing these courses, it should be beneficial to bear in mind that prospective teachers should be in a supportive environment during these experiences (Knoblauch & Woolfolk Hoy, 2008). When they do not have support from their mentors and instructors from the university, and when they do not have enough resources to improve their practices, mastery experiences do not become an improving source for prospective teachers' teaching efficacy beliefs as seen in Knoblauch and Woolfolk Hoy's study (2008).

Although mastery experiences consistently predict teaching efficacy beliefs in the related literature, there is not a consistent result for the other three hypothesized sources (Usher & Pajares, 2008). In the current study, it was seen that physiological and affective states significantly predicted teaching efficacy beliefs of prospective teachers unlike some findings in the related literature. For instance, in their study Mulholland and Wallace (2001) stated that physiological and affective states contributed to a novice teachers' efficacy beliefs less than the other three hypothesized sources. Similarly, Poulou (2007) mentioned that physiological and affective states are the least influential source among other sources. Such findings cause to neglect the possible contribution of physiological and affective states on teaching efficacy beliefs. However, such results might derive from the difficulty of measuring physiological and affective states source and some measurement errors rather than the nonsexist contribution of this source on teaching efficacy beliefs (Usher & Pajares, 2008). The findings of current study and the study of Morris (2010) support this claim and show that physiological and affective source should be taken into consideration while investigating teaching efficacy beliefs.

While investigating physiological and affective state source, feelings of anxiety and/or stress are generally interpreted as negative for the development of teaching efficacy beliefs. However, Bandura (1997) stated that some degree of such feelings might positively contribute to teaching efficacy beliefs. The study of Morris and Usher (2011) also supported this claim and showed that even award winning professors feel some anxiety and stress before their lessons, but they are able to overcome this feelings during lessons which in turn enhance their teaching efficacy beliefs. In this study, the calculated mean score for the physiological and affective state source indicates that participants occasionally feel anxiety, stress and tension while teaching. However, whether these feelings affected their teaching efficacy beliefs in a positive way or in a negative way remained unclear. Therefore, qualitative follow up studies might be beneficial while investigating how physiological and affective states affected teaching efficacy beliefs (Usher & Pajares, 2008).

According to multiple regression analysis results, vicarious experience and verbal persuasions did not significantly predict teaching efficacy beliefs of prospective teachers. However, Bandura (1997) stated that hypothesized sources are highly related with each other. In such situations, independent contribution of verbal persuasion and vicarious experiences in multiple regression might be over shaded by mastery experience (Usher & Pajares, 2008), which might be the case in the current study. Therefore, there is a need for the further study in order to clarify how the vicarious experiences and verbal persuasions are internalized by prospective teachers in terms of teaching efficacy.

Limited number of participants and non-random sampling method limits the generalizability of the observed results in the present study. Therefore, it is suggested to replicate the study with different samples in order to improve the generalizability. Furthermore, when the issue is how the hypothesized sources contributed to teaching efficacy beliefs, supporting quantitative analyses with qualitative analyses would be beneficial to undermine the limitations of quantitative measurement on hypothesized sources (Usher & Pajares, 2008). Therefore, mixed method studies are suggested as a further study. Future research studies may also focus on possible teacher education program experiences in which prospective teachers' teaching efficacy beliefs are likely to improve and how these beliefs and their sources change during the teacher education program.

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Improving children's perseverance in mathematical reasoning: Creating conditions for productive interplay between cognition and affect

Alison Barnes

University of Brighton, Brighton, UK, a.barnes2@brighton.ac.uk

This paper reports on a small-scale intervention that explored perseverance in mathematical reasoning in children aged 10–11 in an English primary school. The intervention facilitated children's provisional use of representations during mathematical reasoning activities. The findings suggest improved perseverance because of the effect the intervention seemed to have on the bidirectional interplay between affect and cognition. This initially created affectively enabling conditions that impacted on cognition and then created cognitively enabling conditions that impacted on affect. A tentative framework describing this interaction is proposed.

Keywords: Perseverance, mathematical reasoning, affect, cognition, provisional.

INTRODUCTION AND THEORETICAL BACKGROUND

The development of mathematical reasoning is not straightforward; reasoning processes can trace a "zig-zag" route (Lakatos, 1976, p. 42) which necessitates perseverance to navigate cognitive and affective difficulties. The cognitive processes relating to mathematical reasoning have been well documented over the last seventy years (for example, Polya, 1945) and in more recent decades there have been significant theoretical developments in the interpretation of the affective domain in relation to learning mathematics (for example, Hannula, 2011). However, pedagogies to develop children's mathematical perseverance are not yet articulated in the literature. This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The significant interplay between cognitive and affective domains during mathematical learning has been noted at previous CERMEs (Di Martino & Zan, 2013; Hannula, 2011) and this interplay provided the framework for analysing and interpreting the findings in this study.

The importance of reasoning

The central importance of reasoning in mathematics education has been widely argued. For example, Yankelewitz and colleagues (2010) assert that reasoning is crucial in the formulation and justification of convincing mathematical argument. Ball and Bass (2003, p. 28) make a connection between reasoning and the development of mathematical understanding, arguing that in the absence of reasoning, "mathematical understanding is meaningless". They further argue that reasoning has a significant role in the recall of procedures and facts as it is the ability to reason, and not memory that enables a child to reconstruct knowledge when needed. The capacity to reason is therefore a significant factor in children's learning of mathematics and there is value in framing a study with reasoning as its focus.

Mathematical reasoning can be considered to include deductive approaches that lead to formal mathematical proofs and inductive approaches that facilitate the development of knowledge; Polya (1959) broadly interprets these two types of reasoning as demonstrative and plausible reasoning respectively. In this study, my interpretation of mathematical reasoning was based on Polya's (1959, p. 7–9) "plausible reasoning" and includes the use of processes detailed by Mason et al (2010) such as: random or systematic specialising by creating examples; noticing patterns to formulate and test conjectures; generalising and convincing.

Perseverance in reasoning

In this study, I have interpreted perseverance in accordance with common dictionary definitions to mean "persistence in [mathematical reasoning] despite difficulty or delay in achieving success" (OxfordDictionaries, 2014). Lee and Johnston-Wilder (2011, p. 1190) identify perseverance as one aspect of the construct mathematical resilience and argue that it is needed to overcome "mathematical difficulties". Such difficulties arise from the "zig-zag" route that mathematical reasoning typically traces (Lakatos, 1976, p. 42) and can be cognitive or affective in nature.

Overcoming cognitive difficulties necessitates the use of meta-cognitive self-regulatory approaches. For Mason, Burton and Stacey (2010), this is characterised by developing internal monitoring to facilitate deliberate reflection on reasoning processes and their outcomes. Such monitoring might result, for example, in changes in approach or use of representation, or rejection of ideas. This fosters a fallibilistic approach (Charalampous & Rowland, 2013; Lakatos, 1976) to engaging with mathematics and mathematical uncertainty. Mason, Burton and Stacey (2010) emphasise the value of considering three phases of work when engaged in activities involving mathematical reasoning: entry, attack and review. The entry phase, characterised by the making of random trials, and the back and forth movement between phases, exemplifies and facilitates a fallibilistic, self-regulatory approach to mathematical engagement.

Navigating Lakatos' (1976, p. 42) zig-zag path also has affective impact and this necessitates affective self-regulatory responses. Goldin (2000) proposes that affective pathways, comprising rapidly changing emotional states, arise during mathematical problem solving. Malmivuori (2006, p. 152) argues that these emotion responses "direct or disturb" mathematical thinking and activate either active or automatic self-regulatory processes. During active regulation of affective responses, an individual consciously monitors affective responses to inform cognitive decision making. By contrast, automatic affective regulation describes self-regulatory processes that act at a sub-conscious level in which negative emotions can act to impede the higher order cognition involved in reasoning.

Successful engagement with mathematical reasoning can be rewarding and impact on an individual's sense of self-worth. Debellis and Goldin (2006, p. 132) describe mathematical intimacy as an affective structure which portrays an individual's potential "deep emotional engagement" with mathematics. They argue that intimate mathematical experiences can give rise to emotions such as deep satisfaction that impact on self-worth. However, positive mathematical intimacy could be jeopardised by experiencing failure. Debellis and Goldin (2006, p. 138) reason that coping with swings in mathematical intimacy is a "meta-affective capability", the development of which characterises successful problem solvers; this is a further presentation of the perseverance needed to be able to reason mathematically.

THE STUDY

In this study, I sought to improve children's perseverance in mathematical reasoning by applying an intervention that provided children with opportunities to use mathematical representations in a provisional way.

The importance of representation in mathematics learning has been extensively documented and this study draws significantly on Bruner's (1966) modes of representation and Dienes' (1964) Dynamic Principle. However, the notion of provisionality is less widely interpreted within mathematics education.

Provisionality is an idea that is drawn on in information technology (IT) education (Leask & Meadows, 2000). The provisional nature of many software applications enables users to evaluate and refine a product as it is being created. Papert (1980) utilised the provisional nature of programming in designing the LOGO environment. LOGO enables a child to create instructions to move a turtle dynamically on the screen. It facilitates children to conjecture, make trials and use the resulting data to make improvements. Hence, this software enables children to construct understanding through a trial and improvement, conjectural approach to mathematics; the intervention in this study sought to impact on children's cognitive responses by applying a similarly provisional approach to children's use of mathematical representations.

Papert (1980) also notes how the provisional nature of programming impacts on the affective domain. It fosters an attitude that mathematical thinking is fallible (Charalampous & Rowland, 2013), that it concerns trial and improvement and conjecturing rather than the singular pursuit of right or wrong answers. Such an approach, he argues, makes children "less intimidated by a fear of being wrong" (Papert, 1980, p. 23). Hence, by constructing an intervention that enabled children to work provisionally, this study also sought to impact on children's affective responses.

This research took place in an English primary school using an action research approach. The study comprised one Baseline Lesson in which the intervention was not applied, and two Research Lessons in which the teacher applied the intervention to her teaching approach. The teacher selected four children to form the study group based on her assessment that their perseverance in mathematical reasoning was limited and would benefit from improvement. Prior to each of the lessons, the teacher and I selected a mathematical activity that presented opportunities for mathematical reasoning. For the Research Lessons, we discussed how the children could use representations in a provisional way and the teaching strategies that might facilitate this. The teacher then created the detailed plans and taught the lessons.

The fieldwork comprised collecting data from the three lessons, post-lesson interviews with children and an evaluation meeting with the teacher. During the Baseline and Research Lessons, I collected data on the four children relating to the cognitive and affective domains through non-participant observation and by taking photographs of the representations that they made. Audio recordings were made of the children's dialogue during the lessons and I used these to augment the observation notes post-hoc. During observations, I used an approach similar to that used by Schorr and Goldin (2008) in their analysis of filmed lessons to gather data relating to key affective events. For example, I noted the children's manner of engagement, their body position and the speed of their speech. I interviewed the study children immediately after each observation. The focus of the interview was threefold: to check my understanding of what I had observed; to gain the children's interpretation of what had happened and why, and to explore the extent of the children's mathematical reasoning.

This paper reports on the thick data arising from the second Research Lesson pertaining to two of the study group, Lucy and Emily.

FINDINGS AND DISCUSSION

Bidirectional interplay between cognition and affect (Di Martino & Zan, 2013) was evident during Lucy and Emily's mathematical engagement in Research Lesson 2. However, it seemed to operate in different directions at different stages of their thinking. Hence, I have used Mason, Burton and Stacey's (2010) entry and attack phases of problem solving as a temporal framework for the presentation and discussion of findings.

During Research Lesson 2, Lucy and Emily engaged as a pair with the problem:

> A square pond is surrounded by a path that is 1 unit wide. Explore what happens to the path length for different sizes of pond.
>
> Resources available: Cuisenaire rods, pencils, A3 plain paper.

The impact the intervention during the entry phase

During the entry phase (Mason, Burton, & Stacey, 2010), Lucy and Emily used Cuisenaire rods in a provisional way to get a feel for the problem; they explored how the criteria given in the activity could be represented and began to explore how the path size related

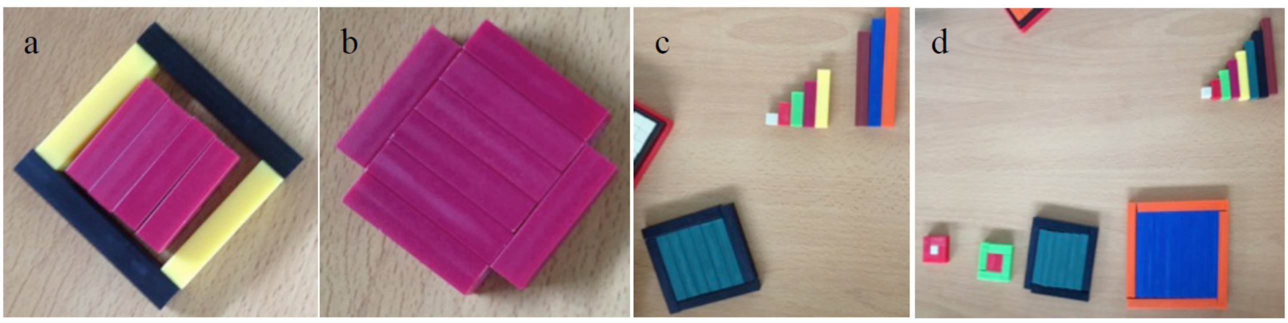

Figure 1: Entry phase trials

to the pond size. In their first three trials (Figure 1a-c) they focused on what it meant for the path to *surround* the pond. They used the information from the first two partially successful trials (Figure 1a-b) to inform their third trial (Figure 1c). This is similar way to in which Papert (1980) described children using the outcomes from their programming in LOGO to fix bugs in code.

The girls' provisional use of representation during the entry phase seemed to impact on their capacity to work with mathematical uncertainty and to adopt a fallibilist approach. Any trials that resulted in failure to meet the criteria set out in the activity, for example those depicted in Figure 1a and 1b did not appear to decrease their engagement or persistence with the activity. Their capacity to work with mathematical uncertainty facilitated their self-regulation and the application of their learning from apparently unsuccessful trials. Emily and Lucy showed no indications of fear, anxiety, bewilderment or reticence that can accompany the beginning of mathematical exploration, when least is known and understood about the problem. Conversely, they seemed highly engaged; they were leaning forwards, constantly exploring the parameters of the problem through their manipulation of the Cuisenaire rods and they alternated between quiet individual construction of examples and paired dialogue to share and develop thinking. The girls portrayed a relaxed appearance during the entry phase; their approach had a sense of playfulness and exploration that could be likened to the unstructured play that Dienes (1964) describes in his Dynamic Principle and this seemed to enable them to experience mathematical uncertainty in a positive way.

During the construction of their third trial, the pair created an ordered arrangement of all ten Cuisenaire rods to serve as a reference of relative lengths and support selection (top right of Figure 1c). In so doing, they noticed that they had selected consecutive rods to create the 6^2 pond and its path. This led them to form the conjecture that began to articulate the relationship between the two dependent variables:

Lucy: I think it will be if you use 1 [for the pond] then it will be 2 [for the path], if you use 2 then it's going to be 3, so it's [the path] going to be 1 higher than your square number

By the end of the entry phase they had constructed and ordered four examples (Figure 1d). They appeared to create each example by randomly selecting a Cuisenaire rod and using this as the basis to create one example; this use of random specialisation typifies the entry phase trials (Mason, Burton, & Stacey, 2010). This facilitated cognitive developments that enabled the girls to notice and formulate conjectures about the emerging patterns between the width of the pond and side length of path and to begin to articulate this relationship.

Hence, during the entry phase, the provisional way in which the girls used representations seemed to foster the emergence of affectively enabling responses and this enabled cognitive developments in mathematical reasoning. The impact of the girls' provisional use of representation during the entry phase is depicted in Figure 2.

The impact of the intervention in the attack phase

The transition to the attack phase was indicated by the girls' use of systematic specialisation (Mason, Burton, & Stacey, 2010). Having organised the data generated through random specialisation into an ordered sequence (Figure 1d), the girls then used the provisional nature of their representations to create gaps between the examples, apparently to identify and accommodate missing data. They then represented all the ponds in an ordered sequence from 1^2 to 9^2 using Cuisenaire rods (Figure 3).

The girls then switched to a more permanent representation in the form of a table (Figure 4). This representation does not simply illustrate total amounts relating to pond size and path lengths. Rather, it includes significant detail relating to the mathematical structures that underpin the relationship between the dependent variables of pond size and path length. Each example of the pond described its width squared, its total value and the odd/even property of this to-

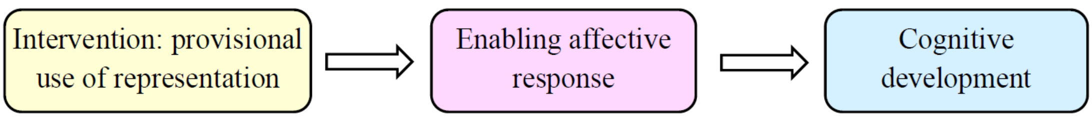

Figure 2: Impact of the intervention during the entry phase

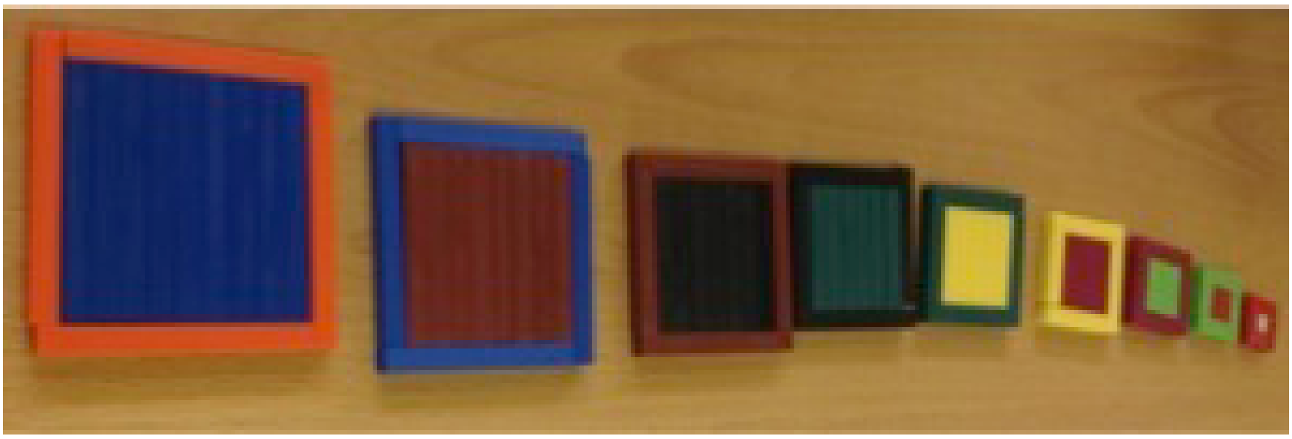

Figure 3: Systematic representation of ponds with widths 1–9

tal. Each example of the path is similarly described by side length multiplied by 4, the total value of the path length and the even nature of these totals. The girls also noted that each total was a multiple of 4. Interestingly, they realised that their recording had not been totally consistent in representing the x4 aspect of the path side length and this led them to underline the x4 component. Whilst there was no evidence in this lesson that the girls became overtly stuck, and hence no necessity to overcome this, they did persevere in formulating and articulating the reasoning for the patterns they observed. Emily's original response to the challenge of explaining the patterns they had identified resulted in a sentence that she was initially unable to complete:

Emily: All the paths are in the four times table. They have to be in the four times table because...

The girls persisted and utilised their understanding of the structures they had identified to formulate their reasoning for the observable patterns. This is captured on the right of Figure 4. In the post-lesson interview, the girls re-visited this:

13 Emily: We noticed about the path, because there's 4 sides to the path, we need 4 sides of the path, so you need to times it by whatever number the length of the path is. So then it's the 4 times table because there

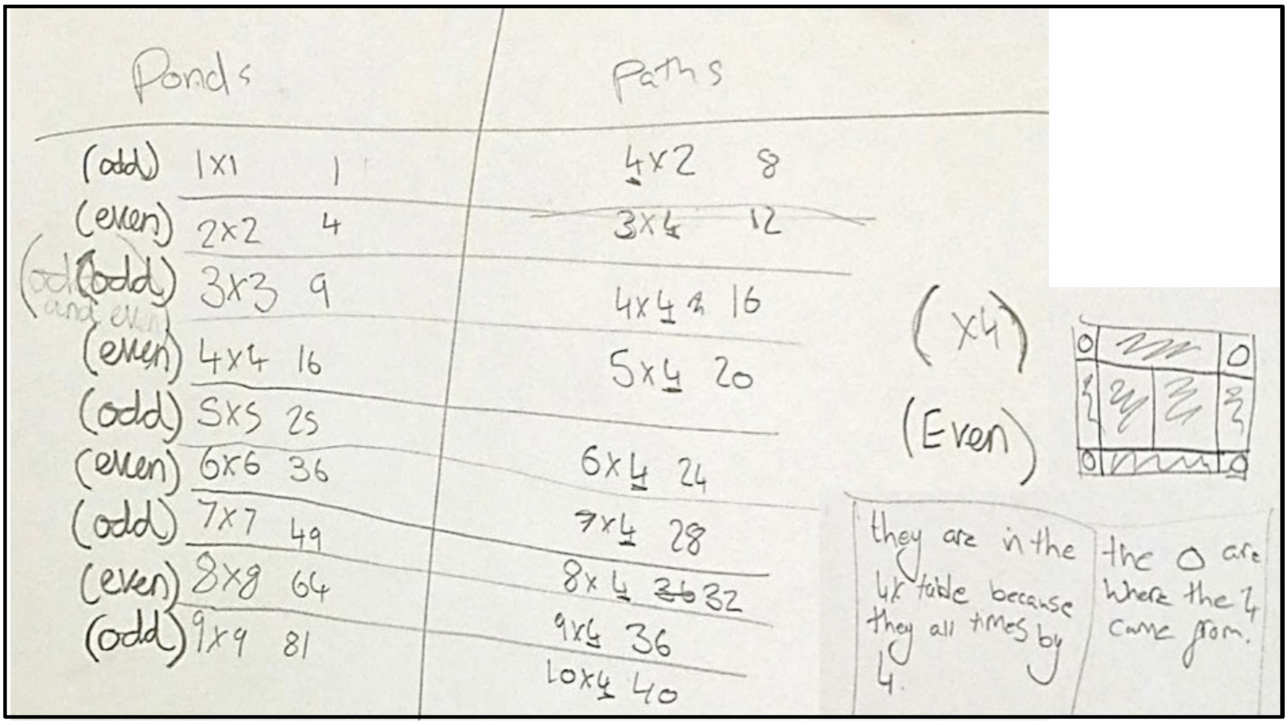

Figure 4: Lucy and Emily's table of findings

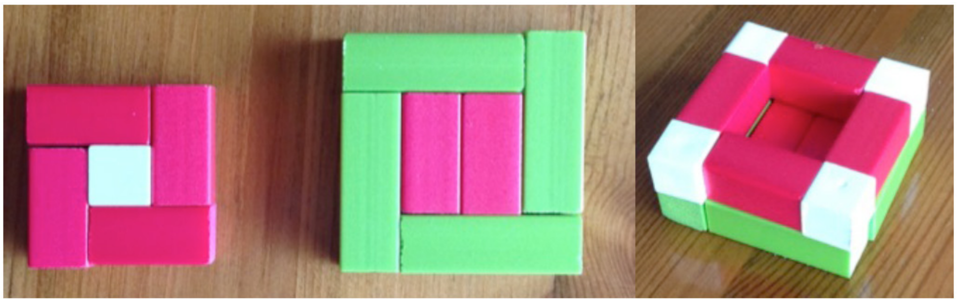

Figure 5: Representations created to support reasoning in line 69

are 4 sides and all of them, the numbers are even because they are all in the 4 times table

69 Lucy: Because it expands so you need to add 4 each time you go up

The diagram on the right of Figure 4 supports the reasoning expressed in line 69. In the interview, the girls re-created this image using Cuisenaire rods; Figure 5 shows how the path surrounding the 12 pond is positioned on top of the path surrounding the 22 pond with the gaps at each corner filled by four rods, each of length 1. There are similarities between the representations drawn in Figure 4 and constructed in Figure 5 and the girls' second trial (Figure 1b); the initial provisional explorations using the Cuisenaire rods, and in particular the example in Figure 1b seems to have helped the girls to understand the structures underpinning the growth of the path size. This understanding enabled Lucy to articulate the reasoning in line 69. The depth of understanding and the extent of the reasoning that the girls achieved resulted in positive affective responses. As in the entry phase, both girls remained highly engaged in the activity throughout the attack phase and took every opportunity presented to talk with the teacher about their findings and seemed eager to share the reasoning that they were constructing.

In the evaluation meeting following the Research Lesson, the teacher reported the impact of the girls' provisional use of representations during the attack phase on their cognitive and affective domains:

18 Teacher: I think [the provisional use of representation] helped them explain their reasoning more and therefore that helped them sustain their interest because they could explain more, because they had something to work from, to explain with. Their level of reasoning was amazing.

96 Teacher: [Lucy's] very proud of the work she's done [in the project]. I only have to mention it and a smile spreads across her face.

108 Teacher: I have seen some improvement in [Emily's] perseverance and resilience [...] in the past she would very much continue to follow a path even though it was wrong [...]. She's been able to stop mid way and realise it's wrong and have to go back to the beginning.

In line 18, the teacher exclaims about the level of the girls reasoning. In the baseline lesson, the girls were able to notice and articulate patterns, but not reason about why these occurred, hence there was a significant contrast with the extent and depth of their reasoning between the baseline lesson and the second research lesson.

The teacher also makes two connections in line 18. First, she makes a link between the girls' provisional use of representation and their articulation of mathematical reasoning. Second, she perceives that the positive cognitive developments contributed to the girls' sustained engagement and curiosity. The impact on Lucy's affective domain appeared to continue beyond the Research Lesson. Lucy's apparent sense of pride (line 96), suggests that she may have experienced developments in mathematical intimacy; that she was emotionally engaged and achieved a sense of satisfaction and self-worth through her cognitive mathematical activity (DeBellis & Goldin, 2006). Line 108 suggests that Emily may have increased her capacity to actively self-regulate (Malmivuori, 2006); this perhaps arises from developments in her capacity to work with mathematical uncertainty which may have arisen through working in a provisional way.

It appears that the provisional use of representations in the attack phase impacts first on the cognitive domain and second on the affective domain; a reversal

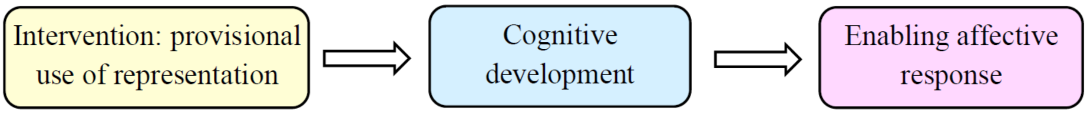

Figure 6: Impact of the intervention during the attack phase

of the processes emerging in the entry phase. This relationship is depicted in Figure 6.

CONCLUSION AND NEXT STEPS

This study sought to develop a practical intervention to improve children's perseverance in mathematical reasoning. The girls' provisional use of Cuisenaire rods appeared to have an enabling affective impact during the entry phase. This facilitated cognitive developments in reasoning as it supported them to behave in an exploratory way, to make and learn from trials, work with mathematical uncertainty and begin to formulate conjectures. In the attack phase, their provisional use of representation seemed to enable the girls to develop systematic approaches to their creation and organisation of trials. This led to their noticing patterns, understanding the underpinning mathematical structures, and using this to persevere in formulating reasoning. It seems that positive bidirectional interplay (Di Martino & Zan, 2013) between affect and cognition, facilitated by the intervention, resulted in improved perseverance in mathematical reasoning. A tentative analytic framework detailing these interactions and synthesising Figures 2 and 6, is depicted in Figure 7.

In the next phase of this research, I plan to work with two classes of children aged 10–11 in different schools to further test the impact of the intervention on children's perseverance in mathematical reasoning.

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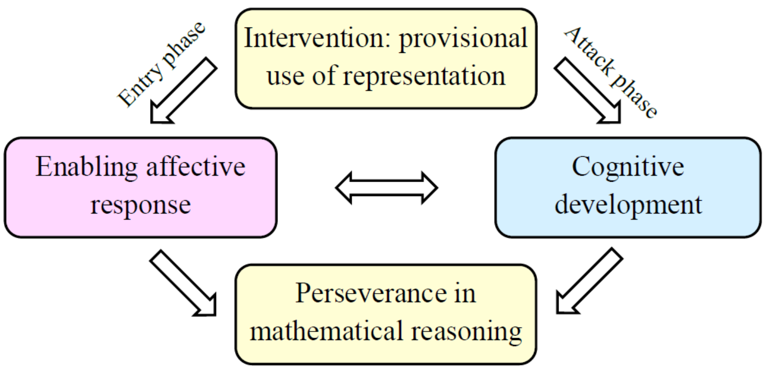

Figure 7: Tentative analytic framework

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Analysing teachers' belief system referring to the teaching and learning of arithmetic

Katinka Bräunling[1] and Andreas Eichler[2]

1 University of education Freiburg, Freiburg, Germany, katinka.braeunling@ph-freiburg.de

2 University of Kassel, Kassel, Germany

In this paper, we want to discuss the structure of teachers' belief systems. Firstly, we discuss teachers' belief systems from a theoretical perspective including characteristics of beliefs systems like its cluster structure, the centrality of beliefs or the hierarchy of beliefs. Afterwards, we analyse the beliefs of one primary teacher emphasising particularly the structural aspects of this teacher's system of beliefs concerning the teaching and learning of arithmetic. Finally, we discuss potential benefits of investigating a belief system in detail. We conclude the paper with a brief summary and suggestions for further research.

Keywords: Arithmetic, belief system, central and peripheral, primary and derivative.

INTRODUCTION

Teachers individually define their way of teaching. Thus teachers decide what mathematical content they bring to the classroom and they decide how they teach a specific content. A teacher's individual reasoning why he/she selects specific content and why he/she prefers a specific teaching style could be understood as a notion of a teacher's system of beliefs (Eichler & Erens, 2015), in which beliefs represent the intersection of the cognitive and motivational aspects of a teacher's mathematics related affect (Hannula, 2012). Lerman (2015, p. viii) states that, "exploring teachers' beliefs and their development are important topics in themselves". However, the importance of gaining knowledge towards mathematics teachers' beliefs has also been emphasised by many researchers since teachers' beliefs about mathematics and the teaching and learning of mathematics potentially have a high impact on the teachers' classroom practice (Philipp, 2007).

As a prerequisite of analysing both, the development of teachers' beliefs and the enactment of teachers' beliefs, there is a demand to investigate the complex structure of teachers' beliefs in detail. For example, referring to the change of teachers' beliefs, Liljedahl, Rolka and Rösken (2007, p. 280) state that a "deeper analysis of beliefs in the context of mathematics teachers' professional growth is needed to penetrate the surface stories of the data and reveal the nuanced and situated belief structures that are often hidden". The mentioned belief structures are also understood to potentially unfold the relation between teachers' professed beliefs and the teachers' enacted beliefs; this relation is not completely explained yet (Furinghetti & Morselli 2011). For example, Wilson & Cooney (2002) suggested that only those professed beliefs would be enacted that are central for a teacher and, thus, they suggested that unfolding the structure of a teacher's belief system could explain inconsistencies. As an important aspect of an in-depth analysis of mathematics teachers' beliefs, research results imply to consider the discipline-specifity of teachers' beliefs (Franke et al., 2007). Consistently, our own research yield considerable differences of teachers' beliefs about different disciplines like e.g. arithmetic (Eichler & Erens, 2015).

Two aims of this project are to analyse the development of mathematics teachers' beliefs and the extent in which these beliefs are enacted in the teachers' classroom practice. For this reason, following the considerations outlined above, we firstly tried to "penetrate the surface stories" (Liljedahl, 2007, p. 280), i.e., to analyse the structure of the teachers' beliefs in depth. After outlining our theoretical framework, we discuss the method of analysing teachers' belief structures or rather teachers' belief systems in depth. This analysis is the main focus of our paper. We further outline the results of our analysis for one teacher. This teacher is part of a sample of 20 mathematics teachers of primary schools involved in our research project. We conclude this report summarising our findings and suggestions for further research.

THEORETICAL FRAMEWORK

Following Pajares (1992) the term beliefs represents an individual's personal conviction concerning a specific subject, which shapes an individual's way of both receiving information about a subject and acting in a specific situation. The specific subject in our research is the teaching and learning of arithmetic. We decided to focus on this specific subject due to two reasons: Firstly, arithmetic is the main subject for the primary teachers of our sample. Secondly, a focus on a specific mathematical discipline is in our research an important aspect for facilitating an in-depth analysis of mathematics teachers' beliefs. As a specific form of beliefs we regard the teachers' beliefs about content, beliefs about ways of teaching or beliefs that represent a teacher's teaching goals (c.f. Eichler & Erens, 2015).

The internal organisation of the mentioned beliefs is called a teacher's belief system (Green, 1971). A belief system is mainly characterised by three aspects:

1) "Beliefs can be either *central*, which means strongly held, or peripheral, which means less strongly held" (Philipp, 2007, p. 260).

2) A belief system could consist of different clusters that are connected in at least quasi-logical ways. This means that different clusters of beliefs could be isolated, but different clusters of beliefs could also be contradictory (ibid.). For example, central beliefs are not necessarily connected to peripheral beliefs. Further beliefs referring a mathematical discipline like arithmetic could be contradictory to beliefs referring another mathematical discipline (Eichler & Erens, 2015)

3) Beliefs systems could be organised hierarchically including primary beliefs and derivative beliefs (Green, 1971). If teaching goals are regarded (see above) a primary goal could be to prepare students to solve problems in their future life. A derivative goal could be to follow a cognitive guided instruction (Franke et al., 2007). A relation between primary and derivative goals must not be necessarily logical in an objective sense (quasi-logicalness, see above). Further, it is noteworthy that "primary beliefs might not necessarily be more central than the associated derivative beliefs" (Philipp, 2007, p. 260).

Research in mathematics education has reported specific clusters of beliefs that refer to different features of the perception of mathematics in general (Dionne, 1984). Based on these two reports Grigutsch, Raatz and Törner (1998) distinct four views that could describe teachers' beliefs about mathematics in general but also teachers' beliefs about the teaching and learning of arithmetic:

— A formalist view stresses that arithmetic is characterised by a logical and formal approach. Accuracy and precision are most important.

— A process-oriented view is represented by statements about arithmetic being experienced as a heuristic and creative activity that allows solving problems using different and individual ways.

— An instrumentalist view places emphasis on the "tool box"- aspect which means that arithmetic is seen as a collection of calculation rules and procedures to be memorized and applied according to the given situation.

— An application oriented view accentuates the utility of arithmetic for the real world and the attempts to include real-world problems into class.

Further we refer to a global distinction of two different ways of teaching mathematics or arithmetic, i.e., a "cognitive constructivist orientation", and a "direct transmission view" (Staub & Stern, 2002, p. 344). We assume that these two different orientations are two ends of a continuum with three points of orientation: constructivism, co-constructivism and transmission (Strohmer et al., 2012).

METHOD

The sample consists of 20 primary teachers. However, in this paper we restrict the discussion to one teacher, i.e. Mrs. A (a young teacher from south of Germany), and her beliefs referring the teaching and learning of arithmetic. This restriction is not based on specific characteristics of Mrs. A, but is based on the main aim of this paper, i.e. to discuss an in-depth analysis of the structure of teachers' belief systems.

We collected data in two different ways: Firstly, we used a semi-structured interview including clusters of questions referring to arithmetic content and goals of teaching arithmetic as well as goals of teaching mathematics, students' learning of arithmetic or materials used, e.g. textbooks. In addition, the interviews incorporated prompts to evaluate given arithmetic tasks or fictitious statements of teachers or students that represent one of the views mentioned above.

Secondly we used a questionnaire referring to teachers' views (Grigutsch, Raatz, & Törner, 1998). Participants were asked to rate every item (e.g., item 5: Everyone is able to invent mathematics or rather to re-invent mathematics) using a 4-point Likert-scale. We adapted this questionnaire by changing the focus to beliefs referring arithmetic (new item 5: Everyone is able to invent arithmetic or rather to re-invent arithmetic), but also used Likert scale. Finally, we used the existing scale of Strohmer and colleagues (2012) to measure the teachers' teaching orientation that we also adapted for teaching arithmetic. We conducted the following three steps for analysing the data.

First step of data analysis

We used a qualitative coding method (Kuckartz, 2012) that is close to grounded theory (Glaser & Strauss, 1967) to analyse the data of the verbatim transcribed interviews. We used deductive codes derived from a theoretical perspective like 'application oriented' belief and inductive codes for those beliefs we did not deduce from existing research (Kuckartz, 2012). We will describe different inductive codes later in the result section.

Second step of data analysis

We weighted the deductive codes with 1, 2, -1 or -2 due to the following rules:

- If a teacher mentions a goal without a precision we weighted the code with 1.
- If a teacher explains a belief more deeply we weighted the code with 2.
- If a teacher refused a belief without explanation, we weighted the code with -1.
- If a teacher refused a belief with explanation, we weighted the code with -2.

One aim of this step of the data analysis was to develop quantitative evidence for the results that we gained through interpretation of the interview transcripts. For example, if the weighted sum of codes referring to a specific view is much higher than for another view, this could serve as evidence for a different grade of centrality of these two views. Further the sum of the weighted codes facilitates the comparison of the analysis of the interview and the analysis of the questionnaires. The deductive codings as well as the inductive codings were conducted by at least two persons and we found the inter-rater reliability to show an appropriate value.

Third step of data analysis

The four subscales of the adapted questionnaire of Grigutsch, Raatz and Törner (1998) yielded four sums of ratings referring to the application oriented view, the process oriented view, the instrumentalism view and the formalism view. We compared the distribution of these four subscales to the sum of weighted codes referring the same views. To facilitate the comparison we used standardised distribution of the rating sums and the sums of the weighted codes. To compare both standardised distributions we used a correlation coefficient and other measures on association (e.g., Kendalls Tau-b) in an exploratory way and, further, used a U-test for proving differences between the two distributions.

RESULTS

As mentioned before, we restrict the focus to one teacher, Mrs. A, and her belief system to give a comprehensive picture of the structure of one belief system.

Process orientation as a central belief of Mrs. A

Mrs. A expressed coherently a process oriented view. For example, to the question of her favourite style of teaching arithmetic and her preferred methods she answered:

> "Truly, it is important that they are able to find the solutions on their own, that they can work individually (...) that they can solve problems, that they can work on open tasks, that they can find their own strategies."

Statements of pupils:

2 I like mathematics because there is a connection to real life problems.

1 I like mathematics because hard nuts must be cracked and difficult problems can be solved.

4 I like mathematics because many exercises can be solved by similar procedures/patterns.

3 I like mathematics because the logic is clear and it follows strict mathematical rules.

Figure 2: Prompt: What would you like for pupils to answer?

Later, nearly the same answer ensued when she was asked about pupils and their way of learning arithmetic:

> "It is always important for me, that it comes from the pupils themselves, that it includes a problem, I like giving pupils problem statements."

Again, being asked to the question, which goals she would like to reach with her arithmetic lesson, she answered:

> "And then there are strategies, i.e. to be flexible, to adapt oneself to something new. Therefore, you need the right attitude that you have the confidence to try something you don`t know and to put effort into it."

The three quoted episodes referring to different topics of teaching arithmetic, i.e. the teaching style, students' learning and teaching goals give evidence that beliefs representing the process oriented view are central in the belief system of Mrs. A.

Further, the prompts given during the interview contain a process orientation. For example, Mrs. A was asked to arrange eight given teaching goals into a hierarchy. Figure 1 shows her arrangement of these goals for arithmetic lessons, where Mrs. A valued problem solving and process orientation as the most important goals.

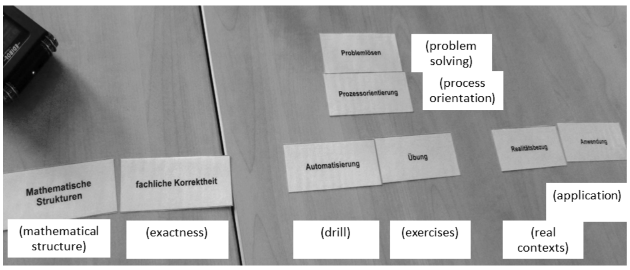

Figure 1: Arrangement of goals for arithmetic lessons by Mrs. A

In Figure 2 we show a further prompt consisting of students' statements representing the four views towards mathematics. The teachers were asked to arrange the statements from most desired (1) to least desired (4) if the statements represent arithmetic. Mrs. A preferred the second statement representing the process orientation.

Just as the professed beliefs the responds to the prompts referring teaching arithmetic give strong evidence that process orientation is central for Mrs. A.

The results of the second and third step of analysis (sum of weighted codes; questionnaire) are shown in Figure 3 where both distributions are standardised.

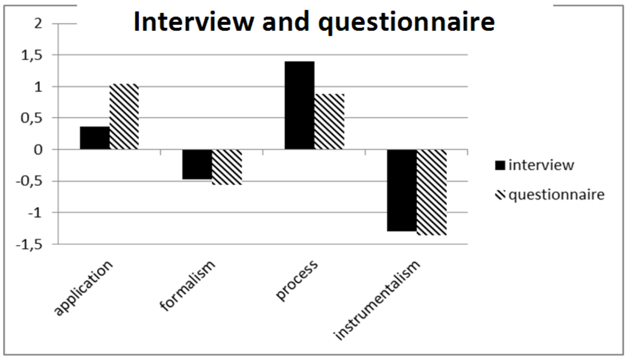

Figure 3: Weighted sum of codes and questionnaire scores

Looking at the figure it is obvious that application and process are more accepted than the other two orientations. The interview results as well as the questionnaire results imply this assertion. In conclusion application and process are central in Mrs. A's belief system. The high degree of coherence in different parts of the interview, the sum of weighted codes and, finally the questionnaire underline that the different instruments all measure the same.

Application as a derivative belief

Although application oriented beliefs are central for Mrs. A, however, her answers concerning the application oriented view gave evidence that application oriented beliefs are derivative beliefs. Thus they seem to be subordinated to process oriented beliefs. That means that application is in some sense a central teaching goal but rather a means to an end for another primary belief:

> "The relation to reality is important too, as I said before referring to money and time, but it doesn´t has to be highlighted all the time. Today, for example, I just gave them a mathematical problem..."

Application oriented goals seem not to be derived from process oriented goals in a logical way. However, both (clusters of) beliefs are central and application oriented beliefs are subordinated to process oriented beliefs.

Formalism and instrumentalism as peripheral beliefs

Although formalism is mentioned in some parts of the interview, it just has a peripheral meaning in the belief system of Mrs A. She names the importance of mathematical correctness once. Further, she emphasises that students should understand what they do in mathematics and why they do things. However, formalism seems to be a peripheral belief of Mrs. A and in some sense a desirable but peripheral result of process orientation as the following quotation illustrates:

> "Apart from that the process oriented training is more useful, because there is a little bit more reflection and control, so you can see what you are doing and why you are doing it!"

While Mrs. A talks positive about process, application and formalism when she regards her teaching of arithmetic, she often refuses instrumentalism. It is mentioned and it is denoted as important, but instrumentalism often receives a negative connotation, e.g. when Mrs. A talks about "mindless practicing" or when she says "you just have to do it this way":

> "I realize this drill is just mindless practicing, but in some ways it makes sense, for example when you want to establish an algorithm, yes, then you just have to do it this way."

The critical comments referring the instrumetalism view of Mrs. A show that this aspect is peripheral in her belief system. Referring to instrumentalism, we did not find evidence for a clear relation to another view.

Concluding the analysis of the belief system of Mrs. A the process orientation is central. Also application orientation is central in her belief system but, however, subordinated. Thus, application oriented goals could be understood as a means to an end to reach process orientation. Formalism is a positive connoted peripheral belief that represents a desired but peripheral result of process-orientation. Instrumentalism is also a peripheral but isolated belief with a negative connotation (Figure 4).

The belief cluster of process orientation

Mrs. A's central belief, i.e. process orientation, is closely connected to a set of further beliefs and, thus, could be understood as a belief cluster consisting of several defining beliefs. These defining beliefs are a result of inductive codes which were formed during the analysis of the interview transcript. We illustrate only two beliefs that constitute the belief cluster of process orientation. The first belief concerns *comprehension* that Mrs. A explains the benefit of open word problems - so called Fermi tasks - that are based on individual models:

> "Comprehension must stand on the top and you can reach it while giving tasks with a problem solving context. This has not to be a big Fermi task but also you can just confront the pupils with something and then see what they do.

The second quotation represents the belief *flexibility*. Here, the word strategies illustrate the closeness to process orientation:

> "...they [the children] should not be afraid of numbers. They should be flexible and fit in their head. And this brings us back to the strategies, they can learn with the help of the "half-written" calculation."

As well as process orientation also application, formalism and instrumentalism are belief clusters. In Figure 4 we show some of the beliefs that for Mrs. A define the four belief clusters.

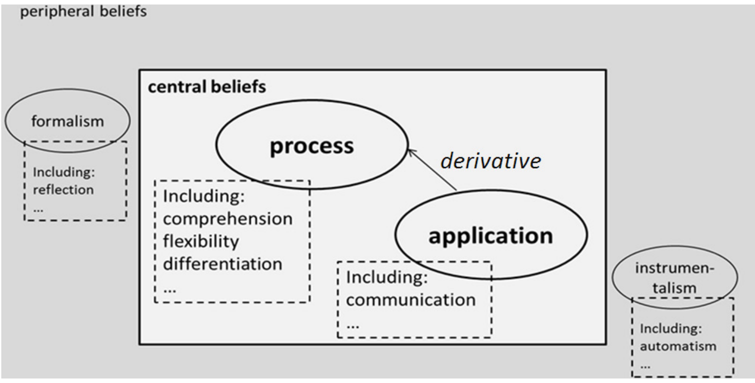

Figure 4: The belief system of Mrs. A

Statement of the pupil	Wish of the teacher (1 is best, 3 is worst)
„I can learn best, when I can choose my work completely on my own and when I am allowed to work independently on it!"	1
„I can learn best, when my teacher tells me what to do and when he helps me immediately when I have any questions!"	3
„I can learn best, when I can choose from different offers and when my teacher supports me when I need help!"	2

Figure 5: Prompt "wish of teacher"

(Co-)Constructivism as a central belief

In the same way as it was shown above, we investigated beliefs that concern more general the process of teaching and learning. Referring these beliefs, a (co-)constructivist orientation is central for Mrs. A:

> "Well, it should be oriented on the pupils, the teacher should abstain himself, so that there exists a high amount of work the pupils do, it should not only be training but also the teacher should give them credit so that they can work independently..."

The centrality of this orientation is also shown in the following prompt referring to learning arithmetic. Here Mrs. A should rate three different statements of pupils. The figure illustrates that it is important for her that the pupils work independently.

As discussed to the results referring the arithmetic related beliefs, the qualitative interpretation, the sum of weigthed codes and, finally, the results of the questionnaire yield very similar results also referring to the different ways of teaching (Figure 6).

Direct transmission view as a peripheral belief

As Figure 6 illustrates for Mrs. A, a direct transmission view only appears peripheral. This is also shown in her statements to her experience in sitting in on classes:

> "... and I have seen this in secondary school, [...] but it is so different and boring. The older teacher stand in front of the class, they lead the way and the pupils replicate and practise. And we learned quiet different things, for example at the university – much more pupil activity, much more problem solving and open lessons."

Most of the time Mrs. A emphasises that she wants the pupils to make their own experiences and that she does not want to stand in front of the class and tell the pupils how things work. Still there exist a few episodes where she talks about the relevance of teacher-centred teaching:

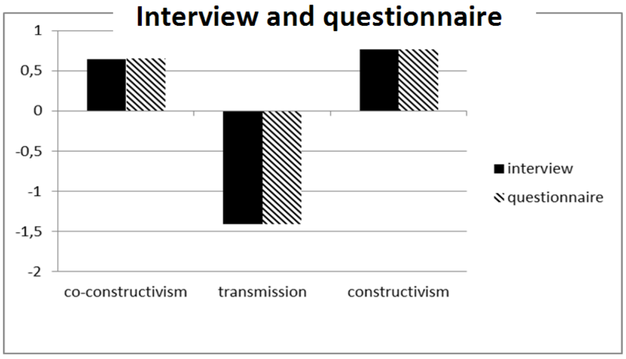

Figure 6: Weighted sum of codes and questionnaire scores

"...so teaching should be effectively and therefore it also is important to teach the pupils sometimes in teacher-centred teaching."

These examples show that a direct transmission view is not rejected but peripheral in the belief system of Mrs. A.

DISCUSSION

Looking at the results we can approve some of the aspects which characterise a belief system referring to Green (1971). Thus the structure of a mathematics teacher's belief system referring to the mathematical subdomain arithmetic contains beliefs with different centrality. The beliefs can be central (here: application and process orientation) or peripheral (here: instrumentalism and formalism). Further, the beliefs are hierarchically arranged as primary beliefs and derivative beliefs. In the case of Mrs. A the application orientation is a means to an end to facilitate process orientation. Finally, we identified different belief clusters, e.g. the cluster process orientation which contains for example comprehension or flexibility (cf. Figure 4). The theoretical statement that central beliefs are not necessarily connected to peripheral beliefs (Philipp, 2007) was pointed out in the connection between the central and peripheral beliefs of Mrs. A. Formalism could be understood as a peripheral result of process orientation, instrumentalism however has no connection to this belief.

Our results can be used to compare teachers from different type of schools and with different background concerning their professional career. However, the main reason of the in-depth analysis of a teachers' belief system that we discussed in this paper is to provide a basis that facilitate further research, i.e. investigating both the relation between professed beliefs and enacted beliefs and the development of teachers' beliefs.

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Identity and rationality in group discussion: An exploratory study

Laura Branchetti[1] and Francesca Morselli[2]

1 University of Palermo, Palermo, Italy, branchetti.laura@gmail.com
2 University of Torino, Torino, Italy, francesca.morselli@unito.it

In this contribution we study group work in mathematics adopting a socio-cultural perspective and combining two theoretical lenses: the construct of identity and that of rational behavior. More specifically, we show how individual identity and dimensions of rationality in interaction may influence conceptual change. Data analysis is performed on excerpts from a group work (age of the students: 12) on negative numbers.

Keywords: Group work, identity, rationality.

INTRODUCTION

Group work as a methodology is often advocated in mathematics education, and its value is often taken as obvious by researchers and also teachers. Nevertheless, working in group does not immediately turn into a search for a common solution. What happens to the students when they see the others doing something different from them? Does every student care about the agreement with classmates? Research may help figuring out the possible causes of failure of group activities. Conversely, research may also show to those teachers, who ask their students to provide a common solution for group activities, how difficult is for students to reach on their own an agreement and how far is a solution from being "common" and accepted by all the members of the group. In this contribution we study group discussion adopting a socio-cultural perspective and combining two theoretical lenses: the construct of identity and that of rational behavior. By means of a networked analysis, we aim at better understanding what happens during group interaction, in particular what makes a group interaction efficient or not for students involved in a task-based activity designed with the aim of stimulating a conceptual change.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Group work and sociocultural perspective

The pioneering analysis carried out but Vygotskij (1978) concerning the crucial role of social activities mediated by signs and language in the development of mathematical thinking traced a path in which a whole thread of researchers placed their roots. The effects of students' interactions in classroom activities have been studied, described and interpreted in mathematics education since its origins, see Radford (2011) for an overview. We rely on a sociocultural perspective, according to which the learning of mathematics takes place in a social context through interactions and is deeply affected by culture (Radford, 2006; 2011). Radford (2006, p. 58) affirms:

> Certainly, the students were actively engaged in what has been termed a "negotiation of meaning". But this term can be terribly misleading in that it may lead us to believe that the attainment of the concept is a mere consensual question of classroom interaction. [...] meaning also has a cultural-historical dimension [...]. It is in fact this cultural object that shapes and explains the teacher's intervention [...] classroom interaction and the students' subjective meaning are pushed towards specific directions of conceptual development. Cultural conceptual objects are like lighthouses that orient navigators' sailing boats. They impress classroom interaction with a specific teleology.

Students are involved in a double-faced problem: they meet at the same time the culture and the others and have to find a place in both the cultural and the classroom discourses, that are related but not necessarily equal. In particular we will focus on the classroom discourse side. Even if the attainment of a concept is not a mere consensual question, the agreement between students is very important in mathematics group activities, as we will show. We refer to Radford's interpretation of cultural-historical activity theory (Roth, Radford, & Lacroix, 2012). This theory is rooted in Leont'ev and Vygotskij's dialectical psychological theories. The keyword *activity* is defined as a common place in which cognition and consciousness arise; through activity individuals relate not only to the world of objects but also to other individuals. Learning is the result of a shared common practice that involves students' subjectivities and in which subjectivities moves towards others and culture to find and transform themselves. In Radford (2008) it is pointed out that students' interaction in a small group is a complex process in which students are involved at many levels, not only at the cognitive one. The processes of objectification (students align their thoughts with culture) and subjectification (a *thinking and becoming* process of being-with-others mediated by alterity) that take place in the teamwork are mediated by culture.

Conceptual change

In our perspective, learning is a *thinking and becoming* process in which students are involved not only at the cognitive level, and a continuous negotiation of meanings between one individual and another mediated by culture (Radford, 2008). We conjecture that conceptual change, as part of mathematics learning, cannot be seen just as "change of concept" at cognitive level: it is a social act deeply related to the subjectivities of students involved in group activity. Drawing from the definition of concept as an emergent object that condenses or generalizes the previous practices (Sfard, 1991), we speak of *group concept* as an object emerging from the individual or shared activities, whose validity is recognized and accepted by all the students.

Identifying and subjectifying

In the thread of Leont'ev and Vygotskij's dialectical psychological theories presented before, we analyze group activity by means of one analytical tool that was introduced by Heyd-Metzuyanim (2009). The author presents a useful tool to distinguish the different ways of interacting of each student in terms of individuality, in particular in a mathematics group work. The tool "allows to point out how identity and emotional processes influence the effectiveness of learning. Subjectifying may help in mathematizing or obstruct it" (Heyd-Metzuyanim, 2009, p. 2). The subjectification process is linked both theoretically and operationally to the identity construction process and to the mathematizing activity in group work. The starting point is the definition of identity (Sfard & Prusak, 2005, p. 1): "Identity is a set of reifying, significant, endorsable stories about a person." This definition is deeply related to the *commognitive perspective* (Sfard, 2008), whose cores are the notion of thinking and communicating. Since thinking is a form of human doing, it can only develop as a collective patterned activity: "Thinking is an individualized version of (interpersonal) communicating." (Sfard, 2008, p. 81). Heyd-Metzuyanim frames mathematizing and subjectifying in the commognitive perspective: mathematizing is communicating about mathematical objects, subjectifying is communicating about participants of the discourse. In this perspective the construction of identity takes place as an internalization of collective discourses that make stories about the self arising. These stories can talk about the way in which a person relates to the mathematics and so can influence the participation in the teamwork, the engagement, and definitively, success or failure in mathematics activities. In her work, Heyd-Metzuyanim (2009) looks at verbal and non-verbal acts of subjectification, distinguishing participation and membership. She operationalizes the notion of *resistance to participation*, seen as a type of subjectifying action always interpreted according to context, especially to the reactions of other participants, especially the teacher. Then she analyzes these acts, deciding if they are identifying processes or not. Identifying utterances (verbal or non-verbal) are "those that signal that the identifier considers a given feature of the identified person as permanent and significant." (Heyd-Metzuyanim, 2009, p. 2). The prototypical cases of different aspects of the relation between subjectifying, mathematizing and identifying are exemplified in Table 1.

Rationality

The construct of rationality was presented by Habermas (1998) in reference to discoursive practice and later adapted to mathematical activity (see Morselli & Boero, 2009 for the special case of mathematical proving). According to Habermas, rational behaviour may be seen as three interrelated dimensions: epistemic dimension (related to the control of the propositions and their chaining), teleological dimension (related to the conscious choice of tools to achieve the goal of the activity) and communicative one (related to the conscious choice of suitable means of communication within a given community).

Identity, rationality and conceptual change

We hypothesize that the conceptual change is a social act and that acceptance of this change may depend on a group-coherence, i.e., a sort of agreement reached using cultural tools provided by the teachers or personal tools. Subjectification and participation can be considered at the same time stimulus for engaging in a group conceptual change process and an obstacle in the individual conceptual change process. In order to describe individual contributions within the group work, we add the construct of rational behavior. Since we will deal with peer interaction, the communicative dimension will have a relevant role. Moreover the epistemic rationality, encompassing the possibility of changing opinion, seems to be linked to group work and conceptual change:

Someone is irrational if she puts forward her beliefs dogmatically, clinging to them although she sees that she cannot justify them. In order to qualify a belief as rational, it is sufficient that it can be held to be true on the basis of good reasons in the relevant context of justification - that is, that it can be accepted rationally. The rationality of a judgment does not imply its truth but merely its justified acceptability in a given context. (Habermas, 1998, p. 310)

RESEARCH QUESTIONS

We wonder whether identity and rationality may have a role in influencing (positively or negatively) the conceptual change that may occur during group problem solving activity. The initial research focus on group work can be turned into the following research questions: 1) Is it possible to describe group interactions in terms of identity and subjectification? 2) Are there cases of resistance to participation? 3) Is group conceptual change an act of social agreement, and this necessity of agreement influence the individual conceptual change? 4) What is added by the analytical tool of rationality?

METHODOLOGY

The task

We present and analyze data from a teaching experiment carried out in grade 6 (age of the students: 12). The teaching experiment concerned the concept of negative numbers. At first, students were asked to answer individually these questions: *1) What is a number? 2) What is it possible to do with numbers?* Afterwards, they worked in group on a task to be solved on the Cartesian axes. The negative part of the axes was used by the students and such a solution was after institutionalized by the teacher. Finally, the students were asked to answer in group to the following questions: *3) You said that numbers are [reference to their former individual answers]... and with numbers you can do [reference to their former individual answers]... Do you confirm your opinions now? 4) Negative numbers are numbers in the sense you intended before?*

The aim is to analyze the conceptual change that occurs when moving from natural numbers to negative numbers to whole numbers, that is to say a wider set that contains both positive and negative numbers. Here we confine to one episode referring to the group work on questions 3 and 4.

Using the analytical tools

We perform a networked analysis, combining different theoretical tools with the aim of reaching a fuller understanding of the episode at issue. The lens of conceptual change allows to characterize the acts of mathematizing analyzed in our research. The first analysis aims at detecting verbal and non-verbal acts that are signal of participation or resistance to participation, membership or non-membership, mathematizing, identifying. The lens of rationality can provide information about the rationality dimension of these sentences. The joint analysis addresses the topic of relating mathematizing and subjectifying acts to rationality.

At first we analyze the episode in terms of identity, subjectification and conceptual change, as derived from Heyd-Metzuyanim's paper (2009). Some sentences from the transcript are interpreted following the criteria proposed by the author in Table 1 and labeled with the codes: identity (I), subjectification (S) and conceptual change (C). Also the specific codes for subjectification used by Heyd-Metzuyanim will be used. Afterwards, we add the analytical tool of rational behavior. We refer to the epistemic dimension when one sentence is linked to a mathematical fact, and we speak of lack at epistemic level if some assumption is taken per se, without the need for a justification. For instance, in the very first part of the working group one student says that negative numbers are to be considered numbers ("*Because she (the teacher) spoke us of negative and positive*"): she is just relying on the authority of the teacher, without paying any effort towards a real understanding. We may say that her statement lacks in terms of epistemic rationality. We refer to the teleological dimension when the action is clearly linked to a goal (and we report a lack in teleological rationality when the reference to the final goal is missing). We refer to communicative rationality when a special care is paid to the organization of the discourse, so as to make the listener to understand. For instance, one student's wide use of drawings and diagrams may be linked to her effort in making her positions understandable to others, thus to a communicative dimension.

DATA ANALYSIS

At the beginning of the group work, students discuss to decide whether it is possible to deal with negative number as numbers. The selected episode refers to a connected crucial issue: whether it is possible to perform operations with negative numbers.

Analysis in terms of identity (I), subjectification (S) and conceptual change (C)

In the subsequent part, we present an example of analysis of one episode in terms of identity (I), subjectification (S) and conceptual change (C), with more specific codes used by Heyd-Metzuyanim (2009). Some labels are assigned even if in the transcript the utterances are not recurrent, because they are repeated many times in the whole transcript.

Other isolated episodes allow to characterize students' behaviors in terms of identifying and subjectifying acts and to link students' way to attend the group work and conceptual development. *Nor* always works on her own, never speaks, even if the teacher asks to do it. Every non-verbal act performed by *Nor* may be read in terms of *resistance to participation*. 6 on 8 students look for agreement but with different aims: some of them to convince the classmates, some to make the work of others coherent to their personal discourse, other to be accepted from the classmates rather than thinking at mathematical contents (as student *Ari*). From the analysis we grasp different ways to participate in the group, that affect personal concept development: *Mar* and *Nor*, working on their own, develop personal concepts; *Ari* and *Luc* abandon their conceptual change to find an agreement; *Bea* and *Er* follow the group conceptual development and only ask for clarifications; *Giu* imposes her point of view on all the group affirming her personal opposition to conceptual change. She identifies clearly herself as good in math and influences the whole process, even when she's not right. The agreement is not reached and the group conceptual change doesn't occur. This failure drives all the students but *Giu* to abandon

	Type			Example	Identifying?
Mathematizing	Mth				No
Subjectifying	Pe: participation evaluating	Sp: related to a specific performance		Saying "I don't understand"	No
		Ge: generalizing		Saying "I hate doing this"	If consistent with other data
	Me: membership evaluating	Vb: verbal	Di: direct	Saying "I am a math person"	Yes
			Id: indirect	Changing the subject of discourse, which can be interpreted as "I don't want to talk about this"	Depends on the nature and frequency of the utterance
		Nv: non verbal	Di: direct	Raising one's hand, which may interpreted as "I wish to speak about this"	Only if recurrent
			Id: indirect	Groaning at a given task, which may be interpreted as "I don't' like this"	Only if recurrent

Table 1: Prototypical cases and labels (Heyd-Metzuyanim, 2009)

11 *Giu*: it must be for all the operations, and then -3 times -2 is equal to?	*Giu* is mathematizing but also identifying as good in math. (I) [Me Nv Di] *Giu* provokes a change in the discourse generality: "If it's true, it has to be always true, for every kind of operation" (C)
12 *Ari*: just a minute, I wrote: This means that it is negative number, but anyway you can do -2+-3 [in column] you get -5.	
13 *Teacher*: and how do you get it?	
14 *Ari*: the sign means that it is a negative number, then I can do... that sign does not mean anything! It just means that it is a negative number, so...	*Ari* is mathematizing and suggesting a path for a conceptual change from numbers as natural to numbers as positive and negative (C).
15 *Giu*: no, A, what you are saying is meaningless.	*Giu* introduces the question of the sense of the operations with negative numbers. This change the status of the following statements (C). She also talks about *Ari* (S) [Pe Sp] and confirms *Ari*'s identity as person not good in math, as she did for the whole discussion saying that as usually she doesn't understand (Is) [Me V Di]
16 *Ari*: yes.	
17 *Giu*: -16:-4? [laughing]	*Giu* laughs when she doesn't agree with *Ari*'s proposals, showing self-confidence in math (I) [Me Nv Di]
18 *Ari*: -16? You must do 16 -4 [after, she adds the minus before the numbers, in column].	
19 *Giu*: try and do the division. The division.	*Giu* is mathematizing but also identifying as good in math and saying to *Ari* what she has to do (I). [Me Nv Di]
20 *Ari*: ok.	*Ari* interrupts her activity accepting *Giu's* request (S). [Pe Nv Sp]
21 *Luc*: wait a minute! [she takes the pen].	*Luc* wants to find a place in the discussion (S) [Me Nv Di]
22 *Giu*: let me speak!	*Giu* wants to participate (S) [Me Nv Di]
23 *Luc* [she does -4 divided by -2, and she writes -2]. Here you are! -4 divided by -2 is -2.	*Luc* tries to answer to *Giu*'s provoking question but doesn't satisfy *Giu*'s request (C).
24 [*Ari* does again 16:4].	

Table 2: Transcript and analysis

their personal path to conceptual change because of group disagreement. So neither the group conceptual change nor the individuals occur and the first event seems to be cause of the second.

ANALYSIS IN TERMS OF RATIONALITY

The theoretical lens of rationality is used to gain understanding of interactions. Here we confine our analysis of *Giu* and *Luc's* excerpts, with a special focus on the part of discussion on the possibility of performing operations with negative numbers. A first result is the crucial role of teleological rationality. *Giu's* initial position is that negative numbers are not numbers as positive ones and she does not want to modify her position. According to this goal (*teleological rationality*), she challenges the propositions of her groupmates. Her interventions are mainly on the *epistemic* level: she suggests that, in order to have negative numbers a numbers, it must be possible to perform operations, and she asks for justification and meaning for such operations (interventions 15, 19). *Giu's* requests to the groupmates are at epistemic level, she pretends justification and meaning. *Luc's* initial position is that negative numbers are not numbers like positive ones (she sees the number and the minus as separate objects). Nevertheless, she is keen to change her opinion: her priority is to gain a group solution. According to this aim (*teleological rationality*) she puts great effort in sharing her ideas with the mates, for instance using diagrams (*communicative rationality*) (intervention 26). When challenged by *Giu*, she is ready to find solutions (e.g. how to perform operations; see intervention 23) and she is rapidly satisfied. She does not put much effort in justifying or giving meaning to the methods. In her exchanges with *Giu* the communicative dimension is prevailing; she does not move to the epistemic level, then she does not come to an agreement with *Giu*.

DISCUSSION AND PRELIMINARY CONCLUSIONS

The first part of the analysis shows that the students participate in the group work in different ways. There are even two cases of *resistance to participation*. We argue that different participation sometimes affects personal concept development and, of course, hinders conceptual change as a group. At the end of the session, there is not a general agreement. We wonder why it was not possible, in spite of individual good ideas, to reach an agreement and why some students abandon their attempt if groupmates don't agree. We turn then to the theoretical lens of rationality. The analysis in terms of rationality shows that teleological rationality may refer to different goals and that some interventions are clearly on communicative or epistemic level. Combining the two analysis, we can state that individual *participation or resistance to participation* and also *membership or non-membership* may be described in terms of *dimensions of rationality*: if individual interventions are on different levels (epistemic vs communicative), it seems very difficult to reach an agreement. If a dimension prevails, some students can avoid to participate. Moreover, individuals may have different aims and act accordingly (teleological rationality), may consider the epistemic dimension or not, and this may affect individual/collective conceptual change. We hypothesize that group work, in order to be efficient, should take into account the three dimensions (in particular, the epistemic dimension can not be neglected); moreover, group interactions are not fruitful if the groupmates focus on different dimensions.

The preliminary results of this study suggest further research. From one side, we plan to analyze other data (including long-term observations of the students), in order to test our working hypothesis concerning the link between identity, conceptual change and rationality. Moreover, we see other issues that need further exploration:

1) The link between identity and teleological rationality brings to the fore the relationship between identity and goals. This could also be linked to the work of Gómez-Chacón (2011), who draws from Camilleri and colleagues (1990) the idea of identity strategies as "processes or procedures set into action (consciously or unconsciously) by an agent (individual or group) to reach one or more goals (explicitly stated or situated at an unconscious level); procedures elaborated in function of the interaction situation, that is in function of the different determinations (socio-historical, cultural, psychological) of this situation" (p. 24).

2) The role of the teacher is crucial in helping students to interact at the same level; furthermore, we hypothesize that other kind of tasks, for instance aimed at comparing individual solutions rather than providing immediately a group solution, could be more efficient.

3) Finally, we wonder whether there is a link between identity and rationality: more specifically, we wonder whether the resistance to participation depends on the dimension of rationality that most characterizes the identity.

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Invented or discovered?

Eleni Charalampous

University of Cambridge, Faculty of education, Cambridge, UK, ec455@cam.ac.uk

Does mathematics pre-exist and hence is discovered or is it invented and owes its being to humans? What do students believe and how does this interact with their beliefs about the production and the meaningfulness of mathematical knowledge? This paper presents results based on 18 Greek students' interviews about their relationship with mathematics through an epistemological lens. The findings diverge from what the literature suggests especially with respect to whether mathematics is perceived as a meaningful human activity and to what extent it produces certain and fixed conclusions. Ideally educators could foster beliefs which promote students' engagement and understanding of mathematics.

Keywords: Mathematics ontology, epistemology, existence.

INTRODUCTION

The ontology of mathematics is a hot debate in the philosophy of mathematics. The key question is whether mathematics pre-exists or comes into existence through human activity. Does mathematics transcend humans or is it simply yet another sector of human knowledge. The question is complicated with respect to mathematics because it is entangled with its epistemology. Although mathematical concepts may not appear to be materially substantiated – at least not in the same sense that a table is – mathematical conclusions have long been endowed with a certainty that would be strange to assume for any creation of the human mind (Hersh, 1999).

Moreover, it seems that mathematicians and mathematics educators do not share the same views on this issue. Most mathematicians tend to embrace the belief that mathematics is independent of the human mind. On the contrary, most educators advocate the belief that mathematics is constructed by humans (Sfard, 1998). Research has generally associated the belief that mathematics pre-exists with traditional teaching practices. Teachers who view mathematics as an independent entity would present mathematical knowledge as fixed. Consequently, their role is to transmit it to the students while the latter's role is to passively absorb it. Educators, of course, opt for an active engagement of the students (Lerman, 2002). However, there have been great mathematicians (e.g., Hardy, Gödel) who have been actively engaged with mathematics and who have done wonders holding the belief that educators dread.

Consequently it is contestable what we would like students to believe about mathematics' ontology. Should they follow the steps of great mathematicians or will this render them passive learners? Nevertheless, before aiming at such a question, we need to know more about students' beliefs on this issue and how they affect the student's relationship with mathematics? Although there has been abundant research in students' beliefs about mathematics (e.g., Schoenfeld, 1992) the issue of ontology seems to have been neglected. This paper focuses on it, investigating the second of the above mentioned questions in the traditional teaching context of Greece.

THEORETICAL FRAMEWORK

The distinction between finding something that already exists and something that is novel is captured by the verbs 'discover' and 'invent'. We discover something that already exists the same way that Columbus discovered America. To the contrary when we invent something it owes its existence to this very process of invention[1].

The predominant opinion in the history of the philosophy of mathematics speaks of discovery. This tradition may be traced back to Plato and has been called Platonism[2] after the philosopher. Platonism is nicely captured in the words of the mathematician G. H. Hardy who maintains that

> mathematical reality lies outside us, that our function is to discover or observe it and that the theorems which we prove ... are simply our notes of our observations. (1967, pp. 123–124).

This is an ontological assertion related to the 'mode of existence' of mathematics. However, it has been this ontological assertion that underlain the predominant epistemological conviction about the certainty of mathematical knowledge. Mathematical truth is absolute and objective since the truth of any mathematical statement is judged against an extra-human mathematical reality.

Nevertheless, many modern philosophers reject Platonism as an absurd idea; we can see and touch the physical reality, but where is this purported mathematical reality (Hersh, 1999)? If Platonism is rejected, then mathematics can no longer be discovered. Mathematics is now claimed to be invented, and again an ontological conviction is coupled with an epistemological claim. Mathematics does not exist and mathematical knowledge becomes fallible. Lakatos (1976) argues that no proof guarantees the truth of the theorem it proves; there is always the possibility of a hitherto unknown counterexample which will refute the theorem's generality. Moreover, Paul Ernest (1991) presents mathematics as a socially constructed field of knowledge; there is no longer a need to assume an external mathematical reality and no longer a need for this craving for certainty.

Paul Ernest also relates this to mathematics education. If mathematics is invented it acquires a human face. It is not a timeless, unerring entity which imposes itself on students. It is only a human creation and students can re-invent it through the process of learning. Consequently, mathematics could become meaningful for students as a product of a human activity. Nevertheless, mathematics seems to retain this potential even if it is discovered. According to Galileo 'the book of nature is written in the language of mathematics' and understanding the world around us has always been meaningful to humans.

In any case, philosophy of mathematics suggests that it is hard to disentangle ontological from epistemological beliefs about mathematics. Therefore, in the following I also discuss epistemological beliefs of the students, but only in relation to the main question of ontology.

METHODS

This article reports some preliminary results of a study investigating epistemic beliefs of Greek students at the last grade of upper secondary school (17–18 years old). The study follows a qualitative interpretivist paradigm. Twenty eight students were interviewed twice. The interviews investigated their relationship with mathematics through an epistemological lens touching upon subjects such as truth, certainty, logic, rules and usefulness and comparing mathematics to other courses or to life in general.

Before the second interview was conducted, the first one was transcribed and used as a stimulus for a further and more detailed discussion. Effectively, generating questions for the second interview with a particular student was influenced both from that student's first interview and earlier first interviews; while later first interviews were also affected by this process. The duration between the two interviews varied between 10 days to one month and on average each interview lasted 70 minutes.

All students come from the same middle-class school of Athens. Practical reasons limited the research to this school where access was easily granted. However, the interviews revealed such a variety of beliefs that including other schools in the sample was not judged necessary.

The analysis is still in progress. All interviews have been transcribed and the two interviews of each student have been paired. The second interview is regarded as a continuation of the first one and each pair is analysed as a whole. So far I have worked with the paired interviews of 18 students in a chronological order. As a first step each of them was read as a story trying to identify the main factor or factors which marked the student's relationship with mathematics. This initial reading revealed that the main points of each interview could be organised as a cohesive narrative around these factors. The factors were very diverse (e.g. doubt, theory, mistakes, fiction). However there were broad themes which appeared repeatedly in most of the narratives. The factors may be seen as different ways to colour such themes.

One of the themes is the ontological status of mathematics. This paper focuses on it in connection to epistemological issues of mathematical truth and certainty, and to meaningfulness of mathematics for the students. The results that follow are organised around the concepts of discovery and invention. They are based on the interviews of eighteen students, who here have been given pseudonyms.

FINDINGS

Discovery

Some students maintained that mathematics exists. For example, Platonas, maintained

> When in the past, they tried to interpret a phenomenon … they needed mathematics, in a sense they, not created it, in a sense mathematics was there, but they, that is, they discovered it, yes.

Of course, most students had a difficulty explaining how mathematics exists. Nonetheless, their belief was usually not shaken and even when it was, they still found it hard to coordinate this with their experience.

> Yes, mathematics isn't something ordinary that you can say you discover, it is a way of reasoning. . . . It's invented, now that you mention it, but it isn't that we came up with mathematics, now you'll ask me who did? (Aspasia)

A dubious concept was imaginary numbers. However, although most of them admitted that they are invented, they retained their Platonistic beliefs.

> Yes, imaginary numbers are called imaginary exactly because we invented them. However, in general mathematics is discovered. (Xenofontas)

> But mathematics hasn't been created. It's been discovered in the sense that, okay apart from some things which we have made in order to help us, in general mathematics is something that exists. (Foivos)

Mathematics was perceived to exist around us. It started from observing objects around us and it ends in explaining phenomena around us.

> It's just that based on . . . numbers, humans defined that a certain object, this is the 1, this is the 2, and so slowly they discovered that around them there are groups of identical objects. So then they started doing operations, and this led after many years in the invention[3] of theorems in order to justify phenomena that occurred around them. (Filia)

The paradox is that although discovery implies that mathematics is independent of human beings it also brings mathematics close to human beings. If mathematics is out there in the physical world then it is something quite intimate and not just some weird figment of imagination.

> I know that it isn't impersonal and that everything is based on it. . . I've thought about it. In order to construct something the mathematics which made it is needed . . . so I'm grateful to mathematics. (Foivos)

None of the Platonists doubted that mathematics has applications in our lives.

> the exercises, for example, they have applications on things that we want to find. . . for example, we have an, an equation and we want to know the result … for something that will help in our daily lives. (Filia)

Mathematics was important exactly because it explains our world and otherwise it wouldn't have been so developed.

> No, [mathematics] would exist, but … we wouldn't have discovered it to the extent that we have discovered it now. (Ermis)

In all, mathematics was meaningful. Moreover, human agency was not absent with respect to mathematical discovery. After all, it is people, mathematicians, who produce mathematics. This could justify why students, who generally endorsed Platonism, sometimes utilised phrases which would hint at invention while describing mathematics as a human activity. Further justification is provided by the fact that invention succeeds anyway in penetrating mathematical activity. At least we did not find symbols in the world; we only agreed to use them in order to denote what we did find in the world.

> As you go backwards you'll eventually reach the basis, an axiom of the kind 1+1=2. . . This is so because *you have defined it so*. (Foivos)

> I believe that it was *an initiative and an inspiration* of those who started all this. (Patonas)

Other common beliefs were that mathematics may change, but the change is incremental. Essentially, change is better perceived as development, an enlargement of mathematics when new data are discovered.

> Yes I believe that if some needs lead to an extension of mathematics, then new rules will be discovered . . . on the basis of the old ones, of course. (Platonas)

> No, this is a development . . . and complex numbers, which they didn't know, they discovered them. And it emerged through, now I remember. . . I think through physics, the issue of light. (Ermis)

New propositions complement the old ones. All of them believed that mathematics essentially comprises one system.

> I don't know [if we could have defined things differently] because whatever we have defined we have defined it based on our universe, based on some things that we observe. (Foivos).

> No, [it can't be different]. Mathematics is in a way the explanation of what we see. It's something natural, that is, you have one apple and another apple, so you have two apples, it can't be something else. (Xenofontas)

Different sub-systems may exist but they do not cancel each other; they co-exist as different models of the same reality. The old models suffice for certain cases, while the new ones explain new data which cannot fit the old model.

> No, [Euclid] wasn't wrong. It's just that when they examined it deeper and with more cases . . . they suggested that other things may also happen. (Platonas)

The belief in one system and incremental change of mathematics is also reflected in their belief that there is a unique absolute truth which we may not be able to find, but which we slowly approach. Mathematical conclusions are part of this truth.

> Truth is one-sided. . . I believe that new things are continually discovered. That is, soon we'll have learned much more; now we're still in the darkness. (Aspasia)

> [The proof] is essentially the tangible evidence that a proposition that you have assumed is true. (Platonas)

Interestingly though, Platonism did not exclude verification of mathematics through fallible social processes.

> Somebody says an idea, 500 people agree, 600 disagree and in the end one of the 600 finds something else or they simply agree because one of the 500 proves that it holds for additional reasons which the first one had not found. (Foivos)

This is reminiscent of Lakatos' *Proofs and Refutations* rather than Plato. However, it is not in opposition with Platonism per se. If mathematics is external to humans it can remain infallible even though their attempts to discover it are not. So, Platonism allows for certainty in mathematics even if people are not entirely certain about it.

> When I think about mathematics and somebody shows me something, that this must be done, [then] I'll think why it mustn't, I will examine it. . . Therefore, so far: yes, I'll accept the results of mathematics, but always having also in mind the doubt that something else may hold. (Ermis)

Invention

Most students suggested that mathematics is invented.

> [Mathematical conclusions] are unshakable because they are stable, that is, they don't change. You'll tell me that some of them change, but they have been checked, as I mentioned before. It has been supported that they are unchangeable, that is, their value is permanent. (Platonas)

> Generally, I don't believe that mathematics exists as a material idea, that is, you can't touch it. (Diomidis)

> In mathematics there is 'if this holds then it's done so'. That's all there is. Or 'let', 'let this be'. . . Assumptions of the mind. (Evyenia).

> It's a human creation. . . I think that when you prove something, you essentially make the rule. (Pelopidas)

The paradox in this case is that although invention implies that mathematics is part of the human intellect it may also create a gap between mathematics and the individual. This depends on whether the invention of mathematics is meaningful to the student. There were students for whom mathematics was deeply meaningful, students for whom mathematics had some worthwhile meaning and students who struggled to find any meaning in mathematics.

> Yes [mathematics] is standardised . . . but this has another beauty. (Loukianos)

> Yes, I belong to the couples who though separated I still love [mathematics]. (Litha)

> Mathematics is completely theoretical, that is, the logic that it has, it won't produce . . . something crazy, that is, it won't be something that I can use in my everyday life, that's why I don't hold mathematics in great estimation. (Kosmas)

In the first case students had at least a feeble idea of axioms and perceived mathematics as something that humans have invented based on initial assumptions in order to suit their needs.

> It doesn't mean that they hold necessarily, we just have created things so that they . . . improve our everyday life. (Lysimahos)

> the world of mathematics is as we define it, that's why there are different geometries . . . And geometries, all that exist, they were created with the intention of solving some problems. (Kleomenis)

In the second case mathematical invention was perceived as some sort of experimentation. Mathematics was invented as applications corroborated some assumptions.

> basically everything has an experiment. Because in order to find something new, for example, you must try it out. This is called experiment. (Lida)

> I think that they solved many times an exercise or type of exercise . . . that they were reaching at the same conclusion repeatedly, so . . . then they said to make it a rule . . . Not that they deliberately tried to make a rule, I believe that it just appeared. (Diomidis)

Finally, in the third case invention appeared to be the result of the lack of meaning.

> I'd say pre-existed, pre-existed? It didn't pre-exist, it's all human investigation, I believe. (Kosmas)

> That is, someone would have imagined all these, to someone all these came; it can't be just like this. (Evyenia)

Some students in the third group seemed to perceive mathematics as some people's personal views. These were students who held a highly relativistic view about life.

> They should ask Pythagoras. . . [Me having an opinion on his theorem], essentially it's like me going and saying something with respect to a view of Socrates. (Klio)

> Okay now, it would be somehow [strange], if we said for each [person] that they don't think correctly (Evyenia)

In all, only two students who chose invention believed in a unique truth, and even these did not believe that we had access to it. Moreover, they were both students who did not find mathematics meaningful.

> we are just people, each of us is just a unit, If we could see the world from above then we would be able to judge that this is a definite truth, this is a definite lie. (Kosmas)

Certainty was much more moderate among students who maintained that mathematics is invented. However, it was present especially in the cases when mathematics was also meaningful – even moderately – to them. Some of them found certainty in the exact process of invention, but almost all of them grounded it on social reasons too. Nevertheless, the process of invention itself was excluded from certainty.

> Because it's theory . . . basically there's no chance. . . in life, something may hold or may not hold . . . Well, no [it isn't strange that you don't find this in

> mathematics] because mathematics is theoretical. (Kleomenis)

> [We accept the first assumptions] because we get used to them . . . I think that there haven't been attempts to change . . . the foundations. . . So since they have results and validity in everyday life [we] continue using them. (Lysimahos)

> [What's proven] usually doesn't change . . . all the mathematicians have seen them, and they have been considered. . . but I think that within the university context . . . I think that there is more room to doubt them and to be demolished by someone. (Diomidis)

Certainty was absent only in cases when mathematics was not meaningful to the students. This could simply be due to under-confidence, but sometimes was inherent of a subjective view of mathematics. If certainty persists in this group then it is genuinely social.

> I wouldn't say that something said by mathematics is always true. . . you take cases and you assume, essentially, as we said before, 'let this be' or 'if that'. (Evyenia)

> I haven't seen anything different, only what I have been taught . . . they haven't shown to me something else in order to believe that it may not be this way. (Pelopidas)

What is special about social certainty is that it can remain intact even in the face of change because each time it includes exactly these truths which are believed to be certain.

> So until someone demolishes it, it's right, it's true. If it's demolished, then it's wrong. . . because it's truth, we accept the truth, but truth may many times be reversed with the presentation of new evidence. (Kosmas)

Nevertheless, certainty was not absolute, but the result of the scarcity of change or of the lack for necessity of change. Moreover, although past content was generally viewed as stable, it was not entirely safeguard against invention.

> Okay, there is a chance of mistakes, but I believe that most of them won't change. (Diomidis)

> If it changes then all the rest should change too . . . I'm not absolute about this not happening. I just don't think that it's possible to happen. (Danai)

Therefore, change is not necessarily incremental. Nevertheless, mathematics remained a unified system apart from the cases of utter subjectivity and of one student whose knowledge of axioms was more developed. Otherwise the system was one: what they have been taught in school.

> The most typical example is geometry. Euclid organised it anyway, but afterwards Riemann? Who was it? He didn't like it; he wanted to use, to show other things, and so he changed it. (Kleomenis)

> [Definitions may] not have the exact same words, they simply have the same sense. . . It can't be [that they don't have the same sense]. (Diomidis)

CONCLUSION

Although the students had learned mathematics within a traditional setting, most of them believed that mathematics was invented. However, it was within the context of invention that mathematics could appear meaningless to students. Contrary to what would be expected according to the literature (e.g. Simon et al., 2000), students who believed that mathematics is discovered also viewed it as a human activity. Their account of the discovery was given in social terms and echoed *Proofs and Refutations* (1976). Most importantly, the fact that mathematics existed was coupled with mathematics' ability to explain the natural world and it made mathematics meaningful. On the other hand, some of the students who saw mathematics as a human invention failed to find meaning in it. Moreover, it seemed that this failure almost forced the idea of mathematics as an invention; it was just somebody else's invention and they could not see themselves in it.

Furthermore, Platonism is also associated with the belief that mathematics is a static body of knowledge (Charalambous et al., 2009). Nevertheless, all students regarded mathematics as something that evolves. A static element appeared indeed among Platonists, but referred to past knowledge and it did not prevent new data amending this knowledge. Additionally, this belief was not restricted to students who believed in discovery of mathematics. It was generally endorsed by most students though it was socially tinged when mathematics was seen as an invention. Mathematics brings results so there is no reason to change it; old claims have been already checked by myriads of mathematicians and so on. Doubt, when present, seemed to be a general trait of the student's personality and the greatest doubter among my sample was a Platonist.

This picture of discovery or invention of mathematics as painted of the students of this study is quite different from the one usually forwarded by mathematics education. However, what seems more important is not what students believe about the being of mathematics, but whether they find meaning in it. If they do, then they will be willing to engage with it. It seems that Platonism may help towards this goal. It would also be interesting though to find the reasons which lie behind the divergent views of invention of mathematics. Some students do not find the invention meaningful. However, invention appeared to allow for a clearer view of the organisation of mathematical knowledge into axiomatic systems and thus a better understanding of mathematical epistemology.

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ENDNOTES

1. Invention and discovery do not represent a strict dichotomy neither in the literature (e.g. Livio, 2011) nor in my interviews. However, for issues of space I will focus on the two extremes and on the predominant view in each student's interview.

2. I use 'Platonism' as an umbrella term for all theories which postulate that mathematics somehow exists.

3. Filia used invention meaning 'we were aware of it, that we wanted to find something'.

Discursive positionings and emotions in a small group's learning of geometric definitions

Wajeeh Daher[1,2], Osama Swidan[3] and Juhaina Shahbari[1,4]

1 Al-Qasemi Academic College of Education, Baqa al-Gharbiyye, Israel, wajeehdaher@gmail.com
2 An-Najah National University, Nablus, Palestine
3 Kaye Academic College of Education, Beersheba, Israel
4 Sakhnin College of Education, Sakhnin, Israel

This research examines the discursive positionings and emotions related to them of a group of three seventh class students. We videoed the group of students' discussions regarding the definition of terms associated with the circle topic and interviewed them regarding their emotions during the process of defining the geometric terms. We used the discursive analysis of Evans, Morgan and Tsatsaroni to analyze the participants' positionings and emotions. The research results indicate that the learning atmosphere in the group was positive due to type of leadership that prevailed, as well as to the collaborative working with a technological tool. This atmosphere resulted in the students having positive emotions about their learning.

Keywords: Discursive affect, positionings, emotions, geometry, small group.

INTRODUCTION

Students' emotions have become a growing research area in educational psychology (Knollmann & Wild, 2007). Ingleton (1999) describes emotions as a powerful factor which encourages or inhibits effective learning. Furthermore, the affective aspect of students' mathematical learning (including beliefs, attitudes, and emotions) has a mutual relationship with the cognitive aspect of this learning (Sriraman, 2003). This puts studying the affective aspect of students' learning on the agenda of mathematics education research. Here we study students' positionings and emotions when learning geometry. To do that, we use the discursive positionings and emotions framework (Evans, Morgan, & Tsatsaroni, 2006; Morgan, Evans, & Tsatsaroni, 2002). The use of this framework enables us to analyse two aspects of students' learning that affect other aspects of this learning, namely the social and the emotions aspects. Moreover, using the framework, we can examine how emotions are related to positionings, an issue that has not been attended to widely. More specifically, the social aspect is studied through looking at students' positionings and their expressions in language, as the use of pronouns, which indicates whether the group member feels insider or outsider to the learning taking place. This feeling could also affect the member's emotions. For example, the insider probably feels content and satisfied by the learning taking place.

The discursive positionings and emotions framework draws on social semiotics, pedagogic discourse theory and psychoanalysis, and studies emotion as discursive positioning. The analysis of learners' mathematical positioning and emotions, according to this framework, takes into consideration positionings available to the mathematics learners through their learning practices, where those positionings enable and constrain the learners' emotions, and where emotions are considered as shaped by power relations. Few studies used this framework that takes care of two primary aspects of students' mathematical learning (the social and the emotional) to analyze students' positioning and emotions in geometric situations. We will attempt to do that, specifically, when a group of seventh grade students works with Geogebra to discuss geometric terms associated with the circle's topic. The discursive analysis, its basis and its phases are described in more detail below.

A discourse is a system of signs that provides resources for participants to construct social meanings and identities, experience emotions, and account for actions. Evans (2006) names the following functions of discourse: First, it defines how certain things are represented, thought about, and practiced; second, it provides resources for constructing meanings, and accounting for actions; and third, it helps construct identities and subjectivities that include affective characteristics and processes.

The discursive analysis of students' emotions and positioning has two phases: the structural and the textual. In the structural phase, learners' positionings are analyzed. Evans (2006) defines positioning as a process where a participant takes up and/or is put into one of the positionings made available by the discourse(s) at a specific context. This explains the mutual influence of the social and the individual, where the social setting makes available specific practices and thus positionings, and individuals retain a degree of agency that enables them to position themselves in available or created positionings. According to this framework, a person's identity, which includes more durable components of affect such as attitudes and beliefs, comes from repetitions of positionings, as well as their related emotional experiences that occur in the history of the participant (Evans, ibid). Evans, Morgan and Tsatsaroni, in their writings about discursive analysis describe the positionings taken care of in the structural analysis: Helper and seeker of help (helper positioned more powerfully), collaborator and solitary worker, director of activity and follower of directions (the latter less powerful), evaluator and evaluated, insider and outsider.

There is more than one available positioning for a participant, if in one discourse or in several discourses. Moreover, positioning is not permanent, not completely determined, nor freely chosen, where participants are constrained and enabled by their personal histories and the discursive resources available to them (ibid). Furthermore, in the 'progressive classroom', the positionings of the collaborator and insider are encouraged because they help advance students' learning of mathematics.

The second phase of discursive analysis (the textual analysis) has two functions (Evans, 2006): (a) showing how positionings in social interactions are actually taken up by the participants, and (b) providing indicators of emotional experience. Furthermore, in the textual analysis, indicators of interpersonal relationship and emotional experience are considered (Tsatsaroni, Evans, & Morgan, 2007). This analysis has two stages. In the first stage, the focus is to identify the interpersonal aspects of the text that establish the positions of the participants. Indicators at this stage include reference to self and others, reference to valued statuses (e.g. claiming understanding or correctness), modality (indicating degrees of un/certainty), hidden agency (e.g., passive voice) or repetition. The second stage of the textual analysis attends to (a) indicators of emotional experience generally understood/used within the (sub) culture: direct verbal expression (e.g., 'I feel anxious'), use of particular metaphors (e.g. claiming to be 'coasting'), emphasis by words, gesture, intonation, or repetition (indicating strong feelings), body language (e.g., facial expression or blushing); (b) indicators suggested by psychoanalytic theory, as indicators of defenses against strong emotions like anxiety, or conflicts between positionings (as 'Freudian slips'), surprising error in problem solving, behaving strangely (as laughing nervously), denial (e.g., of anxiety).

Using the two phases of the discursive framework, we analyzed the positionings taken by seventh grade students and their related emotions when developing collaboratively, with the help of GeoGebra, the definition of geometric terms associated with the circle topic.

Research questions

1) How are positionings taken up by middle school students, working in a group to define geometric concepts in the presence of technology?

2) How are students' emotions associated with the positionings that they take up when they define geometric concepts with technology?

3) How does technology affect students' positionings and related emotions?

METHODOLOGY

Research setting and participants

We analysed in the present research the affective aspect of the learning of a group of three grade 7 students. Following is a description of this group, where the description is based on the evaluation of the students' mathematics teacher.

The group consisted of Haya (A high achieving student in mathematics with strong personality), Janan (A high achieving student in mathematics with a sociable and friendly personality), and Rana (A middle achieving student in mathematics, who encountered learning difficulties due to family circumstances).

The three participating students did not work with GeoGebra before, and they were introduced to it in two hours' time. Furthermore, the students had learned the topic of the circle in the sixth grade, but they learned it then without GeoGebra.

Our analysis of one group's learning of geometric definitions attempts to shed light at students' positionings and related emotions, when they learn geometry with a technological tool. This analysis of just one group learning is consistent with previous studies that analysed different aspects of students learning (See for example Yerushalmy & Swidan, 2012). Nevertheless, we are aware that further research is needed to verify the results we arrive at.

Data collecting and analysing tools

We collected our data using observations of the group learning and interviews with its members. The group's learning was videoed and at the end of each lesson, the three students were interviewed individually regarding their positionings and emotions during learning. We analysed the two types of collected data using the discursive analysis framework presented above. Moreover, we combined the analyses of the data collected by the two tools.

Learning material

The group of seventh grade students worked with different activities, where they discussed the definitions of geometric concepts associated with the circle's topic. It was expected that performing the activities, the participants would deepen their knowledge regarding the concept of circle and its related concepts: circle's center, chord, radius, diameter, circle tangent, circle circumference and area. The activities were written keeping the explorative and discursive learning in mind. Below is an example on a question in the unit.

(a) We want to draw a circle using Geogebra.

(b) We want to draw a diameter in the circle.

(c) Manipulating the diameter, how can we define it?

(d) How many diameters are there in a circle?

(e) What is the relation of the diameter and the chord?

FINDINGS AND ANALYSIS

We describe here the different learning events of the geometric concepts associated with the circle topic, together with students' positionings and related emotions that prevailed in these learning events. We start from the learning events as we consider students' positionings and emotions associated with these events.

Difficulty in defining the circle's center in spite of the group members being collaborators

The first requirement of the activity was to define the circle's center. Haya initiated the exploration of the group (1), by telling the group's members (she and two other members) that they should follow the directions of the activity (1 and 5), and by using GeoGebra to drag the circle. Then she addressed Janan and Rana (the other two members of the group), and started to discuss the circle's center, but soon the conversation turned to be about the chord (6–11), the diameter (6–11), the secant (12–17) and the tangent (12–17).

Note: When describing the learning events, silence for m moments will be denoted by [..m..].

1 Haya: The circle's center is ....
2 Janan: it is the point lying in the middle of the circle.
3 Haya: the middle ...
4 Janan: It is the center.
5 Haya (again): The circle's center is ....
6 Janan: every chord that passes through it becomes a diameter.
7 Haya: a diameter? [..15..] What is a diameter?
8 Janan: it is this that passes through the circle.
9 Haya: it is this that passes through the center and the circle.
10 Janan: it is a line that passes through any part of the circle.
11 Haya: if it passes through the center it becomes a diameter [Haya uses the mouse to drag the circle and watch how the diameter and radius change] ... the secant is like ... it intersects the circle in two points.

12 Rana: the tangent surrounds the entire circle [Rana and Janan were looking at GeoGebra interface).
13 Janan (vehemently): Yeh [Haya dragged the tangent again and again].
14 Rana: the tangent is this that touches the circle.
15 Janan (again vehemently): it does not intersect the circle. It touches the outer line. [Haya continues to drag the tangent].
16 Janan (looking at GeoGebra interface with interest): when the secant touches the circle it becomes a tangent.
17 Haya: the secant is like … it intersects the circle in two points.

Haya played the role of the group director, though the whole conversation and actions seemed to be of collaborators more than of a director and two followers of directions. The collaboration occurred through asking questions and answering them, and through frequent attempts to agree on the definitions of the circle's center and other concepts associated with the circle. Haya seemed to be directing the activity, by two means: her persistence to ask questions and her use of GeoGebra to get new examples of the circle and its components. Haya's questions and actions led the group to improve their definitions of the concepts associated with the circle.

The facial expressions of the group members showed that they were enjoying their learning with GeoGebra as a group. This learning enabled them to improve, as a group (collaboratively) and on their own account (independently), their knowledge of the circle topic, which was represented in better statements about the diameter, the secant ant the tangent [interview]. Furthermore, the improvement in the group knowledge empowered them, which made them content and happy [interview].

The group turned again to discuss the concept of the circle center, as the following learning event shows.

The group's director effort to come back to the original activity

Haya continued acting as the group leader. She declared they need to write the answer of the first question (18). She repeated the center's definition given earlier by Janan (19), and advanced the discussion further by asking another question to make that definition clearer (20). As a response to the question, Janan once again tried to describe the center (21). Haya tried to overcome the group difficulty in defining the center by investigating further the issue through dragging the center of the circle using GeoGebra (22). She announced again the mission of the group. So, Janan added another property to her definition of the center (23). Rana, contributed to the discussion by repeating Janan's first description of the center (24).

18 Haya: We have not answered the first question yet. What is the circle center?
19 Rana: a point. [..15..] [Haya wrote: a point lying in the middle of the circle].
20 Haya: how can we assign a point in the middle of the circle?
21 Janan: what? [..15..] before the radius.
22 [Haya dragged the circle center and the group described what happened to the circle and its components] Haya: We want to define the center of the circle.
23 Janan: it is the base of the circle.
24 Rana: It is a point lying in the middle.

Janan's behavior indicates her interim positioning as a follower of directions (21, 23).The silence indicates the group difficulty in defining the circle's center. In the interview, the group members said that their silence hid their frustration and uncomfortability due to feeling powerless because of their difficulty, as a group, in defining the center of the circle. Haya's work with the applet emphasizes her leadership. Her use of the pronoun 'we' (18, 20, 22), indicates she was an insider, and Janan's immediate answer (21, 23) indicates that she too was such. Rana's participation (24) also indicates her interim positioning as a collaborator. It seems that Rana, being not a strong student in mathematics, lessened her collaboration in the group discussion and mathematical work, which made her at the beginning less of an insider than the other two girls. This also made her feel neither content nor comfortable [interview]. Nevertheless, this did not prevent her from interfering and correcting the other members of the group when needed, as the following learning event shows.

Trying to be an insider and get involved in discussing the circle's radius

Once again Haya moved the mathematical talk away from its focus - the circle center (25), this time to answer the next question in the activity about the radius of the circle. Probably she did that to change the negative mood of the group, becoming uncomfortable and frustrated because of their feeling powerless due to their difficulty to arrive at an accepted definition of the circle center. Janan tried, as before, to participate in the group's discussion (26, 29, 31), indicating she continued to look at herself as an insider.

25 Haya: How can we set the radius of the circle?
26 Janan: we extend a line from the center.
27 Rana: No, the radius.
28 Haya: the radius. How can we set the radius?
29 Janan: we extend a line from the center to the line of the circle.
30 Haya (pointing at the circumference in GeoGebra interface): to the circumference.
31 Janan: to the circumference, Yeh.
32 Rana: according to the diameter length. [The facial expressions of the group members showed frustration]
33 Haya: what is the radius of the circle?

Rana seems here trying to get power in the group, correcting Janan (27), and participating in the answering of the question about the setting of the radius (27, 32). Rana's contribution was not evaluated highly by the director of the group who again asked about the radius (33). This little acceptance of Rana's answer by the director of the group made her frustrated of her positioning in the group [interview]. Nevertheless, she tried to contribute again to the discussion, as can be seen from the following event.

Collaborating to define the diameter and discuss its relation with the chord: The power of knowledge

Haya tried to define the radius (34). She stated her definition hesitantly, as if not confident of it. In spite of her hesitation, the two other members accepted her claim by repeating it (35–36). Haya then turned to read the next question in the activity (37). The three girls collaborated to define the diameter and describe its relation with the chord (38–43).

34 Haya (pointing at the circle in GeoGebra interface and saying in a hesitant voice): we write like this: "the radius is a line that starts at the center and extends to the circumference".
35 Janan: the radius is a line that goes from the center to the circumference.
36 Rana repeats: From the center to the circumference.
37 Haya read the next question: what is the diameter of the circle? [The group members looked at the diameter that they drew].
38 Janan: It is a straight line that passes through the circle's center.
39 Haya read the next question: what is the relation between the chord and the diameter? She answered: the diameter is in origin a chord passing through the circle's center.
40 Janan: If it does not pass through the center it will be a chord, but if it passes through the center it will be a diameter.
41 Haya: the diameter is a straight line that starts at the circle's circumference and passes through the circle's center.
42 Rana: it ends at the second side of the circle.
43 Haya wrote: the diameter is a straight line that starts at the circle's circumference, passes through the center and continues to the other side of the circle.

Again we see that the three students worked as collaborators, which led to their agreement on one of the definitions of the circle's diameter. Here, Haya maintained her firstness through reading the question and writing its answer. The two other students also contributed to the common knowledge of the group, which gave them more power. This power resulted in the two girls' satisfaction, as their facial expressions showed, which encouraged them to keep participating in the group discussions.

DISCUSSION AND CONCLUSIONS

In this paper, we analysed the positionings and related emotions of a group of students when defining geometric concepts associated with the circle's topic. Doing so, we used the discursive emotions framework suggested by Evans, Morgan and Tsatsaroni. This framework is a promising one since it is appropriate for complex context. This is so though it is difficult to incorporate some important constructs in it like previous students' experience and current learning beliefs.

The research results indicate that the director of the group's learning claimed her positioning by means of different behaviors: initiating the exploration work of the group, telling the group members what should be done to answer the activity questions, demonstrating persistence in asking questions that investigate the geometric topic, and in manipulating the geometric objects in GeoGebra. At the same time, the director of the group claimed her positioning by regulating the group members' emotions to avoid their negative emotions associated with their difficulties to define geometric concepts and to facilitate their engagement with their learning (Fried, 2011). As a result of this emotion regulation strategy of the group director, the interpersonal functioning of the group improved (Gross & John, 2003). It can be said that the actions of the group director, in the frame of her positioning as such, are related not only to the emotional aspect, but also to the different aspects of the group learning: the cognitive (asking questions related to the circle topic), the meta-cognitive (regulating the group's advancement through changing the discussed topic), the social (advancing the discussion of the group), the behavioral (manipulating GeoGebra), and the meta-affective (regulating the group's emotions) aspects. So, the group director claimed her positioning by means of administrative means more than by means of knowledge, though she asked and answered questions, and tried her most to contribute to the group process of defining the circle concepts. Thus discursive power could be claimed by administrative means, in addition to other means described in (Evans, Morgan, & Tsatsaroni, 2006), as knowledge and giving help.

To direct the group learning, the group director generally used the pronoun 'we' to initiate a journey with the group (Dafouz, 2007) regarding the learning of one of the concepts associated with the circle topic. This use of the plural personal pronoun indicates that the group director was an insider (Evans, Morgan, & Tsatsaroni, 2006) who took the lead in making the group succeeds in investigating the geometric topic.

Overall, the group members worked as collaborators. The collaboration of the group was facilitated by the group conversation, so it could be said that the leadership in this group was conversational leadership, where this conversation was seen by the group members, especially the director of the group, as a core process for effecting positive change (Hurley & Brown, 2010), in our case learning change.

Being collaborators, the group members worked with GeoGebra to further their study, were curious to move forward with their geometric investigations and were content and satisfied due to the power they acquired as a result of their collaborative knowledge advancement with the help of GeoGebra. Thus the technological tool facilitated their collaborative investigations of geometric concepts, empowering them as mathematics learners, and, as a result, causing them to have positive emotions about their learning of geometry. Moreover, in spite of these positive emotions, the group members had negative emotions when they had hard time defining the geometric concepts associated with the circle. These negative emotions were caused due to their feeling powerless not able to agree on the definition of the geometric concepts. The group members overcame their negative emotions by manipulating the geometric objects in GeoGebra and thus arriving at agreed definitions of the geometric concepts. Thus the technological tool empowered the group members, changing their negative emotions to positive ones. Furthermore, the technological tool not only empowered the group as a whole, but also empowered members who controlled the work with it, like Haya.

The group members' level in mathematics influenced the acceptance of the member to be an insider or outsider. Thus the two relatively strong members in mathematics accepted being insiders, while the middle achieving member did not accept at the beginning of the activity to be an insider regarding the learning happenings of the group, which made her frustrated from her positioning in the group. This situation changed as she tried to participate in the group's discussions and contributed to the knowledge development of the group about concepts related to the circle's topic. This change in her involvement with the group's work could be related to the group's perceived atmosphere, for the "members of a group will tend to behave according to the way they perceive the prevailing atmosphere" (Douglas, 1978; as in Gunn, 2007). This perceived atmosphere was characterized by being a positive learning atmosphere maintained by the group director. Moreover, this positive atmosphere gave power and freedom to the group members, enabling them to express themselves freely, and, as a result, causing them to have positive emotions: content, satisfaction and being happy.

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The mathematics teacher: An emotional rational being

Marina De Simone

Università di Torino, Dipartimento di Matematica "G. Peano", Torino, Italy, marina.desimone@unito.it

Starting from the "theory of rational behaviour", introduced by Habermas in 1998, I attempt to demonstrate how affective factors are entwined with those of rationality in in the decision-making processes of the mathematics teacher. This type of analysis has been carried out by developing an adaptation of the concept of "emotional orientation", offered by Brown and Reid in 2006. In particular, I will present the case of one teacher with her grade 9 class, involved in the explanation of linear equations.

Key-words: Rationality, emotional orientation, expectation, mathematics teaching, beliefs.

INTRODUCTION

In this paper, I focus the attention on the discursive activity of the teacher when she is introducing linear equations. At the end of the twentieth century, the philosopher Habermas developed the "theory of rational behaviour", where he discusses how people discursive activities show their rational behaviour. In the last years, this philosophical and sociological framework has been re-elaborated and adjusted to mathematics education by a working group constituted by many researchers from Italian, French and Spanish universities. This collaboration has produced two different research forums, presented during the PME Conferences in 2010 (Boero, Douek, Morselli, & Pedemonte, 2010) and in 2014 (Boero & Planas, 2014). This paper is an expansion of the research presented within the last RF by F. Ferrara and myself and that I developed for my PhD dissertation.

Many of the educational studies about rationality according to the Habermas framework, were centred on the students in the mathematics classroom (e.g., Morselli & Boero, 2011). However, if, from the one side, this theoretical framework seems to be suitable to examine decision-making processes of a 'rational being', from the other side, also the study of the discursive activity of the teacher seems to be crucial examining the interactions which happen in the classroom. It is important to consider the decision-making of the teacher, because one peculiarity of the teacher is making decisions within the classroom. Several authors have recognized an essential role to the decision-making of the mathematics teacher. For example, Bishop considers it as the activity "... at the heart of the teaching process" (Bishop, 1976, p.42).

THEORETICAL FRAMEWORK

Habermas defines a rational being as a human being who "can give account for his orientation toward validity claims" (Habermas, 1998, p. 310). He speaks about the concept of discursive rationality of the subject, explaining that it is not only referred to the discourse – as it could seem from the term at a first glance – but it has three different roots: the knowledge, the action and the speech, or, said in a different way: knowing, acting and speaking. Then, starting from the Habermas' assumption that for a rational being the discourse and the reflection on it (not necessarily explicit) are entwined, "the three rationality components - knowing, acting, and speaking - combine, that is, form a syndrome" (Habermas, 1998, p. 311) at a holistic level where reflection and discourse live together. Knowledge, action and speech constitute what he calls the epistemic, the teleological and the communicative components of rationality. They are inseparable, since a rational being acts in a specific manner to achieve a goal, on the basis of a specific knowledge, communicating in a precise way with the aim of being understood by the community. Hence, within the discursive activity, they are always present at the same time. According to Habermas, the epistemic rationality is connected to the justification of the knowledge at play: "We know facts and have a knowledge of them at our disposal only when we simultaneously know why the corresponding judgments are true" (Habermas, 1998, p. 312). Concerning the teleological rationality, Habermas states that "all action is intentional", that is, every action is originated from an intention of the subject with the aim of the realization of a result. He speaks of teleological rationality when "the actor has achieved this result on the basis of the deliberately selected and implemented means" (Habermas, 1998, p. 313). Finally, Habermas states that the communicative rationality "is expressed in the unifying force of speech oriented toward reaching understanding" (Habermas, 1998, p. 315).

In his speculation about rational behaviour, Habermas seems to lack any reference to emotion or passion. He seems to avoid any emotion by claiming that the force of a good argument should be free of emotional tags. Several philosophers and social theorists complained that, in the development of his theory, Habermas doesn't take in account the emotional side of human beings. For example, Rienstra and Hook, quoting the philosopher Heller, posed the question that "Habermas leaves no room for "sensuous experiences of hope and despair, of venture and humiliation", accusing him of completely avoiding the "creature-like" aspects of human beings" (Rienstra & Hook, 2006, p. 13). Therefore, basing on the assumption that rationality is deeply linked with the emotional sphere, I looked for researches that confirmed this hypothesis in particular in mathematics education and human neuropsychology.

In the last years, research in mathematics education has progressively perceived the existence of a mutual interaction between the affective sphere and cognition in mathematics learning (Zan, Brown, Evans, & Hannula, 2006). As highlighted by Hannula (2012), many research studies, focused on mathematics-related affect, have been dated from the eighties. An important grow in the theory on mathematics-related affect was due to McLeod (McLeod, 1992), whose main goal was to build "an overall framework of mathematics-related affect that would be consistent with research that is cognitively oriented" (Hannula, 2012, p. 138). In McLeod's framework (McLeod, 1992), which is considered a cornerstone for the literature on mathematics-related affect, emotions occupy a fundamental place, because of their unstable or less stable nature than that of beliefs and attitudes. Unfortunately, the aim of constructing a general theoretical framework that embrace all the research on mathematics-related affect has not yet been achieved. The most relevant problem is related to the terminology used in this field, because it is not universal. For example, as Di Martino and Zan discussed deeply (Di Martino & Zan, 2010), "some define attitude as positive or negative degree of affect, others identify emotions and beliefs as two components of attitude, while yet others define attitude as consisting of cognitive (beliefs), affective (emotions), and conative (behaviour) dimensions" (Hannula, 2012, p. 140). The recent research in mathematics-related affect has considered different affective concepts from those of the McLeod's (1992) framework such as values, identity, motivation, and norms. Zan and colleagues (Zan, Brown, Evans, & Hannula, 2006) have recognised the limited use of emotion in mathematics education research, even if it should be one the essential concept. They pointed out "how repeated experience of emotion may be seen as the basis for more 'stable' attitudes and beliefs" (Zan, Brown, Evans, & Hannula, 2006, p. 116). For Schoenfeld, emotional aspects are included in the wider category of beliefs, while goals is a motivational concept (Schoenfeld, 2010).

Human neuropsychology is another important field of research that studies the relationship between the affective and the rational sphere from a neurological point of view (Damasio, 1994, 1999). Specifically, Immordino-Yang and Damasio have shown the connection among emotion, social functioning and decision-making as a turning point for understanding the role of emotion in decision-making, the relationship between learning and emotion, and how culture shapes learning (Immordino-Yang & Damasio, 2007). While educational research often considered decision-making, reasoning and processes related to reading, language and mathematics as detached from emotion and body, these authors have shown that "learning, in the complex sense in which it happens in schools or the real world, is not a rational or disembodied process; neither is it a lonely one" (Immordino-Yang & Damasio, 2007, p. 4). For them, emotion is "a basic form of decision making, a repertoire of know-how and actions that allows people to respond appropriately in different situations" (Immordino-Yang & Damasio, 2007, p. 7).

This neurological research is becoming applicable also in the field of mathematics education. For example, Brown and Reid have developed and adapted the hypothesis of somatic markers (Damasio in 1994) for studying the decision-making processes (Brown & Reid, 2006).

EMOTIONAL ORIENTATION

Brown and Reid (Brown & Reid, 2006) analysing the processes of teachers and students' decision-making, have considered the notion of "emotional orientation" (Maturana, 1988). In particular, they have focussed on the "decision-making that happens before conscious awareness of the decision to be made occurs." (Brown & Reid, 2006, p. 179). Maturana (1988a, 1988b) referred the notion of "emotional orientation" to the criteria for acceptance of an explanation by members of a community, and considered emotions as being the foundation of such criteria. Reid adapted the concept of emotional orientation to the mathematics field, defining the "mathematical emotional orientation". The criteria for accepting an explanation in the particular case of the mathematical emotional orientation involve "the use of deductive reasoning, a basis in agreed upon premises, and a formal style of presentation" (Reid, 1999, p. 1). Moreover, there are many shared experiences and assumptions in mathematics, like the language used to talk about it. In the end, there are also many "actions" when someone does mathematics, like "drawing diagrams, generalizing statements, making conjectures" (Reid, 1999, p. 1). Emotions are still at the basis of these criteria. The concept of "emotional orientation" allows me to speak of the interconnection between rationality and emotion. In fact, as the words themselves suggest, the "orientation" of a subject oriented towards validity claims is "emotional", that is, affected by the emotions in a certain way. But there is still a methodological problem of how, practically, this entanglement can be analysed. Hence, I sketchily present an adaptation of the theoretical framework of the emotional orientation in order to speak practically about these two sides of the same coin. I define the "emotional orientation" of a subject (e.g. a teacher) in terms of "the set of her expectations": the term "expectation" is connected to her "emotions of being right" when she uses specific criteria for accepting an explanation by a community (e.g. a class) rather than other ones (Ferrara & De Simone, 2014). The most difficulty encountered in studying emotions is their "visibility" and, then, their "certain" identification. In this context, when I speak of emotion of the teacher I will refer to her *emotionality*, namely the set of "behaviours that are observable and theoretically linked to the (hypothetical) underlying emotion" (Reber & Reber, 2001).

METHODOLOGY

The study presented in this paper is part of the research for my PhD thesis whose focus is on aspects related to rationality of the teacher in the teaching of linear equations at secondary school. The participants were 3 teachers and their grade 9 classes, in a scientifically oriented secondary school in Western Italy. The teachers were selected assuming that rationality and emotions are proper of human beings and with the purpose of having different emotional orientations. Each teacher was first interviewed and asked about her personal beliefs on the topic of linear equations, on algebra in general and on how she uses the didactical materials. Each interview lasted roughly twenty minutes and was videotaped with one camera facing the interviewer and the subject. Then, the whole class activities conducted by the teacher and the students' working group activities were also videotaped. All voice and bodily movement during the interviews and the classroom activities were recorded. The videos were transcribed for data analysis. For the identification of the emotional orientation of the teachers I paid attention also to some indicators that allow me to say something about the emotions of the teacher. In particular, I considered as indicators the tone of voice, the words, the repetitions, the emphasis and the body language (facial expressions, gazes, gestures…). So, I identified the expectations of the teacher – that constitute her emotional orientation – starting from what she explicitly declared in an *a-priori* interview. Then, I tried to find them again, reflected in the class activities, through the indicators I listed above. At last, I analysed the transcriptions from both the emotional and the rational point of view, at the same time, because they are naturally entwined. Due to space constraints, I present the case of one of the three teachers involved in the whole research, whom I call Lorenza.

AN EXAMPLE: THE EMOTIONAL ORIENTATION OF LORENZA

I identified different expectations that constitute the emotional orientation of Lorenza, but for the limited space, I show just one of them. From the interview, I identified her *expectation about the validity of the previous knowledge of the students that can be used for* *constructing the new one*. With "previous knowledge", I mean what students have learnt both in the middle school and with her. In order to highlight this expectation, I collected the moments of her interview from which this expectation could be detected. Lorenza explicitly declared: "Usually, I begin to treat linear equations starting from their previous knowledge *in order to see whether it is valid, or whether the students have misinterpreted the various procedures that they have been taught in the previous years. Anyway, I begin a new topic starting from the knowledge that the students already have*". During the interview, I asked Lorenza when she introduces for the first time the letters in algebra. She answered that she usually uses letters for the first time in physics, but she commented: "*they are already able a little bit to manage it*" and, then, she repeated the same concept: "*even if when we speak of sets, the letter represents already something for them or also in the logic language the logic variable, then there is already a formalization from this point of view, we say*". In another passage, it was asked to Lorenza when she speaks for the first time of equations and if she links the concept of linear equation with that of function. She stated that she makes this link for the first time in physics: "*in physics we have already said something about the equations, but just basics because I wanted to put them to work on inverse formula, then I said: "What do you know?" they know already something and they know to deduce or, in theory, they should be able to deduce an inverse formula given a formula*". An interesting thing is that, during the interview, she explicitly made just the same question that she asks to her students, perhaps, because she is used to make it to her students for testing what is their knowledge and if it's valid. Lorenza added that students have already known something about equation, but "*in a very naïve way*", so they have to go in depth with her become aware of the link between the equations and the straight line.

From these pieces of the interview, it becomes quite clear that Lorenza believes that it is important to recall the previous knowledge of the students during all the lessons, not just when a new mathematical topic is introduced.

After detecting Lorenza's expectations, I analysed the transcriptions of her lessons in which they are actually reflected. I will show how her "orientation" towards validity claims is "emotional", that is affected by emotions. Using a metaphor, rationality and emotions of the teacher can be seen as the weave and the warp of the fabric. As the weave and warp entwined constitute the fabric, the rationality and the emotions entwined shape the teacher as she actually is.

The first example I propose is taken from the first lesson after Easter holidays, during which the teacher was recalling the concept of identity – explained in the last lesson before Easter – with the aim of introducing, formally, the concept of equation.

1 T: before holidays, I hope that someone remembers just something, we have spoken about [pronouncing] identities, then, is there anyone who wants to give, for now, [tone of voice of a statement not of a question] the definition of identity and to do only an example of identity? Don't be shy! [smiling] (Figure 1) [lifting up her chin and biting her lips] (Figure 2). Please [she addresses to S1 who is raising up his hand]

Figure 1

Figure 2

2 S1: it is an equality that is verified for each value that it is replaced to the letter

3 T: fine, it is an equality between two expressions, that contain letters, that is verified for each value we go to ascribe to

the unknown. One example, we have done an example within the classical ones [smiling]

4 S1: (a+b)2=a2+b2+2ab

5 T: for example, the development of a special product is an equality between two expressions that contain letters and then it can be considered an identity and each value we go to give to the unknown a or to the unknown b, the result on the left and on the right of the equality sign must be the same and, conversely, what can be considered as an equation, do you remember? [speeding up] you have already seen them in the middle school, partly, no? yes [she is answering herself], we have already reviewed in physics since at the beginning of the year they serve us for working with formula etcetera, so we have already given indications. In the light of this path that we have done, any of you would like to hazard a definition of equation [tone of voice proper of a statement not of a question and, then, she lifts up her chin]? Try to hazard, Andrea!

The action of Lorenza of asking something that students already know (the definition of identity and an example of it) is aimed at constructing the concept of equation (# 5). This action comes along with a particular tone of voice not proper of a question, but rather of a statement (# 1). The affirmative tone of voice of the question and that facial expression (# 1: she lifts up her chin after speaking) could show her expectation that someone remembers the concept of identity and will answer to her, because the class has already seen it a short time before. Waiting an answer, she laughs (Figure 1) and she bites her lips (Figure 2), probably, because she wants a feedback from the class. The action of asking something that the students should know is full of emotional hues linked to her expectation about the validity of the previous knowledge. This passage of the transcript highlights an *emotional teleological rationality* of Lorenza. It is not just a matter of what she is doing, but rather of *how* and *why* she is acting in that way. From the beginning, her speech seems to be charged by emotions (#1: "I hope", "just something", "Don't be shy!"). These emotions are related to her expectation ("I hope") that students remember the concept of identity, even "just something" (she can be easily satisfied, as long as, they are able to say something). She seems quite confident about their knowledge, thinking that her students don't answer because they are shy, indeed, she incites them into doing, using the imperative phrase "Don't be shy!". This "emotion-soaked" speech highlights an *emotional communicative rationality* related to her expectation about classroom culture. There is not only what she is communicating, but also *how* and *why* she is doing it that way. Requiring again the example of an identity (#3), after the answer of S1, could be interpret as a way of involving more students in the discussion and to evoke the classroom culture as much as possible (#3: "One example, we have done an example within the classical ones"). Another time the teacher's speech comes along with an emotional element (she smiles), because she seems to feel that students need to be comfortable for answering, even if they already should know the example.

Lorenza recalls just the term of identity to shift easily to that of equation: the former is an equality true for every value of the unknown, while the latter is an equality that may not be satisfied or, in the case it is satisfied, it can be undetermined or determined. This epistemic shifting comes along with an insistence of Lorenza on the fact that they already learn first degree equations both in the lower secondary school (grade 8) and with her in grade 9 (#5: "you have already seen them in the middle school, partly, no?", "yes [she is answering herself], we have already reviewed in physics since at the beginning of the year they serve us for working with formula etcetera, so we have already given indications", "In the light of this path that we have done any of you would like to hazard a definition of equation"). In addition, she asked her students "to hazard" a definition of equation, but with the tone of voice proper of a statement and not of a question (#5), probably because she is expecting that students construct new knowledge starting from the previous one. The insistence in the speech, the facial expression, the tone of voice linked to the knowledge into play could inform us about an *emotional epistemic rationality*.

Then, the discussion goes on as follows:

6 S2: it is an equality between two literal expressions in which the value of x is replaced by a unique value to make it true

7 T: we say that it is satisfied just f(or)

8 S2: for a single value of x

9 T: always?! (Figure 3), do we always find it?!for you this value or (Figure 4), let's try to think a little bit

Figure 3

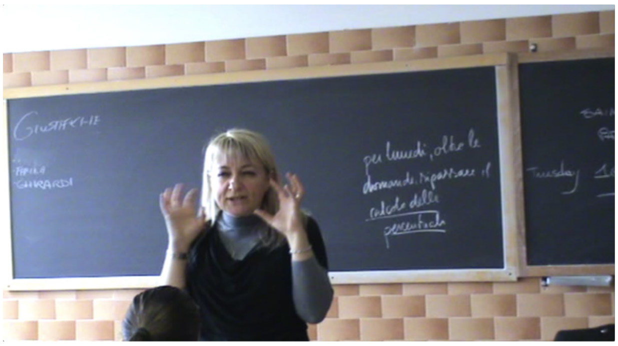

Figure 4

10 S2:		sometimes it's impossible
11 T:		It could be, right.

After the answer of S2 (#6), Lorenza clarified his definition of an equation. This action is aimed at a first introduction of the different types of equation (determined, undetermined, impossible), which she will develop in the next lesson. The rhetorical questions (#9: "always?!", "do we always find it?!"), the facial expressions in Figure 3 and Figure 4 and the general involvement in the discussion (#9: "let's try to think a little bit") accompany the actions she made on the basis of a certain knowledge into play, communicating in a specific manner. Particularly, in Figure 4, she seems to "catch" with her hands what they are in mind about the concept of equation. This frame brings out an *emotional teleological, epistemic and communicative rationality* of Lorenza.

DISCUSSION

I presented an emblematic example of the coexistence of the emotional rational aspects in mathematics teaching. In particular, I showed how this merger outlines the decision-making processes of the teacher. As I highlighted in the analysis of the example, all the teachers' decisions – about knowing, acting and speaking – are "visible" in language, but, mostly, in her emotional aspects. This doesn't mean that emotions explain the decisions, but, rather, that decisions are very often "visible" through emotions. I tried to study this complexity through an adaptation of the concept of "emotional orientation", used by Brown and Reid in 2006, drawing on the work of Maturana (1988).

Referring to the transcript, from one side, the emotions of Lorenza are linked to her expectation about the class culture she developed from her own beliefs. These emotions can become clear from her tone of voice, her way of speaking, her body language. From the other side, the choices of Lorenza (starting from the identities to introduce equations, recalling explicitly with insistence the previous knowledge of the students...) are connected to this expectation of constructing new knowledge, basing on the validity of the previous one. Then, the emotions are strictly related to the choices and this gives the meaning of how the "orientation" of Lorenza can be "emotional".

Hence, the analysis of the discursive activity of the teacher has naturally led to propose an enlargement of the Habermas components of rationality. As shown in the analysis of the excerpt, I tried to highlight the *emotional epistemic rationality*, the *emotional teleological rationality* and the *emotional communicative rationality* of Lorenza. In this context, I consider the *emotional epistemic rationality* as related to *why* the teacher uses that specific justification of the knowledge at play; the *emotional teleological rationality* as related to *why* the teacher makes that actions to achieve a goal and the *emotional communicative rationality* as related to *why* the teacher uses that speech oriented towards validity claims. These three adapted components of rationality are always present in the discursive activity of the teacher. Obviously, as testified by the example, during specific moments of the classroom activity, one component could emerge more than the others.

The role of the *a-priori* interview results particularly significant for this kind of analysis since it enabled to scrutinize the teacher's beliefs and orientations for the teaching of the concept at stake.

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How a local oral examination considers affective aspects of knowing mathematics

Anne Birgitte Fyhn

UiT – The Arctic University of Norway, Tromsø, Norway, anne.fyhn@uit.no

To measure whether and to what extent pupils know mathematics is complicated. The test situation will influence the pupils' work and in addition there are aspects of knowing mathematics that are difficult for a written test to assess. Since the early 1990s, Norwegian pupils undergo a local oral examination at the end of the ten-year compulsory school. Rules and guidelines for this examination has developed over time in accordance with curriculum changes. According to the 1992 national guidelines, the examination has to be based on the pupils' project work or similar. In addition, pupils' creativity and imagination is highlighted. The 2014 national rules are different, the only thing here that a written test cannot reveal, is skills in mental calculation.

Keywords: Assessment, motivation, attitudes, beliefs, creativity.

INTRODUCTION

"Different groups of people can have very different views of what "counts", or should count, in mathematics" (Schoenfeld, 2007a, p. 3), so what it means to know mathematics is far from a simple question. The question of how to assess what pupils know, is even more complicated. Tests or examinations constitute one form of assessment; other forms are for example feedback on daily work. One aim of an examination is to assess what the pupils know; another aim is to assess whether the teaching is successful. Morgan (1999) identified two main strands in research related to assessment in mathematics education. One strand focuses on the design of the tests. The other focuses on critiquing traditional forms of assessment, and often proposes new forms of assessment that are better aligned with the curriculum aims. Norway introduced a local oral examination in 1990 (Ministry of Education, Research and Church Affairs, 1992). This examination highlighted local curricula and pupils' project work; this was a new form of assessment aligned with curriculum aims. The national guidelines for the local oral examination have evolved since the first written guidelines appeared in 1992. The research question is: How do the 1992 national guidelines and 2014 national rules for the local oral mathematics examination consider affective aspects of knowing mathematics?

Because the rules and guidelines reflect the curriculum, an analysis of the curriculum development is presented as an introduction to the further analysis. This paper focuses on the rules and guidelines for the local oral examinations from two perspectives: a) how to facilitate opportunities for the pupils to show what they know, and b) what is assessed. A common problem with tests and surveys is that there are aspects of knowing mathematics that are difficult to reveal in a written test. Affective aspects are among these. The paper first discusses motivation, attitudes, beliefs, and creativity, elements that reflect affective aspects of knowing mathematics and which are found in central documents for Norwegian mathematics education. The following section reveals how Norwegian curricula consider affective aspects of knowing mathematics. Then assessment in mathematics is presented, before the analysis of the 1992 national guidelines (Ministry of Education, Research and Church Affairs, 1992) and the 2014 national rules (Norwegian Directorate for Education and Training, 2014) for the local oral examination.

ATTITUDES, BELIEFS, CREATIVITY AND MOTIVATION

The Ministry of Education, Research and Church Affairs carried out a large project, Quality in Mathematics Education (Brekke, 2002/1995), between 1995 and 2002. The aims were to develop test materials, conduct a survey of pupils' attitudes and beliefs towards mathematics and mathematics teaching, and describe the spectrum of pupils' performances in different subject areas. During the 1990s, pupils' attitudes and beliefs were accordingly considered important in Norway.

When individuals are doing mathematics, the affective system is not just supporting cognition, but it is playing a central role. Affect as a representational system is intertwined with cognitive representation (Goldin, 2002). Goldin divides affective representations into four sub-domains: a) *emotions*: rapidly changing and usually local or connected to context; b) *attitudes*: moderately stable predispositions towards certain sets of classes of situations; c) *beliefs*: often highly stable, involving the attribution of some sort of external truth or validity; and d) *values*, including ethics and morals, and which refers to deep "personal truths". This paper pays less attention to emotions and values because the Quality in Mathematics Education project highlights pupils' attitudes and beliefs.

Sriraman (2009/2004) identifies creativity as part of mathematicians' work. He points out that creating original mathematics requires a very high level of motivation, persistence and reflection, all of which are considered indicators of creativity. For example, one could be in an environment that is non-supportive of creative efforts, but a high level of motivation may possibly overcome this and pursue creative endeavours. According to Sternberg and Lubart (1999), creativity may not only require motivation, but also generate it. Thus, given the chance to be creative, pupils who might otherwise lose interest in school instruction might find that it instead captures their interest. According to Liljedahl (2013), creativity has more affective aspects than just motivation, because illumination is important in the creativity process. Illumination occurs in the context of trying to work something out.

Self-determination theory proposes that all human beings have fundamental psychological needs for being *competent*, *autonomous* and *related* to others. People are assumed to proactively initiate engagement with their environments. The basis for this activity is intrinsic motivation (Deci & Ryan, 2012). The foundations of self-determination theory reside in a dialectical view, which concerns the interaction between an active, integrating human nature, and social contexts that either nurture or impede the organism's active nature. *Relatedness* concerns the psychological sense of being with others in a secure communion or unity. *Autonomy* refers to being the perceived origin or source of one's own behaviour; it concerns acting from interest and integrated values. Autonomy enables individuals to deal with novelty and generate creative products (Diezmann & Watters, 2000), so autonomy is part of creativity. According to Hannula (2006), motivation is observable only as it manifests itself in affect, cognition and behaviour, for example as beliefs, values and emotional reactions. Pupil-centred classrooms with much teamwork going on may rely on pupils' exhibiting their autonomy and social interactions. According to DeBellis and Goldin (2006), each person constructs complex networks of affective pathways and competencies. These networks have more or less mathematical problem-solving power and their meanings are context-dependent for the individual.

THE THREE LATEST MATHEMATICS CURRICULA IN NORWAY

Three different curricula have been in effect in the Norwegian School during the period from 1990 to 2015. These curricula provide different perspectives on pupils' affect. The 1987 curriculum (Ministry of Church and Education, 1987) focuses on pupils' project work and the schools' development of local curricula. This is interpreted to mean that the curriculum highlights the pupils' autonomy and relatedness, so it is in line with Deci and Ryan (2012). The overarching aims for the subject mathematics in this curriculum, claim that the teaching shall take care of and develop the pupils' logical thought, responsible decisions, imagination and creative enthusiasm. In addition, problem solving is introduced as a separate domain. The curriculum is thus in line with DeBellis and Goldin's (2006, p. 133) point: "The meanings of our emotional feelings are highly context dependent; far more, even, than the meaning of words and phrases." The focus on imagination and creative enthusiasm, shows that the curriculum considers affective aspects of creativity.

The 1997 mathematics curriculum (Ministry of Education, Research and Church Affairs, 1996a) introduces *mathematics in everyday life* as a separate domain. This is interpreted as an explicit focus on context and autonomy. The curriculum points out six main aims for the subject mathematics. One aim is that pupils should develop positive relations with mathematics, experience the subject as meaningful, build self-respect and have self-confidence. This is in line with self-determination theory (Deci & Ryan, 2012). Another aim is that pupils are stimulated to use their imagination, resources and knowledge to find solution methods and alternatives through investigative and problem-solving activities, as well as conscious choices of tools and instruments. This is interpreted as meaning that the curriculum focuses on the pupils' autonomy, creativity and motivation.

The Danish project Competencies and Learning of Mathematics, (Niss & Højgaard Jensen, 2002) constitutes the basis for the interpretation of mathematical competence in the recent mathematics curriculum (Norwegian Centre for Mathematics Education, 2014). *Competence* is someone's insightful readiness to act in response to the challenges of a given situation. The Danish project describes a set of eight delimited dimensions that together generate mathematical competence: Mathematical-thinking competence, problem-tackling competence, modelling competence, reasoning competence, aids-and-tools competence, communicating competence, symbol-and-formalism competence, and representing competence. No affective aspects of competence are explicitly listed, but Niss and Højgaard Jensen (2002) point out that mathematics-teaching competence includes the ability to motivate and inspire pupils. The overarching aims of the 2006 mathematics curriculum (Norwegian Directorate for Education and Training, 2013) emphasize that both girls and boys must get opportunities to gain experiences that create positive attitudes towards the subject.

Table 1 shows how the perspective on affect has developed according to the overarching aims of the latest three mathematics curricula. The 1997 curriculum's (Ministry of Education, Research and Church Affairs, 1996a) aims, explicitly include the pupils' attitudes and their affective aspects of creativity. "Meaningfulness" is left out in the 2006 curriculum. This is interpreted as suggesting that the pupils' autonomy and self-determination are less important. The greatest difference between the three curricula is that while the two previous curricula present a pupil-centred perspective on the teaching, the 2006 curriculum presents a teacher-centred perspective; the 2006 curriculum has no teaching aim regarding the pupils' affects.

The Core Curriculum (Ministry of Education, Research and Church Affairs, 1996b) states the overarching aims for the education. This part of the curriculum elaborates on the preamble to the Education Act, and it is continued in the 2006 curriculum. The Core Curriculum focuses on creativity, autonomy and relatedness. However, the word "motivation" occurs only once, successful learning depends on the teacher as well as on the pupil. The Core Curriculum highlights the pupils' attitudes by claiming that knowledge, skills and attitudes develop in the interplay between old notions and new impressions. The 1997 mathematics curriculum is in line with the Core Curriculum's perspectives on creativity, autonomy and relatedness, while the 2006 mathematics curriculum is not.

ASSESSMENT

Schoenfeld (2007a) claims that assessments can serve useful purposes for the pupils, but the challenge is to make them do so. According to Wiliam (2007), the use of assessment should support learning in any assessment regime; classroom assessment must first be designed to support learning. Schoenfeld (2007b) discusses how to assess mathematical proficiency: what a pupil knows, can do, and is disposed to do mathematically. He describes four aspects of mathematical proficiency: *Knowledge base* (what does it mean to know a content), *strategies* (ability to formulate, represent, and solve mathematical problems), *metacog-*

	1987 curriculum	1997 curriculum	2006 curriculum
Aims for the pupils	Experience mathematics as meaningful	Develop positive relations with mathematics Experience mathematics as meaningful	Get opportunities to gain experiences that create positive attitudes towards mathematics
Aims for the teaching	Take care of and develop pupils' imagination and creative enthusiasm	Contribute to the pupils' building of self confidence Make the pupils experience belonging	Include playful and creative activities

Table 1: How pupils' affects are considered in the curricula's overarching aims

nition (using what you know effectively), *beliefs and dispositions*. Pupils who experience skill-based instruction tend to succeed on tests of skills, but they do not succeed well when tested in problem solving and conceptual understanding. On the other hand, "[s]tudents who study more broad-based curricula tend to do reasonably well on tests of skills" (p. 63), while on tests of conceptual understanding and problem solving, these pupils succeed much better than those who just exercise on skills. *Beliefs* are important, because if you believe that mathematics is not supposed to make sense, your work will reflect this. The pupils pick up their beliefs about the nature of mathematics from their experiences in the mathematics classroom. That mathematical problems have one and only one answer and that mathematics is done by individuals in isolation are typical pupil beliefs.

Boesen, Lithner and Palm (2010) investigated relations between task characteristics and the mathematical reasoning pupils use when solving tasks in a test situation. Their results show that when solving tasks similar to those in their textbooks, the pupils were mostly trying to recall facts or algorithms. The pupils did not have to construct new reasoning or consider any intrinsic mathematical properties. By contrast, the tasks that were not similar to those encountered in the textbook were mostly approached with creative mathematical reasoning.

THE LOCAL ORAL MATHEMATIC EXAMINATION

As a result of the 1987 curriculum's (Ministry of Church and Education, 1987) focus on local curricula, project work and problem solving, Norway introduced a local oral examination that encompassed these three fields. Pupils may hence undergo two different examinations in mathematics at the end of the compulsory school: A national written mathematics examination and a local oral mathematics examination. Only some pupils undergo each examination.

According to the national guidelines, the 1992 oral mathematics examination aimed to: "... assess aspects of the teaching aims, which may be difficult to show in a written test" (Ministry of Education, Research and Church Affairs, 1992, p. 14, author's translation). The examiner (most commonly the mathematics teacher) leads the talk/discussion with the pupil, while the external examiner determines the grade afterwards. The Ministry designed and published booklets with guidelines and guiding materials for teachers. These guidelines are difficult to access, so they are listed here:

— The test has to include tasks from at least three of the ten main subject areas in the syllabus.

— The test has to give room for use of different methods, creativity and imagination.

— The test has to include tasks where the pupil may explain procedures and rules that she/he uses in solving the tasks. It might be satisfactory that the pupil just sketch how she/he will solve the task.

— The test has to include tasks involving mental calculation and approximation.

— The test might include tasks where the pupils are free to use technical artefacts such as calculators and computers.

— If there is information about project work or similar, then the test has to include questions related to this work (p. 14, author's translation).

In 2014, the responsibility for the local (oral) examination guidelines belongs to the school owners. The Education Act (Lovdata, 2013) regulates the oral examination, and the Norwegian Directorate for Education and Training (2014) has elaborated regulations for the national rules. There is a 24-hour mandatory preparation time, which starts with one day at school with all kinds of aids permitted. In the beginning of the preparation time, the pupil gets a theme or a problem. What goes on in the preparation time is not included in the assessment. Each pupil has a right to pedagogical aid during the preparation day at school. The pupils present their theme or problem during the examination. The examiners then use this presentation as a basis for a mathematical discussion, for which the teacher has prepared questions for the pupil. The examiners cannot ask questions just from a narrow part of the subject. The discussion has to cover at least 2/3 of the examination time. The examination has to be organized so that the pupil can show to what extent the competence aims in the curriculum have been reached. The mathematics curriculum clarifies the meaning of "oral skills":

> Oral skills in mathematics involve creating meaning by ... participating in discussions, communicating ideas and elaborating on problems, solutions and strategies with other pupils ... this development starts with a basic mathematics vocabulary that leads to precise professional terminology ... (Norwegian Directorate for Education and Training, 2013, pp. 4–5)

The schools can make local guidelines for how to carry out this examination. Two examiners assess the pupil, and one of them needs to be a teacher from another school.

ANALYSIS OF NATIONAL GUIDELINES AND RULES

Schoenfeld (2007a) points out that there is more to mathematical proficiency than being able to reproduce standard content on demand. He warns against what he calls "the illusion of competence" by asking: "Have you learned the underlying ideas, or are you only competent at things that are precisely like the ones you've practiced on?" The local oral examination has developed from 1990–1992 and into the 2014 examination form. In order to investigate how the rules and guidelines for these examinations consider affective aspects of knowing mathematics, a framework is built on theories from Deci and Ryan (2012), Hannula (2006), Sriraman (2009/2004), Liljedahl (2013), DeBellis and Goldin (2006) and Goldin (2002). This generates four affective aspects of knowing mathematics: motivation, creativity, attitudes and beliefs. The data in this study consist of the 1992 national guidelines for the local oral examination (Ministry of Education, Research and Church Affairs, 1992) and the 2014 national rules for the local oral examination (Norwegian Directorate for Education and Training, 2014).

The framework leads to three points in the 1992 guidelines: the conversation form, the use of creativity and imagination, and the inclusion of local curricula and pupils' project work. The guidelines explicitly emphasize that the examination shall aim at having a conversational format. This opens up for the teacher to focus on each of the framework's four categories. The use of creativity and imagination directly points at creativity and indirectly points at motivation. The inclusion of pupils' project work is an important aspect of the oral examination; according to the national guidelines (Ministry of Education, Research and Church Affairs, 1992), project work is based on the pupil's interests, ideas or experiences with social practice. Interests are part of the pupil's intrinsic motivation (Deci & Ryan, 2012). Our emotional feelings are highly context-dependent (DeBellis & Goldin, 2006) and thus related to social practice. According to the 2014 national rules (Norwegian Directorate for Education and Training, 2014), the most main point is that the pupils can show their competencies. The examination conversation may consider all the framework categories, but it is explicitly stated that the teacher and the external examiner choose the context for the examination. So the pupil's autonomy is not considered important. The examination aims to assess how the pupils have achieved the competence aims for grade ten in the national mathematics curriculum (Norwegian Directorate for Education and Training, 2013). It turns out to be one single competence aim that a written test cannot assess: to develop, use and elaborate on methods for mental calculations. Based on the competence aims, the teacher can provide each pupil with problems so that the four categories in the framework are covered, but the national rules leave it to the schools to make this choice. Creativity and the pupils' project work are not explicit issues for assessment in the 2014 examination. The teacher may hence provide the pupils with problems that concern the pupils' attitudes, beliefs, motivation and creativity, but the 2014 national rules have no explicit requirement for this, unlike the 1992 National guidelines.

CLOSING WORDS

The overarching aims in the mathematics curriculum have changed since the local oral curriculum for all was introduced. The overarching aims in the mathematics curricula from 1987 (Ministry of Church and Education, 1987) and 1997 (Ministry of Education, Research and Church Affairs, 1996a) present a pupil-oriented perspective, which highlights that the teaching shall provide the pupils with opportunities to show what they know. The overarching aims in the 2006 mathematics curriculum present a teacher-centred perspective with no teaching aims concerning the pupils' affects.

The 1992 national guidelines for the local oral examination (Ministry of Education, Research and Church Affairs, 1992) emphasize that the test has to assess aspects of the subject that are difficult to reveal in a written test. The 2014 national rules (Norwegian

Directorate for Education and Training, 2014) have no similar requirement. In 1992, the pupils' project work was the basis for the examination; in addition, the pupils' creativity and imagination were highlighted. This means that affective aspects of knowing mathematics were considered important. The guidelines point out that the test has to provide the pupils with opportunities to show what they know. In 2014, the teacher provides the pupils a problem or a task, and they get one school day to prepare a presentation of this task. At the examination the pupils discuss this presentation with the teacher and the external examiner. The pupils' social practice is hence not highlighted the way it was in 1992. The 1992 national guidelines explicitly points out that the test has to provide opportunities for the pupils to use creativity, and "to show creativity" was one requirement for achieving the highest grade. The 2014 national rules do not consider creativity; nor do they focus explicitly on affective aspects of knowing mathematics. However, the main point in these rules, is that the pupils can show their competencies. Further research is necessary to provide a more thorough analysis of this issue.

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Students' emotional experiences in high school mathematics classroom

Gustavo Martínez-Sierra

Autonomous University of Guerrero, Research Centre of Mathematics Education, Chilpancingo de los Bravo, Mexico, gmartinezsierra@gmail.com

The aim of this qualitative research is to identify Mexican high school students' emotional experiences in the mathematics classroom. In order to obtain data, focus group interviews were carried out with 53 students. Data analysis is based on the theory of cognitive structure of emotions (Ortony, Clore, & Collins, 1988), which specifies eliciting conditions for each type of emotion and the variables that affect intensity. The participants' emotional experiences are: satisfaction, disappointment, hope, fear, joy, distress, boredom, interest, pride, reproach, self-reproach, like and dislike with different eliciting conditions. These results show that all students' emotional experiences are based on their appraisal in terms of a goal structure present in the mathematics classroom and in the school setting.

Keywords: Emotions in mathematics education, students' emotions, emotions in mathematics classroom, appraisal structures, theory of cognitive structure of emotions.

INTRODUCTION

In the field of mathematics education, most of the research on students' emotions focuses on their role in mathematical problem-solving (e.g., Goldin, 2000; Op' T Eynde, De Corte, & Verschaffel, 2007). Among other results, these studies have confirmed that people tend to experience similar emotions in the process of problem-solving. By example Op' T Eynde and colleagues (2007) found that students experience different emotions while solving a problem. They can be *annoyed*, *frustrated*, *angry*, *worried*, *anxious*, *relieved*, *happy* or *nervous*. First, for instance, a student can be *worried* during the process of finding a strategy to solve a problem (this is evidenced by students' use of descriptions such as "brow lowering" and "not feeling well"). The student becomes *frustrated* if the solution to the problem does not seem to appear after 10 seconds ("I don't want to use the calculator", "it does not help me", "but I still want to reach the goal"). Later, *panic* appears and finally *anger* ("come on, what is this all about").

Research on emotions in mathematics education highlights the necessity to move beyond the view of distinguishing between positive and negative emotions. It is also suggested to go beyond analysing emotions in problem solving and investigating emotions routine activity (Hannula, Pantziara, Wæge, & Schlöglmann, 2010). So we have assumed the task to identify emotional experiences in routine activities in mathematics classes. In order to go beyond a consideration of positive and negative emotions we use the cognitive structure of emotions theory (Ortony et al., 1988). This is the main reason to focus on the following research question: *What are the students' emotional experiences in high school mathematics classroom?*

We are aware that the analysis of narratives of emotional experiences is quite different from the direct analysis of emotions but, like Ortony and colleagues (1988, p. 8), we are willing "to treat people's reports of their emotions as valid, also because emotions are not themselves linguistic things, but the most readily available non-phenomenal access we have to them is through language".

THE THEORY OF THE COGNITIVE STRUCTURE OF EMOTIONS

We have chosen the *theory of the cognitive structure of emotions* (by the initials of the surnames of the authors OCC theory from now) to identify the students' emotional experiences. For Ortony and colleagues (1988) emotions arise as a result of interpretations of situations by those who experienced them: "[Emotions can be taken as] valenced reactions to events, agents or objects, with their particular nature being determined by the way in which the eliciting situations is construed" (Ortony et al., 1988, p. 13). Thus a particular emotion experienced by a person on a specific occasion is determined by his interpretation of the changes in the world: "When one focuses on events one does so because one is interested in their consequences, when one focuses on agents, one does so because of their actions, and when one focuses on objects, one is interested in certain aspects or imputed properties of them *qua* objects" (Ortony et al., 1988, p. 18).

Different types of situations that elicit emotions are labeled in classes according to a word or phrase corresponding to a relatively neutral example that fits the type of emotion (Ortony et al., 1988). For example, to refer to the emotion type "pleased about the confirmation of the prospect of a desirable event" they choose the emotion word *satisfaction* because it represents an emotion of relatively neutral valence among all those that express that you are happy about the confirmation of something expected.

The characterizations of emotions in the OCC theory are independent of the words that refer to emotions, as it is a theory about the things that concern denotative words of emotions and not a theory of the words themselves. From the distinction between reactions to events, agents, and objects, we have that there are three basic classes of emotions: "Being *pleased* vs. *displeased* (reaction to events), *approving* vs. *disapproving* (reactions to agents) and *liking* vs. *disliking* (reactions to objects)" (Ortony et al., 1988, p. 33).

Reactions to events breaks into three groups: one, the Fortunes-of-others group, focuses on the consequences for oneself of events that affect other people. The other two, the Prospect-based and Well-being groups, focus only on the consequences for oneself. Reactions to agents are differentiated into four emotions comprising the Attribution group. Reactions to objects lead to an undifferentiated group called the Attraction group. There is also a compound group of emotions, the Well-being/Attribution compounds, involving reactions to both the event and the agent simultaneously. It seems to be a general progression that operates the different groups of emotions in order: first reactions to events, then to agents, and finally to objects. From the previous considerations, the OCC theory specifies 3 classes, 5 groups and 22 emotion types. To illustrate in Table 1, we present the corresponding emotions to the Prospect-based group.

To interpret emotional experiences in mathematics classes we have added two types of emotions in the Well-being group of emotions to the OCC theory (Martínez-Sierra & García González, 2014). We call them *boredom* and *interest*. These emotional experiences are elicited by the appraisal that the students made of their own cognitive state: 1) states of *alertness* and *concentration* that produce understanding and learning in the case of *interest*, and 2) states of *distraction* and *deconcentration* that prevent understanding and learning in the case of *boredom*. Thus, we consider *boredom* emotions like "Displeased about an undesirable cognitive state of distraction" and *interest* like "Pleased about a desirable cognitive state of attention".

Class	Group	Types (sample name)
Reactions to events	PROSPECT-BASED	Pleased about the prospect of a desirable event (hope) Pleased about the confirmation of the prospected of a desirable event (satisfaction) Pleased about the disconfirmation of the prospect of an undesirable event (relief) Displeased about the disconfirmation of the prospect of a desirable event (disappointment) Displeased about the prospect of an undesirable event (fear) Displeased about the confirmation of the prospect of an undesirable event (fears-confirmed)

Table 1: Emotion types according to the OCC theory (an extract)

OCC theory specifies three global variables that affect the intensity of different emotions types, three central variables and nine local variables. They are briefly laid out in Table 2.

OCC theory defines *goals* as what one wants to achieve. There are three kinds of goals: active-pursuit goals (A-goals), interest goals (I-goals), and replenishment goals (R-goals). A-goals represent the kinds of things one wants to get done, like passing a course or finishing university. I-goals are more routine goals and are necessary to achieve A-goals or support them, like passing a test. R-goals are those that should be satisfied from time to time in a cyclical nature, like attending a class. Furthermore, it is important to distinguish between all-or-nothing goals, like passing a test, and partially attainable goals, like solving a problem. These distinctions allow oneself to determine the intensity of the different experienced emotions.

METHODOLOGY

Context

The high school where the study was carried out lies to the west of Mexico City. Most of the students live in municipalities bordering the metropolitan area of Mexico City located in the State of Mexico, they come from low economic extraction and most of their parents did not attend college-level. Most students' mothers are housewives. Due to the inflexibility of the curriculum, all students have the same mathematics schooling path composed of six courses (one per semester) with five hours each class per week: 1) Algebra, 2) Geometry and Trigonometry, 3) Analytical Geometry, 4) Differential Calculus, 5) Integral Calculus and 6) Probability and Statistics. Generally, there is a traditional process of teaching and learning mathematics because mathematics classes focus primarily on the teacher's explanation and the subsequent resolution of exercises by the students.

Participants

We selected 53 regular students for the study (aged between 16 and 18 years, 29 men and 24 women). They were in their fourth semester. Participation was voluntary. We chose this type of participants because given his age we believe they would be able to verbalize their emotional experiences. And since they have completed more than one year in high school, they would be able to inform us their experiences in math class in high school.The participants were officially registered in their fourth semester in the Differential Calculus course, which focuses on developing algebraic skills to study elementary Differential Calculus. The topics of this course are: (1) functions, limits and continuity, (2) algebraic functions derivatives and (3) transcendental functions derivatives.

Data gathering procedure

Methodologically, we decided to access to the students' emotions from their reports of experienced emotions because the focus of the research is on the students' subjective experiences of emotions. Thus, we carried out nine focus group interviews of approximately one and a half hours during the mathematics classes in a regular classroom. We decided to use it because

<table>
<tr><th>Class of emotions</th><th>Group of emotions</th><th>Local variables</th><th>Central variables</th><th>Global variables</th></tr>
<tr><td rowspan="3">Reactions to events</td><td>Fortunes-of- others</td><td>Desirability-for-other
Liking
Deservingness</td><td rowspan="3">Desirability (evaluated in terms of goals)</td><td rowspan="5">Sense of reality
Proximity
Unexpectedness
Arousal</td></tr>
<tr><td>Prospect-based</td><td>Likelihood
Effort
Realization</td></tr>
<tr><td>Well-being</td><td></td></tr>
<tr><td>Reactions to agents</td><td>Attribution</td><td>Strength of cognitive unit
Expectation-deviation</td><td>Praiseworthiness (evaluated in terms of standards)</td></tr>
<tr><td>Reactions to objects</td><td>Attraction</td><td>Familiarity</td><td>Appealingness (evaluated in terms of attitudes)</td></tr>
</table>

Table 2: Variables affecting the intensity of emotions according to the OCC

we observed during previous research at the same school that students feel confident and comfortable to express their thoughts, feelings and emotions about various topics in focus group interview.

For some researchers, individual interviews are more appropriate because emotions are personal. For others, focus group interviews are better because the interaction with others, who potentially feel the same, allows more free expression (Krueger, 1994). In this regard Krueger (1994, p. 20) mentions that "the focus group presents a more natural environment than that of an individual interview because participants are influencing and influenced by others just as they are in real life". Thus we consider that, in our research, the conducted focus group interviews are the correct choice. This is because our research goal is to identify whole participants' emotional experiences; it is not our goal to identify the emotional experiences separately for each of the participants.

The questions asked in the focus groups were: 1) What feelings or emotions do you experience about mathematics? Why do you feel this? 2) What feelings or emotions do you experience in the mathematics classroom? Why do you feel this? 3) What feelings or emotions do you experience just before a mathematics class? And later? Why do you feel this? 4) What feelings or emotions do you experience when you learn mathematics? And when do you do not learn? Why do you feel this? 5) What feelings or emotions do you experience when you solve a mathematical problem? And when you cannot? Why do you feel this? 6) What feelings or emotions do you experience in a good mathematics class? And in a bad class? Why do you feel this? 7) What feelings or emotions do you experience when a mathematics teacher is explaining? Why do you feel this, 8) What feelings or emotions do you experience for a good mathematics teacher? And for a teacher that is not good? 9) What feelings or emotions do you experience in a mathematics assessment? Why do you feel this? 10) What feelings or emotions do you experience in a mathematics test? Why do you feel this?

Two collaborators of the author conducted interviews. One is a PhD student in the field of mathematics education with experience in data analysis using the OCC theory (she is coauthor of the research of Martínez-Sierra & García González (2014)). The other interviewer is a research assistant with experience in conducting individual interviews and focus group interviews. Both interviewers are outside the everyday context of students. The analysis was conducted by the author and discussed with the PhD student. Considering that triangulation the author wrote this version of the analysis.

Data analysis

The videotaped interviews were fully transcribed. In the transcript, students were identified as M*n*-G*k* or F*n*-G*k*: M and F indicate that the participant is male or female; *n* (1 to 5 or 6) is the participant identification number; G*k* (1 to 9) indicates the focus group number. We included explanations in square brackets in order to clarify some of the students' expressions. According to OCC theory to identify a type of emotion we consider three specifications:

1) **Concise phrases** that express all the eliciting conditions of the emotional experiences. We highlight with italic bold letters the concise phrases that shows the eliciting conditions of an emotion in the evidence.

2) **Emotion words** that express emotional experience. We highlight with italic letters the concise phrases that show the emotions in the evidence.

3) **Variables** that affect the intensity of emotions. We underlined phrases that express intensity of the variables in the evidence.

RESULTS

Table 3 shows the students' emotional experiences in the mathematics classroom.

To illustrate in the following we show in detail the evidence related disappointment emotions.

DISAPPOINTMENT EMOTIONS

Not being able to solve problems

Disappointment emotions are triggered when the interest goal of solving problems is not attained.

> **M4-G8:** *I get angry, stress and with a headache* ***if I am not able to solve a problem, because I cannot reach a solution.***

There are two local variables that affect intensity of disappointment emotions: effort and probability.

The effort variable reflects the number of resources that the student uses to solve a problem. These resources depend on the kind of problem; for example, the student can ask their teacher or classmates for help during classwork, but this is not possible in a test.

> **F2-G8:** ***When I am not able to solve a problem*** then I ask myself *what to do because I didn´t* understand anything, <u>so I ask for help from my teacher or a classmate</u>.

A student can choose whether or not to solve a problem in a test. The student who chooses to solve it spends more time on it in order to reach a solution. This effort is linked to the interest goal of passing the test. If this goal is attained then it could trigger future active goal attainments like passing the course.

> **M1-G8:** ***If the problem is in a test,*** <u>then I will go to the extraordinary test.</u>

The probability variable reflects the degree of a student's belief that they will pass a test due to the solution of a problem. Disappointment emotions are more intense when the student believes that not passing the test is a consequence of not being able to solve a problem.

> **M1-G4:** *I feel bad* ***when I cannot solve a problem in a test,*** and I am constantly thinking of it until I get my grade. *I get angry and depressed.*

DISCUSION AND CONCLUSIONS

We found statements about 12 of the 24 types of emotions in our extended OCC theory. Except for *fears confirmed* emotions, we confirmed the presence of all emotions in the prospect-based group (*satisfaction, relief, disappointment* and *fear),* well-being group *(joy, distress, boredom* and *interest),* attribution (*pride, self-reproach* and *reproach*) and attraction group (*liking* and *disliking*).

There are three variables that intensify the prospect-based emotions triggered by the confirmation (satisfaction) or refutation of prospective situations (displeasure, hope and

Group	Type of emotion	Triggering situations	Variables that affect intensity
PROSPECT-BASED	Satisfaction	Being able to solve a problem	
	Hope	Not being able to solve a problem	
	Fear	Not understanding Fearing mathematics class Not passing a test	Desirability
	Joy	End of class Being able to solve a problem at the blackboard	Desirability
WELL-BEING	Distress	Not being able to solve a problem in class Not being able to solve a problem in a test Going to the blackboard	Desirability
	Boring	Not understanding the teacher's explanation Being in a non-dynamic class	
	Interest	Being able to understand teachers' explanations Having a positive attitude towards the teacher Being motivated to pay attention	Desirability
	Pride	Passing a course Being able to solve a problem	
ATTRIBUTION	Reproach	Reproaching the teacher	
	Self-reproach	Not being able to solve a problem	Expectation-deviation
	Liking	Understanding mathematics Being able to solve a problem	
ATTRACTION	Disliking	Not being able to solve a problem	

Table 3: Students' emotional experiences in a mathematics classroom

fear): (1) the desirability of a situation based on the attachment of a goal (such as solving problems), (2) the degree of belief that a prospective situation will actually occur (such as passing a test) and (3) the number of resources used to obtain or avoid a prospective situation (such as solving a problem in a test).

Well-being emotions are intensified by the desirability of the achievement of goals. Each goal is valued according to the goal structure of the student. The degree of this desirability is related to the degree to which the person expects positive consequences from the event. So, intensity of joy and interest emotions increases with desirability. On the other hand, intensity of distress and boredom emotions increases as desirability decreases.

In attribution emotions, pride is intensified in the event that the student follows the rules in their context. For example, a student will feel proud for the effort put into solving a problem or passing a course if this effort is worthy by itself. Intensity of reproach and self-reproach emotions increases depending on the deviation of the expected roles of students and teachers, as in the case of failing a course due to the teacher's actions.

Attraction emotions are intensified by the amount of time students have been attending mathematics courses. The number of courses attended affects students' emotional response: some of them consider mathematics to be a difficult but nice course and others simply don´t like it. This is not a transitory appraisal; it is the consequence of their academic life, which is still in formation and which influences their beliefs about mathematics.

We only found three types of goals that trigger all students' emotions in this context, even when OCC theory states different appraisal structures for each type of emotion. These goals are: active-pursuit goals (A-goals), interest goals (I-goals), and replenishment goals (R-goals). Their final structure is shown in Diagram 1. So, each situation has an implicit or explicit goal that triggers an emotion. This emotion will be positive or negative depending on whether the goal is achieved or not.

In Diagram 1 we express the different relationships between goals with arrows. An arrow from one goal to another means that the first goal may directly affect the achievement of the second goal. For example, passing a test is a goal that affects both passing a course and finishing high school. The diagram also expresses different ways to achieve a goal. For example, understanding teacher's explanations, paying attention in class or attending classes can help with the achievement of solving problems. Furthermore, the letter N above an arrow denotes that the first goal is necessary to achieve the second goal; the letter F denotes that the first goal facilitates the achievement of the second goal. For example, passing high school is a necessary goal to study at university and it facilitates getting a job.

Altogether the goal structure is an inherent part of the students' high school tradition. Goals can be explicit or implicit. The goal structure can be taken as part of the "didactic contract" (understood as "the set of specific behaviours of the teacher which are expected by the student and the set of behaviours of the students which are expected", Brousseau, 1997, p. 31), because it influences, along with the emotional reactions, the

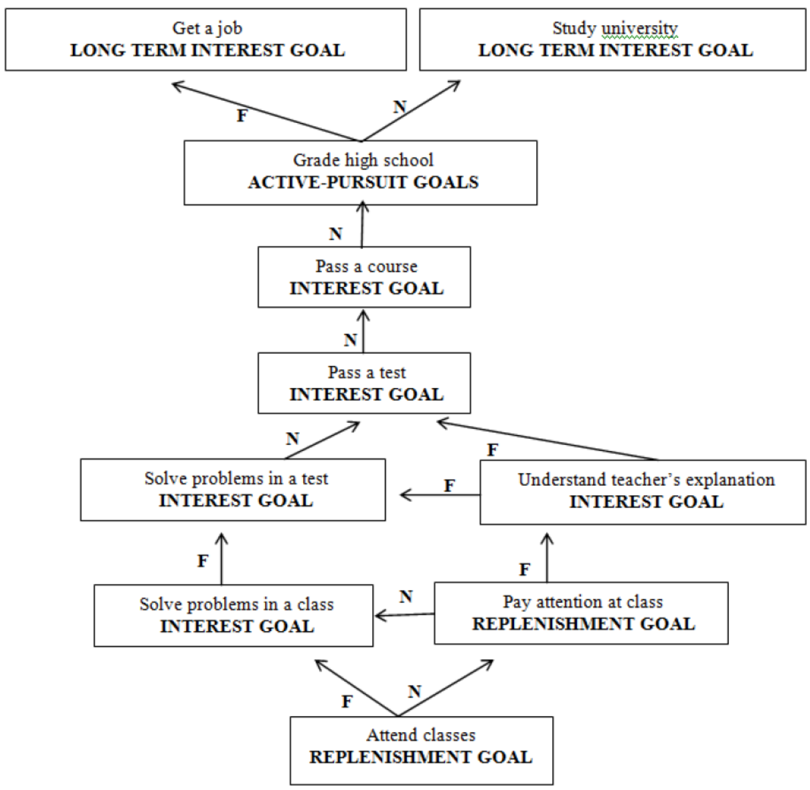

Diagram 1: Students' structure of goals

expected behavior of students and teachers in class. So, students direct their emotions in order to stimulate and guide their conduct to achieve goals that are implicitly or explicitly established in the mathematics class. This is consistent with the perspectives that highlight the complementary relationship between emotion and motivation in learning and performance (e.g., Kim & Hodges, 2011).

Finally, we consider that it is necessary to keep investigating students' emotions (and teachers') in different academic settings and at different school levels. Appraisal theories could help to identify the specific appraisal structure for each academic setting and school level. In this sense, we consider that people experience the same emotions but with quite different appraisal structures even if each individual values an event depending on their own appraisal structure.

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Mathematics teachers in preservice teachers' metaphors

Çiğdem Haser[1], Okan Arslan[1] and Kübra Çelikdemir[2]

1 Middle East Technical University, Ankara, Turkey, chaser@metu.edu.tr

2 Gazi University, Ankara, Turkey

This study explored preservice mathematics teachers' beliefs about mathematics teachers through asking them about their metaphors for mathematics teachers. Preservice teachers' (N=249) metaphors and explanations for mathematics teacher were analysed considering the categorizations developed for the NorBa-TM project based on the extended framework of Beijaard and colleagues (2000). Most of the preservice teachers described mathematics teachers as didactics expert and through their personalities. The findings are discussed and implications for field of mathematics teacher education are presented.

Keywords: Preservice teachers, metaphors, beliefs.

INTRODUCTION

Teachers' beliefs can provide us an insight on how and/or why they teach in a certain way (Cross, 2009). Beliefs about the mathematics teacher are likely to provide information about the roles teachers might assume in the mathematics classroom and how they might enact these roles. Despite the constructivist reform in mathematics education in Turkey since 2005, many teachers continued to conduct more traditional teaching (Avcu, 2014) and many preservice teachers come to teacher education programs with beliefs developed in these classrooms. However, teacher education programs focus on more constructivist roles for preservice teachers and try to initiate related beliefs (Haser & Doğan, 2012). Understanding the image of teacher in preservice teachers' minds might provide clues about the effectiveness of teacher education programs in helping preservice teachers develop beliefs which will help them in their future career.

One way to understand preservice teachers' beliefs about mathematics teachers is to analyse their metaphors for mathematics teachers. The word "metaphor" derives from Greek term "metapherein" which means "to carry over" (Green, 1971). Basically, when describing something with a metaphor we somehow transfer the characteristics and experiences of one thing to another by considering the similarities between these two things (Lakoff & Johnson, 1980). Metaphors could be interpreted as blueprints of people's thought (Martinez, Sauleda & Huber, 2001) and they work as creative instruments to deeply understand a complex phenomenon when it is rather difficult to describe it (Oksanen & Hannula, 2013). Therefore, metaphors are used by researchers in order to investigate preservice and in-service teachers' beliefs about teaching and teaching environments which could be interpreted as complex constructs (Massengil-Shaw & Mahlios, 2008). Teacher identity frameworks could be helpful in analysing metaphors for teachers.

Beijaard, Verloop and Vermunt (2000) identified three distinct knowledge bases of teacher knowledge reflecting teacher's professional identity. Accordingly, teacher identity could be expressed in terms of teacher as *a subject matter expert*, *a didactic expert*, and *a pedagogical expert*. *Teachers as subject matter experts* have deep knowledge in their discipline and transmit information to their students. *Teachers as didactic experts* have knowledge on planning, implementation, and evaluation of teaching and learning process to facilitate understanding for students. *Teachers as pedagogical experts* focus on caring and nurturing students' holistic development (Beijaard et al., 2000). These three aspects of the model are connected with Shulman's (1986) ideas of teacher's content knowledge, pedagogical content knowledge and pedagogical knowledge respectively. However, teacher identity is beyond what teachers should know; rather it focuses on what teachers consider as important in their professional work (Beijaard, Meijer, & Verloop, 2004).

Löfström, Anspal, Hannula and Poom-Valickis (2010) have investigated and categorised preservice teachers' metaphors for teachers by using this model of teacher identity. They further added *self-referential metaphors* to address teachers' personality and suggested that metaphors could be labelled as *contextual* when they described characteristics of the environment teachers worked in. Metaphors including more than one characterisation in equal emphasis were considered as *hybrid*.

The above categorizations were investigated in preservice and inservice teachers' metaphors in Finland. Findings suggested that the most frequent metaphor category referred by inservice mathematics teachers was *didactics experts* (Oksanen & Hannula, 2013; Oksanen, Portaankorva-Koivisto, & Hannula, 2014), whereas preservice teachers mostly preferred *self-referential* metaphors (Oksanen et al., 2014). This difference might be due to the way participants were asked about their metaphors; inservice teachers were asked to complete "teacher is like..." statement and preservice teachers were asked to complete "as a mathematics teacher I am.... " (Oksanen et al., 2014).

The purpose of this study was to explore preservice mathematics teachers' beliefs about mathematics teachers by asking them about their metaphors of mathematics teachers. Another aim of the study was to pilot categorization of teachers' metaphors suggested by Löfström and colleagues (2010) based on Beijaard and colleagues' (2000) model.

METHODOLOGY

Context and participants

The study was conducted at Elementary Mathematics Education (EME) Programs at four Universities in Ankara, Turkey. These four-year programs train teachers for teaching middle school mathematics (grades 5 to 8) and courses are determined by the Higher Education Council (YÖK), the governing body of all universities in Turkey. Although the courses are distributed differently in programs, two-semester methods of teaching mathematics courses are offered in the third year and practice teaching courses are offered in the fourth year of the program across the universities (YÖK, 2007).

A total of 249 preservice teachers were accessed at the end of the spring 2014 and 226 of them (33 male, 193 female) were the participants of this study. Participants were 3rd year (123) and 4th year (103) preservice teachers because they were the participant group relatively close to the mathematics teaching profession. Differences in metaphors due to year level in the programs were not the focus of the study.

Data collection and instruments

The study was a part of a more comprehensive international comparative study NorBaTM (New Open Research: Beliefs about Teaching Mathematics, formerly known as NorBa) conducted in over 15 countries in order to investigate mathematics teachers' beliefs. The questionnaire used for the present study was elaborated from the more comprehensive scale used in NorBaTM study.

The metaphor questionnaire was composed of three parts. In the first part, participants' age, gender, and year level in the EME programs were asked. Then, a brief description of the word "metaphor" was provided as a way of describing a concept by using similarities to another concept. This description was given because the participants might not be familiar with the term "metaphor" or what it actually meant. A similar word used in Turkish language was also reminded. In the second and third parts, participants were asked to describe mathematics teacher and mathematics teaching respectively through metaphors and explain their metaphors. In this paper, their responses to the following statement are reported: "Mathematics teacher is like .... Because, ..."

Researchers contacted the EME Programs at participating Universities, after necessary ethics permissions were granted. They were allowed to collect data towards the end of classes. Pre-service teachers who were at the provided place of data collection (classes) at the time of data collection were surveyed by the questionnaire. They were informed about the study by the researchers and given 20 minutes to complete the questionnaire.

Data analysis was performed by employing the categorization explained below through a manual developed for the NorBa project (Löfström, Poom-Valickis, & Hannula, 2011). First, all three researchers carefully read and discussed about the metaphor categorization. Then, a randomly selected 20% sample of data was coded by the researchers individually. Researchers compared their codes and discussed about the minor differences in coding. This process helped the researchers to make more sense of the categorization and the possible examples in data for these categories. This pilot coding was completed with almost 100% agreement. Then, all data were coded by the researchers individually through the specified categorization. Three researchers compared their coding of data case-by-case. A total of 23 cases including 5 no-response, 9 invalid (cases which researchers could not code), and 9 undecided (cases which researchers could not agree on the final categories) cases were removed from the data. Remaining 226 cases were considered as the data of the study. The disagreements that appeared during the comparison of researchers' coding were discussed to avoid over-interpretation of the explanations.

RESULTS

Distribution of metaphors used by preservice mathematics teachers is presented in Figure 1 below. While the didactics (29.6%) and self-referential (26.5%) categories were seen as the highest categories, only a small portion of the participants preferred to use contextual metaphors (2.2%).

Teacher as didactics expert (29.6%)

Teacher as didactics expert was the most frequent metaphor in the current study. Preservice teachers in this group mostly described mathematics teacher as a guide who assisted students to discover and understand the world of mathematics, and helped students when they had difficulty in mathematics, with metaphors such as guide (3 times), light (2 times), candle, star, and map. Participants also mentioned that mathematics teacher used different ways and methods to facilitate mathematical learning of students. For instance, one participant considered a mathematics teacher like an enzyme:

> Mathematics teacher supports students to discover mathematical ideas through questioning, making inferences and evaluations. Like an enzyme in chemical reactions, a good mathematics teacher can facilitate the mathematical learning of students, whereas a bad mathematics teacher might cause to slow down this learning process.

Some of the preservice teachers who emphasized the importance of using different methods also described mathematics teacher as a creative artist who performed different roles in a mathematics lesson in order to gain attention of students and implement non-routine mathematics instruction.

Another common characteristic for mathematics teachers was providing a basis for students' mathematical knowledge. Preservice teachers who stressed this issue generally associated mathematics teaching with constructing a building:

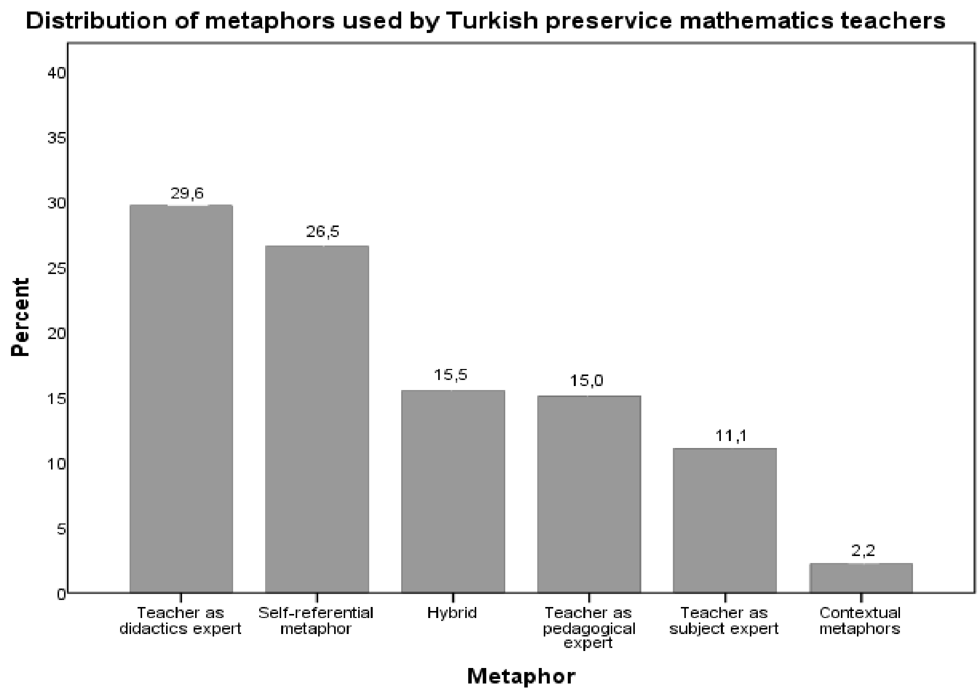

Figure 1: Distribution of metaphors used by the preservice mathematics teachers

> Mathematics teacher is like a contractor. If a contractor uses high quality material and properly constructs the base, the building becomes strong. Similarly, if a teacher teaches the topics by enabling meaningful learning instead of rote learning, students' mathematical knowledge becomes strong.

Self-referential metaphors (26.5%)

In self-referential metaphors, there was an emphasis on mathematics teachers' characteristics which were not related with teaching or mathematics teaching profession. Some of these metaphors were used to appreciate mathematics teachers, whereas some to criticize them. Common positive characteristics of mathematics teachers were stated as being smart, hardworking and practical. Some metaphors describing these positive characteristics were an ant (often used to describe hard working people in Turkish culture), small house appliances, and human brain. Participants appreciated mathematics teachers by stating that mathematics was difficult and only smart people could do mathematics.

Surprisingly, some preservice teachers mentioned mathematics teachers' negative characteristics by portraying them as insensitive machines such as robots or computers. Often, they were not friendly with students:

> Mathematics teacher is like a gravestone. They get tough with the class and do not ever smile.

Some other negative characteristics attributed to mathematics teachers were being arrogant and scary. These negative characteristics were rather stated as how mathematics and mathematics teachers were perceived by students:

> Mathematics teacher is like a doctor. A child who is afraid of getting a shot does not like the doctor. [Similarly,] when a student does not like mathematics, s/he does not like mathematics teacher. When s/he is afraid of mathematics, s/he is also afraid of mathematics teacher.

Participants also frequently stressed that mathematics teacher should be patient while teaching mathematics, which was a difficult job. Patience stone (a phrase used to describe a very patient person in Turkish culture, 3 times) and gardener (2 times) metaphors were used to emphasize this characteristic.

Hybrids (15.5 %)

Hybrids were metaphors including more than one category. The most common category in hybrids was didactical expert (27 times). Didactical expert category was generally expressed together with subject expert (11 times), pedagogical expert (10 times) and self-referential (9 times) categories. The following metaphor included both didactics and subject expert aspects:

> Mathematics teacher is like a chess player. For a chess player, it is not enough to know how each chessman moves. The player also needs to know how to develop tactics to win the game. Similarly, mathematics teacher needs to know all of the details of the topics and apply this knowledge to the lesson considering the students' needs and backgrounds.

Another observable issue about hybrids was that the number of hybrids including contextual elements (7/35) was more than the uniformed contextual metaphors (5/191). It seemed that preservice teachers preferred to use contextual elements by considering other characteristics of a teacher.

Teacher as pedagogical expert (15%)

Preservice teachers who stated metaphors in this category generally mentioned the guiding mission of mathematics teachers in students' lives. Mathematics teachers should support the development of students as human beings and thus, they should enlighten students' lives such as the sun (3 times), light, lighthouse, and pole star. For instance, one participant portrayed the mathematics teacher as the sun and the moon:

> Mathematics teacher enlightens his/her students. S/he tries to guide them and helps students to realize the things around them. Then, s/he observes what they can do by themselves. This is the time when mathematics teacher is like a moon. S/he does not leave them alone, s/he supports them like how moon looks after the night.

Another common issue was the caring and nurturing characteristics of mathematics teachers where they were described as merciful and helpful. Metaphors for the caring characteristics were mother (3 times) and father.

Teacher as the subject matter expert (11.1%)

Preservice teachers who described mathematics teachers as subject matter experts generally focused on two characteristics of mathematics teachers: (i) being knowledgeable and (ii) performing operations without making mistakes.

In the first group, there was a clear emphasis on the knowledge of mathematics teachers as indicated by metaphors such as book, journal, and encyclopaedia. In these metaphors, mathematics teachers were characterized as having accurate knowledge of mathematics similar to a book. Some participants indicated that a mathematics teacher was not only knowledgeable in mathematics, but also knowledgeable in the other content.

In the second group of metaphors, there was an emphasis on the calculation skills of mathematics teachers. Mathematics teachers could successfully perform operations and solve problems in a short time without making mistakes. Metaphors such as smart phone, calculator, and computer were stated:

> Mathematics teacher is like a calculator. A mathematics teacher should perform operations very efficiently otherwise, s/he might be interpreted as weak. S/he is expected to answer questions immediately.

Contextual metaphors (2.2%)

In the current study, only five preservice teachers uniformly mentioned the contextual factors while describing mathematics teachers with metaphors. Two of these preservice teachers focused on what it meant to be a mathematics teacher in Turkey. They indicated that it was difficult to be a teacher and a mathematics teacher in Turkey, and it was not appreciated enough:

> Mathematics teacher is like a slave. Because, a teacher has no value in this country. Furthermore, it is not a well-paid profession and thus, I just consider him/her as a slave.

Another stressed issue was related with how mathematics was seen in the society. Negative bias and fear of mathematics were the main foci:

> Mathematics teacher is like a bogy. In our country, teachers shout and get mad at students. According to my observations, students are especially scared of the mathematics teacher.

DISCUSSION AND CONCLUSIONS

What do these metaphors tell us? They seem to address that preservice middle school mathematics teachers prioritize didactics knowledge and skills when they consider a mathematics teacher as evidenced in their *didactics expert* and *hybrid* metaphors. For the Turkish context, the frequent reference to *didactics expert* might be a reflection of the courses on mathematics teaching and learning offered in the 3rd and 4th year where the guiding characteristic of mathematics teachers for students' learning was emphasized. EME programs had a major change in 2006 and the number of pedagogical content knowledge courses were increased without decreasing the number of mathematics content knowledge courses. A study conducted in the previous version of the EME program with less number of pedagogical content knowledge revealed that preservice teachers mostly believed that a mathematics teacher should have mathematics content knowledge, then pedagogical content knowledge, and then pedagogical knowledge (Haser & Doğan, 2012) in Shulman's (1986) terms. This corresponds to being *subject expert*, then *didactics expert*, and then *pedagogical expert* in the current study. Considering these findings, we cautiously speculate that the emphasis on pedagogical content knowledge courses in the last two years of the EME programs might affect preservice mathematics teachers' metaphors towards *didactics expert*. However, since we did not collect data from 1st and 2nd year students, we do not have information about preservice teachers' metaphors in the first two years of the program and this interpretation is very limited. On the other hand, Finnish preservice teachers described teachers mostly by their personality (*self-referential* metaphors) when they were asked in a rather personalized or subjective way (Oksanen et al., 2014), which could be the case in Turkish context if we had asked in a personalized way.

Preservice mathematics teachers who described a *subject matter expert* mostly emphasized procedural knowledge of mathematics rather than conceptual knowledge. Preservice mathematics teachers in EME programs have been reported to have both constructivist and traditional beliefs about the nature of mathematics (Kayan, Haser, & Işıksal, 2013). It seems that metaphors provided a different window for us to gain more knowledge about beliefs that preservice teachers might have carried from their precollege education.

Self-referential metaphors were stated mostly in relation to mathematics. Being a mathematics teacher was valued by emphasizing that teaching mathematics required being hardworking, smart, and patient. Surprisingly, some participants criticized mathematics teachers for being rather unfriendly and scary. It was not clear whether participants described the mathematics teachers in the eyes of the students or society, or what a mathematics teacher meant for them in their explanations. Therefore, what these metaphors communicated in terms of preservice teachers' beliefs about mathematics teachers remained inconclusive.

Oksanen and colleagues (2014) state that *hybrid* metaphors might reflect the complexity of the teaching profession. *Hybrid* metaphors might provide information about how different sides of the mathematics teaching profession are internalized and integrated. We argue that the effectiveness of teacher education programs might be traced by the *hybrid* metaphors that preservice teachers could develop. If preservice teachers would be able to explain their metaphors by referring to different types of teacher characteristics, could this be a reflection that they have developed a more comprehensive image of a mathematics teacher in their minds? This might be an issue for a further discussion and research in which preservice teachers' metaphors could be investigated through their studies in the teacher education programs and also based on the nature of the programs.

Contextual metaphors were the least mentioned metaphors in this study. This could be a reflection of the insufficient school experience in teacher education programs. Preservice middle school mathematics teachers spend 4 hours per week in the first semester and 6 hours in the second in their senior year in the program. This experience focuses on observing and generally includes 1 or 2 hours of teaching for the whole year. Therefore, they might not be experiencing the contextual elements about being a teacher as inservice teachers do. However, Finnish inservice teachers also did not state *contextual* metaphors much (Oksanen & Hannula, 2013). Yet, it might be the case that crowded classrooms and lack of sufficient instructional materials in Turkish schools (OECD, 2009) could result in more *contextual* metaphors if the study had been conducted with inservice teachers, compared to the Finnish case.

Using metaphors to gain insight about preservice teachers' certain beliefs also revealed evidences about other beliefs. Preservice teachers stated beliefs about the nature of mathematics in their explanations. Many explanations referred to society's views about the nature of, teaching, and learning mathematics. It seemed that asking metaphors might offer more or other than what was intended in the beginning. Asking specific experiences, significant person, or events that have caused to state their metaphors could provide more windows into participants' mathematics related beliefs.

The metaphor framework based on Beijard and colleagues (2000) model was effective in analysing Turkish metaphor data in this study. The eliminated data were difficult to conclude on a category due to the content they included. It should be noted that Turkish data for metaphors do not reveal participants' gender preferences (unless asked) in referring to a teacher because Turkish language does not have gender difference in referring to a person.

Certain limitations should be considered in making sense of the findings of this study. First, written data might not be as detailed as verbal data. Interviews conducted on these metaphors could have provided more insight into preservice teachers' beliefs about mathematics teacher. Participants might have written more about their explanations if they had been given more time.

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Basic emotions of primary school pupils in mathematics lessons

Ineta Helmane

Riga Teacher Training and Educational Management Academy, Faculty of Education, Riga, Latvia, ineta.helmane@gmail.com

The article describes and analyses theoretical and empirical materials the experienced emotions by pupils when learning mathematics in primary school and highlight the factors arousing emotions in learning mathematics in primary school. The article analyses the data obtained in empirical research on the emotions experienced by pupils during mathematics lessons in primary school. In questionnaires and semi-structured interviews pupils reveal what gave them positive emotions in mathematics lessons, as well as what made them experience negative emotions. According to the analysis of empirical data we highlighted the factors of learning mathematics in primary school which caused positive emotions.

Keywords: Basic emotions, pupils, mathematics, primary school.

INTRODUCTION

The great pace of life in the contemporary society demands emotionally powerful people, able to maintain a positive attitude towards life, overcome the fast increasing tension and stress, sustaining appropriate self-esteem, healthy self-confidence (Helmane, 2010). One of the meaningful components creating a harmonious individual is emotions. Emotions mobilize each individual's psychic and physical strength for the further activity to reach the goal, or prevent and hinder the individual's activities, participation in the diverse life- activity processes (Izard, 1991; Thompson & Meyer, 2007). Therefore, it is necessary to be aware of the factors which evoked positive emotions in pupils when learning mathematics at school. The aim of the research is to investigate the experienced basic emotions by pupils when learning mathematics in primary school and highlight the factors arousing emotions in learning mathematics in primary school.

MATERIALS AND METHODS

The essence of basic emotions

Basic emotions appear already in the first years of life through all cultures and with slight or without variations (Izard, 1991; Miezitis, 1992), they are based on the demonstration of a person's movements and expressions (Carlson, 1990). All basic emotions possess the following characteristic features: they attract a clear, strong feeling which a person is aware of; they develop as a result of evolution - biological processes; they have an organizing and motivating affect on the person, serve for his adaptation (Izard, 1991) and manifest themselves in expressive and specific configuration of the movements of facial muscles - mimics (Izard, 1991; LeDoux, 1998).

The physiological base of such positive basic emotions as joy, interest, surprise (Kagan & Havermann, 1980; Ekman, 1984; Carlson, 1990; Izard, 1991; Miezitis, 1992) is mainly excitation process. Positive emotions tone up the body's activities and activate the person, generate strength and energy as well as enhance the person's capability of mental work, increase energy, promote heuristic or creative processing (Carlson, 1990; Ambady & Gray, 2002; Fredrickson, 2004). However, negative basic emotions: anger, hatred, disgust, fear, shame, fault, sorrow (Kagan & Havermann, 1980; Ekman, 1984; Carlson, 1990; Izard, 1991) are based on retention process. Negative emotions usually depress, even paralyse a person as well as decrease a person's activity and also reduce energetic resources (Carlson, 1990; Selighran, 1995). Negative emotions are associate with reduced accuracy on tasks that tap memory, intelligence, and executive functioning (Hartlage et al., 1993; Veiel, 1997). Therefore, emotions may have a positive or negative affect on a person's life processes, becoming a determining force of a person's action in crucial moments of life.

Emotions in teaching-learning process

Emotions are involved in almost every aspect of the teaching and learning process (Schutz & Lanehart, 2002). Pupils' emotions are formed at the interface of personal, contextual, and social aspects of learning (Volet & Järvelä, 2001; Ainley & Hidi, 2002; Schutz & Pekrun, 2007). The teaching/learning process based on positive emotions proceeds more successfully. If we do something with pleasure, we will try to do the same in future (Frenzel, Perkun, & Goetz, 2007). Within positive emotions a more profound approach to the acquisition of various skills and knowledge develops which facilitates openness to new things, creativity and energy to be productive (Olsonm & Torrance, 1996; Gorman, 2001). The positive emotions become a strong motivating, suggestive factor for future actions (LeDoux, 1998; Linnenbrink & Pintrich, 2000; Fredrickson, 2001). If successful, emotions positively motivate and reinforce several extremely successful activities, guesses and ideas which came up during the completion of tasks.

However, negative emotions direct attention and cognitive processing in a negative way (Power & Dalgleish, 1997; Linnenbrink & Pintrich, 2004), reduce the effectiveness of learning, the working memory, the ability to acquire various types of knowledge, skills as well as creativity (Olsonm & Torrance, 1996; Linnenbrink & Pintrich, 2000). If pupil is doing something with negative emotions, he/she will try to stop doing it in all possible ways in future (Frenzel, Perkun, & Goetz, 2007). Also monotonous, boring learning process and failures cause negative emotions (Selighran, 1995).

Learning mathematics is connected with the pupil's individual experience in mathematics and applying it in everyday life, the perspective of his individual learning where emotional factors are as significant and important as cognitive factors in learning process (Tosse, Falkencrone, Puurula, & Bergstedt, 1998). Emotions also include and sustain pupils interest in learning material (Ainley, Corrigan, & Richardson, 2005; Krapp, 2005), in teaching/learning content. Paris and Ayers (1994) underline the value of emotions in learning process - nobody can develop mathematics or intellectual values without emotions, especially in mathematics. The positive learning experience can help to change negative thoughts and feelings and raise pupils' motivation in learning process (Paris & Ayers, 1994). Pupils who experience more positive emotions may generate more ideas and strategies. In addition, emotions can have an impact on different cognitive, regulatory and thinking strategies (Pekrun, 1992), affect categorising, thinking and problem solving (Sutton & Wheatley, 2003). In contrast negative emotions may trigger the use of more rigid strategies, such as simple rehearsal and reliance on algorithmic procedures, thus leading to reduced attention and more superficial processing of information (Pekrun et al., 2002). If a pupil feels sad, she/he may be preoccupied with thoughts about negative emotions and unable to refocus her/his attention on educational information. This would have implications for children's academic performance (Davis & Levine, 2013). Negative emotions also commonly disrupt mathematics learning. Some children have a condition termed math anxiety that is characterised by fear of mathematics (Ashcraft, 2002; Hinton & Miyamoto, 2008). This emotional state disrupts cognitive strategies and working memory (Ashcraft & Kirk, 2001).

Research of pupils' basic emotions in mathematics lessons

The research of emotions experienced by pupils in mathematics lessons in primary school was carried out in Grade 3 in four Riga schools, total of 107 pupils (age 8–9). The selection of the schools involved into the research was done by intentional assessing of the school environment descriptions, based on the similarities of the following qualities: school social economic environment, ethnic environment, school's physical environment, time-table, the number of pupils in the school, as well as the length of the teachers' pedagogical experience. The selection of the classes involved into the research was done by intentional assessing of the mathematics lessons descriptions, based on the similarities of the following qualities: National Basic Education Standard (2006), mathematics text books according to List of Confirmed and Published text books (2009), 4 mathematics lessons per week. The research was done within the academic year over the period from 2012 to 2013 in which pupils self-evaluate experienced basic emotions while learning mathematics in primary school were explored.

The data were obtained applying such empiric research methods as questionnaires with open questions, semi-structured interviews and test as Dembo's methodology for self-assessment of basic emotions (Helmane, 2010). The aim of empirical research methods was to select and to specify the experienced basic emotions by pupils in mathematics lessons in primary school and the factors evoking emotions in general learning mathematics situations in primary school.

The pupils were asked to fill in a questionnaire about the experienced basic emotions while learning mathematics. The pupils of Form 3 involved in the research marked individually on Likert-type scale positive emotions (joy, interest, surprise) and negative emotions (anger, disgust, fear, shame, fault, sorrow) experienced while learning mathematics in primary school. When marking every basic emotion, the pupils took into consideration Likert- type scale where 1 point corresponds to the answer – never experienced the given emotion, 3 points correspond to the answer – the emotion has often been experienced, but 5 points meant that the emotion has always been experienced.

During the further survey the pupils individually reflected and pointed out at least 3 factors, stimuli, situations which evoked positive basic emotions at school as well as pointed out at least 3 factors, stimuli, situations when they felt negative basic emotions. The data obtained from questionnaires were specified in semi-structured interview where the pupils supplemented the answers to the questions about the factors evoking basic emotions. The peculiarity of this semi-structured interview was that the questions previously were not formulated precisely, and also their succession was not strictly determined, however, during the interview it was clarified to what extent, on what conditions, in which situations the pupils experienced positive or negative emotions while learning mathematics in primary school as well as specified the factors evoking these emotions, their exposure according to the criteria set out for the research. Each individual interview was about 15 minutes long, it was recorded, transcribed and coded.

RESULTS

The data obtained according to Likert-type scale as a result of questionnaires by 107 pupils testify that during school time pupils' experienced positive and negative basic emotions at similar intensity. The situations when the pupils experienced or not experienced distinct positive emotions during learning mathematics are not in majority. In contrast, when evaluating basic negative emotions, the pupils indicate that they did not feel disgust or were ashamed in math lessons (see Picture 1).

It is characteristic that the pupils often experienced such positive basic emotions as joy and interest in mathematics lessons. In most cases, the pupils' interest and joy were aroused by the opportunity to use visual aids, play didactic games and the teacher's positive attitude in mathematics lessons. However, a positive evaluation of the given tasks and activities in mathematics has been a precondition which aroused joy, interest and surprise in pupils. In most cases, the pupils experienced such negative basic emotions as fear, shame and sorrow in mathematics lessons. These negative emotions caused the situations related to a pupil's incompetence, failure in doing a certain mathematics task as well as the cases when pupils encountered with a negative assessment of their work and the comparison of their work with that of the other pupils.

The majority of pupils (86%) confirmed in semi-structured interview and questionnaires that in mathematics lessons, it was interesting to work with small countable material. Joy, interest and surprise in pupils were aroused by the opportunity to do mathematics tasks with the help of sticks, fingers, coins and banknotes. As a result of manipulation, the experi-

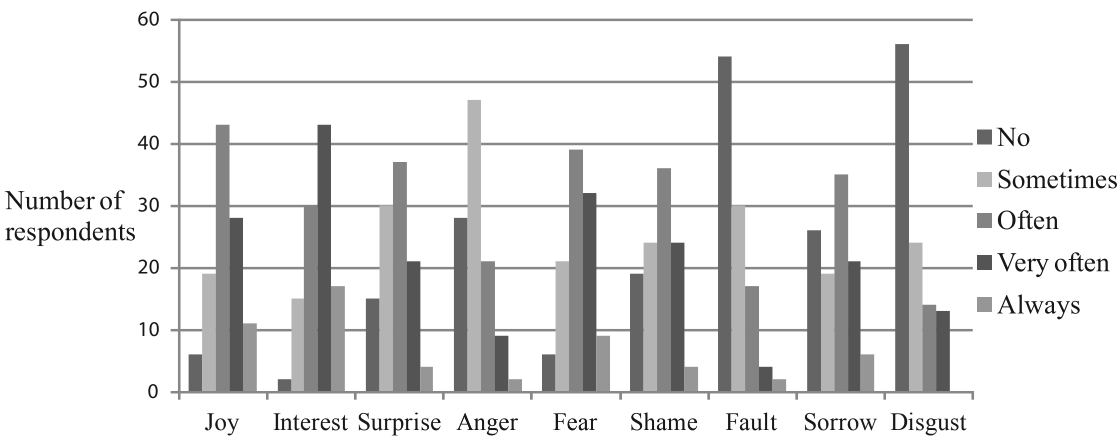

Figure 1: Pupils experienced basic emotions in mathematics lessons

enced positive emotions by pupils are related to the opportunity to get acquainted with mathematics skills more successfully, solve problems. Frequently, pupils mention varied didactic games with small countable material as a factor which generates positive emotions. It is testified by pupils' comments on the same question:

Interviewer: What is interesting in maths lessons?
Pupil: For me... to count with sticks and money banknotes.
Interviewer: What surprises you in mathematics?
Pupil: The teacher allowed me to count with fingers.
Interviewer: Count with fingers?
Pupil: Yes, it is cool, I can count everything... but mum does not allow to do it at home...

According to pupils' (92%) answers a factor evoking positive basic emotions is drawings, pictures in mathematics textbooks which attract pupils' attention, as well as facilitate the perception of the essence and conditions of the task. Pupils are happy about the objects seen in the pictures of the tasks, their reinforcing function in perception of the conditions and essence of the mathematics task. The pupils have interest in the tasks which are visualized in textbooks with the help of pictures and drawings, for example:

Interviewer: What surprises you in maths lessons?
Pupil: I have a colourful and beautiful maths textbook.
Interviewer: What is colourful and beautiful in it?
Pupil: A lot of pictures, I like that.
Interviewer: What do you do with pictures?
Pupil: I look at them, then I understand and... do correctly.

The teacher's personality as a factor evoking emotions can initiate both basic positive and negative emotions. The majority of pupils (44%) comment that positive basic emotions have been evoked by a sensitive, creative teacher, who has a positive attitude towards the pupil, interest in mathematics as a subject. The pupils experienced positive basic emotions in such mathematics lessons which were exciting, interesting, not boring. In these lessons, the pupils had joy about varied organizational forms of the teaching process (group work, games, research, manipulative activities), when the pupils were provided with the opportunity to be active participants of the study process. The interest in pupils was aroused by the explanation of theoretical concepts and the essence of mathematics skills with the help of real objects, thus the study content to be obtained was explained visually, for instance:

Interviewer: What is interesting in your maths lessons?
Pupil: I have the best teacher; it is so cool to learn together with her, she shows everything.
Interviewer: How does she show everything?
Pupil: With money notes, drawings...we also play games...and she smiles all the time, she does not yell.

In the cases, when pupils (36%) experienced negative basic emotions during school years, the teacher of mathematics had not listened to pupils' thoughts, had not allowed them to be active participants of the study process, sometimes the teacher's working style had been authoritarian. Pupils experienced fear of the teacher's reaction about the incorrectly solved problem, anger about the teacher's intolerant attitude to the pupil's incompetence and failure in mathematics lessons. Negative emotions anger and disgust in pupils were also initiated by monotonous, uniform mathematics lessons where pupils were passive performers of the teacher's instructions. For instance:

Pupil: I hate to go to the lesson, where it is boring and not interesting.
Interviewer: How is it – boring?
Pupil: Every day the same – sit and do tasks.

Very often in questionnaires pupils (88%) mention their personal achievements and success in mathematics lessons as a prerequisite initiating positive emotions when each success allowed to experience positive emotions. Also, the recognition of the achieved (69%) has often evoked positive emotions. Pupils experienced joy and surprise if they could solve a mathematics task correctly and received a positive assessment for their work according to each pupil's individual contribution and growth. Joy was also aroused by such situations where pupils were able to solve different problems of higher difficulty level. Failures in most of the cases caused negative emotions in pupils (74%). The pupil's mistakes were not perceived as an opportunity to develop correct mistakes and master the skill. In mathematics lessons, pupils felt angry about the inability to do a task or could not successfully solve the problems given to them in mathematics test. Fear and shame are evoked in pupils in such situations when they are made to demonstrate their inability in front of other pupils, for instance, when solving a problem unsuccessfully at the blackboard. Pupils feel ashamed when they do not understand a mathematics task, if it is compared to the positive achievement of other pupils in mathematics. For instance:

Interviewer: What are you afraid of in maths lessons?
Pupil: I am afraid to solve problems because I can solve them incorrectly.
Interviewer: Do you need to be afraid of that?
Pupil: If there is a mistake, it will be bad... others will get to know.

When characterizing mathematics content, pupils in most of the cases mention the acquisition of word problem (78%) and multiplication within the table (72%) as a factor which evoked negative emotions while learning mathematics. In a questionnaires and semi-structured interview pupils clarify the experienced negative emotions as incomprehension about the necessary activities for doing the task, an insufficient skill to read a word problem. Most of the pupils emphasize that negative emotions anger, sorrow and fear while solving word problems were experienced also because the word problems did not arouse their interest and did not have connection with real life and the surrounding processes. When mastering multiplication within the table, pupils name the main reason for having negative emotions shame and fault i.e. learning multiplication table by heart without the comprehension about relationships in the multiplication table:

Interviewer: What do you feel ashamed of?
Pupil: That I have to memorize multiplication table.
Interviewer: Is it difficult to memorize?
Pupil: I cannot remember so much by heart... I cannot count as fast as it is necessary, I want to think a little...Multiplication is terrible, why must I memorize it?

Pupils (73%) experienced positive emotions most often while mastering addition and subtraction skills at school. Pupils point out that they willingly did arithmetic operations, it was easy and understandable. The joy experienced by pupils is mainly characterized as a qualitative application of mathematics skills during the solution of mathematics problems, for instance:

Interviewer: What are you most of all pleased in mathematics?
Pupil: About numbers and their addition. It is terrific because I have to add numbers and I can make it because I can do it easily.

Partly experienced negative emotions sorrow, anger are in the acquisition of the following mathematics content: mastering fractions (64%), mathematical variables and measures (57%), pupils consider boring such tasks where they mechanically have to perform mathematical transformations and express relationships between variables and measures, and it is not shown in which life situations and how these acquirable skills could be applied.

CONCLUSION

The mathematics learning process needed to include and use that sort of positive emotion-causing stimuli: the manipulation activities with objects, based on the practical independent activity with diverse visual aids, rational work modes acquiring by manipulation with objects; the purposeful system of exercising, where the pupils clearly recognize the exercise aim, understand the execution of the exercise, the exercises are arranged in a well-considered system, disseminated in time, the exercises also include the revision of the mathematical skill and they are miscellaneous; the use of the skills in diverse life-activity situations where the diverse work forms, methods and approaches are applied, integration possibility into other subjects, miscellaneous exercises according to their contents, forms (Helmane, 2010).

As a result of the research, it is possible to select the factors facilitating positive emotions while teaching mathematics in primary school: curriculum which is easy for a pupil to understand and perceive, which is encouraging, in good arrangement, intensity and appropriate difficulty level; efficient techniques of work and visual aids are used in the acquisition of skills; also the pupils' activities are practical and independent; diverse forms of work, methods, especially the method of play are used in the acquisition of skills which encourage a pupil's active participation and mobilization of potential in order to achieve the desired objective; communication with peers and the teacher; evaluation of a pupil's progress and achievements by the teacher, peers, also self-esteem.

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Development of mathematics-related beliefs scale for the 5th grade students in Turkey

Nermin Kıbrıslıoğlu[1] and Çiğdem Haser[2]

1 Hacettepe University, Faculty of Education, Ankara, Turkey, nkibrislioglu@hacettepe.edu.tr

2 Middle East Technical University, Ankara, Turkey

The aim of this study was to develop a valid and reliable scale for 5th grade students in Turkey to investigate their mathematics-related beliefs. For this purpose, a mathematics-related belief scale was constructed by the researchers based on Op't Eynde, De Corte and Verschaffel (2003) framework and piloted by 390 5th grade students from 2 middle schools in Ankara, Turkey. Data were analysed by descriptive and inferential statistics. Factor analysis revealed three factors: beliefs about mathematics and mathematics learning, self-efficacy beliefs, and beliefs about the role of the teacher. The overall reliability of the scale was .77, indicating a good reliability. The mathematics-related belief scale will be used in a future study where 5th grade students' mathematics related beliefs will be identified.

Keywords: Scale development, mathematics-related beliefs, 5th grade students.

INTRODUCTION

Beliefs are important components of mathematics teaching and learning process (Kloosterman, 1996; Philipp, 2007). Students' beliefs about mathematics influence the effort they will spend for the tasks, their interest in mathematics, and enjoyment with the task (Kloosterman, 2003), and they have the potential to determine how students connect real life activities and school mathematics (Lester Jr., 2002). Their beliefs about achievement have a considerable influence on their success at school (Wittrock, 1986). Students' learning experiences affect their beliefs and their beliefs about learning influence their approach to the new learning experiences (Spangler, 1992). Therefore, there seems to be a reciprocal relationship between mathematics learning and mathematics related beliefs (Schommer-Aikins, Duell, & Hutter, 2005).

Beliefs are formed by students' direct or indirect experiences (Lester Jr., 2003), which makes their classroom experiences important. However, in order to help students form beliefs which will enhance their mathematics learning, it is substantial to identify their existing beliefs. Identifying younger students' beliefs is specifically important because they are more open to be influenced by classroom experiences.

As a result of the recent change in school system in Turkey 5th grade is included in middle schools with some revision in mathematics curriculum. This revision did not change the emphasis on problem solving and building relationships in middle school mathematics and teachers' facilitating and guiding role for students' learning (MEB, 2013) that the previous curriculum stated. However, to what extent these emphases have influenced students' beliefs at this level has not been investigated yet. Investigating 5th grade students' beliefs when they start middle school will enable us to (i) understand the effectiveness of the elementary school mathematics instruction on students' mathematics related beliefs and (ii) determine the possible mathematical experiences in middle school grades which will help them learn meaningful mathematics. Therefore, the current study aimed to construct a valid and reliable instrument to investigate 5th grade students' mathematics-related beliefs.

There are several scales addressing students' mathematics-related beliefs. However, the psychometric properties of these scales, including some popular scales, are problematic (Walker-Wheeler, 2007) and their Turkish versions either lack validity evidences or target older students' beliefs. Therefore, there is a need to for a valid and reliable scale addressing younger students' mathematics related beliefs.

LITERATURE REVIEW

Several researchers have defined beliefs. For example, Fishbern and Ajzen (1975) define beliefs as information that a person has about an object or idea. According to Richardson (1996), beliefs are "psychologically held understandings, premises, or propositions about the world that are felt to be true" (p. 2). Kloosterman, Raymond and Emenaker (1996, p. 39) refer beliefs as "the personal assumptions from which individuals make decisions about the actions they will undertake." Schoenfeld (1998) indicates that "beliefs are mental constructs that represents the codification of people's experiences and understandings" (p. 21). These definitions address that we have beliefs about the world around us and we also use these beliefs to make sense of the world.

As beliefs are subject specific (Philipp, 2007), there is also a need to define mathematics-related beliefs. There are several definitions of mathematics-related beliefs and there hasn't been an agreement on a common definition (Furinghetti & Pehkonen, 2002). This study employed the definition proposed by Op't Eynde, De Corte and Verschaffel (2003) which addresses mathematics-related beliefs as "the implicitly or explicitly held subjective conceptions students hold to be true, that influence their mathematical learning and problem solving" (p. 28) because it focuses more on students' mathematical learning and problem solving.

Theoretical framework

Different approaches to mathematics-related beliefs have been discussed in the field and several researchers have introduced different categorizations, with common and distinct aspects (Philipp, 2007). A comprehensive framework given by Op't Eynde and colleagues(2003) is mainly based on Schoenfeld's (1983) view that cognitive actions are determined by the nature of the task, social environment, and the perception of the individual. Hence, beliefs reflect the effects of self, belief object and the context (Schoenfeld, 1983). Op't Eynde and colleagues (2003) proposed that mathematics-related beliefs were determined by the context, personal needs, and mathematics education. Their framework had three main categories of beliefs about: (i) mathematics education, (ii) the self, and (iii) social context. This framework was employed for the current study.

Beliefs about mathematics consisted categories of beliefs about mathematics as a subject, mathematics learning and problem solving, and mathematics teaching. Beliefs about mathematics were about the answer of the question "What is mathematics?" in students' minds. Beliefs about self included students' motivational beliefs such as self-efficacy, control, task value, and goal orientation beliefs. Beliefs about social context consisted of beliefs about social norms in students' classrooms including role and the functioning of the teacher and student; and beliefs about socio-mathematical norms such as beliefs about what is accepted as mathematical justification in the class (Cobb & Yackel, 2014). Table 1 summarizes the framework.

This framework is more comprehensive and it contains other classifications which provide a wider perspective for investigating students' beliefs. Therefore, it was employed as the theoretical framework in this study. However, its comprehensive nature makes it difficult to investigate these beliefs a single study with 5th grade students. Therefore, beliefs about nature of mathematics, learning mathematics, role of the teacher, and self-efficacy beliefs were investigated in this study.

SCALE DEVELOPMENT PROCESS

The scale was developed in three main steps. First, the related literature was reviewed in detail and items

Beliefs about mathematics education	Beliefs about self	Beliefs about the social context
-Beliefs about mathematics as a subjects -Beliefs about mathematical learning and problem solving -Beliefs about mathematics teaching in general	-Self-efficacy beliefs -Control beliefs -Task value beliefs -Goal-orientation beliefs	-Beliefs about social norms in their own class The role and functioning of the teacher The role and functioning of the students -Beliefs about socio-mathematical norms in their own class

Table 1: The framework of students' mathematic related beliefs

were written. Then, experts' opinions were gathered and items were revised. Last, five students were interviewed in order to ensure the clarity of the items for the students.

Literature review

An extensive literature review of belief frameworks in the literature were conducted and a comprehensive belief structure framework suggested by Op't Eynde and colleagues (2003) was employed for the study. A detailed examination of this framework, previous studies, and the characteristics of the age group resulted in four sub-domains (factors) to be considered for the study: beliefs about mathematics as subject, beliefs about learning and problem solving, mathematics self-efficacy beliefs, and beliefs about teacher role and functioning. Then, an extensive literature review was conducted for each factor. In general, these factors were studied individually in many studies through several scales in order to understand the types of items which explained the specific belief domain. An examination of these items showed that there were both common and different aspects, and these aspects were taken into consideration to get a more comprehensive instrument. After these studies, the first version of the instrument was developed with 68 items.

Experts' opinion

The first version of the scale was shared by two researchers working on beliefs in the field of mathematics education to examine the content and comprehensibility of the items. They were asked whether the items and factors were coherent, statements were clear, and expressions were appropriate for the 5th grade students. After they reviewed and suggested changes in the items, revised items were shared by two specialists in the field of educational measurement to ensure the properness of the scale in terms of measurement principles. Then, two middle school mathematics teachers reviewed the items with respect to clarity for students as they had more interaction with the students and they suggested certain changes about wording of the items. At the end of the experts' revision, scale consisted of 32 items.

Student interviews

After the revision of the scale, five 5th grade students from a public school in Ankara were interviewed about clarity of the statements. They were asked about what they understood of each item and whether there were any words that they did not know about their meanings. Students had difficulties in three items. It appeared that they considered "lecturing" and "guiding" the same. Therefore, the item "Our teachers guide us when we are learning" was deleted. Students also struggled with the item related to the relationship construction between old and new knowledge. While some students were able to understand the meaning of building relationship, others couldn't understand the item and preferred to respond as undecided. This item was revised as "I remember previous knowledge when I am learning new things" in order to make the statement clear.

In order to ensure that one class hour will be sufficient to complete the scale for students, the time students spent in answering items was observed. Students finished responding 32 items in approximately 20 minutes during the interviews. After the items were reviewed once more with respect to the interview results, the fourth version of the questionnaire was constructed which consisted of 34 items with 3 point Likert scale as agree (3 points), undecided (2 points), and disagree (1 points). Negative items were scored reverse. The maximum score one can get from the scale was 102 and the minimum score was 34. Three-point scale was preferred because 4^{th} and 5^{th} grade students might have difficulty in understanding "partially agree" or "partially disagree" phrases.

DATA COLLECTION AND ANALYSIS

Data for this pilot study were collected from two conveniently selected public middle schools in Ankara. Although there are different suggestions about the sample size for getting proper factor analysis results, it is indicated that larger sample sizes produce more proper results. Tabachnick and Fidell (2007) argue that sample size should be 10 times of the item number or at least 300 for proper factor analysis. These suggestions were taken into consideration and data were collected from 390 students (201 male, 182 female) from fifteen 5^{th} grade (ages 9–10) classrooms.

The implementation was conducted by the first researcher. Before distributing the scale, she explained students that there were no correct answers for the items and their thoughts were important for the research. Students were also informed that no information would be shared with their teachers, their answers would not affect their grades, and there was no need for writing their names. Data were collected in courses other than mathematics in order to reduce the teacher effect.

Principal Component Analysis (PCA) was conducted in order to determine subscales and validity of the scale. As the sample size was ensured during the data collection, other preparations were performed before conducting PCA. First, negative items were scored reverse. Second, Bartlett Sphericity Test and Kaiser-Meyer-Olkin (KMO) were checked to ensure that data set was factorable which means that some correlations should exist (Tabachnick & Fidell, 2007). In order data set to be appropriate for factor analysis, Bartlett Sphericity Test should be significant, which means p value smaller than 0.05 and KMO value should be at least 0.6 (Tabachnick & Fidell, 2007). The analysis indicated that data set was appropriate for factor analysis ($F=0.80$, $p < 0.05$).

Certain principles suggested by Tabachnick and Fidell (2007) guided the PCA: Items should have factor loadings 0.3 or above to fit the factor structure in the selection process. If items which are loaded in more than one factor have difference between factor loadings smaller than 0.1, then it is better to exclude them from the scale. When deciding the number of factors, the factors with eigenvalues more than 1 are initially taken into consideration. However, only eigenvalues may not be sufficient for the final decision. Another estimate can be made by interpreting scree plot but there is still need for more operations for proper factor solution. Based on these criteria, the factor analysis was repeated until reaching a proper factor solution.

RESULTS

The results of the first analysis showed that there were 11 factors whose eigenvalues were more than 1. However, the scree plot seemed very complicated to reduce the factors and factor loadings were not appropriate. In order to get a more appropriate factor solution, items with communalities less than 0.2 were removed because small communalities indicated that the variable was not related to the other variables in the data set (Tabachnick & Fidell, 2007). Therefore, 11 items were removed from the analysis. In order to reach the best factor solution, the Promax rotation method, an oblique rotation, was employed because gives better results in identifying the correlating and noncorrelating factors (Tabachnick & Fidell, 2007). After this reduction, the analysis was conducted once more and the most appropriate factor solution appeared as 3 factors solution. The scree plot in Figure 1 also supported 3 factors solution. Then, items which were loaded on unrelated factor, items whose factor loadings were smaller than 0.3, and items which were loaded almost equally to more than one factor were removed from the scale.

After deciding the factor structure, 5 items which were indicated as important in the literature were added one by one as they didn't conflict with the factor structure. Two items whose factor loadings were smaller than 0.3 were also added to the scale with the same reason. Wordings of two items were changed

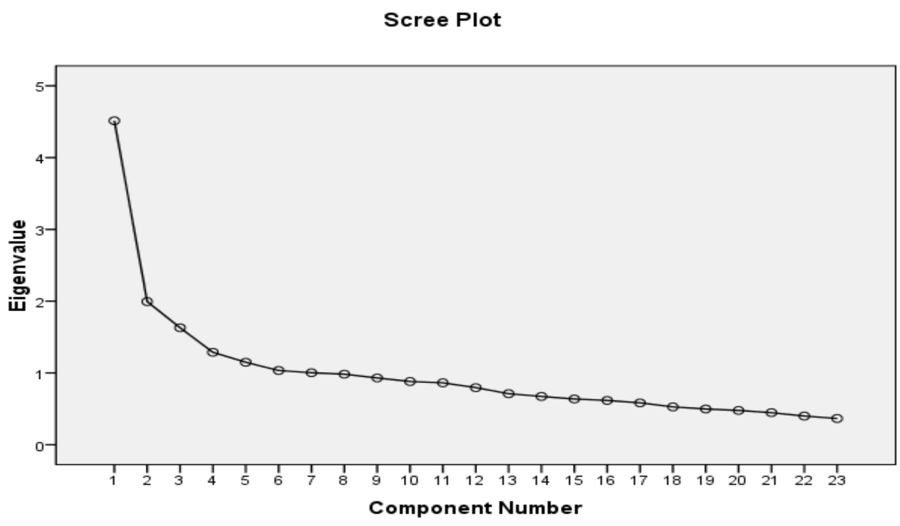

Figure 1: Scree plots of eigenvalues of pilot version of the scale

because they might be confusing for the 5th grade students.

According to the results of the analysis, 6 items were loaded under the first component (*beliefs about teacher's role*), 7 were loaded under the second component (*self-efficacy beliefs*), and 10 were loaded under the third component (*beliefs about mathematics and learning mathematics*). This structure explained the 35% of the total variance in the dependent variable, the total score gained from the scale. The mean and standard deviation of each item are given in Table 2.

Cronbach Alfa coefficient, was computed for the reliability measure and the pilot study of the scale resulted in 0.77 Cronbach Alpha reliability coefficient, which indicated a satisfactory reliable measure (Tabachnick & Fidell, 2007). The Cronbach Alpha value for components of *teacher's role* was 0.48, *self-efficacy* was 0.73, and *mathematics and mathematics learning* was 0.6. Students' mathematics-related beliefs were examined for each component and item by item.

The mean score for *beliefs about teacher's role* component was 14.3 (out of 18). This factor was about how students perceived the role of their mathematics teacher in the class. They rather portrayed their mathematics teachers as the authority for knowledge in the class showing them how to solve problems step-by-step and transferring knowledge to them. On the other hand, they believed that their teachers listened to them carefully and made mathematics lessons fun. Their teachers also seemed somewhat to encourage them to discuss mathematics problems.

Students' mean score in *self-efficacy* component was somewhere between undecided and agree (x=16.7 out of 21). They seemed to believe that they had ability in mathematics. They felt confident while studying mathematics and mathematics was not a difficult

Items	Mean	SD
Teacher is the one who transfers knowledge to us.*	1.07	.340
Teacher shows us how to solve mathematics problems step by step.*	1.10	.376
Our teacher enables us to discuss mathematics problems with our classmates.	2.32	.806
Our teacher behaves us friendly.	2.77	.559
Our teacher teaches mathematics lessons in fun way.	2.68	.630
When we ask questions, our teacher listens to us carefully.	2.85	.450
When we don't understand a mathematics concept for the first time, we cannot understand it later.*	2.18	.885
Mathematics is a difficult subject for me.*	2.39	.778
I think I don't have ability in mathematics.*	2.46	.740
I can make mathematics homework easily.	2.70	.536
While studying mathematics, I feel that my self-confidence is decreasing.*	2.53	.747
Mathematics is easy for me to understand.	2.48	.690
I panic when I come across a different mathematics problem.*	2.05	.865
Mathematics concepts are related to each other.	2.58	.666
We use school mathematics concepts in our daily life.	2.86	.417
Knowing mathematics makes our life easier.	2.86	.449
Mathematics homework helps me understand mathematics better.	2.78	.536
Studying mathematics increases our mathematics ability.	2.82	.467
Making mistakes in mathematics helps in learning.	2.16	.856
Understanding is important while learning mathematics.	2.90	.383
There may be more than one solution path for mathematics problems.	2.86	.430
While learning mathematics, I need to remember my previous knowledge.	2.71	.604
Mathematics problems can be solved correctly only by our teachers' solution methods.*+	1.99	.887
It is important to develop different solutions while solving a mathematics problem.	2.74	.537

* indicates negative items. All items are translated by the authors.

\+ This item was added to the final scale although it did not appear in this factor.

Table 2: Descriptive statistics of each item

subject for them. However, students were undecided about whether they could understand a concept which they couldn't understand in the first time later and that they would panic when they see a mathematics problem for the first time.

Students' mean score in *mathematics and mathematics learning* component was 27.3 (out of 33). Students agreed that knowing mathematics will ease their life and school mathematics could be used in real life. Their responses indicated that they considered mathematical concepts as related to each other, rather than unrelated facts. They agreed that there might be more than one solution for a mathematics problem and it was important to develop these solutions, but they were undecided for whether problems could be solved only by their teachers' method. Students believed that studying mathematics and working on homework enhanced their mathematics learning but they were undecided about the role of making mistakes in their learning.

DISCUSSION

The main purpose of this study was to develop a valid and reliable scale. The results showed that although the scale was constructed in four sub-domains, the items related to beliefs about nature of mathematics and beliefs about mathematics learning were loaded under the same factor. It might be the case that students' beliefs about learning mathematics are closely related to what they believe mathematics is about. *Self-efficacy* items and items related to *teacher role* appeared in separate factors as designed. The overall reliability of the scale was high, but *teacher's role* factor had lower reliability measure. The reason might be related to the number of items. Cronbach's Alpha value is very sensitive to number of items and when it is fewer than ten, it may take lower values (Tabachnick & Fidell, 2007).

Students' responses indicated that they agreed with rather authoritarian teacher roles in the classroom, but their classroom experiences seemed to help them develop beliefs about the connected nature of mathematical knowledge and existence of multiple solutions for mathematical problems. They also believed in the usefulness of mathematics, a belief that elementary school students tended to develop as they progressed towards higher grades (Kloosterman, Raymond, & Emenaker, 1996). Students' responses indicated that their teachers might be supporting discussion in the mathematics classroom, although the nature of the discussion was unknown. When students believed that they could learn mathematics through discussion, they engaged in discussion in the mathematics class (Jansen, 2008). These findings addressed that guiding mathematics teachers for effective discussion in the classroom could be considered in order to help students develop beliefs about and practice effective discussion in the middle school mathematics classrooms.

Students seemed to believe that spending effort in mathematics resulted in learning. Fifth grade students might not have developed beliefs about quick learning which relates learning quickly to ability rather than hard work. This might have resulted in their beliefs about the usefulness of mathematics, as observed in 7th and 8th grades students in other contexts (Schommer-Aikins, Duell, & Hutter, 2005).

Fifth grade students in this study had considerably higher self-efficacy beliefs and mathematics was not a difficult subject for them. These findings addressed that there might be a promising cumulative influence of elementary school and 5th grade experiences the mathematics class on students' beliefs. Understanding the nature of these experiences could provide middle and high school mathematics teachers with ideas for their practices resulting in higher efficacy beliefs in students.

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'Two things I like, maths and chocolate': Exploring ethical hedonism in secondary mathematics teaching

Elizabeth Lake

University of East Anglia, Norwich, UK, e.lake@uea.ac.uk

What motivates mathematics teachers to remain in the profession when disengagement from mathematics is common? I would suggest one of the reasons is deriving pleasure from engaging in teaching mathematics. As part of research into how teachers communicate their enjoyment of mathematics, eight experienced teachers were interviewed, observed during teaching and then engaged in discussion of lesson extracts. This paper takes interview samples to illustrate intellectual pleasures, with a thematic analysis of the narrative generated from the interviews. The analysis suggests that the deeper pleasures experienced by mathematics teachers derive from experiences of dependability, deviation or success, whilst underlying pleasures in teaching mathematics is a living of self, motivated by service to others.

Keywords: Affect, pleasure, ethical hedonism, teachers, mathematics.

INTRODUCTION

Any interaction inevitably has an affective dimension. In this paper, affect is defined from a social perspective as a stable trait as opposed to a more transitory emotional state (Hannula, 2012). In the teaching of secondary school mathematics, emotions play a central role in forming an affective orientation (Drodge & Reid, 2000). If an experienced mathematics teacher has a strong positive affective orientation towards mathematics, then this orientation guides their practice, satisfies desires, and gives pleasure. Frenzel and colleagues (2009) suggest that not only is affect important within the learning of mathematics but that enjoyment supports effective student engagement.

This paper is part of a larger study exploring affective responses to mathematics for experienced teachers of mathematics and how these responses are communicated to students in the classroom. Currently there is little research on how experienced mathematics teachers communicate in mathematics teaching. This paper uses *ethical hedonism* (Onfray, 2007) as a frame to focus on what makes the teaching of mathematics pleasurable for a teacher. The assumption is that, to teach effectively, teachers must gain some affective reward from their teaching.

After defining pleasure in the mathematics classroom context, I review some relevant literature that provides insight into the pleasures of teaching mathematics. Acknowledging the inaccessibility of subjective experiences is important, so the data is teacher stories, and the analysis uses examples of teachers expressing pleasure. Conjectures are constructed from the literature and compared with selected data, enabling a presentation of espoused pleasures via themes with illustrative examples. I summarise, framing with the components of ethical hedonism, the elements that give teachers the greatest pleasure in their teaching of mathematics. The paper concludes with a brief discussion of implications of these interpretations of teacher pleasure.

PLEASURE AND ETHICAL HEDONISM

Pleasure is often defined as a feeling of satisfaction and enjoyment. Yet pleasure is also a subjective mental state experienced as enjoyable or worth seeking, a state that is satisfying more than basic needs or biological drives; on the contrary, experiencing pleasure is a psychological feedback mechanism that forms a potentially addictive feeling of positive anticipation. Pleasure generates a desire to recreate an experience found to be entertaining or aesthetically pleasing, which then impels seeking of similar sensations (Damasio, 2006). This process may then, by repetition, lead to fixed beliefs since even apparently unemotional everyday logic is conditioned by underlying impressions left by formerly intense affects (Schlöglmann, 2006).

In this paper, and this research, the term pleasure is used in an aesthetic and ethical sense, unlike, say, the French term ‘jouissance’ that has stronger physical connotations. Damasio (2006) discusses the interlinking of pain and pleasure, suggesting two parts: a bodily experience and the attached emotional experience. Both parts inform subsequent reaction and or action, subconsciously and consciously. Litman (2005) suggests pleasure links mathematics with an appetite for food or sex, where satiation or reward comes from positive feeding of social identity and community recognition. This social reward is associated with recognition and service.

Onfray (2007) defined ethical hedonism as an introspective attitude to life based on taking pleasure yourself and pleasing others, without harming yourself or anyone else. In other words, ethical hedonism is a joyful utilitarianism that gives moral pleasure. This perspective may be useful in exploring pleasure as it divides pleasure into self, others and an ethical yet purposeful context, where stronger, memorable pleasure is experienced when all three elements are addressed. Yet there is a temporal element to pleasure, unlike happiness which is more stable. Whilst one can again experience pleasure on recall, a person cannot always be in this state. Yet this very temporality may effectively aid the re-creation desire and compensate for any associated unpleasantness. Similarly to Utilitarianism, the ‘greatest good’ of ethical hedonism engages cognitively and viscerally whilst aligning one’s own pleasure with that of others. The pleasure then has personal and social value as the greatest good, is relatively damage free and hence is ethically acceptable. This is this multi-levelled view that accords to pleasure the dimensions of self, others and ethical.

To summarise, pleasure is taken as having emotional, psychological, physiological and social dimensions. Exploring whether there is evidence for ethical hedonism within the teacher narratives, whilst identifying what the teachers suggest as giving them pleasure in their roles, is also necessary. The next section locates pleasures for teachers within relevant research, and seeks themes to support data analysis.

STRUCTURING PLEASURES FOR MATHEMATICS TEACHERS

Research into affect specifically for experienced secondary mathematics teachers is limited, although there has been more mathematics student orientated research (McLeod, 1992). Given the interactive nature of teaching, this research is relevant to teachers as well. Smith (2010) examines pleasure in the form of happiness in learning mathematics, including the role of success and the dependability of mathematics within social and mathematical identities for students. Research on the role of the teacher, power relationships in the classroom (Walkerdine, 1988) and pleasures from deviation (Drodge & Reid, 2000) can also inform understanding of pleasure.

One of the more stable traits associated with pleasure is happiness. Smith (2010) found for mathematics students that happiness, especially subject pleasure, came from the dependability of mathematics. This pleasure from dependability may also prove to apply to teachers. Happiness as an affective trait requires constant high self-efficacy which combines confident knowing, doing and feeling. Smith (2010) suggests that, by choosing mathematics, people express something about themselves which is viewed as positive. By using and reproducing a mathematical identity they derive pleasure and hence happiness. Smith’s research explores finding pleasure in your own work, pleasure which includes working for and with other people. Smith suggests that pleasure is often equated with success and that work and happiness can coexist in contemporary society where happiness is the sole purpose of life, and society promises to provide conditions for you to obtain this happiness. This balance of work and happiness can exist for teachers too, but with additional complexity from the role of a mathematics teacher in regards to vulnerability and risk.

Between freedom and compliance, within the autonomy of a mathematics classroom, lie experiences of deviations from ‘norms’ that can give pleasure. Vulnerability, in the context of teaching (Davies, 2006), derives from conflict of mastery and submission. In this model, the risk or vulnerability comes from, for example, exposure to judgement, whilst pleasure derives from a sense of connectedness. The degree of emotion a teacher is comfortable showing may come from balancing these two limits. This perspective is important, as it may be that the exercise of freedom, self-regulated within constraints, is what gives pleasure. Yet simultaneously, social recognition from being seen as compliant can also give pleasure.

If a teacher’s own learning was empowering, and pleasure derives from successful teacher and student interactions, then revisiting an environment, such as a classroom where success was experienced may trigger re-creation desires. If pleasure deriving from repeated experiences of success is significant, then it is reasonable to expect that teacher interviews would have frequent examples of positive educational experiences. We can also assume that teachers of mathematics would be good at mathematics and hence experienced more pleasure than pain in transition from learner to teacher. This positive imbalance may have led to investment because of feeling pleasure, with subsequent development of trust in the dependability mathematics.

Mathematicians also obtain a sense of pleasure from mathematical discovery which may also apply to a teaching context. Similarly, ‘ah-ha’ moments, for example when a method or solution becomes suddenly clear, are known to be pleasurable for learners and bring power along with the experience (Liljedahl, 2005). The same pleasure for teachers could be the revelation of a student misconception, or through being a witness to student pleasure from their ‘ah-ha’ moments. If pleasure is associated with deviations from the norm, either a discrepancy (Drodge & Reid, 2000) or ‘ah-ha’ moments, then deviations can elicit emotional reactions and hence are likely to be recalled in interview, but also may elicit strong identifying statements that are directed at the deviator, for example a student. Robert and Wilbanks (2012) suggest that “we experience pleasure if the sudden resolution involves an unexpected connection. Making that connection has been likened to cognitive ‘play’ and to the feeling associated with solving a puzzle” (p. 1073). An enticement to engage in more similarly rewarding activities. Deviation may appear come in many forms; difference from routine, deviation from ‘normal’ behaviour, as in the unexpected, or from the pleasure of making unexpected mathematical or social connections.

To summarise, the data will be considered through the lenses of the themes of pleasure in success, pleasure arising from deviations and from dependability of mathematics as described above. The social context is important, as is the role and experiences of the teacher, especially pleasures which help form positive emotional disposition or attitude (Di Martino, 2011) associated with a mathematics teacher.

DATA COLLECTION AND ANALYSIS METHOD

The data used in this paper is from two audio-recorded unstructured interviews with eight UK secondary teachers (A-H, Adam to Helen) who have been teaching in school from three to nearly thirty years. There are equal numbers of male and female teachers, all but two trained at the same university at different times. Six teach in rural schools and two are from a larger urban school. In the first interview the teacher relates their life history, talking about their mathematics and their teaching. The second interview, closely following observation of the teacher in action, is a stimulated recall of an extract from the lesson using video extracts from their teaching. The extract selection is guided by the use of a galvanic skin response (GSR) sensor worn by the teacher in observed lessons. The sensor measures visceral response to intense emotions such as excitement or anxiety. In the stimulated recall the teacher evaluates and explains their thinking and how they felt during the observed lesson. Both interviews are audio-recorded and the transcripts analysed for examples of pleasure. The research design has been approved by an ethics committee and the participants have consented for the data to be recorded and used for research.

This paper draws on the articulated pleasures for each teacher. Analysis of the interview data is influenced by the first two parts of ethical hedonism, a division of self and others (social interaction).

These divisions provide structure for examining teacher pleasures. These categories (Table 1), for the teacher comments that relate to pleasure, are then subdivided into two further categories. The teacher comments are themed by pleasure for self; as their re-

Pleasure for self	Relationship with mathematics	Self-identifying stories
Social interaction	Role of a teacher as professional	Significant others

Table 1: Analysis categories for examples of pleasure

lationship with mathematics, and secondly self-identifying stories. The ‘others’ category, (social interaction), is divided into the role of a teacher as the professional self and then significant others. These are selected as all the teachers speak about these four categories. Once classified, each category is illustrated and interpreted before being discussed in the final section. The intention in the next section is to reveal through examples any similarities or differences within the pleasures of these mathematics teachers and to further explore their sensory and intellectual pleasures.

EXAMPLES AND INTERPRETATION OF TEACHER PLEASURES

The analysis presented here explores what gives pleasure to teachers of high school mathematics structured by the categories in Table 1, but discussed in *deviation*, *success* and *dependability* terms, followed a discussion of other pleasure examples.

In all the interviews, teachers talk about their relationship with mathematics, for example, ‘….my relationship with the subject, it’s stronger than ever’ (Edward). The relationship is presented as positive, and comments relate to personal *success* in the subject, ‘…just absolute joy. Nobody ever told me maths was hard…maths was just like breathing… I thought maths wasn’t important because it was easy’ (Gus), but also the *dependability* and certainty that mathematics represents for them, ‘I realised that I quite like maths, cos it’s nice, you can have a… there’s always a right answer, or most of the time…I really liked that…’ (Debbie) or ‘maths was something that was really important to me, and I enjoyed it’ (Freddie).

From these examples, I would suggest teaching mathematics is perceived as safe and *dependable*, the participants experiencing personal *success* in mathematics as their identities were forming. The intensity and frequency of the comments, as illustrated above reflects the importance of subject within the teacher’s identity. Re-living such an experience within a mathematics classroom allows constant recall of a positive experience. The pleasure in teaching a subject where they have experienced personal *success*, which effectively pleases the self, combined with actively sharing such pleasure with others in a morally good way, meets the criteria of ethical hedonism.

Similarly, all participants tell self-identifying stories of their self as student, ‘I have just always excelled at maths I could always do everything in maths lessons and I found other lessons quite hard’ (Adam). These comments show how important and pleasurable mathematics was to each teacher, ‘Maths has always been my favourite subject when I was at school, and I enjoyed it, and was fairly good at it, and found it interesting and I kind of…I liked being able to solve problems and I don’t know really, I enjoyed algebra’(Helen). But many stories report change during higher education, with examples showing critical points where their pleasure in the *dependability* of mathematics was shaken, ‘…by the end I kind of like lost the love a bit for maths…’ (Adam). ‘I enjoy my maths but I didn’t enjoy my degree’ (Carol). ‘…started and just, just hated it’ [laughs] (Edward). Yet, all of these participants are now teachers and have a professionally *successful* outcome.

It is common for the teachers to speak about their positive and *successful* school experiences in terms of mathematics. They talk about themselves as individual students, or embed subject comments in the social of school experience. All mention transition, especially negatives associated with transition into university, mainly assigning the difficulty of mathematics as the reason. All but one, who took a combined teaching and mathematics university course, speak very positively about this experience. One difference is illustrated by Debbie. She was ill during school and talks of mathematics as reliably accessible during her illness, unlike other subjects; her ‘horrid’ transition occurred earlier than university. However, all comments suggest that these teachers experienced positive emotions through *success* in school mathematics. Carol used mixed tenses in the interview, often bringing the past comments into links to her present role, suggesting reflection, but also that her wider identity aligns with her current role. The comment on mathematics and chocolate in the title comes from Carol whilst discussing a significant event in her school life, providing an example of aligning sensory and intellectual pleasures. The pleasure that social recognition can give is strong in the stories of this teacher, suggesting that for her, teaching offers a continuation of pleasure as socially recognised *success*.

Pleasure from the role of a teacher as professional, combined with the social dimension of teaching, is mentioned in association with *deviation*. Pleasure seems to come from changes to daily practice, 'Maths can become incredibly boring... bore you to death by making you practice it forever, and that's not what I do. Cos maths has got to be exciting, it's got to have something in it other than sheer boredom' (Gus). Or pleasure from surprises, or being creative and making change or challenges, a motivator for several of the teachers. '...they couldn't do it... I sort of then showed them, and they clapped me...I didn't expect it... it was just sort of "Oh well done miss"...' (Carol). or, similarly to a famous line in the film Forrest Gump's, 'Life was like a box of chocolates, you never know what you are going to get,' Debbie comments that '...teaching is different every day, it's a challenge, different challenges every single day...never dull, its lots of things but it is never boring...'

But *deviation* through challenge seems to be important, as without it boredom emerges, and excitement is lost,

> '...I was saying on Friday to my husband that I didn't really feel excited about maths, we were talking about the emotion of it... and perhaps I'm not conveying it because a lot of it is the day to day of it, I've done it 20 times, 50 times, 100 times...' (Carol).

However pleasure from deviation through challenge or creativity can be viewed as a lure into mathematics for students,

> '... I go with the philosophy of fun. If I'm not having fun in the lesson, if I'm not enjoying myself, then the kids aren't either...there's no hook...so it's trying to make it fun, trying to let my personality come through a bit, have a bit of a laugh with them...' (Debbie).

In general, the teachers use many terms to describe their teaching, from interesting, pleasurable (including social pleasure), creative or transformative, through to labour, challenge, routine, or a part of life. The descriptions illustrate the complexity of pleasure within a teaching context, and that pleasure is individual and social. Yet their comments do not necessarily distinguish between sensory and intellectual pleasures. Often there is a strong experiential aspect to what gives the teachers pleasure. I would suggest that pleasure emerges from aligning a socially located identity with their individual view on the purpose of being a mathematics teacher.

Yet simply identifying with the mathematics teaching role can give pleasure, '...other stuff that kind of takes you away from teaching...teaching's the fun bit...' (Adam), or personal pleasure in mathematics, '...and I sort of enjoy that freedom to explore my subject... and take students along with that...' (Edward), or in comparison with other life pleasures, '... I was playing teaching... I was teaching and it was more fun than playing rugby and therefore I didn't need the rugby anymore...' (Gus).

All the teachers mentioned one or more significant others, such as a teacher, 'My maths teacher for my GCSE, O level years was brilliant'(Bertha), 'I got on very well with him... someone I am still in contact with...' (Carol), 'I just really clicked with him and that style suited me and could just practice, practice' (Adam) '...and I had this wonderful nun who was my mentor and tutor, and she was magic' (Gus). A friend or a family member as a significant other, '...my dad... used maths a lot... I can remember him sitting down and helping me with maths....' (Carol). A stronger degree of pleasure from engagement with others as well as self is evident here. Especially as, in addition to all the teachers mentioning at least one significant other, all positively mention their mathematics teachers in high school.

Pleasure for others appears in interviews as indirect pleasure in the *success* of students, suggesting that pleasing others is perhaps a stronger driver for these teachers rather than balanced by pleasure for self. One of the strongest examples of pleasure combines sensual and intellectual pleasure, '...at the end of primary school there was a competition...we [dad and I] won a bar of chocolate [laughs], two things I like, maths and chocolate' [laughs] (Carol). Not only is eating chocolate a sensual pleasure, Carol uses chocolate to recall a story that illustrates the reward of early social recognition as someone who could do mathematics, an intellectual pleasure. Significantly, the story also relates *success* in relation to a significant other. I would suggest this pleasurably recalled experience supports establishment of her belief that mathematics gives pleasure; as does chocolate. The combination has all the characteristics of ethical hedonism, harmless pleasure for self, others, as well as social recognition and acceptance. The importance of the social within pleasure is emphasised by several of the teachers, such as the pleasure of helping others,

> ‘I remember somebody saying to me in the first lesson, where is the fraction key on the calculator. I think that was a bit of a “oh, maybe I do know a bit more than you, I can help you. That’s good.”’ (Helen).

And Carol reports ‘[I was] more comfortable talking to that group...cos they are weaker, so I feel I can help them more. They engage better’. There are examples of social pleasure from a common love of mathematics, ‘...there’s a little group and they’re like, yeh, I really like maths [laughs]...students I teach get that enthusiasm from me, and they like the subject... they like me as well’(Adam). Or social pleasure from interaction with a class, ‘But actually, that’s how I teach. Complete, complete enjoyment. And that class is an absolute joy to teach’ (Gus). Social pleasure by proxy appears in a number of interviews, including teachers articulating as if they were students. Such as ‘...I teach quite a lot of lower sets and somebody in the class might say, “oh right yeah I get that now”, and actually properly get it, and that will change... change their lives...’ (Bertha). Bertha is also describing her experiences of student ‘ah-ha’ moments by proxy as discussed above.

Overall, the stronger of the articulated pleasures appear when pleasure is experienced by teacher, students and further, is associated with satisfaction in fulfilling the role of a mathematics teacher. There is an individual, social and moral dimension to pleasure from fulfilling a role. For example,

> ‘...you can get the kind of challenge from that as well. I like the way that you can sort of relate it to real life. I like the fact that kids value it’ (Helen).
> ‘That’s me, the more you laugh the more you learn’ (Gus).

To summarise the selected data examples and interpretation, examples from teachers illustrate how pleasure is located within *success*, *dependability* and *deviation*. The categories of self and others enable the importance of social as a thread to emerge in different forms, such as indirect *success*, and how *deviation* appears to change between the role of student and teacher.

DISCUSSION AND CONCLUSIONS

In this section the categories are briefly discussed in terms of the themes of dependability, success and deviation in context, before considering intensity of pleasures in relation to ethical hedonism. Teachers find pleasure within their roles, something evident when talking about their professional lives, sharing their emotional relationship with mathematics, invariably in an open, cheerful manner. This paper focusses on the pleasurable aspects of teaching, although there are few negative examples. Just talking about their professional lives appears to be a pleasurable experience, and I was honoured to hear and engage, albeit briefly, in their stories.

As Smith (2010) found for mathematics students, subject pleasure and happiness, often in the context of success, comes from the dependability of mathematics. The data here suggests the same for teachers, where mathematics is, for example, ‘analogous to breathing’ (Gus). But most of the teachers report that their trust in the dependability of mathematics was shaken in their transition through university mathematics and the evidence from these teachers infers that dependability is less important when discussing teaching roles. I would suggest that deviation is especially associated with pleasure through power, that a teacher can decide to deviate from norms of their classroom, to be creative, accept the associated risk and use deviation as a teaching strategy; one which draws and engages their students and hence gives reward, a form of pleasure in teaching through intellectual play, as suggested by Robert and Wilbanks (2012). This idea of play or fun is used by Adam, Edward and Gus when talking about gaining pleasure from their teaching role.

Experiences of joint completion of hard work or overcoming difficulties may conjoin success, deviation and dependability, all in a pleasurable memorable experience for the teacher. The interview excerpts highlight the importance of the social aspects of teaching mathematics, including pleasing significant others. The ‘brilliant’ or ‘magic’ person described by these teachers. Research suggests significant others is important in a professional career (Zeldin & Pajares, 2000). Pleasing others as well as self is important, but the intense, memorable pleasures align different forms of pleasure and combine self, others and some form of value as in ethical hedonism. Yet additionally, pleasure can be by proxy, recall or from social interaction in a purposeful context. Pleasure in teaching derives through mastery shared (where learning occurs), and where mirroring or synchronisation of values occurs, and reciprocity of pleasure in the subject induces stronger pleasure. This reciprocity is particularly evident for Carol, telling her story of the pleasure of maths and chocolate.

Defined as pleasure at a deeper level, meta-satisfaction occurs at a personal, social and ethical level, a resolution through ethical practice (Hobbs, 2012), that shapes future action. This shaping develops the emotional orientation of a teacher, and the degree of investment, or risk they are able to take in a classroom, acting as a limit for positive emotional exchanges. Pleasure comes in many forms, especially from interaction that revisits an arena where the teachers themselves, as learners, were successful. Teachers would, from experience, anticipate and plan for such reward pleasure by proxy. Teachers seek and recognise revisiting as something that will give emotional reward. Gratification comes when repetition continues to give pleasure, strengthening their emotional relationship to mathematics in a teaching context.

Knowing more about contextual pleasures in mathematics, for teachers as well as students, may support countering any circumstances of displeasure within a mathematics classroom. Although this paper is focussed on what teachers say, the fuller research also explores observed classroom practices, especially how observed pleasure links to the pleasures identified in this paper. One intention is to explore how teachers share their emotional relationship with mathematics in a classroom and what limits positive emotional exchanges. This may have implications for teacher training.

ACKNOWLEDGEMENT

I would like to thank the teachers for their cooperation, and my supervisors, Professor Elena Nardi and Dr Paola Iannone, and UEA for PhD studentship funding.

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Patterns of motivation and emotion in mathematics classrooms

Gareth Lewis

University of Leicester, Leicester, UK, gl116@leicester.ac.uk

A case is made that the study of state, as opposed to trait, is necessary to give a more rounded picture of affective aspects of students' experience of learning mathematics. This paper is based on a study of disaffection with school mathematics. The study was a primarily qualitative study conducted from a constructivist and interpretivist perspective, using Reversal Theory as the main design and interpretive framework. The paper reports findings of some patterns in the students' experience of motivation and emotion in mathematics classrooms. Sequences of motivation and emotion are described, and the notion of dominant narrative is introduced. Two common motivational pathways are described: 'mastery for a reason versus what's the point', and 'engagement and pleasure through interest versus I can't be bothered.'

Keywords: Motivational state, emotion, motivational pathways.

INTRODUCTION

Research in affect in mathematics education has been dominated by attention to stable and semi-stable constructs such as attitude, and as such these are often treated as trait. Twenty years ago McLeod (1994) pointed out that the predominant paradigm in mathematics education research is taken from psychometrics. He pointed out further, that:

> complicated statistical analyses of questionable questionnaire data were not necessarily reflecting accurately what students were thinking and feeling. (McLeod, 1994, p. 640)

Students' experience of learning mathematics involves a dynamic shifting of motivation, emotion and other affective constructs. Nonetheless, emotion has not been studied in great depth in mathematics education, as has been pointed out by Evans (2000) and others. Hannula and colleagues (2009) talk about 'a need to move beyond simplistic positive/negative view of emotions and distinguish different types of negative emotions.'

The equivalent importance of state, as opposed to trait has been pointed out by Hannula (2012). In that paper Hannula sets out a metatheoretic structure of affect, and one of his three key dimensions, seen as important, is the distinction between state and trait. He notes the long history of attention to this distinction, and traces important sources of research into issues of state. Goldin and colleagues (e.g., Goldin, et al. 2011) make a similar distinction to state and trait between what they term local and global affect, and point out the importance of research into local affect. What is important for the present paper is Hannula's statement:

> There is a clear imbalance in favour of studies that focus on traits over studies that focus on states… (Hannula, 2012)

He goes on to say:

> In particular, studies that focus on the dynamics of emotional or motivational states in a classroom or other learning community are still rare. (ibid)

This is not to claim, of course, that no such studies exist, and indeed, Hannula, in his paper, cites a number of examples, including his own work with Peter Op't Eynde. Goldin and colleagues have developed the notion of engagement structures, which are constructs related to state (Goldin, 2011), and more recently, colleagues have studied and reported on the dynamic of the movement of experience through the lens of such structures (Sanchez-Leal, 2014).

The study reported here focusses on the state rather than trait aspects of experience, and, further, emphasises aspects of experience related to motivation and emotion.

THE STUDY

The data reported here derives from a study of disaffection with school mathematics which seeks to understand more deeply the subjective experience of disaffection beyond the notion of positive or negative attitude and other trait-like constructs. The study centres the focus on motivation and emotion, and was conducted from a constructivist and interpretative perspective of the learning of mathematics. In this way, it was primarily qualitative. Reversal Theory (Apter, 2001) was used as the main design and interpretive framework. The theory is a comprehensive account of personality, motivation and emotion and as such affords a basis for understanding affect centred on the subjective experience of the individual.

The theory proposes that there are eight such motivational states, organised into four oppositional pairs, which each represent quite different ways of experiencing the world. The eight states are:

- Serious (telic) versus playful (paratelic)
- Conforming versus rebellious (negativistic)
- Mastery versus sympathy
- Self versus other-oriented

Reversal Theory also gives an account of 16 primary emotions, that are related to motivational states and our experience of the world. The theory further proposes that we are in one of each pair of states at any one moment, but that one (or sometimes a combination of two) states is focal in our phenomenal field, in that it (they) dominate our experience at that moment. The theory also introduces a new construct to psychology – that of reversal itself. This says that we switch or reverse between states frequently.

The primary method for data collection in the study was the individual interview, although a number of instruments and procedures were adopted in and around the interviews to aid the elicitation of data. One instrument, called TESI-ME, adapted from the Tension and Effort Scale (Svebak, 1993), was used to survey 150 pupils and students on their experience of negative emotions. Individual responses were available for discussion at interview. The methods and instruments, together with some preliminary results, are described in more detail in Lewis (2013a). Forty nine young people (aged 14 to 18 years) were interviewed from two further education colleges and one school in the UK. The data from 15 of these young people has been analysed and presented as a series of case studies and also as an aggregate analysis.

Among the findings of the larger study were:

- The volatility of young people's relationship with school mathematics – affect is not trait
- It also shows the complexity and ambiguities within students relationship to school mathematics – balancing positive with negative aspects
- Evidence of the whole range of motivations and emotions being operative in mathematics classroom
- Disaffection ≠ unmotivated. Helen, for instance, is clearly disaffected, but she is highly motivated – a paradox explored very little in current literature
- However, negative affect does disable or de-limit learning
- Unexpected but significant evidence of self-regulatory and metacognitive skills in disaffected low achievers

PATTERNS OF MOTIVATION AND EMOTION

As indicated above, the results of the main study have been reported elsewhere, including to the working group at CERME 8. Here, and below, I will draw attention to a number of theoretically-based and empirically grounded notions and propositions that have emerged from the research, which can be collectively thought of as patterns of motivation and emotion. Focussing on state, as this study does, demonstrates that students' subjective experience of learning mathematics is highly volatile (at all levels of temporal granularity), dynamic and complex. However, like all complex systems, evidence of pattern does emerge, and it is the intention here to indicate a number of aspects of those emerging patterns.

Evidence of sequences and shifts in motivational state, including reversals, are found throughout the accounts, and these come with the associated and expected shifts in emotion. And although there is extensive evidence of all 16 primary emotions being active in the experience of these young people, including evidence of positive emotions such as pride, excitement, the evidence from the TESI-ME suggest that the negative emotions of anxiety, boredom and anger are more prevalent than any other emotions. This is interesting in that, whilst anxiety has been widely studied, boredom and anger have not.

Shifts in motivational state and emotion are in evidence in all of the accounts in this study, and these sequences can be quite individual. A 'typical' sequence (although there are many such sequences, and variations on them), often starts with the need to succeed at a task, coupled with a strong need to understand the context or correct procedure. A subsequent lack of progress can lead to frustration, which in turn leads to anger brought about by a reversal from the conforming to rebellious state. Typically, the performative and the affective aspects operate in parallel in this way.

Dominant narrative

The flavour of such sequences can be illustrated by the case of Liam, who is a disaffected year 9 pupil.

> *I panic quite a lot when I don't get it.*

By his own account, the sequence sometimes also ends in anger.

> *L - if I don't get it...if people are getting ahead...I'm always behind....and it gets quite annoying...I just wanna..be where everyone else is*
>
> *Gl – why is it annoying?*
>
> *L – cos everyone else can get on and I just don't understand...but some people get quite annoyed as well*

Perceiving oneself to be behind the others is a strong source of negative affect. In Liam's case it leads to anger. There is a sense that 'this is not right – it's not the way it's supposed to be.'

In motivational terms, his desire to be 'keeping up with the others' is an expression of competitive self-mastery. To serve this need he has to be able to perform the tasks set as well (and as quickly) as the others, and so he approaches tasks in a serious-conforming frame of mind. When the goal is threatened, because he doesn't understand or can't do it, the arousal increases and he feels anxiety or panic. Not being able to 'do it' disrupts his sense of conformity, and he reverses into rebelliousness, and this switches the emotion from anxiety to anger. Sometimes he asks for help, but being able to 'do it' appears to result in little pleasure in itself. The benefit appears to be that he is able to keep up with the others. At one stage he says "I'm always behind (the others)." Even if it is not literally true, it reveals how he often feels about the situation in his mathematics classes.

Liam's guiding motivational statement is:

> *I just wanna be where everyone else is.*

Since the notion of 'keeping up with the others' recurs time and again in Liam's account, it can be considered as a *dominant narrative*. The dominant narrative itself is located within the motivation-emotion nexus of the experience of the individual. In Liam's case the notion of 'keeping up with the others' is based on the motivational state combination of competitive self-mastery, and in the 'losing' mode, the associated emotion is anger leading to humiliation.

A number of subjects have demonstrated strong recourse to such a recurrent theme, which represents something of a behavioural attractor to which they return again and again in their narrative, which suggests that this theme has some strong significance in their subjective experience of school mathematics. Examples would include Anna's apparently ever-present need to redefine herself from a 'D' to a 'B' (see Lewis, 2013b) ; Scott's inescapable characterisation of his relationship to mathematics as 'struggle' (a word he mentions 17 times in a half-hour interview), and so on. Such dominant narratives seem to touch every aspect of affect, from emotion to attitudes, beliefs and even their identity as mathematical learners, thus demonstrating the complex and intimate relationships between affective constructs within their subjective experience.

Not all of the sequences in the data relate to performative-focussed states and emotions such as anxiety and anger. Helping others is seen by many students as a motivationally rich activity, which often provides some of the rare positive experiences for disaffected students.

For instance, here is Anna:

> 'it makes you feel good...because you get it and someone else doesn't...so you feel a bit proud...but if you're helping them...you feel a bit nice about it...does that make sense?'

When asked about the possible contradiction about feeling good when someone else is struggling, she replies:

> 'It's not like I can do it and they can't.....it's like I *can* (my emphasis) do it...and they can't...it's not like they're bad...not like ohmigod I got better than her.'

From her account, the progression of states appears to be:

— Self-mastery – competitiveness, and an attempt to do better than the others, leading to a sense of pride (if successful)

— She then notices the lack of progress in others (other-sympathy), which may trigger a sense of guilt

— This causes her to help her classmates (other-mastery), which in turn results in her feeling 'nice' (self-sympathy).

In this way, helping others can be seen to be a motivationally rich activity for Anna, since it satisfies a range of motivational needs – even though some of them are contradictory.

MOTIVATIONAL PATHWAYS

Motivational states, or combinations of states represent qualitatively different ways of experiencing and engaging with activities in a mathematics classroom. There appear to be a number of identifiable motivational pathways which, whilst they play out in highly individual ways, have some features in common.

A number of contexts have emerged that seem to characterise much activity and learning in mathematics classrooms, and each of these involves a typical or predominant *motivational pathway*. I describe two of these here in more detail.

The serious-conforming pathway

The first of these can be described as the serious-conforming pathway. It is characterised in the Table 1 below.

Nature of activity	Mathematical context	Motivational state	Motivational requirements	Positive affective correlates – if successful	Negative affective correlates – if unsuccessful
Procedural tasks	Procedure to be learnt and performed correctly. Speed and right answers rewarded	Telic Conforming Self- Mastery	Telic – that meaning, purpose or utility are established. Conforming – sense of duty. 'Correct' procedure to be followed Mastery – I understand, so I can do	Need to apply effort to overcome negative feeling in order to learn. Performing correctly results in a sense of relaxation/relief. A sense of achievement (if it is perceived to be important, or if it took effort to learn). A sense of pride at overcoming and winning	Anxiety, fear or panic when faced with challenge. Possible switch to anger if effort is unsuccessful. Sense of detachment or being excluded in case of failure, resulting in mastery-losing and sense of humiliation

Table 1: Mastery for a reason versus 'what's the point?'

Since this pathway is dominated by the serious (telic) state, it is the outcome, and not the doing of the task itself that is the source of satisfaction, and as such successful engagement and performance is maintained by metacognitive skills such as effort and persistence. Students engage in this mode because they perceive the outcome to be valuable and worthwhile as well as achievable. Developing a narrative of significance will enhance motivation in this mode.

This mode of engagement is by far the most usual mode referenced in classes in this study, and reported by subjects. For some, it appears there seems to be little variation. Where it becomes institutionalised, it can become the dull 'textbook and worksheet diet' of drill and exercises described so vividly by many of the subjects of this study.

It is also worth pointing out that any possible pleasure and satisfaction here is deferred. In the serious (telic) state, this is only likely to be available on successful completion of a task. The satisfaction (e.g. relaxation/relief) is a low arousal emotion, and thus is unlikely to be experienced intensely.

The playful-rebellious pathway

The second pathway or mode is the playful-rebellious route.

The notions of fun and excitement appear often in many of these young peoples' accounts, it seems clear that it is an important motivational need that can provide a source of motivational satisfaction. The evidence here is that it is not trivial, and not an add-on to make difficult tasks more palatable. Yet the mathematical education community has rarely had the language or the theoretical frameworks to discuss its importance. It has thus been trivialised and so relegated in importance. In many classrooms described in this study, it is shut off as a motivational opportunity. At the same time, there is substantial evidence in the data that students will often be in the playful state.

Here is Adnan:

> when I'm not just listening to a teacher going on and on... I'd rather have a teacher involved in some sort of humour... or not just doing maths but for example using maths in everyday life.... so it would make you think... do you know... the next time I come across the situation I can do it like this... and that's where excitement comes in and curiosity... like it's a real life situation and they're making it exciting by giving you this real life situation... (Adnan)

The mention of humour, excitement and curiosity are all characteristic of the experience of the playful state. In fact, humour can trigger the playful state, and it is interesting to note that real contexts can create the excitement that is so enjoyable in this state.

Nature of activity	Mathematical context	Motivational state	Motivational requirements	Positive affective correlates – if successful	Negative affective correlates – if unsuccessful
Open-ended task or problem to solve or investigate	Mathematical thinking and heuristic skills required. No necessary single right answer. Methods are undetermined and in control of student. Possibility of group working. Making connections, reasoning and understanding are privileged.	Paratelic Rebellious Mastery	Curiosity, real-world interest, challenge, game, unfamiliarity or novelty. Activation of exploration and discovery. Need to share within the group	Fun, excitement, enjoyment and immersion in the task at hand. Sympathetic satisfaction from working in a group.	Boredom if the challenge is too easy. If too hard, a reversal to telic could induce anxiety, or shift of focus to mastery-losing resulting in humiliation

Table 2: Engagement and pleasure through interest versus 'I can't be bothered'

Here are other examples that emphasise the point:

> 'at some point I felt a bit of curiosity because... I wanted to know... because there was so much important stuff...like dealing with money obviously... and then... I knew that would help me at some point in my life... I knew... I'd get a job... or if I'd get in a career like dealing with maths or... so I did get curious... and some of the lessons were interesting as well.....depending on the teacher in the classroom (Nadia)

> Miss B... she taught it as a game... for different people she put it in different contexts... like for me it was always football... like angles... and stuff... she used football to help me understand it... but for different people it was different things... she knew a lot about us... what interested us... some people who liked to have questions... then work it out... but other people want to do it a different way... she taught us all a different way... how we liked (Harry)

When people are in the playful (paratelic) state, goals and outcomes are not in the phenomenal frame. People engage in activity for the momentary pleasure it affords them, and thus they are more likely to be creative and spontaneous, and to be unmindful of risk. This is a highly desirable state in which to engage with mathematical contexts. There needs to be pedagogically sound and justified ways of incorporating this into the life of the classroom.

It enables creativity and pattern-spotting, particularly in a problem solving context. It is precisely where the 'rich learning' described in ACME (2012) comes from. Indeed it is possible to go further and to say that to be paratelically engaged with mathematics is a sign of affection for mathematics, since our interest and curiosity will enable us to engage without fear. The inability to do so is a litmus test of disaffection. Further, the poverty of mathematics curricula and pedagogy reported by these young people means that paratelic activity and engagement is not encouraged and enabled in so many mathematics classrooms.

That there is more evidence in this study of boredom than of any other negative feeling or emotion related to school mathematics, suggests that these young people spend a lot of time in the playful state, but are unable often to find legitimate outlet or expression of it in the context of the activities of the mathematics classrooms. It seems that many teachers don't appear to understand that they could harness this playfulness into productive activity by introducing elements of curiosity, gaming, unfamiliarity, something out of the ordinary, to increase the arousal in a positive way, thus prompting the interest and engagement of the students.

In the context of teaching practice, it is interesting to note that there is a paradox at the heart of the experience of playfulness. As Apter puts it:

> in order to experience excitement, then, we need both the possibility of danger and something we believe will protect us from it. (Apter, 2007, p. 31)

This sense of danger or threat is psychological and can be brought about by uncertainty and ambiguity. The sense of safety, which Apter labels the 'Protective Frame', can be created by a classroom climate which reduces the focus on right answers, and a humane and supportive environment, among other means.

SUMMARY

The study shows that a deep account of the experience of learning (or not learning) mathematics can be provided by focussing on motivational state, and the associated emotions. Although the data is qualitative, and although stability as used in the notion of trait has been disregarded here, it is still possible to discern pattern in the accounts of these young people. In this paper, I have focussed on three theoretically-based but empirically grounded notions. Dominant narrative is exhibited by a number (but not all) of the participants in this study. It is an interesting phenomenon which deserves further study, and which may have implications beyond mathematics education. The two motivational pathways described here can be thought of as qualitatively different modes of engagement with learning mathematics. The serious-conforming pathway describes an emphasis on the development of competence (mastery), and outcome-focussed disposition which is consistent with much current research. On the other hand, the distinctive playful-rebellious pathway not only represents a qualitatively different mode of engagement, but at the same time is much less recognised or studied. This is not to claim that it is new. It is possible to recognise the ideas of Dienes in the notion[1], and, more recently, Goldin and colleagues (2011) have described a phenomenon with some similar characteristics in his engagement structure 'I'm really into this'. It would be interesting to examine further the presence and efficacy of this pathway in learning mathematics.

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1 I am grateful to a reviewer for pointing this out.

Emotions as an orienting experience

Peter Liljedahl

Simon Fraser University, Burnaby, Canada, liljedahl@sfu.ca

In this article, I first introduce data collected from 38 prospective elementary teachers after an intensely negative emotional experience preparing to play a game called Around the World. From these data emerges a picture of these prospective teachers being deeply affected by this experience. To try to understand these changes I present a theory by Leont'ev. This theory, based on Vygotsky's cultural historical theory, looks at the relationship between motives, activity, and emotions. Using this theory, I argue both theoretically and empirically that what has actually changed for these prospective teachers are their motives. More specifically, the hierarchy of their motives. The results contribute to work in mathematics education that anchors emotions in a theoretical framework and links them to other constructs in the affective domain.

Keywords: Emotions, participationist, motives.

INTRODUCTION

In the spring of 2010, Kim Beswick visited my EDUC 475 class for a day. EDUC 475 is the mathematics method course for prospective elementary school teachers. Each section of the course usually has 30–35 students, 90%-95% of whom are female. On the particular day that Kim visited we were discussing basic operations on single digit numbers – addition, subtraction, multiplication, and division. The goal of the lesson was to get the students to experience methods of teaching these operations other than memorization and rapid recall, which is the only method familiar to many of them.

Although the lesson has this goal, this only defined the general direction I wanted to go in. During the actual lesson I draw on a large repertoire of activities and discussion points that tumble out in a, more or less, improvised order. This allows me to more effectively respond to my perceived needs of the specific group of students at that specific time.

As it was, many of prospective teachers I was teaching the day Kim visited, although seeing the merit to the many alternative methods I was modelling, were still not ready to abandon the 'drill' method of teaching fluency of the basic facts. Many had mentioned at the beginning of this lesson, as well as in the previous lesson, that they had regularly used *The Mad Minute* during their practicum. This was problematic to me. The *Mad Minute* is a test, usually given once a week, where students are challenged to answer 30 questions in on minute. Their scores on these tests are often recorded in some public fashion and the top achieving students are rewarded for their achievements. The possible negative consequences of this method are many, yet it continues to be practiced for its efficiency, simplicity, and tradition ... and parents like it.

To emphasize the potentially negative consequences of this method I did something I had never done before. After the pre-service teachers returned from a break I gathered them around me. I told them that we were going to do a basic facts activity. The way this activity would work is that I would point at one of them and ask them a basic multiplication question (3 × 4, 6 × 8, etc.) and they would have two seconds to respond. If they responded correctly in that time they would be allowed to sit down. If they failed to give response, or their response was incorrect, they would remain standing and I would come back to them after I had gone all the way around the class. This would continue until all the students were sitting.

This game, as it is referred to by practicing teachers, is called *Around the World*, and is often used, in conjunction with *The Mad Minute*, as a way for students to practice their basic facts. Unfortunately, it has the same sort of public shaming qualities that the *Mad Minute* does.

The pre-service teachers gathered around me were, as a group, visibly uneasy. There were a few who seemed excited at the prospect of playing a 'game' and the thrill of competition. But the vast majority were horrified at what was about to happen. When the tension had built to a crescendo I pointed at the first prospective teacher and, instead of asking a basic multiplication question, asked, “How are you feeling right now?” And then, to the whole group, “How are all of you feeling right now?”

The relief in the room was tremendous, and the ensuing conversation was beyond anything I had expected. The experience of almost having to play *Around the World* was transformative for these soon-to-be teachers who talked about how they NOW understood how negative this game—and *The Mad Minute*—could be. For over an hour they talked about their past experiences, sharing the negative impact these types of ‘games’ had on them as learners. A few of them shared their positive experiences with these types of activates, but even then quickly acknowledged that their enjoyment was not worth the price of misery that the rest of the students had to pay. We discussed why parents liked these ‘games’ and ways, as future teachers, to deal with that. In the end they vowed, individually and as a group, that they would never do this to their future students.

After the class, in debriefing the activity with Kim, we both concluded the obvious – the prospective teachers had had a powerful emotional experience and that that experience had caused wide sweeping changes in their intended practice (Liljedahl, 2008). But, we also concluded that we currently had no theoretical framework to make sense of this experience.

In mathematics education research in general, and in affective research in particular, emotions remain a largely unresearched and not-well-understood construct. The little research that exists are “sidelights rather than highlights of the studies” (McLeod, 1992, p. 582).

As such, we decided that we needed to recreate the phenomenon and to gather data on it.

METHODOLOGY

So, in the spring of 2012, working with a new group of 38 (35 female and 3 male) EDUC 475 students, I recreated the *Around the World* activity. As mentioned, EDUC 475 is an elementary mathematics methods of teaching course. It runs for 13 weeks and is comprised of 13 lessons – one each week. Each lesson is four hours long and is typically designed around a number of activities and resultant discussions. Between lessons, students are assigned readings and prompts to be responded to in a reflective journal. As with the previous class, many of the prospective teachers in the current class had acknowledged that they had used *The Mad Minute* or the *Around the World* activities during their practicum, either on their own initiative or at the urging of their sponsor teacher.

As such, in the fourth week of classes I once again ran the *Around the World* activity. This time, however, instead of immediately going into a discussion I did something different. As the tension built to a crescendo I pointed at a student and asked her how she felt, and then I immediately asked her, and all her classmates, to sit down and write in their journal how they felt at that moment. The students wrote for 10–15 minutes. We then had a whole class discussion much as I had led for the class two years prior.

At the end of the class they were assigned a further journal prompt:

> Discuss your experience in today’s class around the issue of multiplication. What did you feel when I sprung the “stand up and get ready to answer multiplication facts” activity? What sort of self-reflection did you go through? How do you feel now after we debriefed it?

Towards the end of the course, the students were given a further writing prompt potentially related to the *Around the World* activity and discussion.

> Now that this course is almost over what is something that you will NEVER do in the teaching of mathematics? Why? What is something that you will ALWAYS do? Why?

Taken together, data consists of the relevant entries from the written journals of these 38 prospective teachers. These data were analysed using a constant comparative method (Glaser and Strauss, 1967) to emerge themes pertaining to their emotions and the effect of those emotions, both short term and long term.

RESULTS

For the most part, the game of *Around the World* created a very negative emotional experience for these prospective teachers.

Fear

Fear, in one of its many forms, was one of the most commonly expressed emotions immediately after the activity.

Misha — Terrified! I can't do mental math very quickly and I don't like being the centre of attention when under scrutiny. The only thing I could think was "I'm going to be the last one standing". I don't want to look slow in front of my peers and teacher. Through my education career I sit in my seat praying not to be called on.

Allison — Mortified. I don't like to be wrong or feel embarrassed in front of my peers. It can be extremely difficult to get the answer right as I'm too busy thinking about me, or what they are saying to care about the problem. Eventually I feel I'd just guess to get it over with.

Anxiety

The other emotion frequently expressed is anxiety.

Beth — Heart racing anxiety! The thought of being picked on and not knowing gives me the heebie jeebies, especially in a subject that is probably my weakest. Being that it is multiplication and is something that I probably would get right doesn't really help shake the feeling you get when you know that there is pressure to perform. [..] If I feel like this at 23 how would a kid feel?

Jocelyn — I am feeling really anxious and nervous. I am worried about being embarrassed about not being able to answer the multiplication question in front of the class and I am also really worried about being the last person standing.

Nervousness

Nalah — I felt nervous because I might not know the answer to the multiplication question he might ask. [..] While we were standing there waiting for Peter to ask, I was thinking back to grade two and three and how we played the game *Around the World*, and how nerve racking it was.

Defeated

Anne — It also reminded me of a time when my grade three teacher called me to the front of the class to answer a question. She knew I wouldn't know it, but I had to do the walk of shame to the board only to admit to the whole class that I didn't know the answer. I dreaded going to class. I just remember being in class and feeling defeated by math.

TEACHER CHANGE

These very negative emotions were not fleeting. Despite the fact that during the course we engaged in over 50 activities and discussions, and read over 400 pages mathematics education literature, six weeks after the *Around the World* activity, 24 of the 38 prospective teachers in the course chose to discuss this specific activity, and the emotions it triggered, when responding to the prompt about something they would never do in their teaching.

Misha — Something that I will NEVER do in the teaching of mathematics is put students on the spot and force them to answer questions. Like many other people, I have experienced embarrassment from being put on the spot and answering incorrectly. I understand how low it can make; a student feel and I don't want to be the one to make my class feel that way.

Sofia — In teaching math I will never use the Mad Minute to drill students on their multiplication tables. The costs to many students outweigh any benefits to a minority of students.

Jocelyn — Now that the course is over, I have discovered that I will never make my students do any sort of drill or mad minute that may deflate their confidence and cause them to want to avoid mathematics. I realize what effect 'mad minute' exercises had on me as a math student and when Peter simulated a mad minute situation, I felt terrified and extremely anxious. I would never want my students to feel that kind of panic and fear. As a teacher, I hope to foster a love for learning mathematics and want to create an environment whereby my students feel confident and safe.

Even those who were originally excited by the game talked about the negative emotions they say their classmates experience.

Alison To be honest, I was excited to play the game. But I can see how an activity like this could bring high levels of anxiety for students in a class that are insecure about their amount of knowledge or skills with respect to what is being quizzed on the spot. I did not feel panicked because I am confident that my multiplication skills are fine. [..]Something I will NEVER do when I teach math will be multiplication drills. It traumatizes children that are not finding this activity successful, and it could give them a bad taste for math for the rest of their life.

Of the 14 who did not speak of the *Around the World* activity explicitly in their response, 12 made commitments that were tangential to some of the ides that cascaded from subsequent discussions on the learning of basic facts in general, and assessment in particular.

Anne I will NEVER use assessment as a way to rank students.

Khaly I'm not afraid of mathematics any more, to learn or to teach. I also think that mathematics can actually be fun. I am excited to teach my new students (when I get my first class). Show them that math is not as scary as it seems.

Taken together, 36 out of the 38 prospective teachers, despite many having used it in their practicum, vowed to never use *Around the World* (or the *Mad Minute*) in their future practice as teachers. For them, their own experience with this activity had triggered very negative emotions, sometimes reminding them of similar activities and emotions from when they, themselves, were children. These emotions were not only enduring, but also instrumental in changing things in the prospective teachers' practice.

But what exactly is it that has changed for these teachers? This is my research question. Given that they don't actually have a classroom in which to enact these changes we cannot say that it is their practice that has changed. Perhaps it is their intended practice that has changed? But what is backstopping this intention? Intentionality is a reification of deeper constructs. The question is, what is the construct that grounds these intentions, that was deeply affected by the emotional experience that these teachers had when being placed in a position of having to play *Around the World*? To answer these questions we need to look more closely at emotions.

EMOTIONS

Emotions, as theoretical construct in mathematics education, are seen as the fleeting and unstable cousins of beliefs and attitudes (McLeod, 1992). They are either a reaction to an experience (McLeod, 1992) or a reaction to an interpretation of an experience (Mandler, 1984). Regardless, emotions are acknowledged to affect learning in general (Zan, Brown, Evans, & Hannula, 2006) and cognitive processing in particular (Hannula, 2002). Over time, negative emotions can reify into more stable and disassociated manifestations of fear, phobia, and hatred (DiMartino & Zan, 2012; Tobias, 2009), each of which will have an effect on actions (Hannula, 2002; Tobias, 2009).

That emotions exist, and that they simultaneously emerge from, and shape experience, is clear. That these emotions then regulate future actions is also clear. What is not clear, however, is how this happens. What psychological mechanisms link emotions to actions? The answer to this lies not in the abstract.

> The variety of emotional phenomena and the complexity of their inter-relations and sources is well enough understood subjectively. However,

> as soon as psychology leaves the plane of phenomenology, then it seems that it is allowed to investigate only the most obvious states. (Leont'ev, 2009, p. 168)

That is, emotions must always be considered in the context of the phenomena in which it occurred.

EMOTIONS AND ACTIVITY

Consider a wolf in the wild. This wolf has a vital need to eat, and this need to eat drives him to hunt. These hunts result in him catching mice, rats, and rabbits. This then shifts the abstract need to eat into a concrete need to eat mice, rats, and rabbits. Then one day, he catches, for the first time, a duck. This, in turn, changes his need to include ducks in his menu of things he eats. And so on. Each time the wolf, through his hunt, encounters a new animal that he can eat, his needs change.

For Leont'ev (2009), such is the relationship between needs and activity. As humans, our vital needs, abstract and unrefined, drive our activity to satisfy these needs. These activities, grounded in phenomena, in turn gives an object to the needs.

> The fact is that in the subject's needy condition, the object that is capable of satisfying the need is not sharply delineated. Up to the time of its first satisfaction the need "does not know" its object; it must still be disclosed. (Leont'ev, 2009, p. 161)

This recursive relationship between needs and activity, each driving the other, expands and refines both the object of need, and the need itself, forming what Leont'ev (2009) refers to as *concrete-objective needs*. This, in turn, changes the subsequent action.

> [..] it is understood that changing the concrete-objective contents of needs leads to a change in methods of their satisfaction as well. (Leont'ev, 2009, p. 162)

Sometimes, however, this recursive cycle shifts the need from the object to the activity itself, forming what Leont'ev (2009) calls an *objective-functional need*. These needs, such as the need to work, to be productive, or to be creative, for example, do not displace the original needs that spawned them, but come alongside them as additional new needs. It is important to note, however, that not all objective-functional needs come from a newly acquired focus on activity. Likewise, not all vital needs are based on objects. In a cultural historical framework, action oriented needs can be part of the milieu. For example, the need to be subservient, to strive for physical perfection, or to always clean, can be *a priori* embedded needs within a person's specific cultural upbringing. Regardless, activity and "the satisfaction of the need" helps to delineate it.

For Leont'ev (2009), these delineated concrete-objective and objective-functional needs, in their ideal and reflected forms, are what he calls *motives*. And despite the fact that the language on needs and activity are shot through with willfulness and implied consciousness, our motives are not always known to us. Further, our activities are multi-motivational.

> Such breaking down is the result of the fact that activity necessarily becomes multi-motivational, that is, it responds simultaneously to two or more motives. (Leont'ev, 2009, p. 169)

They organize themselves in hierarchies, and these hierarchies define, to a great extent, an individual's personality.

> A division of the function of sense formation and simple stimulation between motives of one and the same activity makes it possible to under-stand [sic] the principal relationships characterizing the motivational sphere of personality: the relationships of the hierarchy of motives. This hierarchy is not in the least constructed on a scale of their proximity to the vital (biological) needs in a way similar to that which Maslow, for example, imagines: The necessity for maintaining physiological homeostasis is the basis for the hierarchy; the motives for self-preservation are higher, next, confidence and prestige; finally, at the top of the hierarchy, motives of cognition and aesthetics. (Leont'ev, 2009, p. 170)

Finally, for Leont'ev (2009), *emotions* act as an internal signal within this relationship between our motives and the actions that work satisfy them. That is, despite the fact that motives could be unknown to an individual, when they are realized there is an emotional response that signals that success has been achieved.

> Here we are speaking not about the reflection of those relationships but about a direct sensory reflection of them, about experiencing. Thus they appear as a result of actualization of a motive (need), and before a rational evaluation by the subject of his activity. (Leont'ev, 2009, p. 166–167)

These emotions have the potential, then, to reorganize the hierarchical order of these motives.

For example, a businessman has a goal to earn more than $100,000 in his job as a sales manager. One day, his boss calls him into his office and tells him that he is receiving a raise and will now be earning $110,000 per year. The man is elated. Later on that same day he overhears that his colleague has also been given a raise and will now be earning $120,000. Suddenly, a feeling of dread comes over him. Reflecting on this negative emotional response the man comes to see that what he really wanted was to be the best sales manager in the company. Earning over $100,000 a year was not the primary goal. The primary goal was to be the best sales manager in the company. But that goal was hidden from the man.

In this example, the man's surprising emotional reaction to hearing that his colleague was making more than him left him with an "emotional residue" (Leont'ev, 2009, p. 172) that moved him to engage with his hierarchical structure of motives, and to try to figure out what it is that is really driving him. In so doing, a motive that he was not previously aware of revealed itself. This results in a re-orientation of motives, which is tantamount to a re-orientation of his personality – all of which is triggered by his emotional response to an experience.

As such, in Leont'ev's (2009) framework, emotions serve as the orienting mediator between action and motives, and between motives and personality. In short, an emotional response to a specific experience draws the attention of the individual to their motives and allows them to begin the cognitive process of re-orienting their motives hierarchy.

EMOTIONS AROUND THE WORLD

Leont'ev's theory of emotions, motives, and personality, situated within the cultural-historical paradigm of individualized activity theory allowed me to look anew at the data from the prospective teachers playing *Around the World*.

All of these teachers wanted to be good teachers. This was one of their many motives. But they also wanted to please parents, have their students be good at basic multiplication facts, and to not make their students anxious or fearful, to name a few. These many goals were organized into hierarchies, unique to each prospective teacher. For the most part, these prospective teachers were not aware of many of their motives. Instead, they were fixated on their current goals of learning how to teach mathematics, getting good grades, and/or having their knowledge experience acknowledge. The "emotional residue" left from their experience playing *Around the World* helped them to see some of these motives. And it helped them to re-orient them.

In what follows I provide a brief case study on one of the prospective teachers—Tara—selected for her clear articulation of motives in her writings. Tara's case is analysed through the lens of Leont'ev's theory on motives, emotions, and personality (2009).

Tara

Immediately after the *Around the World* activity Tara wrote that she was feeling a little anxious.

> Tara I'm feeling a little anxiety, because [she] did not want to look stupid if [she] got it wrong.

However, she also saw merit in this activity.

> Tara As a teacher I see the value in this activity. Students must be 'switched-on' and engaged. It forces them to use their brains and everyone must participate. The likelihood of everyone getting the correct answer is unlikely so no one will feel bad if they don't get to sit down. It also creates a competitive environment and opportunity for kids to shine.

The hierarchy of motives from Tara's post-activity journal indicates that students being 'switched-on' is one of the primary motive for her as a teacher (see Figure 1).

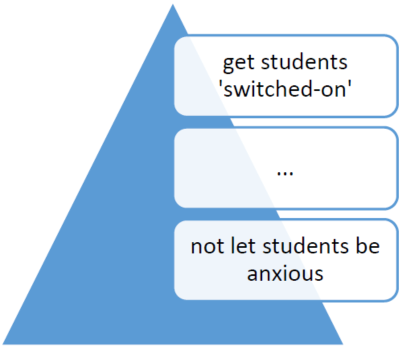

Figure 1: Tara's initial hierarchy of motives

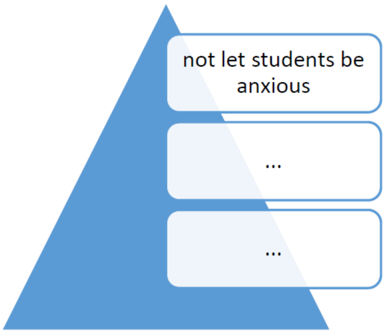

Figure 2: Tara's final hierarchy of motives

Later that night, when responding to the journal prompt her motivations had changed somewhat.

> Tara Today when Peter sprung the "stand up and get ready to answer multiplication facts" activity, my first feeling was fear and anxiety. I was worried I'd get the answer wrong and look stupid in front of the class. [..] This exercise made me think about my own classroom, how or whether I would use an activity like this. I think I would, that being said, would my students feel the same anxiety I did? Most likely they would, but I think after getting into the game they would enjoy the competition. The environment I plan to create in my class would assure them that I was not having them do this activity to humiliate them but to use mental calculation and practice their skills.

Tara's new hierarchy of motives (see Figure 2) can now be seen as having concern for her student's anxiety at the top and the motive to 'switch-on' students has dropped away. The few hours that she had to reflect on her experience with the activity and the *emotional residue* it left has seemingly caused her to re-orient her motives.

And this residue endures. At the end of the course Tara continues to talk about her motive to not let students become anxious, although not quite as explicitly.

> Tara I will never just stand at the front of a class and 'teach'. [..] Basically, I won't be afraid to teach outside the box ... the traditional box that I learned in and that we all know so well. I want to inspire my students. [..] I now know math doesn't have to suck ... the way it did for me in grade school.

CONCLUSION

So, what is it that changed for Tara and for the rest of the prospective teachers who 'played the game' of *Around the World*? One possible explanation is that it was their motives. More specifically, their hierarchies of motives were re-oriented – re-oriented by the emotional residue left after the intensely negative experience of being told that they would be playing *Around the World*. For 36 out of 38 of these prospective teachers, such a re-orientation resulted in a motive to not cause their students anxiety took its place as the primary motive at the peak of the hierarchy. This was not a new motive, but rather a motive that promoted up the ranks as a result of their emotional experience. And even after six weeks, and after 50 activities and 400 pages of literature, the concern for student anxiety remained as the primary motive. Leont'ev's (2009) theory allowed us to view emotions, not as fleeting abstract notions, but as robust and powerful contributors to the motives and future action cycle.

ACKNOWLEDGEMENT

The results of the research in this paper were presented, in part, as a keynote at the Mathematical Views (MAVI) conference in Falun, Sweden in October 2014.

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"Let's play! Let's try with numbers!": Pre-service teachers' affective pathways in problem solving

Francesca Morselli and Cristina Sabena

University of Turin, Torino, Italy, francesca.morselli@unito.it, cristina.sabena@unito.it

The paper focuses on the intertwining between affect and cognition in pre-service primary teachers, with the twofold goal of studying teachers' affect when facing problem-solving activities and of exploring paths for teacher professional development. 81 future teachers were proposed problem solving activities with specific "affective documentation" requests. We present here the first results of data analysis, and describe the main affective pathways emerged. Results are then discussed also with respect to methodological issues and to implications for teacher education activities that include the affective factors as relevant variables.

Keywords: Emotions, affective pathways, problem-solving, teacher education.

INTRODUCTION AND THEORETICAL BACKGROUND

This paper originates from our double interest as researchers in mathematics education and as teacher educators and investigates the intertwining between affect and cognition in problem solving processes carried out by future primary teachers.

Starting from the seminal work of Polya (1945), problem solving is a crucial issue in mathematics education (Schoenfeld, 1992), and constitutes also a fertile ground for studying the influence of affective factors in mathematical thinking processes (McLeod & Adams, 1989; Gómez-Chacón, 2000; Furinghetti & Morselli, 2009; Pesonen & Hannula, 2014). In fact, the solving process encompasses cycles, deviations and stops, and is influenced by resources, control, methods, heuristics, and affect (Carlson & Bloom, 2005). In particular, DeBellis and Goldin (2006) speak of the *affective pathways* as "established sequences of (local) states of feeling that interact with cognitive configurations" (ibid, p. 134). They underline also the importance of meta-affect, defined as "affect about affect, affect about and within cognition about affect, and the individual's monitoring of affect through cognition" (*ibid.*, p. 136). From a methodological standpoint, studying the interplay between affect and cognition is quite a complex issue. Pesonen and Hannula (2014) use screen recordings and emotional states automatic recognition, to study upper secondary students problem solving with dynamic geometry software. Other approaches are based on self reports or self study. Gómez-Chacón (2000), for instance, proposed the Problem Mood Map "for the diagnosis of emotional reactions and the subject's self-evaluation" (p. 152). In her study, local affect is self evaluated and registered by students during and at the end of the mathematical activity, by means of a special iconic code. The aim of such an instrument is to provide teachers information on students' affect during problem solving, but also to "foster in the pupil an awareness of his own emotional reactions" (p. 153).

The affective dimension plays a crucial role in future mathematics teachers, because it can endanger the success of their professional education processes and their future teaching processes (Hannula et al., 2007). Many studies have been arguing that what teachers believe and feel have a clear influence on what students believe and feel (e.g., see Hodgen & Askew, 2011). This issue is especially crucial for teachers at the primary school level, since in most countries they are not specialist in mathematics, and may have developed negative experiences with the discipline in their past experiences as students (Di Martino & Sabena, 2011; Lutovac & Kaasila, 2014). Coppola and colleagues (2013) have focused on future teachers' *attitude towards mathematics and its teaching*, stressing the importance of considering emotional disposition, view, and perceived competence—components of attitude, following Di Martino and Zan (2010)—both with respect to mathematics and to its future teaching. In this research, great attention is given to the link between the *past experiences* of pre-service teachers as students and their *future perspectives* of becoming mathematics teachers: also pre-service teachers with negative past experiences may show positive attitude towards its teaching (Coppola et al., 2013).

In our view, it is crucial promoting in future teachers the reflection on the intertwining between affect and cognition, and to work on the meta-affective dimension, as important steps in their professional development path. In this paper we present the first analysis of one activity conceived within this perspective, i.e. the problem-solving with "affective documentation" requests.

METHODOLOGY

The context

The study was carried out on the university students attending to the first year of the master degree to become kindergarten and primary teachers of the University of Turin (academic year 2013–14), in which the authors are in charge of the Mathematics Education courses. Throughout the math education course, pre-service teachers were asked to engage in various kinds of activities (problem posing, problem solving, analysis of children's solutions, classroom discussion transcripts analysis, and school textbooks analysis), individually or in group-work, as well as in collective discussions mediated by the teacher educator.

A preliminary study on prospective teachers' affect, performed according the model of the attitude towards mathematics and its teaching, suggested that pre-service teachers education should act in two ways: in *continuity* with respect to the need for a personal reconstruction with the discipline ("math redemption" in Coppola et al., 2013) and with the beliefs about the importance of affect in the teaching and learning processes, but also in *discontinuity* with the widespread procedural view of mathematics (Morselli & Sabena, 2014). On the base of these results, we set up a series of three problem-solving activities, coupled with specific "affective documentation" requests.

The affective documentation requests

We set up problem-solving tasks with specific additional questions about the *description of the solving process* and the *description of the emotional states* (affective documentation request) perceived in three different moments: at the reading of the text, during the solution phase, and reconstructed at the end of the activity. To this aim, specific open questions were inserted on the working sheet. The first question was immediately after the text of the problem: *"Describe how you feel when reading the text"*. Then, the solvers were asked to report their solving process *and* emotional state: *"Now try and solve the problem... Don't use another sheet, try to write down here all your attempts, thoughts and FEELINGS when solving the problem (your reasoning, emotions, blockages...)"*. Finally, once reached a solution, they were asked to write down the story of the solving process: *"Now, tell the story of your solving process, describing as much as possible the emotions you felt during the process"*.

This methodology has some common points with Gomez Chacon (2000)'s instrument, since the final aim is to gather information on local affect and also to promote meta-affect during and after the solving process. Also, our approach has some links with the teacher education interventions performed by Chapman (2008), where self-study has a key role. One way of fostering self-study is asking students to narrate stories of their solving process. Affective documentation can be considered also as a special case of story.

Our approach has the double aim of studying the intertwining between affect and cognition, and of making pre-service teachers more and more aware of the influence of affect during mathematical activities, in order to increase their ability to fruitfully manage it. Thus, asking solvers to document their own thoughts and emotions, although a demanding and non-neutral requirement, was valued also for its meta-affective outcome. It is important to underline that prospective teachers were not used to such a kind of request.

The 1089 task

During the course we presented three problems with an increasing complexity as regards the expected solving processes; also the requests of affective report are made more and more explicit. Here we focus on individual reports to the 1089 problem, the third presented to the future teachers (text inspired by Coles, 2013):

> Pick any three digit number (e.g. 752) with 1st digit bigger than 3rd. Reverse the number (in

the example, 257) and subtract (in the example, 752-257=495). Reverse the answer and add (in the example 495+594). [*the example is also written in column; the final result, 1089, is written*]

a) Now, try with another three-digit number (with the first digit greater than the third one) and do the same procedure: what result do you get?

b) Can you find a result that’s is NOT 1089? Why?

To answer question a) it is sufficient to carry out the same procedure as in the worked example. Question b) is the real challenge. A first exploration on numerical examples may lead to a counterexample (for instance, starting form 423 one gets 198 as final result). Once found the counterexample, the problem is solved; nevertheless, expert solvers could go on with the reflection, in order to see whether it is a sort of “isolated” counterexample or there is a regularity in the set of counterexamples. Indeed, all 3-digits numbers where the first digit is equal to the third digit plus 1 are counterexamples. Moreover, for all the counterexamples the result is 198. One could even come to a general conjecture: if the starting number of the type “first digit is equal to the third digit plus 1” the result is always 198, otherwise the result is always 1089. In the case in which the solver finds all numerical examples that end up always with 1098, she/he is expected to move from an explorative phase to the formulation of a conjecture, and the search for a justification on the general plane. This passage from numerical examples to mathematical argumentation was crucial to the aims and contents of the teacher education course.

Research questions

We explore the potentialities offered by the “affective documentation” requests to research affect in mathematical problem-solving. At the same time, we investigate whether such kind of tasks for pre-service teachers may have professional development value.

The research questions guiding the study are: 1) What affective pathways emerge? How do they intervene in the solving process? 2) Is it possible to establish some link between attitude towards mathematics and affective pathway? 3) What indications for teacher education come from the intertwining of solving and affective pathways?

In this paper, we will face the first question, and try to get some insight on how to direct future research in order to tackle the other (more ambitious) questions.

ANALYSIS

The 1089 task was faced by 81 students (all those who were attending the specific lesson). As usual in the lessons, students were allowed to collaborate in small groups: actually most of them chose the group-work; however, they had to fill the sheet individually and to write down the names of the groupmates. The subsequent qualitative analysis is performed on the individual reports.

The first approach to the problem

Considering the emotional reactions at the reading of the text, 48 future teachers (59%) declare only positive emotions, while 13 (16%) declare only negative emotions. The most quoted emotion is *curiosity*, mentioned in explicit way by 35 students (43%). Curiosity (but also incredulity) comes often along with the *cognitive needs* of understanding and discovering, which push towards undertaking with trust the resolution of the problem. An example is Schirry: “*I feel curiosity and desire of discovering, which push me to engage in solving the problem as a challenge*”.

Among positive emotions we remark also *astonishment*, indicated by 11 students. In some cases, astonishment is due not so much to the content of the problem, but to the novelty given by the typology of the problem (Betty: *“I feel astonished because I never focused on these types of problems”*). Among the students that signal mixed emotions (i.e. both positive and negative) we find mostly curiosity and interest combined with puzzlement (FedePeri: “*At the first reading I feel a bit of puzzlement, curiosity about the outcome and desire of trying with other numbers to discover if the results is always the same*”). In these cases, puzzlement, if accompanied with positive emotions such as curiosity, can constitute an important engine for the solving process, pushing towards the exploration phase. Some students welcome the activity even with enthusiasm, underlying its disruptive character with respect to the mathematical activities of their past experience (Amy: “*Let’s play! Let’s try with numbers! No proof for which it is needed to have studied so much!*”). Though rare (3 cases out of 81), this kind of enthusiastic welcome to a complex problem out of classical schemas can be considered as another sign encouraging us in continuing in this direction, on an intervention plan. On the other hand, also the few students who indicate only negative emotions do undertake the solving process, with some success (possibly due to the collaboration with their mates during the group-work).

Finally, some students, rather than indicating an emotional aspect, underline their cognitive needs when facing the problem. In particular, some of them express the necessity of reading the text several times. Sometimes we can trace the reference to interiorized schemas, which guide the solving process (Lia: “*I had to read the text several times, because during the first reading I try essentially to seize the general sense of the problem*”), and even the description of the kind of performed reading, as G.C.: “*I had to read the text several times lingering especially on the key-words and on the example. Once understood the text, I thought how to solve the problem*”. Students who quote the necessity of reading carefully and several times the text, typically show also strong elements of meta-cognitive control over the solution.

All the aforementioned examples refer to an intertwining between text reading (as a first step in the problem solving process) and *emotional disposition* (one component of the construct of attitude). Other examples refer to the role of *perceived competence* (another dimension of attitude) in text reading: a sense of tranquillity is juxtaposed to the awareness of knowing how to face what is requested (perceived competence), while restlessness and discouragement are quoted when the students foresee, by the first reading, not to be able to solve the problem. Some students make an analytical separation between the two requests of the problem, declaring opposite feelings in relation with opposite competences perceived to face them; an example is Chi: “*For what concerns the first part (the first question) I can say I am quite tranquil, because the request is clear and manageable. The problems come out with the second request, because I do not feel I am able to give a valid argument for that*”.

The solving process

The specific formulation of the task, providing a precise indication on how to start the solving process, appears to prevent “initial blocks”. The first solving steps engage the students a lot, on the one hand thanks to their procedural nature (a kind of mathematical activity in which the students feel at easy, as it emerged from the questionnaires); on the other hand, the task allows for a certain freedom for the exploration. This latter aspect is underlined as a positive factor by some students, as Lavixx94: “*It is a problem which allows us initially a certain freedom in our choices and this can be a positive factor that does not create anxiety in solving it, providing already given numbers to solve. One can so get to solve the passages in a quiet way*”. The second request (*Can you find out a number…*) is the actual core of the task. Some students start the exploration without any well-defined goal; some others are instead oriented since the beginning towards finding out a counter-example. As concerns the emotional aspects, we remark that outcomes that are *analogous* from the mathematical point of view are accompanied by very *different* emotional reactions, ranging from tranquillity to frustration, as we detail below.

Let us consider first those who obtain always 1089 (32 out of 81). Some of them are comforted from the regularity of the result, and as a consequence pushed to exploring and understanding. Others are satisfied with finding out other examples giving 1089, but feel stuck in proving it. In few cases (5) the reference to the need of a proving phase is completely missing, while a sense of amazement is the final step of the process. It is the case of Elis94, who checks only one example and comments “*Amazement, we did not think that it turned indeed into 1089!*”, but also of Estestest, who explores with six numerical examples. A third category is formed by those who feel sad, frustrated, or even angry for not finding out any counter-example (Martitea93: “*I remain astonished because I cannot realize of not being able of finding out a starting number that does not give 1089. I felt so to say also some anger*”). In the last two categories we can signal a lack of *knowledge at a meta-level on the mathematical activity*, which would lead to recognize a regularity in the absence of the counter-example, and hence to searching for a general explanation. Probably those who feel frustrated are too much focused on the question “Can you find out…”, as if it were a rhetoric question (“*Surely I have to find it out!*”). In this case, the emotional reaction would be strictly related to the *problem formulation* or to the *problem interpretation*: this appears to us a viable route to work on with the students, in an intervention perspective.

The majority of the students (59 out of 81) found out one or more counter-examples (leading to 198). An analogous range of emotional reactions, from surprise to puzzlement, can be detected. In other terms, we want to stress that it is not the result *per se* to provoke a certain emotional reaction, rather its interpretation within the mathematical activity, which in particular is determined by one's own *expectations on the problem* and *meta-mathematical knowledge*.

Dually, the same emotion may be linked to very different behaviours in the solving process. The most meaningful case relates to *astonishment*: it is sometimes indicated as final emotion, not stimulating further reflections (especially in students who obtain always 1089), whereas in other cases it is the engine for questioning and continuing the mathematical work (DadiLuca: *"With this example I don't find 1089 and I feel a sense of astonishment. Will it maybe be because the 3rd digit is less than the 1st only of 1?"*).

Generally, we can observe that the positive emotions, when are the *only* mentioned emotions, are not necessarily associated with a good solving process or high quality mathematical activities. In our data we can trace a serious *dark side of positive emotions as the only emotions*: students indulge on their positive feelings, do not receive boosts to further verify their results, to question their process, to check it. Their solving process burns out early, with little control at meta-cognitive level. Furthermore, it often happens that the negative emotion is felt later, when the students talk with their mates or the teacher, and realize their low performances.

On the contrary, those who indicate only negative emotions during the solving process do not always carry out poor or uncorrected mathematical activities. In many cases, an initial *negative* reaction is followed by a *positive* reaction associated with a discovery. For instance, Kika narrates in this way the story of her solution: *"At first we took numbers at random, seeing that it always came out 1089, hence we were resigning ourselves to the fact that it always came 1089. Then we tried to take numbers in sequences (e.g., 421-422-423) and we noticed that the results changed with 423, where the last digit was smaller than the first of 1 only. Hence we chose other two numbers with this structure (e.g. 524 and 726) and we transformed the hypothesis in a theory! What a satisfaction!"*).

The cases with more complete and effective solving processes are also those that document *quick up-and-downs of emotions* linked to the different solving moments (e.g. quick sequences of discomfort-new exploration-discomfort- astonishment). An example is given by Giuly who writes down three examples, finding 1089 in all cases and commenting "*Astonishment, wow, happy*", then finds an example with 198 ("*Ops I did something wrong*"), finds another case of 198 and writes "*OK, maybe I understood the rule, we feel yet more accomplished*".

This emotional swing ends up sometimes with a negative emotion, especially frustration or dissatisfaction, which may be linked to three different reasons:

1) feeling unable to communicate properly a certain conjecture, explanation, or argument, as Rubinorosso: "*curiosity-get lost-illumination-frustration in not being able to express well our discoveries*";

2) feeling unable to understand why there is a certain result, as Vios: *"But we cannot understand why it is always 198"*;

3) feeling unable to prove a certain result, as Chi: *"The discomfort is due to the fact that I am not able, in logical-mathematical terms, to prove the reason why, considering a number with the first digit greater than the third one, and the second and third the same, I got a number different from 1089"*

Generally, the final dissatisfaction is indeed related to greater mathematical and meta-mathematical competences. The typical cases are given by those pre-service teachers that express a global sense of success (for being able to find out examples) but also regret, delusion and resignation for not finding a general rule to distinguish the two sets of cases 1089 and 198. Asking where the counter-example comes from, and why only with some classes of numbers you find 198 is a typically mathematical curiosity: as future teachers educators we interpret in a very positive way these final students' dissatisfactions.

DISCUSSION AND CONCLUSIONS

A preliminary reflection concerns our method for studying the affective pathways in problem solving, i.e. the series of affective documentation requests. Even if prospective teachers were asked to report their emotions after text reading, during the solving process, and once solved the problem, many of them did not report emotions *during* the solving process, they rather commented immediately after having accomplished the task, or wrote twice the same comments. As a consequence, we organized our analysis in two sections (emotions after text reading and affective pathways in problem solving), without distinguishing between documentation within and after the process. Actually, writing down emotions and feeling when solving the problem is a demanding task, since the solver should stop the process in order to comment it. In light of these findings, in future interventions we would skip the request of documentation during the process and just ask for a report after the solution: in terms of meta-affect, this should be a valuable request as well. Also, we did not focus on the possible effect of group work on the affective and solving pathways. Future research could be organized so to distinguish between the two levels of “individual emotions” (reported when working individually) and “group emotions” (reported when working in group).

Concerning the results of the study, data analysis confirms the deep intertwining of affective and cognitive factors in problem solving, and reveals some patterns in the affective pathways (first research question): rich and complex problem solving processes are characterized by emotional pathways with a “continuous swinging” of emotions. Such emotional states are both cause and effect of the exploration, conjecture and proving steps. On the contrary, “fixed” emotional pathways, with a more stable (positive or negative) emotional state that persists throughout all the process, occur when solving processes are poor and non efficient. In order to interpret the different affective behaviours, we need more research on the possible link between awareness of the emotional pathway, meta-affect and attitude towards mathematics and its teaching.

As a provisional answer to the third research question, we discuss some possible routes for teacher education. A first result concerns the generally positive emotional disposition at the first reading of the text. It confirms the opportunity of engaging prospective teachers in such a kind of activity, which may be seen also as an occasion to experience new ways of doing mathematics. Another reflection comes from the reported emotions at the end of the solving process. We highlighted that different solvers report the same emotions in different situations and that, finally, it is not matter of positive or negative states. For instance, dissatisfaction and frustration may be linked to the need of better understanding or to the awareness of difficulties at epistemic and/or communicative level. Thus, negative emotional states may be read in terms of competence at meta-mathematical or mathematical level. Conversely, positive emotions such as satisfaction may be linked to a lack in meta-mathematical competence. The 1089 problem is a highly demanding task for pre-service primary teachers, also because of its theoretical nature: the core is constituted by conjecturing and proving, which we value as fundamental mathematical activities. However, pre-service teachers with low competencies at mathematical and meta-mathematical level happened to live the experience of solving the 1089 problem—with correlated positive emotions—and to discover their failure only later through the confrontation with more expert mates or the teacher educator: in this case, a negative emotion is likely experienced, risking to be the most persistent result of the overall activity. The crucial point, for us as teacher educators, is to find out efficient ways to intervene on the latter cases. This also opens the discussion on what kind of mathematical activities are worth to be proposed, and how to push the reflection at meta-level. We need more research on the design of suitable tasks for promoting prospective teachers’ awareness of their emotional pathways during problem solving.

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About relationships in the affect domain

Manuela Moscucci and Cecilia Bibbò

University of Siena, Department of Information Engineering and Mathematical Sciences, Siena, Italy, manuela.moscucci@unisi.it, bibbo@student.unisi.it

During the last decade, learning/teaching environment has assumed an important role in affect research due to the socio-constructivist account of learning/teaching mathematics. Despite this, the concept of interpersonal relationship, which is an important variable of the environment, has not yet been considered explicitly in affect research. The first aim of this paper is to show the importance of studying explicitly this concept in affect. The second aim is to underline a first characteristic of the interpersonal relationships. In the light of neuroscience results about mirror neurons, the authors observe that a relationship is characterized not only by the perceptible senses communication, but also by a hidden communication that takes place through mirror neurons sets and perhaps through other means not yet been discovered.

Keywords: Affect, relationships, neuroscience, mirror neurons, theoretical framework.

INTRODUCTION

Until the '90s, researches about affect in mathematics education have always been rooted in the historical constructs, beliefs, emotions, attitudes (McLeod, 1992) and values (DeBellis & Goldin, 1999). However, during the last decade, many scholars underlined the need of expanding and reorganizing the field, which appeared to be not sufficiently complete (Hannula, 2002; Hannula, Evans, Philippou, & Zan, 2004) and clear about the correlations among the constructs (Hannula, 2011; Hannula et al., 2004; Leder & Grootenboer, 2005). During the same period, other constructs, such as motivation, self-esteem, anxiety, were studied in affect domain. At the same time, some researchers (e.g., Goldin & DeBellis, 2006) switched from the consideration of an individual itself to an individual in his environment, analyzing both beliefs, emotions and attitudes of a person, student or teacher, (intro-affect), and the external/interpersonal ones of others belonging to the same environment (inter-affect). In conclusion, during the last decade, affect researchers have highlighted two requirements: the extension of the theoretical framework and the necessity of taking into consideration the environment of growing and learning/teaching of people holding emotions, beliefs, attitudes, values etc.

TOWARDS RELATIONSHIPS STARTING FROM AFFECT LITERATURE

Affect constructs' social component has been studied by many researchers (e.g., De Corte, Op't Eynde, & Verschaffel, 2002; Evans, 2000; Goldin & DeBellis, 1997) who argued that context matters. Already at the beginning of the 90s, some researchers (e.g., Bishop, 1988; Halliday & Hasan, 1989; Henriques, Holloway, Urwin, Venn, & Walkerdine, 1984) were aware of the fact that affect variables are socially constructed in the educational environment. However, only years later, it was claimed the importance of context in affect research. Op'T Eynde, De Corte, and Verschaffel (e.g., 2006) and Evans, Morgan, and Tsatsaroni (e.g., 2006) analyzed in depth affect in the complex environmental system. This means that thanks to affect literature we realize that social context has a key role in the quality of emotions, beliefs, attitudes, values of people. But now, we would highlight that the characterizing element of an environment are the interpersonal *relationships* that all individuals, of the considered context, establish among each other, and with all components of that social environment. The relationships guide the ability of regulating emotions and developing beliefs about ourselves, others and the environment itself. Creating, renovating, modifying beliefs and regulating emotions depends on relationships. The aim of this paper is to underline that in affect there is a hidden key element that has never been explicitly recognized: the interpersonal relationships and relationships, in general (in the following, named only relationships). This 'new' perspective could gain clarification of the notion of belief, emotion and attitude toward mathematics. We believe that relationships could be studied both as what links together affect constructs in a single framework, and what adds new meaning to all constructs. However, as we will see, there is much more. The relevance of relationships comes out not only from the affect literature, but also from the neuroscience one.

ORIGIN OF THE IDEA

The embryonic idea arose a long time ago. In the ‘80s Moscucci noticed a strange circumstance regarding a primary school teacher, who we will call Rose. Despite not having relevant professional skills, Rose was getting surprising results with her students. Rose’s students were generally quite wealthy, but not always of high cultural caliber. Rose, as it used in Italian schools, was the only class teacher of all subjects and taught her students from the age of 6 years old up to age of 11, for 5 consecutive school years, in a small Italian city. Moscucci was astonished by the fact that Rose’s students were almost all good students when they went to the high school. This statement does not derive from a statistical survey, but only from Rose’s recognized reputation in the city. Moscucci began to look at the case of Rose after a first cycle of Rose’s students. Moscucci was struck by the fact that, right at the beginning of the following cycle of teaching, Rose expressed her satisfaction to Moscucci for the skill of her students in the new cycle: children of 6 years only and that Rose had met only a few months before but had already defined as talented and very gifted kids. At the same time Moscucci knew that Rose’s students from the first cycle, who were also defined by Rose as exceptional students, were very good in high school. EvenE Moscucci did not study Rose’s case, it remained in her mind, as a researcher in mathematics education, the curiosity to formulate a hypothesis to explain a fact so strange.

Another story. During the last fifteen years Moscucci has conceived and directed many school projects, in Italian schools, aimed at overcoming students’ difficulties in mathematics, as, for example, those described in Moscucci, Piccione, Rinaldi, Simoni, and Marchini (2005). In the first part of their realization, the projects were conducted according to a particular educational path, called MBSA, Meta Beliefs Systems Activity (Moscucci, 2007), which aims to rebuild students’ relationship with math. Here it is not important what the whole structure of MBSA is, but only the activity “My story with math” planned in MBSA, when the participants are asked to tell their story with math, with the explicit request to focus their attention above all on emotions experienced during math activities. This activity is widespread in research (e.g., Di Martino & Zan, 2010). So, in those circumstances, Moscucci gathered many significant sentences spoken or written by students with difficulties in mathematics and reflected on them. Beside every sentence, there are questions that arose spontaneously.

> *“Even if the teacher told me I was good at math, she seemed to think the opposite”* (Ada, 15 years old). “In what sense “she seemed”? What did Ada mean?”

> *“I know that my teacher believes I am not good at math”* (Lisa, 15 years old). “How did Lisa know what her teacher believed?”

> *“Even if my teacher always smiles at me, I never feel sure and relaxed about what I’m doing”* (Lea, 16 years old). “Why didn’t Lea feel sure and relaxed, if her teacher was so nice to her?”

> *“When I do math homework together with my brother, I feel like he thinks I am slow”* (Sara, 14 years old). “How did Sara realize that her brother thought that she was slow? What did Sara mean when she said “I feel”?”

> *“When my father is in the kitchen, I am not able to do math homework”* (Giulia, 14 years old). (From the whole interview we knew that usually Giulia did her homework in the kitchen and that Giulia thought of her father believing she was not good at school). “What is it that stopped Giulia?”

> *“Even if the teacher told me I was good in math she seemed to think the contrary!”* (Amy, 16 years old). “Did Amy really believe that her teacher was thinking the opposite of what she told her?”, “Why?”

> *“When my teacher gave me the report card, I heard her voice in my mind saying ‘You will never really understand mathematics!’”* (Deb, 15 years old). We have reached the pinnacle! This girl not only made assumptions about the thoughts of others, but even claimed ‘to hear a voice in her mind’, her

teacher expressing an opinion. Was it just a way of saying or did she really hear a voice?

We want to point out that the reported sentences were said by students during a school project organized on a vocational school to overcome difficulties in math. So, all the involved students were not good students in math.

NEUROBIOLOGICAL BASIS TO SUPPORTING RELATIONSHIPS

The importance of context in learning processes concerns, of course, any discipline, not just math. However, regarding affect, such importance is justified not only as a sector of mathematics education, but it is amplified by the choice of its constructs, emotions, beliefs, attitudes and values. They are just the product of the interaction between the person with his/her environment. Hence, in our opinion, the necessity of including relationships in affect research studies. Relationships are widely studied and investigated by neuroscience. However, we intend to bring to the attention of affect researchers that is not enough to use the neuroscience results, since it appears to be particularly relevant in this context. Science considers human beings as in continuous interaction with their environment, and neuroscience has already stated the essential role of human interpersonal relationships in the construction of any kind of thought and knowledge. The brain is a 'social organ' of the body, it is exquisitely social, and emotions are its fundamental language (Siegel, e.g., 1999, 2000, 2001). Learning is a relational process that encloses regulation of emotions, that is fundamental in the development of all knowledge. So, what are exactly relationships from a neuroscientific point of view? Neuroscience, at this time, does not give a definition of relationship. However, affect researchers must not retrace the beaten track for decades in search of a definition of beliefs. We may see relationships as a primitive concept and settle for partially characterizing the relationships and study their characteristic elements even if not fully characterizing. We mean, maybe, in the future, other characteristics of relationships might be discovered and studied and we might arrive at a completely satisfactory characterization. The neuroscience result that, in our opinion, is a matter of great interest for affect research, concerns mirror neurons, that are a specific type of neurons. They are involved in a kind of communication unknown until now, at least from a scientific point of view. Mirror neurons, discovered by Gallese and Rizzolatti (University of Parma, I) in 1995, are one of the most important relatively recent discoveries in neuroscience. Mirror neurons "appear to play a fundamental role in both imitation and action understanding" (Rizzolatti & Craighero, 2004, p. 169). The activation of mirror neurons is, indeed, able to generate an internal motor representation (potential act) of the act observed, from which learning ability through imitation depends (Rizzolatti & Sinigaglia, 2006). In particular, these neurons should seem to play an important role in imitation learning. They are activated subconsciously, allowing people to trigger processes of imitation and communication without awareness. *The human brain has many and many mirror neurons that specialize in carrying out and understanding not just the actions of others but also their intentions, behavior and emotion, through direct feeling* (e.g., Gallese, Keysers, & Rizzolatti, 2004). According to researchers, in social relationships, the functions of mirror neurons allow an immediate understanding of what others are doing, without any kind of mediation or interpretive reasoning. Mirror neurons constitute a scientific proof of the existence of a sort of not perceived communication. So, we may claim that communication has visible (or at least sensitive perceived) and conscious components, such as body languages and other sensitive human expressions, but also unconscious and invisible components. We will refer to the last type of communication, with the diction *hidden communication*. The discovery of hidden communication between two people influences the view of the quality of relationships, and it is clear how the consideration of this element of interaction between people intervenes in studies about human relationships. Let's say even more explicitly. Communication through the mirror neurons sets shows that a relationship between two persons is not constituted only by the set of the interactions between them, perceptible by senses, as, for instance, verbal communication. There is much more. We mean that, at the moment, a relationship between two people is an entity which has to be described fully. In fact, we have said: "*Communication through mirror neurons sets*", but, despite us not being neurobiologists, we believe we can say, as a highly probable hypothesis, that almost certainly mirror neurons are nothing more than a kind of transmitting and receiving antenna. As a matter of fact, knowledge of the functioning of mirror neurons determines a new vision of interpersonal relationships and an absolutely innovative approach to the study of the nature of the relationships. In addition to this, recent evidences suggest that mirror neurons are involved in what is usually called *empathy,* which is the capacity of feeling the same emotions that others feel. Indeed, we can say that this gives a scientific link to something that, up to now, was guessed, imagined, perceived too, but had not a scientific root. *"Empathy, or the ability to share feeling states with other individuals, is an important aspect of affiliative, prosocial behavior in modern-day humans"* (Nelson, 2012, p. 179). Empathy is thus a form of hidden communication. "At a neurobiological level, empathic responding is thought to reflect activity within distinct neural circuits subserving other social processes such as understanding person-specific experiences (theory of mind) and reflexive activation of observed experiences in others (mirror neurons)" (ibid, p. 1). Mirror neurons allow us to feel the mind-state of another person. They automatically and spontaneously pick up information about intentions and feelings of those around, creating emotional resonance and behavioral imitation as they connect the internal state with those around, even without the participation of a conscious mind. All these means that mirror neurons enable us to connect with each other. They dissolve the wall between one and the others. These results have a significant impact on the study of beliefs and emotions in general. In fact, through mirror neurons, we continuously receive information from all those who take part to our environment. This information, of which we are unaware, greatly contribute to the construction of beliefs or patterns of interpretation of the reality, and therefore the quality of our emotions. Then beliefs, emotions, attitudes and values depend on the relationships that a person has structured over the time, in every context of life, family and school in the first place. Therefore, the consideration of these findings in neuroscience is, in our opinion, a starting point for collaboration between neuroscientists and education researchers, and particularly between neuroscientists and affect researchers. Indeed, there is much more from the point of view of affect research. Indeed, all this not only means that there is an interconnection among emotions and beliefs of people, but also that there is a link rooted in neurocerebral mechanisms. All this stresses the importance of the analysis of the relationships of the person, student or teacher of mathematics, in order to understand deeply the quality of his/her emotions, beliefs, attitudes and values. And, all this is not only due to the importance, for affect, of the context from a social constructivist point of view, as it has been highlighted above, but on the basis of new scientific results from related fields of research.

A last observation about the case of Rose. When Moscucci first met Rose she was particularly intrigued by some Rose's personal characteristics: her enthusiasm for her students, her firm belief that all her students were good at learning and very intelligent, and her joy about working with children. The results about mirror neurons allow to hypothesize that Rose transmitted to her pupils her enthusiasm that affected positively their self-esteem and their confidence whose role on learning is well known.

ABOUT RELATIONSHIPS

The discovery of mirror neurons states that there are hidden information shared between people, which are absolutely independent from people will. We must take in account that when we work with the relationships we are working with the brain structure. Scientific results about mirror neurons constitute the first step to characterize the concept of relationship also from a neuroscientific point of view. They allow us to talk about *interaction between individuals* not only as communication, in the usual sense of the word. In particular, communication is an exchange of information based on sensory perception. But researches show that the interaction between two individuals is not just communication. A relationship is something much more complex than the set of information of interpersonal communication, even if we look at it in all its forms: verbal, gestural, mimics, tactile, postural, kinesthetic, etc. Then we might refer to these types of communication such as characteristics of the *rapport* between two people, while the *relationship* is characterized by the *hidden communication*, due to mirror neurons, and, perhaps, due to other things yet to be discovered, but, without doubt, due to the hidden communication, that is proved existing. We might say that *sensory communication gives rise to a rapport, while the hidden interaction, or hidden communication, gives rise to a relationship*. The relation between rapport and relationship deserves to be investigated scientifically. For the moment, we can make some considerations based on observation of what happens in the usual social contexts. For example, the experience leads us to believe that there may be a rapport, maybe for many years, between two people between whom there is a weak relationship, while there may be a strong relationship between two people who have just met, and between whom there is a very weak rapport. This may be one reason for distinguishing rapport from relationship. We may say that usually, a relationship and a rapport are co-present, but now we cannot exclude any relation between relationship and rapport. In fact, it is well known that particularly linked people can communicate with each other some moods that are affecting them emotionally, even at a distance. In the literature (e.g., Segal, 2000) there are reports about identical twins, who never met, so without having any rapport, one could perceive fear, for example, experienced by the other. However, although we cannot exclude that there are people between whom there is a relationship, but not a rapport, we must admit that these are really *limit cases*. Hence, when we talk about relationship we mean the set of rapport and hidden communication. To sum up, the study of relationships must take into account their dual nature of communication: explicit and hidden, and the hidden communication will be the component that, henceforth, will contribute more to the advancement of research. The hidden communication that occurs through mirror neurons and on which, at least in part, is based empathy, affects beliefs, emotions, attitudes, values. The consideration and inquiry of the relationships in their entirety can really help affect researchers to progress about their studies.

RELATIONSHIPS IN THE AFFECT DOMAIN

Affect researches identify beliefs and emotions as essential elements to understand how students build their mathematical knowledge. Learning processes have their origin in the experience, and experience depends on the relationships established between a person and every single other person in his/her environment, and the environment itself. In fact, through relationships, a human being, but surely even animals, begins, early in his/her childhood, to interpret his/her experiences and, in turn, through these experiences of interpretation, s/he builds models of interpretation. These models constitute the basic references to interpret his/her next experiences and, in particular, his/her emotions, along all his/her life. So, emotions are a result of the use of models built through the relationships. In addition, beliefs are shaped through relationships and they are involved in the interpretation of experiences. Then, emotions and beliefs are closely linked and influence each other, because of their nature and their construction which occur through the relationships. As a consequence, affect constructs have their roots in relationships, and should be seen in absolute dynamicity and synergy. It is really difficult to deal with one of them independently from the others. These are the main aspects which have led us to consider relationships as a key element in the affect domain and, in our opinion, just these arguments are absolutely compelling for considering the relationships in the affect domain. However, we claim that there is another reason, perhaps even stronger, requiring researchers to extend their studies about the relationships and, at the same time, strengthening them. The study of the principles of quantum physics supported again the ideas that have allowed this work to take shape. As we said, many affect researchers have emphasized the importance of the correlation between the constructs. We believe it is useful for the research climate in affect domain, to borrow a principle of quantum physics. Quantum physics shows that the properties of objects manifest themselves as such, only when the objects interact with each other. Then, it is important, in affect, to study not only what are the constructs themselves, but also the relations between them, just to understand their nature. In fact, constructs' nature may emerge completely from the study of the relationships which contributed or contribute to the construction of a certain belief or value, to the manifestation of a certain emotion, to the structuring of a given attitude, or motivation etc.

CONCLUSIONS

The findings of neuroscientists about mirror neurons will certainly have many repercussions in the educational research. The suggestions that we deducted from the research on mirror neurons, convinced that much more come out from research in neuroscience. For example, studies on the contagion of yawning in primates, may allow to make the assumption that the importance of the hidden communication between teachers and students is a possible research hypothesis, and we will make it explicit and deal with in future papers. Studies about the "hidden communication" might really give great impetus to the research in the field of human communication and in all fields in which communication plays an important role, such as education, and, particularly for us, affect. And we absolutely intend to face this issue in the near future. For the development of affect research we support the importance of bringing to light *relationship*s as a real construct and studying it explicitly. The processes of teaching/learning are about people and how people interact. The relationships among people, students and teachers, are involved in these processes. Relationships intervene substantially in these processes and, therefore, must be considered a variable to evaluate the processes of teaching/learning. So far we have studied the relations between emotions, beliefs and attitudes of the student and, to a minor extent, of the teacher. Now affect researchers have to study emotions, beliefs and attitudes of the student closely connected with emotions, beliefs and attitudes of the teacher, namely as a result of the quality of the interpersonal relationship between student and teacher! Which kind of teacher's beliefs, emotions and attitudes could be inhibitory to the student learning process? And which ones could affect positively? How could we act on teacher's inhibitory beliefs, emotions and attitudes? This is just to give some examples of research questions that could really be addressed in affect research, in mathematics education and, more in general, in education. Investigating the profound influence on us of those around us, or what Siegel (e.g., 1999) calls "the neurobiology of we", it will represent another step forward in the development of a comprehensive view of human learning processes, even if these further investigations could not meet our original expectations. At this stage, we may say that the way in which we perceive students regarding the development of their potentialities and abilities in mathematics, could likely affect their success in mathematics and the development of their general potentialities. Within the realm of affect, taking into account relationships means considering not only belief systems and emotions of students, but also those of their teachers, and not only that. It means to consider these belief systems and emotions as a complex structure, not as static, but dynamic and continuously evolving.

This paper aims just to gain the attention of affect researchers regarding the possible implications that a deeper investigation about relationships could have in the understanding of affective variables in mathematics education.

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ENDNOTE

Cecilia Bibbò worked about the study during the preparation of her Master's Degree Thesis, discussed at the University of Siena, on October 25th 2013.

On the role of affect for sense making in learning mathematics – aesthetic experiences in problem solving processes

Eva Müller-Hill[1] *and Susanne Spies*[2]

1 University of Cologne, Cologne, Germany, eva.mueller-hill@uni-koeln.de

2 Siegen University, Siegen, Germany

Sense making is one important goal of learning processes in school mathematics. Empirical studies on genuine sense constructions of mathematics students show the importance of sense construction categories with a subjective and an inner-mathematical orientation at the same time. From a theoretical point of view, we investigate two ways of fostering such sense constructions within school mathematics, by means of aesthetic experiences on the one hand and through reflection of problem solving processes on the other. In both cases, affective aspects play an important role. We also discuss the merits of intertwining aesthetic experiences and problem solving processes regarding sense construction. We finish with some remarks on possible ways of employing our theoretical results practically.

Keywords: Sense making, mathematical beauty, problem solving, beliefs, affect.

INTRODUCTION – THE ROLE OF SENSE MAKING IN LEARNING MATHEMATICS

Sense making is a general, and at least from the viewpoint of constructivism of course quite natural goal for the learning of mathematics. Nevertheless, the meaning of "sense" within the educational debate is underdetermined, sometimes vague. A great variety of different aspects of sense have been brought up in this debate, sometimes explicitly articulated, sometimes more implicitly. Such aspects are, e.g., purpose, benefit, intention, or merit. From an analytic point of view, most of these aspects have at least two dimensions: an objective-subjective and an inner-extra-mathematical dimension. With regard to these dimensions, a large part of the discussion in mathematics education focusses on rather objective, inner- and extra-mathematical issues of "sense making" (e.g., what is or what should be the "sense" of negative numbers, which lifeworld contexts are adequate to teach them). Subjective issues of sense making oriented to the learning subject, on the other hand, are usually discussed with regard to rather extra-mathematical themes, like incentives, personal relation to teacher, beliefs about the subject's own abilities in learning mathematics etc. However, there are also popular positions emphasizing subjective inner-mathematical aspects of sense making as of great importance for the learning of mathematics, as the work of Skovsmose (e.g., Skovsmose, 2005), or Ruf and Gallin's work on "core ideas" (e.g., Ruf & Gallin, 1998).

The guiding question of the following considerations is: How can subjective, inner-mathematical aspects of sense making in learning mathematics be addressed, and triggered within learning processes, and what role does affect play in this regard? To our mind, this question is less a question about methods of teaching. It is rather meant as a question about contents of teaching mathematics, on two different levels: In second order, it addresses the object level of mathematics itself (negative numbers, triangles, functions and graphs, variables etc.). In first order, it shall deal with "meta-units" of doing mathematics (problem solving, proving, discovering new connections, reconstructing solution processes and so on) which can be a basis for subjective inner-mathematical sense making in a learning process.

Our approach is to take inert sense making categories of learning subjects as a starting point. Vollstedt (2011) investigates such inert categories of sense making, "sense constructions" in her terminology, for the case of students of the lower secondary level (15 till 16 years old) in a qualitative, comparative empirical interview study (34 participants, 17 from Hongkong and 17 from Germany).[1] In a first step, she classifies sense constructions into some 3 times 3 matrix (Table 1).

The category "high individual orientation–high inner-mathematical orientation" (cell 3) corresponds to our "subjective inner-mathematical" dimension:

> **High intensity of inner-mathematical orientation**: The sense construction is in immediate relation to mathematical contents [and not to other aspects of learning processes in school, as, e.g., social interactions].
>
> **High intensity of individual orientation**: The individual itself is the focus of the sense construction, intra-individual relations dominate [and not social institutions or demands]. (ibid, pp. 130f., German in original, added brackets give short descriptions of value "low")

Vollstedt found that this category is empirically contentful, that is, sense constructions could be reconstructed from the interview data that fall into that category. In the following, we will concentrate on two of these sense constructions: "experience of autonomy" and "mathematical purism" (ibid, pp. 142–148). Vollstedt defines these two categories with reference to a "criterion of personal relevance" (ibid, p. 129, German in the original):

> *Mathematical purism*: Doing or learning mathematics is personally relevant to the individual if it appreciates the purism of mathematics stemming from its formality and logical composition and benefits from this appreciation regarding its understanding of mathematics.
>
> *Experience of autonomy*: Doing or learning mathematics is personally relevant to the individual if it experiences self-reliance in doing or learning mathematics, e.g., in terms of learning autonomy and original development of solutions to mathematical problems.

Answers coded as "mathematical purism" deal with a special fascination of pure mathematics, and describe a positive affective relationship to a special mathematical discipline (e.g., geometry or algebra) or express personal relevance of clear and logical mathematical structure (ibid, p. 146). Answers belonging to the category "experience of autonomy" describe positive experiences by, e.g., choosing exercises autonomously or finding individual ways of solving problems (ibid, pp. 142–144).

In the following sections, we discuss two kinds of "meta-units" of doing mathematics in relation to these two sense constructions. Both have a certain affective potential, and therefore specific merits regarding the initiation of sense making in learning mathematics. First, we argue that *aesthetic mathematical experiences* provide good opportunities for the learning subject to appreciate mathematical purism. For the case of autonomy, aesthetic judgments are distinctly different from right-or-wrong-judgments. Aesthetic mathematical experiences are often closely related to so called AHA! experiences, a highly affective type of individual understanding processes. Second, we look at *mathematical problem solving processes*, and, in particular, the affective components of those, in relation to the sense constructions "mathematical purism" and "experience of autonomy", and then turn to discussing the value of intertwining both perspectives to arrive at a particularly fruitful basis for sense construction within learning processes at school.

AESTETHIC EXPERIENCES AND PROBLEM SOLVING PROCESSES AS SOURCES OF SENSE CONSTRUCTION IN LEARNING MATHEMATICS

Sense construction and mathematical beauty

Relying on Spies (2013), we adopt the premise that though mathematical beauty isn't definable entirely, a concept of beautiful pieces of mathematics, cover-

Intensity of inner-mathematical orientation of sense construction	Intensity of individual orientation of sense construction		
	Low	medium	high
High	1	2	3
Medium	4	5	6
Low	7	8	9

Table 1: "Typology of sense construction" (ibid, p. 133, German in original)

ing for instance theorems, proofs, argumentation schemes, and heuristic strategies, can be explicated. Informed by mathematics as a scientific discipline it is also fruitfully employable within discussions of mathematics and beauty in mathematics education. The concept of mathematical beauty developed in (Spies, 2013) is explicated by four relevant attributes: *range*, *economy*, *epistemic transparency*, and *emotional effectiveness*. At least three of these attributes, range, epistemic transparency, and economy, are directly related to the sense constructions "mathematical purism" and "experience of autonomy" investigated by Vollstedt. Regarding the sense construction "mathematical purism" in detail, the following relations hold:

Range: There are two aspects of range often referred to when it comes to aesthetic value judgments in mathematics. On the one hand, a beautiful piece of mathematics connects different parts or branches of mathematics, for example the use of algebraic tools to solve a geometrical problem.

> A beautiful proof often makes unexpected connections between seemingly disparate parts of mathematics. A proof which suggests further development in the subject will be more pleasing than one which closes off the subject. (Stout, 1999, p. 10)

On the other hand, an argumentation owns a certain kind of beauty if the idea at the core of the argument is applicable to a variety of other cases, if the chosen heuristic is paradigmatic in some sense. Becoming aware of the range of an argument or a result may deliver *insight* in the system of mathematics itself. The awareness of broad connectivity of an argumentation or of the paradigmatic character of a heuristic may help to establish an appreciation of mathematics also as a self-contained system besides its applicability in extra-mathematical contexts.

Epistemic transparency: This attribute of beautiful mathematics underlines the subjective character of aesthetic experiences within mathematics. It explicitly stresses the importance of a subjective *understanding* of mathematical structures in connection with an aesthetic mathematical experience: A beautiful proof offers a special kind of deep understanding of *why* the result is true. Often, this is described as an illumination, as a spontaneous grasping of the whole argument from one moment to the other, as an AHA! experience, together with strong positive emotions.

> The mathematician's "aesthetic buzz" comes not only from simply contemplating a beautiful piece of mathematics, but, additionally, from achieving insight. (Borwein, 2006, p. 25)

Accordingly, an aesthetic mathematical experience is not only a product of positive feelings, but also linked to a special kind of deep individual understanding why.

Economy: Under the sense construction "mathematical purism", Vollstedt subsumes statements concerning the shortness of mathematical arguments, or the number of formulas you have to remember to solve a certain range of mathematical problems (Vollstedt, 2011, p. 147). The properties expressed by these statements correspond to economy as an attribute of beautiful mathematics, and link the attributes of range and epistemic transparency. As G.H. Hardy points out in his famous *Apology*:

> In both theorems [mentioned as examples for great mathematical beauty; author's remark] there is a very high degree of unexpectedness, combined with inevitability and economy. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail–one line of attack is enough in each case. (Hardy, 1940, p. 113)

Hence, it is not shortness of an argument as an extrinsic property that releases an aesthetic value judgment. It is the impression of economy, that is, shortness of argumentation in relation to its range and with regard to epistemic transparency for the judging subject. There is a necessary connection between individually oriented and inner-mathematically oriented aspects to trigger aesthetic judgments, and thus, possible sense constructions.

Regarding the sense construction "experiencing autonomy", we take into account that aesthetic experiences of beauty undergone by an individual, by becoming aware of them, lead to conscious judgements of beauty, stated or not. These aesthetic value judgements are in a specific way opposed to the prominent excluded middle character of mathematics, and the focus on formal mathematical correctness (Müller-Hill & Spies, 2011). While the latter are rather related to experiences of special conformity and authority of mathematics, aesthetic value judgements transcend formal correctness and are, at least in part, subjectively justified. Nevertheless, they can usually be explained on the basis of inter-subjectively graspable criteria by the judging individual. In this sense, aesthetic mathematical experiences give room for autonomous, but negotiable judgements of beauty, and as a consequence, for responsible, self-relying decisions and actions based on these judgements.

The fourth attribute of mathematical beauty according to (Spies, 2013), *emotional effectiveness*, explicates the affective character of aesthetic mathematical experiences. When mathematicians talk about the aesthetic pieces of mathematics, they use a highly emotional and affective language. Leone Burton reports from an interview study with practicing mathematicians:

> The mathematicians discussed aesthetics [...] in terms that were emotive, full of expressed feelings. (Burton, 2004, p. 63)

Often, the expressed emotions are used to qualify one of the other attributes of mathematical beauty described above.[2] Terms of special relevance according to aesthetic value judgement seem to be *unexpectedness* and *surprise*. A beautiful argument may evoke the feeling of surprise about its (economical) form, of a "surprising twist" in the argumentation, or of unexpectedness regarding the heuristics employed in the argumentation. The feeling of *inevitability* of an argumentation seems to be another aspect of the emotional effectiveness of beautiful proofs (see also the above quote from Hardy, 1940, p. 113), as the mathematician Gregory Chaitin states vividly:

> After the initial surprise it [a beautiful proof] has to seem inevitable. You have to say, of course, how come I didn't see this! (Chaitin, 2002, p. 61)

Sense construction and central types of problem solving processes

Of course, problem solving is a widely discussed issue in mathematics education, and in turn there are a number of different conceptions of problem solving and ways or conceptual tools to describe and investigate problem solving processes. In the following, we discuss problem solving processes as a basis for sense construction in learning mathematics, and ask for the role of affect regarding this relation.[3] To this end, we lean on the synopsis given in (Schoenfeld, 1992), which explicitly embraces and explicates "beliefs and affects" as an important (cognitively effective) aspect of problem solving processes (see ibid, p. 348). The aspect "beliefs and affects" contains students beliefs, teachers beliefs, and also "general societal beliefs" about the nature of mathematics and doing mathematics. We will focus on the "belief system" (Schoenfeld, 1985) of the problem solver, encompassing beliefs, attitudes and opinions about mathematics itself (about „formal mathematics", his „sense of the discipline"; Schoenfeld, 1992, p. 359), about mathematics as a subject matter, and about doing and learning mathematics. Schoenfeld emphasizes the extraordinary powerful impact of the belief system on mathematical problem solving.

In the following, we distinguish two ways in which problem solving processes can generally be connected to sense construction in the above sense, that is, in which they can be meaningful and personally relevant to the problem solver. The first way refers to the *decisions* that are made throughout the process by the problem solver, and the motivations behind them. The second way considers *changes in the belief system* as responsible for subjective sense making. Such changes may be initiated by actual performance of, or by later reflection on, problem solving processes.

Decisions guiding the course of action throughout a problem solving process are potentially meaningful components of these processes. At least to a certain degree, they allow to infer something about what is personally relevant to the problem solver in terms of guiding motives and reasons for these decisions. Potentially subjectively meaningful decisions, and actions in turn, have to be autonomous at least to a certain degree, intentional, and goal oriented. Usually, decisions are also affectively driven, which strengthens their subjective relevance and, in the case of several alternatives, can even be the last instance to decide. Undergoing a process of problem solving also induces an interaction with the individual belief system of the problem solver. The belief system, being the network of conceptions, opinions, attitudes and beliefs related to mathematics, is per definition the basis of all subjective sense making in learning mathematics. Sense constructions on the basis of problem solving processes will therefore necessarily incorporate changes in the belief system, in the form of expansion, overwriting, or readjustment.

Regarding both decisions and changes in the belief system, it is not sufficient for fostering sense making just to get learning subjects involved in concrete problem solving processes. Sense construction in the way described above will usually not take place automatically. We suggest that *explicit reflection* of problem solving processes including decisions, action guiding motives, and affective aspects like emotions felt during the process or conscious changes of certain beliefs is necessary.

Reflecting on decisions as part of a problem solving process obviously encourages sense constructions of the type "experience of autonomy". But problem solving processes also have a specific sense making potential with regard to „mathematical purism", by focusing on recognizing *structural types* of mathematical problem solving processes. Reflections on characteristic elements of problem solving processes of a certain type can both lead to an appreciation of the formal structure of mathematics (regarding corresponding changes in the belief system), and lead to a better subjective understanding, as it may promote the ability to transfer problem solving approaches and strategies (regarding the orientation on a certain type of problem solving processes as a guiding motive for decisions in a concrete process). The latter also provides the opportunity for experiences of autonomy. The inner-mathematical focus can be increased by highlighting *central structural types* of mathematical problem solving processes. Mathematical problem solving processes of a central type employ certain, structural elements that are characteristic for working with central concepts of (branches of) mathematics like "function", "gauge", "number", or "area". [4]

Now in turn, we will argue that there are at least two ways in which a combination of problem solving processes and experiences of mathematical beauty can be particularly fruitful with regard to sense construction in learning mathematics, exploiting their affective elements in a specific way.

AN INTERTWINED PERSPECTIVE

As stated above, the decisions made during a problem solving process can be meaningful and relevant to the problem solver in terms of ideas, reasons or motives guiding them and the corresponding actions. These guiding ideas, reasons or motives are usually not merely rational, but decisions and actions are also guided by affect. Goldin (2000), e.g., investigates the relation between affective states (including aesthetic experiences) and chosen heuristics in problem solving processes. Though criticizing Goldin's subsumption of aesthetics under what he conceives as affective states, Sinclair in her (2008) also stresses the importance of aesthetics as coupled both with affective and cognitive aspects of mathematical problem solving. However, she emphasizes that

> the aesthetic and the affective domains each *function* differently in the problem-solving process: the aesthetic draws the attention of the perceiver to a phenomenon, while the affective can bring these perceptions to the conscious attention (ibid, p. 55, our emphasis).

In this sense, we consider decisions and actions driven by experiences of mathematical beauty as important, potentially sense-constitutive affective elements of mathematical problem solving processes. This is additionally underpinned and emphasized by a number of famous practicing mathematicians. For example, in his famous essay *The psychology of invention in the mathematical field,* Jacques Hadamard reaches the following "double conclusion" after reviewing psychological and philosophical literature on general and mathematical invention:

> That invention is choice. That this choice is imperatively governed by the sense of scientific beauty. (Hadamard, 1954, p. 31)

This observation from scientific mathematics can at least partly be adapted for problem solving processes in school mathematics: Choice is a meaningful element of problem solving processes. Choices are i.a. governed by subjective motives like perceptions of mathematical beauty.[5] To foster sense construction sustainably, the conscious attention of such aesthetic perceptions should be supplemented by explicit reflection on their role for the course of concrete problem solving processes.

With regard to the second form of sense making in problem solving processes, on the other hand, aesthetic experiences can help to initiate *changes* in the inner-mathematical component of the belief system due to their specific affective character as described above. This might be necessary when students fail in completing a problem solving process on their own because of holding certain constraining beliefs. Even if they succeed to finish the problem solving process with some help, the constraining beliefs might be abstracted from years of classroom experience (Schoenfeld, 1985). Hence, rational reflections of such single problem solving processes will usually have only little effect on their belief system, and therefore hardly trigger new sense making. Aesthetic experiences might catalyze the reflective impact in this regard. In particular, aesthetic mathematical experiences are often conceived as AHA! experiences. Empirical studies as (Liljedahl, 2005) show that there is a strong relation between the reflection of AHA! experiences and (even drastic) changes of student‘s beliefs about their learning of mathematics because of the affective, especially the emotional potential of AHAs (ibid., 231). AHA! experiences of mathematical beauty work in a quite similar way regarding unexpectedness, surprise, sudden inevitability (compare ibid., 226), and emotional effectiveness, but they are directed to attributes and relations of mathematical argumentations, formulas, diagrams, theorems, etc. The four attributes of mathematical beauty described above, range, economy, epistemic transparency, and emotional effectiveness, can be experienced jointly or in different combinations as aesthetic aspects of a certain piece of mathematics dealt with in a concrete problem solving process. Therefore, it seems promising to assume a similar impact of reflecting on aesthetic AHA! experiences in problem solving processes on student's beliefs on mathematics itself.

SHORT OUTLOOK ON DESIGN ISSUES

When we now direct the focus to designing issues in a short outlook, we rather have in mind developmental questions and the design of supporting learning material, not measurement issues. Our contribution must obviously be conceived as rather theoretical with regard to the design of concrete learning environments or teaching material. One reason for this is that from our point of view, sense construction is strongly tied to individual parameters of learners and learning groups. This makes it difficult to argue in favor of concrete designs in detail. Nevertheless, we think that we have generally identified good candidates to direct concrete designing attempts to. Problem solving is standard to school mathematics today. What might not be standard is guided reflection of concretely undergone processes with an explicit focus on action guiding motives, decisions, beliefs, or affects. Also still non-standard is the upgrading of aesthetic experiences as admissible and negotiable justifying reasons for choosing between alternative courses of action within problem solving processes. Future work will expand the view onto the scope of content didactics, aiming at a specification, e.g., of mathematical concepts, appropriate mathematical problems, and (types of) corresponding problem solving processes and beautiful pieces of mathematics as the subject matter of possible learning environments to foster subjective inner-mathematical sense construction. A point of practical interest in the design of learning environments will be the degree of guidance and instruction accompanying the problem solving activities of learners. Through a theoretical lense, this will link the perspective opened up here to questions about the role of creativity.

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ENDNOTES

1. Vollstedt uses grounded theory methodology, not operationally defined items or coding categories. Due to space, we refer to the original source for more details on Vollstedts shaping of the categories based on her data.

2. Similarly, "emotional relation to mathematics" is described as part of "purism" (Vollstedt, 2011, p. 146).

3. See also (Presmeg, 2014) for another most recent, programmatic emphasis.

4. "Central types of problem solving processes" corresponds to "fundamental mathematical ideas" famously introduced by Bruner in the 1960s and discussed under a variety of readings. We will not enter this discussion here, because the explication of „central" is not necessary for the general arguments discussed. Further qualification of "central" will be necessary for future work, esp. from a content didactical point of view.

5. According to Hadamard, this assumption is quite obvious: "Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature." (Hadamard, 1954, p. 104)

The role of multiple goals in students' motivation and achievement

Marilena Pantziara[1] and George N. Philippou[2]

1 Cyprus Pedagogical Institute, Latsia, Cyprus, marilena.p@cytanet.com.cy
2 Univeristy of Nicosia, Egkomi, Cyprus

Studies in the area of achievement motivation have made a distinction between mastery and performance goals. Many studies investigated the positive and negative outcomes from the adoption of each goal in students' behaviour and achievement. Recently, there is an ongoing discussion concerning the role of multiple goals in understanding students' behaviour and achievement. This paper addresses the role of mastery goals, performance goals and multiple goals, in students' mathematics achievement and motivation. Data were collected from 620 6th graders (study A, N=299 and study B, N=321). The results of both studies were consistent regarding the effect of the different multiple goals profiles on students' achievement and motivation.

Keywords: Multiple goals, motivation, achievement.

INTRODUCTION

In everyday school one can realize that some students perform better than others, tend to work harder, ask for help, are eager to participate in school activities and use more sophisticated learning strategies than other students. Researchers, mainly in the Educational Psychology domain, examine the role of motivation in the learning and teaching context in an attempt to interpret students' behavior and learning outcomes (Pintrich, 2003). Motivation has been found to play a key role in students' current and future academic success (Pintrich, 2003; Pantziara & Philippou, 2014).

In this respect a number of socio-cognitive frameworks have been developed and used in research on students' motivation in the school context (Pintrich, 2003; Elliot 1999). One of the most applicable and predominant frameworks is achievement goal theory (Elliot, 1999). Achievement goal theory focuses on goals, as the reasons and purposes for engaging in achievement tasks (Elliot, 1999). While there is a great amount of studies that examine the role of each of these goals in students' cognition, affect and behavior, as opposite constructs, few of them refer to students holding simultaneously multiple goals. Multiple goals and their interactions have been investigated mostly in secondary and college level (Mattern, 2005; Pintrich, 2000). The current study investigates the role of multiple goals and their interaction with elementary students' motivation and achievement.

BACKGROUND AND AIMS

Achievement goal theory

Over the past three decades, achievement goal theory has emerged as an important theoretical prospect on students' motivation in school settings, as it provides a satisfactory framework that emphasizes the importance of how students think about themselves, their tasks and their performance (Midgley, Kaplan, & Middleton, 2001). Rather than considering individuals' with low or high level of motivation, achievement goal theory focuses on why individuals are motivated. The theory posits that individuals engage in academic activities to accomplish different goals. Achievement goals are defined as the competence-relevant objectives that individuals attempt to achieve and these different objectives are associated with different quality of engagement in schoolwork and different cognitive, affective, and behaviour consequences (Elliot et al., 2005; Kaplan, Middleton, Urdan, & Midgely, 2002).

Within this framework, two achievement goals are described, mastery and performance goals. Mastery goals refer to an individual's objective of developing personal competence and growth (Kaplan & Maehr, 2007), while performance goals refer to demonstrating ability, focusing on attempts to create an impression often through the comparison with others

(Kaplan & Maehr, 2007). There has been noteworthy consistency over a large number of studies about the relation between mastery goal orientation and adaptive patterns of cognition, affect and behaviour. Indicatively, mastery goals were found to evoke positive processes like effort, expenditure, persistence self efficacy, self-regulated learning, positive affect and well-being (Elliot et al., 2005; Kaplan & Maehr, 2007). On the other side, research findings concerning performance goals were found to be inconsistent. A number of studies found that performance goals are associated with positive processes like positive affect, effort, persistence and graded performance (e.g., Elliot, 1999; Zusho et al., 2005). Other studies, however, reveal that performance goals are less adaptive and are related to negative affect, strategy use, and performance (Elliot & Church, 1997; Kaplan & Maehr, 2007).

In a revised goal theory perspective, researchers have distinguished between performance approach and performance avoidance goals. While performance approach goals focus on doing better than others, performance avoidance goals focus on the possibility of failure and on the attempt to avoid it (Kaplan & Maehr, 2007). Studies investigating mastery, performance approach and performance avoidance goals have consistently found that performance avoidance goals are related to maladaptive patterns of motivation, affect and performance (Pintrich, 2000).

Achievement goal theory and multiple goals

In an attempt to explain inconsistent effects of performance goals, research investigates whether students pursue these goals while at the same time pursuing mastery goals. It may be that in the classroom context, students can endorse both mastery and performance goals in different levels of each of these goals. Research so far examining mastery and performance goals, has often distinguished the effects of these two goal orientations without exploring how these two goals may jointly influence students' behaviour and performance. Qualitative studies revealed that students expressed multiple goals for engaging in school activities (Kaplan et al., 2002).

According to Pintrich (2000), there are two views as to how these two goals can be combined to give the best outcomes concerning students' behaviour and achievement. The first view, under the revised goal theory perspective, argues that having high levels of both of these goals could be the most adaptive. If there are positive effects for mastery and for performance goals, then a focus on mastery and a focus on trying to be better that classmates at the same time would result in positive consequences. The second view, under the normative goal theory, suggests that holding performance goals could have negative effects due to the distractions fostered by students' attempts to be compared with each other or to their negative judgments concerning themselves. In this perspective students' involvement fostered by mastery goals would be diminished and lead to less positive outcomes. Under this view the most adaptive pattern would be high mastery goals and low performance goals.

The effects of multiple goals have been examined using different methodologies like cluster analysis, median splits (examining individuals with different patterns of high and low goals) or by using regression analysis to investigate interactions between goal variables. Research examining the effects of different profiles of mastery and performance goals presents evidence that the low level of mastery goals in combination with the low level of performance goals is almost always associated with negative outcomes. In addition the combination of low mastery goals and high performance goals has also been shown to lead to negative outcomes, although less negative than the combination low mastery and low performance goals (Kaplan et al., 2002).

Concerning the profiles high mastery/high performance goals and high mastery/low performance goals, earlier studies have produced mixed results. Particularly, Meece and Holt (1993) investigated ele-

Intensity of inner-mathematical orientation of sense construction	Intensity of individual orientation of sense construction		
	Low	medium	high
High	1	2	3
Medium	4	5	6
Low	7	8	9

Table 1: "Typology of sense construction" (ibid, p. 133, German in original)

mentary students' (5th and 6th grade) different profiles of multiple goal orientation in science and revealed that the group of high mastery/low performance goals showed the most positive achievement profile. Pintrich and Garcia (1991) also found in their study with college students that the specific profile of multiple goals had the most adaptive profile. Similarly, Mattern (2005) found in her study within a foundational teacher education course, that students holding simultaneously high mastery/low performance goals had the highest achievement, followed by the high mastery/high performance group. In contrast Bouffard, Boisvert, Vezeau and Larouche (1995) in their study with college students, found that the group of students with high mastery/high performance goals had the highest level of motivation, cognitive strategy use, self-regulation and achievement, followed by the group with high mastery/low performance goals.

In one of the very few longitudinal studies, Pintrich (2000) examined the changes in the behaviour, affect and achievement of four groups of junior high school students (based on the combination of high and low mastery and performance goals) in their math classroom following them from the 8th to the 9th grade. The study revealed that the two groups (high mastery/high performance and high mastery/low performance group) did not differ significantly regarding changes in motivational beliefs, self-efficacy, task value and test anxiety in the 8th and later in the 9th grade. In terms of task value, the high mastery/high performance group reported higher levels of task value than did the high mastery/low performance group. Yet, both groups ended in the same level of achievement.

In addition some researchers have suggested that holding both high mastery and high performance goals could be more adaptive that holding only high mastery goals (e.g., Harackiewicz, Barron, & Elliot, 1998). They argue that students holding high levels of both goals may be able to motivate themselves to succeed in various achievement contexts. The mixed results of the investigation of multiple goals suggest that more research is needed in this direction. Specifically, more research might be needed taking into account students' age, context and task domain. In this respect, the purpose of the current study was to investigate 6th grade students' multiple goals in the mathematics classroom. Specifically, the study aimed:

— To examine the relation between the four profiles of multiple goals (high mastery/high performance, high mastery/low performance, low mastery/high performance and low mastery/low performance) and student's achievement and motivation.

— To investigate if there is consistency in the results of two studies regarding the outcomes from the adoption of different multiple goals profiles.

METHOD

Participants and instruments used

Data were collected from two studies in Cyprus. In the first study (Study A) the participants were 299 students (164 females and 135male) and in the second study (Study B) the participants were 321 students (185 females and 136 males). All students came from the 6th grade (average age, 11.5 years old).

In both studies the participants completed a questionnaire measuring their motivation in mathematics and a mathematics test assessing their performance in the concept of fractions. In both studies the data were collected in the mid of the second semester of the school year so as to allow for the evolution of certain motivational constructs and goals with the specific classroom context and the mathematics teacher.

The questionnaire measuring students' motivation was constructed for the needs of these studies and it comprised of 35 Likert-type five-point items (1-strongly disagree, 5-strongly agree). The questionnaire comprised of six subscales measuring: (a) mastery goals (e.g. It is important to me that I improve my mathematics skills this year), (b) performance approach goals (One of my goals is to show others that I'm good at my mathematics work), (c) performance avoidance goals (It's important to me that I don't look stupid in mathematics class), (d) self-efficacy (I' m certain I can figure out how to do the most difficult mathematics work), (e) interest (I am enjoying mathematics lessons very much), and (f) fear of failure (When I am tackling a challenging task, I find that I am reminded of my previous failures). The first four subscales were adopted from the Patterns of Adaptive Learning strategies (PALS) (Midgley et al., 2000). Students' fear of failure was assessed using nine items from the Herman's fear of failure scale (Thrash & Elliot, 2003). Interest was defined in terms of intrinsic motivation, the enjoyment of and interest in an activity for its own sake (Elliot & Church, 1997). Students' interest was measured using seven items from the Elliot and Church's study (1997).

The mathematics test measuring students understanding of the fraction concept in both studies comprised of fraction items from published research and they assessed students' understanding of fraction as part of a whole, as measurement, fraction equivalence, fraction comparison and fraction addition. More information about the tests can be found in Pantziara and Philippou (2012). Students' achievement was based on their total score in the fraction test; each of the tasks was graded with 0 (wrong) or 1(correct).

Data analysis

In study A we conducted exploratory factor analysis for the six motivational variables using the software SPSS and in study B we conducted a confirmatory factor analysis for the six factors using EQS software. Performance goals used in this study reflected on approach performance orientation to classroom work.

To examine the interactions between mastery and performance goals we dichotomized the two scales using median splits. For the mastery goals, students scoring below 4.6 in Study A and students scoring below 5 in Study B belonged to the low mastery group while the rest of the students to the high mastery group. For the performance goals, students scoring below 3 in both studies were classified to the low performance group and the rest of the students to the high performance group. The results of the procedure are presented in Tables 1 and 2.

Then an ANOVA analysis with multiple dependent variables was contacted in each study (A and B) to investigate for differences in groups with multiple goals concerning their motivation and achievement in mathematics.

RESULTS

A detailed description of the extraction of factors concerning the achievement goals and the motivational variables (fear of failure, self-efficacy beliefs and interest) can be found in Pantziara and Philippou (2014). Students' achievement in the fraction test was calculated regarding their total score on the tests. Specifically for Study A, the total score was 23 and for Study B the total score was 21. Tables 3 and 4 present the Mean, Standard Deviation and Cronbach's alphas for all affective variables for Study A and Study B respectively.

Low mastery N=122 (40.8%)	High mastery 177 (59.2%)
Low performance N=130 (43.5%)	High performance 169 (56.5%)

Table 1: Groups of Study A

Low mastery N=159 (49.5%)	High mastery 162 (50.5%)
Low performance N=150 (46.7%)	High performance 171 (53.3%)

Table 2: Groups of Study B

Factors for Study A	Mean (1–5)	SD	Cronbach's a
Mastery goals	4.52	.46	.71
Performance (approach) goals	3.08	.93	.80
Self-Efficacy	4.09	.62	.71
Interest	3.85	.89	.89
Fear of failure	2.20	.78	.66

Table 3: Means, Standard Deviations and Cronbach's alpha for each of the five factors-Study A

Factors for Study B	Mean (1–5)	SD	Cronbach's a
Mastery goals	4.62	.56	.68
Performance (approach) goals	3.02	1.08	.81
Self-Efficacy	4.03	.69	.66
Interest	3.84	.97	.84
Fear of failure	2.43	.80	.73

Table 4: Means, Standard Deviations and Cronbach's alpha for each of the five factors–Study B

As tables 3 and 4 present, both samples seem to have a positive view of mathematics. In both studies the mean for mastery goals is above 4.50, self efficacy above 4, interest above 3.5 and fear of failure below 2.5. Students in primary school in the specific educational setting seem to get involved in mathematics more for mastery reasons and not so for performance approach goals.

Regarding the specific objectives of the study, a one-way between groups analysis of variance was conducted to explore the impact of the different multiple goals profile on students' mathematics achievement and motivation (self-efficacy, interest and fear of failure). For study A, the results show a statistically significant difference at the $p<0.5$ level in scores for the four groups regarding achievement $F(3, 295) = 3.5$, $p=0.015$, for self-efficacy $F(3, 295) = 15.93$, $p<0.001$ and for interest $F(3, 295) = 18.36$, $p<0.001$. No statistically significant difference was found in scores for the four groups regarding fear of failure. Table 5 presents the scores for each group (mean and standard deviation) in each affective factor and achievement. Means within a row with the same subscript are significantly different from one another.

Post-hoc comparisons using the Tukey HSD test indicated that students in high mastery/low performance group had the highest achievement from all other groups. This difference was statistically significant only for the low mastery/high performance group. This group had a higher mean achievement than the group of students with high mastery goals (Mean=13.65). Students in high mastery/high performance group declared the highest interest following by the high mastery/low performance groups. These two groups had statistically significant difference from the low mastery/low performance group concerning interest. Students' in the low mastery/low performance group had the lowest self-efficacy from the students in the other three groups, and this difference was statistically significant.

For study B, the results show a statistically significant difference at the $p<0.5$ level in scores for the four groups regarding achievement $F(3, 317) = 6.35$, $p<0.001$, for self- efficacy $F(3, 317) = 19.06$, $p<0.001$, for interest $F(3, 313) = 23.32$, $p<0.001$ and for fear of failure $F(3, 311) = 10.77$, $p<0.001$. Table 6 presents the scores for each group (mean and standard deviation) in each affective factor and achievement. Means within a row with the same subscript are significantly different from one another.

Post-hoc comparisons using the Tukey HSD test indicated that students in high mastery/low performance group had the highest achievement from the students in the other three groups even though the difference

Dependent Variables	High mastery/low performance		High mastery/high performance		Low mastery/high performance		Low mastery/Low performance	
	M	SD	M	SD	M	SD	M	SD
Achievement	14.32^{a}	4.11	13.17	4.61	12.03^{a}	4.12	12.70	4.17
Interest	4.06^{b}	0.75	4.20^{c}	0.82	3.52	0.94	$3.35^{b, c}$	0.80
Self efficacy	4.19^{e}	0.54	4.33^{d}	0.56	3.90^{f}	0.62	$3.74^{d, e, f}$	0.60
Fear of failure	3.21	0.85	3.22	0.88	3.29	0.71	3.37	0.89

a= p<0. 05, b=p<0.001, c= p<0.001, d= p<0.001, e= p<0.001, f= p<0.05

Table 5: Scores for each group (mean, standard deviation)-Study A

Dependent Variables	High mastery/low performance		High mastery/high performance		Low mastery/high performance		Low mastery/Low performance	
	M	SD	M	SD	M	SD	M	SD
Achievement	12.88^{a}	4.20	11.64	3.90	10.14^{a}	4.00	11.39	3.81
Interest	$4.00^{d, e}$	0.92	$4.33^{b, c}$	0.69	$3.63^{c, e}$	0.94	$3.33^{b, d}$	1.02
Self efficacy	$4.06^{f, g}$	0.68	4.47^{f}	0.45	3.84^{f}	0.67	$3.70^{f, g}$	0.70
Fear of failure	$2.06^{h, i}$	0.63	2.33^{j}	0.89	$2.69^{i, j}$	0.75	2.63^{h}	0.75

a= p<0.001, b=p<0.001, c= p<0.001, d= p<0.001, e= p<0.05, f= p<0.001, g= p<0.05, h= p<0.001, i= p<0.001, j= p<0.05

Table 6: Scores for each group (mean, standard deviation)-Study B

in students' achievement was statistically significant only from the group of low mastery/high performance goals. This group had a higher achievement than the group of students with high mastery goals (Mean=12.21). Students in high mastery/high performance group had the highest interest from all the students in the other three groups and the difference in scores was statistically significant for students in low mastery/high performance and low mastery/low performance group. Concerning self-efficacy, students in low mastery/low performance group had the lowest self-efficacy and this difference was statistically significant from the groups with the highest self-efficacy (high mastery/high performance, high mastery/low performance). As fear of failure is concerned, students in low mastery/high performance group had the highest fear of failure and this difference was statistically significant from the groups with the lowest fear of failure (high mastery/low performance, high mastery/high performance).

With respect to the second aim of the study, as it can be seen from tables 5 and 6 the two studies show remarkable consistency in the characteristics of the specific four groups regarding their motivation and achievement. Specifically, and concerning the groups with the most positive outcomes, the group with the highest achievement is the high mastery/low performance goals even though the mean difference was not statistically significant from the group high mastery/high performance goals. The group high mastery/high performance goals had the highest interest from all the other groups in both studies. Again the mean difference was not statistically significant for the group high mastery/low performance goals. The same group had the highest self-efficacy in both studies from all the other groups. In contrast, the group with the most negative outcomes related to achievement is the low mastery/high performance group and the group with the most negative outcomes related to motivation (self-efficacy and interest) in both studies is the group low mastery/low performance goals.

DISCUSSION

The aim of this study was to investigate the role of multiple goals and their interaction with elementary students' motivation and achievement. A general conclusion of the study is that both views - the one under the revised goal theory perspective and the normative goal theory - concerning multiple goals, are found in this study to be applicable to the development of elementary students' achievement and motivation. Specifically, in line with the first view that holding high levels of both, mastery and performance goals could be the most adaptive, in both studies (Study A and Study B), students in the group of high mastery/high performance goals had the highest interest and the highest level of self-efficacy beliefs. These findings are in line with the results of Bouffard and colleagues (1995) study with college students, who found that this group of students had the highest level of motivation.

In line with the normative goal theory, that the most adaptive pattern would be high mastery goals and low performance goals, results of both studies (Study A and Study B) indicated that students in the group with high mastery/low performance goals had the highest achievement from all other groups even though the mean differences from the group high mastery/high performance group was not statistically significant, a result similar to Pintrich's study (2000). An important finding is that in both studies (Study A and Study B) this group had higher achievement that the groups of students with single high mastery goals. From these findings it can be concluded that performance goals when combined with mastery goals does not diminish the positive effect of mastery goals. Students who are concerned at the same time about their mastery and about their performance in comparison with others seem to have an adaptive pattern of achievement and motivation.

In line with the normative goal theory and parallel to other studies (Pintrich 2000; Kaplan et al., 2002) both studies (Study A, Study B) showed that the groups low in mastery goals both in combination with low or high level of performance goals are associated with negative results. Students in both of these groups reported low levels of achievement, interest, self efficacy and the highest levels of fear of failure.

The results of our studies may lead to the conclusion that mastery goals lead to the most adaptive patterns. Even though the results could inform teachers to work for the development of mastery goals in the mathematics classroom, the combination of high mastery goals with high performance approach goals may also lead to adaptive patterns regarding students' achievement and motivation. It was found that performance approach goals alone usually do not have a positive effect on students' interest (Zusho et al., 2005), a key factor in students' life-long learning. In this study's results, the group with high mastery/high performance declared the highest interest.

In conclusion, we suggest that more research is needed in the domain of multiple goals in relation to students' age, different learning contexts, and different measures of students' achievement.

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Affective transgression in learning mathematics

Barbara Pieronkiewicz

Pedagogical University of Cracow, Institute of Mathematics, Cracow, Polan, basiapieronkiewicz@o2.pl

Mathematics as a school subject raises a lot of negative emotions within students. It is seen as a difficult, detached from reality, full of useless in everyday life definitions and theorems field of knowledge. Not surprisingly, it causes a lot of anxiety, emotional tension and internal discord. Nowadays, many students declare their humanities preferences in purpose to justify the lack of involvement in learning mathematics. This article looks at this state of affairs through the lens of psychological concept of transgression. The first and foremost cornerstone of this paper is the notion of "affective transgression". It is introduced after a brief exposition of transgressive concept of a man. This theoretical approach sheds some new light on teaching and learning mathematics.

Keywords: Transgressive concept of a man, affective transgression in learning mathematics, beliefs, meta – affect.

INTRODUCTION

A lot of attention has been paid to students' achievement in mathematics so far. Although researchers and teachers' efforts seek to improve the quality and efficiency of mathematics education, there are still many students who – for several reasons – achieve low scores in this school subject. What is more unsettling, is that many students declare humanities preferences in order to explain and legitimize their lack of engagement for learning mathematics and even among those who reveal their potential, there might be a significant number of underachievers (Rimm, 2008).There is a general agreement amongst mathematics educators that, regarding students' achievement, neither affect nor cognition should be underestimated (e.g., McLeod, 1992; Vinner, 2013). In that sense, many researchers have been examining how affect influences the field of mathematics education (e.g., DeBellis & Goldin, 1997, 2006; Schlöglmann, 2005). Fortus (2014) goes even further, stating that all science educators ought to pay more attention to affect. He claims there is a great urgency to lift students from boredom and indifference, because "without engagement, learning is partial, at best".

It is thus, of fundamental importance to focus research on understanding the role that students' beliefs about the egalitarian nature of mathematics and the importance of mathematics education in their lives, in the process of learning mathematics. Considering the transgressive concept of a man within mathematics education, brings new ideas that address these issues. The last part of this paper, attempts to compress (Thurston, 1990) efforts and considerations made by affect researchers so far, into a thinkable concept (Tall, 2004) of *affective transgression*.

MATHEMATICS FOR ALL OR FOR MATHEMATICALLY GIFTED

Many students hold the belief that to be good at mathematics, one needs to have some innate predispositions to grasp it, and that ordinary students cannot be expected to understand this school subject (Schoenfeld, 1992). In contrast, some psychologists or educators try to convince the broad audience to the egalitarian nature of mathematics. Already in 1973, Piaget mentioned that fact:

> Any normal student is capable of correct mathematical reasoning, if attention is directed to activities of his interest, and if by this method the *emotional inhibitions*, that too often give him a feeling of inferiority in lessons in this area *are removed*. (...) There is no field where the "full development of the human personality" and the mastery of the tools of logic and reason which insure full intellectual independence are more capable of realization. (Piaget, 1973, pp. 98–99, 105)

Aligned with those thoughts, also Krygowska referred to the nature of mathematics:

> There are different levels of mathematical activity and with the exception of extreme cases, we can find an appropriate level of activity to any normal student. (...) When it comes to the development of mathematical thinking, we *must not write any student off*. (...) The student has to *take a fancy to mathematics*, *find pleasure* in solving mathematical tasks, even though it requires effort and difficult concentration. (Krygowska, 1975, p. 243)

Although these statements were given in purpose to emphasize that mathematical skills can be developed by any student, they also refer to affect and explicitly ascertain that it impinges on learning mathematics. It is reasonable to expect that students who are afraid to expose their emotional or intellectual shortcomings will tend to avoid doing mathematics (and consider that they are not able to learn it). Many students tend to have a distracting behaviour in classrooms in order to try to deviate their colleagues and teacher attention from their own mathematics difficulties. As part of my research, I have interviewed a high school student having low scores in mathematics. According to the information given by his teacher, the boy usually had side conversations, made sniping remarks and was playing around on every lesson. In an individual talk we had, the student confessed that he was showing off to get the rest of the class laughing. The real reason why he was doing that, was that he wanted to divert his colleagues' attention from his mathematical misunderstanding. He said that it seemed easier to hold on when they were all laughing with him, under his control, on the jokes he made, than to hear the laughter on his failure. Thus if we want to convince young people, especially reluctant and disobedient low-achievers, that they may become 'good at math', we need to take into account students' vulnerability to affective stimulation coming from mathematics. Among many features this discipline has, one of the most remarkable is that mathematics, like no other field of knowledge, evokes human's affective responsiveness. For example, in the scope of the theoretical review one can find many references to math anxiety, and only very few references to anxiety outside mathematics (e.g., "chemistry anxiety" or "biology anxiety"). Nobody likes to experience negative frustration, fear or helplessness, being, in this context, the teachers' responsibility to develop their work allowing students to overcome such experiences. That's why some students prefer to avoid mathematics, rather than take the challenge and confront the constraints. Paradoxically, all of these negative experiences related to mathematics, could be seen as a positive phenomenon. Intentional reversing negative affective patterns (Rimm, 2008) may be recognized as much more than just about the achievement in mathematics, being linked with the reinforcement of a widely understood personal development and fulfilled life.

(GOOD) REASONS FOR LEARNING MATHEMATICS

In light of the above remarks, it is no surprise that one of the most frequently encountered questions that student pose to their teachers is "Why do we have to learn mathematics?". Every teacher should have the answer well considered in advance. This problem opens up a long list of questions collected by Posamentier and colleagues (2013), who recognize this question as a symptom indicating that students do not appreciate mathematics. They deem convincing answers like, for instance, mathematics is useful in everyday life, it provides a wide range of career opportunities. However, the most important reason they give, refers to the remarkable nature of the discipline:

> Mathematics is a huge, logically and deductively organized system of thought, created by countless individuals in a continuous collective effort that has lasted for several thousand years and still continues at breathtaking pace. As such, mathematics is the most significant cultural achievement of humankind. It should be a natural and essential part of everyone's general education. (Posamentier et al., 2013, p. 3)

The reasons one can present for learning mathematics are wide different and can inclusively be controversial. Some of the possible reasons could be by the fact that mathematics is "beautiful", and it's worth learning this subject for its own sake (e.g., Davis, 1993; Lockhart, 2009). From another point of view, learning mathematics can be seen as a stepping-stone to further education at all levels of the academic studies (e.g., Vinner, 2013). Some authors emphasize the assumption that mathematics trains the mind, and provides universal mental tools that enable us to reason correctly (e.g. Dudley, 2011). Others pay attention to the role that mathematics plays in our daily life or in STEM-related professions (e.g., Fortus, 2014). Several publications consider mathematics as a source of social empowerment, a central element of culture, art and life, and the driving force for the development of civilizations (e.g., Ernest, 2010). Some of the reasons given above are controversial. For example, there is no consensus neither within the case of usefulness of mathematics in everyday life (e.g., Lockhart, 2009; Wu, 1997) nor the demand for mathematical skills in the workplace (e.g., Dudley, 2011; Vinner, 2013). As part of my research, involving high school mathematics teachers, participants were asked for reasons for learning mathematics. Most of the given answers, akin to taglines, were related to mathematics usefulness and importance for daily situations (e.g. counting money, shopping) and shaping logical reasoning. But there were also answers as:

> I tell my students they should learn mathematics today, so they could teach others mathematics in the future.
>
> I tell my students, there is no escape from mathematics! Mathematics is everywhere!
>
> Look at lawyers for example. Do you think they could do the same work as a simple worker does? [The answer was "Yes"]. And now, look at the worker. Do you think that he could do the same things as the lawyer does? ["No."] Hence, you see, it's worth learning more because it gives you a wider range of possibilities.

What are possible responses a student could give to refute these arguments? They are easy to figure out: "I'm not going to teach mathematics to anybody", "Oh, it sounds scaring! Though you say there is no escape from math, ... I will try the best I can". A friend of mine told me an authentic story about a woman who was trying to encourage her son to doing mathematics, so he could have better life opportunities than she ever had. The boy said: "There is no sense in learning. Dad is a scavenger, you are a cleaner. What future do I have? I'm sure I will be doing exactly the same".

However, there is one undisputed answer, that no student could ever debunk. The transgressive concept of a man, discussed further, provides a new reason justifying the value of learning mathematics. Moreover, the concept yields a new perspective on students as both learners and humans concerned about their growth. Finally, the idea of *affective transgression in learning mathematics* emerging from this psychological concept, may be successfully implemented in school practice and result in improving "weak" students' achievement.

TRANSGRESSIVE CONCEPT OF A MAN

The term *transgression* is defined in different contexts (e.g. geology and genetics). In geology *transgression* is the spreading of the sea over land as evidenced by the deposition of marine strata over terrestrial strata; in genetics it means a peculiar case of heterosis - the increase in growth, size, fecundity, function or other characters in hybrids over those of the parents. In its transposition into psychological ground, Kozielecki (1987) uses terms of an intentional and deliberate overcoming of physical, social or symbolic boundaries. *The concept of psychological transgression* is devoted to the importance of the role that crossing over personal boundaries and subverting limitations play in everyone's life. From this standpoint, a man is a self-directed, expansive creature who intentionally crosses the *boundaries* understood as demarcation lines separating what he is and what he owns, from what he may become.

Kozielecki (1987, 1997) has outlined four worlds of transgression wherein the exceeding boundaries can be taken towards: 1) material objects - territorial expansion in the physical world, 2) other people - expanding the control over other people but also altruism and extension of individual freedom, 3) symbols - intellectual expansion; going beyond the information given, development of knowledge about the world and 4) oneself - self-creation, self- development, unlocking one's potential, coping with one's weaknesses. In that sense, transgressions may be of different kinds: psychological or historical, individual or collective, constructive or destructive, but also, in other level, it can be creative or inventive and expansions (e.g., material, interpersonal, intellectual).

The human being is assumed to be able to carry both the *telic* (goal-oriented) and *autotelic* (intrinsically rewarded) actions. In the former, he acts in pursuit of a variety of goals and creates new values that satisfy his needs. In the latter kind of actions, the goal is less important than the satisfaction and pleasure simply coming from carrying out activities. Moreover, regarding autotelic actions indicated by high level of involvement, Kozielecki states that *goals emerge from activity* not conversely, because goals in this case have no distinguished status. The author notices that goal-oriented activities become exhausting and boring in a short time, hence when the motivational tension relieves and the goal is achieved, a person ends up the task and refuses further actions. On the contrary, those who are totally committed to any kind of activity, do not feel tired. They forget about the lapse of time and even experience the state of *flow* (Csikszentmihalyi, 1991). Kozielecki (1997) emphasizes that thanks to the commitment we become more self-governing instead of being just human – robots.

However, from the viewpoint of transgressionism, another distinction is of higher importance. Kozielecki (1987, 1997) focuses on two kinds of actions that entities undertake: *protective* - designed for the maintenance of the *status quo* and *transgressive* ones – exceeding the boundaries and enabling the development of personality. The juxtaposition of these two types of human activity is presented in the table below.

Kozielecki puts forward the view that personality is equipped with a kind of internal *comparator* (a part of human's will), which allows comparing plans with achieved state of affairs. It is also the comparator that decides whether to stop the action or continue. The salient feature of the protective actions is that they are directed by the *principle of negative feedback* – reaching the goal (namely restoring or maintaining the status quo) ends up the activity taken by a man. On the contrary, transgression is directed by the *principle of positive feedback* which works reversely: not only isn't the motivation reduced, but also it is sustained or even increases during the activity. The notion of *affective reallocation* is introduced to name the positive correlation between adaptation and negative emotions on the one hand, and between transgression and positive affective experiences on the other. *Hope* may serve as a good example of such a positive experience. It is defined as a multidimensional cognitive structure, in which the central factor is the belief that in the future one will be offered the good (achieve an important objective), and the degree of certainty, or probability, is stated (Kozielecki, 2006).

These two kinds of behavior exposed briefly above, differ also in terms of the motivation involved. Two kinds of human's motivation are distinguished by Kozielecki (1987): *homeostatic* – a typical motivation for protective actions, (however, sometimes transgressions could also be stimulated by this kind of motivation) and *heterostatic* – a specific motivation for transgressive actions. The former arises if and only if in human's brain there are two independent information at one time: one concerning the desired state of affairs (S) and the second one, involving the actual state (A). When the comparator ascertains the existence of discrepancy D(S,A), the organism engages in behaviors designed to reduce the psychological imbalance. To get back to homeostasis, considered as a preferable state, a man undertakes actions intended either to dismantle the deficits or to remove the excess. This process leads to satisfaction and relief.

Protective actions	**Transgressive actions**
play key role in adaptation and survival	satisfy higher needs of a human being
regulated by the needs of deficit	regulated by the needs of growth
undertaken to maintain the status-quo	orientated toward a meaningful change
other-directed; depend more on the changing external environment	inner-directed; depend on the components of personality, for instance, creativity, knowledge, motivation, courage, perseverance
necessary	Possible
"I know I have to"	"I know I am able to"
repeatable	non-recurring
planned	Spontaneous
often predictable	harder to predict
accompanied by negative emotions, especially fear	accompanied by positive affective experiences, especially hope
performed similarly to following an algorithm	inherently heuristic, fallible, underspecified

Table 1: Protective actions vs. transgressive actions

Homeostatic theory focuses on the maintenance of the internal physiological environment. However this theory doesn't describe all human's behaviors adequately. For example: it is not sufficient to explain why people sometimes explore their environment and intentionally seek for arousal disrupting the equilibrium. What underlies human's motivation in this case is the driving force of growth. The discrepancy D(L,A) between the level of aspiration (L) and the actual state (A) raises internal tension that leads to actions oriented on growth and satisfaction. However, a man seen as an insatiable creature can never reduce D(L,A) completely. This discrepancy exists permanently. The role of the comparator is then twofold: it detects the existence of D(S,A) and evaluates the extent and the content of persisting D(L,A). Overall, then, we can state that homeostatic motivation serves to minimize annoyance, whereas the role of heterostatic motivation is to maximize the pleasure.

There are two specific types of the heterostatic motivation that lead to transgression. The first one, which has been coined by Kozielecki, is the *hubristic motivation*, "conceived as a cluster of motives that make people assert and enhance their self-worth" (Kozielecki, 1987, p. 177). It is the major driving force of transgression. *Hubris* (also: *hybris*) is a term derived from Greek literature and philosophy. In the past it meant pride, insolence and arrogance, but here it is deprived of pejorative meanings. Transgressive concept of man takes into account that every human being has the desire, at some point, to be distinct from others, to be important, to shine the spotlight on others. The hubristic motivation manifests itself as striving for superiority or striving towards perfection. It is insatiable, very affective, sated with egocentric and hedonistic drives.

The second type of the driving force specific for transgression, is *cognitive motivation*. It is nonegoistic, instinct to master and competence, governed by the principle of growth. It can be stimulated by the novelty or complexity of the subject, uncertainty or lack of information, as well as by the cognitive conflict raised when two or more contradictory beliefs, ideas, or values are held at the same time, or when existing beliefs etc. are confronted by some new directly contradicting information.

AFFECTIVE TRANSGRESSION IN LEARNING MATHEMATICS

In this section the focus is on providing a clear link between transgression and mathematics education.

A wide range of affect literature from around the world provides a considerable number of research reports exploring, in its depth and breadth, affective conditions of learning mathematics. The common reason why all these efforts are taken, is the more or less implicit assumption that identifying obstacles for effective learning, will contribute to a meaningful change in the quality and effectiveness of teaching. Paraphrasing Thurston, we can say that mathematics education is amazingly compressible, and "once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression" (Thurston, 1990). Attempting to compress previous considerations made within the affect domain, into a thinkable concept (Tall, 2004), I herby introduce the notion of *affective transgression in learning mathematics* (short: affective transgression).

By *affective transgression* I mean an intentional process of overcoming personal affective barriers that preclude one's mathematical growth and development. The process is a psychological, individual and constructive transgression toward oneself. It is, by definition, highly recommended for low- and underachievers. Affective transgression might occur if and only if a person a) has insight into emotions (s)he experiences, b) is aware of the belief systems (s)he holds and c) has the will to make changes, believing they are good and possible. Meta-affect, considered to be the most important component of affect (DeBellis & Goldin, 2006), is inevitably required here. It should be developed to make all emotional experiences productive for learning and accomplishment.

Learning mathematics, seen through the lens of transgressive concept of a man, becomes an activity leading to inner growth and personal development. This argument seems to be a good and irrefutable reason that provides powerful meaning to the learning of mathematics. From the concept of hubristic motivation, we can deduce that even if students reject mathematics (as a school subject or as a domain where they are expected to be active), they will never write themselves off. Hence, until students (especially low- and underachievers) don't see doing mathematics as an autotelic activity enhancing their self-worth, they might refuse commitment. What is of the utmost importance, transgression accompanied by positive affective experiences (i.e. hope, faith), has a great impact on the language being used to describe students' relationship with mathematics. Every *problem* becomes a *challenge* now, *impossible* turns out to be *achievable*, or *not easy to achieve* and *the harder to achieve, the more wanted*. Protective actions that students take (Vinner, 1997), are replaced by transgressive ones, they are encouraged to. Thanks to this approach, external goals (like passing exam or having good grades) are eclipsed by the pleasure and satisfaction coming from student's endeavour.

Teachers have to meet two general prerequisites to make the affective transgression possible:

1) establishing growth-promoting climate in the classroom – which requires genuineness, unconditional positive regard and empathetic understanding. What counts most in effective treatment (Rogers, 1995) seems to be not a particular technique, but the personal relationship between teacher and his student,

2) transgression – oriented teaching, comprising both (meta)cognition and (meta)affect (DeBellis & Goldin, 2006).

Meta-belief systems activity (Moscucci, 2007) and *diagnostic teaching* (Schoenfeld, 2011) enriched with affect and meta-affect, may serve as transparent examples of practices that might evoke student's affective transgression.

As belief systems constitute the *weak* element of affect structure (Moscucci, 2007), they need to be identified first. This may come through observation or individual interviews. There are also scales, constructed and validated in purpose to measure beliefs (i.e., Kloosterman & Stage, 1992). It is the role of a teacher to recognize students' beliefs and bring them to their attention. If beliefs turn out to be a hindrance for one's growth and achievement in mathematics, they need to be restructured. Counterexamples as well as class or small group discussions of beliefs can be effective for reflecting on negative beliefs (Kloosterman & Stage, 1992). This is already a stepping-stone towards overcoming them.

FINAL REMARKS

This paper has offered an explanation of what transgressive concept of a man is. It is my attempt to bring closer and describe the phenomenon of *affective transgression*, I have observed many times in my teaching practice. The notion of *affective transgression*, introduced in this paper, is hoped for to be powerful inspiration for researchers in the affect domain.

Given the paper length restrictions, I shall conclude by few brief personal comments. First of all, it is very hard to reach out to low- or underachievers. Whoever remains just a mere teacher instead of being a whole person, will surely fail. And the second remark (last but not least) is that *teaching to transgress* stands in relation to *ordinary teaching mathematics*, much as *giving a fishing rod* stands in relation to *giving a fish*. A student who has experienced his personal "mathematical transgression", will never stop hungering for more. Just like once turned into a beautiful swan, the ugly duckling felt neither ugly, nor the duckling.

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Teachers' individual beliefs about the roles of visualization in classroom

Angela Schmitz[1] and Andreas Eichler[2]

1 University of Education Freiburg, Freiburg, Germany, angela.schmitz@ph-freiburg.de

2 University of Kassel, Kassel, Germany, eichler@mathematik.uni-kassel.de

This paper investigates mathematics teachers' beliefs regarding the roles of visualization for teaching fractions and algebra. The paper discusses the construct of beliefs and theoretical roles of visualization. We further give rationale for the qualitative approach of our study. In the results we compare two teachers' beliefs from a larger sample. We work out that, although both teachers use visual representations in a similar way, they assign different roles to their respective use. Their beliefs appear to be stable across subdomains. We discuss our impression that the teachers emphasize other roles of visualization than educational research implies.

Keywords: Teachers' beliefs, visualization, domain-specific, fractions, algebra.

INTRODUCTION

The use of visualization is regarded as important for the teaching and learning of mathematics. Possible *roles* of visualization are for example to promote students' understanding (Presmeg, 2006; Duval, 2006), to enable problem solving (Heinze, Star, & Verschaffel, 2009; Duval, 2014; Arcavi, 2003), to discover, to describe, and to explain (Rivera, Steinbring, & Arcavi, 2014).

The demands on teachers to appropriately teach using visualization are high, e.g. to use visual representations in a conscious and active way (cf. David & Tomaz, 2012; Presmeg, 2014). Accordingly, nowadays it is emphasized to incorporate strategies on how to use visual representation into teachers' education (e.g., Presmeg, 2014). Thus, on the one side, we gained a lot of knowledge about a theoretical framework of visualization and a framework for teaching with visualization. On the other side, teachers seem to have complex attitudes regarding visualization (e.g., Gómez-Chacón, 2015). Yet we gained scarce results regarding the question of how mathematics teachers define the *role of visualization* for their classroom practice, or why they define the role of visualization in an individual way. However the teachers' thinking or rather their beliefs are understood to crucially impact on both these teachers' classroom practice (Calderhead, 1996) and their willingness to receive new information about teaching and learning with visualization (Chapman, 1999).

For this reason, our study aims to investigate teachers' beliefs about visualization in more detail. We seek to identify individual perspectives and reasons as to how and why teachers do or do not use visualization in classroom and which roles the teachers assign to visualization. In this report we try to give first answers to the following research questions, by means of a comparison of two teachers' beliefs:

(1) What are secondary teachers' beliefs about the roles of visualization in fractions and algebra? (2) To what extent do these roles differ in specific domains?

For this, we first outline our theoretical framework, including definitions of the constructs of beliefs and visualization and including an overview of the roles of visualization. Afterwards we discuss our method. The main part of this paper refers to results relating to the beliefs of two mathematics teachers from a larger sample. These teachers point out their understandings of visualization and their beliefs regarding the role of visualization. The results represent an empirical counterpart to our theoretical knowledge about teaching with visualization.

THEORETICAL FRAMEWORK

Definition of beliefs

In order to answer our research questions the multifaceted term "beliefs" has to be clarified. We chose a description given by Philipp (2007) who combines many similarities of existing definitions in the context of teaching and learning:

> Beliefs - psychologically held understandings, premises, or propositions about the world that are thought to be true. Beliefs are more cognitive, are felt less intensely, and are harder to change than attitudes. Beliefs might be thought of as lenses that affect one's view of some aspect of the world or as dispositions toward action. Beliefs, unlike knowledge, may be held with varying degrees of conviction and are not consensual. Beliefs are more cognitive than emotions and attitudes. (p. 259)

For this study, several aspects are important:

— As individual judgments about the validity of statements beliefs differ from knowledge, but they contain a cognitive component (Baumert & Kunter, 2006).

— Beliefs can be understood as lenses in the way that they form a specific world view. They are assigned to influence perception and action (Philipp, 2007).

— Being gradual incorporates that a belief regarding a specific role of visualization can individually be more or less important for a person (cf. the term of central beliefs; Thompson, 1992).

— Beliefs are classified as rather stable (cf. also Hannula, 2012) and refer to specific content (Eichler & Erens, 2015). Regarding visualization, we look for the specificity of beliefs referring to a mathematical domain, but also for similarities if different domains are regarded.

Calderhead (1996, p. 719) identifies "five main areas in which teachers have been found hold significant beliefs". Of these, "beliefs about learners and learning" as "assumptions teachers make about their students and how their students learn", "beliefs about teaching" as "beliefs about the nature and purposes of teaching", and "beliefs about the subject" as "epistemological issues - what the subject is about" are of interest when investigating beliefs about visualization (ibid.).

Definition of visualization

This study uses a broad definition given by Arcavi (2003, p. 217):

> Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings.

The definition integrates many facets of visualization. The following are especially relevant for this study:

— It integrates process and product. In everyday language, which is likely to be used by teachers, both terms are often used synonymously.

— It integrates different ways of how to handle visualization, especially the meaning of making something visual for oneself (often: to visualize) as well as to make something visual for somebody else (often: to use a visual representation). Both aspects are important in classroom.

— It mentions diverse kinds of pictures. Thus, everything not being completely symbolic could be considered as visualization.

— It defines visualization as goal-oriented and enumerates purposes like advancing understanding. Yet the purposes are not considered to be exhaustive.

Roles of visualization in theory

The purposes of visualization lead to the different roles of visualization: Quoting different researchers in the introduction, we mentioned a list of possible roles of visualization, i.e. promoting understanding, enabling problem solving, facilitating discovery and explaining. Partially overlapping, Arcavi (2003) enumerates in his definition the purposes "depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings". From an epistemic point of view the roles which Giaquinto (2008) mentions are also specific for mathematics: proving, discovering, heuristic aid, augmenting understanding, and enabling calculation.

A comparison of the different definitions of visualization yields that it might be common to these roles that they could be grouped in (1) understanding, explaining (2) discovery, problem solving, heuristic aid, and developing ideas, (3) describing, depicting, and communicating information, (4) enabling calculation, (5) proving.

Empirical studies on teachers' beliefs regarding the role of visualization

However, there are few studies about teachers' beliefs regarding the *role of visualization*. For this report we chose three studies. Stylianou & Silver (2004, p. 353) report in a study about problem solving that "experts and novices perceive visual representation use as a viable strategy. However, the two groups judge visual representations likely to be useful with different sets of problems."

Biza, Nardi, & Theodossios (2009) investigate how teachers use visual arguments in proving in the domain of functions. Teachers who at first glance appreciate visual arguments reveal differences in the acceptance in follow-up interviews.

Research results from text-picture-integration suggest that teachers for different domains have different views on the use of visualization (McElvany et al., 2012).

These studies suggest that the role of visualization from a teachers' point of view might be related to a specific content. They also support the idea of a qualitative approach, as all of them show that it is important to investigate the teachers' thinking in detail.

METHOD

The research questions are part of a qualitative study about how teachers want to use visualization in classroom. The study is carried out with twelve secondary teachers. Further, the study covers the domains fractions, algebra, functions, and calculus, since these domains cover secondary mathematics of all ages, and visualization can be used in different ways, e.g., as a typical means of illustration (like the pizza in fractions), as a mathematical object (e.g. the graph of a function), or with a rather structural focus (e.g., drawings in algebra). The teachers are chosen by a "theoretical sampling" (Glaser & Strauss, 1967) which aims at contrasting cases (e.g., regarding type of school, age).

In semi-structured interviews lasting three-hours the teachers talk about how and why they use visualization in classroom. The interviews address the same questions for each domain, e.g. regarding the teachers' use of visual representations, their aims, their perspective on students' learning and their use of technology. It is completed by general questions about mathematics and mathematics teaching.

The interview transcripts are analyzed according to Grounded Theory (Glaser & Strauss, 1967). In open and axial coding we developed categories concerning the function and the relevance of visualization for the teachers. The objective is to generate theoretical concepts. In part, the concept aims at a consistent perspective on teachers' beliefs regarding the role of visualization in classroom. The roles might be different from existing theoretical frameworks.

The study does not investigate how teachers act in classroom. To explain teachers' practices, many other factors, e.g. pedagogical orientation and context (Philipp, 2007, p. 275), are to be considered. However, the study is based on the assumption that teachers' beliefs impact their classroom practice.

RESULTS

In this paper, we compare the roles of visualization in classroom for two mathematics teachers. Alan is a teacher at a comprehensive school with about ten years of teaching experience. Claire has been teaching at an upper secondary school for about twenty years. We develop the roles of visualization regarding fractions and algebra (quest. 1), and we analyze in how far their beliefs regarding these roles are domain-specific (quest. 2). First, we analyze Alan. The part regarding fractions is rather detailed in order to provide also an insight into the development of the codes. According to grounded theory the resulting hypotheses influence the analysis of Alan's statements about algebra. Then we analyze Claire in the same way, but more briefly.

Alan: Fractions

For Alan it is very important that his students first develop a conception of a fraction as a "part of the whole". From his point of view it is paramount that they can relate fractions with images, see for instance the following quotation:

> Alan: There [in fractional arithmetic] it is very important that, at the beginning, students have an idea of what a fraction is in the first place. [...] And this is something which they always have to keep in mind. [...] They really have to have this image in their heads.

He stresses that students develop his intended conception of fractions if they can use many different geometric forms. Also as regarding calculation in fractions, Alan considers visualization helpful. Expressions like "see", "have pictures of something", and "understand" are used synonymously, as he did already for the conception of a fraction. Alan believes that *visualization supports students' understanding*:

> Alan: For many, this [addition of fractions with a common denominator] is not clear at first. If you just draw it, they see it immediately.

At the same time, visualization is for him a methodical aid for *explanation.* Alan appreciates good reasons. Visual means can support him with this:

> Alan: Visualization also helps with improper fractions and mixed numbers [...] This you can somehow also do well with pieces of pizza [...].
>
> Alan: Reducing and expanding with rectangles, [...] this can be done perfectly with horizontal and vertical lines, like here. And then, once they have figured it out, I move on quickly to the purely algebraic.

In addition, the preceding quotation, and also the following, highlights his objective of withdrawing from visualization and his orientation towards algebraic objectives, for example the mastering of calculation techniques:

> Alan: You see, when calculating with fractions, we do not always think of anything graphic, either. And that is where I want to guide the students, too, of course.

At first glance, a contradiction seems to arise: Alan regards it as very important that his students have a visual image of a fraction. He also uses visualization for explaining calculation. However, at the end, for Alan calculation is a symbolic manipulation. Taking into account the above quotes, for Alan it might be sufficient to show the deduction of constructs or methods with the aid of visualization. This hypothesis *visualization supports justification* is supported by the following quote:

> Alan: In this case, it is really a lot easier to have remembered that you divide by a fraction by multiplying by the reciprocal, and so on. Nevertheless, it is important for them, of course, to have first understood the principle why this is so. And this just works well with easy visualizations. But then they quickly have to get past this.

Alan expresses the value of using visualization when introducing a subject as a basis for the following:

> Alan: That is why this introduction phase with the drawings always takes quite a long time. As far as this is concerned, I'm of the opinion that this is the basis for everything else which comes later.

But he only gives justification for an operation if he thinks that this is feasible with drawings that are easy to use (cf. reducing and expanding fractions). On the other hand, when he does not know an easy visualization he does not give a justification:

> Alan: This [division by a fraction] is difficult, because this is about how often one fraction fits into another fraction. You can visualize that wonderfully if the result is a natural number. But if a fraction fits four seventh times into another fraction that becomes difficult to visualize.

Concluding this paragraph we can say that Alan values visualization for *understanding*, for *explaining* and as *justification*. He only uses drawings which he finds easy to understand. Mathematics seems to have a deductive character for him: New things have to be logically derived from the well-known. As soon as they are justified they can be used without knowing the reason or the visualization.

Alan: Algebra

Alan's beliefs regarding visualization regarding algebra are very close to his beliefs expressed regarding fractions. For example, for solving linear equations Alan uses a drawn beam balance to support students' *understanding*:

> Alan: None of them know a beam balance any more, but intuitively they grasp it immediately.

The aim of using visualization is that the students get a plausible reason for the later procedure (*justification*):

> Alan: This will enable them to see what the effect of what we are illustrating is in algebra. And this is intended as a first explanation as to why or how equivalence transformations work.

He further consistently expresses the belief that for him the deduction is very important. If a plausible deduction is easier with other means than visualization, he uses other means. Visualization is not automatically related to deduction, but when he uses it, then often for deduction.

Interim summary 1

Alan regards the role of visualization as an aid to *understanding*, *explaining,* and *justifying* (quest. 1). These beliefs can be identified in two domains. This could be an indication for a belief that is similar in different (mathematical) domains (quest. 2).

Claire: Fractions

We compare the results regarding Alan with Claire's beliefs. She also uses visualization intensively when introducing a subject. Afterwards her aim is to establish schematic calculation. We would like to know if she assigns the same roles to visualization. Claire, too, develops the conception of fractions using many drawings:

> Claire: That is very important, so that they just get this idea of a fraction. Divide something in five equal parts, take three of these parts. That you also just visualize it.

Similarly she uses visualization for the illustration of most calculation operations. But she is not as convinced as Alan that visualization is helpful for understanding:

> Claire: Four fifth divided by three, however [...] How does it suddenly get into the denominator? Thus, I again made a drawing. [...] But this is one which is difficult to draw. I doubt that this really helps the students. I rather did it because of the requirement that I once learned that you always have to visualize, and how important this is.

She rather uses visualization because she learned it in that way. To visualize seems to be a *norm*: "Teachers use visualization when they introduce a new theme." On the other hand, Claire sees a high benefit for her students in the fact that they can remember the contents later because of the visualization:

> Claire: I always try to select something that still is easy to remember.

Although Claire shows quantities and development of the use of visualization in fractions similar to Alan, she associates different roles with it. She rather uses it as a *mnemonic aid*. Even if she is not convinced, she uses visualization for explanation. She rather seems to conform to an educational *norm* she learned at university.

Claire: Algebra

In algebra, too, Claire - like Alan - uses a balance for solving equations. She appreciates the possibility to *reactivate* knowledge once the subject is manifested:

> Claire: That is also something you can well remember later; once this has been understood, you can always come back to the balance [...], standing there with your hands on the left and right side.

She does not claim to present every subject in a visual way, as algebra seems to be a rather abstract domain, not related to visual elements:

> Claire: Algebra is something which is incredibly abstract, actually. [...] To get somewhere

in algebra, it is not always visualizations that does the trick.

Interim summary 2

One important reason why Claire uses visualization is a *mnemonic aid*. In fractions, she additionally associates visualization with a *norm*, in algebra she does not (quest. 1). The role of a mnemonic aid seems to be similar in different domains. Also her orientation on perceived norms, which might influence her beliefs about the role of visualization, seems to be similar in different domains (quest. 2).

DISCUSSION

This paper identified and compared beliefs regarding the role of visualization when teaching fractions and algebra, analyzing two secondary teachers, Alan and Claire. Both intensively use visualization as part of the introduction of a new subject, and both teachers tend to emphasize schematic calculation. However, in a deeper analysis, they show different beliefs about the roles of visualization. This result is in accord with Biza and colleagues (2009) who found teachers' beliefs about visualization to be similar at first glance, but to be considerably different upon closer inspection.

The identified beliefs about the roles of visualization are about learning (*understanding aid, mnemonic aid*), about teaching (*explaining aid, norm*), and about the subject mathematics (*justify*). The investigated beliefs appear to be rather stable across domains.

In comparison to existing frameworks, we find *understanding* as a role for both teachers, with high intensity for Alan. Both teachers use understanding "in the sense of one's grasp of a definition [...]" (Giaquinto, 2008, p. 36). Alan also uses it "not only [for] grasping the correctness [...], but also [for] appreciating why it is correct" (ibid.), which is related to the role of *justification*. Both teachers do not express much consciousness about the "conversion of representations" (Duval, 2006, p. 121) as a source of understanding. *Explaining* and *justifying* could be beliefs that are in a cluster (Thompson, 1992) with *understanding*.

From a teaching point of view the aspect of classroom practice as learned in teacher education, the *norm*, is important for Claire. This could be related to sociological and cultural aspects of using visualization in mathematics teaching (cf. Arcavi, 2003).

Finally, the *mnemonic aid* is a role which does not seem to be very important in the literature about teaching and learning of mathematics, but possibly in practice.

For these two teachers we found only a few indications related to discovery and problem solving (Heinze et al., 2009; Duval, 2014; Arcavi, 2003), especially when making a sketch related to an application-oriented task. There are also indications that visualization can serve to "enable calculation" (Giaquinto, 2008, p. 39); both teachers recognize the "visio-spacial nature" (ibid) of some algorithms they teach.

In our study, we did not prove that the teachers enact their professed beliefs. However, "under specific conditions the teachers' espoused beliefs could explain the teachers' enacted beliefs" (Eichler & Erens, 2015, p. 197).

Prospectively, we will compare our results to more domains, e.g. also functions and calculus, to see if the teachers' beliefs are stable across other domains. We further expect to find other beliefs regarding the role of visualization like using visualization as a method for overcoming students' fears, seem to be relevant. We hope to develop a theory of beliefs regarding the role of visualization for teachers.

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Is boredom important for students' performance?

Stanislaw Schukajlow

University of Münster, Münster, Germany, schukajlow@uni-muenster.de

This experimental study of 192 ninth and tenth graders was conducted to investigate a connection between performance on different types of problems and boredom using task-unspecific and task-specific questionnaires. Students were randomly assigned to one of two groups and were asked about their boredom either before (Group 1) or after (Group 2) task processing. In Group 1, the relation between performance and boredom was different for different types of problems. In Group 2, students who achieved higher scores reported lower boredom across different types of problems. The connection between performance and task-unspecific and task-specific boredom did not differ significantly and ranged from 0 to -.36.

Keywords: Emotions, performance, boredom, modelling.

INTRODUCTION

In the current study, I focused on students' boredom and on the connection between boredom and performance as students solved different types of problems. The problems that were selected as content for the current study either had or did not have a connection to reality and could be solved by applying linear functions or Pythagoras' theorem. The research questions pertained to (1) the connection between performance and boredom, (2) the correlation between students' performance and task-unspecific boredom in comparison with the correlation between performance and task-specific boredom, and (3) the relation between performance and boredom compared across three types of problems.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

Boredom as a negative emotion

In educational research, emotions are defined as complex phenomena that include affective, cognitive, physiological, motivational, and expressive parts (Pekrun & Linnenbrink-Garcia, 2014). One important dimension of emotions is their valence. Researchers have often distinguished between positive and negative emotion/affect without specifying the kind of emotion they were interested in. Hannula and colleagues (2009) underlined the importance of overcoming such a simplistic view on emotions and suggested that researchers should identify which positive or negative emotions they are focusing on. For example, anxiety, frustration, and boredom have a negative valence and enjoyment and happiness have a positive valence. Another dimension that illustrates the importance of specifying emotions is the degree of activation or deactivation. This dimension describes the psychological states (activating excitement vs. deactivating relaxation) that humans report about emotions (Pekrun & Linnenbrink-Garcia, 2014). Negative activating emotions include anger, anxiety, and frustration. Hopelessness, and boredom are typical negative deactivating emotions.

Students' emotions influence their career aspirations and thus also their current and future lives. Self-perceived levels of boredom depend to a large extent on students' general experience at school and in particular on their experiences in specific school subjects (Jablonka, 2013). A control-value theory of achievement emotions assumes that the value of learning materials and the controllability of learning activities are important for students' emotions (Pekrun, 2006).

Boredom is one of the most frequently reported negative emotions in the classroom, and some researchers see boredom as a key problem of modern society (Klapp, 1986). For several decades, research efforts in education were focused on the negative emotion of anxiety, whereas the role of other negative emotions (e.g., boredom) in educational contexts and their relations to other emotions, learning goals, motivational variables, and performance were not yet well understood. However, in the last 20 years, theoretical models of emotions have been improved considerably. As boredom is a deactivating emotion that decreases human activity, a negative connection between boredom and performance or academic achievements can be expected. The few studies on the connection between boredom and performance often used students' final grades as an indicator of performance. These studies identified negative correlations between boredom and grades at school and at university (-.24 and -.64, respectively) (Goetz, Frenzel, Pekrun, Hall, & Lüdtke, 2007). Similar results were also found for the relation between boredom and grades or performance in elementary school (Sparfeldt, Buch, Schwarz, Jachmann, & Rost, 2009). However, as far as I am ware, in the only study conducted on students in early secondary school to investigate the connection between boredom and performance, no significant correlations between boredom and ninth-graders' argumentation, reasoning, or proof were found (Heinze, Reiss, & Rudolph, 2005). These contradictory results indicate the importance of enhancing research on the relation between boredom and performance using different approaches to the conceptualization of boredom in order to clarify the value and valence of this relation.

Measurement of emotions

The most commonly used measures of students' emotions are questionnaires, but analyses of self-reports given in interviews and analyses of students emotions during problem solving are also widely used in mathematics education (Jablonka, 2013; Pesonen & Hannula, 2014). Questionnaires used to assess students' emotions have shown high reliability and validity in previous research (Pekrun, Goetz, Frenzel, Barchfeld, & Perry, 2011; Sparfeldt et al., 2009). Questionnaires allow researchers to access data from large samples and to distinguish between different emotions such as hope, enjoyment, pride, anger, anxiety, boredom, and others (Pekrun et al., 2011). However, because of the complexity of emotional reactions, a multi-method approach can be helpful for accessing affect (Hannula et al., 2009; Schukajlow et al., 2012; Zan, Brown, Evans, & Hannula, 2006). One way to increase the coverage of questionnaires may be to take object-specific aspects of affect into account. Following this idea, two types of questionnaires for the measurement of boredom were applied in the current study (c.f. for enjoyment and interest Schukajlow & Krug, 2014a): task-unspecific affective scales, which were validated in other studies (Pekrun et al., 2011), and a new task-specific questionnaire applied in recent studies (Schukajlow & Krug, 2014a; Schukajlow et al., 2012). Another important factor that may influence students' boredom is task processing. Thus, we measured students' boredom before and after they solved problems in two randomized groups in order to compare the stability of the relation between performance and boredom.

The development of task-specific questionnaires is based on distinguishing different objects (or subjects) of students' affect. Similar approaches can be found in educational psychology, where achievement, epistemic, social, and topic emotions are separated according to their object focus (Pekrun & Linnenbrink-Garcia, 2014) or in mathematics education in the beliefs area, where the question of the subject-specific structuring of beliefs was suggested by Törner (2002). Object-specificity varies from very general such as "learning" or "mathematics" to specific ones such as "mathematical topic" or even "mathematical problem" (cf. Figure 1).

Sample statements for boredom illustrating the different levels of object-specificity are: "I get bored in classes", "I get bored in mathematics classes", "I get bored solving equations", and "I get bored solving the equation $3 + 2x = -4x$", respectively. Measurements for which statements with a high level of object-specificity are used (1) provide exact information about the kind of mathematics the researcher is interested in, (2) allow the investigation of new research questions that focus on the comparison of affective measures regarding different mathematical topics or kinds of problems, and (3) reveal high sensitivity to the changes in students' affect that can emerge from interventional programs. Empirical research has shown the importance of the differentiation between different domains and thus indicates the importance of object-specificity in measuring emotions (e.g., differentiating between boredom in mathematics and physics classes) (Goetz et al., 2007). As task-unspecific and task-specific questionnaires assess the same construct, I did not expect

Object	learning	mathematics	mathematical topic	mathematical problem
Level of object-specificity	low →			high

Figure 1: Objects and levels of object-specificity for affect

performance to be more or less strongly correlated with task-specific measures than with task-unspecific measures. This supposition was confirmed for interest and enjoyment (Schukajlow & Krug, 2014a), but it is an open question for boredom.

It is essential to distinguish between prospective affect (measured before task processing), current affect (measured during task processing), and retrospective affect (measured after task processing) (Ainley, 2006; Efklides, 2006; Schukajlow & Krug, 2014a). Each point of measurement reveals information about affect with regard to problem solving, and it can be important for past or future achievements.

Problems with and without a connection to the real world

To measure students' task-specific boredom, three types of problems that differ in their strength of connection to reality and are typically distinguished in research in modelling and application (Blum, Galbraith, Henn, & Niss, 2007) were selected. The types of problems were modelling, "dressed up" word, and intra-mathematical problems. All problems could be solved using the Pythagorean theorem or linear functions as mathematical procedures. To solve modelling problems, students need to understand the situation described in the task and must be able to construct a situation model of the task. Then they simplify the situation model by structuring and mathematizing, and they generate a mathematical model. The mathematical model can be transformed using mathematical procedures to create mathematical results, which have to be interpreted and validated. In the "dressed up" word problems, a mathematical model is "dressed up" by the situation; thus, students need to "undress" the problem, mathematize it, and use mathematics to solve it. Therefore, the problem solving process is not as complicated for this type of problem. As intra-mathematical problems are not connected to reality, students begin their problem solving process directly by using a mathematical model.

On the basis of the results of our previous study (Schukajlow & Krug, 2014a), we expected that there would be no significant differences in correlations between different measures of performance and boredom. Students with higher scores on performance tests were expected to be less bored when solving the problems.

Research questions

The research questions we addressed were:

1) Is students' performance connected to task-unspecific and task-specific boredom in mathematics measured before and after problem solving?

2) Is students' performance connected more strongly to task-specific than to task-unspecific boredom?

3) Are correlations between performance measures and task-specific boredom different for different types of problems (modelling problems, "dressed up" word problems, and intra-mathematical problems)?

METHOD

One hundred ninety-two German ninth and tenth graders from 4 middle-track and 4 grammar school classes (53.6% female; mean age=16.1 years, SD=0.86) participated on the present study. Students in each class were randomly assigned to two groups. In Group 1, the participants solved the problems first and afterwards reported on their task-unspecific boredom and on their boredom with regard to each problem. Students in Group 2 were asked about their task-unspecific and task-specific boredom and then worked on the performance test. The same tasks and questionnaires were administered to both groups (see Figure 2), and the students in these groups were given the same

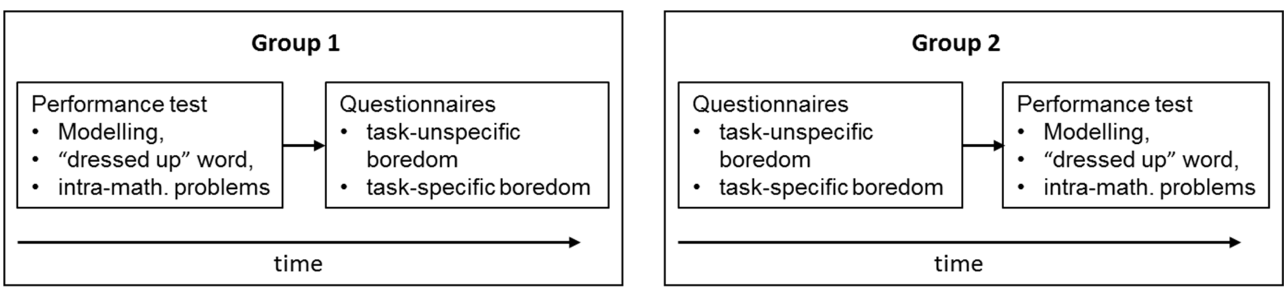


Figure 2: Design of the study

Maypole

Every year on Mayday in Bad Dinkelsdorf there is a traditional dance around the maypole (a tree trunk approx. 8 m high). During the dance the participants hold ribbons in their hands and each ribbon is fixed to the top of the maypole. With these 15 m long ribbons the participants dance around the maypole, and as the dance progresses a beautiful pattern on the stem is produced (in the picture such a pattern can already be seen at the top of the maypole stem).

At what distance from the maypole do the dancers stand at the beginning of the dance (the ribbons are tightly stretched)?

Figure 3: Modelling problem "Maypole"

amount of time to solve the problems and to fill out the questionnaires.

Sample problems

Eight modelling, eight word, and seven intra-mathematical problems were selected for this study. Sample tasks with and without a connection to the real world that could be solved using Pythagoras' theorem are presented in Figures 3 and 4. The maypole, football pitch, and side c were classified as modelling, "dressed up" word, and intra-mathematical problems, respectively (for more sample tasks and detailed analysis of classification see Krug & Schukajlow, 2013; Schukajlow et al., 2012).

Performance tests

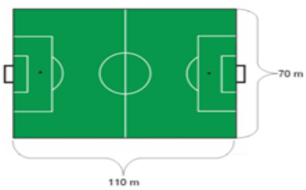
A performance test was developed for each type of problem. All tasks were examined within the framework of other projects. The Cronbach's alpha reliabilities were .59, .67, and .52 for the modelling, word, and intra-mathematical tests, respectively, and were acceptable for the small number of items and their diversity (across different contexts and/or different mathematical procedures).

Task-unspecific and task-specific boredom

Task-unspecific boredom was examined with a scale used in other studies (Goetz et al., 2007) and consisted of 4 statements that were answered on 5-point Likert scales ranging from (1=strongly disagree) to (5=strongly agree). A sample statement is "I am bored in mathematics classes". Cronbach's alpha was .85.

On the task-specific questionnaire, each of the 23 problems was followed by a statement about students' boredom. Both groups (cf. Figure 2) were instructed: "Read each problem carefully and then answer some questions." Group 2 was then told: "*You do not have to solve the problems"* because they were going to solve the problems after the boredom ratings, whereas Group 1 had already solved the problems, so they were told: "*You do not have to solve the problems (again)!*" After task processing, students in Group 1 were asked to rate the extent to which they agreed or disagreed with the statements "It was boring to work on this problem." Students in Group 2 were asked before task processing to rate the statements "It would be boring to work on this problem." A 5-point Likert scale was used to record their answers (1=not at all true, 5=completely true). 3 scales that measured task-specific boredom were formed across eight modelling problems, eight "dressed up" word problems, and seven intra-mathematical problems. The Cronbach's alpha reliabilities were .91 for boredom with the modelling and word problems and .85 for boredom with the intra-mathematical problems.

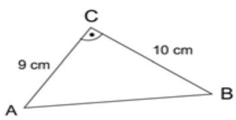
Football Pitch



Trainer Manfred would like to carry out a diagonal run with his team. To do so he would like to know how long the diagonal of the football pitch is. Can you help him?

Calculate the diagonal length of the football pitch.

Side c



Calculate the length of the side c = |AB|.

c= ________

Figure 4: "Dressed up" word and intra-mathematical tasks "Football Pitch" and "Side c"

Treatment fidelity

To examine the differences in the implementation of the treatment in Groups 1 and 2, a 5-point Likert item: "Before I agreed or disagreed with the statements (about task-specific boredom), I solved the problems" (1=not at all true, 5=completely true) was administered. Means and standard deviations were 4.3(1.17) for Group 1 and 2.19(1.01) for Group 2. The comparison of students' responses using an unpaired *t* test showed a significant mean difference between the two groups ($t(179)=13.07$, $p<.0001$, Cohen's $d=1.93$). This result shows that the students in Group 1 solved the tasks significantly more often than students in Group 2 *before* they reported their task-specific interest or enjoyment.

RESULTS

First, the connection between students' performance and boredom was analysed (correlations for Groups 1 and 2 are presented in Tables 1 and 2). As expected, students who solved the word problems better reported lower boredom on this type of problem. A similar result was also found for the relation between modelling and task-unspecific boredom. However, correlations of zero were observed for intra-mathematical problems and a weak and nonsignificant relation for the connection between performance and task-specific boredom on modelling problems.

In Group 2, in which students reported their boredom before solving the problems, negative correlations that ranged from -.24 to -.36 were found for all types of problems. Thus, students who felt low task-specific and task-unspecific boredom showed better results on the performance tests.

In order to answer the second and third research questions, Fisher's z transformation was applied, and then the z-scores were compared using a statistical procedure from Cohen & Cohen (1983, p. 54). This procedure provides information about the statistical significance of the difference between two correlations. The analysis of correlations for different types of problems presented in Table 1 and Table 2 showed that the largest difference between correlations was for the modelling problems in Group 1 (-.13 vs. -.33). However, the difference between correlations was not significant (z-score = 1.45, p = .14). Thus, students' performance was comparably related to task-specific and task-unspecific boredom.

The third research question addressed a comparison of correlations across different types of problems. The analyses of the values presented in Table 1 showed valuable differences between correlations for intra-mathematical and for "dressed up" word problems (0 vs. -.25) for task-specific boredom and between correlations for intra-mathematical and modelling problems (-.09 vs. -.33) for task-unspecific boredom. Both differences were significant at the 10% level (z-score = 1.74, p = .08; z-score = 1.72, p = .08, for task-specific and task-unspecific boredom, respectively). Thus, the correlation between performance and boredom with regard to the intra-mathematical problems tended to be weaker than the correlation between performance

		boredom			
		ma	w	mod	task-unspecific
performance	ma	0			-.09
	w		-.25*		-.29*
	mod			-.13	-.33*

Note: *p<.05; ma intra-mathematical, w word, mod modelling problems; sample size N=100

Table 1: Pearson correlations between performance and task-specific and task-unspecific boredom in Group 1

		boredom			
		ma	w	mod	task-unspecific
performance	ma	-.36*			-.34*
	w		-.30*		-.28*
	mod			-.24*	-.28*

Note: *p<.05; ma intra-mathematical, w word, mod modelling problems; sample size N=92

Table 2: Pearson correlations between performance and task-specific and task-unspecific boredom in Group 2

and boredom with regard to the word problems for students who solved the problems before reporting on their boredom. Similar differences were also found for the correlation between performance on the intra-mathematical problems and task-unspecific boredom and the correlation between performance on the modelling problems and task-unspecific boredom. However, another pattern of correlations was revealed for Group 2. Higher levels of boredom were connected with lower levels of performance across all types of problems for Group 2.

DISCUSSION

In this paper, the relation between performance and boredom was analysed using task-unspecific and task-specific scales. The results were not univocal. When students reported on their boredom before they solved the problems, their level of boredom was negatively connected to their performance (see similar results by Goetz et al., 2007). Somewhat different results were found for students who estimated their boredom after task processing. Students who achieved low scores on the intra-mathematical problems reported about the same value for boredom as students who achieved high scores. Similar results were found for the connection between performance on argumentation tasks and boredom (c.f. Heinze et al., 2005).

The correlations between performance and boredom were comparable between task-specific and task-unspecific boredom. The analysis of this question with regard to interest and enjoyment in the previous study showed the same result for enjoyment and interest (Schukajlow & Krug, 2014a). However, the correlations for task-specific boredom deviated across the different types of problems more than they did for enjoyment and interest. This result confirms the importance of differentiating between different affective measures as called for by Hannula and colleagues (2009).

The comparison of correlations across different types of problems showed that the correlations tended to be lower for intra-mathematical problems than for word or modelling problems in Group 1. Thus, the type of problems may be an important factor that has to be taken into account in future studies. According to our findings, teachers should put more effort into decreasing their students' boredom when presenting word or modelling problems because boredom on these tasks is negatively connected with students' performance.

One important future research question is about the direction of connection between performance and boredom. Longitudinal and interventional studies need to be conducted to answer this question. More research has to be done for the development and validation of research instruments for the measurement of boredom. An interesting approach may involve using software to identify students' emotions (Pesonen & Hannula, 2014). Research on developmental models of emotions is another future area of research. Such research should examine whether general affective changers emerges from changes in task- and situation-specific affect. Finaly, we need more research on instructional elements which could deacrease boredom. A promicing teaching approach could be prompting students to find multiple solution for problems with missing information, which found to affect students' expirience of competence and interest (Schukajlow & Krug, 2014b).

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Promoting cognitive engagement in secondary mathematics classrooms

Karen G. Skilling[1] and Gabriel J. Stylianides[2]

1 King's College London, Department of Education and Professional Studies, London, UK, karen.skilling@kcl.ac.uk

2 Univeristy of Oxford, Department of Education, Oxford, UK

Cognitive engagement (including self-regulation) is crucial for promoting student learning, but research suggests that teacher beliefs about cognitive engagement are less refined than their beliefs about other kinds of engagement. We used surveys and interviews from 40 teachers across 8 secondary schools to investigate teacher beliefs and practice that the teachers report using to promote cognitive engagement in their classes. Participants responded to questions about two fictitious teacher scenarios. About half of them identified with Teacher A, believing in the importance of completing practice questions and providing students with a list of revision topics. Those who identified with Teacher B favoured encouraging students to self-assess their competency, monitor their progress, and develop individual revision plans.

Keywords: Cognitive engagement, self-regulation, mathematics.

LITERATURE REVIEW

Promoting student engagement, interest, and participation in mathematics is considered important for students' learning and subsequent study in mathematics. In educational research high levels of student engagement are consistently linked to academic success (Wang & Holcombe, 2010) and are a predictor of students' achievement (Gettinger & Walter, 2012). There is a general agreement that engagement comprises three types—behavioural, emotional and cognitive—operating together (Fredricks, Blumenfeld, & Paris, 2004). Although engagement is considered a multidimensional construct with different types of engagement operating at varying levels of intensity, this research is concerned with the role of cognitive engagement in mathematics teaching and how teachers report promoting students' self-regulated learning in their classes.

Cognitive engagement

Cognitive engagement refers to students' approaches to academic tasks as well as their psychological investment in, and willingness to, master complex concepts (Fredricks et al., 2004). Conceptions of cognitive engagement draw on goal orientation and cognitive strategy use, whereas self-regulation theories that have historically been connected with motivational processes and academic functioning (Cleary & Zimmerman, 2012; Wolters & Taylor, 2012). Cognitive engagement includes thinking deeply and broadly about concepts while using strategies such as organisation, rehearsal, and elaboration as well as regulating and managing the learning process. Students' level of cognitive engagement is influenced by their goal orientations, the range of strategies students use, and students' underlying motivational factors. Components of self-regulated learning are considered particularly relevant to student cognitive engagement as they are both concerned with and are "used to understand students' functioning and performance with regard to academic contexts" (Wolters & Taylor, 2012, p. 635). For this study, frameworks for engagement and self-regulation are considered to be complementary, with the processes of self-regulation being considered important for cognitive engagement and involving a range of motivational factors as depicted in Figure 1. Although the focus of this study is on cognitive engagement and self-regulation frameworks, Figure 1 depicts how cognitive engagement is "dynamically interrelated" (Fredricks et al., 2004, p. 60) with behavioural and emotional engagement, while also acknowledging the influence of motivational and contextual factors on all types of engagement.

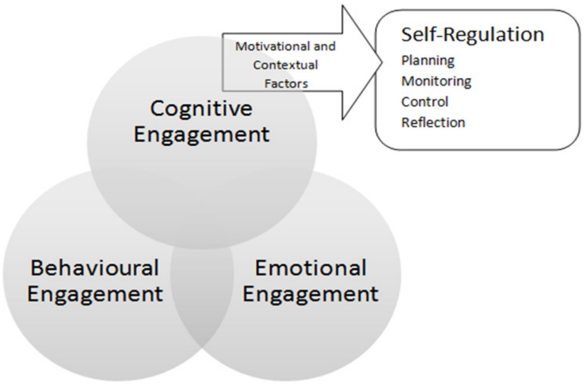

Figure 1: Types of engagement and self-regulation processes

Self-regulation

Self-regulation involves several important processes or phases for promoting deep learning. The phases include: forethought (goal setting and planning), monitoring and control, and reflecting on learning (Pintrich, 2004; Zimmerman, 2002). Self-regulation is centrally concerned with student agency and the degree to which students are active participants in their own learning. Therefore, while engagement frameworks tend to be more concerned about what students do, self-regulation frameworks focus on the processes students use to support their cognitive and behavioural functioning for managing their learning (Wolters & Taylor, 2014).

Self-regulated learners tend to be more aware of their knowledge, beliefs, motivations, and cognitive processes and, this awareness, allows these students to judge how successful or effective they are in their learning (Butler & Winne, 1995). The extent to which students are engaged in academic work and use self-regulatory processes is likely to be influenced by their goal orientation (Anderman & Patrick, 2012). Goal orientation is often determined by the student and influences the amount of time students spend prior to and during tasks on activities such as planning, organising, studying, monitoring, and reflecting. Students with high mastery goal orientations not only tend to use a variety of self-regulatory processes but also display a number of adaptive motivational factors such as self-efficacy, value, and persistence. Additionally, students' emotions play a central role in cognitive processing and engagement and should be considered together in learning settings; this is because student thinking and emotions are intrinsically linked to underlying motivational factors that influence cognitive processing (Hannula, 20006).

However, not all students have high mastery goal orientations or hold adaptive motivational factors. In any classroom the goal orientations of students are likely to be varied, and it is expected that not all students will use (effectively) self-regulative processes. Accordingly, students' ability to plan, organise, monitor, and reflect on their learning will also differ. Apart from individual goal orientations, in school settings sustained engagement in academic work is mediated and influenced by classroom contexts. Classroom contexts that can shape student engagement include the classroom environment and goal structures which are established through explicit and implicit teacher practices and teacher-student interactions (Anderman & Patrick, 2012; Reschly & Christenson, 2012). This means that what teachers do and say in their classrooms can influence students and mediate the use of self-regulatory processes, such as promoting forethought, considering prior knowledge, directing attention to key components, making effective strategy choices and reflecting on achievement to foster student cognitive engagement (Cleary & Zimmerman, 2012).

Teacher beliefs about and practices they use to promote cognitive engagement

In an earlier investigation, Skilling (2013) found that mathematics teachers' beliefs about cognitive engagement were less extensive and detailed than their perceptions about behavioural and emotional engagement. This may be in part because it is more difficult to identify signs of cognitive engagement (e.g., they are less observable than student behaviours) or because mathematics teachers feel less confident about assessing indicators of cognitive engagement. In addition, Skilling (2013) found that teachers reported practices for promoting cognitive engagement that were restricted to completing homework and study strategies focusing on behavioural aspects, such as time management; little did the teachers report about planning, monitoring, and evaluating student learning during lessons. Moreover, similar to a finding by Hardré (2011), the majority of teachers reported using practices that met students' immediate motivational needs such as explaining relevance, future use and application of mathematics concepts compared to few teachers who used practices that met students internal motivational needs, supported autonomous learning and mastery of concepts.

The focus of this study

The study we report in this paper builds on the research we described above by exploring teachers' beliefs about cognitive engagement and how these relate to the practices that the teachers report using to promote cognitive engagement in their mathematics classrooms. For this research, beliefs are defined as "psychologically held understandings, premises, or propositions about the world that are felt to be true" (Richardson, 1996, p.103). Teacher beliefs, therefore, vary according to their bearer and fundamentally reflect the relationship between what the teacher is considering when planning and executing instruction (Mason, 2008).

To conclude, this study focuses on the following research questions:

1) What are teachers' beliefs about cognitive engagement in early secondary mathematics classrooms?

2) What practices do teachers report using to promote student cognitive engagement?

METHODOLOGY

Data were collected from 40 teachers across eight secondary schools in England. The schools included five mixed ability comprehensive schools and three selective schools. There were two phases to this investigation: a teacher survey phase and a teacher interview phase. The designs of the survey and interview questions were guided by the components of cognitive engagement and self-regulation phases described earlier.

The teacher survey phase asked participants to respond to questions about two fictitious teacher scenarios (see Appendix). The scenarios outlined how Teacher A and Teacher B prepared the Year 7 students (11 years olds) in their classrooms for a mathematics test. Each scenario drew on literature from cognitive engagement and self-regulation to embed particular phrases as 'markers' to emphasise the different strategies and processes used by Teacher A and B.

The scenarios differed by the degree to which Teacher A and B promoted student involvement and used particular strategies and processes when preparing their students for a class test. For example, Teacher A handed students a list of topics for revision and told them to revise at home by looking over their notebooks, whereas Teacher B asked students to reflect on how competent they felt about particular concepts, to contribute to a 'class revision list' and to develop an individual revision plan. Teacher A told students it was important to get a high grade in the test, to ask for help if needed, and offered a small number of practice questions in class for those who may not have been revising at home. In contrast, Teacher B told students to focus on mastering concepts they did not understand, checked their revision plans and asked them how they felt about their test preparations. There were limited expectations by Teacher A about the range and depth of self-regulatory processes the students would use and there was an emphasis on performance rather than mastery goals. Teacher B, however, displayed expectations that students would use a range of self-regulation strategies and asked students to set their own goals based on their self-assessment, to make plans for mastering concepts, and to monitor and reflect on their revision preparation.

In the surveys, the participants were asked to respond to eight open-ended questions about the fictitious scenarios by referring to relevant line numbers provided for each scenario that they felt provided evidence for their responses. For example, participants were asked to compare their practices with those used by the teachers in the scenarios, to identify perceived similarities and differences between Teacher A and Teacher B, to list practices in the scenarios that they considered important or not important for student test preparation, to consider if their practices change with different groups of students (e.g., grade level, gender, achievement), and to indicate with which of the two teachers in the scenarios they identified more.

The second phase used semi-structured interviews with 17 participants who completed the first phase. The interview questions were guided by the survey questions and asked the participants to elaborate on selected survey responses. This included questions about their beliefs and the practices they report using to promote cognitive engagement in their classrooms. Questions probed participants' approaches toward setting goals and planning revision for assessments, and about monitoring and regulating learning processes during revision and when completing tasks in their classroom. The participants were also asked about ways they encouraged their students to self-monitor their progress. Finally, participants were asked to describe ways they provided feedback to their students about achievement, setting goals, regulating and reflecting on learning. The interviews were recorded and transcribed.

The analysis of the surveys and interviews drew on the cognitive engagement (Fredricks et al., 2004) and self-regulation frameworks (Pintrich, 2004; Wolters & Taylor, 2012), as components of these were reflected in the design of the two teacher scenarios. The responses to each survey question were listed and coded in association with the phrases or 'markers' that were embedded in the two teacher scenarios. The categories were then enriched with any additional practices that were reported by the teachers, providing data for addressing the research questions (Constas, 1992). This paper reports on key findings from the surveys supplemented by interview data.

FINDINGS

Of the 40 participants surveyed (coded T1-T40), 17 identified with Teacher A and 14 identified with Teacher B. From the remaining participants, six identified with both teachers and the other three identified with neither. This paper reports the participants' responses from three selected survey questions. The three questions were chosen because together they revealed participants' beliefs about (a) which teacher they identified with and why and (b) the practices of each teacher in the scenario that the participants believed to be (not) important in the context of the scenarios. Specifically, Question 1 asked participants: "Overall, which teacher do you identify with the most and why?" Questions 2 and 3 asked, respectively, participants to list up to two things that each teacher in the scenario did that they believed were important / not important for supporting the students' test preparation.

Participants who identified with Teacher A

The 17 participants who identified with Teacher A provided a total of 25 responses explaining their reasons for why they identified with Teacher A. The main reason given was that Teacher A was seen to have greater teacher control/structure/leadership (36%), with a strong "teacher led focus" (T9), and "a more structured approach and more control" (T4). The second most frequent reason was that Teacher A practiced concepts in class (16%) for example, stressing the "importance of practising the concepts" and holding a "revision lesson before a test to support [students]" (T7). The next most frequent reason was attending to student needs (12%), as several participants commented on the students being "needy" (T8) and that Teacher A "gave the students the most information about the test...to highlight individual weaknesses" (T2). The fourth most frequent reason was that Teacher A made better use of time in class (12%). See T6's response, for example:

I tend to like to direct my students towards their problem areas rather than let them take time to find them for themselves. Time always seem to be too much of a factor.

In response to Question 2, the participants who identified with Teacher A made a total of 33 responses to describe teacher practices in the scenarios that they believed were important for test preparation. The following practices were noted as being particularly important: setting practice questions in class (30%), for example slotting "tests in lessons leading to the main test" (T4); supporting students seeking help (24%), for example "making time to help students both in and out of lessons" (T8); and providing a list of topics for revision (21%), such as "handing out a sheet with key concepts (T9).

In response to Question 3, participants who identified with Teacher A provided 14 responses about the practices of Teacher B that were not important for supporting the students' test preparation (several participants did not provide a response). The three main practices believed to not be important were: developing individual revision plans (29%), asking students to self-assess their competency (21%), and contributing toward creating a class list (21%). Some participants questioned the value of asking students to develop their own revision plans because they felt that students "may not assess themselves accurately" (T2) or that students might "identify only what they think may be included" (T5) in the test. Creating a class list was believed by three of the teachers to be a waste of time wasting, such as for T6:

It takes time to develop a topic list as a class, which could have been spent more usefully completing actual questions. Would all students be able to manage their time to successfully draw up a revision plan and fulfil it?

Participants who identified with Teacher B

The 14 participants who identified with Teacher B provided a total of 20 responses explaining their reasons for aligning with this teacher. The main reason was the emphasis Teacher B placed on student self-assessment (40%). For example, these participants believed that students should be "reflecting as much as possible" (T27) on their work and should be "actively engaged in their own learning" (T26). Participants believed that an important feature of Teacher B's approach was that:

More emphasis is placed on students assessing what they can do and what they need to improve on and then going and working on those topics (T30)

The second reason was that Teacher B encouraged independence and student responsibility (30%). For example, several participants believed in encouraging student independence, noting that Teacher B used "strategies to encourage the students to engage with all aspects of their revision independently" (T23) and also handed "over ownership of revision to students" (T31). The third reason was the use of revision skills to plan improvements (25%). One participant believed that the purpose of revision was for "trying to get the students to understand which areas they need to improve on" (T28). Another participant noted:

Allowing students to identify their own areas to improve and implement their own plan is a step towards them improving because they want to, not because I am telling them they need to (T24)

In response to Question 2, the participants who identified with Teacher B made a total of 28 responses to describe teacher practices in the scenarios that they believed were important for test preparation. The most important practice included student self-assessment (32%), such as asking students "to assess their competency in each area" (T21) and encouraging students "to think about what might be in the assessment and self assess their confidence in these areas" (T24). The next most important practice was developing individual revision plans (29%), with one participant reporting that encouraging the "creation of revision lists and individual preparation gives the students' responsibility for their learning" (T24). The third most important practice was monitoring revision plans (18%) with several participants noting that the value of checking "revision plans to make them [the students] self-aware and in control of their revision" (T22).

In response to Question 3, the participants who identified with Teacher B provided 22 responses about the practices of Teacher A that were not important for supporting the students' test preparation. The main practice viewed as not important was "telling students it was important to achieve a high grade" (T20) (32%), as this was seen as "controlling and vague" (T23). It was also reported that "telling students to study more" (T24) was not important (27%) or that students should revise everything for a test (14%), as they "may not need to revise each concept" (T21).

CONCLUSION

Overall, participants who identified with Teacher A felt that this teacher tended to "be more structured" rather than giving students "freedom to think". Students were perceived as "needy" and Teacher A was believed to provide students with the "most information about the test". Additionally, participants who identified with Teacher A did not believe that making individual revision plans, asking students to self-assess, or contributing to a class list were important for supporting student test preparation.

In contrast, those who identified with Teacher B believed that practices such as student self-assessment, making individual revision plans, and monitoring revision progress were important. Participants who identified with Teacher B believed that this teacher's strategies encouraged "students to engage with all aspects of their revision independently". These participants also explicitly referred to specific processes of self-regulation such as planning, monitoring, checking and reflecting when responding to the survey questions (Cleary & Zimmerman, 2012; Wolters & Taylor, 2012), with the majority referring to at least two self-regulatory processes in their responses. Furthermore, participants who identified with Teacher B made comments about the importance of students being actively cognitively engaged in their learning by "allowing students to identify areas for improvement and implement their own plans"; they also emphasised student autonomy and responsibility for learning. The findings also suggest that the participants who identified with Teacher B believed in the importance of fostering student autonomy, independence and strategies for learning by using pedagogical practices that are associated with classroom mastery goal structures (Anderman & Patrick, 2012) and with promoting cognitive engagement and self-regulatory skills in their mathematics classrooms.

In this study just over half the participants identified with Teacher A and just under half identified with Teacher B, indicating that teacher beliefs about ways to promote cognitive engagement are potentially diverse. The results revealed that participants who identified with each teacher had strong beliefs about the practices they believed were and were not important for using in mathematics classes. Subsequently, one may hypothesize that participants who identified with Teacher A – and did not seem to believe in the importance of students developing individual revision plans, self-assessing their competency, or contributing to class revision lists – may be less likely than participants who identified with Teacher B to promote cognitive engagement and self-regulatory practices in their classes.

For this study the participants responded to scenarios that specifically asked about teacher practices for test preparation with Year 7 students. It is possible that different responses would be made for other situations, such as a 'typical' lesson covering a new topic or lessons with students of a different age. Future research can build on the findings of this study to investigate teachers' beliefs and self-reports about cognitive engagement and teachers' use of specific self-regulatory processes in their classrooms. For example, do teachers promote goal setting but not specific planning processes? In what ways do teachers promote students' self-monitoring of their learning? Do teachers ask students to reflect on their thinking and feelings as they work through and master difficult concepts?

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APPENDIX: TEACHER SCENARIOS

Context: Two Year 7 mathematics classes at one school will complete a topic test during second term. The teachers of each class provided students with information a week before the test about ways they could prepare. Below are suggestions by Teacher A and Teacher B. Please read each scenario and respond to the questions at the end. The line numbers for each scenario can help to make references to the text.

Scenario involving Teacher A

Teacher A reminded students about the upcoming topic test and handed out a sheet with an outline of the key concepts that would likely be covered in the test. The teacher suggested that the students set aside time for revision and to make sure they practised each concept, by looking over their notebooks as it was important for them to achieve a high grade on the test. The teacher also mentioned that the students should ask questions in class if they were unsure of the steps to solve questions. Alternatively, they could come and see the teacher during break time to clarify any questions before the assessment. In each lesson before the test the teacher set five practice questions in case students had not been revising at home and students who got three or less correct were advised they needed to study more.

Scenario involving Teacher B

Teacher B also reminded students about their upcoming topic test. The students were asked to look through their mathematics notebooks and textbooks during the lesson and recall specific topic concepts that they thought would likely to be included in the test. Based on their class work, the students were then asked to record how competent they felt about each concept. During the lesson, the teacher also asked the students to draw on their self-assessment notes and contribute to the creation of a ‘class’ revision list, from which examples could be revised during lessons before the test. The teacher also told the class that it was expected that each student would develop individual revision plans. Students would work on their individual plans at home, making time to focus on mastering the concepts they believed they needed to improve on. Throughout the week the teacher checked the revision plans of each student and asked how they felt about their preparations.

The restricted yet crucial impact of an intervention on pupils' mathematics-related affect

Laura Tuohilampi, Liisa Näveri and Anu Laine

University of Helsinki, Helsinki, Finland

Past research clearly indicates that students' mathematics-related affect develops destructively during school years. However, not many efficient interventions have been done. The efficiency of the interventions may become minor if other factors dominate the development of affect structures. Also the methods in order to measure the impact might be insufficient. However, the negative development of affect must be taken seriously. Especially the most harmful consequences, such as girls' unnecessarily poor self-efficacy, needs to be tackled. Here, we present a three-year intervention designed to improve primary school pupils' problem solving skills, and consequently mathematics-related affect. The impact was restricted but crucial: girls' affect regarding mathematics decreased less in the intervention group.

Keywords: Mathematics-related affect, gender differences, development of affect, intervention.

BACKGROUND

Numbers of studies show that students end up having an unnecessary negative affect towards mathematics when they leave school (international results, e.g., in Lee, 2009; Sjøberg & Schreiner, 2010; national results, e.g., in Tuohilampi & Hannula, 2013; Hirvonen, 2012). In addition, affect develops destructively: children tend to have very positive affect (e.g., they view the learning subjects enjoyable, and see themselves very capable) when they come to school (Tuohilampi, Hannula, & Varas, 2014; Harter, 1999), but during the school years the affect turns negative (the enjoyment turns into dislike, the feelings of capability decreases) and harmful for learning (Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014). Especially girls suffer from having negative emotions towards mathematics already after first three years of schooling. Also girls' self-efficacy has been noticed to be unnecessary low: even when performing well, a girl might feel incapable in mathematics (Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014). The presented development is to some extent natural, as it is indeed necessary for children to get social responses, including negative ones, in order to be able to modify their self-concept. After an almost omnipotent view of the self in the childhood (Harter, 1999), a certain number of negative, significant responses contribute to a more realistic self-view. When it comes to mathematics, the worrying thing is that the students do not become only realistic, but also remarkably negative. Unfortunately, having negative affect towards mathematics makes people avoid such future choices where mathematics is included (Tuohilampi & Hannula, 2013). Further, there is some evidence that negative affect connects with poor participation with other students and learning activities (Kirshner, 2014). In addition, students' poor wellbeing, such as a negative self-concept in mathematics or disaffection (see Lewis, 2014), should be significant per se.

Tuohilampi, Hannula, Laine and Metsämuuronen (2014) noticed in their recent study of Finnish students that the deterioration of mathematics-related affect begins very early, already after 3rd school year. It is particularly interesting that this happens in Finland that has a reputation of a remarkable performance level acknowledged by national studies (e.g., Metsämuuronen, 2013), and by international studies (PISA-studies, see e.g. OECD 2010): this makes Finnish primary school pupils an interesting population when it comes to examine how to prevent the deterioration. Letting the affect become negative in the first place is particularly problematic, as repairing it has noticed to be hard work (Hannula, 2006). Also, cumulative disappointments can lead to the cycles of failure, fear, the expectancies of failure and test anxiety (Pekrun, 2006). This is why it would be wise to concentrate on maintaining the affect as positive as possible throughout the school years, with a special focus on the early school years.

Most mathematics-related interventions seem to concentrate on performance or cognitive problems, such as dyscalculia (see a review of mathematics-related interventions in Dowker, 2009). Fewer interventions have been done to improve affect. These interventions have had a focus on students' self-control, and social interaction. For example, in an intervention by Rimm-Kaufman and colleagues (2014) there was a Responsive Classroom approach (RC) in use, aimed to foster relationships in the classroom and support students' self-control to enhance student achievement. This goes in line with the studies of Pekrun (2006), who has introduced control to be one of the defining elements of optimal affect structure and its development. For example, Pekrun (ibid) argues that when the learning demands exceed pupil's individual capacities, she/he loses her/his control over the activity. This may lead the pupil to reduce the value of the activity in question, and make the experience boring. Finally, boredom may reduce the pupil's engagement with the activity by decreasing the effort one puts in an activity, consequently reducing future success.

Having control over the action (action-control expectancies) and trusting that the action will lead to the expected outcome (action-outcome expectancies) are the key elements in Pekrun's (e.g., 2006) control-value theory of achievement emotions. When it comes to mathematics, one cannot always see the path to the outcome at the beginning. Thus, in mathematics a certain degree of resilience and tolerance towards mistakes might be necessary. However, the pupils should experience their actions effective. This can be done by allowing the pupils to proceed through small and various steps. Pupils should have the expectation that their efforts are worth to be done. If the tasks would allow different strategies in order to find the solution, many of the pupils' efforts would be beneficial. That is how they can have action control expectancies. On the contrary, there are less action-control expectancies if the pupils just either know or do not know the only possible solution. The expectancies the pupils have also connect with the amount and quality of responses the pupils get from their significant others. If it is a clear cut that a pupil either knows or does not know the solution, the evaluation the pupils make about themselves may become very polarized. Some pupils can make it, some pupils cannot. If, on the other hand, there are plenty of possibilities to proceed within the tasks, and the steps are small enough, it should be more likely that every once in a while even the weakest pupils succeed, and the strongest pupils make an incorrect effort. In such circumstances, the peer evaluation becomes versatile, and the responses the pupils get from their efforts diverse. That in turn plays a role on pupils' affect structure construction.

In addition to control and social interaction, improving mathematical understanding may be one path to achieve more positive affect: in a longitudinal study of Tuohilampi and Hannula (2013), high performance was the biggest cause of positive affect in future. These three elements connected with the optimal affect structure development suggest that an intervention could, or even should include the following goals: 1) minimize negative responses that are unconstructive, 2) give students possibilities to control their actions and 3) support students' understanding about the content of learning. However, even a good intervention faces a challenge of affect structure's resilience, as the dispositions of the students are noticed to be fairly robust. Chapman (2002) for example has shown that there is a need for open conflict that is meaningful to the holder before a change in the affect structure is likely.

One way to reach the presented three intervention goals is to use open ended problems. In such problems, more than one solution can be possible, and to find a solution, pupils need a linear or a cycling problem solving process where they use their resources, heuristics, beliefs, and abilities of monitoring and self-regulation (Schoenfeld, 2012). Because of the nature of the open ended problems, there are usually many opportunities where to start and how to proceed. Following that, there is typically at least something a pupil can initiate and perform. In addition, because of the several options of how to find an answer (or answers), the pupils' own actions ought to produce a positive outcome in most cases. Thus, using open ended problems should lead to high action-control expectancies, as well as high action-outcome expectancies (Pekrun, 2006), and consequently, the possibilities to control actions and learning is guaranteed to the pupils (the intervention goal number 2). These elements, on the other hand, widen the strategy options and thus decrease the number of "wrong choices". Following that, the negative responses from significant others regarding pupils' actions could be minimized (the intervention goal number 1). Open ended problems may also enhance pupils' understanding as they allow connections to several or untypical contexts. A traditional instruction, wherein specific learning content is mostly connected with the same, isolated context makes pupils' knowledge structures fragmented, and does not help pupils to generalize their thinking. In their study about students' conceptions Saglam, Karaaslan, and Ayas (2010) show that fragmented, isolated knowledge structures, produced by restricted contexts, cause students to fall short in solving problems across contexts. Thus, the use of open ended problems having less limited contexts may help pupils to create deeper and more applicable understanding (the intervention goal number 3). In this study, we report how an intervention that is built around open ended problems, guaranteeing the three intervention goals presented above impacts primary school pupils' affect structure development.

INTERVENTION

Here, we examine a three-year intervention from 3rd to 5th grade which included a monthly activity with a mathematical problem. The problem was in most cases an open ended and they were selected or developed by the research group. The teachers were allowed and instructed to execute the problem solving sessions according to their preferences. In most cases, the teachers used collective activities wherein pupils were allowed to discuss the problems, to move, and to work collaboratively.

We will introduce two of the problems that were used during the intervention. The first one to be presented is "Divide a square: Make such a division to a square that makes the two parts of the square totally equal. How many different solutions can you find?" This problem was implemented in the 3rd grade and it was the second problem in the project. In the pupils' solutions, five levels of thinking were present: level 0 = no solution; level 1 = the two most obvious solutions (two triangles and two rectangles); level 2 = division by a straight line that is not diagonal, nor passes the middle points of the sidelines of the square; level 3 = the thinking of level 2, replacing the straight line with a curve; and level 4 = clearly understanding the central symmetry of the task (Laine, Näveri, Pehkonen, Ahtee, Heinilä, & Hannula, 2012). Because of the five levels of understanding, the active nature of the task (a pupil could easily just use a pen to figure out the solutions), and the collaboration the pupils were allowed to have during the task, the intervention goals presented above were fulfilled. The second problem to be presented here is "*Etana-Elli (= a snail called Elli): Etana-Elli climbs up a wall very slowly. During some of the days she gets up 10 cm, during some of the days 20 cm, during some days she sleeps and does not move, and during some days she is in a very deep sleep and descends 10 cm. The wall is 100 cm high. After ten days of climbing, Etana-Elli is on a halfway of the wall (which means that she has mounted 50 cm).What could have happened during the first 10 days? Describe as many scenarios that are possible.*" This problem was implemented in 4th grade being the 7th problem in the project. Also in Etana-Elli -problem the pupils could easily initiate actions, and several solutions were possible. Thus the intervention goals got fulfilled within the problem.

METHOD

The data used in this study was gathered within a research project that aimed to develop mathematics learning and affect structure among pupils in Finland and Chile (see further description of the project in Laine, Näveri, Pehkonen, Ahtee, Heinilä, & Hannula, 2012). Here, we focus on Finnish pupils' data, wherein the number of pupils that participated either the pre-test, the post-test or both tests was 320. The pre-test data was collected in regions near to Helsinki at the beginning of the academic year 2010–2011 during September-October 2010. The post-test data was collected within the same classes at the end of the academic year 2013–2014 during April-May. The schools are fairly uniform in Finland (see OECD, 2010, p. 87), so the data can be considered representative to urban pupils in Finland. In the pre-test, there were 25 classes involved. 10 out of these classes were intervention groups, the rest of them being control groups. In the post-test, six control groups were not reached and three intervention groups had left the project (they quit doing the tasks, but yet participated in the post-test). Among the three classes that quit, one had participated in the project for two years whereas the other two had participated only one year. We decided to include the class that had participated for two years (i.e. more than 50 % of the intervention tasks) but exclude the classes that had been participating just one year (i.e., less than 50 % of the tasks). Moreover, there was a teacher change in two of the included intervention groups, and some movement regarding the pupils had happened, as there were pupils in the pre-test but not in a post-test and vice versa: those pupils' data were excluded from the analysis. In sum, we included in the data pupils who had participated in all the intervention tasks or at least 2/3 of them, and that had participated both of the measurements, but might have had a new teacher during the intervention.

The following factors of affect were measured in the questionnaire: self-competence, (spice item: "I have made it well in mathematics"), self-confidence ("I am sure that I can learn math"), the difficulty of mathematics, referred to as DoM ("Mathematics is difficult") representing cognitive dimension; the enjoyment of mathematics, referred to as EoM ("I have enjoyed pondering mathematical exercises") representing emotional dimension; mastery goal orientation, referred to as MGO ("On every lesson, I try to learn as much as possible") representing motivational dimension; and effort ("I always prepare myself carefully for exams") representing behavior. The purpose of the instrument was to catch the trait aspect of affect (see discussion on the cognitive, emotional and motivational dimensions, and the state - the trait aspects of affect in Hannula, 2011). The instrument was a shortened and simplified version of the instrument used by Hannula & Laakso (2011) to measure 4th grade Finnish pupils. The instrument worked well in that context and seemed suitable for measuring mathematics-related affect within Finnish population. In the instrument there was a 3-point Likert scale in use ("true", "partly true", "not true"). Bearing in mind that the pupils were just 9-year old in the pre-test it was justified to use only three points, as this makes the instrument simpler. The scale is an ordinal scale, as the middle option, "partly true", may situate differently between the two ends depending on the examinee. In the questionnaire some of the items were direct (e.g., "I have made it well in mathematics"), while some were indirect (e.g., "I am not very good in mathematics"). For the analysis, the items that had an inverse content were recoded to share the same direction with directly stated items.

Before starting the analysis, we constructed a sum variable of all the questionnaire items regarding both measurements. The reliabilities (measured by Cronbach alpha's) were satisfactory: $\alpha = .895$ in the pre-test and $\alpha = .858$ in the post-test. To find out the answer to our research problem, we calculated the distributions of pupils' affect within both measurements. A paired sample t-test was used to compare the means of the distributions regarding the two measurements and an independent sample t-test was used to compare the means of the distributions regarding intervention and control groups and genders.

RESULTS

In Table 1, there are the distributions of all items' sum variable regarding all pupils, intervention group, and control group.

The mean of all items for all pupils in the pre-test was 1,37 (1 = positive, 3 = negative), and the standard deviation was 0,30. In the post-test, the mean of all items for all pupils was 1,64, the standard deviation being 0,29. In a paired samples' t-test there was a statistically significant difference between the pre-test and post-test regarding all the pupils ($t(193) = -11.88$; $p < .000$), the intervention group ($t(108) = -9.72$; $p < .000$), and the control group ($t(84) = -6.98$; $p < .000$). The results indicate that there is a remarkable decline in pupils' affect regarding mathematics from the beginning of the 3rd to the end of the 5th grade in both the intervention group and control group.

When it comes to the differences between the intervention and the control group, no statistically significant difference was found with respect to all items

	Group	Positive	In between	Negative	N
Pre-test, all items	All pupils	168 (75,3%)	54 (24,2%)	1(0,4%)	223 (100%)
Post-test, all items	All pupils	90 (32,4%)	186 (66,9%)	2(0,7%)	278 (100%)
	Intervention group	41 (33,9%)	80 (66,1%)	0 (0%)	121 (100%)
	Control group	49 (31,2%)	106 (67,5%)	2 (0,4%)	157 (100%)

Table 1: Distributions regarding all pupils, intervention group, and control group

in post-test ($t(276) = -.67$; $p = .505$). Looking further at the differences between the groups factor by factor in post-test did not change the picture: $t(287) = -.06$, $p = .954$ regarding self-competence; $t(294) = -.79$, $p = .433$ regarding self-confidence; $t(290) = -1.50$, $p = .134$ regarding the difficulty of mathematics; $t(290) = 0.62$, $p = .533$ regarding the enjoyment of mathematics; $t(294) = -.57$, $p = .571$ regarding mastery goal orientation; and $t(290) = .62$, $p = .536$ regarding effort. Besides the non-significance between the groups, no trend was found regarding the minor differences regarding the different variables, as with respect to one variable the mean could be lower for the control group, but with respect in another the mean could be lower for the intervention group.

When it comes to the gender differences, we still did not find any significant differences in either of the tests (gender difference in pre-test: $t(122) = 1.05$, $p = .295$; gender difference in post-test: $t(140) = 1.57$, $p = .118$). However, when testing the control group's and intervention group's difference in the post-test separately to genders, a statistically significant difference was found regarding girls' development (girls: $t(67) = 2.08$, $p < .05$; boys: $t(87) = .42$, $p = .634$). The mean of the control group girls in the post-test was 1.82, and for the intervention group girls 1.65. This means that the girls had benefitted from the intervention, but not boys. The significance in the development came through two factors: self-confidence ($t(729 = 2.39$, $p < .05$), and EoM ($t(72) = 2.47$, $p < .05$).

DISCUSSION

We have reported the impacts of a three-year intervention aimed to improve primary school pupils' mathematics-related affect through focusing on pupils' control on their learning, social interaction, and mathematical understanding. According to our results, the impact was not as strong and widespread as one would have hoped. For the sake of future interventions sharing the same goal, it is necessary to gain knowledge about why it had such a minor impact. Even the effects of a well-designed intervention may become disguised by other features in school, more significant to the pupils. As Chapman (2002) has shown, a significant conflict is needed to allow affect structure to become reorganized. The pupils in an intervention may get positive experiences, yet those experiences might be less significant than school expectancies, peers' perceptions, or teacher's actions effect.

The other perspective is the method used here. Perhaps a questionnaire based quantitative data does not reveal all the possible nuances that might have been affected during an intervention. Qualitative analysis could perhaps better show less visible changes and thus reveal a stronger result. A mixed method approach could be advisable. However, as there was no significant difference between the whole intervention and control groups, it seems likely that a stronger change in the practices is needed. In our intervention, there was a monthly problem solving class for three years. Maybe the amount of doing was too little for the pupils, or maybe such classes would need different school culture to be more effective. For example, pupils in Finland do not rate their learning environment as positive as their mates in other cultures do (Tuohilampi, Laine, Hannula, & Varas, submitted). Thus, pupils in Finnish classes might need support to become effective with working socially among problem solving. The intervention presented here would possibly have become more efficient if there had been more support for pupils to become socially active.

The benefit for girls in the intervention related to their self-confidence and enjoyment of mathematics. This is extremely critical, as girls suffer poor and unrealistic mathematical self-confidence worldwide (Syzmanowics & Furham, 2011) and in Finland (Tuohilampi & Hannula, 2013). This makes girls avoid mathematics in future (ibid.), so even the impact was restricted on girls, it was extremely welcome. Girls' emotions towards mathematics have also been critical (ibid.), and it is delighting that the intervention could help girls to maintain their emotions more positive. Hannula, Kupari, Pehkonen, Räsänen, & Soro (2004) have presented that collaborative atmosphere and learning methods connect with increasing self-confidence and mathematical performance especially regarding girls. This seems natural, as while girls feel less confident with mathematics in general, they might find it helpful to work in co-operation with others. Thus the benefit for girls might have come through the increase in collaboration. Girls also differ from boys in their interests, as boys are more oriented towards technical aspects of science whereas girls tend to show more interest in human issues (Sjøberg & Schreiner, 2010). It is possible that there are different cognitive styles between genders, and the non-competitive context affected more to girls' style. To give some critique, one has to be reminded that making several t-tests may lead to misleading statistical significances raised just by a coincidence. However, what the girls benefit is in line with their needs, and the p-values were very near to $p < .01$.

This study has given us the insights of the possibilities and the restrictions an intervention may have. We continue to work with the rich data collected during the research project to contribute our knowledge of the development of mathematics-related affect in even more nuanced ways.

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Emma’s mathematical thinking, problem solving and affect

Hanna Viitala

University of Agder, Kristiansand, Norway, hanna.l.viitala@uia.no

This paper aims to understand one pupil’s mathematical thinking through problem solving and mathematics related affect. The results reveal a successful, though quite unsure, problem solver whose affective state (connected to problem solving) seems to tell the same story as her affective trait (view of mathematics). The differences between results on affective state and trait seem to be connected mostly to emotions.

Keywords: Mathematical thinking, problem solving, affect.

THEORETICAL FRAMEWORK AND RESEARCH QUESTION

Developing mathematical thinking is one of the three tasks of instruction listed in the Finnish National Core Curriculum for Basic Education (Finnish National Board of Education [FNBE], 2004, p. 158). Some aspects of mathematical thinking are evaluated through tests at school, at the national (e.g., Rautopuro, 2013; Hirvonen, 2012) and international (e.g., OECD, 2014; Mullis, Martin, Foy, & Arora, 2012) levels. However, we lack a deeper understanding of the mathematical thinking pupils’ take into their lives and possible further studies after comprehensive school.

Mathematical thinking is not defined in the curriculum, but a list of thinking and working skills is provided as evaluation criteria for every age group. These lists include learning objectives such as pupils’ use of logical elements in their speech, judging truth of simple propositions and noticing parallels and regularities between different events (FNBE, 2004, p. 166). For sixth to ninth graders, ‘thinking skills and methods’ is also introduced as its own entity in the learning objectives parallel to core contents such as algebra and geometry (Mathematics curriculum, ibid, pp. 158–167).

When thinking skills and methods are listed in the curriculum, problem solving is repeatedly referred to. The term ‘problem solving’ is not defined. However, in final-assessment criteria for ninth graders, four problem-solving phases are introduced. These phases are similar to Polya’s (1957) problem-solving phases. The process view on problem solving also guides this study where (in line with the curriculum) pupils’ activities, actions and explanations during problem solving are interpreted as visible signs or expressions of their mathematical thinking.

In this study, thinking is considered mathematical when it relies on operations that are mathematical (Burton, 1984). Furthermore, a mathematical task is called a problem if the solver has to combine previously known data in a new way to her to solve the task (e.g., Kantowski, 1980).

When mathematical thinking is described, in addition to cognitive aspects, we should also explore affective factors (e.g., DeBellis & Golding, 2006; Vinner, 2004) and seek to understand the interrelationship between affect and cognition (e.g., Hannula, 2011; Zan, Brown, Evans, & Hannula, 2006). Instead of categorizing affective factors for instance as beliefs, attitudes or values, affect is seen as a mixture of cognitive, motivational and emotional processes:

> Cognition deals with information (self and the environment), while motivation directs behaviour (goals and choices). Success or failure in goal-directed behaviour is reflected in emotions (e.g., shame). These emotions, in turn, act as a feedback system to cognitive and motivational processes. (Hannula, 2012, p. 144)

Affect is seen as a psychological domain with its state and trait aspects (Hannula, 2011). In connection to problem solving, we focus on rapidly changing affective states. The more stable affective traits follow the categorization of pupil's view of mathematics introduced by Pehkonen (1995; discussed also in Op't Eynde, de Corte &, Verschaffel, 2002). These categories are mathematics, oneself as a learner and user of mathematics, learning mathematics, and teaching mathematics.

In an aim to understand the interrelationship between cognition and affect in mathematical thinking, we look at one pupil's, Emma's, problem solving and explanations on affect related to mathematics. The information from this exemplary case can later be combined with other cases to form a more informative view on mathematical thinking at the end of Finnish comprehensive school. So, with the question '*What characterizes Emma's mathematical thinking?*' we try to understand the mathematical thinking Emma takes from comprehensive school into her life and further studies (cf. mission for basic education in FNBE, 2004, p. 12).

METHODS

The data was collected in three cycles. Emma's results from the first cycle are discussed in Viitala (2015). This paper adds both problem solving and affective results and reports on findings from all three cycles of data collection.

Participant

Emma is a high achieving girl (mathematics grade 9 on a whole number scale of 4 to 10) who was selected for this paper based on a previous report (Viitala, 2015). The data collection was organized in the first semester of 9th grade when Emma was 15 years old.

Data collection

The data was collected from mathematics lessons and interviews over the course of three months. In each of the three cycles, one mathematical task was solved in an ordinary classroom situation as a 'main task'. In Emma's case this meant that the pupils solved the tasks individually but they were allowed to talk about the tasks with a friend or ask for help from the teacher. In each of the three cycles, Emma was video recorded while she solved the task(s) in class and her solution on paper was collected.

The interviews took place either on the same day, or on the next day after solving the task in the classroom. The interviews contained two parts. The first part concentrated on affective traits and treated the following themes: pupil's background, mathematical thinking, and pupil's view of mathematics (following the categorization of Pehkonen, 1995; see example questions in Table 1). This part of the interview was semi-structured and focused (Kvale & Brinkmann, 2009).

The second part of the interview was about problem solving. The classroom data was used as stimuli when Emma's problem solving was discussed. Emma explained her actions and thinking and responded to questions such as, 'What are you thinking now?' and 'Why are you doing so?' Also, some affective and metacognitive questions were asked, for instance, 'What did you feel when you read the task?' and 'Did you think about your own thinking when solving the task?'

Finally, Emma was asked to assess her confidence before, during and after solving the problem(s), as well as her confidence in school mathematics using a 10 cm line segment (scale from 'I couldn't do it at all' to 'I could do it perfectly'; cf. estimation of certainty, e.g., in Merenluoto, 2001). All interviews were video recorded.

The tasks used in this paper are released PISA items. PISA tasks are well tested and based on real-life situ-

Theme	Example questions
Background	Tell me about your family.
Mathematical thinking Mathematics	What does mathematical thinking mean? / How do you recognise it? What is mathematics as a science? / Does it exist outside of school? (How? Where?)
Oneself and mathematics	Is mathematics important to you? / Does it help you think logically? (How?)
Learning mathematics	How do you learn mathematics? / Is it most important to get a correct answer?
Teaching mathematics	Does teaching matter to your learning? (How?) / What is good teaching?

Table 1: Interview themes and example questions

	Task	Given information	Why chosen?
Holiday, Q1	Calculate the shortest distance by road between Nuben and Kado.	Map of the area, Table of distances, Answer given in kilometres.	Complex situation, Combining different data
School excursion	Which (bus) company should the class choose, if the excursion involves a total travel distance of somewhere between 400 and 600 km?	Written explanation of the situation and rates that the bus companies charge.	Uncertainty, Decision making
Indonesia, Q3	Design a graph (or graphs) that shows the uneven distribution of the Indonesian population.	Table of the population of Indonesia and its distribution over the islands.	Open task

Table 2: Descriptions of some of the tasks used in the project

ations (e.g., OECD, 2006, p. 108). The 'main' PISA tasks, Holiday, School Excursion and Indonesia (ibid, pp. 77–78, 87, 111, respectively), were solved in the classroom. These tasks are open with respect to solution strategy (see task examples in Table 2). Indonesia also included two additional questions in which Emma's skill to read a table was tested (referred to as Indonesia Q1 and Q2).

In class, Emma also started to solve Carpenter (OECD, 2009, p. 111) which was added in case of extra time. In the interviews, Emma answered questions from the tasks Distance (OECD, 2006, p. 102; modified to 3 and 5 km), Growing up (OECD, 2009, p. 106; Q's 3 and 1) and Braking (ibid., pp. 128–129; also Q49 from a Web page of the Finnish Institute for Educational Research).

Analysis

The analysis was divided into two sections: Problem solving, and Affect related to mathematics. In problem solving, the main focus is on the cognitive problem solving process written in the curriculum for grades 6–9 as core content or final assessment criteria of thinking skills and methods. These phases follow the problem solving principles described by Polya (1957) and will be reported accordingly.

In connection to problem solving, pupil's thinking about their own thinking as well as control and self-regulation (e.g. keeping track of what is being done during problem solving) will be discussed as a part of metacognition (Schoenfeld, 1987). The pupil's motivation to solve the tasks as well as confidence and feelings during problem solving is reported as part of the psychological affective state.

In affect related to mathematics, pupil's view on mathematics is reported (see Table 1). These results can be referred to as affective traits. The discussion on metacognition concentrates on the aspects listed as learning objectives in the curriculum (e.g., trusting oneself; FNBE, 2004).

The results are descriptive. More information related to methods and methodology can be found in earlier publications (Viitala, 2013, 2015).

RESULTS

Problem solving

Understanding the problem. Emma uses much of her time for understanding problems and the given information (in class 2–5 minutes which is some 30–55 % of total solution time). She seems very thorough and she says that she wants to understand every aspect of a task before starting to plan and solve it. On one hand, this seems to be a key element in her success as a problem solver. On the other hand, this might hinder her to solve a problem (not understanding all the mathematical expressions in Braking Q49) or to give a correct answer (she was prone to give an answer which she can completely understand in Braking Q49).

Emma uses graphs (maps and diagrams from the tasks) to assist her thinking when putting the given information together (e.g., marking routes and distances to the map in Holiday), understanding a problem (e.g., the graph in Braking for Q49) making a plan to solve a problem (e.g., routes in Holiday) and reflecting on the task when solving it (e.g., connecting the question to the graph in Growing up). The use of tools and drawings that assist thinking is part of mathematics learning objectives.

Making a plan. After taking the time to understand a problem and given information, Emma does not need much time to make a plan. Making a plan seems to happen on the third reading of the question, after reading the question quickly through on the first reading and putting together the given information on second reading (according to her explanations, e.g., for Holiday Q2 and School excursion).

If a task feels hard, Emma says she thinks of alternative ways to solve it (in Holiday she calculated a second route to confirm her result). Additionally, if the task feels too simple, she might try to calculate the task further after getting the answer (Indonesia Q2, this calculation was later erased because 'it felt stupid').

Carrying out a plan. Emma seems careful and thorough in solving mathematical tasks. After understanding the task, Emma is fluent in transforming a text to a mathematical form (mathematical expressions e.g. in Holiday and School Excursion). She says she can return to the task description as a confirmation also in the middle of solving a task (e.g. in School excursion). She proceeds step-by-step with the tasks.

If Emma has different options to solve a task, she says that she chooses the one that feels more 'probable' (e.g., routes in Holiday) or has less doubt (e.g., choosing a point where to calculate School excursion). For a task that had more than one answer (Distance), she spontaneously found two answers and a third one after being probed. In most cases Emma was able to justify her actions and conclusions.

Looking back. Emma says she checks tasks only in tests. In the research project, the first class situation felt like a test situation for Emma and it was the only time she checked her answers (Holiday). Other answers to PISA tasks she checked from a friend (Holiday and School excursion) or left it until the interview. In addition, Emma feels that she does not need to check her calculations when a calculator is used.

If Emma is not sure whether she has understood the task or given information correctly, she chooses the interpretation that feels most reasonable and proceeds with that (e.g. table in Holiday, and Q1 and Q2 in Indonesia). This aspect of 'looking back' is done during the process of solving the task.

Affect related to problem solving. Emma feels unsure when faced a word problem. She might 'panic' if the task has a lot of text (Holiday Q2) or numbers (Indonesia), or she cannot understand all the given information (e.g. table in Holiday and mathematical expressions in Braking Q49). When she gets stuck with the task description, she seems to lack efficient tools to overcome the situation (Holiday Q2, Braking Q49, also visible in Emma's explanations about doing homework and learning mathematics; on getting stuck, see, e.g., Mason, 2015). In these cases, asking questions helps her to overcome the difficulties and proceed with the task.

Getting help (Holiday Q2, before solving the task) or asking the correct answer from a friend after solving the task (Holiday and School excursion) seems to have a direct influence on Emma's confidence. Similarly, not checking her answer (Indonesia) seems to make her feel very uncertain and anxious even in the interview (until the results were given). See Emma's confidence related to problem solving in Table 3.

Emma might experience many different feelings when facing, planning and solving a problem (e.g., in School excursion: nervous, unsure, doubtful, 'normal' and relieved chronologically, cf. Table 3) but she agrees that her feelings do not necessarily affect her more stable feeling of confidence. The main motivation for Emma to solve the given tasks was the video camera. However, when a mathematical obstacle was encountered, Emma was motivated to learn from it (e.g., not understanding some mathematical part of the discussion in an interview, such as graphs for companies in School excursion, or percentages when discussing Indonesia or Braking).

	Confidence *after reading* the task	Confidence *while solving* the task	Confidence *after solving* the task
Holiday, Q1	5.5	5	7.25
Holiday, Q2	3.75	6.25	7
School excursion	5	6.25	7.25
Indonesia, Q1-Q3	3.25	4	4

Table 3: Emma's confidence (0–10, ±0.25 mm) for the tasks solved in the classroom

Affect related to mathematics

Mathematics. Emma's view on mathematics is very much tied to a school subject. For her, mathematics is calculating, both as a school subject and as a science. Mathematical knowledge is gained by calculating and correctness of mathematical knowledge can be verified by asking the teacher. Emma thinks mathematics is useful and needed for instance in other school subjects (e.g. civics, physics and chemistry). When asked, it is hard for her to see connections between mathematics and real life. Emma uses mathematics outside of school when she is shopping.

Oneself and mathematics. Emma is motivated to learn mathematics. The feeling of success drives her forward and succeeding with a difficult task makes her feel proud of herself (intrinsic motivation). She values the opportunity to show her skills to her teacher and classmates by going to the black board to calculate tasks. It feels rewarding and it motivates her to learn 'the next thing' (extrinsic motivation).

Emma likes mathematics and thinks it is 'quite fun' and interesting. However, in her own words, she does not feel 'very confident' in mathematics (cf. Table 4). However, she thinks that this might be a good thing: If you are too confident, you might not use that much time for thinking or check the calculations of a task. Emma thinks that confidence and mathematics grades are two separate things. Her grade (9) is the best she thinks she can achieve.

Interview 1	Interview 2	Interview 3
6.25	5.5	5.5

Table 4: Emma's confidence (0–10, ±0.25 mm) in mathematics

Learning mathematics. Emma, together with her friends and family, values learning mathematics and thinks that mathematics is useful. She also agrees that the atmosphere with regards to mathematics in her class is positive. Emma seems to trust that if she studies mathematics she can succeed and get better in it. The feeling of success, the belief that mathematics is worthwhile and useful, and future studies motivate her to learn mathematics.

For Emma, learning mathematics is 'understanding', in addition to 'memorizing' and 'reasoning'. Understanding means you are 'able to use a method'. Learning as well as understanding takes time for Emma. Understanding comes from calculating, asking questions, and proceeding little-by-little from easier tasks to more difficult ones. Rote learning is important. Emma learns new things as independent issues and does not actively seek for connections to previous knowledge. She 'forgets things quite quickly' and an indication of this was seen also in the interviews (calculations with percentages).

Emma's feelings in learning mathematics are also closely connected to understanding. Learning mathematics is fun when she understands or succeeds in mathematics. Not being able to understand is irritating. Sometimes learning mathematics is also tiring. Mathematics is easy for her 'but only a little' (cf. Table 4). Emma seems to take the responsibility of her own mathematics learning. She says that if she succeeds with a test, it is because she has studied for it and learned in class. Failure, on the other hand, means that she has not done enough work.

Teaching mathematics. Mathematics teaching methods and the mathematics teacher play a great role in Emma's learning of mathematics. She believes that without teaching she could not learn mathematics. A good mathematics teacher offers opportunities to ask questions, gives time for (rote) learning, does not proceed too quickly to the next thing, and proceeds from easier tasks to more difficult ones. These all are features that Emma's current mathematics teacher seems to possess (according to Emma's explanations and the researcher's observations from the classroom).

In some respect, Emma seems to connect her feelings and success in mathematics to her teacher. In elementary school she did not get along with her teacher. She was an average pupil (grade 7–8) who was not interested in mathematics, did not succeed in it and mathematics felt like torment for her. In lower secondary school Emma got a new mathematics teacher who she liked, and whose teaching she liked. Since then, Emma says she has liked mathematics and been a high achiever.

SUMMARY AND DISCUSSION

Results from Emma's problem solving and affect related to mathematics seem to give a well-matching picture of Emma's mathematical thinking. Emma is a reflective learner and problem solver who needs time for understanding. Her thoroughness and tendency to ask questions (both from friends and the teacher) seem to be the key to her success in both respects. Emma is not very confident in mathematics or problem solving (though slightly positive, cf. Hirvonen, 2012) but, as a consequence, she seems to be very careful with her thinking and working. Moreover, her uncertainty might be a reason for her success both in problem solving and mathematics.

The results showing differences between problem solving and affect related to mathematics seem to be connected to less stable affective traits. As an example, Emma's confidence has more variance in problem solving than in mathematics and her feelings experienced during problem solving have more tendencies to negative feelings (e.g. unsureness) than her feelings in learning mathematics. What is notable regarding Emma's affect in mathematics is that, contrary to previous research results (e.g., Tuohilampi, Hannula, Laine, & Metsämuuronen, 2014), Emma's feelings towards mathematics have become more positive since elementary school.

Throughout the study, Emma worked in a sustained and focused manner with the problems. Even though she is not very confident in mathematics, she seems to trust herself as a mathematics learner (e.g., aiming to learn more mathematics so she can succeed in advanced mathematics in upper secondary school). She also seems to take responsibility for her own learning (e.g. reasons for succeeding or failing in tests). All these aspects (listed in the curriculum; FNBE, 2004) together with her problem solving skills seem to offer her a solid foundation for future studies.

In addition to preparing pupils for further studies, basic education must also provide opportunities to obtain the knowledge and skills pupils need in life (FNBE, 2004, p. 12) and mathematics teaching should help pupils to see the connection between mathematics and real life (ibid, p. 158). Nonetheless, even though the PISA tasks that were used were situated in the real world, Emma saw them purely as mathematics tasks. Additionally, she struggled to see where she uses mathematics in her own life outside of school (homework, shopping). After analysing all the cases in the project, we can see if this might be a possible trend among Finnish pupils.

The upcoming curriculum (FNBE, 2014; will be implemented in 2016) draws more attention to mathematical thinking and real-life connections. For instance noticing connections between learned concepts and applying mathematics in other school subjects and surrounding society are written as individual learning objectives and as final-assessment criteria (ibid, p. 433–434). It is hoped that this will direct pupils' attention more towards their thinking and connections between mathematics and real life, at the same time making mathematics more worthwhile and enjoyable.

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Investigating sources of pre-service teachers' self-efficacy for preparing and implementing mathematical tasks

Bilge Yurekli[1], Mine Isiksal[2] and Erdinc Cakiroglu[2]

1 Gazi University, Ankara, Turkey, bilgeyurekli@gazi.edu.tr

2 Middle East Technical University, Ankara, Turkey

The goal of this study was to understand pre-service teachers' self-efficacy beliefs for preparing and implementing mathematical tasks and to explore which sources of self-efficacy they wighted and interpreted when judging their capabilities as a result of attending a mathematics teaching methods course designed with a focus on mathematics teaching through tasks. Nine pre-service teachers participated in this study and they were interviewed at the end of the methods course. Findings revealed that methods course provided opportunities for mostly vicarious experiences and physiological states sources, and participants were feeling highly efficacious to prepare mathematical tasks, but they had doubts about their capabilities to implement tasks effectively.

Keywords: Mathematical tasks, pre-service teachers, self-efficacy, sources of self-efficacy.

INTRODUCTION

Learning mathematics with understanding requires active student involvement in the process of learning *through* problem solving where students solve problems, evaluate and explain their solutions, make connections among concepts, create and use different representations of mathematical ideas, and communicate their mathematical thinking (National Council of Teachers of Mathematics [NCTM], 2000). Because problem solving is the center of this learning process, the problems (i.e., mathematical tasks) that are presented to students mainly determine the quality of learning environment in classroom and the level of students' understanding (Stein, Smith, Henningsen, & Silver, 2009). A mathematical task is a problem (or a set of problems) "the purpose of which is to focus students' attention on a particular mathematical idea" (Stein, Grover, & Henningsen, 1996, p. 460). As research showed, to prepare and implement challenging tasks which promote students' higher-order thinking skills, teachers should have the required knowledge (Sullivan, Clarke, & Clarke, 2009). However, having necessary knowledge not always guarantee successful performance because self-efficacy is a strong determinant of performance (Bandura, 1997).

Researchers have reported that teachers' self-efficacy beliefs, or teachers' beliefs in their capabilities to bring about students' learning and achievement (Tschannen-Moran & Woolfolk Hoy, 2001), have the power to influence their instructional behaviors, attitudes toward teaching, their classroom management, as well as their students' motivation, achievement, and self-efficacy (Caprara, Barbaranelli, Steca, & Malone, 2006; Pajares, 1992; Woolfolk, Rosoff, & Hoy, 1990). When teachers hold strong efficacy beliefs, they spend more time with students who have difficulty in learning and try to improve their teaching; teachers with doubts in their capabilities, on the contrary, tend to experience burnout, job dissatisfaction, and leave the profession (Caprara, Barbaranelli, Steca, & Malone, 2006; Klassen, & Chiu, 2010). Thus, it is essential that teachers have strong beliefs in their capabilities to prepare mathematical tasks and enact them effectively with students.

Because of its crucial role in effective teaching, teachers' self-efficacy beliefs were studied in various domains, such as science and mathematics. However, studies were mostly focused on inservice teachers' self-efficacy (Klassen, Tze, Betts, & Gordon, 2011). Since efficacy beliefs are most likely to change during skill development (Bandura, 1997), it is important to study pre-service teachers' self-efficacy and find ways to help them develop strong beliefs in their capabilities through teacher education programs. According to Bandura (1997), self-efficacy is developed through four different sources: Mastery experiences, vicarious experiences, social persuasions, and physiological states. Mastery experience, also the most powerful source of self-efficacy, is the information gained from personal performances where success boosts self-efficacy and failures undermine it. Vicarious experience refers to the information gained from observing or comparing oneself with model's performances. Model similarity determines the effect of this source on self-efficacy, the greater the assumed similarity is, the more persuasive the model's success or failures are. Social persuasion is the feedback received from others about personal capabilities. Finally, physiological states such as stress, anxiety, and mood during performances provide information for self-efficacy development (Bandura, 1997).

In order to help pre-service teachers build strong efficacy beliefs, it is necessary to understand factors that serve as sources for their self-efficacy. However, there is a lack of research on these sources because researchers have often been interested in exploring correlates and outcomes of teachers' self-efficacy (Klassen, Tze, Betts, & Gordon, 2011). Based on hypothesized sources of self-efficacy (Bandura, 1997), there were a few quantitative attempts (e.g., Poulou, 2007) which failed to examine the predictive power of each of the four self-efficacy sources. The quantitative nature of these studies did not allow for the exploration of which sources could be provided by teacher education programs, either. A qualitative approach, on the other hand, can contribute more to our knowledge of how factors related to teacher education program operate through sources for pre-service teachers' self-efficacy. This way, teacher educators can also be provided with a clear guideline for improving their programs to support self-efficacy of future teachers.

Concerned with the gap in the literature on sources of pre-service teachers' efficacy beliefs, this study was designed as an attempt to explore sources of pre-service elementary mathematics teachers' self-efficacy in the context of a mathematics teaching methods course. The focus of this study was on pre-service teachers' self-efficacy beliefs for preparing and implementing mathematical tasks, which was defined as their beliefs in their capabilities to prepare (select, adapt, modify, or create) challenging mathematical tasks and enact these tasks in classrooms effectively.

METHOD

The present study was aimed at exploring sources of pre-service teachers' self-efficacy for preparing and implementing mathematical task after completing a mathematics teaching methods course. In order to reveal the insights of pre-service teachers upon enrolling in the methods course, qualitative case study method was employed. The context of this case study was Elementary Mathematics Education Program, more specifically, the methods course offered by this program at a large public university[1] in Ankara, Turkey. Nine out of 46 junior pre-service teachers studying in this program constituted the cases.

The methods course

As a part of Elementary Mathematics Education Program, pre-service teachers were required to enroll in the mathematics teaching methods course in their third year. This two-semester-long methods course was designed to help pre-service teachers learn to teach mathematics through challenging mathematical tasks, as well as to use and prepare manipulatives and integrate technology in mathematics education. The focus of this course was on both the Elementary Mathematics Curriculum (Ministry of National Education, 2013) covered in Turkey and NCTM Principles and Standards (NCTM, 2000). Methods course was taught by an associate professor for 14 weeks per semester and took place twice a week. On every Monday, there was a meeting for lectures on a chapter from the textbook, *Elementary and Middle School Mathematics: Teaching Developmentally* (Van de Walle, Karp, & Bay-Williams, 2010). Pre-service teachers were expected to read assigned chapters every week before attending lecture meetings. There were also several unannounced quizzes prior to these meetings. Two to three questions related to the weekly subject were asked in each quiz. After this two hours of theoretical study (i.e. lectures), on every Wednesday, there was a lab meeting for pre-service teachers' presentations of tasks they prepared about that week's topic. They worked in groups of 5–6 to create tasks and enact their tasks in the lab with their classmates. The instructor was giving feedback on

1 At this university, English was used as the medium of instruction.

each group's work, following their implementations. Pre-service teachers were expected to provide feedback to their peers as well. There were also a midterm and a final exam.

Participants

Nine junior pre-service teachers accepted to participate in this study. Eight of the participants were women and one was man. Required courses pre-service teachers had to complete throughout the program included mathematics content courses (e.g., calculus, analytic geometry), courses related to educational sciences such as educational psychology and classroom management, as well as Turkish, English, and basic physics courses. Elective courses on mathematics education were also offered by both Elementary and Secondary Mathematics Education programs. Participants were going to enroll in field experience courses in their last year as senior pre-service teachers.

Data collection and analysis

Data were collected through semi-structured interviews which were mainly guided by a list of questions prepared for pre-service teachers to gain an understanding of their perceived efficacy beliefs (e.g., *How confident are you in your capabilities to prepare challenging mathematical tasks effectively?*) and to provide deep information about the sources of their self-efficacy (e.g., *Which components of methods course made you feel confident about your capabilities to prepare challenging mathematical tasks effectively?*). The first author joined class meetings throughout the methods course to build trust with participants and interviewed participants at the end of the second semester of methods course. Each interview lasted around 30 minutes. Interviews were first audio-recorded and then transcribed by the first author as well.

Data were coded through constant comparative analysis method (Merriam, 2009), and data analysis process was two-fold. First, data related to participants' efficacy judgments for preparing and implementing tasks were coded. *Self-efficacy for preparing tasks* and *for implementing tasks* were the two categories used in this part of analysis because participants regarded their efficacy beliefs for preparing and using tasks separately. Then, data related to the efficacy-relevant information which participants referred to when judging their capabilities were coded using hypothesized sources of self-efficacy (Bandura, 1997). That is, *mastery experiences* (when participants talked about the efficacy-relevant information gained through their own performances, like in group work), *vicarious experiences* (when vicarious learning occurred, like through observing peers' presentations), *social persuasion* (when participants mentioned the effects of feedback, like the feedback they received from the instructor on group work), and *physiological states* (when emotions and mood of participants were perceived as sources of self-efficacy, like having fun while creating tasks) were the categories used for coding the sources of participants' self-efficacy beliefs. Subcategories were created regarding components of methods course (i.e. lectures, group work, peers' presentations, feedback on group work, textbook, unannounced quizzes). A mathematics education researcher participated in coding process. A coding sheet was generated and both researchers (the first author and the coder) coded a randomly selected interview transcript. When results of each researcher's codes were compared, a 92% coder agreement was reached.

FINDINGS

Findings revealed that participants completed the methods course mostly with high sense of efficacy for preparing and implementing mathematical tasks. More specifically, 8 of the participants expressed strong self-efficacy for preparing tasks (e.g., "At this point, I feel really really efficacious because, like I said, we have activities about almost every subject in the curriculum, we prepared all of them," Participant 8 [P8]) and one expressed moderate self-efficacy ("I feel so-so [confident], I cannot claim that I can prepare real good activities," P5). Five participants were feeling strongly efficacious for implementing tasks, whereas 4 of them had doubts about their capabilities to implement tasks effectively (e.g. „I wish we had more chance to implement [tasks], so, like I said, I have doubts about putting [tasks] into practice," P1). Participants who were concerned about possible classroom management issues and felt the need for real classroom practices expressed less confidence in their capabilities to implement tasks than their counterparts.

Findings regarding the sources of pre-service teachers' self-efficacy showed that participants used efficacy-relevant information provided by all four hypothesized sources of self-efficacy as a result of their enrollments in the methods course.

Mastery experiences

The methods course was found to support participants' efficacy development by providing mastery experiences through lectures, group work, and unannounced quizzes. Participants expressed two ways of mastery experiences that lecture hours provided. First, the instructor used questioning method where questions were aimed at promoting participants to explain how they would enact tasks from the textbook with their future students and to generate ideas for (accommodation or modification of) those tasks. This method nurtured participants' thinking to enhance and master their knowledge for preparing and implementing mathematical tasks, and participants' successful performances boosted their self-efficacy. Second, the expectations of the instructor from participants affected the level of effort they put forth, and the amount of effort participants expended influenced their self-efficacy.

As a source of mastery experience, participants expressed positive effects of working as a group to design mathematical tasks and implement those tasks in class during lab hours. However, one participant had problems with her group members and she didn't believe that group work contributed to her efficacy development.

When judging their capabilities, participants referred to their performances in unannounced quizzes they had to take before lectures. Findings showed that these quizzes operated through mastery experience source for participants' efficacy beliefs.

Vicarious experiences

Vicarious experiences as another source for self-efficacy of participants were provided by lectures, group work, peers' presentations, and the textbook. The effect lectures created on participants' self-efficacy was perceived as vicarious experience source, where the instructor was transmitting her knowledge and skills. The instructor's lecturing with an emphasis on the features of challenging mathematical tasks and "tips" to use such tasks effectively in classroom context were believed to positively affect participants' efficacy development.

Group work was regarded as a different vicarious experience source. Working as a group was perceived as a more positive influence than working on a task alone, for it gave participants the chance to learn from each other vicariously.

As findings revealed, participants viewed their peers' presentations in the lab as a learning source which contributed to their self-efficacy vicariously. Their peers were also models to whom participants referred when judging their own capabilities. Thus, peers created the opportunity for social comparison. When participants compared themselves with their counterparts, they mainly focused on competent ones. Such successful examples motivated participants to do better.

Additionally, the textbook was perceived as a symbolic model for learning and operated through vicarious experience source. Reading prior to lecture hours also increased participants' understanding from lectures. Participants stated that they could benefit more from lectures, since they were prepared for the class. Participants appreciated the quality of the book, too; yet they experienced trouble with reading it. First, the book was written in English and it required extra time to finish reading a chapter than reading any text written in Turkish. Second, participants thought the second semester's readings were longer and more complicated, which took more time to complete reading assigned chapters and caused them struggle to understand. Thus, participants sometimes did not even finish reading before attending lectures.

Social persuasions

Feedback on group work which participants received during methods course operated through social persuasion source for participants' self-efficacy. At the end of each group's presentation in the lab, pre-service teachers and the instructor provided feedback about each group's work. These feedback (i.e. evaluative feedback) contained information about their capabilities and were perceived as a way of self-assessment. Among the three sources of feedback (i.e. the instructor, group members during group work, and peers during lab hours), the instructor was found to be the most effective because, with her knowledge and experiences, she was more credible. However, participants thought the instructor was being judgmental from time to time.

In addition to feedback provided during lab hours, participants regarded lectures as a social persuasion source. Recall that participants uttered the difficulty of reading long and complicated chapters. Thus, they believed that readings should be supported by lectures, since they sometimes misinterpreted the information in the textbook or sometimes "didn't even understand what the book was saying" (P3). This way, lectures worked as corrective feedback, a social persuasion source for self-efficacy which helped participants learn from their mistakes and improve their knowledge.

Physiological states

Findings showed that the methods course provided physiological states source for participants' efficacy beliefs through group work, peers' presentations, feedback on group work, textbook, and unannounced quizzes. First, group work operated through physiological states source, and participants interpreted their emotions and mood during preparing and implementing tasks when judging their capabilities. Participants mostly enjoyed working together in a group to create and present tasks. On the contrary, one participant uttered that she felt anxious when presenting tasks in the lab, but she was still confident that she would enact tasks with her future students effectively because she believed she would feel more comfortable with students.

Participants expressed positive affective states like having fun when working on peers' tasks, too, but they got bored and showed reluctance to participate when they were assigned to low quality tasks. According to Bandura (1997), positive moods enhance self-efficacy and negative emotions lower it. Since participants did not experience negative emotional arousal (e.g. getting sad, angry, or stressed), it could be asserted that peers' presentations mostly contributed to participants' self-efficacy.

The feedback received from the instructor were perceived as a negative influence for some participants. Like mentioned earlier, there were participants who thought that the instructor could be judging and criticizing. Therefore, when preparing tasks, some participants were affected negatively from the instructor's criticism and either doubted their capabilities or experienced negative arousal (e.g. anxiety).

As a physiological states source, reading the textbook caused boredom for one participant. She explained that she was bored with reading in English.

Even though unannounced quizzes were viewed as a "push" for studying for lectures and in turn actively participating on Mondays, they also caused stress and anxiety. When participants didn't have time to complete reading assigned chapters or when they failed to understand ideas presented in the textbook, they felt distressed because these quizzes were added up to overall grades.

DISCUSSION

In this study, the aim was to investigate sources of pre-service elementary mathematics teachers' efficacy beliefs for preparing and implementing mathematical tasks. In addition to detecting which sources pre-service teachers relied on to construct their efficacy beliefs, this study was an attempt to define components of methods course which predicted pre-service teachers' self-efficacy. Findings revealed that methods course provided all four sources for self-efficacy beliefs of pre-service teachers. That is, various components (i.e., lectures, group work, peers' presentations, feedback on group work, textbook, and unannounced quizzes) of the course operated through different sources and contributed to the development of pre-service teachers' efficacy beliefs. When compared to other sources of self-efficacy, components of methods course mostly operated through vicarious experiences and physiological states sources for self-efficacy. It could be that participants weighted these two sources of self-efficacy more than mastery experiences, since their mastery experiences were limited to group work, involvement in lectures, and unannounced quizzes.

An important finding was that lectures had the power to operate through mastery experiences for pre-service teachers' efficacy judgments, when the instructor used questioning method in her lectures. Participants of this study mainly perceived lecture hours as a vicarious experience source, like one would expect because lecturing is the transmission of knowledge and skills (Bandura, 1997) from the instructor to pre-service teachers, but the instructor was able to take this further by using questioning method. Questioning promoted participants' thinking and generating ideas about preparing and using mathematical tasks, and participants relied on experiences gained through cognitively enacting and verbally sharing their ideas. Teacher educators then should design courses where pre-service teachers are encouraged to actively participate in lectures and share their thoughts on preparing and implementing tasks.

Modeling is a strong influence on self-efficacy (Bandura, 1997), and findings showed that participants gained vicarious experiences through models similar to them (i.e. peers), the instructor as a competent model, and the textbook as a symbolic model. Therefore, teacher educators should promote vicarious learning of pre-service teachers from each other, which might be achieved by letting them work in groups or asking them to evaluate peers' tasks and give feedback. Classroom discussions might also create the environment for pre-service teachers to contribute to peers' learning. Teacher educators should also share either their own experiences or other teachers' experiences on preparing and implementing tasks, by using video recordings for example, to provide pre-service teachers with various vicarious learning opportunities. The quality of textbook used is also found to be an important issue that teacher educators should take into account. Task samples in the textbook and clear and detailed information proposed there can help pre-service teachers gain experiences vicariously and contribute to their efficacy development. Reading chapters as a requirement of methods course, as findings showed, might enable pre-service teachers to benefit more from the lectures. Thus, it is important to motivate them do the readings prior to lectures. One way of providing such motivation could be unannounced quizzes, like findings showed. Yet, quizzes might cause stress and anxiety, if pre-service teachers have difficulty in completing reading assignments. Choosing a textbook easy to read and understand might prevent pre-service teachers from experiencing negative affective states.

Findings also revealed the crucial role of physiological states in the development of pre-service teachers' efficacy beliefs. Even though participants talked about mainly positive emotions and moods, when faced with criticism, they experienced negative affect. Bandura (1997) stated that "moderate levels of arousal heighten attentiveness and facilitate deployment of skills" (p. 108). Teacher educators, therefore, should be cautious in controlling the level of arousal pre-service teachers' experience. Creating a classroom environment in which pre-service teachers will feel comfortable and voice their thoughts without being criticized can boost their self-efficacy.

Upon completing methods course, almost all of the participants (8 out of 9) regarded themselves strongly efficacious for preparing tasks, whereas only 5 of them expressed high level of efficacy for implementing tasks. Participants stated that they had doubts about classroom management and in turn they didn't express strong efficacy for implementing tasks. From this aspect, the lack of mastery experiences in actual classroom contexts might have caused lower level of development in pre-service teachers' self-efficacy. As Bandura (1997) asserted, mastery experience is the most powerful source of self-efficacy. Thus, offering fieldwork experiences as a part of methods course might help pre-service teachers to benefit more from this course.

The importance of this study is its attempt to describe how methods course provided sources for pre-service teachers' self-efficacy than simply defining which sources pre-service teachers used to develop their efficacy beliefs, as in previous research (e.g., Poulou, 2007). However, because this study is limited with 9 pre-service teachers, examining sources of pre-service teachers' self-efficacy with a larger sample can reveal how different elements of methods course support pre-service teachers' efficacy development. One way of achieving this can be using open-ended questions, and findings of this study might help researchers with the design of such questions.

Longitudinal studies can also show how pre-service teachers construct their efficacy beliefs over the time of methods course and the ways sources for their self-efficacy interact. Such an approach can also disclose the influence of other courses and experiences pre-service teachers engage in outside of methods class.

Considering the power of physiological states on self-efficacy, a broader approach to the investigation of pre-service teachers' efficacy beliefs as perceptions of competence (e.g., Coppola, Di Martino, Pacelli, & Sabena, 2012) and the role of emotions on the construction of self-efficacy can illuminate the pathways to support future teachers' beliefs through teacher education programs.

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TWG08

Posters

Teachers' perspective on group dynamics

Chiara Andrà, Domenico Brunetto, Nicola Parolini and Marco Verani

Polytechnic of Milan, Department of Mathematics, MOX, Milan, Italy, domenico.brunetto@polimi.it

In groupwork activities, interaction plays a crucial role. We focus on two affective variables that frame interaction: one's own perceived competence ("I can") and the competence that one recognize to the others ("you can"). In this poster, we show different interpretations of mathematics secondary school teachers about students' "I can" and "you can" from observations of video excerpts of students working in groups. We discuss agreement/disagreement among teachers' interpretations.

Keywords: Group interaction, teacher noticing, perceived competence.

Interactionist research acknowledges that learning in mathematics occurs in and through interaction, which is framed by the students' "I sense" and "I can" (Andrà & Liljedahl, 2014). "I can" may prompt the student to intervene in a conversation. "I can't" may push a student to stay silent (see Andrà & Liljedahl, 2014). Each student may (or not) recognize a competence to each one of her classmates. Therefore, there is another dimension: "you can". "I can" and "you can" are seen as interior states, whilst utterances, postures, etc... are the external expressions of such internal states.

The teacher also plays an important role, intervening and driving the activity (Radford, 2014). Since emotions take a significant part in determining the outcome of the activity, it is crucial for the teacher to detect them and to react accordingly. Specifically, we aim at examining teachers' noticing (Sherin, Jacobs, & Philipp, 2011) with respect to the "I can"-"you can" frame. Our hypothesis is that becoming conversant with this frame allows the teacher to better ground her meaningful actions/interactions within the complexity of group activities. For example, in the case of a non-cooperating leader, it would be necessary to provide feedbacks that temporarily decrease her "I can" and increase her "you can"; indeed, in the case of a student with low "I can", it is important to provide feedbacks that increase her self-confidence, but also increase her mates' sense of "you can" towards her.

Andrà, Brunetto, Parolini, & Verani (this volume) analyse a video excerpts (from a probability course to prevent gambling abuse), codified as follows: a point corresponding to the student under analysis is placed on the one of the four cells of the 2x2 table representing the internal states "I can" – "I can't", "you can" – "you can't". Dwelling time is represented by a circle: the longer the time, the bigger the circle. Transitions are represented by oriented arrows. The result is a trajectory.

The excerpt comes from a group of 4 grade-10 (15 years old) students: Alice (A), Barbara (B), Carola (C), and Dora (D). They are analysing a slot machine (3 rolls, 9 symbols), and have to compute the expected winning. B starts as cooperative-leader ("I can"–"You Can"), then she becomes leader, suddenly she is not able to come up but she trusts her classmates, eventually she comes back to collaborate with her classmates on the task.

We firstly observe a group of teachers interpreting the same data presented in Andrà and colleagues (this volume) and asking them to fill in a 2x2 table like the one presented in Figure 1. We collected also teachers'

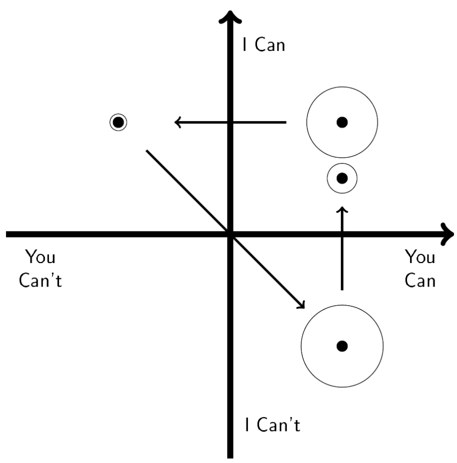

Figure 1: B's trajectory

extended comments. We seek for agreement/disagreement with respect to our interpretations, focusing only on one student per group as an initial step.

There is a certain degree of agreement among teachers, and between teachers' interpretations and ours. In the poster we show some data that allow us to claim that collecting the teachers' views according to the "I can" – "you can" frame goes beyond the purpose of validating a methodology: it is a way of observing, collecting and giving sense to affective moves that may drive the teachers' decisions and behaviour in classroom. There are emotional issues that emerge with respect to the teachers' perceived expectations in the "management" of group activities, as well as their specialised knowledge of group dynamics.

ACKNOWLEDGEMENT

Project funded by Politecnico di Milano through the 5x1000 Polisocial Award 2013 in collaboration with Fondazione Politecnico di Milano.

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Attitudes of secondary school students towards work in learning situations

María del Socorro García González[1] and Rosa María Farfán Márquez[2]

National Polytechnic Institute, Center of Research and Advanced Studies, Cinvestav-IPN, Ciudad de México, Mexico, mgargonza@gmail.com, rfarfan@cinvestav.mx

The purpose of this paper is to characterize the attitudes of Mexican secondary school students (12–15 years old) when carrying out mathematical activities. In order to study attitude, the tripartite model of attitude was adopted and a methodological design was organized based on the adapted model. So far a learning situation focused on the subject of proportionality has been applied, in which we have been able to identify two attitudes: Acceptance and collaboration.

Keywords: Attitude, secondary school, characterization.

INTRODUCTION

Over the years, research regarding attitudes towards mathematics has grown and has studied students ranging from basic level to higher education students and teachers from the same levels. Acknowledging, the relevant role that attitude plays in the teaching of mathematics (Di Martino & Zan, 2010; Gómez-Chacón, 2000; Hannula, 2012; McLeod, 1992), our aim was to characterize the attitudes of a group of secondary school students in relation to proportionality. We considered attitude to be like the assessment carried out by a student of the resolved mathematical activity. We adopted the tripartite model (Rosenberg & Rovland, 1960), which considers attitude to be composed of emotions, beliefs and behaviors.

PROCEDURE FOR DATA COLLECTION

The learning situation regarding proportionality was carried out during a first year secondary school mathematics class. 28 students participated, working in teams. The students' activity was filmed. In order to carry out individual interviews regarding the work carried out and some aspects of a personal and academic nature, 3 teams were selected: a) 2 men, b) 3 women and c) 1 man and 3 women.

RESULTS

We identified a system of attitudes formed of two attitudes: 1) Acceptance of the activity and 2) Collaboration between classmates. Both formed a collaborative work system and were coherent with and related to each other. They were linked to emotions, beliefs, behavior and factors shown in Table 1.

The extent to which attitudes identified throughout the activity were manifested, increased and decreased. There were cases in which, at the beginning of the activity, the level of collaboration was very low, but

Emotional reactions	Beliefs	Behaviors	Attitudes	Associated factors	Associated agents
Shock Confusion Happiness	Problems containing few operations are easy. Division problems require information regarding what is being divided and between how many people it is being divided.	Empathy between classmates Willingness to work as a team	Acceptance Collaboration	Learning contract Goals of the student Interests of the student	Parents Classmates Teacher

Table 1: System of identified attitudes

thanks to interaction between classmates, the level increased, students managed to get involved in the work and to resolve the activity. By carrying out interviews, we discovered that there were agents that indirectly influenced the attitudes of students including classmates who helped to solve problems, the teacher who assessed them and parents for whom mathematics was important and who demanded that their children obtain good grades, even better than for other curriculum subjects.

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Formative scaffolding: How to enhance mathematical proficiency, prevent and reduce mathematics anxiety

Annika Grothérus

Linköping University, Department of Social and Welfare studies, Linköping, Sweden, annika.grotherus@liu.se

I present the Formative scaffolding approach, a method for teaching, evaluation and assessment in mathematics, describing a study, where 22 upper-secondary social science students' perceptions and experiences of using formative scaffolding with respect to a test in mathematics are explored. Results indicated that formative scaffolding might reduce mathematics anxiety and enhance mathematical proficiency. Students emphasise the opportunity for a second chance and that the learning process is visualised.

Keywords: Formative assessment, scaffolding, mathematics anxiety, proficiency.

In this poster a method for teaching, evaluation and assessment in mathematics is presented. The method is referred to as the formative scaffolding approach, see Figure 1, in which the concepts of scaffolding (Wood, Bruner, & Ross, 1976), formative assessment (Black & Wiliam, 2009) and writing to learn (Kågesten & Engelbrecht, 2006) are intertwined with each other (Shepard, 2005). Learning comprises cognitive, social, emotional and cultural embedded processes and involves construction of knowledge (Bransford, Brown, & Cocking, 1999; Shepard, 2005). Allowing learners to actively participate in constructing knowledge may provide them with a deeper understanding, more self-confidence and motivation in using their knowledge (Smith, Maclin, Houghton, & Hennessey, 2000). Enabling learners to actively participate in constructing their knowledge and visualize their learning is a fundamental core in formative scaffolding. The overall objective of this study is to present and

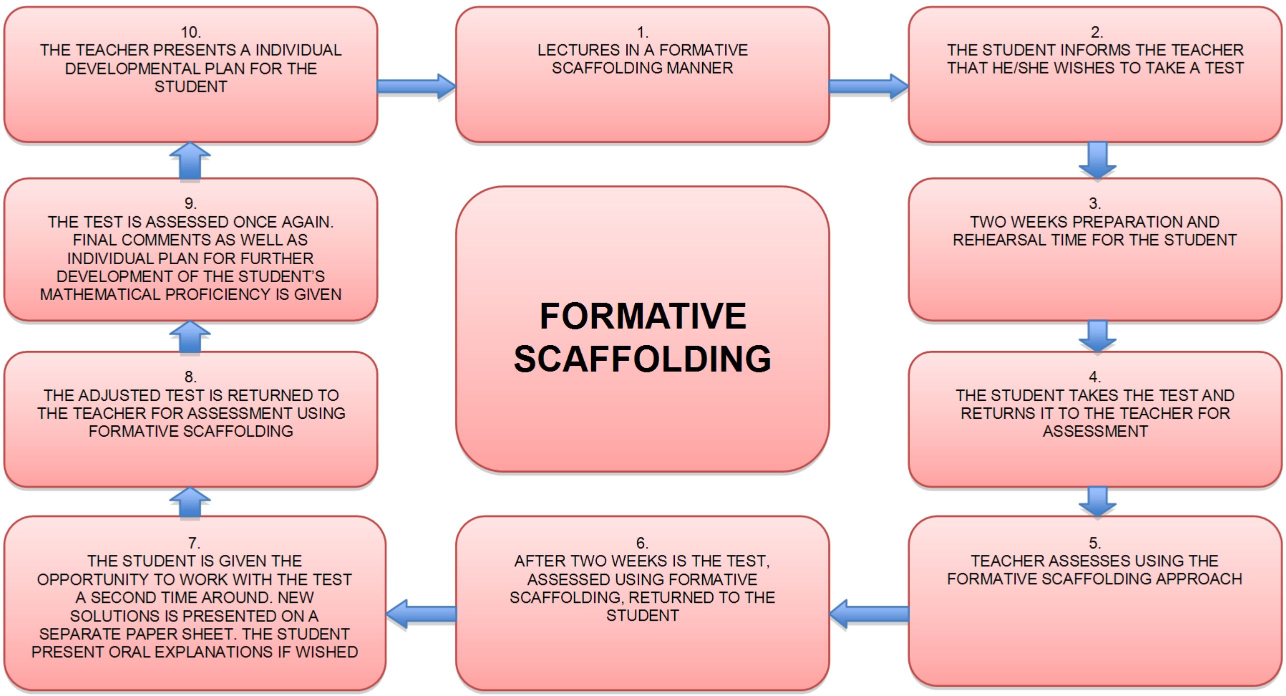

Figure 1: Overview of the formative scaffolding approach process

describe the teaching, assessment and evaluation method – formative scaffolding, see Figure 1 – and explore twenty-two social science students (age 17–18 years) experiences of the method with respect to mathematical proficiency and anxiety in mathematics. The participants constituted one class in upper secondary school. Through an exploratory action research study we investigated: experiences of formative scaffolding in relation to ordinary test in mathematics; effects of formative scaffolding in relation to mathematical proficiency and mathematics anxiety. Possible limitations in the design of the study was that the class it self was acting as a control group. The present study is to be viewed as a pilot study providing important information about the method as such and how to design a larger future study. Data were collected through short written narratives and one Likert type structured question. Students written responses and indication on the structured question were analysed using thematic coding analysis (Boyatzis, 1998). Emerging themes and illustrative quotes of the students' experiences is going to be presented. Results indicated that formative scaffolding might reduce mathematics anxiety and enhance mathematical proficiency. Students emphasise the opportunity for a second chance and that the learning process is visualised. We suggest that the proposed formative scaffolding approach may be added to the list of other potential tools for learning, and, this approach can be used to make summative tests in mathematics to an additional opportunity for learning.

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Why mathematics? Students' narratives about failing in compulsory school mathematics

Oana Hansson

Malmö University, Faculty of Education and Society, Malmö, Sweden, oana.hansson@mah.se

BACKGROUND

This study is concerned with students on an introductory programme where they are supposed to compensate for earlier failure in mathematics in compulsory school. Having worked for several years with students in this category, I have noticed that some of them have passed in other subjects where mathematics is used as a tool. The question then becomes: Why mathematics? Thus, the focus is on students who have passed in other subjects, including physics and chemistry, in compulsory school, but not in mathematics.

According to official statistics, 8.6 % of all students who finished the Swedish compulsory school in 2011 failed to pass in mathematics (Skolverket, 2012). The Swedish compulsory school consists of nine years of schooling. Students who have received a pass in Swedish, English and Mathematics in addition to certain other subjects are eligible for a three-year upper secondary school education. For students who are not eligible, there are five introductory programmes, which lead either to further study or to work.

Generally, the frequency of low achievement in mathematics increases during compulsory school age (Magne, 2006), which makes it interesting explore what goes wrong in those years. As Magne (2006) points out, there is not much research on students with low achievement in mathematics.

In order to research low achievement in mathematics, it seems relevant to consider students' perceptions of mathematics teaching. According to Samuelsson (2007) students' feelings range from satisfaction to anxiety and fear. Gierl and Bisanz (1995) describe mathematics anxiety as a lack of enjoyment in mathematics. When a student with mathematics anxiety is faced with the requirement to work with mathematics, a chain of reactions like panic, frustration, paralysis and mental disorganization could appear (Foire, 1999; Bandalos, Yates, & Thorndike-Christ, 1995). According to Nyroos, Bagger, Silfer and Sjöberg (2012), the presence of test anxiety is significantly present as early as in the Third Grade.

Reasons for students' mathematics anxiety could be the activities that they are supposed to do (Greenwood, 1984) or the teacher's ability to create a positive learning climate (Samuelsson, 2008) or boredom (Ingram, 2009). The study seeks to contribute to research by focusing on school mathematics as the single subject where the student has not passed, and implications of the study could be a better understanding of students' emotional states like affect or anxiety in relation to school mathematics.

PURPOSE AND RESEARCH QUESTION

The aim of the study is to develop an understanding of what may affect students in such a way that passing in mathematics turns out to be impossible. The study is based on students' narratives and the research question is:

— How do students explain why they have failed in mathematics while they have passed all other subjects in compulsory school?

My poster shows students' utterances and visual connections (arrows) to some aspects like affect, anxiety, motivation etc. Background, research question and methodology are presented in text blocks.

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TWG09

Mathematics and language

Challenges and research priorities in the context of TWG09: Mathematics and language

Núria Planas[1], Anna Chronaki[2], Frode Rønning[3] and Marcus Schütte[4]

1 University Autonomous of Barcelona, Barcelona, Spain, Nuria.Planas@uab.cat
2 University of Thessaly, Karditsa, Greece
3 Norwegian University of Science and Technology, Trondheim, Norway
4 Technical University of Dresden, Dresden, Germany

We begin by briefly examining the relevance of the study of language in mathematics education research, as well as some of the collaborative forms in which TWG09 is moving the study of language forward in the field. To provide reasons for the recent contributions by TWG09, we summarise lines of concern and tendencies that come from our reading of the set of papers and posters presented at CERME9. We build on the review work by Morgan (2013) to argue for the consolidation of such a diversity of lines of concern and tendencies in the contemporary agenda of the domain. The discussion of accomplished goals of the agenda points to a number of challenges and priorities in the context of TWG09 – its participants, their interaction and their activity.

Keywords: Language, mathematics education, mathematics education research, understanding, challenges.

OVERVIEW

The study of language in mathematics education research is a relatively young domain, however, with well-organised academic initiatives on an international level. One of the activities promoted by the European Society for Research in Mathematics Education (ERME) is the Thematic Working Group 'Mathematics and Language' (see Rønning & Planas, 2013, for a brief historical overview of the group since its origin). The Thematic Working Group 'Mathematics and Language' at CERME9 (TWG09) has brought together a collection of papers and posters that analyse a wide spectrum of topics, theoretical traditions and analytic approaches in this domain.

The fact that the study of language is at the core of intersections between many topics and theories inevitably has an effect on the nature, composition and evolution of our group. In 2015, for instance, some senior researchers in the field (i.e., researchers who have been active in the study of mathematics education for many years, although not necessarily keeping a regular focus on the study of language) have come to our group to discuss aspects of their work. They have shared expertise with, on the one hand, senior researchers in the group (i.e., researchers who have been active in the 'Mathematics and Language' domain within ERME for many years) and, on the other hand, with a large number of early researchers in the field (i.e., researchers who are new to both the study of mathematics education and the study of language issues with relation to mathematics education).

The purpose of this introductory text is to provide a conceptual and scholarly context for the papers and posters presented at CERME9. The rest of the text is organised into three sections. The first section deals with arguments for the relevance of the study of language in the field. The second section provides a description of how our domain is being constituted in ways that include a number of perspectives on language; following this, we trace a diversity of lines of concern and tendencies which come from our reading of the set of papers and posters. The third section concludes with reflections on how to mould the future activity of participants in TWG09. Collaboration among researchers from different countries, social realities and theoretical traditions are crucial aspects of what is needed to push boundaries further and explore new horizons in the study of language with relation to mathematics teaching, learning and education. One of the problems to be addressed is, in view of what is needed, what can be done and how it can be done in the years to come.

THE STUDY OF LANGUAGE IN MATHEMATICS EDUCATION

While questions addressing the relationship between language and mathematics have been a focus of study for over 30 years (see, e.g., Pimm, 1987), we still understand little about how and why language develops in interaction with mathematics teaching and learning development. Despite the fact that much remains to be researched, substantial knowledge in this regard is being produced inside and outside of ERME. We have arrived at the present destination – in which our domain is well recognized, established and differentiated in the field – after a long journey.

In 1974, an international symposium on "Interactions between Linguistics and Mathematical Education" was held in Nairobi, Kenya, sponsored by UNESCO in cooperation with ICMI and the Centre for Educational Development Overseas. Up to that moment, there had been no international events focusing on the relationship between mathematics and language. The final report of that symposium (UNESCO, 1974) revealed the lack of studies and research experience in this domain; moreover, it urged the international scientific community to adopt an agenda by identifying key issues, questions and needs at the intersection of mathematics education and language. Five years later, a review paper, with potential for founding an early agenda, was published by Austin and Howson (1979) in *Educational Studies in Mathematics*. A total of 240 references on language and mathematics education were compiled to indicate possible areas for investigation as well as areas in which research activity had already commenced.

In 2015, more than four decades after the Nairobi symposium, the examination of key issues, questions and needs at the intersection of mathematics education and language continues. Some of the newest issues and questions reveal, for instance, the increasing interest in a range of language uses in school mathematics contexts. Such uses may vary from one classroom to another, and more generally among school cultures, histories and pedagogies of teaching, but overall they may be dependent on who the involved users are (Planas, 2014). A large amount of evidence for the complexity of language use can be found in *Mathematics Education and Language Diversity*, the abbreviated title for the volume of the New ICMI Study Series edited by Barwelland colleagues (2015), which is one of the latest milestones of the long journey in the consolidation of the study of language in mathematics education.

Another step in the journey toward the maturity of the domain has been undertaken by Barwell (2014) and Morgan (2014), in their respective synthesis of reasons for the relevance of the study of language in mathematics education in two of the entries in the *Encyclopedia of Mathematics Education*:

> The teaching and learning of mathematics depend fundamentally on language. Mathematics classrooms, for example, may feature discussion among students, lectures by the teacher, printed curriculum materials or textbooks, and writing on a blackboard or on a screen. (Barwell, 2014, p. 331)

> While some aspects of mathematical language, such as its high degree of abstraction, may be an obstacle to participation for some people, doing mathematics is highly dependent on using its specialized forms of language, not only to communicate with others but even to generate new mathematics. In making this claim, we need to be clearer about what mathematical language is. (Morgan, 2014, p. 388)

During recent years, many new studies, journal articles and conference papers have contributed to reporting empirical findings and accurate theoretical developments in the domain. The creation and revision of knowledge in these works is being guided, either implicitly or explicitly, by common basic questions, whose origins can be traced back to the early work on language and teaching dilemmas by Adler (1998): *Where is language in this? Why does it matter? How can it be researched in ways that are idiosyncratic to mathematics education?* All in all, the existing variety of contemporary lines of concern in the domain can be thought of as different epistemological and analytical strategies of approaching these questions.

COMMON QUESTIONS, DIFFERENT LINES OF CONCERN

In her CERME8 plenary, Morgan (2013) offered a range of possibilities around what language enables mathematics education researchers to study, and how much the study of language challenges the ways in which we carry out research in the field. Moreover, she provided a thorough literature review concerning the work presented in the Thematic Working Group 'Mathematics and Language' over recent decades. In that plenary, Morgan claimed that the quality of research on language in mathematics education is influenced by a number of studies whose ideas have been at some time presented and discussed in ERME. We interpret the range of possibilities posed by Morgan (2013) in terms of potential lines of concern which map the contemporary domain and whose research has been initiated, inside and outside of ERME, with different levels of intensity. We refer to research concerning:

1) Communication and interaction in mathematics teaching and learning.

2) The representational systems in creating/structuring mathematical knowledge.

3) The role and use of languages in mathematics education practices.

4) The features of codes/registers in the construction of mathematical language.

5) The intervention of discourses in knowing and thinking in/about mathematics.

These five lines of concern illustrate how the agenda regarding the study of language in mathematics education has introduced new complexities and, thus, has evolved toward building a domain much wider than that imagined in the 1970s. In this respect, it is not surprising that the lines of concern suggested by Morgan to show the range of possibilities already opened for research in 2013, have emerged from our reading of the papers and posters presented in TWG09 during CERME9. While the limited collection of papers at CERME9 cannot totally reflect the scope and wideness of the contemporary domain, it confirms that some work is being done in each of these five directions. For the purpose of organising the discussion around the set of papers and posters accepted for presentation at the time of the conference, we used five abbreviated descriptors, respectively aligned with specific dimensions of the indicated lines of concern. These descriptors are:

1) Communication, interaction and gestures.

2) Epistemic, cognitive and structural aspects.

3) Multilingualism and sociocultural aspects.

4) Mathematical language and language use.

5) Discourse, practices and positioning.

In what follows, we summarise some of the lines of concern and tendencies which can be traced in the collection of papers and posters. As a whole, the process of grouping papers and posters has provided a straightforward picture of how and how much language is approached as a bundle of notions that other notions (gestures, registers, diagrams, talk, multilingual, mathematical objects, etc.) transverse, all of them with a diversity of attributed meanings and understandings. This is why the location of contributions is not univocally determined. It reflects our interpretation of the knowledge privileged by the authors in their texts. We may have overlooked ideas that some of the authors consider essential. Nevertheless, as the intention is to map the work in TWG09 and not to give a detailed account of each single paper or poster, we hope that all authors will see their work represented.

Communication, interaction and gestures

Six papers by Reinhardtsen, Carlsen and Säljö; Nordin and Björklund-Boistrup; Boukafri, Ferrer and Planas; Vogler; Farsani; and García Moreno-Esteva and Hannula, along with one poster by Roubicek, constitute the works in the direction of *Communication, interaction and gestures*. All these authors understand verbal (oral and written) and non-verbal (gestures, facial expression and visual contact) forms of language as communication strategies in classroom interaction. Language, in its verbal and non-verbal forms, is conceived as an instrument of communication, also in its various forms: individual, collective, peer-based, etc. In this way, diverse language use is tantamount to communication, learning and teaching.

In the presented works, we see two major tendencies or emphases that can be found at the intersection of several interrelated lines of concern. One is the tendency of the study of interaction through the study of gestures in classroom contexts of mathematics teaching and learning. In Farsani, for instance, gestures are proven to provide direct data for researchers to examine communication between teachers and learners. Here, a joint finding with the work by García Moreno-Esteva and Hannula is the role of gestures in the production of successful instances of mathematical communication, with a number of opportunities for mathematics teaching and learning. Closely related to the analysis of gestures, the production of intertextuality and multimodal texts to represent classroom data is also examined by these authors.

Another tendency in some of these works is the study of communication through and with digital media as part of the language environment for mathematics teaching and learning. The shift from text-based paper-and-pencil communication to multimodal digital communication is viewed as a qualitative change in the practice of mathematics teaching. It is generally noted that multimodal dynamic texts produced by teachers and students should be reproduced by researchers in different kinds of multimodal dynamic transcripts. Here, the potential interplay between multimodal resources, digital technologies and mathematical modes of communication is discussed by Reinhardtsen, Carlsen and Säljö, at a methodological level, through the development, application and evaluation of analytic tools created *ad hoc*.

Epistemic, cognitive and structural aspects

Six papers by Ruthven and Hofmann; Erath and Prediger; Krause; Fetzer and Tiedemann; Meyer; and Mellone and Tortora, together with a poster by Zwetzschler, constitute the collection of works in the direction of *Epistemic, cognitive and structural aspects*. All these papers address the study of ways in which mathematics classroom discourse develops grounds for and access to (school) knowledge and (school) knowledge claims. In this respect, an elusive area of concern regarding epistemic issues in school mathematics has been present in TWG09.

One identifiable tendency in this direction is the conducting of classroom studies. Ruthven and Hofmann introduce the term *epistemic order,* which refers to a system used to describe how ideas are developed and evaluated in the classroom. Their work expands the classical Initiation-Reply-Evaluation and Initiation-Response-Follow-up patterns of interaction (e.g., Sinclair & Coulthard, 1992) by including codes to explain who is responsible for what, and who takes the initiative. The work by Erath and Prediger also deals with how students participate in the configuration of the mathematics classroom discourse. These authors are interested in understanding how students' learning opportunities relate to forms of participation in the discussion of epistemic issues concerning knowledge about the construction and justification of school mathematics knowledge. The paper by Krause investigates the epistemic function of gestures. In a study of tenth graders, she looks for evidence that gestures may support reasoning actions when mathematical knowledge is constructed.

The other papers are oriented toward how language use relates to what is knowable, which indicates another tendency in this direction of papers. Fetzer and Tiedemann look at the interplay between language and manipulatives. They show how a fairly simple language can become mathematically more complete when it operates in connection with both human and non-human objects. Meyer is concerned with transformations of algebraic expressions, and with how learners convey the structure of an algebraic expression to other learners with the use of language. The paper by Mellone and Tortora is of a more theoretical nature, although based on experience from working with learners. These authors discuss how an idea of ambiguity, often implicit in school mathematics, can be explicitly used in the teaching activity.

Multilingualism and sociocultural aspects

Five papers by Barwell; Poisard, Ní Ríordáin and Le Pipec; Ní Ríordáin and McCluskey; Chronaki, Mountzouri, Zaharaki and Planas; and Klose, together with one poster by Šteflíčková, constitute the collection of works in the direction of *Multilingualism and sociocultural aspects*. All these papers share the basic conceptualisation of the students' languages as pedagogic resources and entries to learning identities, though they may take different perspectives on how language(s) and speakers relate to mathematics teaching and learning.

In these works we see two tendencies. One is the tendency of putting together design experiments and case studies. Chronaki and colleagues examine the case of a learner whose dominant language is not the language of instruction in a way that leads to the design, implementation and assessment of strategies for language use in mathematics classroom interaction. This approach is also taken by Barwell in his research with bilingual students in Canada. Barwell relates language use and its outcomes to the distinct language identity of certain groups of students in the multicultural mathematics classroom. In particular, he suggests that the construction of mathematical learning in multilingual settings is often guided by views of languages other than the languages of instruction and their speakers, rather than views of mathematical competence, performance and achievement.

A second tendency comes when examining the number of considerations at a political level made by the authors in their papers. All the presented papers in this direction focus on the perspective of languages as resources while considering the dilemmas involved in the choice and use of languages. These dilemmas indicate the extent to which the deficit perspective, in the study of language and language diversity, is more and more contested nowadays, particularly due to the impact in the domain of studies recurrently cited by some of the participants in TWG09 (e.g., Moschkovich, 2002). Poisard, Ní Ríordáin and Le Pipec, for instance, reflect in their paper on some of the enduring compensatory responses in the interpretation of the needs of students whose home languages are different from the language of instruction or the *lingua franca*.

Mathematical language and language use

Six papers by Edmonds-Wathen; Segerby; Tiedemann; Söbbeke; Albano, Coppola and Pacelli; and Engvall, Samuelsson and Forslund, along with one poster by Arce, Conejo and Ortega, comprise the collection of works in the direction of *Mathematical language and language use*. Most of the research focuses on the learning of mathematics at primary, secondary and university levels, while there is also a study in an indigenous population. It is shared an understanding of language use and language development for mathematical development through writing, reading and talking.

We identify a tendency around the analytical distinction of writing, reading and talking. Segerby focuses on the kinds of writing of young children in mathematics lessons. She finds that children write more in calculations as they progress through primary school, at the expense of transactional and poetic modes. Arce, Conejo and Ortega focus on the diverse uses of written notebooks by students at school secondary level. Albano, Coppola and Pacelli cover writing and reading since the pupils in their study are asked to develop written arguments from reading graphs. They find evidence of the complexity of switching between colloquial and literate registers. In a paper related to talking, Engvall, Samuelsson and Forslund draw conclusions about the relationship between communicative teaching strategies and students' development of procedural and conceptual language. On the other hand, Tiedemann addresses the notion of linguistic norms to examine classroom talking.

All papers in this direction show the extent to which the use of language in mathematical situations is determined by the mathematics, and by the situations and the people interacting within them. The abstractness of mathematics is highlighted, which often influences mathematical language by making visualisation impossible, along with the cultural situatedness of this mathematical language, in which for example the centrality of spatial prepositions is not necessarily true. Taken together, the papers show that there is no common, fixed mathematics constituting the language we use in mathematical situations. From a primary classroom in Germany, where students have to negotiate the linguistic norms of mathematics lessons, to the negotiating of concepts of motion in an Australian indigenous language, language is constituted by specific mathematical situations and by the participants in them. The construction of mathematical meanings as a basis for learning mathematics is therefore in constant interplay with the use of mathematical language.

Discourse, practices and positioning

Six papers by Nachlieli and Tabach; Ingram and Pitt; Wagner, Dicks and Kristmanson; Jung and Schütte; Dooley; and Hess-Green, Heyd-Metzuyanim and Hazzan constitute the collection of work in the direction of *Discourse, practices and positioning*. All these papers address the study of classroom discourse for the identification, production and analysis of interconnected teaching, learning, positioning and identity-work practices. Across papers there is a common tendency concerning theoretical integration and theories networking.

Nachieli and Tabach, along with Hess-Green, Heyd-Metzuyanim and Hazzan, address discourse in mathematics classrooms through a variety of theoretical perspectives such as commognition (Sfard, 2008), discursive psychology and theories of identity, while Wagner, Dicks and Kristmanson develop connections between socio-linguistics, pragmatics and theories of positioning for the purpose of their research. More generally, different authors base their notion of discourse on different theoretical approaches. Some of the papers are focused on socio-cultural approaches where discourse is not seen merely as individual word meaning or as linguistic utterances located in a classroom; here, the authors embrace theories of language where the subject is seen as a social, historical and political interlocutor. Mathematical discourse is related to varied ways of talking, reading, representing, interacting, moving, silencing, believing, valuing, identifying, performing and acting.

The tendency in these papers appears at the intersection of a diversity of lines of concern, whose collective reading provides opportunities to unpack the discourse(s), diversity, ethics and epistemologies of learning mathematics. In Nachieli and Tabach, and also in Ingram and Pitt, we find the study of the construction of competency and success in the course of classroom interactions where different representations of 'competent' and 'successful' students are at play. Moreover, we find the study of the role of vague language and politeness in mathematical conversations in Dooley, of notions of uncertainty in students' discourses related to prediction and conjecturing in Wagner, Dicks and Kristmanson, of young children's participation in decontextualised mathematical discourse in Jung and Schütte, and of emotions in experiences of discourses on achievement and learning in Hess-Green, Heyd-Metzuyanim and Hazzan, amongst other lines of concern.

WHERE TO GO FROM HERE

In this final section, we build on our discussion of the contemporary research agenda of our domain inside and outside of ERME, to shortly reflect on the potential of TWG09 –its participants, their interaction and their activity– in the further refinement and development of this agenda. We see TWG09 not only as a space for researchers who share interests concerning language issues in mathematics education, but more importantly as a space for researchers who may have an influence on each other through their interactions at the time of the conferences and during the in-between periods. This second perspective represents a mature stage in the process of constructing a group with a strong scientific identity, capable of quality group work.

By the end of CERME9, the team of co-leaders asked participants of TWG09 to suggest reflections on how to mould the future activity in the group. An e-mail communication sent by one of the participants, who was attending her first CERME, revealed her vision of TWG09 as a rich space though still in an early stage of group work maturity. This is an instance of what this participant shared with us:

> It is important to get to see, as a group, the enormous challenges waiting for us to work on them. What I have missed though, is explicit consideration of how we are going to organize the near future so that we face these challenges by building on each other. (Participant of TWG09 at CERME9, e-mail communication)

This participant critically raised the issue of building on each other as a group. This is an insightful comment on the importance of building seriously on relevant work and ideas by other participants in order to strengthen the domain and our potential as contributors to its progress. The view by this participant may not be unique. We must continue to learn how academic collaboration within TWG09 can be reinforced in order to jointly face the many challenges posed at present and in the years to come in the study of language in mathematics education, both inside and outside of ERME.

Collaboration and regular work within the context of our group can contribute to moving the international agenda forward in directions that address, for example, the role and use of gestures in the bilingual mathematics classroom, the anticipation of forms of interaction for dialogic-based class discussion, the development of cross-linguistic analysis of mathematical cognition, the evaluation of content- and language-integrated approaches to learning, or the introduction of multimodal resources for the benefit of mathematics thinking. All in all, TWG09 – its participants, their interaction and their activity – can contribute to expanding our knowledge of a number of issues, questions and needs, with implications for the achievement of quality mathematics instruction and equitable learning environments. By building on each other, we can learn more about how talk and interaction develop throughout mathematical activity. We can particularly learn how preferences for exploratory talk and inquiry-based lessons are validated by rigorous empirical work, and how variations of these preferences have an effect at different stages of mathematical learning. Insights from research in this domain can assist in developing interventions to improve mathematics teaching and learning practices. Thus, one of the challenges is to conduct and promote research that increases our understanding of how mathematics teaching and learning develops, but that also allows us to identify effective strategies and ways of improvement in practice.

It is our hope that the following set of papers and posters will stimulate interest in and appreciation of work across a wide number of language aspects. Regardless of the aspects that we decide to highlight in our work, the centrality of language must always be kept in mind. We invite our colleagues in the field, as well as teachers, teacher educators, curriculum developers, policy makers, etc., to think along with us.

ACKNOWLEDGEMENT

Núria Planas' time to participate in this collaboration was funded by the Catalan Government (ICREA-Academia Professorship and Group GIPEAM SGR2014-972).

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TWG09

Research papers

Reading data from graphs: A study on the role of language

Giovannina Albano[1], Cristina Coppola[2] and Tiziana Pacelli[2]

1 Università degli Studi di Salerno, DIEM, Fisciano, Italy

2 Università degli Studi di Salerno, DipMat, Fisciano, Italy, ccoppola@unisa.it

The work described in this paper is framed in a larger study focused on written argumentations produced by university students in tasks regarding graphs, conversions between graphs and their analytic properties and relations among different graphs. Using the functional linguistics framework, we analyse difficulties related to an examination task on functions and their derivatives. The use of colloquial and literate registers in mathematics problem solving is the lens through which we analyse and discuss the errors. We draw attention to the difficulties concerning the use of literate registers and how these difficulties influence the errors in the interpretations of texts and figures.

Keywords: Language, colloquial and literate registers, written argumentations.

INTRODUCTION

In this paper we focus on the analysis of written argumentations, produced by undergraduates to justify their answers in problems concerning graphs, relations among various graphs and their coordination with analytic properties.

The study was born in a first year University context, and, in particular, concerns Biology freshman students attending a basic mathematics course. According to the goals of mathematics instruction in applicative domains, such as Biology, the course primarily is aimed at allowing students to interpret and compare graphs of elementary functions. The need for argumentation comes from the need for preventing them from answering at random and for promoting a more-in-depth understanding.

Our research is framed in the context of the language as the key point in learning processes (Sfard, 2001) and of the importance of writing to learn (Morgan, 1998). In this framework, languages are seen as constructors of the meanings themselves. From this viewpoint, the quality of language influences the quality of thinking and this requires educational attention to the correspondence between semiotic activities and linguistic competency of the participants (Ferrari, 2004). In fact, evidence shows how some learning difficulties in mathematics can be ascribed to poor linguistic competence (Ferrari, 2004).

In the following we are going to investigate the protocols produced by a sample of students that were required to compare graphs and to produce an argumentation to justify their choices. A lot of difficulties emerged at different levels, in particular, in data reading, even with visual data.

We interpret these difficulties, which are linguistic or semiotic in nature, using the functional linguistic framework. In particular, we focus on the use of colloquial and literate registers in mathematics problem solving.

Our research question is the following: How the level of linguistic competence, particularly the difficulties concerning the use of literate registers, influences the way of interpreting texts and figures. More in general, in a Vygostkian framework (Vygotskij, 1934), we are interested in investigating how the language does (or does not) support thinking in the interpretations of texts and figures.

THEORETICAL BACKGROUND

Language is growing as one of the most relevant issues for research in mathematics education. Several authors have studied the interactions among the different semiotic systems in mathematics learning.

From the cognitive point of view, Duval's (2006) investigations have highlighted how the coordination of various semiotic representations is the key to comprehension in mathematics.

According to O'Halloran (2005) three main groups of semiotic systems can be devised: verbal language, symbolic notations and figural representations, which are strongly interwoven in doing mathematics.

Sfard (2001) interprets thinking as communication and regard languages not just as carriers of pre-existing meanings, but as builders of the meanings themselves. So, under this perspective, language should heavily influence thinking.

As already said, there is evidence that a good share of students' troubles in mathematics can be ascribed to improper uses of verbal language (Ferrari, 2004). More precisely, students often produce or interpret mathematical texts according to linguistic patterns appropriate to everyday-life contexts rather than to mathematical ones. The difference is not just a matter of vocabulary, grammar or symbols, but it heavily involves the organization of verbal texts, their functions and relationships with the context they are produced within. This is why Ferrari has assumed a pragmatic approach to interpret students' behaviours. This means focusing on the language use and on the different functions it plays rather than just on grammar. In this respect, a further characteristic of verbal language has to be taken into account, the multivariety, that is the use of language in various registers (intending register as a linguistic variety based on use) (Leckie-Tarry, 1995).

According to Ferrari (2013), in mathematics we need both the colloquial registers, to construct the concepts without paying much attention to the form of their representation, and the literate registers, including symbolic representations, which can be considered as extreme forms of literate registers. The latter ones are essential to express relations among concepts, to elaborate generalisations and to explain solving procedures. Ferrari (2004, 2013) has shown how many of the students' errors can be traced back to the use of typical styles of colloquial registers whereas more advanced styles would be necessary.

METHODOLOGY

The sample taken into account for our research consists of 64 students attending the first year of a 3-year BSc degree in Biology and taking part in a 48-hour

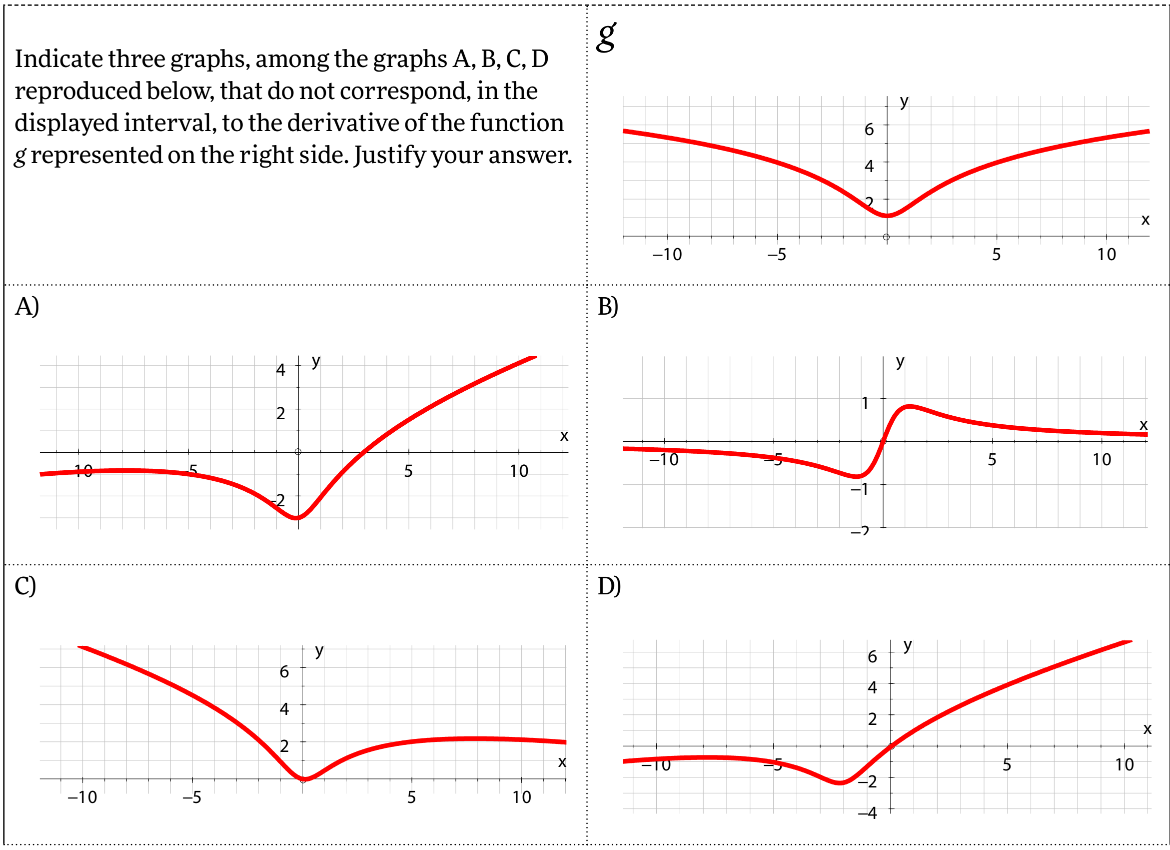


Figure 1: The task in the study

Literate register	Colloquial register
Context of culture: links with other texts and knowledge are activated.	*Context of situation*: the text is linked to the specific situation in which it is interpreted or produced (the exam situation). In this case, students face the problem starting from the text and the images in the task without referring to other knowledge, on the basis of general heuristics (an example is "if f is increasing, then f' is increasing") or looking for analogies (in the graph itself or in other tasks previously carried out).
Lexicalization: words are used according to well-known and shared definitions or descriptions.	*Lack of lexicalization*: everyday meaning of words (that is vaguer) prevails over the one based on definitions (think, for example, of such words as "increasing", "inflexion point", "concavity").
Propositions' explicit structure: subject, complement, conjunctions. Text organization is ruled by grammar and is based on subordination relationships (causative, consecutive, temporal propositions).	*Propositions' implicit structure*: subject or complements are often missing. Text organization is realized by means of spatial nearness of the sentences, or by means of graphical artifices (such as arrows, colours) or textual markers that are vague (for example "and") or randomly (as an example, "therefore" wrongly used).
Stability: text interpretation and production is quite stable, thanks to the explanation and use of verifiable definitions.	*Instability*: text interpretation and production result from factors out of control, e.g. how the students use words, the mental images they activate instead of definitions, previously carried out tasks that they remind of.
Metaphoric-symbolic organization: the link between representations and their content is not natural but conventional or metaphorical.	*Congruent-iconic organization*: organization and form of the representation are not arbitrary but they naturally correspond to organization and form of the meaning.

Table 1: Categories of errors

module in mathematics. According to the aims of the course, the lectures and the tutoring sessions were not based on a deductive approach (that is, definitions, theorems and applications), but they aimed at involving students in activity of coordination of various semiotic systems (Duval, 2006) concerning functions (formulas, graphs, tables, verbal texts).

In accordance with the lectures, in the exams the students were required to master graphical representations of functions and to be able to coordinate graphical and analytic properties of functions and their derivatives. Moreover, argumentation abilities were required to justify their answer.

In the examination task the question was posed in "negative" mode: given a function graph, the students had to indicate three out of four graphs "not" corresponding to the derivative of the given function, justifying their answer (Figure 1).

Our analysis was focused on the argumentations given by the students, even if the answer was correct (i.e., A, C, D), and we looked at them through the lens of the distinction between colloquial and literate registers. We point out that 23 students answered correctly. The errors that emerged have been classified according to typical linguistic categories (Leckie-Tarry, 1995), as listed in Table 2.

ANALYSIS OF PROTOCOLS

We report some protocols of special representativeness, and explain part of our analysis, using the described categories in Table 1. We present the pictures of the original protocols in Italian and our corresponding English translations.

A09: In y =1 the function is increasing, so its derivative has to be increasing, this rules out the graph B. We rule out C because it has x = 0 and at the point 0 is increasing. We rule out D because for y = 2 the function is increasing, and not passing through 0, for this reason we rule out D. The right graph could be A.

In protocol A09 (Figure 2), there is a quite inaccurate and "careless" language and confusion in the reading of graphs. The student refers to a context of situa-

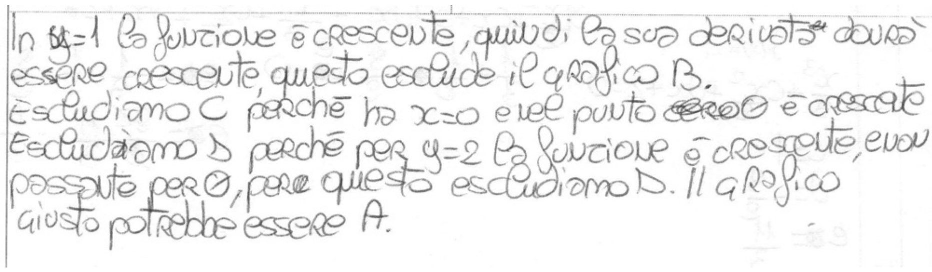

In ~~y~~=1 la funzione è crescente, quindi la sua derivata dovrà essere crescente, questo esclude il grafico B.
Escludiamo C perché ha x=0 e nel punto ~~zero~~ 0 è crescente
Escludiamo D perché per y=2 la funzione è crescente, e non passante per 0, per questo escludiamo D. Il grafico giusto potrebbe essere A.

Figure 2: Protocol A09 – Answer: B, C, D

tion using the interpretation that identifies a function's properties with the properties of its derivative. Moreover the structure of propositions is implicit and there are repetitions. Language appears quite vague in "*We rule out C because it has x=0 and at the point 0 is increasing*". Maybe it is meant to say that the function represented in C assumes 0 as the value for $x = 0$. But being increasing in 0, how can this exclude graph C?

Besides, he/she writes "*In y=1 the function is increasing*", where 1 is the *y*-coordinate of the intersection of the graph of the function with the *y*-axis: an iconic organization seems to prevail. The same happens when the student writes "*for y=2 the function is increasing and not intersecting 0*", reading 2 on the *y*-axis and meaning the point in which the function assumes a value equal to 2.

Students, like the one in protocol A09, do not consistently apply some convention to name the points on the graph (e.g. indicating the *x*-coordinate only, or both the *x* and the *y*-coordinates) but they seem to choose the label that is nearer to the point. In other words, the spatial relationship between signs prevails on their defined meanings as well as on the conventions that regulate their use. As shown in Table 1, iconicity is typical of colloquial registers, whereas in literate ones conventions or metaphors are the standard ways of conveying meaning.

A07: I ruled out the graphs A, B, C. Let's notice that the graph of the function *g* has got horizontal tangent, so $g'(x) = 0$ and I rule out A because it does not pass for the origin; *g* displays an upward concavity so $g'(x)$ is increasing. Therefore I rule out B and C that are decreasing.

Also in protocol A07 (Figure 3) the language is quite inaccurate and there is a strong presence of the characteristics of colloquial register categories (Table 1). The student writes: "*the graph of the function g has got horizontal tangent, so g'(x)=0*", without specifying where there is a horizontal tangent in the graph of function *g*. By writing *g'(x)=0*, does he/she mean that the derivative is identically vanishing? Maybe, in a context of spoken communication, the sentence would be followed by a negotiation with the interlocutor, to explain the exact meaning.

Moreover he/she writes "*g displays an upward concavity*", showing a lack of conceptual control of the image and a lack of lexicalization: maybe the everyday meaning of the word "concavity" influences the answer (it could be that the student is reminded of a "standard" image of parabola). Also in the expression "*I rule out B and C that are decreasing*" a poor conceptual control of the image seems to emerge.

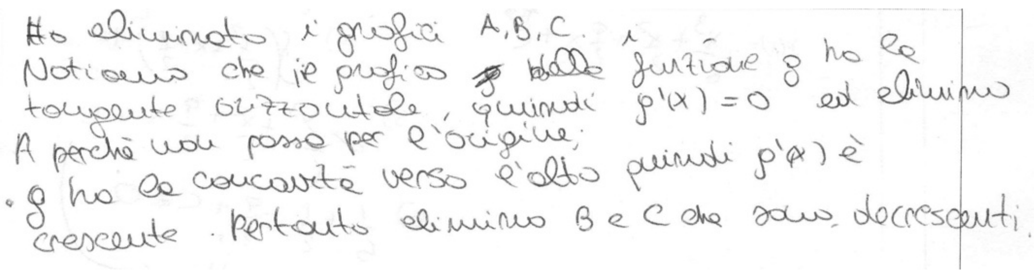

Ho eliminato i grafici A, B, C.
Notiamo che il grafico ~~g~~ della funzione g ha la tangente orizzontale, quindi g'(x) = 0 ed elimino A perché non passa per l'origine;
g ha la concavità verso l'alto quindi g'(x) è crescente. Pertanto elimino B e C che sono decrescenti.

Figure 3: Protocol A07 - Answer: A, B, C

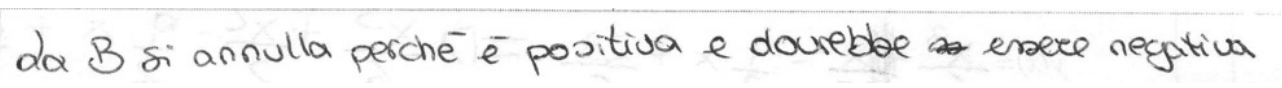
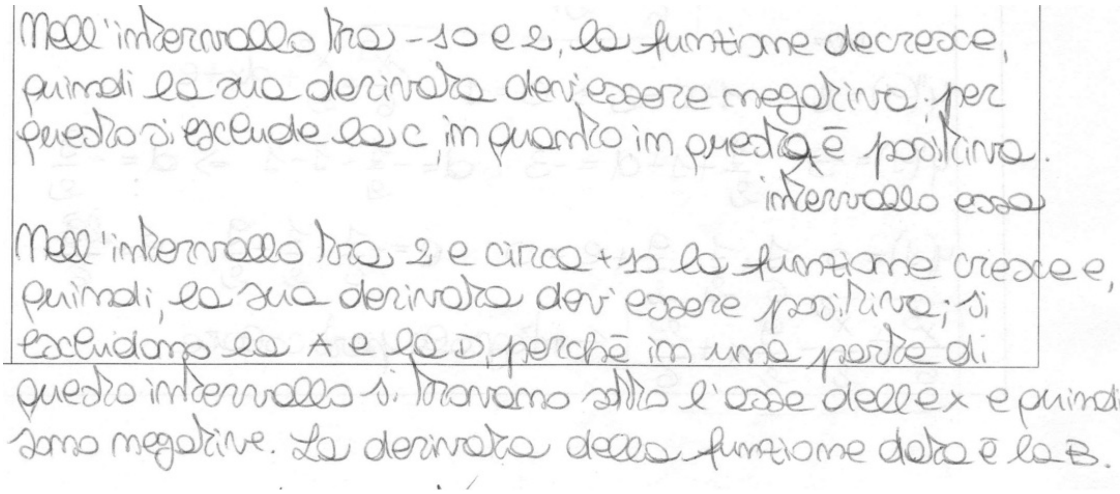

da B si annulla perché è positiva e dovrebbe ~~a~~ essere negativa

Figure 4: Protocol A21 – Answer: A, B, C


Nell'intervallo tra -10 e 2, la funzione decresce, quindi la sua derivata dev'essere negativa; per questo si esclude la C, in quanto in questo intervallo essa è positiva.

Nell'intervallo tra 2 e circa +10 la funzione cresce e, quindi, la sua derivata dev'essere positiva; si escludono la A e la D, perché in una parte di questo intervallo si trovano sotto l'asse delle x e quindi sono negative. La derivata della funzione data è la B.

Figure 5: Protocol A55 – Answer: A, C, D

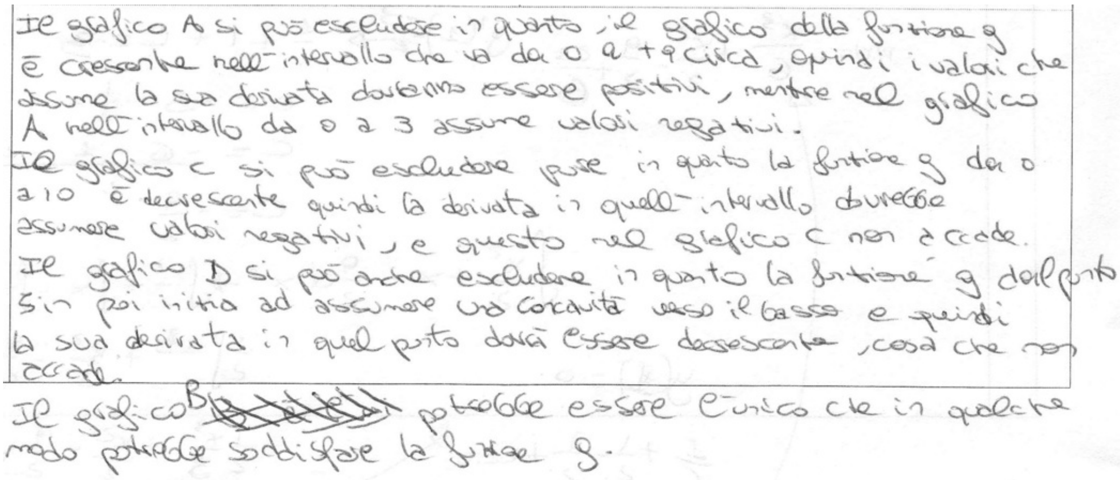

Il grafico A si può escludere in quanto il grafico della funzione g è crescente nell'intervallo che va da 0 a +9 circa, quindi i valori che assume la sua derivata dovrebbero essere positivi, mentre nel grafico A nell'intervallo da 0 a 3 assume valori negativi.
Il grafico C si può escludere pure in quanto la funzione g da 0 a 10 è decrescente quindi la derivata in quell'intervallo dovrebbe assumere valori negativi, e questo nel grafico C non accade.
Il grafico D si può anche escludere in quanto la funzione g dal punto 5 in poi inizia ad assumere una concavità verso il basso e quindi la sua derivata in quel punto dovrà essere decrescente, cosa che non accade.
Il grafico B potrebbe essere l'unico che in qualche modo potrebbe soddisfare la funzione g.

Figure 6: Protocol A06 – Correct answer: A, C, D

A21: B vanishes because it is positive while it should be negative.

Together with the lack in the conceptual control of the image ("*B…is positive*"), we found the protocol A21 (Figure 4) interesting for the use of the verb "vanish" instead of "rule out". The student here uses a word belonging, in some sense, to the mathematical register instead of using a word of everyday language, maybe because the task he/she is carrying out is a "mathematical" task.

A55: In the interval between -10 and 2, the function decreases, so its derivative has to be negative; for this reason the C is ruled out, for in this interval it is positive. In the interval between 2 and about +10 the function increases and, so, its derivative has to be positive; A and D are ruled out, because in a part of this interval they are below the x-axis and so they are negative. The derivative of the given function is B.

The argumentation used in protocol A55 (Figure 5) is quite accurate, but there is a strong presence of iconicity. The student writes "*In the interval between -10 and 2, the function decreases*" reading -10 on the x-axis and 2 on the y-axis which are explicitly written in the diagram (while 0 on the origin is not written). Maybe he/she chooses the most evident label, or the label nearest to the point he/she wants to indicate. He/she writes again: "*In the interval between 2 and about +10 the function increases*", showing once more an iconic interpretation of the diagram.

In ruling out graph D ("*A and D are ruled out, because in a part of this interval they are below the x axis*"), the student shows not only a lack of conceptual control on the image, but, maybe, again iconicity: it seems that the student considers the interval between -2 and +10, instead of the interval +2 and + 10; he/she refers to the label -2 or to the label 2, indifferently, as if both the labels indicate the same point.

A06: Graph A can be ruled out because the graph of function *g* is increasing in the interval going from 0 to about +9, so the values of its derivative have to be positive, whereas in graph A in the interval from 0 to 3 it assumes negative values. Graph C can be ruled out too because function *g* from 0 to 10 is decreasing, so the derivative in that interval should assume negative values, and this does not happen in graph C. Graph D can be ruled out too because function *g* from point 5 on begins displaying a downward concavity so its derivative in that point has to be decreasing, and this does not happen. Graph B could be the only one satisfying in some way function *g*.

The student of protocol A06 (Figure 6) gives the correct answer (he/she rules out graphs A, C and D), produces a detailed conversational argumentation and shows a good capability of reading of graphs and linking data to conclusions. Nevertheless also in his/her protocol, we can find an expression such as: "*function g from 0 to 10 is decreasing*". He/she expresses like one looking at the graph from the origin to the left and from the origin to the right.

Indeed this behaviour emerges from several protocols: a lot of students privilege their own point of view with respect to definitions and to the shared and well-known rules of using representations.

DISCUSSION AND CONCLUSIONS

From the protocols' analysis three macro-groups of students' behaviours seem to emerge. These macro-groups are based on the categories of errors (in Table 1), categories that in students' behaviours are continuously interwoven.

The first macro-group is represented by the protocols of those students that do not apply the usual conventions (the "grammar") of the Cartesian plane but use the image they see. Students belonging to this group do not refer to the *context of culture* evoked by the representations but to the *context of situation*.

In some protocols, as an example, we can observe that, in order to designate the points of the Cartesian plane, the students tend to label the points only by their ordinate value. Probably, they choose this value just because it is the label nearest to the point. Conversely some students use a point of the plane in order to indicate the starting point of an interval on the *x*-axis (see Protocol A55 in Figure 5).

In this macro-group we can find also other students' behaviours: for example students exploring the graph starting from the origin of the Cartesian axis and considering as "positive" the direction from 0 to $-\infty$; other students take into consideration in different graphs points that look the same (as an example, points that are minimum) instead of comparing points that have the same *x*-coordinate. These students remain in an *iconic interpretation of the diagram* and privilege their own point of view (related to the context of situation, i.e. the diagram drawn on the paper as a material object), instead of the recognized rules or conventions for using representations (which are obviously related to the context of culture).

A second macro-group is represented by the protocols of students having some difficulty in the *use of the vocabulary*. For these students some words, such as "positive", "increasing", etc., have a vague and unstable meaning. As a consequence they make errors also in reading the graphs. In such a situation students write sentences such as "*I rule out B because it should be positive for $x > 0$*", probably having in mind the concept "increasing".

Another behaviour related to this macro-group is mixing up the subjects in sentences like "*the derivative vanishes ↔ the function vanishes*".

All these behaviours are typical of the use of colloquial registers.

The last macro-group is represented by the protocols of students having some problems in the *organization of the text*, i.e. the meaning of the words related to their position in the text.

As an example we can observe some protocols in which the students use the expressions "increasing", "positive", "upward concavity" for describing properties related to an interval even if these expressions are true only in a part of the interval.

This behaviour is linked both to that one, above described, characterized by taking into consideration the first image or the first concept that seems to work well without making further inferences, and to how the students tend to organize the text in the colloquial registers. In the last case they are not used to specify the complements because in a context of spoken communication the meanings could be negotiated later. In this way the meanings become unstable and vague and as a consequence it is hard reading the images and making further argumentations.

The outcomes of the analysis seem to suggest the need for a stronger educational attention to linguistic issues in mathematics. It is fundamental the capability of using both literate registers, which support scientific thinking, and colloquial ones, which are indispensable too in the construction of mathematical concepts. Thus switching back and forth between colloquial and literate registers is a crucial process in mathematics learning. These skills are not natural but have to be developed and fostered, by specifically planned teaching activities, from primary school onwards.

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Linguistic stratification in a multilingual mathematics classroom

Richard Barwell

University of Ottawa, Ottawa, Canada, richard.barwell@uottawa.ca

There has been a good deal of research on interaction in multilingual mathematics classrooms, with an increasing emphasis on the socio-political dimension of language in shaping students' and teachers' language choices. I argue that this approach remains under-theorised and offers a limited perspective on the politics of language in mathematics classroom interaction, largely focused on language choice. To deal with this problem, I draw on ideas from the contemporary sociolinguistics of multilingualism, including the notions of heteroglossia and orders of indexicality. To illustrate these ideas and their utility, I present an analysis of an episode observed in a sheltered elementary school second language mathematics classroom in Canada. My analysis shows how the two students are marginalised by the interaction.

Keywords: Multilingual mathematics classrooms, second language learners, heteroglossia, linguistic stratification.

INTERACTION IN MULTILINGUAL MATHEMATICS CLASSROOMS

Research on interaction in multilingual classrooms has increasingly emphasised the socio-political dimension of language and language use (Setati, 2005; Planas & Civil, 2013). I refer to any classroom in which any of the participants uses more than one language in their daily life as multilingual. This definition includes classrooms commonly referred to as bilingual or second language classrooms, as well as those in which the languages of some students are not used and not recognised. This is consistent with sociolinguistic perspectives that challenge rigid separations between languages or language situations (Makoni & Pennycook, 2007). In such classrooms (indeed, in any classroom), language is not simply a means of communication or a tool for thought; the way language is used means that some participants may be privileged in different ways, while others may be marginalised. Such influences are sometimes systemic and reflect wider social forces, such as those of racism or class. I refer to these influences as the socio-political dimension of language.

While research has identified the significance of the socio-political dimension of language, its effects on mathematics classroom interaction are poorly understood. Research on mathematics classroom interaction in multilingual settings dates back at least to the 1990s. Much of this work has adopted a view of language as a 'resource'. Research on teaching practices includes Adler's (2001) identification of dilemmas that arose for several teachers in different multilingual mathematics classrooms in South Africa; Khisty's (1995) comparison of three teachers in Spanish-English bilingual classrooms in the USA and Moschkovich's (1999) study of a Spanish-English bilingual mathematic class also in the USA. These studies highlight the challenges many teachers face in working with students who draw on multiple languages in the mathematics classroom. In all three studies the dilemma, to use Adler's term, of whether or not to give explicit attention to mathematical language or focus on the mathematical ideas emerged a challenge. Research on students' participation, meanwhile has identified several resources on which students may draw in mathematical discussion. These resources include code-switching (Setati, 2005; Planas & Setati, 2009); genre and narrative (Barwell, 2003, 2005); and gestures, writing and diagrams (Moschkovich, 2009).

While the majority of these studies show some awareness of the socio-political dimension of language, this awareness is not always apparent in the design and conceptualisation of the research. In recent years, research has emerged that gives more explicit attention to this dimension. Setati's (e.g., 2008) work, in particular, has highlighted how learners' and teachers' language choices are influenced by the broader politics of language in South Africa. By choosing to study mathematics in English, for example, students hope to get access to better opportunities in higher education or employment. The value of these "social goods" outweighs the challenges that such students may face in studying mathematics in a language that they may only use at school. Similarly Planas and Civil (2013), drawing on data from Catalonia and Arizona, show how the choices of the students and teachers in their study about language use in their mathematics classrooms are politically mediated. In particular, and rather like Setati, they argue that the pedagogical value of students' home languages may be overridden by broader political considerations.

It is clear, then, that the socio-political dimension of language influences what happens in mathematics classroom interaction. The research I have discussed, while making a contribution to the field, has, however, generally avoided a theorisation of this aspect of language use. This work has tended to focus on one single aspect of language use: the choice (if it is a choice) of language. Thus, in Setati's (2008) work, the choice is between English and an African language or languages; in Planas and Civil's (2013) study, the choice is between Spanish and English, or Catalan and Spanish. The socio-political dimension of language is, however, likely to influence mathematics classroom interaction in many other ways than participants' choice of language. To investigate these influences, additional theoretical ideas are needed.

I propose to address this problem by drawing on theoretical ideas from the contemporary sociolinguistics of multilingualism. These ideas are illustrated through analysis of data collected in a second language mathematics classroom in Quebec, Canada. My aim is to demonstrate that these theoretical tools make it possible to develop a more nuanced understanding of how language is implicated in the stratification of students' participation in mathematics and hence how it has an impact on their opportunities to learn mathematics.

HETEROGLOSSIA AND ORDERS OF INDEXICALITY

There have been some significant shifts in how multilingualism (and language itself) is conceptualised and understood in recent years. Many of these shifts can be traced, in part, to the work of Bakhtin (1981) who developed a view of language as situated, dialogic and tension-filled. Bakhtin's ideas have led to a view of multilingualism that, rather than focusing on discrete, clearly defined languages and associated clearly defined groups of speakers, looks at language as social practice situated in social and political contexts (Blackledge & Creese, 2010, p. 25). More specifically, Bakhtin defines the key concept of heteroglossia as "the social diversity of speech types" (Bakhtin, 1981, p. 263). He describes this diversity as follows:

> At any given moment of its evolution, language is stratified not only into linguistic dialects in the strict sense of the word [...] but also [...] into languages that are socio-ideological: languages of social groups, "professional" and "generic" languages, languages of generations and so forth. (pp. 271–272)

Heteroglossia, then, refers to the many patterns that arise within language and which can be associated with some group of people, situation, activity or other social formation. There are two important points to note about this account. First, the many different patterns within the diversity of language overlap and intersect. The language of teachers, the language of mathematics and the language of a region may all be present in the same utterance. Moreover, the distinctions between the speech types to which Bakhtin refers are produced by these practices; they are not pre-given. Thus, what counts as an accent, as 'teacher talk' or even as a language, is locally produced (Bailey, 2007). The way that language practices can 'point to' such associations, allowing as to recognise particular activities, group memberships or situations, is called indexicality. This aspect of language is important in framing particular utterances, so making them interpretable. In previous work, I have examined the role of heteroglossia in the language tensions that have been reported in multilingual mathematics classrooms around the world (Barwell, 2012, 2014).

Second, these ideas make a link between moments of language use and broader social patterns and forces:

> Linguists have increasingly turned to the works of Bakhtin and his collaborator Volosinov because their theories of language enable connections to be made between the voices of social actors in their everyday, here-and-now lives and the political, historical, and ideological contexts they

> inhabit. In familiar terms, Bakhtin's philosophy of language contributes to the means by which we may understand the structural in the agentic and the agentic in the structural; the ideological in the interactional and the interactional in the ideological; the "micro" in the "macro" and the "macro" in the "micro." (Blackledge & Creese, 2009, pp. 237–238)

Ways of talking both reflect the socio-historical dimension of language and create this dimension for the future.

Third, these different ways of talking are stratified; some ways of talking are considered more valuable than others. To explain how this stratification arises in the context of multilingualism, Blommaert (2010) focuses on indexicality:

> Ordered indexicalities operate within large stratified complexes in which some forms of semiosis are systemically perceived as valuable, others as less valuable and some are not taken into account at all, while all are subject to rules of access and regulations as to circulation. That means that such systemic patterns of indexicality are also systemic patterns of authority, of control and evaluation, and hence of inclusion and exclusion by real or perceived others. (p. 38)

This kind of stratification typically maps onto scalar differences in practices, so that local, idiosyncratic practices are perceived as less valuable than more widely used, standardised practices (Blommaert, 2010, p. 35).

Blommaert discusses several examples to illustrate these ideas. For one, he refers to a price list for cold drinks found in London's Chinatown (p. 31). The price list is written in Chinese characters and in English. The English includes "quite spectacular typos" (p. 31), such as "Lced" for "Iced" and "Coffce" for "Coffee". Blommaert points out that for many customers in London, the Chinese characters are "a meaningless design", but which index "Chineseness" and a link with wider Chinatown. He also imagines the sign being printed somewhere in China, where the English would be equally meaningless, simply symbols to be reproduced in printed form. To customers in London, the spelling mistakes might be a source of amusement, but might also index less favoured or less valuable forms of English literacy. Hence, indexicality is itself situated, dependent on who is producing or interpreting language or text, as well as where they are and what they are doing. Blommaert (2010) illustrates this point as follows: "the English spoken by a middle-class person in Nairobi may not be (and is unlikely to be) perceived as a middle-class attribute in London or New York" (p. 38).

RESEARCH SETTING: A SECOND LANGUAGE MATHEMATICS CLASS

From 2008–2012, I conducted an ethnographic study of mathematics learning in different second language settings in Canada, a country with two official languages, English and French. In this paper, I refer to data from one of these settings, located in an Anglophone school in the French-majority province of Quebec. The data come from interactions with a Grade 5–6 sheltered class for students identified by the school as falling behind in both English and mathematics. The students therefore studied these two subjects in a separate class from their regular classmates. I visited the class regularly throughout the 2009–2010 academic year. During that time, enrolment in the class varied quite a bit but never went over 9 students.

For most of the year, all of the students in the class were Cree, one of the original peoples of Canada. The children's families originated in communities in northern Quebec. The students spoke Cree as a first language. They also spoke English, though with a range of proficiency levels. In the move from James Bay to the city, the students went from being part of the majority in small Cree-speaking communities, to part of a minority in a city dominated by French and English.

During my visits to the class, I acted as a participant observer, making field notes during teacher-led activities, and interacting with the students during small-group work. The teacher often asked me to work with small groups of students. I made numerous audio recordings of whole-class interaction and some small-group work, including my own work with groups of students. I collected samples of students' work and photographs of other artefacts, such as posters, work written on the blackboard and anything else that seemed relevant. After each visit, I wrote a brief report summarising my observations.

In the next part of the paper, I present and analyse an episode that I observed during the study. I have described aspects of this episode elsewhere (see Barwell, 2014) but have not previously examined the stratification that arose in the students' interaction.

THE TULIP FESTIVAL PROBLEM

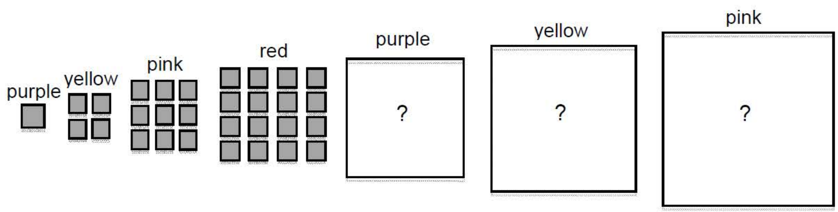
The episode occurred in February 2010, during a part of a class in which the students were working at activity stations. I worked with two students, Curtis and Ben, at a station in which they had to solve the problem reproduced in Figure 1.

> Every year Ottawa holds a world-renowned tulip festival in the month of May. There are different gardens in various locations, one of which is on Parliament Hill. The Canadian Tulip Festival was established to honour Queen Juliana of the Netherlands, in 1953. [...] You are a gardener hired to plant tulip bulbs for the Canadian Tulip Festival in May. You decided to arrange the flowers in a V for Victory format. You decide to use a pattern to make your design. Here is the design you started.

The problem text has a number of indexical features relevant to the students' subsequent work. First, the text is in English. The first part indexes factual registers typical of textbooks or informational texts, such as tourist brochures. Second, the text is an elaborate form of word problem, with a scenario, some information and a mathematical calculation to be carried out. The presentation and structure of the text thus indexes the genre of mathematical word problems. This genre is widespread in Canadian mathematics classrooms and this particular form, known in Quebec as a situational problem, is a common form of assessment item in the province. Third, the text indexes a particular place, Ottawa, and a particular event, the Tulip Festival, with which people in the region might be expected to have some familiarity. More specifically, and crucially for this episode, the text assumes a familiarity with tulips, a flower that is very common in the spring in this region. These associations, combined with the register used in the first part of the problem (e.g. Ottawa as the national capital, "world-renowned"), perhaps also index a form of 'Canadian-ness'. Fourth, the text indexes certain mathematical forms, particularly geometric patterns indicated by the diagram. Thus the text indexes a nation, a region, an event, speakers of a language, a register, a genre and finally, some mathematics.

I began by asking Curtis and Ben to read the problem to themselves and then initiated a discussion about the content [2]:

RB:	okay (.) so what's it about?
Curtis:	its about (.) world's biggest flower=I don't know
RB:	ottawa's biggest
Curtis:	tu (.) lip festival
RB:	tulip festival (.) do you know any of those? (.) do you know what a tulip is? [hm
Curtis:	[flower
RB:	flower right (.) have you ever seen a tulip?
	[...]
Ben:	(...) it's white
RB:	they are lots of different colours white ones red ones
Curtis:	like a rose?
RB:	yellow ones say again
Curtis:	rose
RB:	no it's a bit different from a rose (.) roses yeah (.) tulips just come up in the spring and have a nice flower for about two weeks (.) then they are fin-



How many purple, yellow, and pink tulips do you need to complete the design? Show all your work.

Figure 1: The Tulip Problem [1]

ished (.) there we go (.) let me see your picture
Ben, Curtis: [*laughter*]
RB: have you seen flowers like that
Ben: ^no^
Curtis: yeah (.) in a store

It is apparent from this exchange that the two students have some trouble interpreting the text in the way it was presumably intended. For them, "tulip" initially indexes something rather vague: a kind of flower – they mention roses and, at another point, poppies. And while the text might be designed to index a place and an event, and by extension, some aspects of Canadian-ness, the two students do not make this connection. In this way, the text serves to alienate the students.

Our discussion, which continues in similar vein to clarify what 'bulbs' are and what 'a gardener' does, can be read as an encounter between different "speech types": those of the text, the students and me. Over the next few minutes, the students work at the problem, interacting with the diagram, as I recorded in my notes:

> Ben moved first, drawing in rows of tulip bulbs in the boxes shown in the diagram. He did 5x5 in the first empty box and then moved on to the next box. Curtis looked at what he was doing and then did something similar. At some point, Curtis came up with a solution, fairly quickly. He just wrote three numbers at the bottom of the answer box. I didn't understand his solution but explained that he needed to explain how he worked it out. He wrote a sentence along the lines of 'I added the tulips' – something quite general. So I said he needed to be more precise, to explain what calculation he did. At this point he explained to me verbally and I invited him to write it down. What struck me was that he had little trouble solving the problem, and that most of the time was spent on writing it down in an 'acceptable' way.

My account suggests that the students do relate to the mathematical pattern indexed by the diagram and are able to interact with it and, in particular, to extend it. At some level, then, there is some alignment at this point in the forms of language (including graphic elements) used in this part of the text, and the students' own linguistic repertoires. Their expression of their work, however, remains 'local'; that is, it makes sense to them but does not index more widespread forms of mathematical language (for which, in the interaction, I am positioned as the arbiter):

RB: you have to explain now that you've got these totals okay otherwise if somebody comes along and reads it they will wonder where the number comes from in these kinds of situational problems its quite important that you explain some how how you worked it out

My remarks make it explicit that the goal is to write in a way that is interpretable to some kind of generalised 'somebody' and a generalised situation "these kinds of situational problems" (which are typically used as assessment items). This is an example of what Blommaert (2010) calls 'scale jumping'. The students' linguistic productions index their own locally developed forms; my intervention indexes language forms and communicational requirements associated with people (e.g., teachers) and situations (e.g. assessment) that are more widespread and more valued.

The two students spend a relatively long time working on showing "all their work". Their interactions with me, including the following extract, indicate that this part of their work was quite challenging:

RB: so (.) that's a good beginning (.) but you need to explain like the calculations that you did (.) you need to say what kind of calculations you did
Curtis: times
RB: yup but precisely what did you times what did you add
Curtis: I timesed seven (.) times seven (.) six times (.)
RB: right right
Curtis: seven plus that's it
RB: so like when you worked out for purple
Curtis: I did five times five
RB: uhum
Curtis: plus one
RB: right so I would write purple and then exactly what you just said

The interaction between different speech types is particularly clear in this extract. My use of the word 'need', twice, again indexes expected mathematical ways of talking or writing and, indeed, implies they are a requirement. Through the interaction between Curtis and I, an account of his calculations is pieced together and to some extent endorsed by me. Nevertheless, Curtis's account makes use of relatively local forms of mathematical expression, particularly 'times', rendered as a participle 'timesed'. Throughout this extract, including in my own reflections shown in my notes, my utterances index acceptable ways of talking and writing about mathematics. Through the episode, there is some convergence in the students' utterances towards more conventional mathematical language. Needless to say, these conventions, indexed by the problem text and by me, do not make any reciprocal convergence towards the students' forms of mathematical expression.

DISCUSSION AND CONCLUSIONS

For the students and I, this episode involves an encounter with otherness, with difference, with what Bakhtin calls the alien word (see Barwell, 2014). The point I want to highlight in this paper, however, is that the speech types involved in this encounter reflect prevailing orders of indexicality. The students bring speech types from the periphery: those of Cree-speakers from James Bay for whom English is a second language. Their speech types also include local forms of mathematical language that make sense to the students, either individually or among themselves. The word problem text and I both deploy more authoritative speech types, where this authority comes from an indexing of assessment, of the requirements of the genre and of communicating one's work to a generalised other ("someone"). The encounter between the students, the word problem and me is filled with indexical complexity, but this complexity is ordered; the language of the encounter is stratified, with a hierarchy apparent in which local and peripheral speech types are less valued than more widely standardised forms of mathematical language.

The theoretical sociolinguistic ideas I have drawn on in this paper make it possible to see how the interaction in this second language mathematics classroom episode marginalises the two students. The concept of heteroglossia highlights the way specific utterances in the class are linked to broader, stratified, social patterns and the notion of indexicality facilitates a detailed analysis of this stratification as it plays out in the classroom. This is not to say that the students should not learn 'standard' forms of mathematical language in English; the indexicality of language is part of what makes communication possible. Blommaert, however, distinguishes between the indexical order, which refers to the patterns of language that allow us to recognise references to groups, activities or situations, and orders of indexicality, which refers to the stratification of language and which is implicated in processes of marginalisation. It is through the indexical order that the students recognise the geometric pattern and are able to work on it and find a solution. It is the ordering of indexicalities, however, that marginalises much of the students' repertoire of language practices. Unfortunately, there is no neat way to decouple these processes. This analysis sheds some light on how the socio-political dimension of language influences these students' participation in a mathematics classroom activity.

ACKNOWLEDGEMENT

The project was funded by the Social Science and Humanities Research Council of Canada, grant 410-2008-0544. I am grateful to the school and students for their generous participation. The transcription was prepared by Jennifer Chew Leung.

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ENDNOTES

1. To save space, I have not preserved the layout and abbreviated the problem, omitting additional information about the festival. The diagram is my reproduction of the slightly more elaborate version given to the students.

2. Transcript conventions: short pauses are shown by (.), overlaps are shown by [, rising intonation is shown by ?, emphasis is shown by bold type, whispered speech is enclosed by ^ ^.

Mathematics learning in whole class discussion: A design experiment

Kaouthar Boukafri, Miquel Ferrer and Núria Planas

Universitat Autònoma de Barcelona, Barcelona, Spain, kaouthar.boukafri@uab.cat

In this report, we examine the interacting use of whole class discussion and concrete manipulatives in the learning of geometry in a secondary mathematics classroom. We draw on our prior works on the generation and exploitation of learning opportunities in order to elaborate an example of opportunity for further reflection. In the analysis, two basic aspects are considered: (i) actions of collective argumentation and (ii) types of orchestration involved in the production of the students' learning. We illustrate the analysis through a moment of classroom talk in which a student is the initiator of an opportunity that leads to a situation of mathematics learning. We argue that the teaching activity is decisive in the joint exploitation of the opportunity.

Keywords: Whole class discussion, students' mathematics learning, problem solving, collective argumentation, concrete materials.

CONTEXT, QUESTION AND GOALS

The reported work needs to be situated in the context of a project strategy that includes the study of various mathematics lessons with different mathematical contents in use. All the work in the project is expected to contribute to the knowledge of mathematics learning by broadening the understanding of how mathematics learning opportunities may be created and exploited in classroom talk. Within this context, we address results from a research guided by the following question:

What are the actions involved in the creation of mathematics learning opportunities in whole class discussion with problem solving and manipulatives?

In the classroom for the conduction of the experiment, concrete manipulatives that could be physically handled by students were offered to explore and investigate mathematical concepts and processes for finding solutions to geometry problems. The potential role of whole class discussion and problem solving with manipulatives in the generation of mathematics knowledge was a key assumption in the design of the three-lesson intervention. In particular, a concern to this work was that by examining the role of manipulatives as mediating artefacts in classroom talk, certain learning opportunities in mathematics might be better examined and characterised.

For the time devoted to whole class discussion in each lesson, the same procedure for the analysis was considered. We searched for moments of classroom talk where an approach to the resolution of the problem was being discussed. For each moment and when possible, we identified mathematics learning opportunities and related them to particular mathematical contents. What we present in this report is the analysis applied to one of the identified learning opportunities. Elsewhere (Ferrer, Morera, & Fortuny, 2014) we have detailed our procedure for the detection of learning opportunities and we have examined the role of the teaching activity. In what follows we outline our theoretical orientation, describe the experiment, provide our methods, and discuss our data, findings, and future directions.

THEORETICAL PERSPECTIVES AND NOTIONS

A major problem in mathematics education research has to do with understanding and framing learning. Nevertheless, researchers in the field have reached important agreement on the fact that evidence of learning cannot be gathered in isolation, neither at the level of individuals nor at the level of groups (Sfard, 2001). Without denying the importance of the individual, social theories take the system of actions and practices as the starting and explanatory main component of learning.

In the line of social theories of mathematics learning (Goos, 2004), our work is placed within the tradition of design experiments in mathematics education research. As said by Cobb, Confrey, diSessa, Lehrer and Schauble (2003), this type of experiment aims at identifying and explaining successive patterns emerging from the study of connections between the students' learning and the classroom circumstances in which it is developed. For the conduction of an experiment, the phases are planning, implementation, evaluation and iteration. In this report, we refer to data coming from the first round of implementation of the planned teaching sessions.

As part of our project strategy, design experiments are supported to develop knowledge on mathematics learning, and in particular around mathematics learning opportunities. The notion of mathematics learning opportunities is central to our research as a way to link the social aspects of classroom activity to the students' development of mathematical ideas (Planas, 2014). In our analysis of practices that potentially foster learning opportunities during classroom talk, we give priority to actions of collective argumentation and types of orchestration, which actually are specific types of sequenced actions. To this respect, we plan design experiments in which conceptual and procedural forms of mathematics learning are expected to be facilitated by means of an interacting system involving inquiry-oriented tasks and pedagogical resources such as physical artefacts.

As claimed by Miranda and Adler (2010), there is little literature on the role and use of concrete manipulatives in the teaching and learning of mathematics. From the perspective of the teaching activity, it is argued by these authors that the presence of manipulatives in the development of a task needs explicit and reasoned justification so that students are told the importance of using those resources. This is why we collaborated with the teacher to foster responsibility for a form of teaching activity that privileged the use of manipulatives not only during the time for group work but also during the presentation of the task and its discussion in whole-class talk.

THE CLASSROOM EXPERIMENT

The experiment consisted of three lessons in a classroom of 12 years-old in a school of Barcelona, Catalonia-Spain. What was first selected was the teacher, on the basis of her expertise in teaching mathematics for several years and her active involvement in our research team. She was given in advance the sequence of three geometry problems and she made relevant contributions in order to adapt the wording and content to the particular group of students. The lesson dynamics was also negotiated with the teacher. The students were first asked to read the problem, to work in small groups, and finally to participate in a whole class discussion. During group work, the students were provided with problem-based concrete materials and had to produce written individual responses; after thirty minutes of group work, the teacher took the materials away in order to encourage finishing the responses. The teacher had the materials for manipulation in her guiding of the interaction with the students during whole class discussion. It was possible, and in fact it was promoted, to complete, revise or modify responses up to the end of the lesson. The students in this classroom were used to similar dynamics but the work with manipulatives was new to them.

Packing glasses

We have 12 glasses, each of them measuring 92mm (height) and 74mm (diameter of the major circle). We want to find the cheapest box for all glasses, that is, the box requiring the least material. In addition we want:
– The base of the box to be rectangular.
– All glasses to be facing up in the box.
– No glasses inside each other.
Under these conditions:
What are the minimum dimensions for the box to contain all the glasses?
Which data will you give the shop owner to order the box?

Figure 1: The problem of the second lesson

'Packing glasses' (see Figure 1) was the problem for the second lesson. It was adapted from a problem created by the Millennium Mathematics Project at the University of Cambridge (for the original wording see http://nrich.maths.org/880). The problem was thought of as useful for dealing with early geometrical modelling and optimization of area and perimeter. Several approaches and resolutions are possible, as well as follow-up questions depending on the evolution of the students' talk. The teacher was asked to first introduce the problem and the material, and then to handle the final class discussion with attention given to the mathematical talk of the students. There were two main objects: glasses and boxes. All small groups were given 12 plastic glasses. In the facilitation of whole class discussion, the teacher had three scaled boxes that represented the three possible solutions (with glasses being aligned in one row of twelve, two rows of six, and three rows of four) and the box that represented the case for one glass (one row of one). It was assumed that the manipulation of glasses and boxes would help better understand the problem and the required optimization processes to solve it.

METHODS OF A TWO-SIDED ANALYSIS

Lessons were video-taped and whole class discussions were transcribed. Each transcript was organized into shorter transcripts around moments of class discussion with students exploring an approach to the resolution of the problem. The difficulty of determining the exact turn on which the discussion of an approach started and finished, was addressed by including the turns that were dubious for some reason. On the other hand, participants commented on the same approach at different stages of the discussion; this is why the transcript of a moment did not necessarily consist of consecutive turns. In fact, the moment for illustration in this report is an example of non consecutive turns having been grouped together on the basis of the resolution strategy being under discussion. Its transcript stands for the explicit talk around the required quantity of material for any of the solution boxes.

The construction of transcripts was followed by the search for learning opportunities arising from the interaction among participants in classroom talk. Drawing on the notion of learning opportunity, special attention was paid to reactions of students that serve for clarification, exemplification, generation of new questions…, and which might be explained as provoked by prior interventions of other participants in that lesson. In case of differing interpretations within the team, we went back to the videos until we agreed on a decision. Actions and reactions were initially associated with opportunities to participate and interact in classroom talk, and only when mathematical content was at focus, they were regarded as mathematics learning opportunities. It was during the observations and analyses conducted in other school settings (Ferrer, Morera, & Fortuny, 2014), that the relationship between opportunities to interact in classroom talk and opportunities of mathematics learning was decided as an effective way to approach the detection of mathematics learning opportunities.

Actions of collective argumentation and types of orchestration

After having linked the transcript of a moment to a mathematics learning opportunity by reflecting on the mathematical contents in use, we split the analysis into two parts according to the characterisation of actions of collective argumentation, on the one hand, and types of orchestration, on the other. The selection of this two-sided analysis responds to the idea that the kinds of talk in the mathematics classroom can be interpreted in terms of the quality of the collective argumentation (Krummheuer, 2007) and the variety in the orchestration (Ferrer et al., 2014). Elsewhere (Planas & Morera, 2011) we have commented on collective argumentation among students in the construction of mathematics knowledge in two secondary classrooms.

The distinction of actions of argumentation was carried out at the micro-level 'within-the-moment'. All turns of talk were studied in order to decide whether mathematical reasons for particular statements were being provided. Further, we observed how the different contributions were taken up, either individually or collectively in public classroom talk. There may be a wide range of mathematical reasons depending on the complexity of the inquiry activity (Goos, 2004): from the mere description of an answer, a property or a fact, to the thoughtful proof of a conjecture. Mathematically wrong reasons in the context of the considered statement were not excluded as they helped to understand the coherence in the progression of mathematical talk. We drew on the idea that mathematics learning occurs in the coordinates of the combined potential movements between mathematically wrong and correct actions of collective argumentation in classroom interaction.

The distinction of types of orchestration was also carried out at the micro-level 'within-the-moment'. We used the following six types: exploring the artefact, explaining through the artefact, connecting artefacts, discussing the artefact, discovering through the artefact and experimenting the instrument; and we added the type 'replacing the artefact' to refer to situations in which participants pointed to different artefacts (blackboard, boxes, applets, glasses...), but did not establish connections among them. The importance given to the use of artefacts in the broader project (for findings about mathematics learning in whole-class discussion with dynamic geometry software, see Ferrer et al., 2014) comes from our interpretation of who the relevant others in the interaction are. Classroom talk is developed in interaction with subjects and objects, and consequently the others are not always individuals, students or teachers, but may be represented by sorts of artefacts.

The two-sided analysis applied to moments in which learning opportunities had been identified became a powerful tool for the detection of mathematics learning. It allowed us to conclude that certain opportunities had been exploited into mathematics learning. To this respect, convincing evidence was found, either in classroom talk or in students' written responses, of changes in the understanding of mathematical contents that had been involved in actions of argumentation in interaction with the orchestration of specific artefacts. Exhaustiveness in the identification of learning opportunities and learning was not attempted through the application of our two-sided analysis, and in fact it was in some cases deliberately avoided, since the extent to which mathematical talk had evolved in some turns was difficult to interpret.

EXAMPLE OF LEARNING OPPORTUNITY

We have selected an example of mathematics learning opportunity where it can be acknowledged the role of a student, Aloma, in the introduction in classroom talk of ideas that lead to cohesive reasoning and progress in the resolution of the problem (see Figure 1). It is an example in which a diversity of actions of argumentation and types of orchestration are involved with respect to three main artefacts: blackboard, glasses and cardboard. We draw on the analysis of this example as evidence in support of the potential of the teaching activity in interaction with classroom talk, collective argumentation, and problem solving with manipulatives. We begin by reproducing most of the transcript of the moment related to the opportunity (Roman numerals are included for organization of cross-referenced contents in Table 1):

1 Teacher: Come on, Aloma.

2 Aloma: A way for knowing it would be, for instance, to take into account that each glass, it touches up and down...

3 Teacher: That is, if we put each glass in an individual box [working on the blackboard], it would touch all four [pointing to the drawn lateral] walls; but in this one [indicating the drawn one row of twelve], the second glass would touch here and here [pointing to two drawn lateral walls] and those in the corner would touch three [pointing to three drawn lateral] walls.

4 Aloma: [I] This way we use plenty of cardboard, [II] but with the box of three per four we need less cardboard for the glasses in the corners.

5 Teacher: [I] This glass, for instance, it only touches one wall, while this one does not touch any [pointing to the box with three rows of four]. [II] Here we have built a model for the individual box with the real measures as said by Aloma. (...) Aloma's reasoning is great! [III] She is trying to reduce all these walls as much as possible. Thus, if I want to build a box for the twelve glasses and place the individual small boxes within the three per four, then when two walls get in contact, we can eliminate them because we only want those that are external [in the video Figure 2]. [IV] That is, the glasses in the four corners contribute with two pieces, the others with only one piece, and those in the middle with none. (...)

6 Teacher: What Aloma explained, is it clear? What she said about saving walls, eh? Here [showing the box with three rows of four] we need a total of fourteen walls. But with this [showing the box with two rows of six] we need sixteen walls, and finally, this [showing the box with one row of twelve] needs a total of twenty-six walls. Yes, everyone? Clàudia?

7 Clàudia: The less glasses touching the walls, the best.

8 Teacher: The less glasses touching the walls, the less quantity of cardboard.

Figure 2: Teacher explaining through cardboard

The eight turns that constitute the moment in this transcript were analysed according to the detection of actions of argumentation and types of orchestration. Table 1 situates the turns in relation to actions and types, and summarises some of the curricular mathematical contents involved in the explicit talk. The joint analysis of the transcript and Table 1, along with the video of the lesson indicates the creation of at least a learning opportunity around the optimization of the geometric variable corresponding to the surface of a rectangular volume. Aloma is the student who acts on the glasses and the cardboard to justify the need for establishing the individual box for only one glass as the surface and volume unit of measurement (i.e., the small box is presented as a box of capacity one glass), and simultaneously as the unit of counting on the basis of how many individual boxes can be placed in the considered solutions and how they can be placed. Through talk, manipulation and interaction with the teacher, Aloma grounds her reasoning on the quantity of cardboard for the different considered boxes by comparison with the unit box.

The talk initiated by Clàudia provides evidence of exploitation of the identified opportunity into learning. The teacher also provides further evidence of exploitation of the reasoning introduced by Aloma and expanded by Clàudia. This happens, however, later in whole class discussion, when the teacher modifies

Turns		*Action of argumentation*	*Type of orchestration*	*Curricular content*
1 Teacher		Communication of empirical evidence	Experimenting the instrument (glass)	Bidimensional representation of a rectangular prism Optimization of the perimeter of the prism base given its surface
2 Aloma		Communication of empirical evidence	Experimenting the instrument (glass)	
3 Teacher		Particularization and study of a possible solution	Explaining through the artefact (blackboard)	
4 Aloma	I	Affirmation with empirical support	Replacing the artefact (blackboard → cardboard)	
	II	Comparison among possible solutions	Discovering through the artefact (cardboard)	
5 Teacher	I	Particularisation and study of an alternative solution	Explaining through the artefact (cardboard)	
	II	Establishment of unit of measurement and counting	Exploring the artefact (cardboard)	Units of measurement and counting
	III	Validation and inference of reasons grounded on the unit	Linking artefacts (blackboard & cardboard)	Relative positions of a unit Optimisation of the lateral surface of a rectangular prism
	IV	Classification of space positions relative to the unit	Explaining through the artefact (cardboard)	
6 Teacher		Emphasis and conclusion on a solution from all options	Explaining through the artefact (cardboard)	
7 Clàudia		Affirmation with empirical support	Discovering through the artefact (cardboard)	
8 Teacher		Formalisation of an informal reasoning	Explaining through the artefact (cardboard)	

Table 1: Actions, types and contents in a moment of the second lesson

the condition of a rectangular base for the box. It is a modification that leads to different options for the solution boxes, but still keeps valid the conjecture, "The less glasses touching the walls, the best." From there the mathematical discussion moves toward the problem of isoperimetric space figures and the visualization of cylinders.

By examining how mathematics learning opportunities can be promoted through whole class discussion with problem solving and manipulatives, we have detected opportunities for our conceptual understanding of mathematics learning and the classroom environments in which it may occur. Our research shows that certain actions of argumentation in interaction with certain types of orchestration are positively related to the creation and exploitation of learning opportunities around mathematical contents that are not strictly procedural (see Table 1). We are striving to understand, however, features of the teacher's teaching activity that are decisive in the process that goes from classroom talk to learning opportunities, and from there to mathematics learning. Although important, we have not started looking at whether differences in teaching activity explain differences in learning moves.

FINAL CONSIDERATIONS

We have examined the relationships between actions of argumentation and types of orchestration in the generation of mathematics learning opportunities. We still know little about why and how mathematics learning occurs. Nevertheless, our study provides some light on lessons with problem solving and manipulatives in which learning relates to the development of collective argumentation in interaction with the orchestration of particular materials. The progressive sophistication of argumentation is probably fostered by the role and use of materials, but in the empirical context of our study it is too early to conclude on this. What we already know is that these issues, actions of argumentation and types of orchestration are clearly important for the understanding of the dynamics of learning. First, the notion of collective argumentation has been used to illustrate the importance of interaction among participants –subjects. Second, the notion of orchestration has been used to illustrate the importance of interaction with materials –objects. Other two-sided analyses would be possible under the same idea of keeping balanced the interaction with subjects and that with objects in the mathematics classroom.

Researchers interested in the role and use of artefacts may focus their questions on processes and tools through which students elaborate their thinking strategies in interaction with dynamic geometry software, physical manipulatives, virtual manipulatives, etc. Other researchers concerned with the role and use of talk may focus their questions on outcomes of pair work, small group, whole class discussion, etc. Similarly to Fetzer and Tiedemann (2015), our conceptualization of learning as learning-by-talking-and-doing seeks for integrated approaches. We are aware of the risks of misrepresenting any approach, and more generally, of the few empirical studies in the field taking a balanced subject-and-object perspective. By means of the example in this report, we have attempted to explain that the integration of subjects and objects, instead of its distinction, is at the root of our social view of mathematics learning. Manipulation of objects (either Platonic mathematical objects or concrete manipulatives) and talk with subjects (either one self or others in the class) are expressions of the same basic realization called mathematics learning.

ACKNOWLEDGEMENT

To the Spanish Government, for funding grants EDU2012-31464, EDU2011-23240, BES-2013-063859, BES-2012-053575. To the Catalan Government for funding grants GIPEAM-2014SGR-972, ICREA-Academia. To Laura Morera for her collaboration in the teaching experiment.

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Number words in ‘other’ languages: The case of little Marram

Anna Chronaki[1], Georgia Mountzouri[1], Maria Zaharaki[1] and Núria Planas[2]

1 University of Thessaly, Department of Early Childhood Education, Karditsa, Greece, chronaki@uth.gr

2 Autonomous University of Barcelona, Barcelona, Spain

The present paper, based on a small scale critical ethnographic study, explores the process of experimenting collaboratively with multiple language use for number words as part of young children's mathematical learning activity. Data from a teaching experiment called 'Number words in 'other' languages' is utilized to illustrate the creation of a culturally responsive context with children aged 4 to 6, their parents, the classroom teacher and the researchers. The focus is on the case of little Marram, a Pakistani girl who lives in Greece and who experiences participation by sharing number knowledge in her dominant home language, Urdu. It is highlighted how discourses on gender and language determine Marram's learner identity-work in the multilingual preschool classroom.

Keywords: Early mathematics, language use, gender, discourses, classroom practice, identity-work.

INTRODUCTION

In a number of previous studies we have denoted the benefits of diverse language use in the mathematics classroom as a matter of resourcing learning (Planas, 2014), creating dialogicality (Chronaki, 2009) and, also, troubling essentialist identities embedded in school mathematics discursive practices (Chronaki, 2011). We now focus on an experimental collaborative project, *Number in 'other' languages*, where the design of learning for number and number word is orchestrated around the utilization of the participants' languages in a Greek classroom. Multiple language use can be interpreted as a resource for engaging children from ethnic minority groups, and as a space for reconfiguring subject positions in processes of learner identity-work. Taking into account that ethnic minority learners experience marginalized subject positions in school, the shifting relations encouraged via opening the learning of numbers in multiple languages may create conditions for breaking the certainty of hegemonic discourses and provide space for accessing dialogicality. As a Bakhtinian concept, dialogicality weaves a way to critique language formation in literary texts. Bakhtin in a series of texts (1929/1981) negates language as an abstract system and emphasizes its intersubjective consciousness. By perceiving language use as social in nature he conceives language, communication and identity as interactive multi-voiced phenomena rooted is specific localities. As such, the idea of teaching number words in multiple languages becomes a gesture of how mathematical knowledge can be approached as culturally and politically situated. In this way, mathematics becomes a space for providing access to children's own trajectories as part of their own identity-work and subjectivity configuration (Chronaki, 2009, 2011).

During the development of the present collaborative project, the first two authors observed, depicted and interpreted the complexity of the teaching activity as experienced by children coming from socially marginalized groups. The experiment was designed to serve goals of counter education (Gur-Ze'ev, 2005), in ways that challenge stereotypes and affective positions toward dialogical participation. For the case of little Marram, a Pakistani girl aged 5 years, we analyze how the chance she is given to express out in her mother-tongue the number words, offers a chance to explore her position in the class. It helps us understand complexity throughout actions in the classroom, the family and the school community. Our analysis searches for opportunities to access mathematical knowledge, identity-work and subjectivity that place learners into positions of participation in the classroom. Some opportunities, from Marram's position as a Pakistani girl in a mathematics lesson, are examined.

Educators and researchers who are, like us, interested in focusing on the complexity of the social, cultural and political contexts of mathematics classrooms face the challenge to develop tools and interventions in order to analyze, discuss and re-configure issues connected to relationships amongst authority, knowledge, identity-work and subjectivity. In our research contexts, we draw on critical approaches for mathematics teaching and learning in multilingual classrooms where children are taught the norm of school knowledge while they co-configure mathematical subjectivities as part of broader social requirements for identity-work as learners of mathematics (Chronaki, 2009, 2011; Planas, 2011). Below, the methodology of the study is outlined and the data analysis is organized around three sections.

METHODOLOGY: CRITICAL, COLLABORATIVE AND EXPERIMENTAL ETHNOGRAPHY

The reported work is situated in the qualitative tradition of educational research, and more specifically in the context of critical, collaborative and experimental ethnography. This study is critical in the sense that it intends to critique what is taken as 'reality' in the mathematics classroom, to disrupt taken for granted 'truths' about who is the competent learner and what is valuable mathematics, to trouble discourses that tend to support the construction of hegemonic interpretations and, also, to imagine how things could have differently happened (Thomas, 1993; Skovsmose, 2014). It is a collaborative study in that it emerges from the need of dialogue amongst participants. The project was held in close collaboration with children and their parents who contributed with information concerning their own ways of using language for number words and counting (Lassiter, 2005). It was, also, organized as a teaching experiment in collaboration with the classroom teacher as it aimed to expand the formal mathematics curriculum on counting and to provide links with children's funds of knowledge. Children coming from different ethnic minorities could perform early number activity (e.g. counting, related number word and number symbol in both oral and written genres) in their mother tongue and share number knowledge amongst them. Multiple perspectives of an issue were discussed, debates were constructed and all these helped data interpretation and inference of conclusions.

The methodology is experimental in the sense that the research process was organized as a series of interactive events in a teaching experiment on number that took place for two weeks during the school year 2011–2012 in a class of a public nursery school in Athens. The experiment encountered the urge to teach number in relation to opportunities for intercultural education as has been required by reforms of the Greek curriculum (Department of Education, 2011). The successive arrangement of the teaching experiment covers the sections on which the Greek curriculum focuses on number and aims to investigate its conduction in a multilingual classroom in order to achieve intercultural contact, mathematical participation and the subversion of stereotypes concerning curricular contents and who is capable of expressing opinions about mathematics and of arrogating mathematics as a cultural or symbolic commodity. Specific material was devised as seen in Table 3, but also Tables 1 and 2. During the implementation phase, the teacher was allowed to modify parts of the interventions, while one of the researchers in the team was integrated as an active participant and internal observer in the classroom.

Participants were the twenty children of the class, seven of whom had not Greek nationality at the time of the study, their parents and the classroom teacher. The home languages of these children were Albanian, Russian, Armenian, Filipino, Turkish and Urdu. Many of them had been born in Greece and their parents are either economic immigrants or political refugees. Marram came from Pakistan and her first language is Urdu although her mother uses English as a means of communication as well. In what follows, we will focus on an event when little Marram is encouraged to speak out loud the number words in Urdu. This event points to a number of difficulties, dilemmas, challenges and achievements of Marram, in a situation of struggle amongst local and globalized discourses about school mathematics, girls, and ordinary languages. Pereen (2007) based on Brandist (2002, p. 179) explains that for Bakhtin 'personal identity' transforms into 'an intersubjective dynamic' that is continually renewing itself. Subjectivity, instead of personal identity, is captured in terms of 'being-as-event' in Bakhtin's words and as such it needs to be conceived as situated in the ongoing eventness of everyday practices in the school classroom.

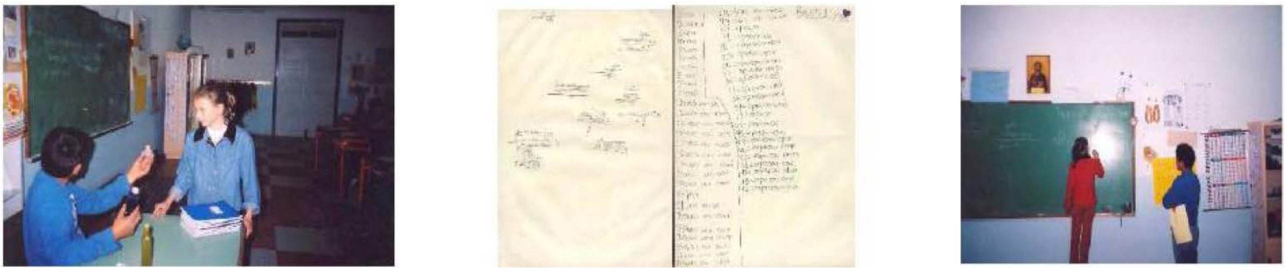

Table 1: Romany and Greek number words (Chronaki, 2009, 2011)

Table 2: A playful outdoor activity on number (Chronaki & Mountzouri, 2009)

DIVERSITY OF NUMBER WORDS: GETTING TO KNOW THE 'OTHER'

The project *Number words in 'other' languages* is part of our attempts during the last years to approach mother-tongue use in classroom mathematical activity as interlinked with learning, identity-work and dialogicality. In Chronaki (2009, 2011), and Chronaki and Mountzouri (2009) we have discussed this perspective first as part of project work in a primary classroom with Roma and non-Roma students (Table 1) and then as playful outdoor activity based on the explorative use of the languages of Greek, Romany and Arabic (Table 2).

The experiment presented here expands into seven languages so that to include all children in the lesson whilst focusing on number counting through number words. A board with number words in all languages was prepared (Table 3).

Children were asked to read words on the board in collaboration with the teacher in whatever manner was feasible for them providing their competence. Although children were familiar with reading, reading and interpreting numbers was a collaborative process with all children involved with teacher's help. Children recognized number digits from 0 to 10 in the vertical axis and country names in the horizontal axis. It was aimed to have all children watching the number words in each of the spoken languages and making comparisons based on both oral sounds and visual stimuli. In parallel, children had the chance to spot specific countries on a given world map. Later, children were asked to refer to the board (Table 3) and speak out the numbers in their mother-tongue. After some discussion they arrived at questions and conclusions like: Which number words are common in languages other than Greek? (0 was linked to all three languages) Which ones have a similar sound? (tre-tree-treea and u-do) Which ones do we use today in Greek? (dort, dortia as used in the dice).

Table 3: Reference material for 'number words in 'other' languages'

NUMBERS IN MY LANGUAGE?! INTEREST AND EMBARRASSMENT

Most children experienced both interest and embarrassment when they were asked to rehearse counting in their mother tongue. Children from ethnic minorities had experienced 'mother tongue' as forbidden in the monolingual school context where Greek is the only language of instruction. In the case of little Marram, although shyness predominates, the co-presence of interest and embarrassment was evident, as can be seen in the event below, and provided the opportunity to discuss parts of how her mathematical subjectivity evolves (Episodes 1 and 2).

Episode 1. First steps into oral counting in Urdu

Researcher: Which child in our classroom is from Pakistan?

Christina: (raises her hand smiling) Marram!

Researcher: Marram, come here! Marram, I would like you to tell us numbers in Pakistani (*invites her to stand in the centre of the circle; she uses the word Pakistani because children know that Marram comes from Pakistan*). I am asking her in English because possibly she cannot understand in Greek. (*to Marram*) I want you to tell us the numbers in Urdu (*Marram smiles, touches her cheeks with her hands, twirls a lock of her hair but does not answer*). Let's do this, sifr, ek... sifr, ek... I'm gonna help you, sifr, ek, do, let's go!

Episode 2. Marram's oral counting in Urdu

Researcher: She has told us before. Do you remember it? Come on! Come on, Marram! Sifr...

Marram: Sifr, ek, do, tin, char, panch, che, saat, at, no, das... (*counting fast; when she finishes she covers her eyes with her hands [Figure 1]*)

Researcher: Well done! But I want you to do it slower. Slowly, okay? Let's do this! You will do this as so: sifr, ek, do...

Arian: Sifr. (*Arian from Albania reminds her the word for zero*)

Researcher: Let's do this! Sifr...

Marram: Sifr, ek, do, tin, char, panch, che, saat, at, no, das. (*slowly*)

Researcher: Well done! Please, applaud her! Well done Marram, thank you!

Attempting to interpret Marram's position in the above two episodes, we denoted at least two significant issues that determine how her subjectivity becomes reconfigured as part of this specific activity and mediate her efforts towards performing counting; shyness and the use of English language. Both of these issues are discussed below.

Being shy: A gendered position for Marram?

Marram showed shyness and intense embarrassment when asked to express loudly the numbers in Urdu -her mother tongue. Embarrassment or shyness are, often, embodied through a variety of facial expressions and awkward body (especially hand) movement. They are also linked to reluctance of taking an initiative attributing thus a weak, passive or idle subjectivity. However, in the words of Marram's mother, shyness needs to be also considered as a 'female' feature of virtue in Pakistan and its expression is something not only normal but highly desirable for young girls. During an interview with Marram's mother and whilst discussing the significance of the above episodes, she explains:

> "Women in Pakistan, we are shy, it's our nature, in our culture. Even here, most people do not know the meaning of shyness... Many children in countries like ours feel shy and do not often participate in every single activity of the classroom. I want my children to make it not only in

Figure 1: Facial expressions by Marram whilst counting

> mathematics but also in all fields.” (Interview with Marram’s mother)

Shyness from the mother’s perspective is a female attribute in Pakistan denoting that young girls grow into women and they become sensible of their femininity. It is performed by women, especially young or non-married. In Pakistan shyness is considered an ethically proper behavior and is linked to virtues like rationality and modesty. Marram’s mother, however, was worried during this part of the interview and asked the interviewer if her daughter was behaving too shy in the classroom. Her concern was based on her fear that Marram’s shyness might not be suitable or compatible with modern cultural norms such as competition, assertion and reclaim. She showed awareness on how Marram’s reluctance could create difficulties for negotiating her position and presence in the public place and believed that her daughter needed to grow confident, assertive and participatory. In Civil, Planas and Quintos (2012) the relevance of considering the family contexts of the learners to better understand some of the classroom events and its participants has been argued. For the case of Marram, the conversations with her mother became crucial for making sense of the girl as participant in the nursery classroom and, specifically, on how she experiences learner subjectivity as she is being caught among various cultures and maybe in the margins of all of them. Civil et al. write about identity issues that immigrant children confront in feeling caught amongst the parents’ culture and the culture in their new country. From this perspective the theme of shyness needs to be interpreted in the case of Marram as a matter of both culture and gender.

English language mediation: What is the status of ‘mother’ tongue?

One can argue how Marram is growing up bilingual as she speaks Urdu at home and understands Greek at school. Talking with her mother, we realized that she needs to be considered as trilingual since she also uses English, mainly with her mother. According to Skourtou (2001), bilingualism refers to the “alternative use of two or more languages from a single person” (p. 199) - a definition that can be expanded to trilingualism. The use of English by the researcher in the class was made to accommodate Marram’s needs and competences. It was a subtle wish by Marram, expressed during initial interviews where she asked the researcher to speak in English. Marram often switched codes not from Urdu to Greek but from Greek to English and vice-versa. By code switching we mean “the practice of using two or more languages in the same communicative act” (Tsokalidou, 2000, cited in Skourtou, 2001, p. 184). When Marram could not describe something she wanted in Greek, she spontaneously used English in the interviews, but she did not do so in the class where her peers and teacher were present. She might think English was not proper in the lesson with other children who could not understand English.

With respect to this second theme, the family context also plays an important role. For Marram, the use of English is symbolically reinforced by the fact that her mother used to work as an English teacher and was head of an English-learning centre in Pakistan (country in which English is still considered an official language for instruction based on the national curriculum); when she came to Greece, however, she began to work at a wax factory. Her mother values English as ‘international’ language, acknowledging its colonialist influence in Pakistani educational system and its globalized hegemony. She wants Marram to reach better access to English language in order to upgrade socially through her studies and future profession. An important point for the mother is the issue of English language learning in the school context. She stated that the fact that the other teachers of her daughter could not speak English caused her difficulties in the communication with them and was a main reason why she did not often visit them at school. Also, in her words it was an ‘insane’ educational choice, as she argued in one of the interviews:

> “English should be spoken at school apart from Greek. But here, at schools there is not an English system. This is bad due to the fact that English is an international language.” (Interview with Marram’s mother)

In the described event, the use of English by the researcher played a significant role in the process of code switching between Greek and Urdu and empowered Marram to participate more fully in the public space of the classroom. More generally and during several classroom events, it was seen that the use of a third language (here, English), which Marram had learnt (and valued highly) from her mother, played an intensively mediating role in this learner’s mathematical participation. Despite Marram knew the number words in her dominant language, for instance, she needed encouragement in a third language which she could easily use, and with which she feels familiar to express them out loud. Thus, English appears to act as a tool facilitating her public presence in the class. It was not only constant encouragement but particularly the language selected by the researcher at different times that acted positively in her occasional overcoming of shyness to mathematically participate by counting numbers in Urdu.

CONCLUDING REMARKS

In this report, the case of little Marram rehearsing numbers in her mother-tongue has served as an event that exemplifies how mathematics teaching and learning in the school classroom is mediated by participation issues that are not directly concerned with curricular content or with didactic methods. According to our research data (teaching experiment observation and interviews) as well as our interpretation of Marram's case as a whole, it seems necessary to draw on a broader perspective of mathematical subjectivity as related to the complexity of children's participatory experiences, identity-work and learning that is, simultaneously, personally, culturally and socially shaped.

Our study has identified the important role of using multiple languages (Greek, Urdu and English) related to a child's worlds in forming the background of the mathematical experience of a single learner (Marram) through exploring her classroom interaction with an adult who acted as a teacher and who was particularly concerned with the fact of language issues intervening in mathematics teaching and learning. Taking into account that the mediation of English is fundamental in the Pakistani school system, as reported by an informed adult and, in this case, a relative of the child (Marram's mother) secures and increases awareness. The adult's attention to this very fact comes to facilitate the orchestration of English as a tool for the expression of ideas but also for the realization of strategic movements of code switching between languages other than English (Greek and Urdu).

The project *Numbers in 'other' languages* discussed here has forwarded the launching of mathematical ideas of others to all children in a classroom. Such an activity facilitates and bridges the connection of informal knowledge that children acquire through their family environment or home languages and formal knowledge that is being taught in the school context. In this way, all children's self-confidence is boosted as they are bestowed with the chance to perform their learning abilities, skills or competences as part of their own sociocultural experiences. The above are consistent with a broader perspective of mathematics as a cultural, social and political asset that teachers are willing to accept through various interpretations and ways of expression from children.

A similar project can be potentially implemented using not only language but also artifacts, objects or materials related to mathematics and emanated from 'other' cultures and social practices. Presmeg (1999) mentions that the use of objects originated from the specific cultural positions they come from, facilitates the connection amongst mathematics in school and society. Although the perils of exoticization, mysticism and idolization of mathematical knowledge still lurk, and dilemmas, challenges regarding learning process are many, their inclusion in pedagogical design continues to be necessary and important to explore further. The present team of researchers shares a mutual understanding of these emerging themes being fundamental for interpreting Marram's experience and accounting the complexity and multiplicity of mathematical subjectivity in modern times. The first theme –being shy– deals with the emotional and affective dimension of the learner and reveals how far she has progressed throughout the discursive practices of school mathematics in order to speak out the numerical sequence in Urdu. The second theme –mediation of English in code switching– deals with the communicative dimension of the participants in the lessons and reveals the role and use of language diversity and language hierarchies in mathematics teaching and learning.

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Vague language and politeness in whole-class mathematical conversation

Thérèse Dooley

St. Patrick's College, Dublin, Ireland
Dublin City University, Dublin, Ireland, therese.dooley@dcu.ie

Politeness theory is based on the notion that individuals in a conversation are endowed with face – positive face is concerned with a desire for social approval and negative face is concerned with a desire to be unimpeded. The theory is relevant in the context of whole-class discussion of mathematics where a teacher has to facilitate development of students' disciplinary understanding and, at the same time, reduce their threats to face as they make contributions in public. In this paper, it will be shown that a teacher's use of vague language can play a role in protecting threats to student face and thereby facilitate participation in argumentation and reasoning. It will also be shown that the competing claims on a teacher's attention in this context render his/her role highly complex.

Keywords: Politeness, face, vague language, whole-class discussion, primary mathematics.

INTRODUCTION

At a time when there is an emphasis in mathematics teaching and learning on the co-construction of meaning by teacher and students, it can be challenging for a teacher to take a supportive role in the classroom and, at the same time, steer students towards increasingly sophisticated understandings of mathematics. This challenge is exacerbated in the context of whole-class discussion where the teacher has to take account of the vulnerability that students might feel when they make a contribution in the public forum of a classroom. The strategies that he/she often employs to meet both of these demands are generally indirect, e.g., the use of questions, or revoicing (Brodie, 2010). The use of such indirect strategies relates to politeness theory – a theory that was constructed by Brown and Levinson (1987) to describe pervasive features of social interaction. Politeness theory has been used as an analytical framework in a range of contexts, including the teaching and learning of mathematics (Bills, 2000; Rowland, 2000; Weingrad, 1998). Rowland (2000) has focused on the role that vague language plays in supporting politeness in mathematical conversations between teachers and individual (or small groups of) students. In this paper, the role of vague language is extended to politeness in whole-class mathematical discussion. It will be shown that a teacher can use it as a means of developing a learning environment where children take intellectual risks and develop a view of mathematics as a subject that is a human activity and a social phenomenon (Hersh, 1997).

POLITENESS THEORY

One of the ideas upon which politeness theory is built is that each participant in a conversation is endowed with *face* (Brown & Levinson, 1987). Face, a term used metaphorically to represent respect, esteem and sense of self, takes two forms: positive face, a desire to be appreciated and valued by others, a desire for approval; and negative face, a concern for freedom of action, a desire to be unimpeded.

However, certain acts threaten face. Such face-threatening acts (FTAs) can be directed towards positive or negative face. For example, criticism and disagreement threaten positive face whereas orders and requests threaten negative face. Moreover, the seriousness of FTAs is influenced by factors such as the power relation or social distance[1] between speaker and hearer.

Threat to face can be mitigated by use of redressive actions which include positive politeness (oriented to positive face), negative politeness (oriented to negative face) and use of hints and metaphors. It is in the mutual interest of persons involved in a conversation to maintain each other's face, as part of a strategy for maintaining their own face. Brown and Levinson present a range of strategies that are available to a speaker in order to protect, or not, the face of a hearer. For example, if an individual solves a mathematical problem incorrectly, a colleague might deal with it in one of the following ways (see Rowland, 2000, p. 86):

1) Don't do the FTA – simply agree or keep quiet.
2) Do the FTA by
    a) going 'off record', that is implicating the FTA rather than doing it directly (e.g., 'I wonder if we have done a problem like this before...')
    b) going 'on record' either
        i) baldly – making no attempt to respect face ('That is not correct')
        ii) positive politeness ('You have come up with a really interesting way of solving that problem but I thought that ...')
        iii) negative politeness ('Would you mind showing me how you applied this formula here...')

While there is a variety of ways in which teachers endeavour to protect the face of students in small-group conversation (Bills, 2000) or in whole-class discussion (Weingrad, 1998), of particular interest to this paper is the way in which vague language used in mathematical (and other) contexts – in particular, (a) the pronoun *we* and (b) linguistic hedges – can be exploited by a teacher to serve this purpose.

VAGUE LANGUAGE

Rowland (2000), amongst others, maintains that coming to know mathematics is imbued with uncertainty and that the use of vague language – for example, hedges and pronouns - points to these uncertainties.

Hedges

Linguistic vagueness is encoded by *hedges* which are words "whose meaning implicitly involves fuzziness - ... whose job is to make things fuzzier or less fuzzy" (Rowland, 2000, p. 471). Rowland (2000) developed a taxonomy of hedges with reference to the discourse of mathematical conjecture. The first major type of hedge, *a shield* indicates some uncertainty in the mind of the speaker in relation to a proposition. There are two types of shield: (a) a *plausibility shield* and (b) an *attribution shield*. A plausibility shield (e.g., "I think", "probably", "maybe") can suggest some doubt on the part of the contributor that the statement will withstand scrutiny. For example, in the statement, "I think that the sum is twenty", the speaker injects a level of vagueness into his mathematical assertion and thus implicitly invites feedback on his solution. The attribution shield implicates some degree or quality of knowledge to a third party (e.g., "Ann got an answer of twenty"). The second major category of hedges are termed *approximators*. The effect of the approximator is to modify the proposition rather than to invite comment on it. One subcategory of the approximator is the *rounder* which comprises adverbs of estimation such as "about", "around" and "approximately" (e.g., "The answer is around twenty.") The second type of approximator is the *adaptor* – it indicates vagueness concerning class membership such as "somewhat", "sort of", "pretty much", e.g., "I am pretty sure that twenty is an even number."

In the interviews conducted with individual (and small groups of) students, Rowland (2000) found that teachers used shields and adaptors in recognition of the face wants of students, whereas students used rounders and plausibility shields to serve their own face wants. He found that, in general, young students seemed to be less sensitive to the face wants of a teacher than to their own. This could be explained by a perceived power difference between a young student and his/her teacher. In the context of whole-class conversation where argumentation is encouraged, a teacher might have competing demands on his/her attention in terms of addressing face wants of different students. Moreover, students might well decide to protect or not the face of their peers. In the excerpt that follows, consideration is given to how vague language is used by a teacher to deal with such complexities.

The pronoun "we"

Bills (2000) says that one of the strategies for positive politeness used by teachers is the use of "we" or "us" to infer inclusion, e.g., "Let's try starting with this one". Rowland (2000) expands on the uses of pronouns (particularly, "it", "you" and "we") in mathematical learning contexts. He suggests that while the pronoun "we" can often be used to indicate a teacher's solidarity with a student (or group of students), the term can also be used to convey distance – "to associate the speaker with a select and powerful group ... to urge acquisition of the 'proper way' of doing [mathematics]" (p. 98). In a classroom situation it can also serve to assuage a command, thereby mitigating threat to negative face (e.g., "We add the units and then the tens...").

METHODOLOGY

In order to investigate the construction of new mathematical ideas by pupils in the context of whole-class discussion, I conducted a classroom design experiment in three different primary schools in Ireland (that is, a series of lessons in each school consecutively). This approach has its roots in the teaching experiment, the central elements of which include instructional design and planning, ongoing analysis of classroom events, and the retrospective analysis of all data generated (Cobb, 2000). Because of its focus on theory development, the teaching experiment has been subsumed into design-based research and, more recently, has been termed a "classroom design experiment" (Cobb, Gresalfi, & Hodge, 2009). I taught 32 lessons (some of which extended over more than one class period) in all to pupils aged 9 – 11 years. I, as researcher-teacher, taught the lessons but the class teacher assisted in planning, teaching and post-lesson analysis. The main data collected were audiotapes of whole-class and group interactions – video recordings were not used due to ethical constraints. Data collection and data analysis were interwoven. Retrospective analysis was conducted on micro- (between lessons) and macro- (between and after cycles of research in the three classrooms) levels. The analytical approach I adopted was microethnography (Erickson, 1992) in which I first considered whole events such as lessons and gradually filtered them to explore the construction activities of individuals, focusing particularly on the sequential emergence of talk and action. This construction sometimes happened within a short period – at other times, it occurred in a zig-zag fashion over the course of a lesson or indeed a few lessons. The use of vague language by children was a crucial element of construction activity in mathematics lessons (e.g., Dooley, 2011). In particular, such language allowed them to engage in the conjecturing activity that is central to the development of novel ideas. Furthermore the follow-up actions by me, the teacher, to their contributions was salient, e.g., revoicing or press moves allowed for pupils to build on each others' thinking (e.g., Dooley, 2009). Re-analysis of the data to explore my use of vague language revealed that I and, to a lesser extent, the pupils used vague language as a means of being polite. In this paper, I examine the issue of politeness in whole-class conversation – in particular, how my utilization of vague language within follow-up moves was another core dimension to children's mathematical constructions. I draw on data derived from lessons that I taught in the first and third schools. I chose these lessons because they exemplify how vague language on the part of the teacher can be used to support contributions by those traditionally excluded from mathematics while, at the same time, move the group towards mathematically correct ideas. Such vagueness encouraged peers to be the arbiter of correctness in the lessons concerned.

EXCERPTS FROM TWO LESSONS

The Grasshopper Lesson

This lesson was one of eleven lessons that I taught in the first school. The school was of middle socio-economic status. There were 30 pupils, 17 girls and 13 boys, aged 10 - 11 years in this class and Mr. Allen was the class teacher[2]. The problem reads as follows:

> A grasshopper is journeying across a mat that is 1 meter long. He starts at the top of the mat, jumps half-way across and takes a short rest. He then jumps half-way across the remaining bit and takes a short rest. He then jumps half-way across the next bit and so on. What are his landing points? Will he ever get to the end?[3]

I drew a line on the blackboard, the initial point of which was marked *0* and the end point was marked *1m*. I asked the group to name his first landing-point and then I drew an overarching loop from 0 to ½. I explained that the grasshopper would next jump half-way across the remaining section. I invited a pupil to mark the second landing-point and again asked the group to name this point (¾). The lesson continued thus. As expected, identification of the fourth landing-point (15/16) challenged some pupils because they had not yet been formally exposed to sixteenths. While some agreed with Jack (a pupil of "average" mathematical attainment on the basis of standardised test scores) that it was fifteen sixteenths, others aligned with an idea proposed by Kate (a pupil of "below average" mathematical attainment) that it was seven and a half eighths. The transcript that follows centres around this episode[4]:

104 TD: What do you think is going to happen next?

105 Chn: It's going to half it//half it//half...

106 Jack: It's half of seven eighths (whisper) [ ]
125 Jack: He's on fifteen sixteenths.
126 //Ch : Seven and a half eighths.
127 Chn: No.
128 TD: He's on fifteen sixteenths, seven and a half eighths or fifteen ...
//Ch talking ... So you think he's on fifteen sixteenths. Where are you getting fifteen sixteenths from?
129 Jack: Cos I think, I think em ... I think a half an eighth is sixteen ...
130 TD: Right.
131 Jack: and eh ...
132 //Ch: I know ...
133 Jack: when ...
134 //Chn: Seven and a half!
135 Jack: and then another sixteenth ... if she went another sixteenth (*other children talking in background*), she'd be there but she didn't go another sixteenth, so she went fifteen sixteenths.
136 Ch: Three quarters ...(*in background*)
137 TD: Fifteen sixteenths ... and who said seven and a half eighths, who said that?
138 Paula: Me.
139 TD: It was yourself, what is your name?
141 Chn: No, it was Kate//it was Kate.
142 TD: I thought it was this girl down here, was it? Yes, Kate ...well maybe both people said it ... that's fine, I thought it was Kate.
143 Kate: It was seven eighths and if he went eight eighths, he would be at the end, so if you go half of it, then it's seven and a half.
144 Mr. Allen: Good girl!
145 TD: Seven and a half eighths and do you think ... Dan? [ ]
159 TD: The jumps ... oh, I know what you mean. So what do we call ... will we call it seven and a half eighths or fifteen sixteenths?
160 Chn: Seven and a half //Fifteenth sixteenths//Seven and a half is easier to manage// No, it's not//Cos you are going two, four, eight//Seven and a half is easier.
161 TD: I think ... you could call it seven and a half eighths but we normally these things ... normally they are brought up to full numbers. But seven and a half eighths would be ok but ... normally it's brought up to something like fifteen sixteenths. (*I write both on blackboard.*)

When Jack proposes fifteen-sixteenths, I ask him to explain his thinking. Although he initially hedges by using a plausibility shield, "I think," in line 129 (l.129) – possibly due to his uncertainty about the fractional name for half an eighth, his reasoning in l.135 is cogently expressed, reflecting his conviction around the argument that he is making. It is interesting that I do not evaluate his contribution in l.137 which, at this juncture, might have had the effect of closing off other contributions (O'Connor & Michaels, 1996). I respond by maintaining his input ("fifteen sixteenths") and then following up on the "seven and a half eighths" contribution. The question that I pose in l.137 ("[W] ho said seven and a half eighths?") may well be due to the fact that I genuinely do not know who has made the input. However, my remark in l.142 ("I thought it was Kate") could be a redressive action in anticipation of an FTA - that is, by not asking Paula directly if she has said it, there is no need for her to admit that she has not done so. In this instance, Paula replies that it is she who volunteered the contribution and immediately other pupils in the class claim it to be Kate's idea. By redressing the threat to Kate's positive face, they make no effort to respect Paula's positive face. In l.142, I make some effort to redress what might be perceived as a combative situation (Kate versus Paula):

> 142 TD: *I thought* it was this girl down here, was it? Yes, Kate ... *well. maybe* both people said it ... that's fine, *I thought* it was Kate.

I indicate that I believe the contributor to be Kate rather than Paula but distance myself from the assertion by use of the plausibility shield, "I thought." In this way I am effecting a double save, that is, with regard to Paula's face (protecting her from possible embarrassment) and to Kate's (if she had not been the originator of the comment). While I give Kate credit for the idea, my "[W]ell, maybe both people said it" represents a further effort on my part to redress Paula's face. Rowland suggests that the particle, *Well,* delays a reply on the part of a speaker (thus inferring refusal or disagreement) – as such, it is one of the ways that threat to positive face can be lessened. In this instance, my use of the term suggests doubt on my part that both people did in fact make the contribution. My subsequent action (giving Kate the floor) suggests that my main motive here is to save Paula's positive face. My gamble – for that is what it is – seems to pay off as, like Jack, Kate justifies her argument ably, indicating that she has indeed constructed a way of naming this point.

In my question in l.159, "Will we call it seven and a half eighths or fifteen sixteenths?," I am probably hoping that the children will resolve the issue. My use of the pronoun "we" in this instance infers solidarity. Failure to agree will offend my positive and negative face wants (as I am the teacher and they are the pupils). On an individual level the children do as required but take different positions on the issue. While this may be due to mathematical preferences, it could also be that the children themselves are engaging in face-saving acts with respect to Kate or Jack (but, not in this instance, both). My discomfort in l.161 is palpable. I am aware that this is a critical event for Kate who does not often make contributions in mathematics classes. I first hesitate – "I think – you could..." referring to the mathematical correctness of her idea. However, I also seem to feel some duty to conventional correctness when I suggest that "we normally ... normally they are brought up to full numbers." In this instance the "we" refers to the general mathematical community – I am suggesting that, while seven and a half eighths is acceptable, the convention is to *round up* denominator and numerator. The "seven and a half eighths would be ok" is a positive redressive action (directed towards Kate). Interestingly, I use the adaptor "something like" with Jack's "fifteen sixteenths" (although I know this to be conventionally correct). In fact, most of my politeness is directed towards Kate rather than to Jack and this may be because I perceive Jack – at this juncture – to have less face want than Kate. My ultimate resolution is to write both suggestions on the blackboard. In the next part of the lesson, other pupils built on Kate's idea to justify their naming of further landing points on the line.

The Gauss Lesson

The Gauss lesson concerned the sum of numbers from 1 - 100 and was a lesson that I taught in the third school. In a previous CERME paper (Dooley, 2009) I reported how one pupil, Anne, used linearity inappropriately to determine the sum. She had suggested that the solution could be found by multiplying 30 (what she thought was the sum of 1- 10) by ten. After some disagreement by others in the class, she revaluated her method. She then made a new estimate and there follows an excerpt of the conversation that followed:

166 Anne: I think the answer would be a thousand.
167 TD: You think it's going to be a thousand. Do you agree with Anne that it's about a thousand? Brenda?
168 Brenda: Eh, no, cos when I em added up forty for it and, em, I got more than a thousand.
169 TD: Oh, wait till we see now, so Brenda is thinking of the problem we were doing yesterday. Brenda yesterday added one plus two plus three plus four plus five all the way up to forty. And what did you get when you added, do you remember when you added up to forty?
170 Brenda: Eh a thousand and something.
171 TD: I think, do you know something, I think it was ... I am not completely sure ... I think it might have been seven hundred and eighty, but I am not sure about that.
172 Brenda: I know it was a thousand.
173 TD: You think it was a thousand ... Anyway Brenda added up, yes Fiona?
174 Fiona: Well, yesterday, forty was seven hundred and eighty.

On the previous day, Brenda had used a calculator to sum numbers from 1 to 39 in order to find a solution for a different mathematical task. When Anne conjectured that the sum to 100 was a thousand (l.166 above), Brenda intimated that this could not be the case by making an implicit reference to the result of this sum. Her recall is inaccurate as she had found the solution to be 780. However, I am interested in drawing pupils' attention to her contribution, as it is a means of moving the lesson forward, that is, of making an accurate estimate for the sum, 1- 100. I use the pronoun "we" (l.169) to do so – "Oh, wait till *we* see now...". I broadcast her input ("Brenda is thinking of the problem we were doing yesterday") although I am inferring this since she has not mentioned the problem of the previous day explicitly. I also rebroadcast the contribution that she made in the previous lesson ("Brenda yesterday added one plus two plus three plus four plus five all the way up to forty..."). I made an error in this rebroadcast by suggesting that Brenda "added up to forty" when she in fact added up to 39. My evaluation of her estimate ("a thousand and something") is marked by vagueness:

171 TD: *I think* do you know something *I think* it was ... *I am not completely sure* ... *I think it might* have been seven hundred and eighty, but *I am not sure* about that.

My recourse to the plausibility shield is interesting – I am quite sure that the solution found by Brenda was 780 and want to indicate this to the rest of the pupils in a way that respects her positive face. I obviously do not wish to engage in the FTA baldly (by telling Brenda that she is incorrect) and thus distance myself from giving the correct solution ("I am not completely sure", "I think it might have been..."). In l.174, Fiona provided the correct solution. She prefaced her input with

"Well" to infer disagreement in a way that lessened the threat to both my and Brenda's positive face. This excerpt, though brief, marked a turning point in finding a solution to the problem concerned - most notably because Brenda drew attention to the lesson that had taken place the previous day where pupils found a formula for adding consecutive whole numbers.

CONCLUDING REMARKS

In a classroom where argumentation and negotiation of mathematical meaning are encouraged, a teacher has to take several factors into account in her moment-to-moment decisions. The excerpts above are interesting not because they are models of exemplary teaching but because they offer an insight into the complexity of teaching in an 'adventurous way' (Weingrad, 1998). Rowland (2000, p. 173) says that:

> Quasi-empirical teaching, inviting conjectures and the associated intellectual risks, is unimaginable if the teacher is not aware of the FTAs that are likely to be woven into her/his questions and 'invitations' to active participation. Redressive action dulls the sharp edge of the interactive demands that the style places on the learner.

In whole-class discussion, there is even more onus on the teacher to 'dull the sharp edges'. In the Grasshopper lesson, children tended to use vague language only to defend their own face. It is true that they did engage in FTAs and some redressive action and this is probably due to their sense of an equitable relationship with their peers. However, they generally engaged in FTAs baldly (e.g., making no attempt to respect Paula's face) in an effort to protect overtly the face of others (e.g., Kate's). There was more politeness in the exchange between Fiona and Brenda – probably because of the task involved. In both instances, my redressive actions were directed towards pupils who would not have been overly confident about their mathematical competence. I, as teacher, had to encourage their participation and, at the same time, focus on the development of mathematical thinking by the whole class, that is, I had to be attendant to the ethical dimensions of teaching mathematics (see Davis, 1997). The use of vague language – because it lessened threat to pupils' positive face and because it seemed to instigate mathematical argument by other pupils – was a means of doing this. The question of how it can best be exploited so that pupils do not develop incorrect mathematical ideas remains to be explored.

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ENDNOTES

1. Social distance is culturally determined and can relate to phenomena such as social class, occupation, religion, sex, age, race etc. (Parrillo & Donoghue, 2005).

2. Gender-preserving pseudonyms are used throughout the paper.

3. The Grasshopper problem is loosely based on Zeno's (490 B.C.) 'racetrack' or 'dichotomy' paradox although Zeno referred to a continuous journey.

4. Transcript conventions are: TD: the researcher/teacher (myself); Ch: a child whose name I was unable to identify in recordings; Chn: two or more pupils making utterance simultaneously; ... : a short pause; [ ]: lines omitted from transcript because they are extraneous to the substantive content of the lesson; //encloses utterances overlapping that of next or previous speaker; (*word*): transcriber's comments.

The grammar and conceptualisation of motion in Iwaidja

Cris Edmonds-Wathen

Charles Darwin University, Darwin, Australia
Umeå University, Umeå, Sweden, cris.edmonds-wathen@umu.se

Different languages have different ways of using spatial language, grammatically and conceptually. This paper reports on aspects of the language of motion in Iwaidja, an indigenous Australian language. The way that Iwaidja groups and separates spatial concepts such as direction, height and movement in relation to another object are briefly described using examples from a route description task. The implications are discussed in terms of how understanding these grammatical features can help teachers of Indigenous students, as well as providing keys to cross-linguistic investigations of mathematical cognition.

Keywords: Spatial language, motion, grammatical structures, Indigenous, cognition.

INTRODUCTION

The language we use for mathematics, the mathematical register, can have words and grammatical structures that are specific or specialised for their purposes in the register, but it is built on or out of everyday language. The linguistic structures of individual languages affect how the languages can be used and developed for mathematical thinking. Grammatical structures can constrain what is possible but they can also offer opportunities. Improving mathematical learning outcomes for Indigenous language speaking students in Australia requires a better understanding by non-Indigenous teachers of how their students think and talk mathematically. Describing the variety of mathematical expression can also assist other teachers who are teaching in multilingual classrooms, as well as enrich researchers' appreciation of the scope of variations in mathematical conceptualisation in different languages.

This paper considers the expression of motion in Iwaidja, an indigenous Australian language spoken in Northwest Arnhem Land, Northern Territory, and how this expression might affect mathematical thinking. Spatial language is an area of everyday language with mathematical impact. This impact is direct in relation to those parts of mathematics that are overtly spatial, such as the description of position and movement in the early years that progresses into the languages of mapping and of coordinate systems. Spatial language is also used in areas of mathematics that at first thought may not be appreciated as spatial, such as the manipulation of numbers. In particular this can involve the use of spatial metaphors (Edmonds-Wathen, 2012).

Iwaidja is an Australian language spoken predominantly at Minjilang, a community of approximately 300 people on Croker Island, Northwest Arnhem Land in Australia's Northern Territory. There are also speakers in other nearby communities. Investigating mathematical features of Iwaidja provides insights into mathematical ways of thinking and speaking that are shared by speakers of nearby and related languages such as Mawng and Kunwinjku, as well as contributing to broader knowledge and understanding about the breadth of how languages can express mathematical concepts.

RESPONSES TO DIVERSITY IN MATHEMATICAL LANGUAGE

Some grammatical features of mathematical language have been described; one is that mathematical language tends to nominalise processes (Halliday, 2004). Barton (2009) notes that the English language copes particularly well with the nominalisation processes of mathematics compared with some other languages. Some languages are less conducive to nominalisation than others because they begin with stronger roles for their verbs than other languages. Lunney Borden (2011) has described dynamic verb-based features of the Mi'kmaw language. She advocates the use of a verb based discourse pattern for teaching mathematics to Mi'kmaw children, for example focusing on the active properties of geometric objects, rather than identifying and naming their parts. It does not necessarily mean using the first language of the students, but of responding to some of the ways meaning is made in their languages. Lunney Borden talks about teaching Mi'kmaw children in English, but targeting her English in a manner that used more verbs to describe things and processes and fewer nouns.

In the Navajo language, shapes are also verbs: there is no circle, there is circling. Pinxten, van Dooren and Harvey (1983) describe the worldview of the Navajo as premised on the dynamic nature of the world. Inspired this description, and working with John Mason, Barton (2009) explored mathematical implications of treating shapes as verbs, as actions, calling it Action Geometry. For example, the static view of a circle is all the points that are equidistant from a centre point, forming a planar shape. In a dynamic view a circle is movement with a constant speed and with a constant rate or turn: "circling is actually a special case of spiralling" (p. 31). Barton stresses that Action Geometry is not an actual practice of the Navajo, but was invented by mathematicians.

Responses to diversity in spatial language thus can include teaching about and within the cultural worldview (Pinxten et al., 1987), developing new mathematics (Barton, 2009) and responding to discourse patterns to bridge to the Western mathematics (Lunney Borden, 2011). The example of Action Geometry shows how investigating linguistic diversity can enrich mathematics and stimulate mathematical innovation.

FRAMING PATH AND MANNER

Languages also differ in how they package meaningful components, in what they put together at a word level and what they put together at a sentence level. An example of this is Talmy's (1985) widely used distinction between verb-framed and satellite-framed languages. Verb framed languages present the path of motion on the verb, with manner as a subordinate addition, as in the Spanish *La botella entró en la cuerva (flotando)* 'The bottle moved-in to the cave (floating)'. In satellite-framed languages such as English, the manner is contained in the main verb, as in *The bottle floated into the cave*, where the preposition *into* indicates the path (examples from Talmy, 1985, p. 69). As Talmy notes, while English also has the verb-framed pattern, it is not characteristic, and the verbs that can be used with it are almost all borrowings from Romance languages.

Possible cognitive effects of this distinction are most likely to involve the differences in these expressions in which parts are compulsory to make a full utterance and which are optional extras. Slobin (2006) found differences in the mental imagery of Spanish speakers from English speakers related to this distinction. Similarly, an investigation of language and event perception using eye movements found while the language used does not affect event perception, it does affect their memory of events (Papafragou, Hulbert, & Trueswell, 2008).

The ways in which different languages express motion events is relevant to mathematics education because of the importance of spatial language in mathematics, both directly and metaphorically. Mathematics educators may have a tendency to interpret the way that their own language uses spatial language for mathematics as necessarily linked to the mathematics. Jorgensen (2010) declares that "coming to learn mathematics is heavily associated with the use of prepositions" (p. 29). One example in English might be the phrase "Two goes into six three times". Note that "goes into" is being used in a metaphorical sense; it would be erroneous to say that two *enters* six. The extensive use of spatial prepositions, of course, is a feature of satellite-framed languages such as English. Jorgensen makes her statement in the context of comparing English and Pitjantjatjara, an indigenous Australian language with far fewer prepositions than English. However, it might be useful to ask what mathematical functions Are performed by these prepositions, and then ask what other ways do other languages express those functions.

Furthermore, not all languages fit within one of Talmy's two groups. Some languages favour a serial verb construction, in which both path and manner can occur in verbs used sequentially, and these verbs can have equal status within an utterance (Slobin, 2006). Serial verb constructions have been noted in various languages such as Ewe, spoken in Ghana (Ameka &

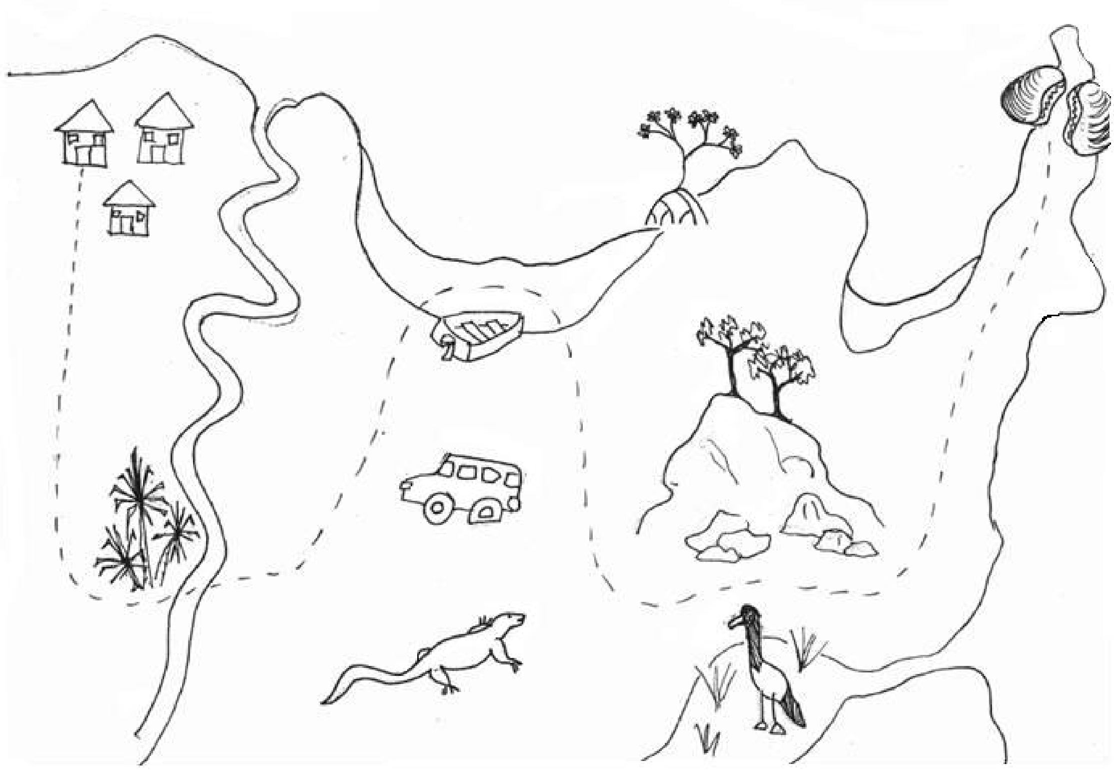

Figure 1: Example of a Director's map showing route

Essegbey, 2006) and Kilivila, spoken in Papua New Guinea (Senft, 2006). Serial verb construction is common in Iwaidja, and the actions themselves can be sequential or simultaneous (Pym, 1985). The remainder of this paper will present some examples of the language of motion in Iwaidja in terms that could enrich our thinking for mathematics education.

THE MAP TASK

This paper reports on some of the results from a Map task designed to elicit route descriptions. It was a barrier task for two participants, derived from the HCRC Map Task (Anderson et al., 1991). One participant, the Director, was given a map with a route marked on it. The other participant, the Matcher, had the same or similar map but without the route. The Director described the route to the Matcher and the Matcher drew the route on their map. I designed the maps to contain items that would be familiar to all participants. An example of a Director's map is shown in Figure 1. The maps differed from those used in the HCRC Map Task by having a coastline and other landscape features such as a creek and beaches. The items on the maps were shown as drawings rather than symbols, as some of the participants would not have been familiar with map conventions.

This task was conducted six times with adult Iwaidja language consultants, with three pairs of participants who each had a turn at directing. The sessions were videoed with an external microphone. The first pair used matching maps where the only difference was that one had a route marked and the other did not. The other two pairs were given different maps, so that some of the items on one map either did not appear on the other or were in different locations. The Director's map still had a route marked on it. The intention was that more complex language would be generated as the participants encountered the differences in their maps. Participants in the sessions where the maps differed were informed that the maps might be different. The task allowed participants to choose what types of spatial language they used to solve it.

The task elicited a rich variety of route descriptions using motion verbs. The current paper focuses on only a few features of Iwaidja that were used in the task – the serial verb construction, the directional distinction and a set of specific motion verbs to describe ways of passing. These features are of interest because of the ways in which they combine or separate mathematically meaningful components as compared to more well-known languages such as English. Other features such as the use of verbs to describe circling, as in Mi'kmaq, are described in Edmonds-Wathen (2013).

Like some North American languages such as Navajo and Mi'kmaq, Iwaidja is a verb-rich language. Many aspects of life are described as processes rather than things. Kin relations, for example, may be expressed using inflected verbs, so that one says "she sisters me" or "he uncles you" (Evans & Birch, 2007). Verbs often take coverbs or adverbs, many adjectives inflect for number, and it is not always immediately clear whether words are verbs, adjectives or adverbs. Iwaidja is a 'head-marking' language, in which affixes, both prefixes and suffixes, provide information including subject, object, direction and tense with regard to a stem.

In general, spatial verbs consist of a stem with a prefix indicating either the subject (in the case of intransitive verbs) or both subject and object (in the case of transitive verbs). Past tense is indicated by suffixes. Future tense is indicated via a separate prefix. In addition, there is an optional prefix indicating directionality, either AWAY from a deictic centre or TOWARDS it (glossed TO). Without this prefix the directionality can be considered neutral. These directions are with respect to the deictic location, which may or may not be the speaker's location. This prefix sometimes combines with subject and object prefixes as well as the future tense prefix to form a single morpheme (Pym & Larrimore, 1979). Examples include *jan-ara,* glossed AWAY.I.FUT-go, 'I will go', and nyan-ara, glossed TO.I.FUT-go, 'I will come'. An example of a prefix which combines subject and object is *r-*, glossed he.to.it, which means 'he (third person singular masculine subject) acts upon it (third person singular object)'.

Examples in Iwaidja are shown in four lines. The first line shows the sentence or phrase in the standard orthography. The second line shows each word. The Iwaidja words have hyphens separating morphemes (meaningful word parts).The third line shows an aligned morpheme by morpheme English gloss. Where the Iwaidja word is translated by more than one English word, but these cannot be separated morphemically in the original, the English words are separated by a period. For example, *artirran* 'he came back' can be separated into *art-* 'towards; he/she/it', *irra* 'come back' and *–n*, which marks the past tense. The third person singular pronoun and the towards direction marker cannot be separated in the morpheme *art-*, nor can the 'come' and 'back' in *irra*. Hence, *art-irra-n* is glossed 'TO.he-come.back-PST'. The final line shows a free English translation of the sentence or phrase. Translations and transcriptions were done with the assistance of a fluent native speaker.

The predominant approach of the six participants was to use specific verbs of motion. Example (1) illustrates many typical characteristics of Iwaidja motion description.

(1) Artirran ararlarrngbung abulakuny awaran. Yabulakuny wardad ba ajbud. Yartirran yarnukbun jumung murrhala ari. Yartirran yarnukbun wuka jumung narrhardi bani. Yartirran barakbarda yariman ba alan ba yawaran, barakbarda jumung birtbani, arrarnarn lda arrayi.

art-irra-n	*a-rarlarrngbu-ng*
TO.he-come.back-PST	TO.he.to.it-turn-PST

a-bulaku-ny	*aw-ara-n*
TO.he-come.down-PST	TO.he-come-PST

ya-bulaku-ny	*wardad*	*ba*	*ajbud*
AWAY.he-come.down-PST	one	the	beach

yart-irra-n	*y-arnukbun*
AWAY.he-come.back-PST	AWAY.he-turn.off

jumung
REL

murrhala	*ari*	*yart-irra-n*
pandanus	it.stands	AWAY.he-come.back-PST

y-arnukbun
AWAY.he-turn.off

wuka	*jumung*	*narrhardi*	*bani*
LOC	REL	goose	it.sits

yart-irra-n
AWAY.he-come.back-PST

barakbarda	*yari-ma-n*
that.one	AWAY.he.to.it-take-PST

ba	*alan*	*ba*	*yaw-ara-n*
the	track	the	AWAY.he-come-PST

barakbarda *jumung* *birt-bani*

that.one	REL	they-sit
arrarnarn	*lda*	*arrayi*
milky.oyster	and	black.lip.oyster

'He came back, he turned, he came down, he came. He went down to one beach. He turned back to where the pandanus tree is. He turned back to where the goose was sitting. He went back there and he took that road, he went along where that thing is, milky oysters and black lip oysters.'

This example includes several uses of the TOWARDS and AWAY prefixes. The basic movement verb *ara* which means 'go' or 'come' depending on the context is seen in the past third person singular forms *awaran* 'he came' and *yawaran* 'he went'. We also see the verb *irra* which means 'go back' or 'come back' depending on the direction, in the forms *artirran* 'he came back' and *yartirran* 'he went back'. *Irra* thus refers to a change of direction. The example also contains both TOWARDS and AWAY forms of *wulaku* 'go down', in *abulakuny* 'he came down' and *yabulakuny* 'he went down'. Hence the use of the directional contrast enriches the semantic scope of the verbs to which it is applied.

We can also see the serial verb construction which is frequent in Iwaidja. The first sentence consists only of four verbs, all taking the TOWARDS prefix: *Artirran ararlarrngbung abulakuny awaran* 'He came back, he turned, he came down, he came.' A colloquial English translation might be something along the lines of 'he turned and came straight back down'. The use of a series of verbs in this context does not necessarily represent a series of actions that follow each other. Rather, the protagonist's single, if complex, act of turning, returning and descending are conceptually packaged together despite each being expressed as independent verbs which could each stand as a sentence themselves in Iwaidja.

In any language, you would expect the language of motion to involve the use of verbs. However, in 'verb-framed' languages, according to Talmy's (1985) classification, manner of motion is expressed outside the verb, for example as an adverb. In 'satellite-framed' languages the path is expressed outside the verb, for example with a prepositional phrase, as in English. The serial verb construction of Iwaidja frequently combines both manner and path in a series of verbs, as was seen in example (1).

Iwaidja encodes specific spatial information into distinct, although related verbs. The range of verbs elicited to refer to passing objects on the map is a good example of this. The most general of these is *marraywung* 'pass.' It is a transitive verb, requiring the specification of what is passed, but does not provide any more detailed information about how the object is passed. Its use is shown in example (2).

(2) Rimarraywung yawarang mangawala ajbud jumung kabala ari.

ri-marraywung		*yaw-ara-ng*
he.to.it.pass		away.he-go-NPST
mangawala	*ajbud*	*jumung*
fast	beach	REL
kabala	*ari*	
boat	it.stand	

'He goes past them and runs along the beach where boat is.'

There are also verbs derived from marraywung for passing in front, behind or to the side of objects. Their use depends upon those objects having an intrinsic front, back or side. The verbs are *marlmarraywung* 'pass behind' (example 3), *wudbarraywung* 'pass in front' (example 4), and *ngunyunmarraywung* 'pass beside' (example 5).

(3) Artirran ararnukbung rimarlmarraywung wuka jumung mudika wulurr.

art-irra-n		*ar-arnukbu-ng*
TO.he-come.back-PST		TO.he-turn.off-PST
ri-marlmarraywung		*wuka*
he.to.it-pass.behind		LOC
jumung	*mudika*	*wulurr*
REL	car	back

'He came back, turned off and passed behind the back of the car.'

(4) Kirrimul warrkarrk aju riwudbarraywung.

kirrimul	*warrkarrk*	*aju*
like	goanna it.lies	he.to.it-pass.

ri-wudbarraywung
in.front

'He passes in front of where the goanna is.'

(5) Kabanangunyunmarraywun baraka dinghy.

kabana-ngunyunmarraywu-n	*baraka*
you.to.it.FUT-pass.beside-NPST	DEM

dinghy
dinghy

'You will pass beside the dinghy.'

Ngunyunmarraywung 'pass beside' contains the body part stem *ngunyuni* 'waist' which also occurs in words such as *mangunyuni* 'beside' and *angunyulunmin* 'side by side'. The derivational origins of *marlmarraywung* 'pass behind' and *wudbarraywung* 'pass in front' are not so transparent.

In addition to the adverb *mangunyuni* 'beside', Iwaidja has *warrwak* 'behind' (an adverb) and *wurdaka* 'in front' (a verb). *Warrwak* and *wurdaka* are common words and were used by the Iwaidja speakers frequently in activities involving description of static location (Edmonds-Wathen, 2014). The examples here of the various verbs of passing indicate more than just choices in the expression of 'beside', 'behind' and 'in front'. Rather, there are differences in their expression depending on whether one is talking about static location or about motion. Similarly, Iwaidja has adverbs *yurrngud* 'on top, above' and *wurrrwud* 'below', which are often used to describe static locations. However, the verbs *bulaku* 'go down, descend' and the opposite *wurti* 'go up, ascend' are used to describe motion up or down.

DISCUSSION AND CONCLUSIONS

Most Australian Indigenous language speaking children are taught mathematics in English by English-speaking teachers. Their mathematics learning is assessed in English. When English speaking teachers talk and think about how to teach the language of mathematics, they tend to package concepts such as spatial concepts in the way that English packages them. They are likely to think of a concept of "down-ness" which applies in the same way to being down and going down, as this is how the English grammar of "down-ness" works. However, it is far from certain that speakers of languages such as Iwaidja link the location of *wurrrwud* 'below' with the motion of *bulaku* 'go down, descend'. Similarly, an English speaker may conflate the "behind-ness" and "in front-ness" conceptualised in the verbs *marlmarraywung* 'pass behind' and *wudbarraywung* 'pass in front', with that in *warrwak* 'behind' and *wurdaka* 'in front' to a greater extent than an Iwaidja speaker. Kriol, a creole language with an English lexicon spoken by over 4000 Indigenous Australian people, and learnt by many Indigenous children as their first language, also expresses path in basic motion verbs. Examples include *guwap* 'go up, ascend' and *gudan* 'go down, descend'.

Educationally, a possible response to this would be for teachers of Indigenous language speaking students to try to use the discourse patterns of their students in teaching the language of location and motion in English (Lunney Borden, 2011). While the spatial prepositions may be emphasised in describing static location, when describing motion the path should not be separated or emphasised: rather than "down, down, go down", the whole verb phrase "go down, go down, go down" could be repeated.

Grammatical structures provide clues to how speakers of different languages structure their understanding of events and to what they are likely to pay attention. Finding out about these grammatical structures can help mathematics education researchers to design investigations cognitive differences between speakers of different languages. The cognitive effects of using a serial verb construction as in Iwaidja could be investigated to add to comparisons between path- and satellite-framed languages. These types of structures are relevant when considering the design of mathematics questions in different languages. Questions that focus attention to one part of the information in one language and to another part of the information in another language may promote different problem solving strategies.

Finally, investigations such as that reported in this paper, into the grammar of basic mathematical concepts, can also provide us with the opportunity to look more deeply into our assumptions about these concepts. This in turn may inspire mathematicians to think creatively beyond the grammatical constraints of their own languages.

ACKNOWLEDGEMENT

This research was supported by a grant from the Australian Institute of Aboriginal and Torres Strait Islanders Studies and an Australian Postgraduate Award scholarship at RMIT University, Melbourne. The author wishes to acknowledge the assistance of Bruce Birch in collecting and analysing the data, and of Joy Malwagag Williams in translating the data.

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How communicative teaching strategies create opportunities for mathematics learning

Margareta Engvall, Joakim Samuelsson and Karin Forslund Frykedal

Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden, margareta.engvall@liu.se

The overall interest in this study is actions of the diverse participants in the mathematics primary classroom. More specifically, attention is put on typical communication strategies that teachers and students, in our research context, use during mathematics lessons. We present some examples with classroom data to illustrate how the strategies are used and what opportunities to learn mathematics are offered as an outcome from the implemented actions by teachers and students.

Keywords: Mathematics education, communication strategies, primary classroom.

INTRODUCTION

The language and how it is used is of significant importance for what is possible to learn in all education (Säljö, 2000). The way the teacher manages to direct the communication in the classroom and how students and the teacher talk to each other is crucial for students' learning in terms of mathematical content as well as how they view themselves as mathematical performers (Franke, Kazemi, & Battey, 2007). The main interest in this study is communicative teaching strategies that are used in the mathematics classroom and what these strategies can offer in terms of students' opportunity to learn mathematics.

The findings in this paper extend the results of a larger study by Engvall (2013), to which Samuelsson and Forslund Frykedal contributed importantly. That study discusses teachers' and students' actions in the mathematics classroom in primary school when written calculation methods are in focus. The theoretical framework is built on activity theory (AT), which here refers to Engeström's (1987) model of the activity system. According to Rezat and Sträßer (2012) this theory is one among other socio-cultural or semiotic theories that has been shown to be successful in mathematics education research. In AT, the theory about tool mediation is a mainstay (Cole & Engeström, 1993). It is pointed out that any meeting with mathematics is mediated through either material tools as for example textbooks and rulers or non-physical tools such as language, visual representations and gestures (Rezat & Sträßer, 2012). A major feature in Engeström's model is that the context has been extended with some additional mediating factors such as rules, community and division of labour in order to allow better analysis (Engeström; 1987; Goodchild, 2001; Rezat & Sträßer, 2012). Yet, in this paper, we focus on verbal language as it is used by teachers and students in some classes during mathematics lessons in primary school, or in other words communication strategies. There are two main questions to be elaborated here:

(1) What kind of communication strategies do teachers and students use in the mathematics classroom?

(2) What do different communicative teaching strategies have to offer the students in terms of opportunity to learn mathematics?

Communicative teaching strategies in the mathematics classroom

In this paragraph some characteristic communicative teaching strategies in mathematics classrooms will be presented. In particular, we pay attention to the language used for teaching, i.e. the kind of language used to demonstrate, explain and exemplify mathematical connections, drawing from Löwing's (2000) instrument for analysis of communication in the mathematics classroom. Löwing makes a distinction between formal and informal language for teaching, where the latter is used e.g. together with manipulatives or when an everyday situation is taken as a starting point for explaining a calculation. In this presentation we focus on the formal language since, according to Setati and

Adler (2000), one of the challenges in mathematics education is to get the students to gradually change from informal to the formal language, which is characteristic for mathematical activities.

According to Löwing (2000), two main categories of spoken language strategies can be distinguished within the formal teaching language, namely *descriptive language* and *conceptual language*. By using descriptive language, teacher and students put attention on procedures, as e.g. when teacher and students together are performing calculations like 53+25 on the board and somebody describes adding the tens by saying "five plus two", which means that the tens are treated verbally as if they were ones. In this case the language is representing the calculation procedure. Conceptual language, on the other hand, is visible when mathematical words with a specific meaning, e.g. "five tens plus two tens" are used in order to explain and not only describe a step of a calculation like this.

The following strategies, presented in the community of international researchers in mathematics education (Mercer, 1995; O'Connor & Michaels, 1993), can be associated with the category of conceptual language. Our first strategy to be presented here is *key questions*. Researchers have demonstrated that teachers use key questions or certain phrases to create structures for supporting students' learning (e.g., Mercer, 1995). Some students pick up these questions or other common phrases from the teacher's spoken language and use them as a kind of support when performing his or her tasks.

Another distinctive strategy connected to the second category is *revoicing*, since revoicing directs the attention towards thinking and conceptual understanding (O'Connor & Michaels, 1993). Revoicing can actually be defined as at least three communicative teaching strategies: (a) *repeating*, (b) *rephrasing* and (c) *recasting*.

With reference to Mercer (1995), repeating can be used for directing the students' attention towards something specific in a student's answer/expression/utterance so this can support the students' learning. When a teacher instead rephrases a student's utterance the class will get another chance to grip what just has been said but in a version that is more consistent with what the teacher wants to point out. Finally, if a student has expressed something almost non understandable in class, this can be further developed by the teacher's recasting so the meaning thereby can be explained for the students (Mercer, 1995; O'Connor & Michaels, 1993).

The content in this section is interwoven with our first research question regarding the use of communication strategies in the mathematics classroom. Our second question is aimed at paying attention to the idea of possible learning, which therefore will be followed up in the next section. This theoretical overview will be concluded by a short presentation concerning different aspects of mathematical knowledge.

Opportunities to learn and mathematical knowledge

One of the most firmly established links between teaching and learning is the idea of "opportunity to learn" (Hiebert & Grouws, 2007). This means that "the students learn best what they have the most opportunity to learn" (p. 378). Although it is impossible to predict learning outcomes in mathematics based on the use of a specific teaching strategy it is still likely to reason about students' opportunity to learn with regard to teachers' and students' strategies during mathematics lessons (Hiebert & Grouws, 2007). Further, this means that it is possible to discuss what teachers' and students' verbal action can offer when it comes to opportunity to learn during mathematics lessons. This is a core message in this paper.

In various frameworks (Kilpatrick et al., 2001; Niss, 2003) mathematical knowledge is presented as something multifaceted which comprises different types of "tightly interwoven" competences. Procedural knowledge, conceptual understanding, communication, reasoning and strategic competence are some examples. In Sweden, as in other countries, this approach has influenced the curriculum in mathematics education. This means that students during mathematics lessons will get more opportunities to be involved in activities of communicating together with thinking and understanding instead of putting big efforts into skill practice and remembering procedures (Anghileri, 2001; Clarke, 2006). For more than 20 years, this approach has also given rise to a common trend for the teaching of arithmetic, which is an essential part of the mathematics content in primary school. Before the students get any instructions on traditional algorithms they will focus on strategies for mental calculation. From a Swedish point of view this means that traditional algorithms gradually have been replaced in the textbooks by other written calculation methods. For example the addition 56 + 28 can be calculated as follows, 56 + 28 = 70 + 14 = 84. The “intermediate” 70 + 14 is written down in order to facilitate the mental work.

In this paper, we focus on teachers’ and students’ communication strategies with respect to procedural knowledge and conceptual understanding when the content is written calculation methods for addition and subtraction.

METHOD

The collected data in this observation study consisted of video-recorded mathematics lessons in five different classrooms. Besides, an audio recorder has been used together with the video camera in order to get a more complete sound reproduction. In addition to the recorded material there were also field notes. The participating classes belonged to four schools with students from areas with comparable socioeconomic status. The number of students in each class was 24–25. Every teacher had the primary responsibility for the mathematics education in her/his class. Most of the teachers did also teach all lessons in almost every subject in the class. Collection of data started during the spring when the students were in second grade and continued during the autumn, when the students were in third grade. The video-recorded material comprises a total of 24 lessons. In all the lessons one and the same content has been in focus, i.e. written calculating methods for addition and subtraction with numbers exceeding 20.

The collected research material has been analysed in two steps. The first step, with inspiration from Braun and Clarke (2006), can be described as empirically oriented and thematic. This part of the analysis has been carried out in order to discern phenomena that form patterns in the material and thus point to the characteristic actions in mathematics classrooms. The result derived from the first step of the analysis has then been used as the basis for the second step, where the analysing tool, inspired by Engeström (1987), has guided the analysing process. Engeström’s model for activity system gives the researcher opportunity to analyse not only actions mediated by tools, but also by other mediating factors such as rules, community and division of labour. Yet, the focus in this paper is on the outcome of the analysis regarding a non-physical tool, spoken language, or more generally, communicative teaching strategies in the mathematics classroom.

ANALYSIS AND FINDINGS

In the following presentation we will report some typical communicative teaching strategies that have appeared in the empirical material. These will be presented in relation to descriptive and conceptual language. Thus, here we put attention to just two aspects of mathematical knowledge, procedural and conceptual knowledge.

The content taught in the classrooms in the study is, as already mentioned, written calculation strategies for subtraction. In the first example below, initially, the teacher invites a student to tell how to calculate the subtraction 55−21. Almost at the same moment the teacher clarifies that the goal is to tell how to write the intermediate but the outcome of the subtraction is not going to be emphasized.

Transcript 1

Teacher	[WRITES 55-21= ON THE BOARD] I don’t want to hear the outcome. I want to hear the intermediate...Peter!
Peter (stud.)	Fifty minus twenty makes thirty.
Teacher	[WRITES 30 AFTER =]
Peter	And the plus.
Teacher	[WRITES +]
Peter	And then it is five minus ten...four ... four...forty... [ADDRESSING THE TEACHER WHO WAITS A MOMENT BEFORE WRITING THE NEXT DIGIT], a four.
Teacher	[WRITES 4 =] And that makes??
Peter	Err, thirty-four.
Teacher	[WRITES 34] Thirty-four. Exactly! Yes, it’s important to keep track of whether you have to use the tens or the ones.

From this sequence we notice that the dialogue is dominated by the number words together with some words which are representing other symbols. Thus, the student’s communication strategy can be defined as descriptive language, since it is mainly used for representing the different steps in the calculation which is performed by the student while the teacher is writing on the board. The student’s third reply indicates some uncertainty regarding the value of the digit one in 21. The teacher’s comments are very limited.

However, in the last lines the teacher mentions some fundamental concepts concerning number value but this does not seem to be aimed at clarifying the meaning of these concepts, rather to give the student an advice. Accordingly, in this example attention is directed towards the procedure.

The next transcript illustrates the first steps of a teacher's instruction to the whole class on how to calculate the subtraction 58–34. Together with the written subtraction on the board, the teacher uses manipulatives, especially adapted for putting on the board, representing the number of tens and ones.

Transcript 2

Teacher /.../ If we are going to calculate fifty-eight minus thirty-four, which numbers are we going to begin with? ... Jenny!

Jenny (stud.) The tens.

Teacher We begin with the tens. In fifty-eight there are five tens [MAKES A RED MARK RIGHT BELOW THE DIGIT 5], In thirty-four there are three tens [MAKES A RED MARK BELOW THE DIGIT 3]. Five tens minus three tens [POINTS AT THE DIGIT 5 AND THE DIGIT 3]... yes, here we have five [POINTS AT THE FIVE MANIPULATIVE TENS ON THE BOARD] and then we take away one, two, three [TAKES AWAY THREE TENS]. How many tens do we have left? ...Mika!

Mika (stud.) Twenty.

Teacher We have twenty left. Two tens or twenty. It's the same thing just different ways to say it. [WRITES 20 AFTER =]....Mm. Which numbers are we going to continue with now?

We can observe that the teacher clarifies the number of tens in each term, e.g. "in fifty-eight there are five tens". By that, the ten's names and the number names synonymous with each other appear at the same time, e.g. two tens and twenty. Here, the strategy of conceptual language is prominent.

In both sequences above there are examples illustrating how teachers use *revoicing* although they do it differently. This is particularly evident in the last teacher sentence in each transcript.

Excerpt from Transcript 1

Teacher [WRITES 4 =] And that makes??

Peter Err, thirty-four.

Teacher [WRITES 34] Thirty-four. Exactly! Yes, it's important to keep track of whether you have to use the tens or the ones.

Excerpt from Transcript 2

Teacher /.../ How many tens do we have left? ... Mika!

Mika (stud.) Twenty.

Teacher We have twenty left. Two tens or twenty. It's the same thing just different ways to say it. [WRITES 20 AFTER =]....Mm. Which numbers are we going to continue with now?

What these last teacher sentences have in common is that the teacher is *repeating the student's reply*. On the other hand, a clear distinction is visible when we compare the teachers' actions in the two examples. In the first example, besides repeating the student's reply, the teacher makes a comment without any further explanation while the teacher in the second example makes a typical rephrasing. This rephrasing, "Two tens or twenty. It's the same thing just different ways to say it", offers the possibility for the class to take part in what has been said, but in a version that is more in harmony with what the teacher wants to point out. In this situation attention is put on the meaning of tens and ones and what the digits are representing in terms of place value. Thus, in the second example conceptual knowledge is focused while that is not really the case in the first one.

In transcript 2 it is also illustrated that the teacher puts attention to the tens and ones, each in turn, by asking questions like "Which numbers are we going to begin with?" and "Which numbers are we going to continue with now?" We recognize this from research as *key questions*. These are typified by a seemingly procedural course of action. However, depending on how these questions are formulated, the repeating character offers opportunity to develop not only procedural but also conceptual knowledge. We can notice that the student in this example uses a place value word to reply on a key question. Therefore, this kind of questions has an important function in that they can help the students to make structures. It is not unusual that the students pick up the teacher's phrases and use them later on when performing similar tasks.

Finally, the following transcript represents the final part of a session when a class is performing the subtraction 45–22 together with their teacher. They have already jointly agreed that the result of the performed subtraction of the tens is twenty. 45–22 = 20 is now written on the board and when students are invited to perform the subtraction by the ones, the teacher gets two different suggestions from students about what to write next, either plus five or minus three. The teacher then repeats the question about how many ones there are left, which is the opening sentence in the transcript below. The way this question is formulated together with the teacher's illustrating the calculation by fingers, makes it almost impossible for the students to give anything else than a correct reply. The teacher's utterance in line four "Now you said minus three" is a comment to the student who just recently suggested that the teacher should write minus three to complete the intermediate.

Transcript 3

Teacher	How many ones have you left then? ... If you have five [SHOWS FIVE FINGERS], and you are going to take away two?
Student	Three.
Teacher	You have three ones left. Now you said minus three, but if you have something left, what sign do you think you should use, when you have something left?
Student	Plus.
Teacher	Plus [WRITES + TO THE RIGHT OF 20]. It sounds like plus is a very suitable word when you have something left, doesn't it? ...If something was missing you shouldn't use it, should you?

This transcript makes visible how the teacher uses a specific expression "have left" in order to put the students on the right track about which sign they are supposed to use when writing the intermediate in performing written subtraction calculations. In the bottom line we can also notice that the teacher makes a contrast by using the word "missing" in order to demonstrate an association with the minus sign. This strategy we call *key words*. This strategy resembles other strategies where focus is on remembering rules, which makes it closely related to *descriptive language* and consequently, attention is put on procedural knowledge.

CONCLUSION

Even if mathematical knowledge is multifaceted, our concern in this paper has been concerned with opportunities to learn in terms of conceptual and procedural knowledge. We have presented a number of communicative teaching strategies that teachers and students use in the mathematics classroom. The two categories, descriptive and conceptual language, are represented to various extents in the five classrooms in the study. Depending on which verbal strategies teachers and students are using, different opportunities to learn are offered to students.

With this paper we want to illustrate teachers' and students' actions when they use these communication strategies. The result indicates that strategies that direct attention to procedures are also focusing on conceptual knowledge and vice versa. As an example, when the teacher repeats a student's reply more or less attention can be directed to concepts depending on what type of formal language the students used. In a classroom where descriptive language is frequent, it is more likely that the strategy repeating a student's reply will put emphasis on procedures. However, this is in conflict with O'Connor & Michaels (1993) who claim that revoicing directs the attention towards concepts and thinking. Furthermore, using another strategy such as key words can be characterized as procedural. On the other hand, the key words carry some meaning and thereby this strategy might offer opportunities to develop conceptual understanding.

What has been presented above can also be illustrated in the following figure. The horizontal line expresses the communicative teaching strategies and the vertical line symbolizes opportunities for developing either procedural or conceptual language. The two bigger crosses illustrate that descriptive language offers possibilities to develop procedural knowledge while a similar relationship can be described regarding conceptual language and conceptual knowledge. Also, conceptual language strategies offer possibilities to develop procedural knowledge but to a lesser degree than from descriptive language strategies. A similar reasoning can be applied to the relationship between descriptive language and conceptual knowledge.

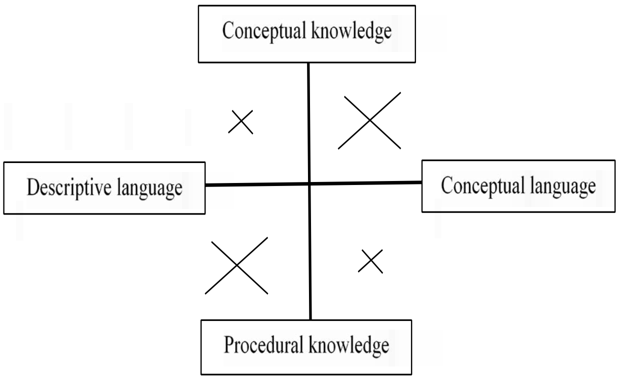

Figure 1: Communication strategies and opportunities for learning mathematics

This paper illuminates how teachers' and students' use of communication strategies influence opportunities to learn with respect to procedural and conceptual language in the mathematics classroom. It can be used for further discussions on pedagogical implications in teacher training programmes as well as in in-service teacher training.

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Diverse epistemic participation profiles in socially established explaining practices

Kirstin Erath and Susanne Prediger

TU Dortmund University, Institute for Development and Research in Mathematics Education (IEEM), Dortmund, Germany, kirstin.erath@math.uni-dortmund.de

Same classroom, same learning opportunity? Although learning to explain takes place while participating in the classroom microculture's practices of explaining, this interactionist conceptualization must be widened in order to account for students' diversity. For analysing not only quantitative, but also qualitative differences between students' participation in explaining practices, we present the construct 'epistemic participation profile' and illustrate how it allows to account for the diversity within a classroom.

Keywords: Explaining practices, interaction, epistemic participation profiles.

DIVERSITY IN PARTICIPATION IN CLASSROOM PRACTICES

Many researchers with social, socio-cultural or socio-constructivist perspectives describe mathematics learning *as an increasing participation in interactively constituted classroom practices*. We join this view by adopting an *interactionist perspective* and account for students' learning processes in explaining practices of classroom microcultures (Prediger & Erath, 2014). These practices are regulated by shared sociomathematical and social norms (Cobb, Stephan, McClain, & Gravemeijer, 2001). Whereas early interactionist approaches assumed the class to be a coherent body in which the mentioned aspects can be "taken-as-shared" by all members (cf. critique in Cobb et al., 2001), later approaches acknowledged diversity among students: students usually participate in diverse ways, and the individual participation shapes the individual learning opportunities which are preconditions for learning achievement (Greeno & Gresalfi, 2008). Learning was then described as increasing from the "legitimate peripheral participation" (Lave & Wenger, 1991) to more acknowledged central participation, hence limited participation was seen as an intermediate state. However, many (especially underprivileged) students seem to stay in a peripheral position (DIME, 2007). In order to deepen this idea of *diverse individual learning opportunities depending on students' ways of participation*, these ways must be described *quantitatively*, but also with respect to their *quality*. Greeno and Gresalfi (2008) give hints for this qualification by describing conceptual practices as crucial for mathematics learning in contrast to purely procedural ones. We present a framework for qualifying students' diverse profiles of participation as one key to understand the reproduction of inequality in classroom interaction where all students have the same formal and curricular learning opportunities (DIME, 2007). This paper deals with *three research questions* from which the first one had to be solved by developing an analysing tool: (Q1) How can we distinguish between students' diverse ways of participating in classroom explaining practices with respect to the epistemic processes of knowledge construction? (Q2) Are there patterns of ways of participating which allow speaking about a consistent participation profile? (Q3) How does students' participation develop over half a year?

We address two points relevant to the TWG as raised by Morgan (2013): First, bilingual learners in mathematics and second, the question what linguistic competences and knowledge are required for participation in mathematical practices.

EXPLAINING PRACTICES IN MATHEMATICS CLASSROOMS

Our case study is embedded in our large research project INTERPASS in which we investigate whole class interactions with joint explaining activities and different research questions. For investigating the mathematical core of explanations and their functions in the process of knowledge constitution, we combine the interactionist perspective on explaining with an epistemic perspective. In the *epistemic perspective*, explaining is defined as aiming at building and connecting knowledge in a systematic, structured way by linking an *explanandum* (the issue that needs to be explained) to an *explanans* (by which the issue is explained). This distinction structured the tool developed for analysing contributions with respect to their epistemic character, the so-called epistemic matrix (Prediger & Erath, 2014), which led to observe the phenomenon presented below.

In the rows of the epistemic matrix (in Figure 1), contributions to a joint explaining activity are distinguished with respect to the *explanandum*. For this, we refine the conceptual/procedural distinction (raised as relevant for participation profiles by Greeno & Gresalfi, 2008) into *seven logical levels*: The four conceptual levels comprise concepts (categories such as maximum), semiotic representations (e.g., a diagram itself), mathematical models (addressing relations between reality and mathematical objects/statements), and propositions (mathematical patterns, statements, or theorems); the three procedural levels comprise procedures (e.g. a general way of drawing a diagram), conventional rules (e.g., "frequencies always on vertical axis"), and concrete solutions (e.g., individual solutions of a mathematical task). In the columns of the epistemic matrix, the *explanans* is distinguished in *six epistemic modes*: ||labelling & naming|| is the only mode that can be addressed by a single word (e.g., "maximum"). The mode ||explicit formulation|| includes definitions and theorems and is a linguistically elaborate way to treat an *explanandum*, it is usually also epistemically more elaborate than the mode ||exemplification|| which addresses examples and counterexamples. The mode ||meaning & connection|| comprises all semantic aspects of an explanandum and those that bridge to another level or mode, for example pre-existing knowledge (e.g., meanings, arguments, reasons), it can have different epistemic degrees of elaboration. The mode ||purpose|| belongs to a pragmatic approach (e.g., "in diagrams, we see pattern more clearly"). The mode ||evaluation|| often appears in context of evaluating solutions.

In our empirical approach, each (complete or partial) explanation that is demanded and given in a classroom interaction is characterized by the navigation through the addressed *epistemic fields* (=the combination of addressed logical level and epistemic mode). Figure 1 contains an exemplary *navigation pathway* of Episode 1 (signs as '#17' refer to the lines in the transcript, circles stand for the teacher, rectangles for the students). In this navigation pathway, the teacher addresses the fields --concrete solutions / models-- ||meaning & connection|| by asking if anybody has a cat at home and knows its weight. Afterwards, he navigates to --models-- ||explicit formulation|| by asking for a complete modelling, thus he navigates from children's concrete experiences to explicit formulations and hence, to consolidated mathematical knowledge.

Whereas two preceding papers investigated by the epistemic matrix the epistemic core of common interactive practices (Prediger & Erath, 2014) and a teachers' profiles setting different implemented curricula (Erath & Prediger, 2014), this paper applies the analysing tool to specifying diverse students' epistemic participation profiles. By focusing on the students, we investigate how different individual students contribute on their own ways within an interactively established social practice.

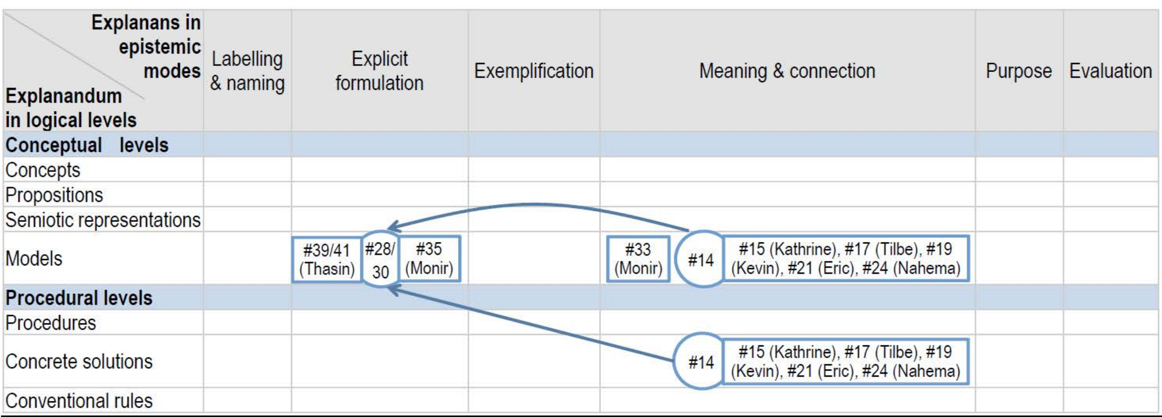

Figure 1: Epistemic matrix for distinguishing explanans and explanandum

We construct a *student's epistemic participation profile* by analysing all her/his contributions in oral classroom explaining practices. The epistemic participation profile is characterized by taking into account (1) the *quantity* of the student's contributions, (2) their *epistemic character*, and (3) their *epistemic potential* for consolidating mathematical knowledge which is determined (3a) by the required level of the epistemic field demanded by the teacher and (3b) by the level of compliance by the individual students. This definition builds upon two assumptions: the epistemic fields play different roles in the collective and individual process of knowledge construction (Vollrath, 2001, pp. 52f), and the individual opportunities to learn also depend on the individual's compliance for contributing to consolidating the knowledge (Greeno & Gresalfi, 2008).

METHODOLOGY OF THE STUDY

Larger data corpus. In the larger project InterPass video data was gathered in 10 times 12 math and language lessons (each 45–60 min.) in five different grade 5 classes (age 10–11 years). The data corpus also comprises all class materials and written products.

Sampling for the case study of this paper. The small comparative case study focuses on 12 math lessons in one higher tracked class (German "Gymnasium", German as language of instruction) in which we compare three students' participation profiles. The students Nahema, Monir, and Thasin were selected due to their similar social and language background (all boys of 10–11 years, second language German learners, living in an underprivileged urban quarter), but contrasting participation profiles. The video corpus for this case study is formed by all episodes of whole class interactions in which one of the three boys was involved in joint explaining activities.

Data analysis. First, the selected video data were transcribed and analysed by means of the epistemic matrix. Second, in order to reconstruct the boys' epistemic participation profiles over time, (1) all contributions above the sentence level of the three boys during classroom interactions of joint explaining were collected, and the *quantity* was determined by counting, (2) their *epistemic character* was operationalized by locating them in the epistemic matrix, and (3) the *epistemic potential* of the contributions (consisting of the required level and the level of students' complying) was identified by an analysis of teachers' navigation pathway and the criterion how the utterance contributed to consolidating mathematical knowledge in the classroom interaction. (For example, explaining algorithms and describing mathematical ways of acting or connecting procedures and concepts demands a high required linguistic and epistemic level of consolidation, whereas naming result or stating everyday experience without connecting it to mathematics is of lower difficulty.) And third, considering the course of the participation during the 12 lessons allowed analysing their stability. The fact that we conceptualize "participation profile" not as dynamic is justified by the empirical outcome that each of the three students' individual way of participation is quite stable over time.

THREE STUDENTS' EPISTEMIC PARTICIPATION PROFILES

The participation of the three boys differs in quantity: Monir and Nahema have 8 and 7 contributions above sentence level (i.e. longer than one sentence) to joint explaining activities in whole class interaction within the 12 lessons (and of course many shorter contributions of one to three words not considered here). Thasin shows a more active participation with 12 contributions above sentence level. The relations are similar for the cases in which the boys raise their hand but are not selected to contribute (Monir 15, Nahema 13, Thasin 32). However, this quantitative information cannot account for unequal learning opportunities in active participation. Only the qualitative analysis of transcripts allows to categorize the differences in the epistemic character and to reconstruct their epistemic potential. We illustrate our analysis procedure by two episodes before describing the results of all analysed episodes.

Episode 1: The meaning of rounded zero in the dot plot

Episode 1 is extracted from a discussion in the class about interpretations for the dot plot in Figure 2, after two students have given opposed interpretations for the 0 kg for cats in Figure 2 ('nothing written' versus 'under ten kilogram, maybe one and a half or two'). The transcript starts when the teacher collects several weights of cats in #14 in order to evaluate the solutions (the translated transcripts use [...] for missing parts, (.), (-) and (--) for breaks of increasing length, CAPITALS for emphasized words):

14 Teacher [...] Does anybody have a cat at HOME, [...] Can you tell how much it WEIGHS, [...]
15 Kathrine um (.) I have now um (.) a little KITTEN at home, well it‘s weighing (.) nearly two KILO;
16 Teacher HM_hm, um (–) TILBE;
...
21 Eric I have a (.) at my grandma's I have a CAT, I once weighed it, (.) it was very THIN, it think its weight was (--) two KILO, and a (.) bigger tomcat I think even SIX kilo;
22 Elif WOW;
23 Teacher HM_hm; NAHEMA;
24 Nahema from my friend who is called KEVIN, (.) the ca (.) um the the cat is like THIS, ((*shows the size with his hands*)) this weighs NINE kilo;
25 Elif NINE; ((*many students expressing scepticism about cats of nine kilo*))
32 Teacher MONIR, (.) THASIN, (.) what do YOU think;
33 Monir well um (-) the um (.) these (.) SCALES of um how much they weigh, is always only (.) after the TENS (.) so (.) always 10, 10 20 30 40;
...
35 Monir but the cat weighs UNDER ten; so (–) um (-) the (.) it does not weigh like (-) um; (–) well it has (-) it has no um (-) it two-digit- ((*indicates digits in the air with his fingers*))
36 Teacher HM_hm; a two-digit NUMBER; or no two-digit WEIGHT;
37 Monir YES;
38 Teacher HM_hm, (.) THASIN;
39 Thasin If a cat would weigh nine KILO on average, one would round it up on TEN;
...
42 Teacher BOTH can be correct by the way; (.) THASIN Monir; [...]

Kathrine (#15), Tilbe and Kevin (in unprinted #17 and #19), Eric (#21) and Nahema (#24) are stating their experiences with the weight of cats, hence they are staying in the epistemic fields for which the teacher asked (#14): the epistemic mode ||meaning & connection|| on the logical levels --concrete solution-- and --models--. After Eric's six kilos were already commented by 'WOW' (#22), Nahema's suggestion of nine kilo (#24) is rejected by other students Nahema's contribution is linguistically correct, but not concise due to superfluous information.

Having collected these concrete values for cats' weights, the teacher moves back to the mathematical core with the next question (#28/30). As the navigation pathway in Figure 1 shows, he navigates into the epistemic field --models-- ||explicit formulation||. He takes on the weight stated by Nahema and calls on Monir and Thasin for giving their suggestions how to model the situation:

30 Teacher one WEIGHT symbolizes about ten ki (.) well symbolizes ten kilo; Now, if a cat REGULARLY, right, (.) well if cats would weigh regularly around nine kilo; would one DRAW a weight symbol there, (.) or rather NOT; [...]
31 class ((*murmuring*))

Monir (#33) first refers to the meaning of the weight symbols in the dot plot and hence addresses the epistemic field --models-- ||meaning & connection||. Then he shifts the mode to ||explicit formulation|| in #35 by stating that since the weight of a cat has not two digits one would not print a weight symbol (see navigation pathway in Figure 1). Monir is struggling linguistically; the multiple breaks indicate how he is searching for suitable words. In #36, the teacher supports him in closing his utterance by translating Monir's gestures into words. Thasin directly addresses the epistemic field to which the teacher referred in his initiating question and models by using the concept of rounding on tens (#39/41). He uses an if-then-clause which is not trivial for second language learners of this age. The teacher evaluates both utterances (#42) by reformulating them and stating that both ways are plausible and not decidable from the dot plot alone.

All three focus boys, Nahema, Monir and Thasin, participate in the whole class explaining activity for the dot plot with a similar length of utterances (Nahema's being slightly shorter). All three contributions are valued in the interaction since the teacher (who is concerned of including everybody) builds upon them in the further interaction. However, we identify a typical difference between Nahema on the one hand and Monir and Thasin on the other hand: Nahema (as all five students in the beginning) only contributes some facts from his everyday knowledge. Although the compliance level is very good, the required epistemic mode of ||meaning & connection|| in this case had only a minor epistemic potential, here reduced to being sensitized for plausible weights. The epistemically deeper work on consolidating students' more general mathematical knowledge comes later in the navigation pathway. In this step, Thasin and Monir are involved (by explicitly formulating how to translate the real life situation of 9 kilo into a diagram). This later step in the pathway has a higher required epistemic level. As Monir's compliance level is not ideal in the beginning, he gets a scaffold with more opportunities to develop his thoughts than Nahema. We will show that the distribution of students on different steps of the navigation pathway has persistent pattern, e.g. the epistemic mode of ||explicit formulation|| seems to be addressed only by some of the (non-focus) students whereas others consistently take more peripheral roles. The epistemic matrix allows describing the peripheral parts of a whole class explaining activity with respect to the epistemic potential for knowledge construction and consolidation.

Episode 2: Multiplication of decimal numbers

Episode 2 took place half a year later in the same class. At the beginning of a lesson, the teacher initiates a recapitulation of the last lesson one week ago. After the girl Tasnim (#10) does not succeed in reporting properly, she calls on Monir to support her.

1 Teacher [...] I would like to know from you, [...] WHAT did we do an eternity ago,

...

10 Tasnim well (.) well three KIDS (.) I believe, were, have (–) on the blackboard (–) well have calculated TASKS? [...] Oh, I can't EXPLAIN it [...]

...

20 Monir [...] so (.) we calculated with (.) DECIMAL numbers, (.) calculated DIVIDING, (.) so um we have (.) at the moment (.) the TOPIC so to speak; (-) um DIVISION and multiplication, (--) with DECIMAL numbers, (-) and we did um (–) um CALCULATIONS, um when to INSERT um the (.) point when dividing (.) um at the result; for EXAMPLE- (.) if you (.) HAVE a number with point, like (-) um ((*2.5 sec break*)) twelve (-) po- um (.) twelve point ((*1.5 sec break*)) seventy-EIGHT, (.) then um you mus- (.) and you have to divide it by (-) um FOUR, (–) um (.) then (.) um (.) it is that are THREE; ((*1.9 sec break*)) um three times four equals TWELVE, (.) then you are calculating twelve MINUS twelve; that equals ZERO; (.) and then you are immediately with the POINT, (--) and um (--) you have this three (–) written DOWN at the result, (-) then you must insert the POINT immediately next to it; because you are (.) NEXT to, (.) um well because you are UNDER the point; with the NUMBER;

21 Teacher HM_hm;

22 Monir you immediately have to put a POINT into (.) the result;

...

25 Teacher THAT was already a bit more DETAILED; right, [...]

After some stumbling, Monir (#20) names the mathematical topic 'division and multiplication of decimal numbers' (i.e. complies the demanded field of --procedures-- ||labelling & naming||), a task of low required level. He immediately continues with a shift to the epistemic mode ||exemplification|| with a potential for later ||explicit formulation|| and explains by an example how the division algorithm works (even though some facts are missing, the idea becomes clear). The teacher evaluates this extensive contribution positively.

Later in the same lesson, after working on tasks in individual seatwork, the second part of Episode 2 starts after discussing the calculation of $19.8 \cdot 0.708 = 14.0184$. Thasin mentions his confusion because his rough estimation $19 \cdot 0 = 0$ does not fit to the result. The class helps him by offering the handier estimation $20 \cdot 1 = 20$. Afterwards the teacher shortly repeats how to put the decimal point at the right place in the result. But Thasin is still troubled and after a classmate points the teacher's attention to Thasin's confusion the conversation starts with him stating what seems strange to him:

2 Thasin um (.) because (.) now if you (.) um (.) MULTIPLY a number, apart from ZERO, (-) the number gets BIGGER (-) ((*silently*)) actually; fourteen is SMALLER than the nineteen;

3 Sina oh yes;

4 Teacher yes (.) to REVEAL that once more; (-) THASIN says; MAN, usually, multiplication means I make something BIGGER [...] EXCELLENTLY seen Thasin; (---) take on (-) one two people who should search an EXPLANATION for that;

Thasin (#2) states his problem by connecting the result of the task to his conceptual understanding of multiplication as an operation that always increases the original numbers. He connects several logical levels in the epistemic mode ||meaning & connection||, namely --concrete solutions / procedures / concepts--. The teacher marks this observation as important by reformulating and positively evaluating it. Several students are asked for an explanation in order to help.

6 Tilbe so (.) um (.) I GUESS so; (--) because it is zero point seven (.) HUNDRED; and if you (.) MULTIPLY the seven hundred times nineteen - so approximately MULTIPLY times twenty, then (.) it becomes FOURTEEN;

...

15 Larissa ((*walks to the blackboard*)) so HERE there are; (-) here there are only three NUMBERS; ((*points to 0.708*)) and here suddenly FOUR ((*points to 14.0184*))

16 Teacher (---) this makes the NUMBER; (--) thereby thus AFTER the point (.) after the point bigger, (.) and in FRONT of smaller; right, (--) does anybody have ANOTHER explanation [...] (---) Thasin, you yourSELF

Tilbe (#6) addresses the epistemic field --concrete solutions-- ||meaning & connection||, here by estimated calculation. Larissa (#15) describes an observation without offering an explanation. The teacher (#16) asks for further different explanations which can be interpreted as implicit mismatch for both. Thasin wants to explain himself:

19 Thasin (--) so- ((*2.6 sec break, walks to blackboard*)) ZERO times nineteen equals zero; (-) but we do have a POINT here, (-) and that makes the zero BIGGER; (.) and AFTER it there is also something written; ((*2.0 sec break*)) and (-) ZERO is always that it gets smaller, (-) and because it's not times ONE, (.) but LESS; ((*1.3 sec break*)) it is SMALLER than nineteen; (--)

20 Tom I didn't understand anything;

21 Teacher try it AGAIN; (-) but one MOMENT; (.) before Thasin (.) starts the EXPLANATION again, (-) I'm putting THIS here; ((*changes 20 · 1 = 20 to 19 · 1=19 at the blackboard*)) maybe you can use THIS Thasin;

22 Thasin Okay; (-) NINETEEN times zero (-) equals zero; (-) and NINETEEN, (.) nineteen times ONE equals nineteen; ((*1.2 sec break*)) and here we got (.) zero point seven hundred and eight ; (-) this is LOCATED (.) um- (-) it's not times ONE, but also not times ZERO; (.) that's why it must be located between the NINETEEN and the zero; (--) it's SMALLER than nineteen, but bigger than ZERO; (.) that's kind of ABOUT,

23 Tom I UNDERSTAND;

24 Teacher YES;

Thasin (#19 and #22) needs two trials to formulate his explanation so that others understand him. The teacher supports him by giving useful referent calculations on the backboard. Thasin takes on this help and explains by connecting logical levels: If $19 \cdot 0 = 0$ and $19 \cdot 1 = 19$ and $0 < 0.708 < 1$, then $0 <$ result < 19. This explanation is marked as understandable by the peers (#23) and positively evaluated by the teacher.

Again, both students, Thasin and Monir, contribute with epistemic potential to the interactive process of mathematical knowledge consolidation. However, we see differences: Whereas Monir stays on the procedural level, Thasin even initiates the connection of the learnt procedure with the prior conceptual knowledge and connects different levels.

Three students' epistemic participation profiles

As Table 1 shows, Episodes 1 and 2 are prototypic for the three students' epistemic profiles that could be reconstructed from the complete material, lessons 1.1 to 1.8 from the beginning of year 5, and from lessons 2.1 to 2.4 six months later (the last number in '1.2.4' indicates a running number for the students' contribution above sentence level). Within the limits of taking only 12 lessons over six months, Table 1 allows first answers to Q3: Students' epistemic participation profiles show certain stability over the time of half a year, visible tendencies in the first episodes can be detectedas persistent pattern.

Explanans in epistemic modes	Labelling & naming			Explicit formulation			Exemplification			Meaning & connection			Purpose			Evaluation		
Explanandum in logical levels	M	N	T	M	N	T	M	N	T	M	N	T	M	N	T	M	N	T
Conceptual levels																		
Concepts												***1.8.6*** ***2.1.10***		1.3.3	1.8.7			
Propositions																		
Semiotic representations				***1.4.5***										1.3.3 1.4.4		***1.4.5***		1.7.5
Models				***1.6.6***		***1.6.2***				***1.6.6***	1.6.5	***1.6.3***						
Procedural levels																		
Procedures				1.1.2 ***2.4.8***	***1.1.1***	1.5.1	1.1.1 ***1.2.4*** ***2.1.7*** ***2.4.8***		**1.6.4** **1.8.8** **2.4.12**		1.1.2	***2.1.9*** ***2.1.10*** ***2.4.12***	1.1.2 ***1.2.4*** ***2.4.8***	1.1.2 1.4.4		1.1.2 ***1.2.4*** ***1.4.5***	1.1.2	
Concrete solutions	1.2.3	2.4.7	2.4.11			***1.8.8***				***1.4.5***	1.1.2 ***1.6.5***	***2.1.9*** ***2.1.10*** ***2.4.12***	1.1.2 ***1.2.4***	1.1.2 1.4.4 ***1.8.6***		1.1.2 ***1.2.4*** ***1.4.5***	1.1.2 ***1.8.6***	1.7.5
Conventional rules																		

Table 1: Students' epistemic profiles (Monir in blue, Thasin in violet, Nahema in green, ***bold italics*** indicate potential for mathematical learning opportunities)

Thasin comparatively often refers to conceptual levels, the mode ||meaning & connection|| and within it the connection across levels. In the second period, he gives conceptual explanations as in Episode 2. As reconstructed for Prediger and Erath (2014), these explaining practices are highly valued in this classroom microculture. The classification of their epistemic potential (Table 1) detected nearly 9 out of 12 contributions as productive for consolidating knowledge since raising meanings is crucial for consolidation.

Monir with limited linguistic resources is specialised on the procedural levels and on the mode ||exemplification|| and the combination of several modes on one logical level. This specialisation allows his contributions to be mostly classified as having epistemic potential for consolidating (mainly procedural) knowledge. Hence, both boys significantly contribute to the individuals' opportunity to learn mathematics according to the microculture's sociomathematical norms. Nahema's seven contributions are mainly procedural and focus on the mode ||purpose|| in which he refers to mostly initial and concrete issues in the teacher's navigation pathways. This concreteness allows activating deictic means and compensating limited linguistic resources. However, in later steps of the classroom's navigation pathways when it comes to knowledge consolidation on epistemically higher levels, he usually keeps silent. This does not mean that he does not profit from passive participation, but he does not contribute actively to consolidating knowledge. Instead of the often assumed participation trajectory of increasing participation, Nahema's participation decreases to one explanation six months later (he raises his hand four times but is not selected to speak). Hence, Nahema's restricted linguistic resources in the language of instruction limit his active participation and due to the limited epistemic potential, seem to strengthen the inequality of his learning opportunities.

DISCUSSION AND OUTLOOK

From our case study for three second language learners with unequal German linguistic resources, we conclude: Students show different ways of participation which can be grasped by means of the epistemic matrix. The location in the fields of the epistemic matrix and within the steps of a navigation pathway makes the unequal individual epistemic potential for consolidating the mathematical knowledge visible.

The developed framework enables us to observe a new phenomenon: The reconstructed patterns show certain stability over time. Rather than talking about naturally increasing participation, we must therefore talk about participation profiles being connected to unequal German resources and learning opportunities within the same class. The relation between the visibly unequal linguistic resources of the three boys and their participation profiles will be the issue for further research since this seems to be one key for understanding the reproduction of inequality in classroom interaction.

ACKNOWLEDGEMENT

INTERPASS ("Interactive procedures of establishing matches and divergences for linguistic and microcultural practices") is funded by the German ministry BMBF (grant 01JC1112). We thank A.-M. Vogler, U. Quasthoff and V. Heller for the collaboration.

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Deictic gestures as amplifiers in conveying aspects of mathematics register

Danyal Farsani

Universidad de Chile, Center for Mathematical Modelling-Education, Santiago, Chile, dfarsani@dim.uchile.cl

This paper investigates communication (verbal and nonverbal) in the bilingual (Farsi-English) mathematics classroom. The study examines the communicative repertoire that interlocutors employed in lessons enabling them to construct meaning and mediate understanding. That is, the ways in which language and gesture can be seen as resources in supporting and conveying mathematical ideas is described. It appeared that the pointing gestures that were produced by the classroom teachers served as a mnemonic device to help remembering the key mathematics register.

Keywords: Mathematics register, code-switching, multimodality, gestures, pointing.

THE ROLE OF POINTING GESTURES IN MATHEMATICAL DISCOURSE

This paper focuses on the communicative functions of gestures in a bilingual mathematical discourse. The term 'gesture' is used in a wide sense, as a physical movement of a part of the body (e.g., hands, arms, eyes and face) (Kendon, 1983; Maschietto & Bartolini-Bussi, 2005) in communicative situations (Streeck, 1988). McNeill (1992, p. 11) used the term gesture to mean "movements of the arm and hands … closely synchronized with the flow of speech". In this paper, I refer to gestures to denote the movements of hands or arms that are synchronously produced in the act of speaking. Along similar lines, Sfard (2009, p. 194) defines gesture as a "body movement fulfilling communicational function" that is co-produced with speech.

Gestures have been categorised into four different groups (see McNeill, 1992). Each gesture category has a different form and function in communication. There are 'deictic', 'beat', 'iconic' and 'metaphorical' gestures. McNeill's (1992) gesture category takes an account of gestures that are dependently produced with their corresponding speech units. Of course, individual gestures can incorporate elements of multiple categories. For example, often pointing gestures trace the outline of the shape that is being pointed at. Therefore in this example, there are indexical and iconic components of a single gesture. In this paper, I will primarily focus on pointing gestures (or deictic gestures). Deictic gestures mostly occur in synchrony with speech. They are employed when interlocutors connect from verbal to visual, in order to index objects, locations, inscriptions that are either present or non-present in the environment. Due to the fact that pointing gestures are so ubiquitous and we interpret them with such ease, pointing can come into view as a trivial phenomenon (Kita, 2003). Pointing often draws on different modalities; for example, a speaker can connect from auditory to visual representation as they point to objects or inscription. Similarly in a classroom when a teacher can point to objects or inscriptions as s/he speaks, those pointing gestures link his/her verbal stream to its physical referents in the environment (Alibali & Nathan, 2012).

Index-finger pointing is the most common deictic gesture but it is interesting to note that there are different variations in index-finger pointing that are correlated with certain discourse factors. For example, the degree of finger closure or openness among the little finger, ring finger and mid-finger while pointing with the index finger is correlated with certain discourse factors. It has been observed that emphatic or first mentions of events tend to be regularly accompanied by the canonical (tightly bunched) index-finger point. In follow-up anaphoric mentions, once the location and the identity of the object have been established, a looser hand is used and the action is executed more quickly (Wilkins, 2003). The forms and functions of index pointing with a looser/tighter degree of closure and its corresponding pedagogical implications within a mathematics lesson have also been explored (see Farsani, 2015).

THE RESEARCH SITE

For part of my PhD research, I carried out an ethnographic-style approach to examine communication in a bilingual British-Iranian mathematics classroom. The students in this study were first and second generations of bilingual (Farsi-English) speakers from a Persian heritage. In this school, the medium of instruction was bilingual and learners were encouraged to value both languages equally. What seemed to be at the heart of the school was creating multilingual spaces (Creese et al., 2006), about using languages flexibly (Blackledge & Creese, 2010), and using a full range of young learners' linguistic repertoires (Creese & Blackledge, 2010). This bilingual school welcomed strategies that supported learning both the content and language simultaneously. I conducted audio-visual recordings of a number of mathematics lessons in order to capture verbal, vocal and visual elements of language to investigate how these modes of communication play a role in making mathematical meaning.

DATA ANALYSIS AND DISCUSSION

This particular interactional recording emerged from a lesson where the classroom teacher (T2) engaged in solving an unknown angle in a regular pentagon. T2 was explaining the solution to how he finds an angle of x which lies inside of a regular pentagon. T2 has divided the regular pentagon into five equal isosceles triangles as a starting point (see *Figure 1*).

I find this extract particularly interesting because T2's instructional talk, gesture and speech convey overlapping information in lines 15–19.

In this transcription, the left-hand – throughout column illustrates the verbal interactions only. The middle and the right-hand - throughout columns signify the multi-modal nature of the classroom and add richness to the verbal discourse. For example, the middle and right column include mathematical notations, Algebraic notations and diagrammatical representations. Based on the idea of what is 'Given' and what is 'New', Kress and van Leeuwen (1996, p. 187) believe that "[t]he elements placed on the left are presented as Given, the elements placed on the right

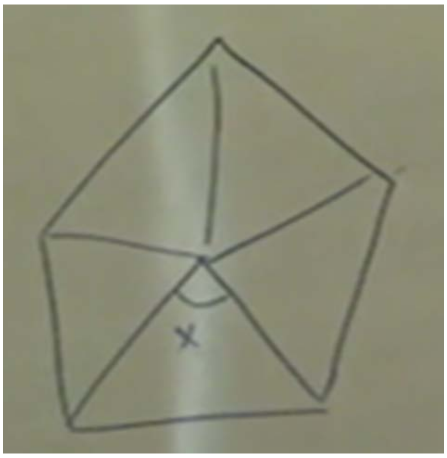

Figure 1: Drawing of pentagon

as New. What is given is what the viewers are already familiar with and agree upon but what is New, on the other hand, is not yet known and viewers must pay special attention to it". Therefore unlike many of the ordinary transcripts, the middle and right hand column are 'new', in that they add clarity and complement the left hand column by bringing visual elements into the verbal transcription.

The multi-modal transcript convention I have used is as follows:

T	Teacher
B	Boy
[ ]	Non-verbal communication
{ }	My translation
Italics	Farsi transliterated into English
Normal font	English language
Dots	Each dot represents one second of silence
Change in font size	Change in volume of an utterance: the bigger the font is, the louder the pronunciation. The smaller the font is, the quieter the pronunciation of the term.

Transcript number 1

T2: *khob, baraye iinke iino*
peida bekonim, {In order
to solve this} that's
a regular pentagon
obviously, and
each side is four ok.

[T2 writes 4 on each side of the drawing of a regular pentagon. He has already connected the centre of the pentagon to every vertex in the pentagon]

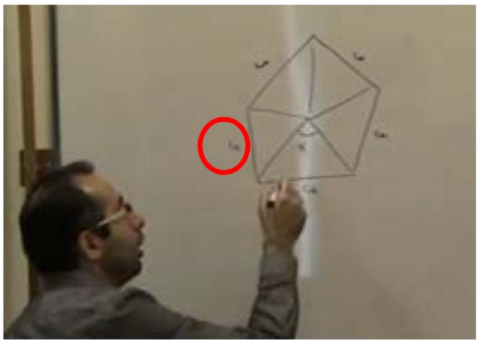

Figure 2: Marks on the drawn pentagon

Chon regular pentagon-*e* centre-*esh age maa*
be behesh vasl bekonim
mitoonim hamash
{Because this is a regular
pentagon, if we connect
the vertices to the centre,
it will all become}

[T2 writes O in the middle of the pentagon which represents the centre.]

isosceles triangle *peida*
bekonim, dorost
bekonim, khob {we will
find isosceles triangles,
ok}

[T2 employs a gesture that incorporates deictic information. He appears to point to his eyes with his index and mid finger as he utters isosceles in his speech]

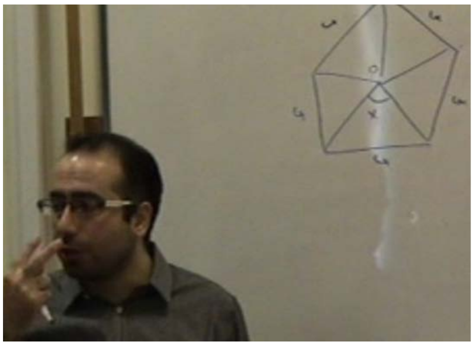

Figure 3: Identified deictic gestures

B1: *motasavi-al-saghain*
{isosceles triangles}
T2: *motasavi-al-saghain*
{isosceles triangles}
khob, iino ke peida
mikonim, {ok, when we
find this angle} angle of x
is equal to angle of x
equal to angle of x and so
on.
Khob, how many angle of
x *darim?* {So, how many
angles of x do we have?}
Bs: five/*panj*
T2: five-*ta. Khob*, {five,
ok} three hundred and

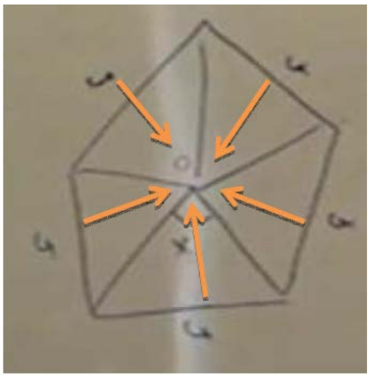

Figure 4: Elements of the drawn pentagon

sixty which is xxx the full
thing
[xxx is inaudible]
B1: divided by five
T2: divided by five

I find this extract of particular interest firstly because of the geometrical transformation that has been made into the regular pentagon; it has been converted into five equal isosceles triangles as a way to proceed. It emphasises the fact that a particular geometrical shape with a specific property can be turned into a number of shapes with a different identity (see Pimm, 1995).

There are different modalities that play a role in this short transcription of a video recording. T2 conveys his instructional message not only in speech but also in the gestures and a number of different modes that he uses as resources in teaching. For example, notice the way in which T2 directs attention by drawing and tapping (see lines 24–29) on the whiteboard. He taps twice on the angle x and moves on pointing to the other remaining four inner angles as he utters his statement “is equal to angle of x equal to angle of x and so on”. Furthermore, code-switching is evident within the technical forms of register. In lines 20–23, both B1 and T2 emphasised key terminology ‘*motasavi-al-saghain*’ in Farsi. In these lines, not only did they showed awareness of other languages (Farsi) in the class but they drew upon their full range of linguistic repertoire ensuring that the equivalent Farsi mathematics register is known and recognised by all students. By doing so, T2 provided an opportunity that acknowledges the linguistic resources that bilingual learners have at their disposal by demonstrating their knowledge and understanding of mathematics in Farsi. It appears that T2 ensures that bilingualism is foregrounded and is at the centre of the teaching and learning that takes place as he develops awareness of the Farsi mathematics register.

What I am more interested to examine in this particular transcript lies on 15–19 where T2 incorporated a form of deictic gesture (with his index and mid finger) indexing his eyes as he uttered 'isosceles' in his speech (see Figure 5). At this stage there could be two possible interpretations of T2's gesticulation; depending on whether the focus of attention is on his fingers or to the eyes which I will now discuss. It is possible that T2's gesticulation could be read as a visual similarity of the two fingers representing two equal sides. Even where the index and the mid finger are not exactly the same size in length, the gesture could serve as a primitive tool to convey the idea of two equal sides of an isosceles triangle. Hence T2's gesticulation represents not only the geometric representation but also conveys its mathematical definition visually.

Alternatively, if T2's gesticulation was indicating his eyes, then the gesture and the accompanying speech (isosceles) did not appear to have/convey any shared semantic meaning. Semantically there is no overlapping information but phonologically there is a strong connection. The relation between T2's verbal message and his deictic gesture is of homophony. Homophonous words are terms that have the same pronunciation as another but different in meaning, origin or spelling[1]. The way isosceles is pronounced is very similar to what can be thought of as 'eyesosceles'. The pronunciation of the term 'eyesosceles' possibly explains the reason as to why T2's gestural representation was directed at his eyes. Isosceles generated a deictic gesture, by means of indexing an object that was phonologically similar to the accompanying speech.

I find it interesting to see how a mathematics register activated a particular gestural representation in teacher's instructional talk. An interpretation of T2's gestural representation of the term 'eyesosceles' in conveying the instructional information reveals that there was no mathematical understanding or meaning assigned to property of isoscelesness. At the same time, T2's gestural enactment shows a great linguistic awareness that helps the remembrance and recollection of an English mathematics register. The enactment of such gesture also increases the emphasis of its verbal counterpart.

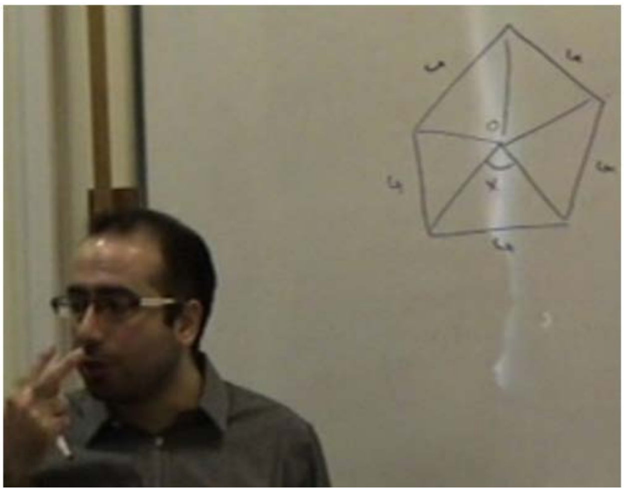

Figure 5: "Eyesosceles" I

Figure 6: "Eyesosceles" II

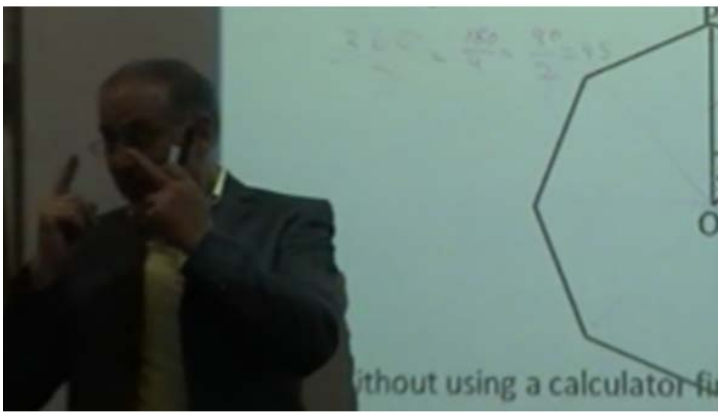

Figure 7: "Eyesosceles" III

The gesticulation for 'eyesosceles' is also evident by another mathematics teacher (see Figures 6 and 7). I would only present a snapshot for each of these gestural representations enacted by mathematics teachers.

In these examples, mathematics teachers' gestural representation for 'isosceles' was beyond deictic because their gestures served more than just indexing as there were also phonological connections to their gestural enactment. There are issues of phonetic and prosodic aspects of language involved in gestural enactment. Both teachers' deictic gestures served to emphasise the verbal language by indexing a visual object that carried a similar sound. Therefore their gestures served as a pedagogic tool to help memorising/remembering 'technical' mathematical words. In other words, teachers' nonverbal message served as a mnemonic device to help remember the terminology and the concept. Therefore, the employment of deictic gestures in instructional talk not only added clarification and richness to the spoken discourse but promoted the remembrance of an English mathematics register.

CONCLUDING THOUGHTS

This paper gave an account of the symbiotic nature of deictic gestures and their accompanying speech, as well as how these two modalities were temporally and phonologically coordinated. I offered examples supporting this symbiotic and the embodied nature of mathematical communication as speakers were engaged in speaking mathematically. The teachers' deictic gestures served beyond 'pointing' as they were not merely used for indexing but played a pertinent role in communication and facilitated phonological purposes.

A question might arise as to what extent deictic gestures facilitate language production in a mathematics classroom. Also, other questions emerge from the bilingual situation of mathematics teaching and learning in my study. Similarly to what was found for the case of homonymy and homophony and their potential confusion in a bilingual mathematics lesson (see Zagorianakos & Farsani, 2012), the reported data points to the relevance of having more than one language of reference.

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The interplay of language and objects in the mathematics classroom

Marei Fetzer[1] and Kerstin Tiedemann[2]

1 Goethe-University of Frankfurt, Frankfurt, Germany, fetzer@math.uni-frankfurt.de
2 University of Cologne, Cologne, Germany

Language appears to be a crucial aspect in mathematical learning processes. At the same time, objects like artefacts or other didactical tools play an important role in those processes. For that reason, we are interested in the interplay of both: We focus on the association of mathematical language and objects in classroom interaction. How does mathematical language vary in relation to objects and their participating in mathematical learning processes? Based on empirical examples, three different forms of interplay are presented.

Keywords: Objects, association, orality, literality.

INTRODUCTION

There is no doubt that mathematics teaching and learning are closely connected to aspects of language. Language is a central learning medium in every class and a content of learning, too. In some cases, it turns out to be a barrier to learning. Empirical research in mathematics education is familiar with research projects about language and mathematics learning (see TWG "Mathematics and language" at CERME over the past years). However, our picture of interactions in mathematics classrooms will be incomplete if we solely focus on oral or written language and, thereby, solely on human beings. There are not only students and teachers who contribute to classroom interaction. Material objects like manipulatives, rulers or diagrams influence the course of action as well. They make a difference in mathematical learning processes. Meanwhile, a sociological perspective that accepts objects as actors in the course of classroom interaction is only on the rise (see Fetzer, 2013, 2015; Kalthoff & Röhl, 2011). Latour introduces a sociology of objects that accepts objects as well as human beings as actors and participants in the emergence of social reality. His actor network theory offers us a new perspective on the interplay of language and objects in the mathematics classrooms (Latour, 2005).

LANGUAGE BETWEEN ORALITY AND LITERALITY

When talking about language in classrooms, researchers often need to specify what kind of language they actually mean. In this regard, Koch and Oesterreicher (1985) offer a helpful distinction that focuses on two fundamental aspects of language: on the form of language and on its function. Firstly, they differentiate the medium of language. Thus, an utterance can be phonic or graphic. For the context of this paper, we only focus on phonic language on face-to-face interactions in mathematics classrooms. Secondly, they identify two different conceptions that an utterance can have. It can be conceptually oral or conceptually written. This affects the question of communication strategies used. More precisely, conceptually oral language is often used when interlocutors are directly related and can refer to a given situation. This is what, consequently, gives a specific form to the language. For example, at any time, the interlocutors can ask questions of understanding, show emotions and influence the course of the interaction. For that reason, sentences may be short and even incomplete. Referring to the given situation, the speakers use deictic expressions and gestures. Thus, orality is characterized by interlocutors who spontaneously negotiate their roles and the course of their interaction. Koch and Oesterreicher call this a language of nearness. Examples are a conversation in the family (medially oral) or a chat among friends (medially written). In contrast, conceptually written language is used when the interlocutors are not necessarily in direct relation and the processes of speech production and speech reception might be separated from each other. Thus, aspects of the situational and cultural context have to be made explicit. As a consequence, sentences are longer and more complex. For example, the writer forms main clauses, but also subordinate clauses to express the relations he wants to inform about. Moreover, he uses more specific terms, e.g. mathematical terms, to be precise and explicit. Koch and Oesterreicher call this a language of distance. Examples are a text of law (medially written) or a scientific lecture (medially oral).

Against this background, one can classify the registers of everyday language and academic or technical language (see Cummins, 2008; Duarte, 2011; Gogolin 2009). Halliday (1985, p. 29) describes a register as "a variety of language, corresponding to a variety of situation". According to Halliday, the use of the term 'register' points to the assumption that individuals usually adapt their use of language to a given situation. Thus, the register of everyday language is rather conceptually oral, irrespective of its medium. It has to fulfil the function of a fast and unproblematic communication in our everyday life. As such, the oral language may be, at any time, supported by gestures or by reference to a context. Words do not have to be clearly defined, sentences may be short or even incomplete. In contrast, the register of academic language is conceptually written, again irrespective of its medium. In academic contexts, language should be as explicit and precise as possible and intelligible without any reference to a specific situation. For that reason, words have to be well defined; sentences might be complex in order to reflect the relations that the author wants to talk about.

REASSEMBLING THE SOCIAL

Latour goes beyond the traditional understanding of the social. His Actor Network Theory (ANT) is a radical change of perspectives proposing a sociology of objects (Latour, 2005). Latour extends the list of actors assembled as participants fundamentally. He does not only accept humans as participants in the course of action. Instead, he integrates and gathers all sorts of actors. "Any thing that does modify a state of affairs by making a difference is an actor" (2005, p. 71). Following Latour, objects participate in the emergence of social reality. Consequentially, Latour recommends a broader understanding of agency. "Objects too have agency" (2005, p. 63), and appear associable with one another, but only momentarily. They "assemble" (2005, p. 12) as actor entities one moment and combine in new associations the next minute. Accepting objects as participants in the course of action, Latour gives up the idea of stable and pre-defined associations and actor-entities. He reassembles the social.

Looking through Latour's sociological lenses, not only the traditional understanding of agency has to be re-defined, but also the notion of action. Objects participate in the course of action and take effect. However, it can be noticed that their mode of action is different from the way human participants contribute to the social interaction.

Mathematics education has to deal with all sorts of (material) objects, didactical tools, artefacts and manipulatives, diagrams and signs. All those objects leave their traces in the emergence of mathematical learning processes and take part in the course of action. Mathematical learning appears to be closely connected to objects and non-human things. Even if Latour himself does not suggest any methods of empirical analysis, Latour's approach proves to be a fruitful background theory for empirical research in mathematics education (see Fetzer, 2013, 2014). Accepting objects as participants in the course of action and following the idea of objects having agency helps us to get a better understanding of mathematical learning processes.

METHODOLOGICAL BASES

The goal of our research is to analyse the relationship between language and objects in everyday mathematics lessons. Probably in every classroom, you can gain insights referring to that topic. To start, we have focused on primary classrooms. However, it was not important to us to observe specific lessons, but rather a wide range of occurring mathematics lessons. Therefore, we observed several mathematics lessons in three different German primary schools. On the bases of videos of the whole lessons, we filtered out those scenes in which humans as well as objects participated. Those scenes were transcribed and became objects of analysis. In order to include a wide spectrum of scenes, we distinguished episodes with and without a teacher.

To get access to interaction processes in mathematics classrooms, our analyses are of a reconstructive manner. They are analyses of interaction (see Cobb & Bauersfeld, 1995). This method refers to the interactional theory of learning and is based on the ethnomethodological conversation analysis (see Sacks, 1996). This method was devised by a working group directed by Bauersfeld. In contrast to conversation analysis, it focuses on the thematic development of a given face-to-face interaction rather than on its structural development. For that reason, it is especially suitable for our research because it allows us to analyse the relationship between language and objects while teachers and students negotiate mathematical meaning. However, reconstructing empirically the way objects participate in learning processes remains unaccustomed. Objects' contributions to classroom interaction become accountable in the process of interweaving. As soon as object-actors become associated with other actors they enter the course of action. Their traces render perceivable and can be captured by a turn by turn analysis (see Fetzer, 2013; Sacks, 1996).

To illustrate our results, we selected four strongly contrasting scenes. 'Diagonal' is a teacher-orientated scene. In contrast, both '1000 dots' scenes are group work situations without a teacher and show an interesting change in the relationship between language and objects. In addition, we chose the very short example 'party hat' in which the object is not present at all, but still a point of reference.

EMPIRICAL EXAMPLE I: 'Diagonal'

The following scene is taken from a lesson about the hundred field. The teacher and the students start by repeating what they already know about the hundred field. Thus, a student says that a column „goes from top to bottom or from bottom to top". This utterance is confirmed by the teacher. Then, another student (Danis) puts up his hand and is asked, but the teacher does not pay attention to him. The scene starts when the teacher turns back to Danis. On the mentioned hundred field, no numbers can be seen. They are all covered by coloured squares, the upper half is red, the lower one is blue. In the transcript, the squares are named by the corresponding numbers in order to facilitate the reading.

26 Teacher: [...] How does the diagonal run?

27 Danis and 100 field: *[going to the blackboard:]* Well, like this.
[on the 100-field from the 60 to the 96] Here for example. Then, I move like this.

Figure 1: Material of the hundred field

28 Teacher: Can you also explain that with words? Now, you have already shown that to us.

29 Danis: *[sits down again]* Well, that works for example like this. .. that I am atthe ten and the 91. I can because the corner and the diagogal runs simply *[moving his finger in front of himself from top right to bottom left]* like this. .. Dialogal. .. But it can't go up and it can't go down or right or left.
[Comment: In German, Danis says "diagogal" and "dialogal" which are, like in English, not the correct words.]

In this scene, Danis and the teacher pick out the diagonal as a central theme. As the teacher is talking about "the" diagonal, one could think that there is only *one*. In contrast, Danis shows two different diagonals, one from 60 to 96 and another one from 10 to 91. One can only guess what his definition of a diagonal might be, but one can see that he identifies more than one diagonal on the hundred field. His two references to a diagonal show a difference concerning our research question. In the first one, Danis points at the hundred field with his finger and, thereby, integrates it into the discourse as a participant. In connection with the object, Danis' utterance gets intelligible. Thus, one can say that the object changes its status and becomes an active participant in the discourse. Danis and the hundred field get interwoven. With the object completing Danis' utterance, the boy's language gets reduced and deictic ("like this", "here", 27). Taking over the turn, the hundred field allows Danis to use an easier, more oral language. He only forms main clauses that are not clear without reference to the concrete situation including the hundred field.

In the second part of the scene, we can reconstruct the opposite. The teacher explicitly asks Danis to give his description in words. Thus, he is confronted with the task to describe the same mathematical phenomenon, but without any direct reference to the object (and maybe even without gestures which do not belong to the category of words either). This interpretation is confirmed when Danis goes back to his chair, sits down again and, thereby, distances himself from the hundred field. Now, the object is no longer a participant of the discourse. Instead, Danis develops a second description of a diagonal solely by the use of words and a gesture. One can see that excluding the object from the course of action means to exclude its actions. Now, Danis has to take over again: He has to compensate the missing object by being more specific in his language. For example, he now names the two "corners" of the figure whose diagonal he is talking about. Furthermore, he starts forming more complex sentences ("because the corner...", "... and...", 29). In summary, one can state that the language is more orientated towards written language.

EMPIRICAL EXAMPLE II: '1000 dots'

In contrast to the first scene, there is no teacher participating in the scene '1000 dots'. Two girls, Martina and Sonja, are sitting at a table. On the table, there is a sheet with the task 'Share 1000 dots fair-mindedly between three children', a pair of scissors and a box containing 100-dots-cards (small cards with each showing 100 dots arranged in lines of ten). First, the two girls try to figure out the task on their own. Once they get a leftover of ten dots they get stuck. This is when the selected scene starts.

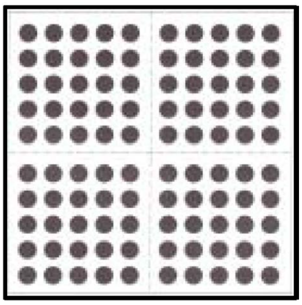

Figure 2: Material of dots-cards

167 Martina and cards: *[reaches with her left hand for the box with the 100-dots-cards]* Shall we make use of them?

Figure 3: Box with the dots-cards

168 Sonja: *[looking at the cards in Martina's hand]* Yes.

169 Martina, box, cards: *[takes the pile of cards out of the box and pushes the box away]*

Figure 4: Student in interaction with the material

170 Sonja: Cause it's difficult, isn't it?

171 Martina and cards: OK. *[puts cards from her hand down on the table like sharing out a deck of cards]* One, two, three.

As soon as the girls get stuck in their solving process, they invite the 100-dots-cards to participate ('cause it's difficult' <170>). These object participants breathe new life into the solving process. Assembling with the girls and the table, the cards embody (at least) one fundamental intermediate step in the solving process: instead of just representing the situation, they *are* the 1000 dots that have to be shared between three children. Their lasting quality contributes to a better orientation within the social solving process. Being arranged in a pile of ten cards they take over the part of a reliable basis. Thus, one can see that objects are not only useful for expressing one's own ideas, but also for developing new ideas that can push the solution process forward. The girls' language changes once the object-actors enter the stage as participants. The association of cards and girls can be reconstructed in the reduced character of the chosen language: short utterances like "of them" <167> or "one, two three" <171> intertwine with the objects and their manifest and structuring message.

Some minutes later, the girls approach a mathematical result: 999 dots lie there separated on three piles. While Martina is cutting the last dot in pieces, Sonja is worrying about the correct result.

917 Sonja Yes, but how much *do* the little ones *count?* How *much?*

918 Martina, pieces, scissors *[laying a little piece of her last cutting action on the middle pile]* A fourth, half of a fourth. *[laying another little piece on the right pile]*

While the solving process is coming to an end on the level of objects, Sonja is worrying about an appropriate language for what the girls can see in front of them (917). The non-human participant was irreplaceable in the solving process. But, subsequently, in order to fix the result and present it to others, mathematical language gets relevant. The girls obviously do not regard the task as completed until they can express their result by means of language, too. Thus, mathematics as a social practice needs a precise language to fix, to communicate and to discuss its results. While the language was relieved during the intensive search for a solution by means of the available object-participants, it now gets more demanding again. Martina forms a description of what the "little ones" count which is comprehensible regardless of the concrete objects. Her language develops from conceptually oral to conceptually written forms. She expresses a mathematically *new idea*, the idea of fraction. By this step in their work, the girls facilitate a precise mathematical communication with others, e.g. their classmates.

Considering both succeeding scenes, they illustrate the development from an orally orientated language in the first scene and the beginning of scene two towards a rather written conception of language including mathematical terms ("a fourth, half of a fourth"). In both examples the object-actors can be reconstructed to be participants. However, their mode of action varies within the emergence of the solving process. At the beginning, they contribute their structuring and lasting quality to the course of interaction. They invite the girls to share the dots out right to the end. Thus, the cards and dots prompt Sonja and Martina to cut the last dot into bits and pieces. Later lying on the table, the mini-objects release the strong requirement for a precise term to label them. They call for language, they demand for mathematical description on a rather written level.

EMPIRICAL EXAMPLE III: 'Party hat'

In the last scene, there is actually no object participating at all. It is only present in the mind of the human participants. In the lesson, the students have worked on their own around the following task:

> Which of the two flatplans fits the three-dimensional figure?
> Why does one of the flatplans match and the other one does not?

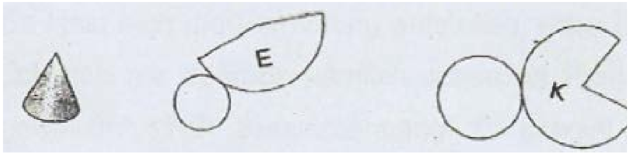

Figure 5: Two flatplans and a three-dimensional figure

Later the students compare their results in the whole class. In the selected scene, it is Dominik's turn to present his findings.

1	Dominik:	K works because the gap is smaller. Cause I have already made myself a party hat of that, you know.
2	Some pupils	*[whisper and nod]*

Dominik puts forward an empirical argument that relies on (his own) experience. He creates the context for his argument exclusively on a phonic level with spoken words. By talking about the party hat, Dominik includes the hat as an actor within the course of action. As such the party hat seems to have the power to convince the other children of the verbally presented decision 'K works'. The party hat is not present at all, but it is replaced by language. In other words, the party hat solely takes part in the interaction in form of its linguistic representative.

EMPIRICAL RESULTS AND CONCLUSION

Objects do play an important role in mathematical learning processes. They may become participants in the course of action and contribute to the emergence of social classroom reality. In our research, we basically differ between two learning situations: Those situations, when objects are to be seen as participants in the course of classroom interaction, and those, when human participants act without reference to objects. In the context of our study the first case might be called 'Language and objects' and the second case can be characterised as 'Language without objects'.

Concerning situations we call 'Language and objects', there are two cases to be observed. Sometimes, object participants and human actors *assemble in their actions*. In these cases, *objects take over part of the turn*. Action is no longer attributed to one single actor but to actor entities, actions are experienced as combined actions. If objects and students assemble in their actions and humans have to take over only part of the turn, language proves to be rather conceptually oral (compare the scene 'Diagonal'). However, the assembled action appears to be 'complete'. Other participants can reconstruct the meaning that emerges. This is the crucial point. If objects participate in the course of mathematical action and take over part of the turn, they thus relieve students on the level of language. This might be an initial opportunity for children with poor language performance to act mathematically.

In other situations, *human actors and objects interact in turns*. In these cases, objects take over whole turns, and students react to those turns. If human actors invite the objects to be a part of the discourse, objects can challenge the students to find a new language (compare the second part of the scene '1000 dots'). We could observe that in these situations language tends to be rather conceptually written. This aspect is of special interest from a didactical point of view. Objects cannot only relieve students on the language level, but also challenge them. They can be 'supporters' in the development of mathematical thinking and of a precise mathematical language.

When looking at situations that we call 'Language without objects', we observe that associations between objects and humans dissolve. Students are obliged to take over each turn on their own. Accordingly, they become responsible for the idea and for the formulation of the intended utterance (see the second part of the scene 'Diagonal'). For that reason, students have to be as explicit as possible in their language. They might use well-defined mathematical terms and rather complex sentences. They might give many explanations about the context of culture and of content in order to be clear about the given topic. Language turns out to be conceptually written.

Summing up, we see conceptually oral language when students and objects assemble in their actions. In contrast, we find conceptually written language, when associations between human and non-human actors dissolve. Finally, we observe a development of language from conceptually oral to rather conceptually written forms, when human actors and objects interact in turns. These situations of language development can be seen as the situations of mathematical learning. Maybe, further research can show how the interplay of language and objects depends on the kind of material that is part of the discourse.

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Using gaze tracking technology to study student visual attention during teacher's presentation on board

Enrique García Moreno-Esteva and Markku S. Hannula

University of Helsinki, Department of Teacher Education, Helsinki, Finland, Enrique.garciamoreno-esteva@helsinki.fi

We present some initial findings obtained from a study on student attention in class using mobile gaze tracking technology. In this descriptive case study, we use a teacher's verbal, gaze, and gestural cues to identify the area on the board she wants her students to focus on, and analyse how one student's gaze location is shifting in relation to this. We found out that the student was actively following the cues during most of the time. However, we also observed moments when the student's gaze location was not in synchrony with the teacher's cues. The gaze tracking methodology seems to be a promising tool for a fine grained analysis of classroom communication and student attention.

Keywords: Mobile gaze tracking, student attention in mathematics classes, teacher gestures and indications.

INTRODUCTION

An important aspect of teaching is the multimodal communication (Radford, 2009; Arzarello, Paola, Robutti, & Sabena, 2009) between teachers and their students. Using mobile gaze tracking technology we will take a close look at the way a student's visual attention responds to cues from the teacher when the teacher is presenting new content on the board.

The teaching act is twofold. There is a pre-planned element of what the teacher does, which may be well planned and rehearsed several times over the years. There is also an improvisational element, reaction to unexpected student questions or comments, which requires the teacher to think 'on her feet'. Both of these modes of action include consciously chosen words, prosody, gestures, and facial expressions. However, some of the visible and audible messages are not consciously chosen, but are enacted more or less automatically, or even unconsciously.

Roth (2012) gives an example of a university professor whose dis-fluency he sees as an indication of the communicative act not being completed 'in mind' beforehand. Rather, it is "an unfolding event of communicating and thinking, which are not ready-made but develop in real time" (p. 237). Radford (2009) claims that "thinking [...] does not occur solely *in* the head but *in* and *through* language, body and tools" (p. 114).

The student side of classroom interaction is more complex than the simple receiving of messages from the teacher. Student behaviour in class is based on the student's personal agency, which is determined by his or her needs, goals, and identity. At the same time, student behaviour is largely reactive to changes in the environment, especially to what the teacher and the student's peers do. There is no research method that can provide a full account of meanings of this behaviour and reasons for it. Clinical interviews and think-aloud protocols distort the social interaction in class and thus lack ecological validity. Interviews done afterwards (including stimulated recall) can only have access to the student's post hoc reconstructions. Moreover, all self-reports are subject to be biased towards socially acceptable responses. Observations of facial expressions, brain imaging, and other physiological measures fail to capture the meanings students associate with their behaviour. Yet, each new methodology has shed light on some new aspects of the complexity of student cognition.

In this research report we shall present results from a pilot study using a mobile gaze tracking device to record students' visual attention during mathematics lessons. Gaze tracking is an established method for the study of attention at the automatic level of processing, in which the person is not consciously aware of fixations and shifts of attention. However, until recently, the technology had only been applicable in laboratory settings. Only now is it becoming available for use in ecologically valid situations such as outdoors (Baschnagel, 2013; Foulsham, Walker, & Kingstone, 2011) or in classes (Yang, Chang, Chien, Chien, & Tseng, 2013; Rosengrant, 2013).

THEORETICAL FRAMEWORK

Human communication consists of more than just words and diagrams; gestures, glances, body movement, prosody are also important aspects of it. Roth (2012) and Radford (2009) claim that there is a debate on the relationship between different communicative modalities. Some contend that both speech and gesture originate from the same psychological structure, while others claim that speech and gesture originate from different psychological structures. Still others claim that speech and gesture are different communicative channels, and that gesture serves a subordinate function.

Goldin-Meadow (1999) points out that the importance of non-verbal aspects of communication such as gestures has been recognized for a long time, at least two thousand years, in theatre, rhetoric, philosophy, and language. She identifies two different types of gesturing: gestures that substitute speech (e.g., sign language) and gestures that accompany speech, often unconsciously. She also points out that gesture enriches communication by providing a different representational format. For a speaker, gesture reduces cognitive burden, helps retrieval from memory, and is a tool for thinking. Gestures have also been observed to be important in the process of forming new concepts (Goldin-Meadow, 1999; Arzarello et al., 2009; Radford, 2009)

In gesture studies, McNeill's (1992) categorisation has been used frequently. He identified four different types of gestures: 1) beats, that do not have content information, but give rhythm and emphasis for talk, focus attention, and coordinate taking turns; 2) deictic gestures (pointing), that point to something concrete or abstract and typically have a verbal counterpart such as 'here', 'there', 'that', 'me', etc.; 3) iconic gestures that pictorially represent the target, for example, by drawing in the air; and 4) metaphoric gestures, that also create an image, but such that the image refers to an abstract concept metaphorically.

When teachers teach mathematical concepts, gesturing – and especially pointing – is common (Alibali & Nathan, 2012). Pointing gestures reflect the grounding of cognition in the physical environment, and pointing can be used to highlight connections between related representations. This exemplifies the need to interpret communication in context, which considers the material and graphical structure of the interaction (Goodwin, 2003; Arzarello et al., 2009).

Sweller and his colleagues have studied the influence of split attention on cognitive load (Yeung, Jin, & Sweller, 1997). Their studies show that when attention is split between two sources of information, a higher cognitive load may impede learning. This effect can be ameliorated when the sources of information are physically integrated. However, the effect is dependent on the student expertise. For more advanced learners additional information may be nonessential and impede learning.

Our research project will study student attention to mathematical diagrams and script on the board during mathematics lessons. Our focus in this paper is how well teacher talk and gestures direct student attention. In our study, we are specifically interested in student navigation when information is presented in two distinct areas, and in how effective is the teacher use of gesturing to help students integrate two sources of information. To our knowledge, no previous study has analysed students' visual attention in classroom situations using gaze tracking.

METHODOLOGY

In order for the researchers to monitor a student's attention in class, the student wears a gaze tracking device, which consists of a glass frame equipped with miniature cameras which produce a video scene and keep track of the direction that the eyes are pointing at. This device allows the software calculation of the direction of the gaze in class, producing a video scene with a dot indicating the locus of visual attention of the person wearing the glass frame (see Figure 3). The gaze tracking glasses are connected to a laptop with two cords, which prevents the student from getting up, but does not restrict movement while seated. The device was developed at the Finnish Institute of Occupational Health in collaboration with Aalto University (Lukander, Jagadeesan, Chi, & Müller, 2013). The prototype of this device is used in this study (see Figure 1). In addition to gaze tracking, there is another video recording of the class from behind, focusing on the teacher and the board, and there is a third video focussed on the subject student wearing the glass frame.

The data is obtained from a Finnish eighth grade classroom in a school where the language of instruction is English and where our subject student is a native English speaker (in this school, the students are bilingual). The subject wearing the glass frame was chosen on a voluntary basis. Informed consent has been obtained from the teacher, the focus student, all other students seen in any of the pictures, and their legal guardians.

The video recording we analyse shows a teacher explaining the topic of linear equations and relating

Figure 1: The gaze tracking device

them to the geometry of the lines using GeoGebra on a Smartboard. The dynamic functions of GeoGebra are not used, but the teacher draws additional lines and symbols on the Smartboard as shown in Figure 2. Specifically, throughout the approximately five-minute clip, a couple of lines appear on a Cartesian coordinate system. The lines intersect at a point (point P1 in Figure 2). Then the following sequence of events ensues. The teacher explains how to find the y-intercept of the first line (line L1, point P2, equation E2), and then, how to find the gradient of the same line by locating a second point (in this case P1) on the line and identifying the rise (segment S1) and its length (N1) and the run (S2) and its length (N2) from the lower point to the higher point. The gradient is calculated (E1). She then identifies the y-intercept and the gradient of the line in the equation of the line (in region Q). The same procedure is repeated for the second line (L2). The y-intercept (P3) is located and its value noted (E3), and a second point (P4) chosen in order to find the rise (S3) and its value (N3) and the run (S4) and its value (N4). The value of the gradient is calculated (E4). Finally, these values are identified in the equation corresponding to the line in region Q. In Figure 2, we identify all the areas of interest to which the teacher makes reference.

We made two segmentations of the video clip in the following way. First, we segmented the clip according to what we think are the areas of the board to which the teacher wants to bring attention, as indicated by the teacher's verbal cues, hand movements, and gaze. We then segmented the clip according to the student's

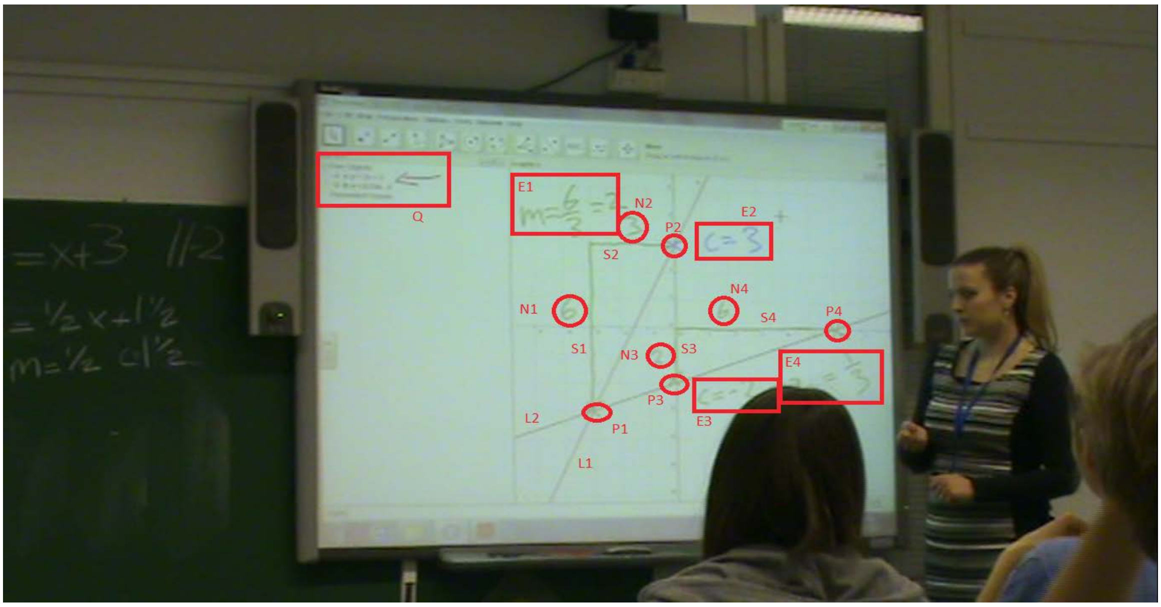

Figure 2: The board with the various regions of attention

gaze location – gaze is typically located for a short period of time at a certain position on the board before moving on to another location. Occasionally, there are glances which move away from a position and back to the original position, or somewhat rapid, but not too rapid shifts back and forth between two spots. We interpret these to indicate that the student attention is split between two areas and in these cases we coded both locations for the gaze. We then analysed the segments in order to find regions where gaze location follows the teacher's cues that direct attention distinctly, and regions where this is not the case.

DESCRIPTION AND ANALYSIS OF DATA

For most of the time during the teacher's presentation, the student gaze location follows teacher's cues quite well. However, there are also moments when this is not the case. We present a detailed analysis of two sections of the clip. Specifically, we shall analyse moments when the student's attention is not matching the intended area of attention. These sections occur after the teacher has finished explaining how to calculate the *y*-intercept and the gradient, and proceeds to link this information to the equations of the lines.

In the first instance (for line 1) the student follows the teacher's indications closely. In the second instance, the student gaze does not follow the pattern indicated by the teacher. The transcriptions follow. We have provided the graphic in Figure 4 to facilitate visualization of the ongoing processes. In the graphic we show how the area of intended attention (AIA) segments interlace in time with the gaze location (GL) segments for each of the two lines discussed by the teacher, line 1 (Line 1) and line 2 (Line 2). The bars representing each segment have been colour-coded according to the area of intended attention or the area where gaze is located: red is the code for the Q area, blue for the teacher's face or hands (T), white for unspecified locations, and other colours for other areas.

For the discussion pertaining to each line we provide the following information. First, we provide the beginning time of the segment and the transcript of what the teacher is saying. We then indicate if there are verbal, hand, or gaze cues that might be used to focus the student's attention onto an intended area of attention and what we perceive to be the intended area of attention. To the right of the transcript of the teacher intervention, we give the same information

Teacher			Student	
Time (ms)	Utterances [actions and non-verbal cues that indicate the area of attention]	Intended area of attention	Time of gaze shift (ms)	Gaze location
742006	and as you can see [glances briefly at equations in area Q]	Not specified	742014	Teacher face
742966	actually [takes another pen]	Not specified	743341	Q
743611	the equation of this line is here [draws an arrow pointing at the equation; gaze and gesture] (Figure 3c)	Q		Q Teacher face Teacher hands Q
748697	you can see that yes, [gaze and gesture]	P2, E2	 749137	P2, E2 Q
	the y-intercept [gaze and gesture]	E2, Q	750496 751959	Teacher face Q
752007	is the number that stands alone [gesture, glance]	Q	753940	Teacher face
754704	and the gradient [gaze and gesture]	E1	755372	E1
756229	is the number [gaze and gesture]	Q	756602	Q
757503	that is the coefficient of x [gesture only]	Q	757743 759268	Teacher face Q
760510	End of sequence		764774	

Table 1: Teacher behaviour and student gaze when the teacher explains the connection between gradient and y-intercept values and the equation for line 1

for the gaze location, except for a transcript. Notice that in some instances, there might be more than one intended area of attention, and in some instances the gaze location information might include two regions if there is a sequence of rapid shifts of gaze location between two spots. First we provide the data that concerns the discussion of line 1.

In the beginning of the segment we see the student focusing on the equation of the line before the teacher gives any explicit indication. Upon closer analysis, we see that when the teacher has finished the previous stage and before she picks a new pen, she casts a brief glance towards the equations. This was very difficult to observe and we noticed it only as we were carefully trying to find the reason why the student moved his gaze into the area where the teacher would move a fraction of a second later. We believe that the student observed the brief glance of the teacher as he was looking at the teacher's face, and therefore was able to react to the teacher's utterance, "as you can see" with a foresight as to where the teacher was likely to focus next.

Next, the teacher is trying to connect the explanation of the *y*-intercept and gradient to the equation of line 1, and the student's gaze location follows the teacher's indications to attend to area Q very distinctly (Figure 3c), and indeed to other areas as well, as can be seen from the top two bars in Figure 4 (in Figure 4, red codes the Q area, and blue codes the teacher's face or hands). What we see in this segment is how the student's gaze shift to the area of intended attention is in slight delay with respect to the teacher's cues. Also note how attention in this section is split between the equations of the line on the left (Q) and the values of *y*-intercept (E2) and gradient (E1) on the left. The teacher uses pointing gestures to successfully guide the student's attention across these two areas of interest.

Now we provide the data that pertains to the discussion of line 2. The student again anticipates the teacher movement to discuss the equations, but he does so substantially earlier (Figure 3a, first red bars in the upper and lower tiers of Line 2 in Figure 4). Moreover, he dismisses the teacher's direction to attend area Q at the end of the intervention (Figure 3b). We speculate as to whether this is partially due to the fact that she moves and turns around and is no longer facing the student; it would seem as if the student wants to see what he thinks the teacher is looking at. In any case, the teacher's pointing gestures are well visible in the student gaze video, but he does not direct his gaze as a reaction to the teacher's actions. This could be an example of self-initiated shifts of attention based on the student's active construction of his own understanding. Also, this might indicate a lack of attention, a blank stare.

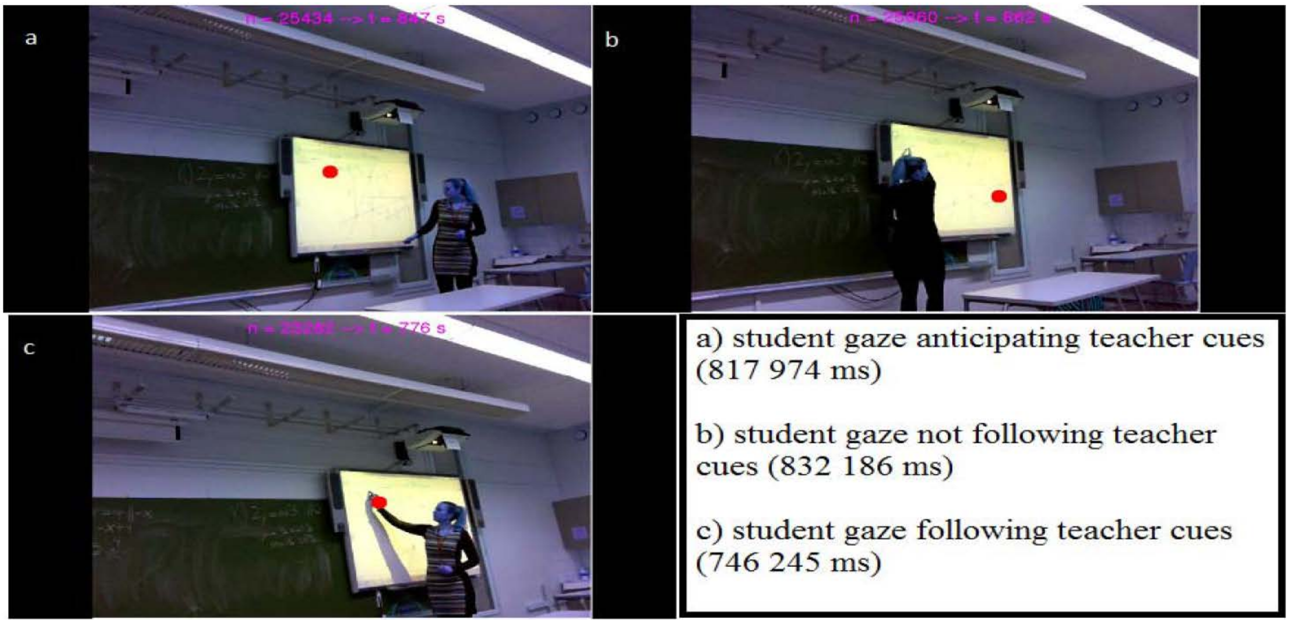

Figure 3: Pictures showing the situations described in the data analysis (Due to calibration error for this distance, the red dot indicating the student gaze location is systematically about 20 cm too much to the right)

Teacher			Student	
Time (ms)	Utterances [actions and non-verbal cues that indicate the area of attention]	Intended area of attention	Time of gaze shift (ms)	Gaze location
817115	pause, changes pen [no gaze, no gesture] (Figure 3a)	not specified	817105 818231	Q teacher hands
818733	this one uses approximate values [gaze, gesture]	Q	819104 819657	teacher face Q
821333	the correct answer there would be one over three (mumble)	Q	821621 822716	teacher face E4
826416	and the same thing again, can I find the [gaze, gesture]	E3		E4
829920	y-intercept [gaze, gesture]	P3	830243	P3
830920	here, the constant [gaze, gesture, teacher turns her back to the student] (Figure 3b)	Q	834352	Not Q, L2
835021	the eeh [gaze, gesture]	E3		Not Q, L2
837102	coefficient of x	Q	837078	S2, N2
838424	is the gradient	Q, E4		S2, N2
840141	End of section		840601	

Table 2: Teacher behaviour and student gaze when the teacher explains the connection between gradient and y-intercept values and the equation for line 2

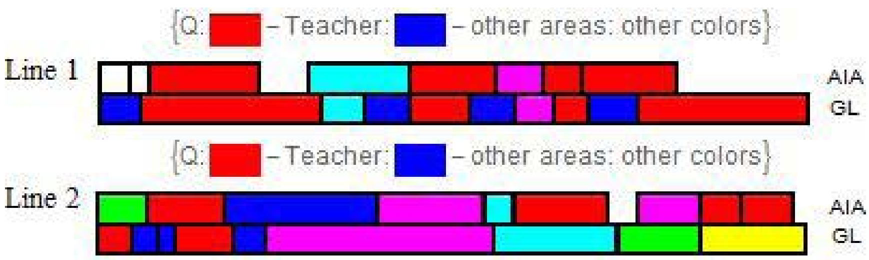

Figure 4: Segment interlacing for the transcripts above

DISCUSSION OF RESULTS

In previous research, the reliability of coding for gestures, speech, and their relationship has typically been high (85%-94%) (Goldin-Meadow, 1999). It seems likely that observing explicit gestures is very natural for human observers, which allows high accuracy. However, when we first watched the video, we did not notice the teacher's glances which, nevertheless, were used as cues by our focus student. This highlights the importance of paying close attention to the teacher's glances in future analysis on student visual attention.

The sections analysed are examples of a well preplanned teacher explanation. Yet, it includes unconscious cues (gaze cues such as glances). The student can follow the explicit cues very well, but is also observing subtle cues. Throughout the whole five minute sequence, the student splits his attention between the board and the teacher's face. In the sequences of closer analysis, there was additional split attention when the teacher connected inscriptions on two separate areas of the board. In both cases of split attention, the student seemed to have no problem following the teacher's cues. In summary, the student seems to be following the teacher's explicit gestural cues and subtle gaze cues quite closely.

However, when the teacher repeats the process for the second line, the student's gaze does not follow the teacher's cues. There are several hypotheses for why this happens. One option is that the student is processing the situation independently and is able to move ahead of the teacher, as indicated by his movement to the equations for the second line well before the teacher's. Another explanation is that the teacher's gaze is such an important cue for the student that when he loses it, his attention begins to drift. There is also the possibility that the student loses interest, due to the repetitive nature of the activity. Although the student's gaze continues to be on the board, it is possible that he is not attending to what he is looking at. We note that just observing the student does not provide information which we gather from gaze tracking data. We see from our data that when the teacher is explaining the first of two examples on the board, the student follows closely, but when she explains the second example, the student is looking towards the board, but is no longer following her explanations closely, lending credence to the usefulness of our technique.

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Emotional interactions of high achieving students in mathematical argumentation: The case of Jasmine

Rachel Hess-Green[1], Einat Heyd-Metzuyanim[1,2] and Orit Hazzan[1]

1 Technion, Haifa, Israel, rachely.hg@gmail.com

2 University of Pittsburgh, Pittsburgh, USA

The study of emotional aspects of interaction may tell us much about the social norms and the meta-mathematical rules according to which participants act during their mathematical activity. To illustrate this idea, we present the case of Jasmine, a 16-year-old student participating in a summer camp intended for exceptionally high achieving students in mathematics. Through a commognitive analysis of the interaction between Jasmine and her instructor (the first author), we examine the implicit meta-rules of discourse according to which the two participants acted and that, at a certain point, led to Jasmine's frustrated disengagement from the discussion. Adding the concept of "framing" shows that this episode could be characterized by the student and instructor's "misaligned frames".

Keywords: Emotion, interaction, students, mathematical argumentation.

INTRODUCTION

In past decades, a growing amount of research has been dedicated to the examination of student affect and emotions in mathematical learning (Hannula, 2012). However, most of this research has been concentrated on students' subjective experience, as elicited by self-reports. More recently, researchers have started looking at the effects of emotion on student-student and student-teacher *interactions* (Heyd-Metzuyanim & Sfard, 2012). These studies have been inspired by a sociocultural lens that sees learning as a form of participation in a discourse. Emotional expressions are an important part of any human communication and thus make up an important and indispensable piece of the puzzle when one wishes to understand how learning takes place in real-life situations.

The socio-cultural view has been used in research that attempts to show the marginalization of students who get disengaged from mathematics (Boaler & Greeno, 2000; Heyd-Metzuyanim, 2013). However, rarely has this lens been turned to the learning of students identified as 'mathematically gifted'. In the present study, we employ such a lens to examine the ways in which emotions, social interactions, and mathematical cognition interact in the activity of high-level mathematical problem solving practiced in a summer camp for mathematically gifted students. We do so by using the *communicational* (commognitive) method for examining mathematical discourse as it intertwines with identity construction in mathematical learning (Heyd-Metzuyanim & Sfard, 2012; Sfard, 2008). Our goal is to explore the analytical tools that may shed light on the affective side of teaching-learning interactions in settings where students engage in high level mathematical argumentation.

IDENTITY, GIFTEDNESS, AND MATHEMATICAL ARGUMENTATION

Studies about mathematical giftedness have mainly examined cognitive aspects of learning (Leikin, Berman, & Koichu, 2009). Whenever studies about giftedness examine affective aspects of learning, they do so from an individual perspective, using concepts like self-concept and self-esteem (Zeidner & Schleyer, 1999). Despite the fact that emotions and self-perceptions have been acknowledged as important for understanding gifted students' learning, rarely have they been studied as they take place in these students' mathematical learning. Similarly, the literature about mathematical argumentation rarely deals with the emotional side of argumentation.

In the present study, we focus on *communication*, including its emotional and non-verbal aspects, to integrate the study of mathematical cognition, social interaction and emotions in the activity of mathematical argumentation. We do this by using the communicational (commognitive) framework (Sfard, 2008), whose main tenet is that thinking can be viewed as an intra-personal type of communication, not qualitatively different from inter-personal communication. Within this sociocultural framework, learning is conceptualized as participation in a specific type of discourse (here, the mathematical discourse). Discourses, claims Sfard, are defined by four characteristics: word use, routines, visual mediators and endorsed narratives. In mathematics, all these are used to create discursive objects such as "2", or "prime numbers". Sfard (ibid) defines learning as a *change in discourse* and differentiates between two types of such change: object-level learning, where students learn new routines for dealing with familiar objects, and meta-level learning, in which the *meta-rules* of the discourse change. Meta-rules define patterns in the activity of the discursants trying to produce and substantiate object-level narratives.

Heyd-Metzuyanim and Sfard (2012) pointed to the fact that while learning mathematics, students do not just participate in the mathematical discourse (or mathematize), they also participate in an *identifying* discourse whose main goal is to produce narratives about oneself and others. In line with Sfard and Prusak (2005) they defined identity as a collection of stories that are reifying and significant, told by a person about herself (1st person identity) or by others about her (3rd person identity). Identifying discourse, or discourse that constructs identities, is made up of *subjectifying* utterances, whose object is people (rather than mathematical objects). However, not all subjectifying utterances can be categorized as identifying. Only those utterances that relate to stable, significant attributes of the person (such as "she is mathematically gifted") are considered to be identifying. Subjectifying (and to certain degrees identifying too) can be, and often is, communicated via non-verbal or indirect means. For instance, emotional expressions almost always communicate some sort of subjectifying message (such as "this is embarrassing for me"). Heyd-Metzuyanim (2013) has shown that focusing on the *disruption* of routines can provide useful insights into the meta-discursive rules that often go unnoticed in teaching-learning situations, in particular to the meta-rules of identification. These are implicit rules governing who is supposed to refer to certain actions (e.g. a student giving a correct answer, a teacher challenging a student's claim), what certain actions convey about participants, and what emotional responses are appropriate for certain situations. We incorporate the concept of "framing" to better articulate the discrepancy that can occur between sets of meta-discursive rules (Goffman, 1974; Tannen, 1993). Tannen (1993) explains framing as the participants' sense of "what is going on" in the interaction. Within a communicational framework, we conceptualize this as participants' sense of the meta-rules that are governing the discourse in which they participate. Disruption in routines, and the emotional reactions that accompany such disruptions (Giddens, 1984) may point to *misaligned frames* (Sande & Greeno, 2012) of the interlocutors. In the present study, we ask: what may emotional reactions of participants in a mathematical discussion tell us about misaligned frames, or different meta-discursive rules according to which the interlocutors are acting?

THE STUDY: SETTING AND METHODS

The research was conducted in a mathematical camp for "mathematically gifted" youth (20 participants aged 15–18) that took place during the summer vacation where the first researcher acted both as a researcher and an instructor. It lasted two weeks and included both mathematical and social activities. Throughout the course of the camp, participants engaged in both group and individual study sessions. The group sessions took place with the guidance of an instructor who is a mathematician (holding at least an M.A. in mathematics). The first author was one of these instructors. Every day the students were given a worksheet of problems that progressively increased in their difficulty. Twice a week students were given the opportunity to present problems on the board that they had previously solved and get their solution 'peer reviewed' by their fellow students.

Lessons and social activities were videotaped, in addition to interviews with students during the camp and recordings of episodic happenings such as casual conversations that took place after the study sessions. The first author held a research diary in which she documented the events taking place during the day including her feelings about these events. Based on this diary, we chose for close examination several events, in which it was clear that some emotional interaction was taking place, along with a rich mathematical discussion. These events were transcribed, including interviews and casual conversations that were relevant for understanding the event. The excerpts presented in this report were translated from Hebrew by the authors.

In this report we focus on one event that happened during a "peer-review" session. In it, one of the most active students in the group, Jasmine, volunteered to present her solution to an advanced problem that was given the previous day. The data for this analysis consisted of the recording and transcription of the lesson, as well as recordings of spontaneous conversations about the incident that took place after the lesson and a semi-structured interview with Jasmine at the end of the camp.

The analysis was based on the communicational framework, as presented above. In particular, we focused on three aspects of the communication between participants: a. the mathematical objects being talked about b. the subjectifying/identifying messages of the interlocutors about themselves and about each other; and c. the meta-discursive rules, or framing implied by these subjectifying messages. For instance, Jasmine's expressions of embarrassment or dismay were interpreted as indicative of an incident in which the happenings were misaligned with what she expected to happen or thought should have been appropriate for the situation. The interpretation of emotional expressions was based on the first and second author's view of the videos, in addition to Jasmine's description of the episode later in a casual conversation with other students. In particular, the interpretation of emotional signals and implicit identifying messages was based on our familiarity with the culture where this incident was taking place. Moreover, though we use Jasmine's report as a source of information, we should clarify that our method for analyzing emotional expressions does not necessitate 1st person reports of their emotions (or feelings). Rather, our method focuses on the communicational work that is done by emotional expressions, that is, how they are interpreted by others (see Heyd-Metzuyanim & Sfard, 2012).

FINDINGS – AN EPISODE OF ARGUMENTATION THAT WENT ASTRAY

The given problem was: *Prove that for each number n! + 2, n! + 3, ..., n! + n there is a prime divisor that does not divide any other number from this set.* None of the other students had solved this problem before. Jasmine was the only one who claimed to have solved it. Whilst starting to write her proof on the board, she noticed one of her peers starting to copy it. Alerted, she said "wait, why are you copying it?... what if I do something wrong?"

Figure 1: One of the drawn diagrams

Figure 2: Another drawn diagram

One of her peers responded "no worries, it happens". This instance of subjectifying already indicated that Jasmine was somewhat concerned with her identity as told by the other participants when walking to the board. Yet right after that she went straight to mathematizing. She began her proof by drawing the diagram seen in Figure 1 explaining it by: "So there is a number N! and then one takes it and adds to it all sorts of things, that's my famous diagram" [13–14]. Again, this rather casual remark about her "famous diagram" indicated that Jasmine was thinking about the way her solution is perceived by others. However, this indirect identifying remark went unnoticed within the rest of the proof, the main argument of which was as follows:

Jasmine's proof

"So that... I added 2, so the number won't be divisible by any number that is smaller than N except 2. OK? Good. And then if I added 3, then it (points to N!+3) won't be divisible by any number smaller than N except 3. And then, once we reach four, that's a multiple of two, then this number (points to N!+4) can also be divided by two. Then, wait, but it *must* be divided also by another prime factor that is greater than N" [19–25].

As she was talking, Jasmine was accompanying her explanation by drawing a diagram of N!+2 and N!+4, visually mediating their divisibility by 2 and another prime factor p or q as seen in Figure 2. At that point, the instructor interrupted, asking "why did you say that?"[27]. Here occurred the first failure in communication. While the instructor referred with the "that" to the claim that there are only two primary divisors of N!+4, Jasmine understood it as referring to the signification of the second prime number with q (and not p). She explained "'cause otherwise it would be the same number" [28]. However, since the misunderstanding of the teacher seemed to be related (at least in Jasmine's eyes) to mere signification of numbers by different letters, this interruption did not produce much stress and Jasmine went on in explaining her solution to the group. At this point, other students started getting involved and asking questions. Some asked only clarifying questions. Others' were more challenging, such as the instance below:

52 Yoav: Why do you say that this is prime (Points to p)?
53 Instructor: Wait, are you saying that it (N!+2) is divisible by 2? OK, but why is the other factor a prime?
54 Jasmine: There must be another prime factor

In the above interchange we see more evidence of a communicational breach. While the instructor and Yoav talked about "the other factor" being "a prime", Jasmine talked about the mere existence of "another prime". Therefore, it is not clear that she saw the necessity of the other factor being itself a prime, it might be that she was content with the factor being made up of several primes.

The instructor, together with some of the students, tried to find counter examples to Jasmine's claim that N!+2 is divisible by 2 and another prime. However, the first few numbers that were used for substitution actually confirmed Jasmine's conjecture. Thus, N=4 produced (24+2)/2=13 and N=5 produced (120+2)/2 =61, both prime numbers. The students and instructor did not get to the next N=6 which would have produced a counter-example ((720+2)/2 = 361, 361 is not a prime). Instead, the following interchange occurred:

The instructor's challenge

84 Instructor: So you say that it's always like that
85 Jasmine: I proved it! (Smiling, in a high, anxious voice). I just don't want to start with all the... (mumbles in a high voice, seems embarrassed)
86 Instructor: Are you sure?
87 Jasmine: Yes! I think so.. (tone of voice moves from assertion to a hint of doubt) I proved it. I just don't remember how.
88 Instructor: No, 'cause it's very interesting. 'Cause if you say this, then you are saying it's very easy to find prime numbers. That means I'll take something factorial, I'll divide it by 2, add 2, um.. no, add 2 divide by 2 and then I get a prime number. That means I found an algorithm for finding a prime number. That's why it seems strange to me.

Several points are worth mentioning in this short interchange. The first regards [85–86]. The instructor started by making a claim that combined subjectifying and mathematizing ("*you say* that <u>it always turns out that way</u>") (*subjectifying*, <u>mathematizing</u>). While she expected Jasmine to respond to the mathematical part (for instance by "yes, it always turns out that way") possibly with another justification, Jasmine responded to the subjectifying part. She referred to her *actions* ("I proved it") rather than to her mathematical claims. This divergence from mathematizing to subjectifying could also be seen in her emotional expressions at this point. Giggling and raising her voice were not going to help Jasmine justify her mathematical claims. Rather, they communicated she was mainly interested at this point in her identity (or the 3rd P identity of hers' as told by her classmates).

Another point regarding the above interaction refers to the meta-mathematical rules according to which the two interlocutors were acting, specifically, what "proving" entails and what may be considered as sufficient justification. For Jasmine, the fact that she did something in the past ("proved" the claim or solved the problem) was relevant and even sufficient for putting forward a mathematical claim [85]. Obviously, the instructor, who was an experienced participant in the mathematical community, did not abide by this meta-rule. However, she did give some credit for Jasmine's hypothetical proof by asserting it is "very interesting" and communicating genuine doubt with regard to her own understanding (see also later in [122]).

Another important point in the above excerpt regards the counter argument that the instructor uses [88]. While up to this point, the counter arguments concentrated on finding a numerical counter example that would disprove Jasmine's claim, this counter argument was of a different sort. It drew on some common knowledge of the mathematical community about the feasibility (or unfeasibility) of finding any prime number. However, this knowledge was not necessarily shared by the students. Jasmine showed she did not share it by replying "It's possible to find prime numbers in this way, it's just that for a calculator it's really difficult to do factorials even with small numbers" [98]. The type of meta-mathematical rule for proving that the instructor was following here is a distinct, rather intuitive meta-rule of the mathematical community, not a formal logical routine. According to it, in order to refute a claim, one can draw on the *implications* of this claim to other problems known to be difficult or unsolved in the history of the mathematical community. Any claim that would imply a simple solution for such a difficult problem (such as providing an algorithm for finding prime numbers) would be regarded with high suspicion. Jasmine did not seem to be familiar with this meta-rule. Rather, she concentrated on checking "very big numbers" (or factorials) on her calculator thus resorting to a familiar routine -proving by giving examples.

At this point in the lesson, the instructor moved to explaining the disagreement between Jasmine and herself to the other students. Once the instructor clarified her suspicion about (N!+2)/2 being a prime, Jasmine started backing off:

Jasmine's retreat

117	Jasmine	No, no. It can be 2 times many things but at the end there's a prime number there.
118	Instructor	Of course at the end you have prime factors, but why are they 2, that's what's bothering me.
119	Jasmine	No, you don't have only 2.
120	Instructor	But this is what you drew here... (points to Figure 2) ...OK, but if you have more (than 2) then you do have a problem.
121	Jasmine	Why?
122	Instructor	Because you assumed you have 2 divisors and then you said that because there are 2, then that's exactly N! + 2 divided by 2, if I understood what you said... (Jasmine looks puzzled). If you get to here (implying N! +N) you'll get to (that fact that) the divisor of this is necessarily different than this denominator (pointing to N!+2). That's how I understood your proof, and maybe I didn't understand anything and that's alright.
123	Jasmine	I didn't understand what you understood from my proof (smiles).

The excerpt above marks a transition in the argumentation of Jasmine and the instructor. Instead of checking Jasmine's claims, the instructor moved to an elaborate explanation of the "holes" or gaps in Jasmine's proof. One should notice that Jasmine's difficulty to follow the instructor [123] is understandable. Despite the fact that the instructor claimed to be revoicing Jasmine's proof, she used quite a different language for its expression. Jasmine talked about N!+2, N!+3,... being divisible into a number (2, 3) and a P (where P referred to a prime factor or simply a factor interchangeably). The instructor talked about P and (N!+2)/2 as interchangeable. In Sfard's (2008) words, she "samed" these two realizations into one object. It is probable that Jasmine did not go through the same "saming" process and therefore had difficulty in following the instructor's argument. Indeed, she remained puzzled, giving her fellow classmates an opportunity to voice their opinions. Reuven commented, with a bit of a humorous tone: "so what you proved here is that every number has a prime divisor". Jasmine retorted, quite excited "No! That's not what I proved here. That's totally not what I proved here! I was relying on it" [144].

At this point, it seems that Jasmine was quite frustrated. She was still convinced her proof was correct, but was not able to follow the instructor's counter arguments. The instructor tried explicating a mathematical meta-rule "so look... like, every claim you make here requires clarification". Yet Jasmine did not understand this simply as a meta-rule of mathematical proving. Rather, she interpreted it as a comment undermining her identity as a successful student, as could be seen in her short, high laugh and her retort "you said I think weirdly, so this is my weird thinking" [154].

From this point on, most of the communication in the class revolved around identity issues. The instructor, trying to alleviate Jasmine apparent distress, said "No, I don't have any problem (with it), it could be true and it's possible you'll find a new claim and write a paper about it" [156]. Jasmine replied cynically "yeah, sure... physics maybe, math no" [159]. Other classmates started giggling and immediately Jasmine's attention turned to them (though they claimed not to have laughed at her). The only mathematical claim that Jasmine still tried to make was that 1,814,401 (by using N=10 in (N!+2)/2) is a prime number. She said so quite excitedly, sticking her calculator in the face of her classmates. Here, not only was she not acting according to mathematical meta-rules of proof, Jasmine was also making a wrong assertion about the mathematical object (1,814,401 is not a prime).

There was no resolution to this argument. The instructor tried to encourage Jasmine to re-enter the mathematical discussion and re-state her proof but Jasmine gave up stating that "she's tired" and that she got "totally confused".

Aftermath

After the session was over, during a casual conversation with the instructor and other fellow students, Jasmine remarked: "I think you embarrassed me in front of the whole class, I stood helpless". To explain her discontent with the instructor's conduct she said to another student "I'm standing at the board, proving a problem, and every second she's interfering with it.... But it was true too. I proved it. I proved it completely and you (instructor) didn't believe (it)!" [9]. It is clear from this description that Jasmine's framing of the situation was very different than that of her instructor or of any experienced participant in the mathematical community. While Jasmine saw counter-arguments as "interferences" that unnecessarily "confused" and embarrassed her, such counter-arguments and rigorous search for the veracity of every claim are the hallmark of mathematical argumentation. Expecting the instructor "to believe" her simply because she said she proved it was another signal that Jasmine's framing was misaligned with that of her instructor.

DISCUSSION AND CONCLUSIONS

In this report we took a close look at an episode of high level mathematical argumentation coupled with intense emotional expressions. Our goal was to examine what may be learned from participants' emotional reactions about their framing of the situation, or the meta-rules of mathematical discussion they were following. The analysis revealed that the interaction included three types of communication failures. One was mathematical, at an object level. Jasmine and her instructor talked differently about "prime numbers" and about the objects signified by a "P". Communicational failures resulting from interlocutors referring to different mathematical objects despite using similar visual mediators or words, are a common cause for ineffective teaching-learning interactions (Heyd-Metzuyanim & Sfard, 2012), yet they do not necessarily involve frustration or embarrassment.

Another failure of communication was at the meta-level of mathematical discourse and related to what "proving" was all about. Jasmine saw "proof" as a process where one outlines her reasoning, not necessarily justifies every step. Her instructor followed the mathematical community's meta-rules according to which every claim has to be examined by searching for counter-examples and by following its' implications. Jasmine's difficulties with proof are not unique. Much literature shows that students often do not understand the role of examples and counterexamples in the process of proving. Such misalignments of meta-mathematical rules have a much stronger potential of producing frustration and heightened emotional expressions, if only for the reason that participants usually are not aware of them and therefore do not have access to the reasons that their communication fails.

Finally, we saw here conflict at the level of meta-rules of identification, or what certain actions and re-actions say about the *identity* of interlocutors. While Jasmine saw any disclaiming of her proof as a cause for embarrassment, and disruptions as mistrust in her ability, the instructor saw the process of making arguments and counter-arguments as a purely mathematical activity without any identifying content attached to it. Such discrepancy with regard to the identity stories that should be constructed as a result of a certain activity is, we conjecture, the most potent fuel for intense emotional agitation that may lead to failure in communication.

Yet, despite the distress this particular episode caused for Jasmine, it was probably an important and useful step towards the ultimate goal of initiating her and her fellow classmates into the mathematical community. In this summer camp, such interactions (though usually not so emotionally intense) were quite common. As could be seen here, the instruction in the summer camp was quite different than in school settings and resembled much more the process of apprenticeship (Barab & Hay, 2001). One of the main indications for this could be seen in the role that the instructor took, that of an equal (though somewhat more experienced) participant in the problem solving process. The importance of such a non-authoritative stance has been noted by many. However, in this setting it seems this role was truly authentic, probably because the problems dealt with were far from being trivial even for the instructor.

Finally, we wish to point to the need, which receives only a partial and very preliminary response in our study, to study gifted or extremely high achieving students' learning from a socio-cultural perspective. Such studies might be able to point to the processes that accelerate these students' learning, thus weakening prevalent notions about mathematical giftedness being a purely biological, permanent and individual trait. Furthermore, they can deepen our understanding about how to enhance instruction in schools with normally achieving students.

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Constructing mathematical competence in interaction: Whose mathematics is it?

Jenni Ingram[1] and Andrea Pitt[2]

1 University of Oxford, Oxford, UK, jenni.ingram@education.ox.ac.uk

2 University of Warwick, Coventry, UK

Interactions in the mathematics classroom affect both the mathematical learning and the identities of those involved. In this paper, we draw upon Discursive Psychology to examine how identities can be developed and altered in whole class interactions. In this sense, identity is not an attribute of a person but is something that is co-constructed through and in interaction. We demonstrate how these identities can shift moment-to-moment within an interaction. Importantly, these identities shift within the same interaction. These changes in identity development have important consequences for mathematical learning and continuing participation and contribute to our understanding of the variance in identities that students self-report.

Keywords: Identity, discourse, classroom interaction, positioning theory.

INTRODUCTION

Identity has become increasingly prominent in mathematics education research and in this paper we build on recent research that focuses on the discursive construction of identity through classroom interaction. We consider understanding and learning mathematics to be an aspect of participating in discourse practices. Furthermore, participating in discourse practices influences, and is influenced by, participants' identities (Esmonde, 2009). In other words, students' mathematical identities are discursively constructed through their interactions and experiences in mathematics classrooms (Grootenboer, Smith, & Lowrie, 2006).

Research has revealed important relationships between students' mathematical identities and their experiences of mathematical practices. For instance, early work by Boaler (1997) showed that students had qualitatively different forms of mathematical knowledge and beliefs about mathematics and the learning of mathematics depending upon the teaching methods they experienced. Her later research then examined how these different teaching methods influenced students developing mathematical identities and their decisions about continuing to study mathematics (Boaler & Greeno, 2000). Cobb, Gresalfi and Hodge (2009) developed this research further by developing an interpretive scheme to explore the relationships between particular classroom norms and sociomathematical norms and the developing identities of the students in those classrooms.

Different conceptions of identity in mathematics education research have arisen since then such as Sfard and Prusak's (2005) narrative work, Solomon's work with figured worlds (2007) and work by Cobb and colleagues (2009) with normative and personal identities. All of these approaches have largely drawn upon data from interviews with students describing their experiences with mathematics. Heyd-Metzuyanim and Sfard (2012) and Wood (2013) have focused on the construction of identity in the moment-to-moment interactions in classrooms. This paper contributes to this body of work.

METHODOLOGY

The conception of identity developed in this paper arises from discursive psychology (DP) (Edwards & Potter, 1992). Discursive psychology is based on the principles of Ethnomethodology (Garfinkel, 1967) and Conversation Analysis (Sacks, 1992). It examines the practical ways in which identity is managed in interaction, which may differ from the narratives or stories individuals may offer in interview situations. The focus here is on how teachers and students discursively co-construct what it means to be a learner of mathematics. From this perspective, identity is something that we 'do', rather than something we 'are'. This is a micro view of identity where identity is conceived as dynamic, complex and situated in the context of the interaction itself. Identities are constructed through these interactions by how the participants orient to each other. Teachers and students construct identities for themselves and for each other through participation in interaction, and also for all the participants and observers of the interactions that occur in the mathematics classroom. This micro approach complements the approaches taken by researchers such as Boaler and Cobb, and those developed from positioning theory (Wood, 2013) which themselves draw from discursive psychology, by giving us a glimpse of *how* these identities are constructed as well as how they shift and change in interaction.

Studies taking this micro approach have illustrated how interactions with different participants within the classroom result in different identities being adopted as the consequences of some of these different identities. For example, Wood (2013) demonstrates how one student's mathematical identity shifts in different classroom interactions with his teacher and his peers within one lesson. Two of the identities constructed, one in interaction with the teacher and one in interaction with a peer, relate to mathematic competence whilst a third, constructed in interaction with another peer, is one of a menial worker needing to be told each step. Yoon's (2008) study also showed that teachers' discursive positioning of non-native speakers affected their participation in lessons but also affected how their peers treated them in the classroom and Turner, Dominguez, Maldonado and Empson's work (2013) offers insight into how teachers can position non-native speakers as mathematically and interactionally competent in a way that supports their peers in positioning them in a similar way. Yet each of these studies examines patterns over time and consistencies in how the different participants orient to each other. In contrast, this study examines moment-to-moment differences in the way identities are constructed in order to contribute to our understanding of the variety of identities that students develop in the mathematics classroom.

The analysis below focuses on three aspects of the talk: the structure of the turns taken by the teacher and his students; and the authorship and ownership of the mathematics (also referred to as the epistemic agency (Ruthven & Hofmann, 2015)). One key feature of talk that is used by teachers related to ownership is revoicing (O'Connor & Michaels, 1993). Revoicing involves repeating students' contributions in a way that attributes the ideas involved to the student. The student is the owner of the ideas involved and capable of offering these ideas. Enyedy and colleagues (2008) emphasise the importance of this attribution in classroom interaction as it "shares the intellectual authority with the students and helps establish their role as one of contributing to the construction of knowledge" (p. 137).

The data discussed in this paper comes from a larger study involving eight mathematics teachers from seven schools in the UK, all working with students aged between 11 and 14 years. Some of the schools serve areas with high levels of social deprivation, whilst others are fee paying independent schools. The teachers all volunteered to be video recorded and the data collected is naturally occurring in that no instructions were given to the teachers about what or how to teach. The transcripts included in this paper are from one of these teachers' lesson with an all-attainment class of 12–13 year old students. The two extracts have been chosen to illustrate how the discursive construction of identity can change within one topic segment within a single lesson. The extracts come from one particular whole class interaction towards the beginning of the lesson where the students are reporting on work they have completed in a previous lesson and at home. The students are preparing for end of year exams and have been working on a worksheet with problems designed by the teacher to support them in their revision. Whilst the majority of studies have focused on small group work in mathematics lessons, whole class interactions are being considered here because the interaction not only positions the students who contribute but also the rest of the class observing and listening to the interaction. Whilst students are generally more agentic in small group discussions than in large whole-class discussions (Turner et al., 2013), the first extract below illustrates an example of where students can be agentic in teacher-led whole class discussions.

IS A MICROCENTURY LONGER THAN A MATHEMATICS LESSON?

In the first extract the students are reporting their work on the question of whether a microcentury is longer than a mathematics lesson. Immediately before this extract there has been some discussion on the meaning of microcentury and the notation associated with the prefix micro.

27 George: well (.) erm (.) I (.) worked it out (.) on erm the calculator and (0.3) it came up as one times (.) ten (1.6) to the power four. and er um (0.9) I times'd it by three hundred and sixty five, to simpl- to make it simpler (.) and um (1.1) and um it still wasn't (0.5) what I wanted so (0.4) I times'd it by twenty four which (0.4) um (1.2) um gave me that nought point eight seven six so it times'd it again by sixty (.) um minutes and it came as fifty two minutes and (1.3) so that's longer than forty minutes so (the answer is it's longer than the lesson) ((teacher is writing down the calculation while it is being said on the whiteboard))

28 Teacher: oh wonderful answer? thank you very much indeed. um you've said it all really haven't you. um this first bit comes up a bit funny on some of the calculators depends what sort of calculator you've got. sometimes when there are numbers that (.) don't fit easily on the display or have lots of noughts in. we represent them in a different way which we'll look at in year nine but um (.) it's basically (.) a ten thousandth. if you um (0.7) press the right button on your calculator and you get ten thousandth. and then with all these timesing what was what was George doing with all this? what was she doing here (1.7) ((turns round to point at a specific part of the calculation on the whiteboard)) yes Lauren

29 Lauren: um she was trying to (re do it) into minutes?

30 Teacher: she was making it into minutes eventually wasn't she I should have written that. (2.8) ((teacher is changing the colour of his whiteboard pen)) she was. so what where- what where the stages she went through. why was she- (0.5) why was she doing each of these steps. Hannah?

31 Hannah: because times three hundred and sixty five is like (.) three hundred and sixty five days in the year, times twenty four because there's twenty four hours in a day, times sixty because there's sixty minutes in an hour,

32 Teacher: but why do you think she stopped when she got to minutes. um I mean she said she got the answer here didn't she but (.) then she carried on and times by sixty Sam?

33 Sam: because it's the same units as what you're comparing it to

34 Teacher: perhaps yes. if you know the lesson is (.) thirty five or forty minutes, then that's what you compare it to...

The extract begins with George explaining her answer to a problem on the worksheet the class had been working on. Her explanation contains multiple pauses and hesitations but she completes her explanation without the teacher or any other student speaking. The teacher, by writing down her explanation on the whiteboard, makes the explanation available to the rest of the class in another representation and also gives weight to the validity of the explanation. This is reinforced further by the teacher's positive evaluation in the next turn.

Several of the pauses in George's turn are longer than the 'standard maximum tolerance' (Jefferson, 1988). Long pauses are often interpreted by the other participants as indicating that there is some trouble in the turn, such as difficulties with the mathematics or difficulties in expressing what the speaker wants to say. It is rare to see pauses of this length in students' turns in teacher-led discussions as the teacher will often step in to speak (Ingram & Elliott, 2014). By the teacher not speaking during these pauses, the student completes the explanation and as such both demonstrates their competence in producing this explanation and their competence in communicating this explanation. This competence is co-constructed with the teacher who neither adds to, rephrases, repeats or revoices the explanation in the following turn. The student is displaying their mathematical knowledge which is structured through the original task and the interaction.

Near the beginning of turn 28 the teacher makes reference to George's comment that the calculator said "one times ten to the power four", which he describes as "a bit funny". He then refers to the different displays produced by different calculators. Whilst George interpreted the calculator display without difficulty, the teacher's comment identifies this as a possible source of difficulty for the other students. This pre-emptive assessment by the teacher constructs the students as not knowing standard form or how calculators deal with large numbers (Barwell, 2013). The difficulty is initially attributed to the way calculators display numbers. The teacher then offers an account for why calculators might do that. The teacher uses 'we' here twice. This emphasises that it's a representation that is used by the general mathematical community (Dooley, 2015) but also accounts for the students not being able to interpret the representation by referring to this idea as something that the students have not yet met, rather than as a deficit in the students themselves. He then minimises the importance of the issue using 'basically' and talking about it as just needing to "press the right button on the calculator". Whilst on the one hand the teacher is constructing the class as lacking the knowledge to work with this representation, the teacher is also treating the difficulty as being outside the responsibility of the students themselves.

The teacher then shifts the focus to the calculation that George was describing in the previous turn. This calculation is attributed to George both by the teacher and by the student that takes the next turn. Lauren phrases her turn hesitantly and answers the question of what George was doing with the calculation as a whole. The teacher partially repeats her response, thereby accepting it, but the addition of the word "eventually" indicates that there was a problem with the response and this is followed by a question that focuses on the stages of the calculation rather than the calculation as a whole. Hannah's subsequent response includes an explanation of what each of the numbers in George's calculation represent. The teacher makes no explicit evaluation or assessment of this explanation but follows Hannah's turn with 'but why' and a reformulation of the question which is answered by Sam. This is then positively evaluated by the teacher, but not in strong terms. "Perhaps yes" indicates that whilst Sam's answer is correct, it is not the answer that the teacher was looking for and the teacher adds additional information in his turn.

In this extract the authority for the mathematics is often given to the students, and they are invited to supply thinking about their peer's strategy for answering the question on the research. The teacher directs each turn to a new student, each of which builds on the explanation offered before and focuses on the calculation initially offered by George. The students are co-constructed as being both mathematically capable and capable of making sense of others' ideas. This also treats the contributions and explanations from the students as important and worthy of consideration. Maintaining the authorship of the explanation with George also implies that she has a mathematical justification for her solution, and therefore evaluates her explanation as valid and treats her as a competent problem solver.

DIVIDING 3.05 BY 2.5

The second extract follows the question 3.05 divided by 2.5 (written as 3.05 as the numerator of a fraction and 2.5 as the denominator) which occurs later on the same worksheet as the earlier question. A student has suggested multiplying the numerator and the denominator by a hundred and the teacher has written 305 over 250 on the board, but the student reported that they had not yet got further with the calculation.

40 Teacher: … I think there's probably an easier number maybe can you see anything to do there Sarah?
41 Sarah: um
42 Teacher: could you use the idea of [a hundred]
43 Sarah: [times by four] yeah
44 Teacher: times by four yep so (0.6) whatever that makes times by four (.) you could probably do that. and then (.) how would you get it finally as a decimal
45 Sarah: um
46 (1.9)
47 Teacher: any ideas. (1.4) if you've got this number here Bella (.) if you got that number d- do you see what's happened here to go from this fraction to this fraction. (1.8) can you see what: (.) um (.) what I've done to convert this into this. what have I done (1.3) I've put a hundred on the bottom and I had twenty five a moment ago
48 Bella: you times'd it by fou[r]
49 Teacher: [ye]s I times'd that by four so I'd have to times the top by four as well and I can do that I just don't know what it is. But whatever it is, how would I get my final answer (0.5) if I wanted it as a decimal. (0.8) this is quite hard actually isn't it. We haven't done much of this …

In this extract the authorship of the calculation shifts and the student contributions are considerably shorter than in the previous extract and often overlap with the teacher's turns. It is a student who first suggests multiplying the denominator by four, though this does follow hints from the teacher in turn 42 and in the turn immediately preceding the extract. The original question invites a range of possible strategies but the hints narrow this range down, so whilst the student has given an appropriate answer of "times by four" (as indicated by the teacher's acceptance of this answer in turn 44) this may not have been a strategy that the student used themselves or would think of themselves without the prompt of "a hundred".

There are several noticeable pauses in this extract in turns 46 and 47. The first two of these pauses offer Sarah the opportunity to answer the teacher's question in turn 44. When no answer is forthcoming, the teacher adapts that questioning towards the calculation that has already been performed. These first two pauses treat Sarah as being able to offer an explanation. However, when these opportunities are not taken up by Sarah, the trouble is treated by the teacher as being with the calculation that has already occurred. The teacher then checks that another student in the class has understood this calculation in turn 47.

The authorship shifts in turn 47 where initially the teacher refers to the numbers involved as belonging to the students, "you've got this number" but then the teacher begins to use 'I' to talk about the calculation that has been performed. The 'you' is used when talking about the strategies that could be used next, the strategies that could be used on the fraction with a denominator of 100. 'I' is used to refer to the calculation that has already been performed when converting between equivalent fractions. This shift from you to I is a further indication that the teacher is treating this calculation as a source of trouble. The teacher is now responsible for the calculation that has been performed and this treats the students as needing to follow his reasoning. The question asked in turn 47 asks a student to explain how the teacher converted the denominator of twenty five to a denominator of a hundred. In Bella's response she also positions the calculation as being the teacher's. In the final turn of the extract, there is another shift in who is doing the calculation that comes next. In turn 44 the teacher asks the students how they would get the final answer and it shifts in turn 49 to how the teacher could get the final answer.

The emphasis in this extract and in the turns that came immediately before has been on the strategies that students can use rather than the final result of the calculation. The teacher is asking for what the students did in turn 40, not what answer they got. In turn 44, the teacher does not perform the calculation involved in changing the numerator once the denominator has been multiplied by four, reducing the importance of the answer to the calculation by referring to it as "whatever this makes". The teacher also positions the students as capable of performing this calculation in this turn before asking for strategies for what could be done next. In turn 49 again the teacher places the emphasis on the strategy but as a response from the students is not forthcoming he then states that "this is quite hard actually" to account for difficulties the students are having in responding to his question. This is then followed by an explanation of "we haven't done much of this" which also accounts for the difficulty by a lack of experiences of working with calculations like this, which locates the issue as outside of the students' competencies.

DISCUSSION

In both extracts the interactional roles of teacher and students are clear. It is the teacher who controls the topic, asks the questions, and decides who can speak when. It is also clear that the teacher has control over assessing the appropriateness of the students' turns. In both extracts the students are treated as capable of explaining and communicating their ideas. They are given opportunities to do so

In this classroom there is a clear focus on the process of doing mathematics. The extracts presented here both focus on calculations but the attention is on the choices made in order to solve the problems and the justification for these choices. Difficulties with the mathematics are also treated in similar ways in both extracts, with the problem being located in the teaching rather than as a deficit in the students. In the first extract the difficulty is attributed to a representation that the students have not yet met and in the second extract the difficulty is attributed to the students not having had enough experience of this type of calculation and to the teacher posing a difficult problem in the first place.

The differences in the treatment of the students in the two extracts arise from the differences in the students' contributions to the discussions. In the first extract the students' responses do not always match what the teacher is looking for but they are accepted as answers to his questions, whilst in the second extract the students' responses are limited in terms of content, length and frequency.

The ownership of the mathematics shifts during the interactions in reaction to the students' contributions. In the first extract the mathematics is attributed to George throughout and this is indicated by both the teacher and the other students. In the second extract the ownership shifts progressively from the students to the teacher. The students position themselves in the first extract as mathematically capable and as capable of inferring George's mathematical reasoning. The teacher could have easily clarified George's answer himself and could also have stepped in when George was hesitating in her turn. By not doing so he enables the students to maintain a position of capability, which is supported further by the teacher in his turns. In the second extract, the students are more hesitant and do not take up the position of being able to suggest a strategy. The teacher initially continues to position the students as capable of suggesting a strategy but as the students do not take up this position the teacher shifts both the ownership of the mathematics and also positions the students as needing to understand the mathematics he is doing rather than as capable of doing it themselves. These different positionings could have consequence for how students come to see themselves as mathematical thinkers (Cobb et al., 2009).

As is evident from the extracts this is a classroom where students are given opportunities to engage in mathematical communication. They are frequently invited to contribute ideas and explanations and to build on other students' contributions. The students also work collaboratively throughout the lessons and for this teacher the majority of time in lessons is spent with the students working in small groups on a variety of tasks. We know from previous research (Boaler & Greeno, 2000; Cobb et al., 2009) that students in classrooms 'like' this are more likely to develop identities associated with mathematical capability and are more likely to continue with their study of mathematics. However, this research does not explain *how* these identities are developed and also it does not explain the variation in students' identities who have similar experiences in their mathematics lessons. The micro approach such as the one taken in this paper and by Wood (2013) begins to give some insight into these two aspects of identity construction. Students are not either positioned as mathematically competent or not according to which teacher they have, but these positionings change and develop moment-to-moment in interaction with all teachers. Students consequently experience different mathematical positionings within each lesson, some of which may be positive but some of which may not. Whilst Wood focuses on the different identities co-constructed in interactions with different participants, teacher and peer, in this paper we have examined extracts from the same topic segment and between the teacher and the whole class.

This paper provides evidence for, and analysis of, the identities enacted by the students and developed by a teacher. The students enact positions where they are mathematically capable and the teacher ratifies and supports these positions. It also demonstrates that identities and positions can change across a short interaction. Paying attention to the influence of minor changes in context may help explain why students may take up different identities within mathematics.

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Discourses in kindergarten and how they prepare for future decontextualised learning of mathematics

Judith Jung and Marcus Schütte

Technical University of Dresden, Dresden, Germany, judith.jung1@tu-dresden.de

Abilities in using decontextualised linguistic forms seem to be of great importance for subject-specific learning in school, including the learning of mathematics. However, the basic elements for mastery of these competences are usually not taught in school, but are assumed. Where can young learners gain these abilities? Based on Cloran and Hasan's investigation of mother-child discourses, the ongoing study presented in this paper analyses pre-school learning in kindergarten and compares this with mathematical learning in primary school Year 4. The goal is to investigate to what extent the linguistic discourse in kindergarten and primary school gives children the opportunity to achieve mathematics-specific discursive competences that allow them to participate successfully in the discourse of the mathematics classroom.

Keywords: Academic language, kindergarten, decontextualised discourse, early mathematics learning.

INTRODUCTION

The results of international comparison studies, such as the various PISA studies and the PIRLS (*Progress in International Reading Literacy Study), a* primary school study, show that in Germany there are significant differences in achievement levels and educational opportunities among children with different linguistic-cultural or socio-economic backgrounds (Bos et al., 2003; OECD, 2006). It is assumed that there is great potential for changing this situation if pupils can master the German language, particularly pupils with a multilingual background. According to this view, a strong connection exists between the linguistic abilities of children and their achievements in school, not only in German lessons but also in other subjects – including mathematics (Schütte, 2009). It is important to emphasise here that the current study does not focus only on children with a background of immigration or those who live and learn in several languages; rather, it concentrates on the influence of linguistic abilities on the learning of mathematics for all children.

Studies in migration research and educational science (Gogolin, 2006), as well as recent studies in mathematics education (Schütte, 2009; Schütte & Kaiser, 2011) have consequently concluded that it is not the mastering of general linguistic competences that is significant for successful (mathematical) subject learning, but competences in a subject-specific academic language. With her concept of "academic language" ("Bildungssprache") Gogolin (2006, p. 82) draws on Cummins's (2000, p. 57) concept of "Cognitive Academic Language Proficiency". Cummins makes a distinction between "academic language proficiency" and "conversational language proficiency", on the basis that children quickly gain abilities in their second language which they are able to use in everyday situations. They need significantly longer to achieve competences in the academic language of the classroom. This academic language, following Gogolin (2006), is characterised by its conceptually written form, which allows it to have high density of information and independence from situations; this means there are fundamental characteristics that do not correspond to children's everyday oral communication. However, the question is how to support children in building up sufficient (academic) linguistic competences to enable them to achieve academic success in (German) schools? Furthermore, in what ways should a school, and the teaching staff working there, seek to adapt to a pupil population entering the school system with extremely diverse abilities?

Both international and domestic research in mathematics education has recently begun to include approaches that have taken up this concept of academic language (Gellert, 2011; Schütte, 2009) and applied it to subject-specific learning, in particular to the learning of mathematics. This study aims to make a further contribution to these subject-specific modifications of the concept of academic language for the learning of mathematics. Section 2 will describe the effects of adopting an academic-language-based perspective on the learning of mathematics. The concept of academic language is discussed in the context of academic discourses, mainly on the basis of theories by Moschkovich (2002) and Krummheuer (1992). The fundamental idea is that children need to learn subject-specific discursive practices in order to be academically successful in their future school careers. The third section outlines the studies on decontextualised language use carried out by Hasan (2001) and Cloran (1994), who were able to reconstruct discursive practices of the socialisation of children in the family in early childhood. These forms of discourse used in the family also seem to be fundamental for the development of mathematical abilities. In section 4 the methodological procedure of the investigation is outlined. Section 5 shows the preliminary results.

SIGNIFICANCE FOR THE LANGUAGE-RELATED LEARNING OF MATHEMATICS

Some studies, particularly in the international literature, have begun to improve the concept of a mathematical academic language and move beyond the deficit-oriented approach. Moschkovich (2002), for example, emphasises the discursive element of the learning of mathematics. She links her situated-sociocultural perspective on the learning of mathematics to a paradigm change away from a view of deficits as preventing learners from mastering the academic language of school to focus on the resources and competences of a diverse pupil population. According to this perspective, the learning of mathematics takes place in a social context in which the participants bring with them different ways of looking at situations, which are negotiated interactively. Approaches centred on academic language, which understand the learning of mathematics as "constructing multiple meanings of words" (Moschkovich, 2002, p. 193) and which relate to Halliday's (1975) concept of "register", often focus on the differences between children's abilities to linguistically act in or with different registers. This deficit-oriented approach leads to the idea of a 'target register' –the academic language– and a register of everyday oral communication, which is of less importance in academic discourses. With this perspective only the mastery of the academic language is seen as sufficient for success in school and the achieving of mathematical understanding. In contrast to these approaches, we suggests that the learning of mathematics always takes place in a public, social and cultural context and represents a discursive activity. There is no single correct mathematical discourse that can be achieved. Learners participate in mathematical discourses in different communities and use diverse resources in different registers in order to successfully communicate mathematically.

Interactional and non-language aspects assume a central role in this perspective on the learning of mathematics. Following this idea we can look at various studies of interactionistic approaches of interpretative classroom research in mathematics education. For example, Krummheuer (1992) shows how children are involved in collective argumentation in the learning of mathematics in primary school and that mathematical learning stems from increasingly autonomous participation in mathematics (Krummheuer & Brandt, 2001). Although, according to Moschkovich (2002), there is no single correct discourse, even while moving away from deficit-oriented approaches we must acknowledge that children will still enter school with different conditions for participation in the collective argumentation on account of their extremely differentiated socialisation in the family. Here we can refer to the studies by Cloran (1994, 1999) and Hasan (2001), who investigated the different discursive practices in familial socialisation in mother-child discourses. The central aspect of these studies is the ontogenesis of decontextualised language as a fundamental factor for successful participation in educational discourse in school.

CHARACTERISTICS OF THE DISCOURSE OF SCHOOL EDUCATION: DECONTEXTUALISATION

Diverse authors have seen decontextualisation as an important characteristic of the pedagogical discourse (Bernstein, 1996; Cloran, 1999). However, what exactly are we to understand by this term? What is the difference between context-dependent and context-independent language? In the last 40 years much has been written about decontextualised language use, for example relating to the difference between oral and written language and the question of language development (cf. among others Olson, 1977), the cognitive development of children (cf. among others Donaldson, 1987), and the concept of academic language and the conditions for a successful educational pathway (Cloran, 1994; Hasan, 2001). However, existing approaches offer no clear definition or description of decontextualised language. The concept is given different emphases according to the frame of reference. This can be seen for example in the use of different but basically synonymous terms such as "context-independent", "situation-independent", "autonomous" and "disembedded" (Cloran, 1999, p. 33). The present paper will engage with the sociolinguistic theories and studies of Hasan (1973, 2001) and Cloran (1994, 1999), based on the ideas of Bernstein. These authors seek to forge a link between the language that is learnt in the family in early childhood and the form of language that is prevalent in school. A fundamental definition of decontextualised language use can be found in Hasan (1973, p. 284): "The term 'context dependent' [...] may be paraphrased as follows: language that does not encapsulate explicitly all the features of the relevant immediate situation in which the verbal interaction is embedded". She emphasises the difference between "material immediate situation" and "relevant immediate situation", and notes that the correct decoding of the linguistic message is dependent on knowledge about the relevant immediate situation, although this knowledge is not linguistic in origin. In contrast, context-independent language is language that "encapsulates explicitly all the relevant features of the immediate situation in which the verbal interaction is embedded" (Hasan, 1973, p. 284).

Cloran (1994) makes use of Hasan's (1973) definition and identifies ten different types of rhetorical unit in the conversations between mothers and their preschool-age children, which she places in an order according to the level of decontextualisation. The ten degrees of decontextualised language use move from the least decontextualised, which is based in the material present moment, to the most decontextualised, based in the relations created by language itself. In order, the rhetorical units are: action, commentary, observation, reflection, report, account, plan, conjecture, recount, and generalisation (Cloran, 1999, p. 37). The rhetorical units of generalisation and conjecture play an important role in the learning of decontextualised linguistic abilities.

The idea of revealing a continuum between context-dependent and context-independent is present in Hasan (2001, p. 53). She uses the concepts "actual" and "virtual" for her investigation of mother-child discourses and the ontogenesis of decontextualised language, which refer to the context of the linguistic expression:

> A context is actual if it can be actually, that is physically sensed by the interactants. [...] A context is virtual if no possibility exists for experiencing it physically: the phenomena are, in fact, not available to the senses. (Hasan, 2001, p. 53)

From this she concludes:

> A discourse is decontextualised/disembedded, not because what it refers to is not physically present to the senses here and now, but because it refers to something that is by its very nature incapable of being present in any spatio-temporal location whatever. It is simply not sensible. (Hasan, 2001, p. 53)

Hasan (2001) also draws a distinction between constitutive and ancillary verbal actions: an action is constitutive when it recreates an actual context that is now spatio-temporally displaced. Constitutive verbal action can also bring into existence virtual contexts, which are entirely text-based. An ancillary action seeks to negotiate some physical action that is ongoing within an actual context. This leads to three different classifications for contexts of discourses: immediate (ancillary and actual), displaced (constitutive and actual), and virtual (constitutive and virtual) (Hasan, 2001, p. 54). Hasan concludes from her observations of mother-child discourses that the best learning environments for the use of decontextualised language are situations where continuity is established between actual and virtual contexts. Children thus have the opportunity to gradually move from speaking about concrete things and experiences to speaking about abstract generalisations. Hasan goes on to suggest that school is not an appropriate place for the learning of decontextualised language use because of the structures of discourse that dominate there; however, these abilities are to a great extent made a precondition for later academic success, as they become a factor for selection. Both actual and virtual contexts can certainly be observed in the classroom, but an individual child barely has any opportunity to autonomously change between them. If we agree that the ability to use decontextualised language forms is assumed rather than taught in schools, this ability must then be learnt in pre-school places of learning, or places outside the school environment.

In her investigation of everyday mother-child discourses, Cloran (1994, 1999) discovered that frequency of use of rhetorical units with a high level of decontextualisation in the interaction was dependent on the social class of the respective family. Cloran defines this by the level of autonomy in the occupations of the child's parents. Parents who have jobs that involve a high level of autonomy, for example teacher, doctor, lawyer and so on, usually have a higher level of professional education, and their families too may have a higher level of education. Cloran found that with mothers and pre-school children belonging to this class of family, the rhetorical units generalisation and conjecture appeared significantly more frequently in the interaction than with subjects from families where the parents had a low level of autonomy in their occupation. Cloran speculates that this could be one reason for the differentiated success in school of children with different family backgrounds.

If we focus specifically on mathematics teaching, and the demands placed on children in the mathematics classroom, we can see that the ability to distance oneself from concrete contexts and to express this linguistically assumes a special importance. Particularly in the primary school environment we can conclude that the successful participation in collective argumentation is made easier for children if they have the ability to make assumptions about the contexts of mathematical discoveries, and to take that extra step to generalise these contexts. According to Donaldson (1987), this linguistic distancing from contexts represents an important aspect of the cognitive development of the individual, enabling abstract thought processes and conclusions.

Gellert (2011) also points to the special importance of linguistic decontextualisation for mathematics teaching in primary school. Mathematics in primary school is characterised by the establishing of distance from concrete everyday experiences while moving towards abstract expressions that are applicable in a general sense (i.e. generalisation). Wagner, Dicks and Kristmanson (2015) examine *children's language repertoires relating to conjecture*. It can therefore be assumed that the gaining of decontextualised linguistic discursive abilities –particularly the use of rhetorical units, which, according to Cloran and Hasan, shows a high level of decontextualisation (e.g. conjecture and generalisation) –before entering school will be helpful for mathematical learning in the classroom. With reference to Moschkovich (2002), before abilities in decontextualisation are demanded of children in the mathematical discourse of primary school, it would be helpful if they could be prepared for these demands through discourses in the family or in kindergarten. In the following we will focus on kindergarten, as educational processes are initiated in this environment by professional teachers. From these points, our research questions are:

1) What is pupils' relationship to generalisations in mathematics lessons in primary school?

2) Can we reconstruct in early discourses with pedagogical experts learning situations that promote children's abilities to change between actual and virtual contexts in the interaction?

METHODOLOGICAL PROCEDURE AND CONTEXT

The study presented here is qualitatively oriented and can be categorised under interactionistic approaches of interpretative (classroom) research in mathematics education (Krummheuer & Brandt, 2001). The empirical basis for the study is provided by video recordings of everyday play and discovery situations, each involving a kindergarten teacher and two children (4,7–5,5 years old). The play and discovery situations can be categorised under the mathematical area of space and form, for example classic situations with building-block constructions that follow a model, or situations that principally engage spatial perception. We contrast this with video recordings of whole-class discussion during mathematics lessons in Year 4. Here the focus is on situations where the teacher introduces new mathematical concepts.

Interactions between the participating children and between the children and the attendant adults are analysed. For the investigation of discourse the transcribed video sequences are examined with the help of interactional analysis and an analysis of the used rhetorical units based on Cloran (1994) and Hasan (2001). The analyses presented by Cloran (1994) were carried out on the basis of linguistic categories. Messages of the chosen transcribed scenarios were identified as belonging to one of the ten classes of rhetorical units: action, commentary, observation, reflection, report, account, plan, conjecture, recount, and generalisation. The study focuses here on the units Cloran highlights in relation to decontextualised language, i.e. generalisation. In examining the subject-specific negotiation of meaning within the interactions, Cloran's method of analysis is linked to the interactional analysis.

RESULTS

In the Year 4 mathematics classroom we find multiple situations where children are asked to generalise content linguistically, shifting out of the here and now onto a more general level. The linguistic forms of the expressions the children use seem on multiple occasions not to satisfy teachers' requirements; however, teachers usually offer no explicit linguistic help. The following example illustrates this.

Birkan:	I know what the "cross-sum" (German: Quersumme, "sum of digits") means.
Teacher:	That's oh. Prick up your ears.
Student:	Cross-sum is plus, you know.
Birkan:	Erm the cross-sum means a erm broken line.
Students:	No, oh I know.
Teacher:	That means you make everything erm total.
Students:	No, not total.
Teacher:	No, I don't know if you weren't onto something. You've, maybe you've expressed it a bit wrong.
Birkan:	Yes erm, if you add everything up.
Teacher:	That's there's. Well you're really onto something there. I think I know what you mean. So now I don't want to torture you any longer. [*The teacher writes an addition task on the board.*]

The teacher makes clear that Birkan's answer points in the right direction but that he has incorrectly formulated it linguistically ("maybe you have expressed it a bit wrong"). After this response Birkan attempts a linguistic improvement of his answer, but this, too, seems to fall short of the teacher's expectations. Without explaining this, however, and without helping Birkan to arrive at a correct formulation, the teacher continues with the lesson and writes an addition task on the board.

The analysis of Year 4 mathematics teaching moreover puts into focus that even in primary school certain mathematical concepts, e.g. a straight line, can only be detached from the immediate context and described linguistically in a virtual context, while visual representations neglect their inherent characteristics. Switching between virtual and actual contexts seems barely possible. Here we can refer to Söbbeke (2015), who investigates the dilemma that mathematical knowledge is abstract in most cases, while using means of visualisation is indispensable for speaking with children about these abstract concepts.

In kindergarten we find above all diverse situations where the participants negotiate meanings in relation to immediate contexts and the teacher seldom generates opportunities for switching from immediate to virtual contexts within the conversation. On the one hand this seems determined by the selection of situations in the area of space and form, and on the other hand hardly surprising, given the age of the children. Only at certain points in the situations with a focus on spatial perception do we see potential for switching from an actual to a virtual discourse. However, this is mostly not on the level of mathematical discussion. One example of this can be seen in the following scenario:

Jacob:	She sees the elephant.
Teacher:	Which elephant?
Jacob:	This one. [*Jacob taps on the big elephant.*]
Heike:	[*Heike taps first on the small, then on the big elephant.*] The big one because the small one is too small.
Teacher:	Why doesn't she see it?
Jacob:	Because it's too small. Because it's a baby elephant.
Teacher:	But you can see baby elephants, too. Why doesn't she see it now?
Jacob:	Because there's a wall here.

At this point the adult confirms Jacob's answer and asks a new question, whose content has no connection to his argument. However, from our point of view this situation shows potential. After Jacob's utterance about the wall, the accompanying adult could lead the discussion in a direction where the children are asked to think about the fact that other small animals, plants or objects could not be seen behind the wall either, and, ultimately, that nothing behind a wall that is shorter than the wall can be seen. This could be a first step towards the use of generalisation and switching from an actual to a virtual context.

SUMMARY AND FUTURE PERSPECTIVES

Our results let us assume that conversational situations that contain various individual possibilities for switching between immediate and virtual contexts appear only rarely, and that children are expected to be able to use decontextualised language without receiving help. In addition we might assume that discourses in kindergarten similarly offer only limited points of access for learning such linguistic competences, and that at least play situations in the area of space and form offer limited potential for introduction to decontextualised language use. This could be because this area is introduced above all through immediate, enactive, and visual means, especially in early learning processes. It may therefore be unnecessary for participants to switch into virtual contexts, because the mathematical area provides very good points of access to immediate contexts. Through looking at other context areas that might evoke a greater need for the use of decontextualized language nuance could be added to this hypothesis and the theoretical construct of the typification of contact with decontextualised language in early education could be extended to different kinds of mathematical content. Furthermore, the question needs to be addressed of whether examples can be found of initiation of switching between contextualised and decontextualised forms of language, and of help offered to children to master decontextualised forms of language, at any earlier point in primary education.

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Use and development of mathematical language in bilingual learning settings

Rebecca Klose

Justus Liebig University Gießen, Gießen, Germany, rebecca.klose@math.uni-giessen.de

Bilingual forms of teaching and learning have become common practice in Germany. However, the idea of teaching mathematics bilingually has not been totally accepted yet. The on-going study aims to investigate how young learners taught bilingually use and develop mathematical language in both of their target languages, with a focus on the usefulness of the communication tool PriMaPodcast. Parts of the research project and the interactive process of producing mathematical audio podcasts are presented.

Keywords: Bilingual learning, mathematical language, audio podcasts.

LEARNING MATHEMATICS IN BILINGUAL SETTINGS

Content Language Integrated Learning (CLIL) stands for a concept related to teaching and learning in two languages. As such, a combination of subject content learning and foreign language acquisition takes place. The extent to which it puts more emphasis on the subject-based components or the language dimension of learning, may vary from one country to another (European Commission, 2006). Bilingual teaching and learning settings have meanwhile become common practice in Germany, yet they are seen far more in secondary than primary education (Elsner & Keßler, 2013). Internationally, there seems to be no clear preference of any particular subject. Besides the social sciences, Geography, History and Economics, Mathematics is included in official recommendations on CLIL provision (European Commission, 2006).

In Germany however, the idea of teaching Mathematics bilingually has not been greatly accepted (KMK, 2006; Viebrock, 2013), neither in secondary school nor in primary education. Pertaining to primary education, concepts and materials required to facilitate mathematical learning in two languages have not been extensively developed nor become available (Küppers, 2013). As said by Rolka (2004) the exclusion of Mathematics depends on certain attitudes of mathematics teachers in Germany who appear to believe that language use has been reduced in this school subject. Yet the German syllabus of teaching mathematics demands the development and implementation of process and language oriented competences such as reasoning, communication and representation. Many show concern for having to learn a technical language for Mathematics, which other subjects do not require.

All these considerations raise the question whether it is really effective to learn mathematical language in two different languages. If so, how would bilingual learners use and develop mathematical language in order to explain mathematical content in both languages of instruction? In the next section some characteristics of mathematical language will be explained further.

CHARACTERISTICS OF MATHEMATICAL LANGUAGE

The acquisition of mathematical language goes beyond the learning of terms and vocabulary (Maier & Schweiger, 1999; Pimm, 1987). Based on Halliday's (1978) idea of mathematics register, Schleppegrell (2007) outlines some characteristics of mathematical language. First, there are multiple semiotic systems such as symbols, oral and written language and visual representations. These systems combined are necessary for the learner to construct meaning. Moreover, there are linguistic features of mathematical language, which include technical vocabulary as well as grammatical patterns. Technical vocabulary does not only consist of mathematical words (e.g., *sum*), but also of words that are not solely mathematical. The latter refers to words of multiple meanings depending on the context (e.g., *product*). As mathematics uses many words that are part of a child's everyday language, technical vocabulary needs to be used in meaningful grammatical patterns in order to be acquired. It typically comes along with 'dense noun phrases' (e.g., *the length of...)* as well as 'being and having verbs', which construct different kinds of relational processes (attribution and identification). While attributive clauses are non-reversible and refer to objects and events (e.g., *a sphere is a solid shape*), identifying clauses define technical terms and are reversible. Schleppegrell lists 'conjunctions with precise, technical meaning' (e.g., *if, when, therefore*) and points out the differences of their use in mathematical and ordinary everyday language. Moreover, the notion of academic language becomes increasingly more important for teaching and learning mathematics. Interdisciplinary language is characterized by precise and elaborated vocabulary and complex sentence structure, but in a more functional manner: It is this academic language, which affords the participation in teaching activities (Cummins, 1984; Schleppegrell, 2012).

There are multiple discourse practices in a mathematics classroom as learners and teachers bring various perspectives to a situation during interaction. Academic mathematical Discourse practices aim for learners to become mathematically literate and refer to practices that generalize, abstract or make claims (Moschkovich, 2007). Different language activities such as discussing and hypothesising are in accordance with the acquisition of a mathematical language, which is "more broadly conceptualised" (Morgan, 2005, p. 103). A communication tool, which implements different discourse practices, will be presented in the following section.

PRODUCING AUDIO PODCASTS FOR MATHEMATICS

In order to investigate how young learners use mathematical language to express meaning, they get to produce audio podcasts (Klose & Schreiber, 2014). Audio podcasts have been a popular medium for several years now. They can also be used for educational purposes such as in teaching Mathematics. Schreiber (2013) calls mathematical audio podcasts produced by primary school children "PriMaPodcasts". By creating audio podcasts for mathematics, the focus is on oral communication and representation, which means that the learners are not able to use visual aids or gestures to bring their message across. To express meaning accurately to an audience, the speakers need to reflect thoroughly on their topic and speech first. On the one hand, the learners have to discuss, research and accumulate mathematical content in a precise and orderly manner. This may foster their mathematics skills. On the other hand, they have to use language consciously in order to transmit their messages. This might support their communicative competence and skills. By producing PriMaPodcasts the students undergo an interactive process.

1. Unexpected Recording
2. Script I
3. Podcast - First Version
4. Editorial Meeting
5. Script II
6. PriMaPodcast

Table 1: Production Steps

The production steps, as summarised in Table 1, are the following:

1) **Unexpected Recording.** The students split into groups of three and answer a mathematical question unexpectedly, e.g., *'What is symmetry?'* or *'What is so special about a square in comparison to other quadrilaterals?'* A voice recorder records their response. It is interesting to observe the kind of mathematical concepts and mathematical language the learners are able to verbalise in this way.

2) **Script 1.** To plan an audio podcast, the students need to create a script. They are free to decide the format, structure and amount of detail they want to include. At this point, they can gather more information through resources such as the Internet, textbooks and given worksheets.

3) **Podcast- First Version.** Based on their script, the children record their first podcast version.

The procedure, material and media used by the groups are of importance as they have an influence on the quality of the first recording.

4) **Editorial Meeting.** Upon presenting the first podcast version to another group and their instructor, the creators receive feedback. After reflecting on it, they can make amendments to improve the quality in terms of content, style, language and performance. The editorial meeting does not only provide others an insight into the learners' thought process but also enables the instructor to clarify misconceptions.

5) **Script II.** The children revise their script of step 1 and enhance it. Despite receiving feedback and suggestions to improve the podcast in terms of content, language and style, the students themselves decide on the changes they wish to adopt in order to produce the final version that gets presented and published.

6) **PriMaPodcast.** Finally, the students produce the PriMaPodcast based on the second script. It will be published on a blog on the Internet. The advantage of publishing the PriMaPodcast in a blog is the ability to categorize all submitted podcasts into main-topics and sub-topics respectively to provide a clear overview. PriMaPodcasts have been produced in various languages: German, English, Spanish and Turkish. PriMaPodcasts recorded in bilingual learning settings are available in this blog: www.inst.uni-giessen.de/idm/primapodcast-bili

In order to investigate the mathematical conceptualizations of young bilingual learners in Germany, we expect them to produce PriMaPodcasts in both languages of instruction (English and German). The overall research questions of the PhD, which have not been fully addressed up to this point in time, are:

1) How do bilingually taught young learners use mathematical language in both target languages (English and German) if asked to present mathematical content?

2) To what extent does their mathematical language develop throughout the entire procedure of producing PriMaPodcasts?

METHODOLOGY

The following research design (see Table 2) was created to investigate the use and development of mathematical language of young German bilingual learners. Before the study, the communication tool PriMaPodcast and its different steps of production emerged during a pilot study at a bilingual school in Frankfurt a. M. Up to now, two rounds of data collec-

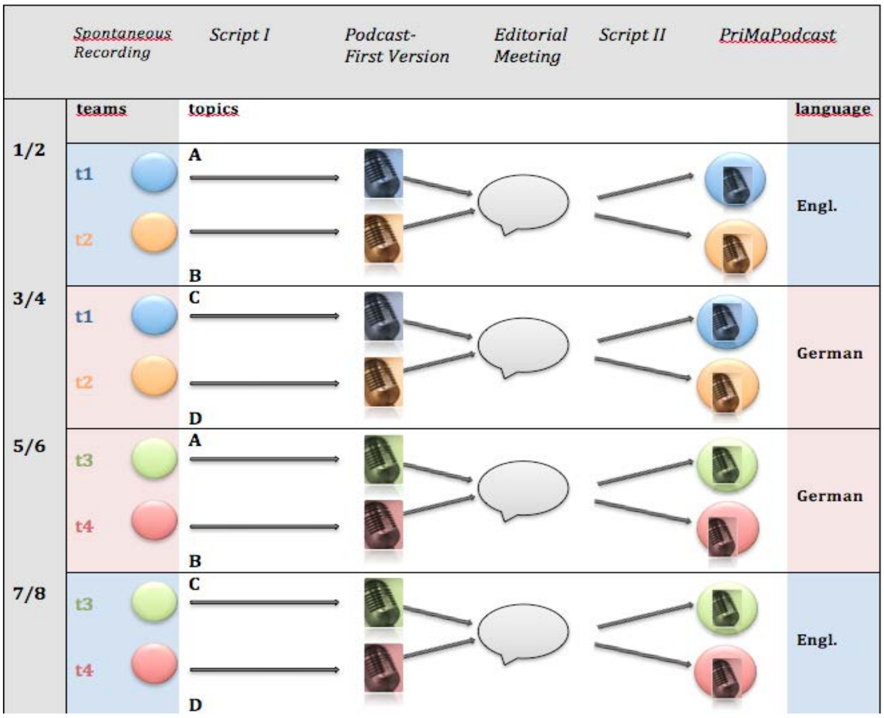

Table 2: Research Design and method of data collection

tion have been carried out. Each time, twelve bilingually taught students of grade 4 were grouped in fours. The groups produced two PriMaPodcasts about two different topics, one in German and one in English. All in all, a total of eight PriMaPodcasts with four different topics in two different languages were developed. In order to capture the different procedures of each team, the entire organisation process was video recorded. The mathematics teachers of both classes did a survey with regards to their personal models of bilingual teaching and learning. The learners' ways of interaction will be examined by the 'interaction analysis' (Krummheuer & Naujok, 1999), which is explained for simplicity by the following English empirical example that depicts the production process.

EMPIRICAL EXAMPLE

The example of the pilot study presented here deals with the question of 'What is symmetry?' Three 4th graders (aged 9 to 10) produced the following PriMaPodcast in a bilingual mathematics classroom in Frankfurt. The transcripts below refer to the following recordings: the unexpected recording, the first podcast version and the final version. All citations of the transcripts are marked in squared brackets <like this>. The following analysis is based on the principles in Schleppegrell (2007) as outlined before. Some of the applied transcription rules are:

. .. ...	Pause: . 1 sec .. 2 sec ... 3 sec
(4sec.)	Pause with given duration from 4 seconds onwards
Text written in *(italics)*	Describes actions for example *(whispering)* or *(incomprehensible expressions)*, ended up with %
Bold	Emphasis
b l o c k e d	Stretched pronunciation
/	Pitch inclination
-	Pitch stays constant
\	Pitch declination
#	One utterance is followed immediately by another

Unexpected Recording

01 Student 1: Explain what is symmetry ... uhm symmetry is w h e n . things are . mh l i k e . mh the same looks like the same on both sides

02 Student 2: Symmetry is uhm l i k e w h e n . it's uh s o m e when it's both the same *(he is mumbling)* be that's what . Kilian said it's . the same

03 Student 3: *(is speaking low)* I don't know what symmetry is\ *(is breathing out)%*

04 Student 2: Yes a symmetry is would be like uhm . a h o u s e and the neighbour has also a house that looks . the same . just the same .. b u t uhm not everything can be symmetry...

05 Student 1: When you draw a picture and . and you wa and you m have a .. mirror it looks symmetric

06 Student 2: Yes like u h m .. m h h how I can say that . symmetry is symmetry *(Student 3 is laughing)*% . like u h like this . like uh a blue order from the same market uhm . yes#

07 Student 3: #the same colour

Student 1 reads out the task and continues giving an answer in <01>. By using an identifying clause combined with the conjunction 'when', he tries to define the technical term 'symmetry'. He talks about things that are the same and emphasizes this fact in the following. Then he reformulates this statement, by saying that something is "the same on both sides". It is difficult to analyse the concept of symmetry he has got in mind because his explanations remain unspecific; technical terms are not used. Moreover, he seems to be uncertain about the topic, which is signalized by having used fillers (mh, l i k e) and pauses. In <02> student 2 alludes "both" to the two things being the same. In doing so, he adopts the grammatical pattern of student 1 and also refers to his answer. In between, his mumbling of words indicates that he has no specific idea yet. Student 3 in <03> answers with a complete sentence and admits to not knowing what symmetry was. Then student 2 makes another attempt by giving an everyday example in <04>. The unknown things are replaced by two houses. He emphasises again that both houses look "the same". Therefore, he begins his statement with an attributive clause. Then he makes a restriction, claiming that "not everything can be symmetry". Being a second language learner he used the noun 'symmetry' instead of 'symmetric'. Student 1 in <05> seems to think of an approach to connect symmetry to mirroring. By using the conjunction 'when' instead of 'if', he tries to describe this procedure. However, he only names two parts of construction, 'drawing a picture' and 'having a mirror'. By having both components, he suggests it would look symmetric. Even though one could understand what he meant, this explanation remains incomplete. Student 2 in <06>

continues with a statement that causes student 3 to laugh. While pondering on an explanation, he simply states "symmetry is symmetry". This attributive clause defines symmetry by itself, which expresses his inability to find a better explanation. Next, he refers to something blue that comes from the same market. This aspect might indicate a translation movement. Student 3 in <07> seems to know what his peer means and puts emphasis on the same colour. At this point the recording ends. The boys seem to have reached a conclusion despite having incomplete statements. They use some grammatical patterns of the mathematical register but they do not use much technical vocabulary and precise explanations. The pupils research their topic and create a script. This serves as basis for their first podcast version.

Podcast – First Version

01 Student 1: Symmetry are things l that look same

02 Student 3: Symmetry means you have a line in the middle and both sides need to are the same like/

03 Student 2: A star has a symmetry . the flag of Great Britain has a symmetry a plane has symmetry - a pentagon a hexagon the White House has symmetry a flower a triangle a guitar a butterfly . uhm . symmetry has also a strawberry a ball . uhm . then underwears has symmetry and p a n t s and the f a c e #

04 Student 3: #and scarfs

05 Student 2: Yes . scarfs and glasses has symmetry .. *(is speaking low)* that's it%

In this version student 1 starts off with a general statement about symmetry by using an attributive clause in <01>. This statement is comparable to the first utterance of the unexpected recording. The emphasis again is on the sameness of things. Student 3 seizes the idea of having "a line in the middle" in <02> which generates two equal sides. This idea, described by a common expression, seems to relate to the line symmetry. In order to express meaning, an identifying clause is used. However, the exact aspect of symmetry and the objects referred to have not been explicitly conveyed yet. Only in statements <03>, student 2 presents some examples concerning line symmetry by using common words as well as technical terms, e.g. "hexagon" and "triangle". It appears that he forgets to mention one example, which student 3 adds in <04>. Student 2 repeats the suggested word and names another example in <05>. In contrast to the unexpected recording, the statements are now better conceived and more structured, except for the rather informal beginning and end. In the editorial meeting the peers praise the chosen examples. The boys receive constructive criticism to improve their script in terms of content, style and language.

PriMaPodcast

01 Student 3: What is symmetry

02 Student 1: Symmetry are things that look same on both sides . when you have a mirror and put it in the middle . of a symmetric thing and it looks same . on both sides it is symmetric ... A mirror is like a line o of symmetry

03 Student 3: These . there are lots of things . that are symmetric\ but not all things are symmetric\ these things are not symmetric\ a radio a door a piano a crane an ocean a sea a river\ these things are symmetric

04 Student 2: Clothes a tie glasses pants and underwears shapes a star the flag of Great Britain a triangle a hexagon a pentagon . nature for example butterflies flowers and strawberries\ . music a guitar/ a violin\ . electricity a plane even words can be symmetric otto

Student 3 reads out the question in <01>. Student 1 gives a definition by using an identifying clause, similar to the beginnings of the other recordings: he equals symmetry with "things that look same on both sides" in <02>. Then he returns to the idea of mirroring which was originally uttered in the unexpected recording. The instruction on how to mirror is more detailed than before. Moreover, the technical term "line of symmetry" is used for the first time. In his explanations student 1 relates to general 'symmetric things'. Before these things are further explained by giving lots of examples in <04>, student 3 emphasizes in <03> that "not all things are symmetric". Having taken up the idea of the unexpected recording, he names seven things, which are not symmetric. He leads over to student 2 who reads out the examples of symmetry in <04>. Some of them, which are presented in the first podcast version, are not mentioned here any more (e.g. the 'White House' or 'a scarf'). Another difference is that the examples are classified into categories like 'clothes', 'shapes' or 'music'. Thus, they are more structured. A new example of symmetry is highlighted in the end: words like 'OTTO' can be symmetric, too.

CONCLUSIONS

As presented in the empirical example, the production of PriMaPodcasts makes it possible to investigate the bilingual learners' use of mathematical language. Each stage of production allows the mathematical language of the group to develop and become more specific as seen in the transcripts. The implementation of mathematical discourse practices, such as explaining and defining, enable the learners to express and discuss their mathematical thought processes. Moreover, the reflection on certain content deepens the students' mathematical understanding. The interactive procedure corresponds with the requirement of a mathematical conceptualization, by going beyond the stringent learning of technical terms and vocabulary. Thus, learners are groomed to become mathematically literate.

In the context of my PhD project, the presented research framework has been implemented twice so far. The analysis of the single stages (audio and video taped) will open up various possibilities to retrack the learners' progress by producing PriMaPodcasts in German and in English. Hoping to get further insights into the learners' understanding of mathematical concepts, the project will continue. As this is an on-going PhD project, further results cannot be disclosed as of yet.

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Gestures as part of discourse in reasoning situations: Introducing two epistemic functions of gestures

Christina M. Krause

University of Duisburg-Essen, Essen, Germany, christina.krause@uni-due.de

In this paper, I present how gestures can contribute to reasoning actions in social epistemic processes. In an empirical study, I investigated possible benefits of students' use of gestures in social learning processes. Integrating an analysis of students' gestures in a reconstruction of epistemic processes led to the identification of epistemic functions of gestures as ways in which gestures contribute to the accomplishment of epistemic actions. Two of these epistemic functions appear to be of special interest when students carry out reasoning in social interaction. Therefore, they will be presented in this paper by means of illustrative examples.

Keywords: Gestures, epistemic processes, social reasoning actions, embodiment.

INTRODUCTION

During the last 15 years, the study of gestures as part of the discourse in the mathematics classroom has increasingly gained attention (see Arzarello & Edwards, 2005). Gestures are considered to be a core resource in mathematical learning processes, a resource that can fulfill both a representational as well as an epistemic function in collaborative working processes (Dreyfus, Sabena, Kidron, & Arzarello, 2014; Krause, in press). Gestures have been identified to simplify the communication of ideas that are not yet fully elaborated (e.g. Reynolds & Reeve, 2002), reducing the cognitive effort needed for finding suitable mathematical words. Other studies point out benefits of collaboratively making use of the shared gesture space as mathematical experimental space in social interaction (Yoon, Thomas, & Dreyfus, 2011). Nevertheless, research investigating the use of gestures in mathematical reasoning is still scarce, although reasoning and argumentation are considered an important mathematical activity (Krummheuer, 2007).

This study is related to my PhD-project on the role of gestures in social processes of mathematical knowledge construction (Krause, in press). Although reasoning situations have not been in the focus of the study, they consituted a particular part of these processes. The investigation of gestures' contribution to epistemic processes within an embodied and multimodal framework led to the suggestion that *gestures may support reasoning actions in different ways when mathematical knowledge is constructed*. Evidence of this hypothsis will be presented more in detail in this paper.

THEORETICAL FRAMEWORK

Gestures are considered "idiosyncratic spontaneous movement[s] of the hands and arms accompanying speech" (McNeill, 1992, p. 37), "being done for the purposes of expression rather than in the service of some practical aim" (Kendon, 2004, p. 15). Underlying all the research on gestures is the assumption that gesture and speech are co-expressive, that is, they refer to the same idea: McNeill sees "gesture and the spoken utterances as different sides of a single underlying mental process" (McNeill, 1992, p. 1) and claims that "speech and gesture must cooperate to express the person's meaning" (p. 11), "each can include something that the other leaves out" (p. 79). According to the *Information Packaging Hypothesis (IPH)* (Kita, 2000, p. 163), gestures can help speakers to "package" spatial information into units appropriate for verbalization (Alibali, Kita, & Young, 2000, p. 593). By the use of gestures, information is parsed into entities more convenient to put into words, "consequently, the collaboration between the two modes [gesture and speech] provides speakers with wider possibilities to organize thought in ways suitable for linguistic expression" (Kita, 2000, p. 180).

Mathematical knowledge is considered to be constructed by individuals in social interaction, not arbitrarily but constrained by our bodily experience as humans in the world (Núñez, Edwards, & Matos, 1999, p. 53). While social-constructivism concerns the social-communicative aspects of learning, embodied cognition adds how individual aspects are shared as being based on bodily experience. It grounds the assumption that in social interaction, not only explicit and conventionalized modes of expression like speech are considered for the interpretation of an utterance, but also implicit ones such as gestures. It is due to our shared bodily experience as humans in the physical world that these implicit embodied modes of expression can be processed similarly by distinct individuals. Hence, the social-constructivist and the embodied approach do not oppose each other: While gestures may embody knowledge that is not consciously accessible, they may contribute to the social interaction implicitly. Reconstructing the epistemic processes within social interaction thus requires considering both implicit and explicit modes of expression.

Furthermore, embodied cognition theory states that bodily behaviour influences the way we think, grounding fundamental mathematical concepts in everyday experience through metaphorical thinking (Lakoff & Núñez, 2000). By expressing something in terms of something else, an entire conceptual environment can be understood via a more accessible or more illustrative domain. The domain referred to is called *source domain* and conveys how the relationships in the *target domain* shall be understood.

The social epistemic process was described by an epistemic action model that has been developed by Bikner-Ahsbahs (2006) based on the interpretation of speech acts. It encompasses the three social epistemic actions of (a) *gathering* single mathematical entities like examples or associations, (b) *connecting* a finite number of them, and (c) *seeing structures* such as generalities or patterns *(GCSt-model)*. The latter takes place when a new entity is built or a known entity is re-built in a new context.

A semiotic perspective has been integrated by understanding gestures as signs in a Peircean sense as "something that stand to somebody in some respect or capacity" (Peirce, CP 2.228) and an analysis within the semiotic bundle. The semiotic bundle is a dynamic structure that consists of different semiotic sets (e.g. speech, inscription and gestures) and relationships between them (Arzarello, 2006, pp. 280–282), both evolving in time. Some synchronous relationships can be described by two features of gestures; one concerning gestures' relation to speech, the other concerning its relation to inscription (Krause, in press). Each is considered to frame the interpretation of the utterance as it is shaped by speech, gesture, and inscription in social interaction:

— Gestures can *specify* aspects of the mathematical object referred to and by this, enrich the verbal utterance. They can specify the *where*, the *what*, the *how*, or *relational* aspects of the mathematical object. 'Where' refers to spatial aspects such as *location* or *direction*, 'what' gives information of the *kind* of the object, 'how' concerns the *style* of a mathematical object or activity, *relations* are specified when the gesture represents relations within an object or between objects in addition to what is explicated in speech. The specification reflects a potential non-verbal influence on the interpretation of an utterance within social interaction.

— Gestures are performed on *three referential levels*: *Level 1* is considered the level of the concrete, when the gesture refers directly to something that is already fixed. On this level, the gesture functions as an index in that its meaning solely derives from the meaning of what it refers to. On *level 2*, the level of the potential, gestures are embedded into an existing inscription. That is, something that is not fixed that way is made visible within an existing representation. The interpretation of gestures on level 2 demands for the material and/or contextual background of an inscription. The gestures in the gesture space, which is the space in the air roughly between the shoulders and the hips (McNeill, 1992, p. 86), are performed on *level 3*. They are free in the sense that they are detached from any inscription. Their disengagement from the concrete reveals a state of conceptualization of a mathematical idea.

Adding a semiotic perspective showed that gestures do not only affect which information may be provided

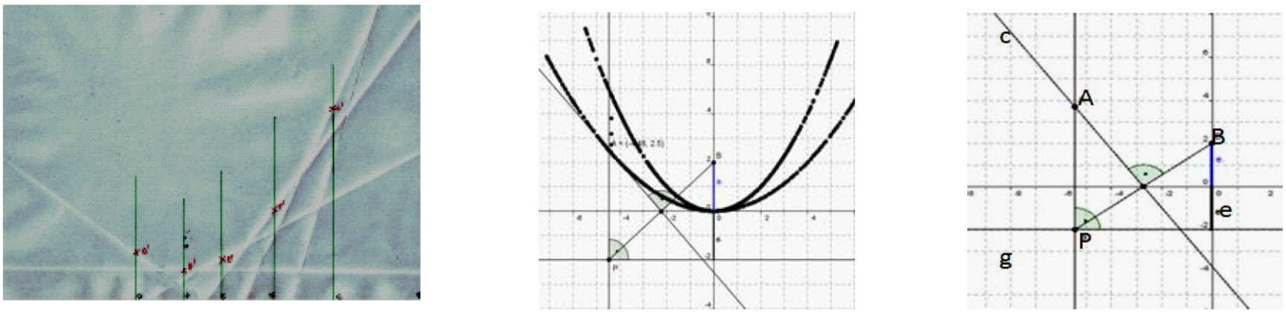

Figure. 1a-c (from left to right): (a) Possible outcome of the folding process, (b) static cut-out of the GeoGebra representation, (c) printout given to the student, representing one possible situation of the GeoGebra environment

by an utterance, but also that gestures can have direct impact on the accomplishment of epistemic actions. They can prepare, support, and also realize epistemic actions in different ways which I call *epistemic functions*. This paper deals with two epistemic functions that I consider to support reasoning actions.

Within the frame of the GCSt-model (Bikner-Ahsbahs, 2006), reasoning actions are a specific kind of connecting actions: Two or more aspects of a mathematical object or idea are linked in order to justify or explain a hypothesis, or the rejection of a hypothesis. This paper focuses on these kinds of connecting actions, providing first suggestions on *how gestures may take part in supporting students' reasoning as epistemic connecting action.*

METHODS

Three pairs of high-achieving students of grade 10 solve three tasks each. The tasks have been constructed to prompt fruitful epistemic processes by initiating epistemic actions. The three tasks deal with different mathematical topics and each provides a different variety of representations to work with in the course of solving the task. The epistemic functions presented in this paper concern a geometric-algebraic task and a task on logical reasoning using the idea of mathematical induction. This choice has been made due to the diversity of mathematical topics and representations provided by the tasks.

The Geometric-Algebraic Task deals with the parabola as geometrical locus. In the course of the task, three different representations are provided to the students: First, they have to construct a folding diagram according to the following instructions. On a given sheet, a point M is marked. They are now asked to (i) mark any point on the lower edge of the sheet, (ii) fold the paper such that the chosen point comes to lie upon the point M, (iii) draw a perpendicular to the lower edge of the sheet running through the point chosen on it, (iv) mark the intersection point of this perpendicular and the folding line with a red cross, (v) keep on proceeding like this by choosing new points on the lower border until they recognize a curve. This process leads to a folding diagram of the envelope curve of a parabola, the folding lines being tangents to the curve and the intersection points being points on the parabola. Figure 1a shows a possible outcome of this process.

Then, a GeoGebra file is introduced that represents a similar situation within a coordinate system. Here, the point corresponding to the one chosen on the lower border is called "P" and can be dragged to the left and to the right. Through dragging, a trace is produced by the points on the curve (Figure 1b). Finally, a printout capturing one possible situation from the GeoGebra environment is given to the students (Figure 1c). A first subtask consists of making conjectures about what can be seen in the folding diagram. Thereupon the folding diagram shall be compared to the GeoGebra diagram before a conjecture about the type of the function shall be stated and justified.

The Task on Logical Reasoning was formulated as a word problem:

An undefined number of persons sit in a circle. Everybody wears a hat. Everybody sees every hat except one's own. Everybody knows that at least one hat is marked. Every five minutes, a bell rings. Everybody who knows that his own hat is marked shall raise his hand with the ring of the bell as soon as he knows. The challenge consists in concluding from the number of hats one sees whether one's own hat is marked or not.

No further representations of the situation have been provided. For the base case of seeing zero marked hats it follows immediately that one's own hat is marked from the condition of knowing that there is at least one marked hat. The induction step can be induced from the reaction of the other persons by reconstructing how many marked hats they see, and then concluding on whether one's own hat is marked from the behaviour of the other's at the n^{th} ring of the bell. Solving the task thus makes use of the idea of mathematical induction and distinguishing cases concerning the reactions of the other consultants at the n^{th} ring of the bell.

The two tasks differ not only in the mathematical topic they deal with but also in the role of representations provided to the students. The rich semiotic variety of the geometric-algebraic environment is suggested to prompt the students to use gestures when reasoning on the kind of function. However, the more important question is *how* gestures may support the reasoning actions when solving this task. The lack of such representations in the task on logical reasoning raises the question whether gestures are used at all, and if so, what these gestures refer to and whether and in which ways they support reasoning as the main mathematical activity of the task.

Data and analysis

The learning processes have been videotaped from three perspectives: One camera filmed from the front to capture the gestures of the students. A second camera was directed to the inscriptions in front of the table. A third perspective was used to record the student's use of GeoGebra visible on a computer screen. Based on these video recordings, transcripts have been written, considering verbal utterances as well as non-verbal actions (see Figure 2 for the transcription key).

For answering the research question stated in this paper, those connecting actions become relevant in which the students justify or explain their conjectures. To reconstruct the role of gestures therein, the gestures are analysed within the semiotic bundle, taking into account also possible metaphorical meaning, grounded in the bodily experience with the physical world.

HOW GESTURES CAN TAKE PART IN REASONING SITUATIONS

In the following section, two examples will be reconstructed. These present the two epistemic functions *contrasting* and *giving visual access to the structure of a reasoning action* as ways in which gestures can support reasoning actions.

Supporting counter argumentation by contrasting to another representation

The two students Mike and Tim work on the task in which the *parabola* is explored and experienced as geometrical locus. They have already constructed the folding diagram and traced the curve in the GeoGebra environment. Preceding the following scene, Mike and Tim have identified the curve to be an exponential function "with some factor in front of it" (235). Prompted by an interviewer, the students are asked

exact.	dropping the voice	exact'	raising the voice
exact	emphasized	,exact	with a new onset
e-x-a-c-t	prolonged	(.), (..), (...)	1, 2 or 3 seconds pause
/S	interrupts the previous speaker	(4sec)	4 seconds pause (for more than 3 seconds)
[squared brackets]	indicate beginning and end of a gesture movement		

Figure 2: Transcription key

Figure 3: GeoGebra diagram as visible on the screen (trace thickened to optimize visibility) and gesture accomplished by Mike in line 288

to make statements about the values of the function represented by the curve for $x = 0$ and $x = -1$. This is when they realise that "it can't be that it is an exponential function' " (284; See Figure 2 for the GeoGebra diagram as visible on the screen at that moment.), explicating the reason in speech and gesture (Figure 1 for transcription key):

284 /Mike: it can't be that it is an exponential function' *(looks at Tim)*
285 Tim: right.
286 Mike: because ,uhm
287 /Tim: then it [would be] (*points towards the screen)* smaller there
288 /Mike: [elsewise the number would be smaller and smaller (*points at the screen from right up to left down)* (.)] the left side normally

Mike's reasoning of the refutation of the curve representing an exponential function consists of three utterances: In line 284, he refutes the conjecture about the type of the function. At this point, the basis of the conclusion he draws after considering the concrete values does not become explicit. In lines 286 and 288 he justifies his conclusion. Although Tim already confirms Mike's statement in line 285, Mike gives an explanation using a counterargument. He starts with "because" (286) but hesitates. This makes Tim start an approach, imprecisely referring to that "it would be smaller there" (287). This mentioning 'something being smaller' in combination with the pointing towards the screen suggests that he has in mind the same reason that in turn is expressed by Mike (288). From Mike's verbal utterance alone, the reasoning is not complete: He uses an undefined reference ("the number") and leaves out aspects that are specified in gesture: The iconic reference to the shape of the curve of an exponential function is combined with the indexical reference to the screen. Through this, two main functions are fulfilled: The gesture allows "the number" to be interpreted as a y-value and the direction of decrease to be from right to left, adding an aspect that is needed to interpret the argument to its full account. Shaping the interpretation frame of the entire utterance, the gesture *specifies* the *where*, the *what,* and the *how* as style of the shape of the curve potentially embedded into the GeoGebra diagram on level 2 in front of the screen. The performance on level 2 superimposes the actual case visible on the screen with the one to exclude as ephemerally shaped in gesture. This depiction of the hypothetical case to be refuted, the shape of the curve of an exponential function, makes apparent the difference in the styles of the two shapes. The curve has been traced in its full extent before and its symmetry is knowledge that is shared between the students. The depiction of the gesture thus represents only that part in which both representations do not fit together, hinting at a difference that is specified and with that, illustrates the counter argument given in speech.

Giving visual access to the structure of the reasoning action

Rosa and Lisa resolve the *task on logical reasoning*. They already hypothesized how to behave after the n^{th} ring of the bell, depending on the number of hats they see and the behaviour of the other consultants. They checked their strategy for some concrete cases and in conclusion, decided it to be valid. In the following excerpt, they investigate whether there may exist another strategy by assuming the case that all consultants wear a marked hat, considering the generic example of five consultants sitting in the circle and consequently seeing n=4 marked hats:

297 Rosa: the at the fourth ring they still see- *(writes: "4th ring: no") (4sec)* none ,oh- *(scratches out "no")* ,there I have always written nobody *(writes "nobody")* (...) *(writes "⇒")* ,that means there have to ,be- *(briefly raises*

Figure 4: Rosa's gesture depicting the path from ***condition*** to ***conclusion***, simultaneous to "more than four"

> *her hand and turns it forward in one fast movement)* [more than four] because they are only five everybody has one. *(writes: "all")*

Rosa concludes on the number of marked hats in the game ("that means there have to ,be [more than four]"), based on the fact that nobody signals at the fourth ring of the bell ("at the fourth ring they still see none"). While this is interpreted as a connecting reasoning action that can be reconstructed from the verbal utterance, Rosa's accompanying gesture metaphorically represents the connection of premise and conclusion as tracing a path from the back forth in an upwards arc by turning the wrist forward once. In this gesture-speech interplay, components of a metaphorical mapping based on the image schema of source-path-goal (Edwards, 2010, pp. 233–234) can be identified: Within this mapping, the "goal" corresponds to the conclusion. The movement of the hand on level 3 metaphorically represents the "path" within the gesture space and embodies what could verbally be expressed by the word "then". The gesture summarizes that the conclusion "there have to ,be- [more than four]" is derived from the condition that nobody signals. It helps understanding how Rosa 'packaged' her utterance into source, path and goal. The metaphorical meaning does not develop within the social interaction such that the gesture is not considered to explicate situated meaning by specifying aspects of a mathematical object. Moreover, it is deeply embodied in the everyday concept of deducing a conclusion from a given premise, not necessarily related to a mathematical topic. The gesture itself implicitly realizes a connection of premise and conclusion and with this, supports the verbal connecting action of logical inference by illustrating it in a more general way. The visual access it provides does not refer to a mathematical object but to the structure of reasoning on a meta-level. This way, it may help to keep the discourse on track by sharing the organization of thought underlying the argument.

SUMMARY AND DISCUSSION

The two epistemic functions of gestures presented in this paper show how gestures may support reasoning as specific kind of connecting action in different ways:

Contrasting-gestures shape a representation of a mathematical object which is reasoned about in comparison to another representation. In this way, the gesture can specify aspects to exclude one of the two possibilities. The contrasting-gesture complements speech and inscription in an *explicit* manner. Without the gesture, it would be left unclear why the verbal utterance justifies the conclusion. Using a gesture this way to support a counter argument becomes possible by the graphical representation of the mathematical object on which is reasoned and the knowledge about it as has been developed within the social interaction.

Gestures can also embody the action of connecting premise and conclusion within an act of reasoning and thereby *give visual access to the structure of a reasoning action.* In this case, the metaphorical character of the gesture is completely detached from the concrete content of the task. The structuring-gesture provides an *implicit* support by not enriching the utterance semantically, but on a meta-level. Using a gesture this way may support the collective act of reasoning as it indicates the logical structure within an argument and may help to keep track. It makes traceable how the argument was organized as logical inference.

Being characterized by a depictive use of gesture and a close and direct relationship to speech and current inscription, *contrasting* can be considered a *situated* function. On the other side, the metaphorical use of gesture while *structuring the verbal discourse* has an implicit meta-relation to speech such that the function may be rather a *universal* one.

This raises the question whether there may be typical ways of using gestures with respect to different types of reasoning actions and about the role of these gestures in learning and in teaching. With regard to the mathematics classroom, Arzarello and Sabena suggest that gestures may foster students' argumentation skills by structuring mathematical arguments (Arzarello & Sabena, 2014, pp. 99–100), similar to what I have presented in this paper. While they adopt an individual approach, the examples presented here indicate how it may also benefit reasoning actions from a social perspective: Adding gestural expression provides visual access to the argument such that it becomes received and processed also in a visual modality, making the understanding of the utterance (and so also of an argument given) more comprehensive.

To test and refine this hypothesis, further individual research on the role of gestures in social reasoning actions is needed, conducted against a more elaborated theoretical background. The epistemic functions of gestures presented in this paper can provide a basis for such an investigation due to their diverse groundings in *graphical aspects* of reasoning on the one hand, and the *logical structure* of discourse as metaphorically *embodied aspect* on the other hand. Furthermore, the presented findings suggest to be aware of the potential of gestures for and in social processes of argumentation.

ACKONOWLEDGEMENT

This project was funded by the Central Research Developmental Fund (ZF) of the University of Bremen (Germany) (grant #03/112/08). The data used in this project is taken from the project "Effective knowledge construction in interest-dense situations", funded by the German Israeli-Foundation (grant #946-357.4/2006).

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Ambiguity as a cognitive and didactic resource

Maria Mellone and Roberto Tortora

Università degli Studi Federico II, Naples, Italy, rtortora@unina.it

Mathematics is often perceived and described as a domain of perfection, where polysemy is seen as dangerous and ambiguity is banned. But if that is true for the "final" dressing of official mathematics, in doing mathematics and, as we claim, in understanding it, things run the opposite way. Contrasting usual didactic practices, suitable forms of ambiguity can be seen and used as powerful cognitive resources. We present reflections on a long lasting didactic activity with students of different grades (from primary to university level), and show how to exploit some ambiguities (in particular linguistic ones) related to the notion of consecutive numbers.

Keywords: Language, ambiguity, polysemy, consecutive numbers.

INTRODUCTION

What is the meaning of *ambiguity*? The 2013 online version of the Webster Dictionary states: "Ambiguity: doubtfulness or uncertainty, particularly as to the signification of language, arising from its admitting of more than one meaning; an equivocal word or expression." In these words we perceive something wrong in any occurrence of an ambiguity. According to common sense, ambiguity is a sort of imperfection or a kind of error; and errors should be avoided. This is particularly true within mathematics. One of the main features of mathematicians' activity is the ability to expunge any inconsistency and every uncertainty from their arguments. However, although there is no doubt that the final aspiration of mathematics (and, for that, of all science and knowledge), is total freedom from errors, it is also true that, in the long path towards this goal, errors often play an invaluable role and make a strong contribution. Many authors have stressed this view, in different philosophical and theoretical domains, from gnoseology to epistemology. In the works by Popper, Kuhn or Lakatos, just to mention a few, errors and ambiguities are viewed as powerful stimuli for scientific development.

In the educational domain, the impact of errors as a didactic resource has also been recognized, albeit more recently. For this we essentially refer to the monograph by Borasi (1996) and to the extensive bibliography therein.

Less widely explored, as far as we know, is a possible positive influence on teaching activities of ambiguity and in particular of polysemy, here considered as a form of linguistic ambiguity. Many authors have noticed how and why the confusion that often arises between a naïve and a technical meaning of the same word (*angle*, *continuous*, *square*, *limit*, *increasing*, and so on and so forth) is the source of many students' difficulties (Tall & Vinner, 1981; Ferrari, 2004; Bardelle, 2010). The usual approach to ambiguities is to identify them in order to avoid, as far as possible, their disturbing consequences.

Our approach is quite the opposite. According to the general view by Borasi (1996), we will try to show how ambiguities, in particular linguistic ones, can play a useful role in the development of mathematics learning. To this purpose, we present an example of an educational activity proposed over several years to students of different grades (from primary to university students). Then we will discuss some real and potential developments of the activity and will make comments on its effectiveness. Finally some reflections will be made on the role of the teacher within a general vision of the educational process.

THEORETICAL FRAMEWORK

One cannot assign a positive value to errors and inaccuracies of students if one has a vision of teaching as transmission. Indeed, our claims on ambiguity and its important role in mathematics learning processes require an active role of learners as conceived in the inquiry approach (Borasi, 1992, 1996):

> Whereas in traditional mathematics classes ambiguity, anomalies, and contradictions are carefully eliminated so as to avoid a potential source of confusion, in an inquiry classroom these elements would be highlighted and capitalized on as a motivating force (Borasi, 1996, p. 25)

Starting from this main premise, the research work we are going to present finds its roots in the Vygotskian sociocultural vision of learning. In particular in our work, the language plays a central role: if errors are of several types, ambiguity is a typical linguistic affair. It is therefore necessary to frame our proposal in the context of studies on the importance of language for mathematics and mathematics education. For this, we refer to general studies of functional linguistics (Halliday, 1985) and to the elaboration of these ideas in the domain of mathematics education, especially (Ferrari, 2004). From there we draw some essential constructs, described below.

The pivotal notion connecting texts to contexts is that of *register*, that is, a variety of language depending on usage. Whenever an individual uses the language for a particular purpose, he selects a register which is suitable for that purpose. Of course, the choice is also bounded by the resources that are available for the individual. For further details on this meaning of register (Ferrari, 2004), where the sense is also contrasted with the same term used differently by Duval (2006). In this framework the key distinction is between *literate* and *colloquial* registers. The difference is functional, in the sense that the same people can use a literate register in a context and a colloquial register in a different one. The literate registers are typically used in communications belonging to scientific, legal, political, literary, etc. domains, in most of the narrative and often in speeches among educated people. The colloquial register is typically used in more or less informal speeches. A fully satisfying definition is impossible. For mathematics educators, what is important is to compare or to contrast the literate registers used within formal mathematics with the colloquial ones used in life and also in the everyday discourses in teaching-learning environments. This comparison represents a tool by means of which it is possible to interpret some students' difficulties. In Bardelle (2010) it is used to analyse the behaviour of a sample of university students involved in the study of the monotonicity of a function and of the properties of its graph.

The ambiguity coming from two different uses of the same word or locution in the colloquial and in the literate (mathematical) register is shown as an obstacle for the acquisition of some mathematical notions. This is true, but at the same time, in many mathematical contexts, ambiguity can be viewed and exploited as a resource, acting as a stimulus for a deeper and more critical advancement of knowledge. More generally, we are convinced that it is quite illusory to try to expunge any ambiguity from the mathematical discourse; or, as it may be, from mathematics itself, according with Sfard's (2008) radical identification of mathematics as discourse.

In the history of mathematics, such intricate paths are frequent (see the analysis of Lakatos (1976) of the Euler's theorem for polyhedra). We do not think that the individual cognitive development reproduces faithfully cultural evolution but want to observe that whenever an epistemologically difficult notion is encountered, both in the history and in a learning environment, many attempts have to be made before the notion can be assimilated. Many of these attempts include shifts from a linguistic register to another and changes in the meaning of some key word.

According to a sociocultural vision of learning and teaching, our way of working with students is based on a continuous interplay between the linguistic components of knowledge and an epistemological analysis of the disciplinary concepts involved. For this kind of analysis, we refer mainly to the theoretical construct CAC (*Cultural Analysis of the Content*) by Boero and Guala (2008). Mathematics is seen as an evolving discipline "with different levels of rigor both at a specific moment in history (according to the cultural environment and specific needs), and across history, and as a domain of culture as a set of interrelated cultural tools and social practices, which can be inherited over generations" (ibid, p. 223). This vision of mathematics forces a conception of mathematics education activities that leads teachers and educators "to radically question their beliefs concerning mathematics in general and specific subject matter in particular" (ibid, p. 223).

A CULTURAL-LINGUISTIC ANALYSIS OF CONSECUTIVE(NESS)

In this study we take into account the notion of *consecutiveness*, in particular its occurrence in the locution *consecutive numbers*. We deeply analyse this notion within the solution and discussion of the following arithmetic word problem, submitted, with some variants, to students of different grades.

> Take four consecutive numbers. Multiply the two middle numbers by each other, then multiply the first one by the last one and calculate the difference between the two results obtained. Repeat the exercise several times, using different numbers. Do you observe any regularity?

The consecutiveness notion is not encountered here for the first time as an argument of discussion within mathematics education research (see Boero, Chiappini, Garuti, & Sibilla, 1995). This notion is suitable to trigger various arithmetic explorations, or to develop argumentation and proof activities in arithmetic, guided by careful teaching mediations. Here, we focus on some subtle ambiguity inherent to the notion of "consecutiveness", trying to show how, if handled with care, it can represent a powerful resource for education and knowledge purposes. With this aim, in this section we analyse some cultural and epistemological aspects of that notion, starting from the above problem, and according to the CAC perspective.

First of all the text refers to consecutive numbers without specifying the number domain. That is not a problem. From one side the use of the word "consecutive" in a literate register, like the text of a word problem, should suggest that the correct domain is the set of natural numbers or perhaps of integers; from the other side the common use in a colloquial register of the word 'number' without specifications, always refers to natural numbers. But what is the meaning of consecutive natural numbers? Even in the scientific register we can recognize (at least) two meanings assigned to these words: i) the first one is framed within the order relation; ii) and the second one comes from the additive structure (one might even distinguish this meaning from the one embodied in Peano's successor operator). Let us give a closer look at these two meanings, and at their formal renderings:

i) Probably the most natural meaning of the term "consecutive" corresponds to the idea "to be immediately subsequent to", like, for example, in "Monday and Tuesday are consecutive days". In mathematical words this requires the presence of an order relation (better, a total and discrete one). So to say in a (partially) formalized language that two elements a and b are consecutive, we have to say that "$a < b$ and there is no element c such that $a < c < b$". From the logical point of view, we have the conjunction of two statements, an atomic one and the negation of the existential of a conjunction (there <u>does not exist</u> any c such that $a < c$ <u>and</u> $c < b$). So, we see that the formal translation of a (relatively) simple notion has a logical structure far from simple. Worse, we could even argue that the notion of consecutive is 'symmetric' (that is, if 2 and 3 are consecutive, also 3 and 2 are consecutive), and so its algebraic translation becomes even more complicated.

ii) The other meaning of consecutive can be expressed as: "Two numbers are consecutive if the second one is obtained adding one unity to the first one", which in algebraic language is written as "a and b are consecutive if $b = a + 1$". Unfortunately, this definition is good for natural numbers (and for integers too), but not for other numerical domains, for example the sets of even or of odd numbers. But one can generalize the above notion of consecutiveness by taking into account the more general 'additive' relationship existing between two closest terms of any arithmetical progression, i.e., with obvious meaning of the variables, $a_{n+1} = a_n + d$.

What happens is that the two above meanings of the word consecutive coincide for natural numbers, but they do not for other kind of numbers: either in the sense that the first meaning disappears (for rational numbers a and b it is meaningless to say that they are consecutive while it makes sense to say that $b = a + 1$); or in the sense that the second possibility vanishes (two even numbers a and b can be consecutive but it can never happen that $b = a + 1$); or in the sense that both conditions are acceptable but bear different meanings (e.g., the case of rational numbers truncated to the second decimal place). The two properties keep their own importance.

We focus on how ambiguity can represent an opportunity for mathematics teaching and learning. In the next section we will show and analyse some students' behaviours and the development of their mathematical knowledge coming out in the attempts to manage the situation produced by the ambiguity of the notion of consecutiveness in the problem presented.

SOME EXPERIMENTAL EVIDENCE

We present some critical points of the experiences that we have been living during these years, working with the problem presented in the previous section. We have submitted the above problem or some variants of it for several years to students of different grades (from elementary school up to prospective elementary and secondary school teachers) with slightly different goals, according to the context and to the age of students (Iannece & Romano, 2009; Mellone, 2011).

According with our inquiry approach, as outlined by Borasi (1992), in the management of mathematical activities, rather than packing ready-made solutions and imposing them to students, we prefer to spend most of the time in mathematical discussions, believing that this increases students' opportunities to build a flexible and critical mathematical knowledge. For this reason we often use open tasks, without posing limits to their developments and, most of all, taking seriously all students' reasoning and feedback. This way of working, besides the outcomes concerning students, allows us to grow in our awareness as educators.

We have collected data regarding students' behaviour when dealing with problems of the type above described. Despite their different ages and experiences, we have observed many common points in their answers and reasoning. Here we report some notable behaviours of about 40 students of a mathematics class of prospective elementary school teachers. We comment on students' typical reactions to a specific articulation of the problem that we have proposed several times.

Most students, after the first arithmetic explorations and the discovery/recognition of regularities, are led to express the notion of consecutiveness in a general form. In their attempts to translate the relationship among the four numbers into the algebraic language, the students often propose to use four letters, typically four "consecutive" letters in alphabetic order, like a, b, c, d, for denoting the four consecutive numbers. Such a choice can be labelled as unproductive, since the letters of the alphabet do not support an algebraic structure, and therefore the circumstance that b follows immediately a does not express adequately the analogous relationship between the two numbers. One could be tempted to push directly towards the more effective algebraic translation a, $a + 1$, $a + 2$, $a + 3$, which is, of course, completely faithful and, above all, suitable for the usual algebraic manipulations, including those required in the problem. We too have done this many times, before becoming aware of the fact that by doing so we were losing a great opportunity. Now, we prefer to recognize the value of this choice, where a sort of isomorphism between two order structures (numbers and alphabet) is clearly glimpsed and exploited. After all, the fact that this attempt is not effective is akin to what normally happens to mathematicians when they reach their results by trial and error.

Giving time and confidence to students who choose the four letters a, b, c, d, for representing the four numbers, allows most of them to implement an interesting bridge behaviour between the initial idea of the four consecutive letters and the final four expressions a, $a + 1$, $a + 2$, $a + 3$: they prefer to maintain the four different letters, but accompanying them with the three conditions $b = a + 1$, $c = b + 1$, $d = c + 1$. This behaviour testifies that often a system of equations is easier to be conceived in comparison to the more concise and effective solution a, $a + 1$, $a + 2$, $a + 3$, that requires the quite sophisticated ability to conceive four numbers and simultaneously their relationship. In other words a, $a + 1$, $a + 2$, $a + 3$ appear as the mental result of a transformation applied to the three above equations [1]. The practice of discussion guarantees that the best representation (a, $a + 1$, $a + 2$, $a + 3$) emerges in any case. But the possibility for the students to discuss about the benefits or the disadvantages of using different algebraic translations represents an invaluable way to converge with full awareness towards the best one.

The typical step that follows in our activity is to ask students to consider the problem for four consecutive even numbers. We have experienced that many students propose without hesitation an algebraic representation like a, $a + 2$, $a + 4$, $a + 6$, and this is true especially for those students who had represented the four consecutive natural numbers as a, $a + 1$, $a + 2$, $a + 3$. Of course the use of the additive representation for natural numbers fosters a similar form also in the case of even numbers. The fact that the students do not manifest any hesitation in shifting from the operator "+1" to the operator "+2", shows that the meaning of consecutiveness associated with the order relation prevails over its additive interpretation in the students' perception of this notion. Indeed they have no problem in abandoning the transcription of "consecutive of n" as "$n + 1$". However, their favourite representation is a, $a + 2$, $a + 4$, $a + 6$, in contrast with $2n$, $2n + 2$, $2n + 4$, $2n + 6$, selected by relatively few students, although the second representation is of course the only one that correctly expresses the evenness of the four numbers. This inaccuracy turns out to be useful. It reveals that students prefer to focus on the arithmetic progression as the real nature of the problem, rather than on the kind of numbers involved (in this case, even). When asked, in the next step, to represent four consecutive odd numbers, the students who had represented the even numbers as a, $a + 2$, $a + 3$, $a + 4$ easily realize that their representation also works with odd numbers, while among the students who proposed $2n$, $2n + 2$, $2n + 4$, $2n + 6$ only a few of them succeed in finding the correct representation $2n + 1$, $2n + 3$, $2n + 5$, $2n + 7$ [2]. The discovery that in both cases (even and odd numbers) the searched difference is always 8, moves the mathematical discourse towards the arithmetic progressions with common difference 2.

The last step of our typical way of managing the problem has the goal of addressing the concept of density of the rational numbers. We request to consider four consecutive decimal numbers. The persistence of the *concept image*, in the sense of Tall and Vinner (1981), of the "discrete" order relation as unique prototype of all the possible order relationships, is the trigger element at the base of our proposal. We have experienced that this condition of cognitive break represents a fertile tool to face the epistemological knot of the density of rational numbers. We have observed that the students often propose to consider decimal numbers truncated at the first decimal digit (for example 2.1; 2.2; 2.3; 2.4), or, less frequently, at the second digit.

This answer is a mistake but this error offers to students the opportunity to start new explorations on arithmetic progressions. They easily discover that for decimal numbers truncated as above the difference in the problem is 0.02, and after that, moving to consider decimal numbers cut at the second digit, they realize that the difference is 0.0002. These are good premises to explore and discover the more general rule according to which, given four consecutive numbers a, $a + d$, $a + 2d$, $a + 3d$ of an arithmetic progression, the difference $(a + d)(a + 2d) - a(a + 3d)$ is $2d^2$.

The path shown is an example of how successive generalizations can be exploited to make evident to students the usefulness of the algebraic language for doing manipulations and seeing relationships. We claim that here we have more, namely the possibility of exploiting the ambiguity of a term like "consecutive" for didactic purposes. Two meanings associated to the term coincide in the case of natural numbers, but they split when passing to other domains of numbers. When moving the problem towards the field of rational numbers, the students have the opportunity to become aware of this double meaning, realizing that, according with the first meaning, there cannot be consecutive rational numbers, while according with the second meaning, but enlarging it to sequences of numbers separated to each other by a constant difference, new fascinating arithmetic structures can be glimpsed and explored. The interesting linguistic-epistemological phenomenon of the splitting of a notion into two different ones, when switching from a source domain to new enlarged environments, already studied from several points of view (Lakatos, 1976, and many others) appears here in a new guise as a learning resource.

EDUCATIONAL OUTCOMES AND CONCLUSIVE REMARKS

Several studies in mathematics education have shown that many students' difficulties in mathematics come from their inability to juggle between the daily life use of a word and the formal use of the same word in mathematics (Bardelle, 2010). On the other hand to have a word with different meanings and different uses depending on the needs of the communication is an advantage for people who are in possession of this variety. In this direction, we are convinced that the ambiguity of words used in mathematics rather than to be hidden, as most education practices usually do, should be exploited as resources for mathematics education. At the same time we are convinced that in order to do this in an effective way, it is necessary to develop deep reflections about the epistemological and linguistic features concerning the use of words in mathematics (and not only). A useful framework for this analysis is the CAC construct by Boero and Guala (2008).

The case of the word "consecutive" examined above offers a particularly rich context, but it is just an example (for an analogous investigation on the word "triangle", see Castagnola & Tortora, 2009). In our research we have understood how the polysemy of such a simple word like “consecutive” is something to explore with our students. This also amounts to doing interdisciplinary work between mathematics and language and to including mathematics fully in human sciences.

During these years, our way of organizing mathematical activities has allowed teachers get involved to grow their awareness as educators, but also to gain deeper competence about the mathematical topics explored with students. The inquiry approach that inspires our work requires from teachers a strong ability to give space and attention to every proposal and idea that students might have and to carefully guide them. It also entails that the teachers be involved in the mathematical work in an atmosphere genuinely oriented to discovery, where even the possible lack of a prompt response to students' questions does not appear as a diminution of their authority. As written by Radford (2014, p. 19): “Teachers and students are in the same boat, producing knowledge and learning together. In their joint labour, they sweat, suffer, and find gratification and fulfilment with each other.”

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ENDNOTES

1. This cognitive behaviour suggests that the usual curricular hierarchy, according to which equations are treated rigorously before systems should perhaps be partly revised.

2. While in $2n$, $2n + 2$, $2n + 4$, $2n + 6$, the term $2n$ is used as a standard way of representing an even number, and +2, +4 and +6 are standard ways of adding two units at a time, in $2n + 1$, $2n + 3$, $2n + 5$, $2n + 7$, the term $2n + 1$ is a common representation of a generic odd number but $2n + 3$ (and $2n + 5$ and $2n + 7$) has to be obtained from $(2n + 1) + 2$, where the two +'s play different syntactic-semantic roles.

Sharing structures of algebraic expressions through language: A transformation gap

Alexander Meyer

TU Dortmund University, Dortmund, Germany, alexander.meyer@math.tu-dortmund.de

Collaborating in the transformation of algebraic expressions results in a need to share the structures of algebraic expressions with the help of language. However, little is known about how such structures are conveyed through language – and adopted by the other participants of the discourse. This paper uses a functional-pragmatic framework to reconstruct the patterns behind such discourses in which structures are shared. With this framework, a transformation gap between conveying and adopting structures is identified: Where the first speaker can only refer to the original expression to propose a transformation, the other participants are able to refer to both the original and to a transformed expression for seeing structures.

Keywords: Structure sense; language; discourse; algebra.

INTRODUCTION

Algebraic expressions can be flexibly interpreted. Arcavi (2005) shows the different ways for students to make sense of symbols and symbolic expressions. Drouhard and Teppo (2004) argue that an algebraic expression, especially one with a distributive structure, can be interpreted in many different ways according to a specific context that can be flexibly activated, e.g. as a function. Accordingly, the students' transformation of an algebraic expression is not a mechanical activity that is aimed at a predefined result, but an activity that is guided by the students' individual interpretations of the structures of expressions. Furthermore, these structures in an algebraic expression do not solely exist „on their own", but come *into existence* in the activity of transforming an expression (Rüede, 2012). For example, when transforming the expression $5ab + a - 5b - 1$, there are different ways to apply the distributive law; students may either relate $5ab$ to a or to $-5b$. Each relation gives way to a different notion of the structure of the expression and, through this, a different application of the distributive law.

In a collaborative task of transforming an algebraic expression, students coordinate their joint activity to a large part with the help of language. More specifically, they explicate their interpretations of the structures of the expression in the discourse with the help of language to other students. Being able to use language to describe structures in algebraic expressions helps students to see structures, as language provides the means to share perceived structures with the teacher and with others. However, little is known about how students convey a structure in an algebraic expression to other participants of a discourse through language - and how language affects how the latter ones can adopt these structures.

STRUCTURE SEEING AND ITS CONNECTION TO LANGUAGE

In order to conceptualize the transformation of algebraic expressions not as a mechanical activity, but as activity that is guided by the interpretations of structures in algebraic expressions, the model of structure sense was introduced (Linchevski & Livneh, 1999) and used and refined in various studies (Novotná & Hoch, 2008). As structures of algebraic expressions emerge in the activities *with* the expressions, structure sense is here regarded as structure seeing, that is, as an interpretative activity of relating identified parts of an expression to each other and to the whole expression in order to decide for a transformation of this expression (Meyer, 2014).

When translating a text into an algebraic expression, students link the structure of the text with the algebraic expression they are constructing. There is evidence that the translation of a task into an algebraic expression is linked to the understanding of language (Duru

& Koklu, 2011). A sequential translation of the words of a text from left to right might be a way for students to structure an expression also in a sequential way (e.g. Clement, 1982). However, other evidence suggests that students do not usually translate this way, but rather first look into the meanings of the words before translating (Capraro & Joffrion, 2006).

The link between the structure of the text and the algebraic expression, however, is not direct, but mediated by meta-linguistic awareness, that is, by the ability of a student to "reflect on and analyze spoken or written language" (MacGregor & Price, 1999, p. 451). Based on their empirical results, MacGregor and Price conclude that "this conscious awareness of language structures and the ability to manipulate those structures may be a manifestation of deeper cognitive processes that also underlie the understanding of algebraic notation" (1999, p. 462).

Caspi and Sfard (2012) give insight into the elements of language that can be used for expressing structures in algebraic expressions, like compound noun phrases and objectified processes. While they do not look into students' seeing of structure, they conceptualize different levels of how elements of an algebraic expression can be expressed with language and what level of generality this language indicates. The first level is that of processes, where language is used to express a calculation. The calculation is presented in the order in which it is executed. The second is the granular level around the description of a calculation; it still describes a process but also contains compound clauses that make procedural elements into an object, like "the sum of…is". The third is the objectified level where complex calculations and processes can be substituted with objects or objectified descriptions, e.g. "A product of a sum of two numbers". Caspi and Sfard find out that 7th graders are, under certain circumstances, capable of talking about algebraic expressions on the third level. Their argument suggests that when students, at a higher level, can express complex structures in a more condensed way with language, then more complex structures are available to participants.

My study is focused on the language that students use to express the structures of algebraic expressions in order to share them with others, and how this affects which structures are adopted by others or put to the fore in an algebraic expression - and which are disregarded. In order to look into this question, I use a functional-pragmatic framework which is based upon an activity-theoretical notion of linguistics. Within a functional-pragmatic analysis, the genesis and transformation of the propositional content in the discourse can be traced back to the linguistic actions and 'linguistic reality' of participants.

METHODOLOGY: A FUNCTIONAL-PRAGMATIC ANALYSIS

The aim of a functional-pragmatics analysis has been formulated by one of its main representatives as follows:

> In short, the fundamental aim of Functional Pragmatics is to analyze language as a sociohistorically developed action form that mediates between a speaker (S) and a hearer (H), and achieves – with respect to constellations in the actants' action space […] - a transformation of deficiency into sufficiency with respect to a system of societally elaborated needs. (Redder, 2008, p. 136)

Functional Pragmatics regards a communicative act as driven by a purpose. Speaker and hearer equally participate through speaker-actions and hearer-actions in this 'purpose-guided' activity. The inner structure of the speech acts of both speaker and hearer are synchronized with respect to „topics, focus of attention, previous (speech-)actions, etc (p. 138)" between an extralinguistic reality (depicted by P), the mental reality of speaker ($\prod_s$) and hearer ($\prod_h$) and the linguistic reality (p) (Redder, 2008).

In this study, the extralinguistic reality (P) is the structures of algebraic expressions, while the aim of the functional-pragmatic analysis is to reconstruct the linguistic reality (p), in this case, the structures that are actually shared in the discourse. The students' joint action of transforming an algebraic expression is a „transformation of deficiency into sufficiency" in relation to the „extralinguistic reality" of algebraic expressions - it is about the speaker transmitting identified structures of algebraic expressions ($\prod_h$) to the hearer ($\prod_h$), so that speaker and hearer arrive at a mathematically acceptable transformation („sufficiency").

Functional pragmatics provides a tool to analyze how structures of algebraic expressions are made available and are picked up in a communicative act for the purpose of finding a mathematically adequate transformation. This reconstruction involves two steps. The first step is the reconstruction of the *surface progression* of the discourse. The second step is the reconstruction of the *pattern* behind this surface progression. In Functional Pragmatics, the pattern is what guides the participants' actions without them being explicitly aware of it.

Reconstruction of the surface progression of the discourse

The linguistic actions of the speaker and hearer are continually influencing each other; at a given time, both participants' actions have an equal potential to influence the other's actions. There are three qualitatively different categories to distinguish these structures of linguistic actions:

- the *organization of the discourse* (Ehlich, 2007, p. 71), that is, how linguistic actions coordinate each others actions;
- the *action potential*, that is, the potential of the speakers action to bring something about in the hearer or vice versa, e.g. bringing about a deeper understanding (Ehlich, 2007, p. 71), this is connected with the purpose behind a speaker's actions;
- the *propositional content* of the linguistic actions.

A functional-pragmatic analysis starts with separating the parts of the transcript in line with these three categories. First, the organization of the discourse is in focus. It is reconstructed by looking into those parts of the transcripts where speaker and hearer coordinate each other's actions (e.g. by expressing interpersonal relations like „I'm writing, you dictate". This is driven mainly by linguistic categories.). Second, action potentials that are realized in a discourse are carved out. In the activity of transforming an algebraic expression, one can think of a situation where only procedural action potentials are realized. Accordingly, the activity would be about a calculation and about arriving at a result - this would relate to Caspi and Sfard's (2012) processual level. In the analysis of action potentials, both linguistic and didactic categories can be put to use. Third, building upon the previous steps, the propositional structure of the discourse is reconstructed. This includes the reconstruction of the linguistic realities of speaker and hearer ($\prod_s$ and $\prod_h$). For this, didactic categories have to be used, as this is on the plane of the mathematical knowledge that is in focus in the discourse. In this step of the analysis, relations, dependencies, references to previous linguistic actions etc. are in focus.

Reconstruction of the pattern that guides the students' linguistic actions

The reconstruction of the surface progression of the discourse is the starting point for analyzing the elementary propositional basis. The elementary propositional basis is the reconstruction of the knowledge that is independent of the speakers and hearers actions. The reconstruction of this knowledge leads to a notion of the linguistic reality of p. In this case, the linguistic reality p encompasses the structures of an algebraic expression that are shared. The relations between p, P, $\prod_s$ and $\prod_h$ constitute a pattern in the discourse, in this case, the pattern of sharing structures of algebraic expressions in a discourse.

In generalizing such a pattern, an apparatus can be reconstructed. An apparatus describes general patterns in discourses; it is assumed that patterns are generalizable to other discourses of the same kind. In this study, the analysis aims at reconstructing discursive patterns of structure seen in regard to the question how structures are conveyed and adopted in a discourse. The pattern presented here, however, can only be regarded as a first approximation of a general pattern. The pattern has to be tested for its generality in other qualitative and quantitative studies.

Background for the study

The case study of Max and Tim presented here is part of a larger design research study. The larger study aims at promoting the students' structure seeing by providing scaffolds and requiring the students to negotiate structural elements of algebraic expressions (Meyer, 2014). Tim and Max are 8th graders from a German middle school; they were chosen for the teaching intervention based on a previous assessment that showed that they possessed a basic understanding of the transformation of algebraic expressions, but a lack of understanding of the underlying structures of algebraic expressions. The teaching intervention consisted of three tasks with subtasks. It took 1.5h and was supervised by trained interviewers.

In the episode presented here, Tim and Max worked on the second task that required them to transform a sequence of algebraic expressions by applying the distributive law. The episode represents the third expression in this sequence; the students thus already applied the distributive law to the two previous – structurally simpler – algebraic expressions. With each expression, the students are given the original formulation of the distributive law $ab + ac = a(b + c)$ as a structural reference. The students already acquired marking strategies and were able to use them to mark structures and to better communicate about structural elements. In the here presented episode, the students start to work on the expression $ab + ac + bd$.

EMPIRICAL RESULTS: SHARING STRUCTURES THROUGH LANGUAGE

As a first step of a functional-pragmatic analysis, a transcript is divided into segments (letters) and sections (numbers); these segments and sections are heuristically refined during the analysis. The already refined segmented transcript is given here; the later analyses refer back to this transcript.

1	Tim:	[1a][*writes down expression ab+ac+bd that is given in the task*] [1b]Hm, [1c]do you want to do this?
2	Max:	[1d]Mhm [*confirming*]
3	Tim:	[1e]Ah, wait, |[2a][*starts to write down the transformed expression a(b+c)+bd*], [2b]I would do it this way, [2c]simply for the reason, [2d]because of course *a* is there two times [*points at the a's in the original expression*], [2e]thus *ab* and *c* [*points at a, b and c in the transformed expression*]
4	Interviewer:	[2f]Mhm [*confirming*]
5	Tim:	[2g]Just taken times [*german for multiplicating*]. [2h]And there [*points at bd*] just only the *b* times *d* is taken. |
6	Interviewer:	[3]Ok
7	Max:	|[4a]Yes, [4b]I believe that too, [5a]because only there is *a* [*points at ab and ac*], [5b]and there is no *a* in front [*points at bd*]
8	Interviewer:	[5c]Mhm
9	Max:	[5d]This is why one just has to put this in brackets [*points at b+d*] [5e]and this other one [*points at bd*] comes behind the brackets. |

The action potential and the organization of the discourse

The *organization of the discourse* that is established by Tim is oriented at Max and at the Interviewer. At first, Tim organizes his actions in relation to Max. In 1a, he writes down the task while at the same time addressing Max: "Do you want to do this?" (1c). On the one hand, Tim wants Max to participate in the solution of the task. On the other hand, he is delaying the discourse, so that he 'squeezes out' some time to think about the algebraic expression at hand. The invitation to participate is held up in 2b, where Tim explains: "I would do it this way" - this discursive action implies that there might be another way and that Max is expected to propose one. Later on, Max directly relates to Tim's transformation of the expression in 2a, confirming it (4a). With this he indicates, that the propositional core of his speaker actions are related to Tim's reasoning. In addition Max organizes his speaker actions in relation to Tim's actions (2b), by suggesting that *he* "believes that too" (4b).

The *action potential* revolves around transforming the expression in a way that is acceptable to the interviewer/teacher, while making themselves understandable to each other. As Tim's actions are directed at the interviewer, who is regarded as a knowledgeable teacher, he likely wants to give correct reasons and to use "mathematical" language (the literate register). Furthermore, Tim needs Max to understand and approve of his transformation, so that they together can arrive at a common solution. Thus, he has to make himself understandable to Max. On the other hand, the purpose of Max' actions is to confirm Tim's transformation of the expression. As shown above, the discourse is organized in a way that requires Max to give an individual perspective. Accordingly, Max has to relate to Tim's actions but also has to give his own perspective on the transformation.

The propositional content of Tim's linguistic actions

The basis of the *propositional content* is the written expression $a(b + c) + bd$ that Tim writes down in 2a. The following speaker actions all relate to this expression (depicted in Figure 1; relation diagrams are an element of a functional-pragmatic analysis). In his first actions, Tim explains the first part of the expression, namely $a(b + c)$, by saying "because of course *a* is there two times" (2d). The following speaker actions depend on this latter speaker action with "thus" and "just".

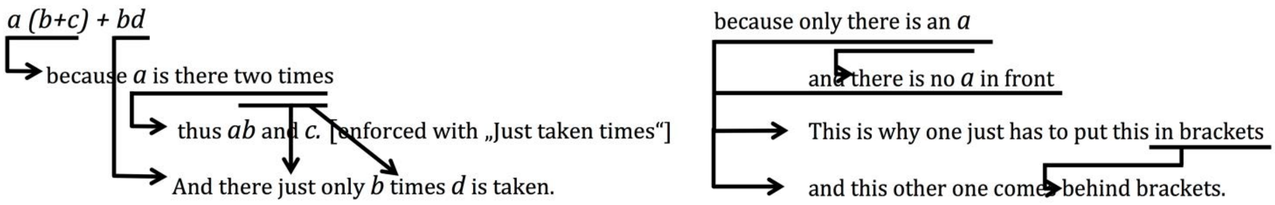

Figure 1: Reconstructed surface progressions of sections 2 and 5

"Thus" refers to the propositional content "*a* is there two times", while "just" is connected to the propositional content "*ab* and *c*" (2g). Tim gives two reasons for the expression $a(b + c)$ that build upon each other. In his later speaker actions, Tim also explains "*bd*". He is connecting this action with "and" to his previous actions, while at the same time he uses deictic gestures/ words ("there") to connect his action to the algebraic expression (2h). The expression "just only" stands in contrast to "two times". In this way, Tim expresses that in case of *bd*, there are no two variables. This is further indicated by using the phrase "taken times" (multiplication) for explaining both $a(b + c)$ and *bd*.

The algebraic expression in Tim's *linguistic reality* has two cornerstones. The first cornerstone is how often a certain variable is there. This is expressed by "*a* is there two times" and "only the *b* times *d*". The second cornerstone is the multiplication; on the one hand, "*ab* and *c* [...][are] just taken times", on the other, "*b* times *d* is taken". Both the first and the second part of the expression rest on this cornerstone. This way it connects the first and second part of the expression.

The relations of the cornerstones show how Tim structures the algebraic expression. "*ab* and *c*" is dependent on "two times". In other words, leaving out one *a* is a result of *a* being there two times, where leaving out *a* is explained by the multiplication. These relations are expressed by Tim as causal relations through "because", "thus" and "just". The second part of the expression *bd* is structured in the same way as the first part. This time, there is only a single variable *b*. The multiplication is not based on leaving out a variable. In summary, Tim structures the expression according to how many times a variable occurs, and how this results in a certain form of multiplication - one time by leaving out a variable (transformation of the first part) and one time by not changing the variables (transformation of the second part).

The propositional content of Max's linguistic actions

The basis of the *propositional content* of Max' linguistic actions is both the original expression and its transformation. In 5a, Max relates deictically to the original expression, pointing at the expressions *ab* and *ac* and referring to them with "there" ([5a]"because only there is a [*points at ab and ac*]"); in 5b, he refers the same way to *bd*. In both 5a and 5b, he argues for the existence/ absence of *a* in the subexpressions of the original expression. In 5d and 5e, Max deictically relates to the transformed expression, but at the same time builds an argument upon the perceived properties of the original expression (indicated by the arrow that encompasses both 5a and 5b in Figure 1), saying "this is why one just has to put this [*points at* (*b*+*d*)] in brackets" (5d). Max uses the same relations in structuring the second part of the expression (5e). Both parts of the expression, $(b + c)$ and *bd* are connected with "and" and by stating the position of *bd* in relation to $(b + d)$ ("behind the brackets").

The structure of the algebraic expression in Max' *linguistic reality* has only one cornerstone. This cornerstone is the (non-)existence of *aa* in the subexpressions of the original expression. However, Max' conclusions that are based on the original expression are, additionally, supported by features of the transformed expression. Thus, his conclusions are circular: Max' justification why one has to transform the original expression into $(b + d)$ requires that $(b + d)$ is already given as a transformation. In this way, Max' justification dissolves the logic *of the process* of transformation. In summary, in Max' notion of the distributive law $ab + ac = a(b + c)$ the two *a*'s on the left side directly result in the bracketed expression $(b + d)$.

Pattern of sharing algebraic structures in a discourse

At first sight, it seems that the linguistic realities of Tim and Max have one element in common, namely the existence of the variable *a*. However, on closer inspection, Max' reasoning is based upon the existence and non-existence of a in the subexpressions, while Tim's reasoning is based on how often a (or b) exists in the original expression. Thus, in the course of this short episode, Max has 'only' picked up one aspect of Tim's structuring of the expression.

At second sight, the linguistic realities of Max and Tim are both based on the original and the transformed expression. However, when comparing the propositional content of Tim's and Max' discursive actions, it becomes apparent that Tim's speaker actions depict a process, in the sense that they relate to his steps of the transformation: Each step relates the transformations to features of the original expression. In other words, the mathematical objects, on which he builds, are located in the original expression, while his deductions are about properties of the transformed expression.

In contrast, Max' speaker actions are abstracted from the process of transformation. While his arguments are based upon objects in the original expression, they require at the same time the existence of the transformed expression. Thus, Max' reasoning is based upon the equation as a whole, that was established by Tim and that links the original expression and the transformed expression. Max' structuring may be a result of the organization of the discourse. As shown above, the discourse requires Max to relate his structuring of the expression to Tim's transformation and reasoning behind this transformation. Perhaps, in the logic of the discourse, Max has no other option as to include the transformed expression into his reasoning.

In a broader perspective, the pattern behind Tim's and Max' negotiation of the structure of the expression can be described as a transformation gap. This transformation gap describes that the one discursive participant who comes up with a first transformation has different means to justify his structuring than the hearer, who can work with both the original expression and the proposed transformed expression. Accordingly, while the speaker needs to connect the original expression with the transformed expression in order to give reasons for his transformation, the hearer can base his reasoning on the original expression *together* with the transformed expression. As a result, at a given point in a discourse about algebraic structures there may exist two different structures of an algebraic expression: One structure that follows the logic of the transformation, and one that disconnects from this logic and focuses on the transformed expression together with the original expression. To the participants, these structures may seem compatible or even equal, while in reality these structures are very different.

DISCUSSION

Although just one case has been discussed in this paper, the reconstructed pattern of conveying and adopting the structures of algebraic expressions in a discourse may yield significant consequences. For example, teaching interventions that focus on promoting students' structure seen by implementing activities of negotiating different structures of algebraic expressions have to account for the transformation gap. One way to do this could be to connect algebraic expressions more strongly to other mathematical objects like functions, so that the different ways of structuring an expression come into light in reference to this object. The possibility of translating the algebraic expression into another representation may further act as a scaffold.

The functional-pragmatic analysis has proven its potential to look into the foundations of the students' ways of conveying and adopting structures in algebraic expressions. The reconstruction of the surface process, resulting in the reconstructed linguistic realities of speaker and hearer, as well as the reconstruction of the pattern behind this, may also be suitable for addressing other research questions, where "deficiency [is transformed] into sufficiency" in students' discourses.

In further studies, the generality of the here presented pattern has to be addressed. It has to be addressed, if it is part of a larger structure-sharing-pattern that is typical for discourses about algebraic expressions. A more systematic view on different discourses of sharing structures may show, if such a pattern is common to discourses about algebraic structures. This might also lead to a deeper understanding of students' resources in regard to structure seeing and of coordinating transformations.

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The discursive routine of personifying and its manifestation by two instructors

Talli Nachlieli[1] and Michal Tabach[2]

1 Levinsky College of Education, Tel Aviv, Israel, talli.nachlieli@gmail.com

2 Tel Aviv University, Tel Aviv, Israel

This paper focuses on teaching processes in the mathematics classroom. We adopt Sfard's communicational framework to address the question: What were the teaching actions that seem to promote learning? To address the question, we analyzed four lessons taken from two courses about functions which were studied by elementary-school prospective mathematics teachers. We identified a teaching mechanism that seems to relate teaching to students' learning. This mechanism includes flexible and dynamic transition between mathematizing (talk about mathematical objects), subjectifying (talk about the participants) and personifying (talk about human participants acting upon mathematical objects). We also identified several types of personifying talk.

Keywords: Classroom communication, personifying, teaching routine.

INTRODUCTION

This paper focuses on teaching processes in a mathematics classroom. Teaching and learning processes are interwoven, both enacted concurrently by different participants. Studying both under a similar lens could provide holistic insights about classroom processes. This study is a part of a larger study about teaching and learning, however, here we mainly focus on teaching. Let us look at the following two turns, uttered by an instructor:

> Please look at the board. Ok, here we have a graph that describes a connection between two variables. The question is, what type of questions we could answer using this graph and which questions we cannot answer by this graph.

> The *y*, yes, it is always *y* that changes according to *x*, ok? We have y that changes depending on, or according to, *x*, ok?

It is obvious that both excerpts are from a mathematics lesson. This certainty stems from the words used (graph, variables), and the articulation (*y* changes according to *x*). However *look at the board* is a general classroom saying about participants' actions, not specifically related to mathematics.

Indeed, during mathematics lessons we could expect the participants to talk about the participants of the discourse (about the students and teacher, as in: *take out your books*, or *could you please write it on the board?*), or about mathematical objects (*this is a graph of a constant function*). Looking back at the above excerpts, we find that some parts are compatible with this division: *Please look at the board* is about human participants; whereas *it is always y that changes according to x* is about mathematical objects. However, we have other sayings that do not fall under those two categories. Rather, they are at the intersection between the two sets, and relate to both –human participants and mathematical objects: *here* ***we*** *have a* ***graph*** *that describes…* We suggest that this type of teacher's interwoven talk about both - participants and the mathematics is important for students' learning.

To learn more about this phenomenon, we closely examined 4 lessons out of two cohorts of data: transcribed video-recordings of two courses about functions that were studied by elementary-school prospective mathematics teachers. The question we asked: *What were the teaching actions that seem to promote students' learning?*

TEACHING PRACTICES AND ROUTINES

Sfard (2008) has defined the teaching-learning agreement, which is:

> a situation that arises when the discursants are unanimous, if only tacitly, about at least three basic aspects of the communicational process: about which is the leading discourse, about the discursants' own respective roles as those who learn or those who teach, and about the nature of the expected change. (Sfard, 2008, p. 299).

From this we (Heyd-Metzuyanim, Tabach, & Nachlieli, submitted) derive that a pedagogue (teacher or instructor) is a person who assumes the role of the leading participant in the discourse while the student is a person who assumes the position of the follower in the discourse. Indeed, these definitions fit the Greek etymology of pedagogy which is *a person who leads the child*.

In that sense, pedagogy is a particular form of communication, and communication, in turn, is a particular type of activity (Sfard, 2008, p. 296). Combining this view with Cultural Historical Activity Theory (Engeström, 1987; Roth & Lee, 2007), one realizes that the main distinction that can characterize a particular activity as pedagogical is its motive. Leont'ev, Roth and Radford (2011) write: "The chief difference in activities is to be found in the difference of their objects or motives. An object/motive (fishing, for instance) is what endows the activity with a particular intent. But activities involve also actions and specific contextual methods and means to carry out these operations" (p. 6).

Combining activity theory with the communicational definitions, we arrive at the conclusion that pedagogy can be defined as the communicational activity which motive is to bring a change in the learners' discourse towards a leading discourse. Such a definition of pedagogy includes all communicational actions (verbal, non-verbal, emotional, etc.).

Contrary to the common division between content and pedagogy (see Leinhardt & Steele, 2005) we offer an alternative lens – that of dividing pedagogical activity into subjectifying (talking about human participants) and mathematizing (talking about mathematics). Subjectifying is thus a communicational activity which motive is to produce narratives about people while mathematizing is a communicational activity which motive is to produce mathematical narratives. Since pedagogy includes both mathematizing and subjectifying, it becomes obvious that different (and something conflicting) motives can be enacted within the same activity.

In this paper, the instructors of the course are the pedagogues and the teaching actions are the pedagogy. We focus on specific teaching-routines (exchange routines, Leinhardt & Steele, 2005), that seem to promote students' learning. For that purpose we focus on the instructor's talk during whole-class discussions.

RESEARCH QUESTION AND PARTICIPANTS

The research question of the study reported in this paper is the following: *What were the teaching actions that seem to promote students' learning?*

The data for this paper are taken from a project that focuses on identifying learning and teaching processes. The data for the project include transcribed video recordings of two mandatory first-year courses about functions for elementary-school prospective teachers studying at a college of education in Israel. The two courses were similar in their content and goals, and were taught in consecutive years, by two different instructors. Each course included 18 students. All of the students have already learned functions and graphs in high school algebra. The students were all older than 19. They vary in age and background, as well as in the level of mathematics that they studied in high school. The two course instructors (Ellice and Talli), hold a PhD in mathematics education and have been teaching at the college for over ten years.

All lessons had a similar structure: opened by a whole-class discussion, followed by group-work (of 3–4 students). The students then reconvened for a summarizing whole-class discussion. The groups were formed by the students and remained stable throughout the course. All 12 lessons of each of the course were videotaped and transcribed. During the lessons, field notes were taken and students' written work was collected. During group-work, the work of two specific groups, chosen randomly, was recorded and transcribed.

For the study reported in this paper we chose to closely examine 4 lessons out of the 24, 90 minutes lessons, the first and last in each course (We chose lesson 2 rather than lesson 1, as during the 1st lesson consent forms were distributed, and the study was explained.) The choice to focus on two different courses did not originate in intent to compare the courses, but to allow us to examine whether a certain identified routine is teacher-specific or could be a part of common teaching routines.

OVERVIEW OF ELLICE'S LESSONS

In the 2nd lesson, the students were asked to solve an algebraic task (see Figure 1).

The following is a sequence of the three first structures in a series of structures. How many cubes are needed to build the nth structure?

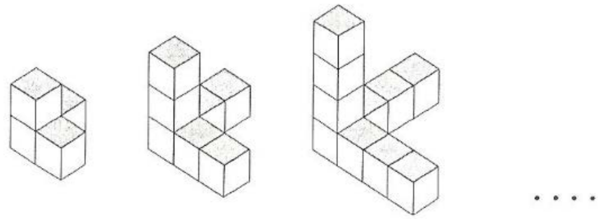

Figure 1: The sequence discussed in class during Ellice's 2nd lesson

While working in small groups, the students were given small cubes to build the structures and answer a sequence of sub-questions aimed at guiding them towards the generalization of the pattern. They then reconvened to discuss the task.

As early as this 2nd lesson, students talked about various representations of the mathematical task: drawing, relation, formula, reversed formula and pattern. These words, although relating to representations of mathematical objects are not the common terms used. A formula and reversed formula were suggested in the form of algebraic expressions (;).

The task given in Ellice's 12th lesson was taken from Friedlander and Tabach (2001). This task included 4 situations each described by a different representation (story, table, graph and algebraic expression). The task included a set of sub-questions that guided the students to first relate to each of the representations alone and then connect between them.

The task given in the 12th lesson was more complex in two aspects: (1) the number of situations described, and (2) the number and type of representations provided. The task in the 2nd lesson included a drawing and tangible representations of one sequence only, whereas that in the 12th lesson included 4 situations described by 4 different representations (story, table, graph and algebraic expression). The students were asked to compare the phenomena. That is, there was a development in terms of the complexity of the given task. The tasks chosen by Ellice in both lessons included various sub-questions that helped students focus on specific representations and relations amongst them.

OVERVIEW OF TALLI'S LESSONS

The task provided in Talli's 2nd lesson included 5 different graphs and 5 verbal situations (e.g. *The amount of fuel in the tank changes according to the distance that the car rides.*) The students were asked to connect between them and support or refute claims by relating to data provided by the graphs.

The task given in Talli's 12th lesson was:

> In your summer job as a salesman, you could choose one of three earning methods: (1) 100 NIS for each working day and 35 NIS for each sale, (2) 80 NIS for each working day and 40 NIS for each sale, (3) 250 NIS for each working day and 5 NIS for each sale. Which would you choose? Explain your choice.

This task includes a verbal description of 3 situations that could be modeled by a function. The students were asked to compare between them. The task did not include reference to any representations. Also, it did not include sub-questions that could guide the students in their attempts to solve the task. This is in contrast with the task given in the 2nd lesson which included a graphic mediator and guiding questions. Also, the task given in lesson 12 asked for comparing alternatives whereas the task in the 2nd lesson asked for describing a situation provided by a graph.

METHOD OF ANALYSIS

To address the research question, we analyzed the entire whole-class discussion in each of the lessons. We focused on the instructors' talk in these discussions. We segmented the instructor's turns into clauses which were then analyzed. Thoroughly examining all the clauses uttered by the instructors, we noticed that they talked either about mathematical objects (mathematizing – y changes according to x) or about the classroom participants (subjectifying – look at the board). However, we identified a third type of communication that is a sub-type of both mathematizing (because it speaks about math objects) and of subjectifying (because it speaks about persons). We call this last type – personifying, and define it as *non-alienated talk about mathematical objects*. Therefore, to analyze the data we categorized the clauses of the instructors' talk according to three categories: mathematize, subjectify and personify. Chi square test was used to determine whether differences are significant.

We further identify types of personification according to what we call *the distance* between the human actor and the mathematical object: one could act upon an object directly (e.g. *what can you say about the interception point?*), could describe her actions upon the object (e.g. *how do we know that the claim is true?*) or talk about how others would or should act upon the objects (e. g. *how do you think a 7th grade student would solve this problem?*). We name these categories as distance 0, 1 or 2.

FINDINGS

In this section we characterize instructors' talk to learn about how the instructor's choice of words promoted students' learning. We refer to three categories, (1) mathematizing, (2) subjectifying, and (3) personifying. Table 1 presents the number of clauses used by each instructor during the whole class discussions which were analyzed, and the (number) and percentages of clauses in each category. This table includes all the analyzed data that is discussed next. From Table 1 we see that both instructors used all three types of talk in each lesson. However, the distribution amongst the three types is significantly different (chi-square test, $p<0.002$).

Moreover, as can be seen from Table 2, in both of Ellice's lessons, the number of personifying clauses

	N	Math	Subject	Person
Ellice lesson 2	316	(53) 17%	(69) 22%	(194) 61%
Ellice lesson 12	157	(48) 31%	(37) 23%	(72) 46%
Talli lesson 2	219	(96) 44%	(28) 13%	(95) 43%
Talli lesson 12	220	(52) 24%	(8) 4%	(160) 73%

Table 1: Categorizing teachers' talk: mathematizing, subjectifying and personifying

(cls.) is significantly higher than that of mathematizing or subjectifying, while there is no significant difference between the number of mathematizing cls. or subjectifying cls. In Talli's 2nd lesson, subjectifying was significantly lower than personifying or mathematizing, while there was no significant difference between mathematizing and personifying. In Talli's 12th lesson, personifying was significantly higher than mathematizing and subjectifying, and mathematizing was significantly higher than subjectifying. In each of Talli's lessons there was significantly more talk about mathematics objects than about participants.

Next, we refer to changes between the two lessons for each instructor. Table 3 presents whether the changes in instructor's talk between her two lessons are significant (chi-square). We find that in Ellice's lessons, mathematizing is significantly higher in lesson 12 than in lesson 2 ($p<0.001$) and personifying is significantly lower ($p<0.001$). In Talli's lessons, subjectifying and mathematizing are significantly lower in lesson 12 than in lesson 2 ($p<0.001$). At the same time, personifying is significantly higher ($p<0.001$).

There is no clear tendency between the two instructors as for the differences in the three types of talk. This will be referred to in the discussion.

Inst.	Lesson	Personifying cls., Mathematizing cls., Chi-square value	Personifying cls. , Subjectifying cls., Chi-square value	Mathematizing cls., Subjectifying cls., Chi-square value
Ellice	2	194, 53, 80.8***	194, 69, 59.2***	53, 69, 2.1
	12	72, 48, 4.8*	72, 37, 11.1***	48, 37, 1.42
Talli	2	95, 96, 0.005	95, 28, 41.7***	96, 28, 41.7***
	12	160, 52, 55.0***	160, 8, 138.0***	52, 8, 32.3***

Chi square test with df=2. *$p<0.05$ **$p<0.01$ ***$p<0.001$

Table 2: Chi-square test for comparing the type of talk within each lesson

Inst.	Mathematizing cls.	Subjectifying cls.	Personifying cls.
	Les. 2, Les. 12, Chi-square value	Les. 2, Les. 12, Chi-square value	Les. 2, Les. 12, Chi-square value
Ellice	53, 48, 11.9***	69, 37, 0.181	194, 72, 10.3***
Talli	96, 52, 20.0***	28, 8, 12.2***	95, 160, 38.8***

Chi square test with df=1. *p<0.05 **p<0.01 ***p<0.001

Table 3: Chi-square test for comparing lesson 2 and lesson 12 for the same teacher

INSTRUCTORS' PERSONIFYING TALK

We identify three types of personifying talk according to the distance between the human actor and the mathematical object. Table 4 presents the number of personifying clauses uttered by each instructor during the whole class discussion and the (number) and percentages of clauses in each category. From Table 4 we see that both instructors used all three types of personifying talk in each lesson. However, the distribution amongst the three types is significantly different (chi-square test, p<0.001). Moreover, as can be seen from Table 5, in both of Ellice's lessons, distance-0 personifying is significantly higher than distance-1 or distance-2, while there is no significant difference between distance-1 and distance-2. In Talli's 2nd lesson, distance-2 personifying was significantly lower than the other distances. In Talli's 12th lesson, distance-0 personifying was significantly higher than distance-1.

Next, we refer to changes between the two lessons for each instructor. Table 6 presents whether the changes in the number of the instructor's personifying cls. between her two lessons are significant (chi-square). We find that in Ellice's lessons, distance 0 personifying is significantly higher in lesson 12 than in lesson 2 (p<0.001), while distances 1 and 2 are significantly lower (p<0.05). In Talli's lessons, the personifying talk remains unchanged related to the three distances.

DISCUSSION

This study is a part of a large-scale study in which teaching and learning processes were analyzed. Here we only refer to the teaching actions and specifically –

Inst.	Lesson	N	Distance-0	Distance-1	Distance-2
Ellice	2	194	(91) 47%	(56) 29%	(47) 24%
Ellice	12	72	(52) 72%	(12) 17%	(8) 11%
Talli	2	95	(55) 58%	(40) 42%	(1) 1%
Talli	12	160	(94) 59%	(64) 40%	(2) 1%

Table 4: Categorizing teachers' personifying cls. according to the distance between the human actor and the mathematical object

Inst.	Les.	Dist. 0, Dist. 1, Chi-square value	Dist. 0, Dist. 2, Chi-square value	Dist. 1, Dist. 2, Chi-square value
Ellice	2	91, 56, 8.33**	91, 47, 14.0***	56, 47, 0.876
Ellice	12	52, 12, 25.0***	52, 8, 32.3***	12, 8, 0.8
Talli	2	55, 40, 0.124	55, 1, 52.1***	40, 1, 37.1***
Talli	12	94, 64, 5.7*	94, 2, 88.2***	64, 2, 58.2***

Chi square test with df=2. *p<0.05 **p<0.01 ***p<0.001

Table 5: Chi-square test for comparing the type of personifying cls. within each lesson

Inst.	Distance 0	Distance 1	Distance 2
	Lesson 2, Lesson 12, Chi-square value	Lesson 2, Lesson 12, Chi-square value	Lesson 2, Lesson 12, Chi-square value
Ellice	91, 52, 13.5***	56, 12, 4.11*	47, 8, 5.51*
Talli	55, 94, 0.018	40, 64, 0.109	1, 2, 0.02

Chi square test with df=2 *p<0.05 **p<0.01 ***p<0.001

Table 6: Chi-square test for comparing lesson 2 and lesson 12 for the same instructor

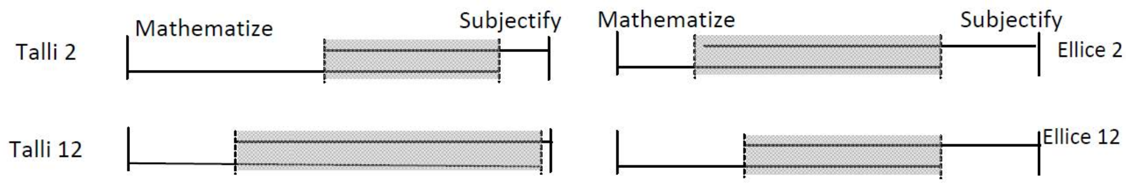

Figure 2: Distribution of mathematizing, personifying (in grey) and subjectifying talk

to teachers' talk due to space limitations. Findings from this study point to phenomena that suggest discursive mechanisms that could connect teaching and learning processes in general.

It seems reasonable to expect that during a lesson in mathematics an instructor would talk either about the participants (e.g., *open your books*, *work in groups*) or about the mathematical objects (e.g., *the function is positive for x<4*). However, from this study it is apparent that there is a third type of talk – about human participants acting upon mathematical objects (e.g., *when we move on the graph*), and that this talk is prevalent. Figure 2 presents personifying as the overlapping between mathematizing and subjectifying.

We did not find a similar pattern in the two courses regarding personifying. In Ellice's lessons, personification was used vastly in lesson 2 whereas in Talli's – in lesson 12. To make sense of this finding, we examined each of the lessons.

In Ellice's 2nd lesson there were several students' sayings that we could relate to Ellice's choice to personify. One is a student's publicly uttered saying of *I got lost*. Several other students joined this statement. Another, is the fact that while working on the task in small groups, at least 2 groups could not find the expression that mediated the given sequence, and therefore felt that they cannot solve the task. That is, the instructor faced a situation in which she had to make the mathematics accessible to her students. The way of doing it included personifying, in general and using distance-2 personifying, in particular. Personifying has the potential to make the mathematics more accessible to the learner as it includes talk about human who acts on mathematical objects. The distance-2 personifying includes others' actions on objects. This could protect the learner as it places someone else in the front, facing the mathematical objects. With that, it provides the learner with a role of judging someone else's actions on the mathematical objects. That is, it has the potential to both empower the student and protect her. Such expressions from the students were not uttered in Ellice's 12th lesson. This is coherent with our finding that Ellice's mathematizing was significantly higher in lesson 12, and her personifying was significantly lower. Likewise, distance-2 personifying was significantly lower in lesson 12.

In Talli's 12th lesson, in which personifying talk was significantly higher, we could not find explicit sayings of students' frustration. However, the students' initial responses to the task given in lesson 12 were local and surprised the instructor at this late stage of the course. Therefore, the instructor had to find a way to help students identify a reasonable strategy with which they could address the task. Thus, make the relevant mathematics accessible to them.

That is, we suggest that the personifying routine enacted by the instructor is strongly related to students' talk, and aims at enabling them to participate in the mathematics discourse by making this discourse more accessible to them.

Earlier, we stated that subjectifying is a communicational activity which motive is to produce narratives about people while mathematizing is a communicational activity which motive is to produce mathematical narratives. What, then, is the motive for personifying talk as a communicational activity? It seems that the motive is related to the activity of participation. That is, it seems that this is a communicational action whose motive is to produce mathematical narratives by first promoting students' participation and engagement in the mathematical discourse.

To substantiate conclusions from this work, larger-scale studies are needed, which will focus on relations between teaching and learning in general, seeking teaching-routines that promote learning. Also, they will focus on the specific discursive-routine of personifying aiming to define the when and the how of this routine.

ACKNOWLEDGEMENT

This study is supported by the Israel Science Foundation, no. 446/10.

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Bilingual mathematics learners, conceptual mathematical activity and the role of their languages. How best to investigate?

Máire Ní Ríordáin[1] and Aisling McCluskey[2]

1 National University of Ireland, School of Education, Galway, Ireland, maire.niriordain@nuigalway.ie

2 National University of Ireland, School of Mathematics, Statistics and Applied Mathematics, Galway, Ireland

The significant role of language in mathematics teaching and learning is not a new phenomenon. Investigating bilingual mathematics learners is complex and research has demonstrated that language switching practices are also complex and involve not only social and cultural aspects, but also cognitive aspects. However, little investigation has been undertaken into the specific role of languages and their influences on conceptual activity at undergraduate level. The framework, and future research directions, presented in this paper aim to investigate further the cognitive aspects of bilingual learners and their use of their languages, when engaged in conceptual mathematical activity.

Keywords: Language, conceptual mathematical activity, bilingualism, framework.

INTRODUCTION

In this paper we will present a theoretical framework to support empirical research investigating bilingual students' use of their two languages as they engage in conceptual mathematical activity at undergraduate level. The need for such a framework has arisen from the authors' previous research findings and development of new research directions within the area of language and mathematics education. Comparisons of the Irish and English languages demonstrates that there are differences between the languages in terms of syntax and semantics and that this may impact on the processing of mathematical text, and advantage those learning through the medium of Irish. However, what is difficult to conclude, without further investigation, is whether differences between the languages, and when/how they are used, have a differential impact upon cognitive processing (Ní Ríordáin, 2013).

Previous studies in the Irish context demonstrate that a significant relationship exists between performance on mathematical word problems and language proficiency, with bilingual students with high proficiency in both languages performing better mathematically (Ní Ríordáin & O'Donoghue 2009). In particular, high proficiency in Irish had a strong correlation with performance in mathematics (through the medium of English) for students in the transition from Irish-medium primary to English-medium second level mathematics education. National testing in mathematics and English at primary level reveals that students in Irish-medium primary education perform the same or better than students in all-English medium education in both mathematics and English (Gilleece, Shiel, Clerkin, & Millar, 2011). Similarly, at third level education, when examining high-ability bilingual students, it was found that some students found it easier to undertake operations and to process ideas in Irish (as opposed to English) and displayed greater comprehension of the mathematics problems, an ability to self-correct, to select appropriate features in the problem and displayed knowledge of their strategies (Ní Ríordáin & McCluskey, 2012).

The significant role of language in mathematics teaching and learning is not a new phenomenon. Given the marked growth of cultural migration, the focus on education for economic development and the emphasis on English as a language for learning, we have become acutely aware of the importance of recognising the significance of language in learning mathematics (Barwell, Barton, & Setati, 2007). However, little investigation has been undertaken in relation to the specific role of learners' different languages when engaged in mathematical learning. There has been a focus more on the social, rather than cognitive functions of code switching/use of languages. In particular, there is a need for the development of a coherent and integrated interpretive framework for investigating whether differences in languages, and their use, by bilingual mathematical learners have a differential impact upon cognitive mathematical processing, while recognising the social aspects of learning. Fundamental to this is the commognitive approach for the study of mathematical learning by Sfard (2008, 2012).

SFARD'S COMMOGNITIVE APPROACH

Sfard's (2008) interpretive framework for examining learning is founded on the premise that thinking is a form of (interpersonal) communication, and that learning mathematics entails extending one's discourse. If assuming the premise that mathematical learning involves initiation into the discourses of mathematics, then learning mathematics involves substantive discursive changes for learners (Sfard, 2008). Accordingly, mathematics teaching involves facilitating such changes. Sfard also emphasises that communication, and being part of a community, is central to facilitating such teaching and learning activities. A discourse is distinctive in terms of a community's practices in relation to *word use*, *visual mediators*, *endorsed narratives* and *routines* (Sfard, 2008, pp. 133–135). Sfard (2012, p. 3) distinguishes between two types of mathematical learning (change in discourse): *object-level learning* (expansion of what is known already, mainly accumulative) and *meta-level learning* (change of meta-discursive rules, more radical and complex kind of change).

Overall, the commognitive framework 'provides a unified set of conceptual tools with which to investigate cognitive, affective and social aspects of mathematics learning.' (Sfard, 2012, p. 1). A key purpose is to help make sense of classroom processes, while being responsive to the intricate nature of complex data generated in a teaching and learning setting. What we aim to do in this paper is to build on this approach within a bilingual university mathematics education context and to potentially assist towards the reification of the framework into tools that can help analyses within such a context.

A FRAMEWORK FOR THE STUDY OF BILINGUAL LEARNERS

The following sections firstly provide an overview of key perspectives and aspects of consideration informing the proposed framework. Key principles of the framework are then presented, taking a commognitive standpoint (Sfard, 2008). The approach, proposed in this paper, is the result of a number of research studies undertaken in the Irish context to comprehend the intricacies of bilingual mathematical learning.

Perspective on discourse

Given the central aspect of discourse to the commognitive approach it is important that we outline our perspective of discourse. We see mathematics as a discourse and a type of communication (Sfard, 2012). Discourse is more than just language. As defined by Gee (1996, p. 131):

> A Discourse is a socially accepted association among ways of using language, other symbolic expressions, and 'artifacts,' of thinking, feeling, believing, valuing and acting that can be used to identify oneself as a member of a socially meaningful group or 'social network,' or to signal (that one is playing) a socially meaningful role.

By employing this definition, Discourses are more than verbal and written language and the use of technical language; Discourses also involve communities, points of view, beliefs and values, and pieces of work. Moschkovich (2012, p. 95) utilises the phrase 'mathematics Discourse practices' to draw attention to the fact that Discourses are embedded in sociocultural practices as they evolve from and involve participation in communities, while also cognitive as they involve thinking, signs, tools and meanings. This concept of Discourse will inform the examination of conceptual mathematical development of bilingual learners, linking both the cognitive and social aspects of use of languages.

Perspective on bilingualism

Our work is with bilingual (Irish and English) mathematics learners and it is essential to incorporate this concept into the proposed framework. Defining bilingualism is difficult, in particular defining whether a person is bilingual or not. Definitions vary between political, social and cognitive perspectives. Grosjean and Moser-Mercer (1997) developed the notion of a "complementarity principle" in which they emphasise that bilinguals use their languages for different purposes and in different domains of life. Dominance in one language over the other is common among bilinguals depending on the use and function of each language. Also, studies involving bilinguals tend to focus on only one language, but due to the complex nature of the issue of bilingualism, aspects of both languages need to be taken into account. For the purpose of the development of a framework for the investigation of bilingual mathematical learners, we utilise Grosjean's (1999) concept of a continuum of modes with monolingual and bilingual at each endpoint. Therefore, by utilising the concept of a continuum of modes (monolingual to bilingual), it facilitates an understanding of bilinguals using their languages independently and together depending on the context/purpose. This is further supported by Cummins' (1980) Common Underlying Proficiency (CUP) model of bilingualism, which is a more apt description of language construction within the mind. Outwardly both languages are different in conversation. However, internally, both languages are merged so that they do not function independently of one another, with a central processing unit that both languages contribute to, access and use. We support a non-deficit view of bilingual learners and view language(s) as a resource and a support for learning.

Language and mathematics

In our research we are primarily concerned with the role of bilingual students' languages in mathematics teaching and learning. We consider mathematical language as a distinct 'register' within a natural language, e.g., Irish or English or French, which is described as "a set of meanings that is appropriate to a particular function of language, together with the words and structures which express these meanings" (Halliday, 1975, p. 65). One aspect of the mathematics register consists of the special vocabulary used in mathematics and it is the language specific to a particular situation type (Gibbs & Orton, 1994). However, the mathematics register is more than just vocabulary and technical terms. It also contains words, phrases and methods of arguing within a given situation, conveyed through the use of natural language (Pimm, 1987). The grammar and vocabulary of the specialist language are not a matter of style but rather methods for expressing very diverse things. Therefore, each language will have its own distinct mathematics register, encompassing ways in which mathematical meaning is expressed in that language. The process of learning mathematics inevitably involves the mastery of the mathematics register (Setati, 2005). Developing a learner's mathematical register provides them with analytical, descriptive and problem solving skills within a language and the communicative competence necessary for successful participation in mathematical discourses.

Conceptual mathematical activity

Given that our research is concerned with undergraduate bilingual students, it is essential to examine the mathematics register and discourse development at this level. The nature of cognitive growth in the development of university-level mathematical thinking has borne considerable scrutiny over the past half-century (Thurston, 1990; Asiala et al., 1997). The associated literature presents a strong case that the maturing over time of the mathematical thinking of professional mathematicians is such that mathematical concepts become distilled and perfectly understood - by them. As a result of such sustained processing, all forms of former struggle or lack of understanding are potentially removed from memory. Thurston (1990) captures eloquently the power and satisfaction of arriving at deep understanding after struggle. In the process, he suggests that "once you really understand it and have the mental perspective to see it as a whole... You can file it away, recall it quickly and completely when you need it, and use it as just one step in some other mental process" (Thurston, 1990, p. 847). At its core, these authors refer to what has been identified and named by Tall and Vinner (1981) as *concept image* for any given mathematical concept such as, for example, function or limit.

It proposes a genetic decomposition for a given mathematical concept. This encompasses knowing what it means to understand such a concept and knowing how such an understanding can be constructed by a student, thus providing a model of cognition for the concept. Such a model plays a key role also in alternative cognitive theories for mathematical thinking and learning such as those due to Tall and Vinner (1981) where it is referred to as *concept image.* A genetic decomposition attempts to identify the layers of meaning, robustness and accuracy that arise as a particular concept is revisited in a variety of contexts. Thus, for example, a student may initially recognize a function only when given a specific (single) formula to compute values. The student restricted to this level will have a narrow range of comprehension. A deeper level will allow the student to appreciate and manipulate the notion that a function can receive inputs, operate on them and return outputs, thus giving greater flexibility. Encapsulation into an object occurs when the student recognizes the process as a whole, namely that a function is a rule between two sets of values with certain properties. Once constructed, objects and processes can be interconnected in various ways. Given that language influences thought and thinking, and that each language will have its way of constructing the concept, insight into the role and effect of bilingualism/languages on conceptual mathematical learning is critical.

Language and learning

Developing a student's mathematics register and participation in discourse is facilitated by language(s). Language is an essential instrument of thought and is necessary for understanding and combining experiences, and is required for organising concepts. The general consensus in cognitive science is to presume that thinking is occurring in some language (Sierpinska, 1994). Vygotsky was one of the earliest theorists to begin researching the area of learning and its association with language. He concluded that language is inextricably linked with thought – "the concept does not attain to individual and independent life until it had found a distinct linguistic embodiment" (Vygotsky, 1962, p. 4). Although a thought comes to life in external speech, in inner speech energy is focused on words to facilitate the generation of a thought. If this is the case, it raises an important question – does the nature of the language used affect the nature of the thought processes themselves? The transition from thought to language is complex as thought has its own structure. Thought is mediated both externally by signs and internally by word meanings (Vygotsky, 1962). It is the use of language as an instrument of thinking that is of importance, as well as its effect on cognitive processing. Therefore, thought is intimately linked with language and ultimately conforms to it. The linguistic relativity hypothesis proposes that the vocabulary and phraseology of a particular language influences the perceptions and thinking of speakers of that language (Whorf, 1956). Accordingly, each language (e.g. Irish or English) will have a different cognitive system that will influence concept formation and development. We support the premise that a language influences our mathematical thinking, but not necessarily to a degree that it determines our entire mathematical thinking (Sternberg, 2003). We propose that there are differences 'between linguistically distinct versions of "the same discourse".' (Kim, Ferrini-Mundy, & Sfard, 2012, p. 2) which correspondingly impact on mathematical learning.

Learning mathematics

Mathematics and learning is arbitrated through mathematical discourse practices, spoken and written language, symbols, gestures, etc. (Forman, 1996). Learning is situated within and involves participation in a community. Within a mathematics classroom, learning involves participation in the discipline of mathematics, in conjunction with the specific type of mathematics associated with the context (e.g. school mathematics, undergraduate mathematics, etc.) (Forman, 1996). When examining bilingual mathematics learners, it is important to address the social use of language within the context, not just its role in cognition. Moschkovich (2012) emphasises the importance of learning being illustrated within the sociocultural practices of a given setting. Importance is placed on describing learners and communities, and seeing culture as a set of practices and actively involving participants (Gutiérrez & Rogoff, 2003). Accordingly, bilingualism is described in terms of participation and use of language(s) by learners for different purposes and particularly in the context of mathematical discourse.

Effective teaching and learning is a complex endeavour. In correlation with a teacher's strategies, the teacher's own philosophical beliefs of instruction are harbored and governed by the student's background knowledge and experience, situation, and environment, as well as the learning goals set by the student and teacher. Moschkovich (2012) emphasises the importance of discerning between the *conditions of* learning and the *processes for* learning, and the importance of describing the curriculum, courses/programmes and teaching and learning approaches utilised that yield successful outcomes for different groups of learners. Therefore, it is important to examine and report on the characteristics of the learning environment such as whether there are opportunities for: speaking, listening, reading and writing; constructing meaning and knowledge; high expectations for all students; rejection of a deficit view of learners (AERA, 2006; García & González, 1995). In particular, we adopt a non-deficit perspective of bilingual learners and focus on the strategies that teachers use in developing conceptual mathematical learning.

Key principles of the proposed framework

Adopting a commognitive approach to research, combined with key concepts discussed in the previous sections, give rise to several key principles and methodological considerations for investigating undergraduate bilingual mathematics learners and their use of their languages. Primarily, the authors' framework is underpinned by a non-deficit view of bilingual learners where languages are viewed as a resource and essential for thinking. Within the framework, thinking can be defined as the activity of communicating with oneself (Sfard, 2012). Accordingly, mathematical thinking can be viewed as a Discourse, which in turn is a form of communication and involves being part of a mathematical community. Taking this view, the language or language(s) in which mathematics is being learned becomes an important issue for consideration. Within the framework, development refers to a change in Discourses (Sfard, 2012). Accordingly, we refer to the development of a student's mathematical Discourse as opposed to the development of the student themselves. Development of Discourses is a product of collective human actions and the context acknowledged. Given that the authors are primarily concerned with conceptual mathematical learning/activity, we are concerned with meta-level developments in Discourses. Since our focus is on bilingual mathematics learners, it is important that an analysis of the language(s) in which the discourse is taking place is conducted. The successive meta-discourses relating to topics of interest, for example functions, need to be documented and compared between languages.

By adopting a commognitive approach (Sfard, 2012), there are a number of key principles that need to be adhered to and which have been adapted to reflect our framework. Firstly, *Operationality*: the purpose of the research is to share useful stories. Therefore, it is important that the researcher's articulation avoids misunderstandings and is unambiguous and clear (p. 9). Second is *Completeness:* the researcher must choose the entire discourse related to the topic as the unit of analysis (p.9). Here, we add to the principle in that when examining bilinguals, we must document this discourse (plausible developmental trajectories) in both languages e.g. the discourse relating to 'limits' or 'functions' in both English and Irish. It should involve an analysis of successive meta-discourses in each language. Third is *Contextuality*: any kind of interaction is an event of learning (p. 9). It is essential that the researcher documents the interactions as fully as possible and analyses utterances within the context of the conversation. We extend this, in the given context, to the need to examine when and how bilingual students/researchers use their language(s). The next principle is that of *Alternating Perspectives*: when analysing data, the researcher alternates between being an insider and an outsider of their own ways of using words (p. 9). This is heightened within a bilingual context in that consideration must be given to both languages, their use in the given context and possibility of significant differences between researcher and participant discourses. Finally, the principle of *Directness*: when describing their study, the researcher presents things said (and done) by the participant first, not their own interpretation of the data (p. 9). It is hoped that these over-arching concepts and principles will foster insights into bilingual mathematics learning and contribute to the development of research.

CONCLUSION AND FUTURE RESEARCH

At the National University of Ireland (NUI) Galway, students have an option to study Mathematics through a bilingual approach (Irish and English) during their first year of undergraduate education. This provides an opportunity to investigate language choices made by undergraduate students and to identify how these choices impact on conceptual mathematical activity. We propose to address the following research questions via an investigation that is supported by the proposed framework.

— In what ways do formal discourses in English and Irish on, for example 'limits', follow different developmental trajectories in undergraduate mathematics education? 'Limits' are chosen as an example given its ubiquitous nature; for example, limits of sequences, of functions at points and at infinity, summation of series, derivatives and integrals. Developmental trajectories refer to identifying all of the discourses related to 'limits' that an Irish-speaking and English-speaking person is likely to encounter (Sfard, 2012). This can be in everyday life or specifically related to a teaching and learning context (e.g. second and third level in this case) – all potential trajectories need to be listed.

- What is the nature of and reasons for meta-level developments of mathematical discourses in bilingual students? Meta-level development refers to a change in the discourse that results in expansion of the discourse relating to a particular topic(s) and is a complex type of change, rather than an 'object-level' change that is more accumulative in nature (Sfard, 2012, p. 3). Therefore, meta-level development is primarily concerned with conceptual mathematical activity.

- How are languages (Irish and English) utilised in, and how do they impact on, meta-level developments in mathematical discourses of, for example 'limits', in undergraduate mathematics education? By detailing the developmental trajectories in each language it is expected to demonstrate how learning can be affected by the characteristics of a language, while also examining when and how bilingual students utilise their languages.

Investigating bilingual mathematics learners is complex and research has demonstrated that language switching practices are also complex and involve not only social and cultural aspects, but also cognitive aspects. The framework, and future research directions presented in this paper, aim to investigate further the cognitive aspects of bilingual learners and their use of their languages, when engaged in conceptual mathematical activity.

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Mathematical reasoning through a broad range of communicational resources

Anna-Karin Nordin and Lisa Björklund Boistrup

Stockholm University, Department of Mathematics and Science Education, Stockholm, Sweden, anna-karin.nordin@mnd.su.se

Mathematical reasoning is examined in this paper that investigates student – teacher communication at the front of the classroom where students gave account of their solutions to a mathematical problem. Adopting a multimodal approach we have discerned how students communicate reasoning through a broad range of communicational resources, such as speech, drawing, hand gestures and the like. We adopted Toulmin's (2003) model of argumentation as means to capture different elements of the reasoning given account for in the communication.

Keywords: Reasoning, argument, multimodality, communication.

INTRODUCTION

The importance for students to participate in mathematical reasoning has been highlighted in international frameworks (e.g., Niss & Jensen, 2002). In the ongoing PhD study, of which this paper is a part, the interest of exploring and understanding reasoning concerns reasoning in public, as a communicative act, not as a thinking process. Specifically, we, in this paper, investigate reasoning when students explain solutions to problems while positioned at the front of the classroom. The teacher takes part in the communication as well.

> Reasoning is a collective and human transaction, in which we present ideas or claims to particular sets of people within particular situations or contexts and offer the appropriate "reasons" in their support. (Toulmin, Rieke, & Janik, 1979, p. 9)

An argument is described by Toulmin and colleagues (1979) as a "train" of reasoning. Various works have studied observable reasoning or argumentation in mathematics education, some focusing on collective or collaborative argumentation/reasoning with a focus on peer interaction (e.g., Bjuland, Cestari, & Borgersen, 2008; Mueller, 2009). When analysing students' collaborative reasoning, Bjuland, Cestari and Borgersen (2008) highlighted the need for paying attention to more than writing and speech when analysing arguments. Another example is Meaney (2007), who considered the role of gestures in strengthening arguments when analysing levels of mathematical literacy demonstrated by students.

Within research on mathematical communication, there is a trend to recognise the multimodal nature of communication even though it continues to privilege language as the primary mode (Morgan & Alshwaikh, 2012). In order to analyse and understand communication in mathematics education, a multimodal approach has been adopted (e.g., Björklund Boistrup, 2015; Morgan & Alswaikh, 2009). While taking on a multimodal approach, Morgan and Alswaikh (2009) drew attention to the duality of a mode, the drawing mode: as the process of drawing and as the outcome, the picture itself. By using a multimodal approach when analysing reasoning expressed in student – teacher communications we hope to contribute to the understanding of reasoning as a sequence of communicative acts. In this study we investigate how students display reasoning in student – teacher communications through a variety of communicational resources when giving account of solutions at the front of the classroom.

ANALYTICAL FRAMEWORK

Drawing on Toulmin and colleagues (1979), we view an argumentation as consisting of ways of giving reasons and hence we understand argumentation and reasoning as strongly connected (Krummheuer, 1995). In order to examine reasoning we adopt Toulmin's model of argumentation (2003). When analysing communication, as constituted through a wide range of resources (such as speech, symbols, pictures and the like), we added to the framework a multimodal social semiotic perspective.

TOULMIN'S MODEL OF ARGUMENTATION

In mathematics education, Toulmin's model of argumentation has been used, sometimes in a reduced form (e.g., Krummheuer, 1995, 2007; Meaney, 2007), consisting of four of the six original elements of the model, where three of them represent the "core" of an argument. The elements are the data/ground, the warrant and the claim/conclusion, as well as the forth element, backing. The model that serves as the basis for our analysis of reasoning is presented in Figure 1.

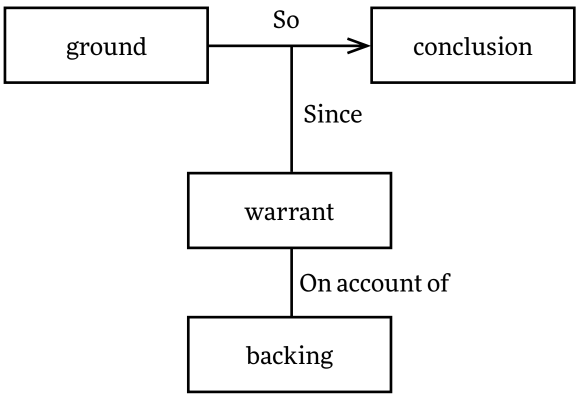

Figure 1: Toulmin's reduced model of argumentation

In a mathematical argument, a *conclusion* (Figure 1 top right), is presented, often in the form of a solution of a solved problem. In order to support a conclusion, some underlying foundation for the conclusion needs to be produced, which is to be seen as the *grounds* (Figure 1 top left), often consisting of facts or information. In order to justify the step between the presented grounds and the asserted conclusion, the *warrant*, can be provided (Figure 1, middle). The warrant is provided to show that starting from the grounds, the step to the given conclusion is appropriate and legitimate. The *backing* (Figure 1, bottom), gives authority to the warrant and can, like the grounds, often be expressed in the form of facts. Toulmin (2003) described the process of argumentation as G because W so C. Participants in a communication do not necessarily structure their contribution clearly according to the elements of the model but they may still be identified through a detailed analysis of interaction (Krummheuer, 1995). We adopted this model as an analytical tool in order to reconstruct reasoning in a situation where students presented solutions to problems while positioned at the front of the class and with the teacher participating in the communication.

Multimodal social semiotics

Focusing on communication, we adopted a multimodal social semiotic perspective (Kress, 2010) in order to describe and understand the mathematical reasoning in classroom communication. Kress (2010) stressed that resources for communication are to be understood as more than writing and speech. They include images, facial expressions, gestures, and the like. The various communicational resources form multimodal ensembles which constitute the communication. In our study we focused on the semiotic resources that communicated mathematical reasoning identified in classroom interaction. Through this perspective we have analysed how the participants used various semiotic resources in their interaction to present reasons in support of a conclusion, which is regarded as the central activity of reasoning when forming an argument (Toulmin, et al., 1979). The multimodal approach affected transcripts as well as analysis and findings, which will be described further on.

METHODOLOGY

The context of this paper is a case study (in the sense of Hammersley & Gomm, 2009) including four classes from grade three to five. In this particular paper we draw on data from two episodes from a grade four class. The episodes are from two presentations at the front of the classroom about solutions to a problem. In order to identify potential reasoning in the presentations we structured and analysed data from the two videorecorded episodes in the following way. The episodes were transcribed taking on a multimodal approach by using the software Videograph. Videograph made it possible to note the semiotic resources the participants were using in terms of our interest in communicated reasoning. With a focus on elements of an argument, the following semiotic resources were identified as being relevant for this study: *speech*, *written text* (including symbolic notions), *drawings*, and *hand gestures* including the use of manipulatives (physical resources). This provided an overview of the communication (see Excerpt 1 as an example), making it possible to identify different elements of an argument according to Toulmin's model and a multimodal approach.

ANALYSIS

The teacher's aim with the lesson which we give account for in this paper, was to involve the students in a problem solving activity, in pairs or small groups, as well as a whole class activity where selected solutions were presented. These presentations incorporated students explaining and justifying their reasoning. The mathematical content of the lesson was fractions and the task was formulated as follows:

> It is a sport/field day and it is sunny and warm. The school will provide food and drinks. Each student is given ¼ of a liter of juice to drink. There are 16 students. How much juice will be needed?

At the end of the lesson the teacher asked some of the groups to present their solutions to the class. We present two episodes from the presentations, including analysis and findings. We chose these episodes since they represent two different solution strategies and, as will be shown, different aspects of student reasoning.

Description of Episode 1: Stina starts with the bottles

In this first episode, Stina and her friend are standing in front of the class in order to present their solution. The teacher asks Stina, who is doing the presentation of her group's work, how she initially was "thinking". Stina tells the teacher that she drew four bottles and then she starts to draw one bottle on the board. Being asked by the teacher regarding the number of bottles, Stina clarifies that they started off with drawing one bottle. Stina continues to talk but is interrupted by the teacher who wants to know what they did with the bottle. Stina starts to explain that they counted (inaudible continuation) and is prompted by the teacher to do that with the bottle on the board. Stina divides the bottle into four parts and clarifies in words what she did. Asked for a clarification as to why they divided the bottle into four parts, Stina responds verbally "For it to be...since everybody could drink one quarter". The follow up by the teacher to Stina's response is yet another question, concerning the meaning of a quarter.

In Excerpt 1 (see Table 1) we give account for the continuation of the interaction. Actions taking places at the same time are beside each other horizontally.

Stina continues to draw a third and a fourth bottle, each time dividing each bottle into four parts, while explaining verbally how many they were enough for, ending up with "and then they were enough for sixteen" with a picture of four bottles each divided into four parts. The student – teacher interaction continues for a while and includes another student as well, ending with a verbal clarification that there are four liters needed for sixteen students.

Analysis of the first part of Episode 1

In Episode 1 we could identify in the analysis the elements of Toulmin's model. Here we give account of our analysis of the first part of Episode 1. Stina's argumentation here is made in relation to the teacher's question regarding how many can drink from the first bottle. The image of the four parts of the bottle visualized how many that could drink from one bottle (Excerpt 1, row 5) and this was also expressed by speech, "four" (row 6). This utterance (picture+speech)

	Speech	Writing/Drawing	Hand gestures	
1	Teacher (T): How many parts should one divide it into then?		T takes a magnetic circle divided into four quarters and puts it on the white board.	Stina (S) points at the parts in the bottle
2	S: Four			
3	T: Four. Ok		T points at the circle	
4	S: It was not enough. We had to do one more	S starts to draw one more bottle next to the first one.		
5	T: How many could drink from the first bottle?	S divides the second bottle into four parts.		
6	S: Four			
7	T: Four. Ok			

Table 1: A multimodal transcript from Episode 1

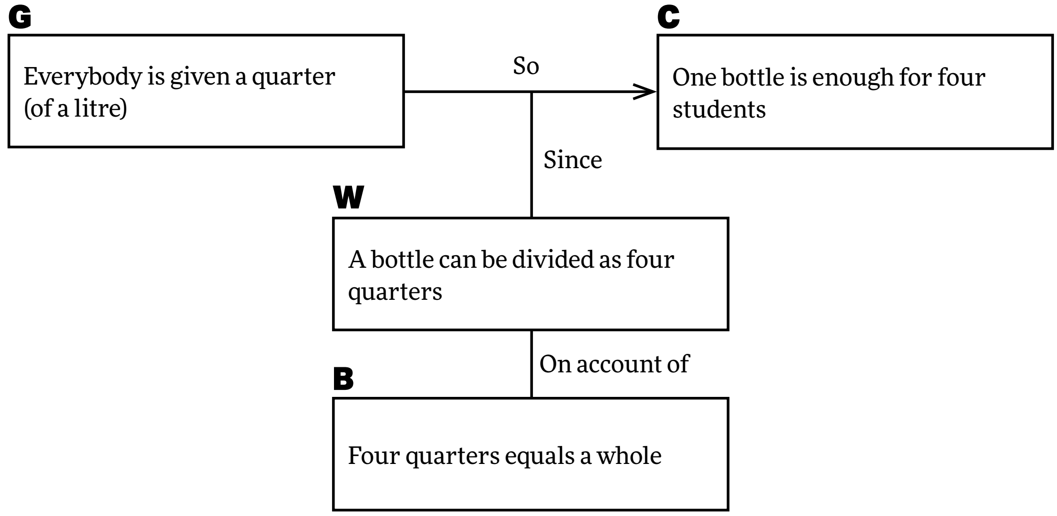

Figure 2: Elements of Stina's argument in Episode 1

has been identified as the first conclusion. In Figure 2 we summarise our analysis according to Toulmin's model.

As we wrote earlier, we could identify a conclusion from Stina's picture of bottle divided into four (row 5) and speech, "four" (row 6). This utterance from Stina was seen as claiming that four students could drink from one bottle. The fact Stina appealed to as the foundation for claiming that four students can drink from one bottle was categorised as the grounds and was identified from her verbal response to why the bottle was divided into four parts "because everybody can drink a quarter".

The warrant in this specific situation has been identified in this argument by taking into account the use of different semiotic resources. The warrant in this case should answer to why one can claim a bottle to be enough to four students (the conclusion) if each person is given a quarter (the grounds). The warrant, which we construed here, was: a whole can be divided as four quarters, which was communicated by Stina through drawing, speech, and hand gesture. More specifically, we construed that she communicated that a whole can be divided into four in her drawing where she divided the bottle (as a whole) into four parts together with her saying "we divided it into four parts". That each part is to be seen as a quarter was stated in her answer to the teacher's question on why they divided the bottle into four parts. Stina also indicated, according to our analysis, that one of the drawn parts of the bottle was to be considered as a quarter when pointing (row 1), a bit vaguely, towards the parts of the bottle when asked by the teacher what a quarter means. The teacher also contributed to the argument by stating, "Four" (row 3) as a verbal reinforcement to Stina's verbal response "Four" (row 2) to her own question regarding how many parts it (the bottle as a whole) can be divided into.

Further evidence, authorizing the warrant, was categorised as backing: a whole equals four quarters, and in this episode was identified in the image of the magnetic circle consisting of four quarters of a circle put at the white board by the teacher (row 1).

Description of Episode 2: Frida starts with the mugs

After two groups had presented their solution, it was time for Frida to present her and her friend's solution. With both Frida and the teacher standing at the whiteboard, the teacher asks Frida to start with how she began to solve the problem. Frida begins to explain how she started off solving the problem by stating "I started to draw each student's mug". While expressing this Frida starts to draw a rectangle on the white board, emphasizing it by also pointing at it. While beginning to draw a second rectangle/mug (from now on referred to as "mug") the teacher asks her how much each mug contains. Frida completes drawing the second mug and writes ¼ in the first mug saying, "There was"... without continuation. The teacher expresses the content in each mug verbally "There was a quarter in each mug" and puts a quarter of a magnetic circle over the first mug drawn by Frida. Frida writes ¼ in the second mug and continues to draw six more mugs on the same row and eight mugs on a row below ending up having drawn sixteen mugs in total (Figure 4)

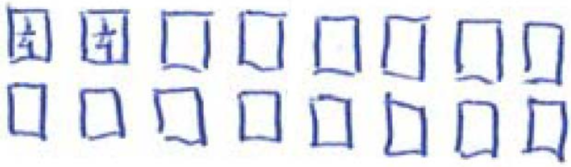

Figure 4: A picture of the way Frida drew the mugs

The teacher asks Frida how many quarters she had but Frida does not respond to that question and continues "I did like this" and shows on her picture of the sixteen mugs how she divided the sixteen mugs into four groups with four mugs in each group by drawing a line after four mugs and after eight mugs in the first row, and after four mugs in the second row and partly after eight mugs in the second row (Figure 5).

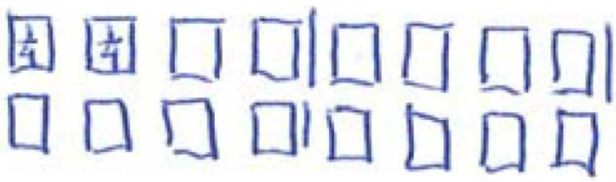

Figure 5: A picture of how Frida divided the mugs

Here Frida is interrupted by the teacher who wants to return to her question regarding the number of quarters. After some elaboration by the teacher regarding the number of quarters, she asks another student how many litres there are and receives the answer "Four litres". As a final contribution in this presentation the teacher is stating. "We do know that in one litre there are four quarters" at the same time as she is putting the four quarters of the magnetic circle into a full circle on the white board.

Analysis of Episode 2

Also in Episode 2 we could, in the analyses, identify the elements of Toulmin's model. We summarise our analysis in Figure 6.

The presentation in Episode 2 ended up with the conclusion that four liters were needed. The conclusion was identified in Frida's final image reflecting the outcome as the four groups of four mugs/quarters in each liter (Figure 5) and was identified in the other student's verbal answer "Four liters" towards the end of the presentation.

In these episodes two grounds, supporting the conclusion that four liters were needed, were identified. These grounds were identified as 1) there are sixteen students, and 2) each student is given a quarter each. The use of the first ground was identified in Frida's drawing of the sixteen rectangles. Each rectangle was explained to illustrate a mug for each student, which is clear in Fridas speech "I draw all students' mugs" when she started to draw the mugs. The number of the mugs/students, sixteen, was expressed in the drawing and the picture of sixteen mugs (as well as by the teacher when elaborating verbally on what Frida was drawing). The second ground – each student is given a quarter – was identified when Frida wrote ¼ in two of the mugs as a response to a question from the teacher regarding how much the mug contained. After Frida had written ¼ in one of the mugs the teacher made this clearer by saying "It was a quarter in each mug". A visual representation of the content was seen in the form of the magnetic quarter of a circle placed over the first mug by the teacher.

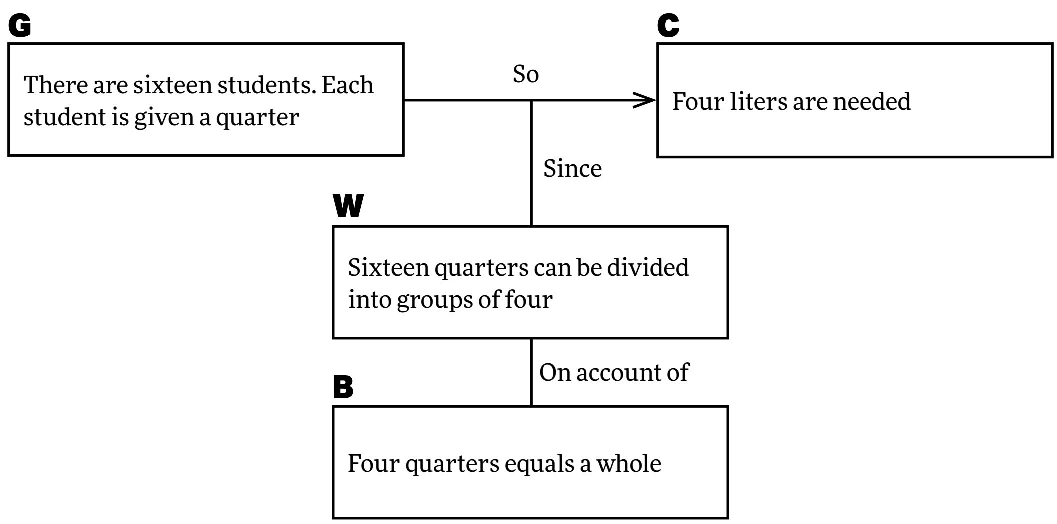

Figure 6: Elements of Frida's argument in Episode 2

Our understanding is that Frida's method, to divide the sixteen quarters into groups of four, leading to the conclusion of four litres, indicates the warrant, justifying the step from grounds to conclusion. We construed the warrant to be: sixteen quarters can be divided into groups of four. The grouping of quarters was indicated by the process of drawing a line after the fourth mug in the first row, after the eighth mug in the first row and after the fourth mug in the second row.

In our analysis we identified that the teacher provided the backing by putting the four quarters of a circle, into a whole circle, showing: four quarters equals a whole. This was also identified in the teacher's, less formal, verbal expression "We do know that in one liter there are four quarters".

CONCLUDING DISCUSSION

The aim of the study was to identify reasoning in student – teacher communication, interpreted by us as presenting support, (that is, reasons) for a conclusion, showing how these reasons can be seen as giving strength to the claim (Krummheuer, 1995; Toulmin, 2003). By using Toulmin's model as an analytical tool together with a multimodal approach we have been able to discern aspects of reasoning.

We investigated how students display reasoning through a variety of communicational resources when giving account of solutions at the front of the classroom. In our findings we discerned how both students in the episodes presented here displayed reasoning through several communicational resources but in different ways. An example of this is how the girls displayed a ground they both referred their argument to: each student is given a quarter. In the first episode we could see Stina displaying the use of this ground by picture and speech whereas in Frida's case we could see how three different resources were identified as being used, interplaying with each other. In the second episode Frida displayed the ground by writing ¼ in a drawn rectangle (she had previously in speech communicated it to represent a mug to one of the students). Another example was when they were justifying the step from ground to conclusion, the warrant in Toulmin's model. Stina communicated this both through drawing, speech and gesture and Frida only through drawing. This illuminates how essential it is for teachers, as well as researchers, to pay attention to what is displayed in various communicative resources in order to capture and illuminate the reasons, mathematical justifications, and to make them accessible for all students in classroom communication.

By taking a broad range of communicative resources into account, this study opened up for capturing students' silent and non-symbolic display of reasoning which would have passed unnoticed if we had only been looking for verbal or written expressions. One example was our identification of Frida's process of drawing lines, which we construed as her justification, providing warrant for the conclusion considering the grounds she provided. Expressed verbally, such as in "A number of quarters can be divided into groups of four in order to get the number of wholes", it would be likely to receive a response from a teacher as a "proper" justification. If it was only expressed in drawing it might go unnoticed. If the warrants, such as Frida's in this case, are not noticed and elaborated upon, they might pass as unnoticed by other students in the classroom as well. Hence, an opportunity for the teacher to highlight and generalize mathematical ideas may be lost.

We want to clarify that as researchers adopting a multimodal approach, we always need to make choices of what resources to pay attention to in the analysis. In this paper we focused on reasoning and the resources identified as being relevant to this. If we had chosen to focus more broadly on the interaction itself, for example on feedback, other resources, such as voice, facial expression etcetera, would have been part of the transcripts and analysis as well.

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Mathematics education in bilingual contexts: Irish-English, Breton-French

Caroline Poisard[1], Máire Ní Ríordáin[2] and Erwan Le Pipec[1]

1 UBO, ESPE de Bretagne, Bretagne, France, caroline.poisard@espe-bretagne.fr

2 National University of Ireland, Galway, Ireland

Irish and Breton are both Celtic languages but unity has vanished resulting in deep linguistic differences. But the common heritage is still at hand when one considers the lexicon, grammatical peculiarities and number. The concept of the structure of a language impacting on thought processes is referred to as the linguistic-relativity hypothesis, which proposes that the vocabulary and phraseology of a particular language influences the thinking and perception of speakers of this language, and that each language will have a different cognitive system. This paper examines the Irish and Breton languages in their bilingual context, their linguistic characteristics and impact on mathematics learning in comparison to English and French, while identifying future research requirements.

Keywords: Bilingual contexts, Irish-English, Breton-French, linguistic differences.

INTRODUCTION

Irish and Breton are both Celtic languages, spoken in the western ends of Europe, namely in Ireland and in the western half of Brittany (France). The Celtic languages are divided into two branches: the *gaelic* one (the native languages of Ireland, Scotland and the Isle of Man) and the *brythonic* one (comprising Welsh, Cornish and Breton). Accordingly, these languages might have once been the same language or at least, two dialects of the same language. Nevertheless, centuries have passed by and unity has vanished, resulting in deep linguistic differences. However, the common heritage is still at hand when one considers the lexicon (roughly 5 000 shared words, those expressing very old notions such as a house: *teach/ti*; weather: *aimsir/amzer*; good: *maith/mat*, etc.) and grammatical peculiarities (mutations of initial consonants to distinguish word gender; mutations in syntactical context; 'declined' prepositions; word order, etc.). Numbers are also a domain where a very old continuity can be traced, as will be shown in this paper. These two languages are spoken in a bilingual context, involving either a Germanic language: English, or a Romance one: French. This is the first investigation of its nature into the Irish and Breton languages and given the rise in Irish-and Breton-medium education it is timely and can contribute to the development of policy in this area.

In this paper we present a preliminary study which builds on two previous research studies in the two commented contexts (Ní Ríordáin, 2013; Poisard et al., 2014), with the aim of identifying future comparative studies. We focus here on languages features and the potential of language as a resource for teaching (Adler, 2001), while examining fundamental questions about relationships between mathematics and language (Barton, 2008). Taking into account that language reflects the way we see the world and that mathematics is a modelisation of the world, our questions are: What are the features of languages that may be of importance for mathematics teaching and learning? How can language be a potential resource for teaching? How to identify mathematical particularities in linguistic expression?

LANGUAGE CONTEXTS

The history of Irish and Breton is marred by stories of decline and persecution. However, both languages have experienced parallel revivals at various times. Since the 90's on, there is more and more concern in both populations to revitalize their languages and avoid complete extinction (Le Pipec, 2013). Strategies have been and are being designed at state or near-state level to revitalize them, primarily by supporting them as medium of education. Table 1 provides a summary

	Type of schools	Number of primary and secondary students	Total of students
Breton-medium education	Bilingual public schools	6 662	15 338
	Bilingual catholic schools	4 971	
	Diwan schools	3 705	
Irish-medium education	Maintenance heritage language	15 546	56 974
	Immersion	41 428	

Table 1: A comparison between Irish and Breton educational contexts

of the number of students learning through the medium of Breton and Irish.

Breton schools are part of the centralised educational system of France and the teaching of Breton is optional. There are over 15,000 pupils learning through the medium of Breton (with an average growth of 4 to 5% per year), ranging from pre-school to secondary level (OPLB, 2014). There are three types of Breton-medium schools that pupils can attend, distinguishable by two ways of utilising the languages. The 'bilingual schools' (Public or Catholic) devote the same amount of time to each language. On the other hand, *Diwan* schools claim to be 'immersive' schools since Breton is only in use during the first four years of schooling. French is then introduced by the age of 7 to 8 and takes up to half of school time by age 11.

Irish-medium education is the norm for those growing up in the Gaeltacht regions in Ireland (maintenance heritage language education). The rise in popularity of primary (Gaelscoileanna) and second level (Gaelcholáistí) immersion education (Irish-medium education outside of Gaeltacht regions) is significant and has seen an increase in excess of 60% over the past decade. Currently, approximately 8% of the primary level population and 4% of the second level population are learning through the medium of Irish (Gaelscoileanna Teo, 2014). In Ireland, there is an adequate supply of suitably qualified teachers at the primary level due to incentives and a strong history of native speakers pursuing a career in primary teaching. However, at second level education there is a shortage of suitably qualified teachers to deliver the specific subject areas of mathematics, science and foreign languages through the medium of Irish. Similar problems arise in Brittany. However, at primary level a shortage of skilled speakers willing to become teachers can be observed. Moreover, no *specific* qualification is required for teaching through the medium of Irish/Breton at primary or second level education, although some specific teacher education programmes exist. Accordingly, teachers in Irish- and Breton- medium schools may not have high standard of Irish/Breton themselves, nor have an understanding of the complexities of teaching and learning mathematics in a bilingual context. Therefore, it is not surprising that teachers report difficulty in supporting the language development element of mathematics teaching and learning and a lack of suitable resources and textbooks to support their work (Poisard et al., 2014).

THEORETICAL FRAMEWORK

Bilingual education research shows that using two or more languages to learn and teach is not a simple addition of languages that enable someone to use one language or another (Cummins, 1984). Indeed, linguistic competencies and learning strategies are involved simultaneously and bi-/multilingualism is seen as a "language-sensitive approach of content" (Bernaus et al., 2012). In our work we see bilingualism as a particular form of multilingualism.

The concept of the structure of a language impacting on thought processes is referred to as the linguistic-relativity hypothesis (Whorf, 1956). The basic premise of this hypothesis is that the vocabulary and phraseology of a particular language influences the thinking and perception of speakers of this language, and that conceptions not encoded in their language will not be available to them. Hence, they are proposing that each language will have a different cognitive system and that this cognitive system will influence the speaker's perception of concepts (Whorf, 1956). Whorf emphasises also that we act according to how we describe things, and accordingly different languages may classify experience in different ways. Therefore, in theory, an Irish speaker/learner should have a different cognitive system to that of an English speaker/learner, influence our actions and accordingly may influence mathematical understanding. For example, Miura and colleagues (1994, p. 410) contend that 'numerical language characteristics (East-Asian languages) may have a significant effect on cognitive representation of number'. However, other researchers have questioned argued for the difficulty in applying the linguistic-relativity hypothesis and the difficulty in testing such claims in relation to mathematical thinking (Towse & Saxton, 1997). We acknowledge that this may be too strong of a way of viewing the influence of language on the mathematical thinking and less severe forms of this hypothesis have been proposed. We support the premise that language may not shape and determine our entire mathematical thinking, but that it may influence it to a certain degree and facilitates our thinking and perception (Sternberg, 2003). Moreover, we are acutely aware of the importance of other factors such as exposure to mathematics, teaching strategies employed and culture as influencing attainment in mathematics, not just language (Towse & Saxton, 1997).

Some research shows the positive effect of teaching and learning mathematics in bilingual and multilingual classrooms. In particular, Adler (2001) considers linguistic plurality as a possible resource for mathematics teaching. Three types of resources are distinguished: material, cultural (including language) and human resources. Adler's work is set in the post-apartheid context in South Africa where multilingual classrooms and lack of material resources are common. She shows how linguistic diversity can gradually constitute a resource for mathematics teachers. At primary level, Setati (2005) explored the language practices in primary multilingual mathematics classroom in South Africa where the complex relationships between English language and home language and mathematics education is confirmed.

In Australia, Edmonds-Wathen (2015) has studied the grammar and conceptualisation of motion in Iwaidja, an Indigenous language. In her paper she discusses how understanding grammatical features to express spatial concepts in Iwaidja can help teachers of Indigenous students in their activity of teaching mathematics.

Moschkovich's (2002) research in the United States demonstrates that language can be a resource if a teacher's focus is not only on acquiring mathematical vocabulary, but also on constructing multiple meanings across registers and on developing participation in mathematical practices. This is possible only if teachers are aware of cultural and mathematical needs to teach mathematics.

The example of New Zealand is also of interest to us (Barton, Fairhall, & Trinick, 1998). In the 1980's mathematical vocabulary was developed in Maori. Several general principles were adopted in making vocabulary decisions and "metaphors were a common method of vocabulary development in both formal and informal settings. [...] An example was the development of *rere* and *arawhata* as early translations of *continuous* and *discrete* as applied to statistical data, the methaphor being that of a flowing stream or one proceeding in a sequence of waterfalls" (Barton, Fairhall, & Trinick, 1998, p. 5).

In the Welsh context, Jones (1993) concluded that there are benefits to studying mathematics in a minority language due to it being developed relatively recently as a language of learning and accordingly the terminology established tends to avoid linguistic complexity and employs a more self-explanatory mode. Furthermore, Dowker (2005) found an advantage for students learning through the medium of Welsh, in comparison to English, in terms of how numbers and arithmetical relationships are expressed in Welsh.

Our work in the Breton context (Poisard et al., 2014) shows that the particularities of the Breton language can be a resource for the teaching of mathematics, for example in the teaching of geometrical concepts and oral numeration. Teachers in this study also identified a lack of suitable resources as a significant issue. In particular, many of the material resources used in class are a direct translation from French to Breton, with no consideration of linguistic and cultural specificities. Mathematical and linguistic competencies are interrelated and these competencies need to be jointly developed by students and teachers.

Research in the Irish context demonstrates that students with high ability in Irish and English perform better mathematically (Ní Ríordáin & O' Donoghue, 2009) and that a significant relationship exists between their performance on English mathematical word problems and their Irish language proficiency at the primary to second level education transition (Ní Ríordáin, 2011). A comparison of the English and Irish languages demonstrates that there are differences between the two languages in relation syntax, semantics and access to meaning. However, what is difficult to conclude, without further investigation, is whether differences between the languages have a differential impact upon cognitive processing (Ní Ríordáin, 2013).

In this paper we discuss such questions in Irish and Breton mathematical vocabulary and the structure of the languages, while demonstrating how bilingual students' languages can be a resource for them in mathematics learning. We examine fundamental questions about relationships between mathematics and language.

LANGUAGES: A RESOURCE TO TEACH MATHEMATICS

Some linguistic peculiarities of Breton and Irish may influence the teaching of mathematics. The facilitating aspects and combinations of key features of Breton and Irish are presented here, and where appropriate comparisons are made with English and French. The findings presented here are hypothesis and further investigation (class observations, interviews, questionnaires, etc.) would be helpful to give some more precise conclusions of the influence of the use of these languages on bilingual learners of mathematics. We will not discuss dialects in this section, but Breton and Irish codification as united languages leave much space for dialect variation. In many places, it may happen that the language of books differs from oral use. General features of the languages (sentence length, topic prominence, mutations) are presented initially, followed by a focus on mathematical aspects (transparent lexicons, oral numerations and numbers). The six aspects we develop here are linked to our previous individual research (Ní Ríordáin, 2013; Poisard et al., 2014) and demonstrate commonalities between both languages.

Sentence length

Short sentences are generally more common in Breton and Irish than in French and English, which demands less concentration for pupils and short-time memory is devoted to side-information. Shorter sentences lend to an easier understanding of mathematical text and are a desirable feature (Austin & Howson, 1979). English and French readers may have a greater cognitive processing load, and this suggests a difference in mathematical processing.

Topic prominence

Breton and Irish are strongly oriented topic-prominence languages, in comparison to French and English. In Irish, the first word is usually the verb, while in Breton, it may be any word (rarely the verb). Indeed, in Breton, words expressing new information should come first in a sentence, no matter of their grammatical status or function. In problem solving, this gives pupils clues about relevant mathematic information, emphasised by their position in texts (Galligan, 2001) in comparison to English and French. Topic prominence may alter the complexity of semantic structures and have an effect on mathematical processing.

Mutations

They are an important feature of Celtic languages, either to indicate word gender or in syntactical constructions. As an example, in Breton after the word *tri* (three), any *p-* standing at the beginning of a word goes to *f-*. When combined with *poent* (point), *three points* is thus expressed as *tri foent*. The permanence of words through surface changes may help to understand the permanence of mathematical relations. Although mutations would cause difficulties for pupils with good language skills, recognising mutated words (especially when there are many of them) may be problematic for those insufficiently familiar with the language. Such mutations are not evident in English and French.

Transparent lexicon

A large part of the mathematical vocabulary is coined out of autochthonous word-roots. The meanings of many words have become easier to understand. For example, to say *parallel* in Breton is *kenstur* (same direction) and in Irish is *comhthreomhar* (equal directionality). Many of the Breton/Irish words describe concepts/objects as opposed to just labelling them. Given that the more easily and quickly the meaning of words is activated, the simpler it is to process mathematical text. It may help to retrieve all the words associated with the concept thus enhancing the total cognitive structure (Galligan, 2001). However, Celtic-made vocabulary has been criticised for various reasons: some words might have been coined too quickly by *amateur* linguists; often these words are not encountered outside of the classroom; in some cases, they were promoted not in an effort to facilitate understanding, but to wash brains of a foreign language; concern lies with if pupils must leave and should adapt to a dominant-language school. This is why an international mathematical lexicon has also been developed in Breton (Kergoat, 2012). Instead of *kenstur*, the teaching authority now recommends to express *parallel* as *parallelenn*. Further investigation is needed in terms of student learning and accessing meaning in more common languages/lexicon of English/French.

Oral numerations and counting

In this area Guitel (1975) gives an historical view of written numeration. One of the particularities of Celtic languages is the use of vigesimal system (base 20) to say numbers. Some groupings by 20 are evident in French also. In old Irish, we find traces of 20 groupings in all tens: 30, 40, 50, 60, 70, 80, and 90. For example 90 is *ceithre fhichead a deich* (four-twenty and ten, 4×20+10). We find traces in French for 80 (quatre-vingts four-twenty, 4×20+10) and 90 (quatre-vingt-dix, four-twenty-and-ten, 4×20+10) where it is literally the same expression as old Irish. Breton refers also to 20 for 40, 60, 70, 80, and 90. Indeed 40 is *daou-ugent* (two-twenty, 2×20), 60 is *tri-ugent* (three-twenty, 3×20)... 90 is *dek ha pevar-ugen*t (ten-and-four-twenty, 10+4×20), etc. In Breton the word order is not the same: +10 is mentioned before and not after as in French and English. We have the equality (the model) 90=4×20+10= 10+4×20 that is shown in a comparison of languages. If we make a link with topic prominence (above), we could argue that old Irish emphasises the grouping by 20 (coming first) and Breton the addition on this grouping (+10 coming first). These characteristics are clearly different from the numerations/counting systems of English and French (base 10). We think that the comparison of number names could be a rich resource for teaching oral numeration, written numeration and the associated mathematical meaning. Oral numerations have been studied concerning other contexts. For example, number words in "other" languages is explored by Chronaki, Mountzouri, Zahakari and Planas (2015) in the Greek context to experiment a mathematical learning activity with young children, and shows that the creation of a culturally responsive context.

Word order to say numbers

In Breton, *32 <u>pupils</u>* is said *daou <u>skoliad</u> ha tregont* (two <u>pupil</u> and thirty, 2 <u>pupil</u> and 30). The common name expressed by the number is in-between tens and units. One can notice that the name is here in singular form (*skoliad*) and not plural (*skolidi*). Indeed plurality is not attached to nouns but it is expressed by the adjunction of a number. For large numbers, the word order refers also to this rule. For example, 32 000 <u>pupils</u> (*daou vil <u>skoliad</u> ha tregont, 2* thousand <u>pupil</u> and 30 (thousands not mentioned), coming from 2 000+30 000) and 3 020 <u>pupils</u> is (*tri mil ugent <u>skoliad</u>*, 3 thousand 20 <u>pupil</u>, 3 000+20). Irish is similar to Breton when using ordinal numbers in that the common name is positioned in-between tens and units. For example, 53 <u>pens</u> is *trí <u>pheann</u> is caoga* (three pens and fifty, 3 pens and 50). Large numbers generally preserve this order also. This can be seen as a difficulty for students, but we also argue that this is a good opportunity to work on number sense and the place of each digit in a number. In Irish, different words are used for counting people: two, three, four etc. is expressed by *beirt, triúr, ceathrar*, to signify that the numbers relates specifically to counting people. The comparison between the languages is a good resource to understand the grouping by three of large numbers.

DISCUSSION AND CONCLUSION

This paper explored specific aspects of language concerned with mathematics teaching and learning in relation to the bilingual contexts of Irish-English and Breton-French. But why does this matter? The importance of language for the teaching, learning, understanding and communication of mathematics cannot be ignored. Features of the Irish and Breton languages presented in this paper demonstrate the importance of investigating languages and their potential impact on mathematical learning. For example, some Breton and Irish words assist in conveying meaning and/or permit the concept to be formed more readily. Similarly the sentence structure allows access to key information. Some promising insights are emerging, suggesting that students who learn through the medium of Irish or Breton may experience advantages in terms of mathematical learning. Further investigation is needed into how a particular language and its syntactical structure may impact on mathematical activity and reasoning.

When investigating the Irish and Breton languages we are cognisant of the fact that we are investigating bilingual learners (Irish-English, Breton-French). The relationship between mathematics learning and a student's language is complex, and further complicated when working with bilingual learners. Moreover, we need to consider mathematics as a discourse and that this is not a singular or homogenous discourse (Adler, 2001). Accordingly, mathematical learners use multiple resources and languages from their experiences (both in and outside of the learning context) and we need to be cognisant of multiple registers co-existing in the learning environment. Bilingual learners should not be viewed in a deficit mode, but rather view their language(s) as a resource for learning mathematics. However, as demonstrated in this research paper, this area is under-researched and under-theorised. Research practices/findings generated from participants from a dominant group (e.g., monolingual speakers) assumes these to be the norm for all learners. We endorse a call for more research in relation to the role of bilingual learners' different languages when engaged in mathematical learning (Barwell, Barton, & Setati, 2007). There has been a focus more on the social, rather than cognitive functions of code switching (Ní Ríordáin, 2013). The authors of this paper purpose that there is a need for a coherent and integrated framework to investigate whether differences in languages, and their use, by bilingual mathematical learners, have a differential impact upon cognitive mathematical processing. The authors also stress the importance of recognising and integrating the social aspect of learning into the framework and seeing language as a resource for mathematics teaching and learning.

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Capturing learning in classroom interaction in mathematics: Methodological considerations

Jorunn Reinhardtsen, Martin Carlsen and Roger Säljö

University of Agder, Kristiansand, Norway, jorunn.reinhardtsen@uia.no

This paper discusses issues of how to transcribe and analyze video-recordings when studying learning in small group work in mathematics. Since bodily features of interaction and the use of artefacts play important roles in mathematical reasoning, a multimodal approach to transcribing is necessary. Thus, the theoretical grounding for transcriptions has to be in accord with the perspective on learning adopted in the analysis. In the paper, the principles for studying what Radford (2000) refers to as knowledge objectification processes when learning mathematics will be discussed.

Keywords: Analytical approaches, knowledge objectification, multimodality.

INTRODUCTION

This paper discusses ways of doing video analyses that are relevant for understanding mathematics learning. Thus, this is a methodological paper. Our particular focus is on multimodality as a resource for learning but also as a methodological challenge for research. Analytical approaches, selection of episodes and a multimodal transcription will be discussed in light of recent developments in the field.

The background of this study is an international, comparative project called VIDEOMAT (Kilhamn & Röj-Lindberg, 2013), which studies teaching and learning of introductory algebra in four countries: Finland, Norway, Sweden and the USA. Students are between 11 and 13 years old. Data include video-recordings of lessons, written materials from student activity, teacher interviews etc. Five consecutive lessons when algebra was introduced[1] in classrooms in the four countries were documented. A group work session with a pattern task with matchsticks was selected for further investigation. This task resulted in a multitude of problem-solving strategies among students, all the way from counting to sophisticated forms of mathematical reasoning (multiplicative/generalizing).

In the literature there are many attempts to make use of multimodal analyses to understand learning processes in the context of mathematics. We will comment on some of these below. Considering that we are at an early stage of advancing knowledge through the use of multimodal approaches, we have formulated the following question for this paper: *In what ways can video recordings be transcribed and analyzed in order to study student's learning processes?*

BACKGROUND

The methodical reflections in this paper focus on classroom interaction in a problem-solving, small-group setting. A particular aim is to understand the knowledge objectification process (Radford, 2000, 2002). The object of activity in the classroom, as the students work with the matchstick task, is to develop algebraic thinking; more specifically to perceive the general nature of a pattern, and to use this insight when solving a problem. The ability to generalize is viewed as one of the most important developments in mathematical thinking.

Our analysis will follow a socio-cultural, Vygotskian view on learning and development. A central idea is that learning results from participation in social and interactional processes. Equally important is that this perspective stresses that learning and knowing are cultural phenomena.

Approaching group work in mathematics classrooms with an interest in the contributions of multimodality, the cultural-semiotic theory of learning, developed by Radford (2000), provides a promising route ahead. Radford (2002) suggests that knowledge objectification happens through semiotic activity, that is through "objects, artifacts, linguistic devices and signs that are intentionally used by individuals in social processes of meaning production, in order to achieve a stable form of awareness, to make apparent their intentions and to carry out their actions" (p. 14). The process of knowledge objectification is understood as the process of placing something at the center of someone's attention. In this study, knowledge objectification thus refers to the process of perceiving generality; the knowledge of the general nature of the pattern having a genesis and a development, and, as a further step, knowing how to express the generality mathematically and to solve the problem.

METHODOLOGICAL DEVELOPMENT IN THE STUDY OF LEARNING

The methodological deliberations by different researchers have been scrutinized. The studies analyzed all rely on naturalistic data and interpretive approaches to method, and they represent different choices in terms of data collection and analysis.

Bjuland (2002) focused on small group problem-solving in mathematics by student teachers. Data were collected by audio recordings, and the theoretical perspective utilized was dialogical, situated and socio-cognitive. The unit of analysis, referred to as an episode, was "conceived as a sequence of verbalizations focused on a special mathematical topic or idea" (p. 64), relevant to the research questions. These were then categorized according to five features of problem solving processes: sense-making, conjecturing, convincing, reflecting and generalizing.

Carlsen (2009), working in a sociocultural tradition, analyzed the appropriation of mathematical tools by students attending the final year of high school. Video recordings were used. The aim was to trace development of the student's mathematical reasoning. Relevant parts of the entire audio recorded material were transcribed in detail and subjected to in-depth analysis. The transcripts included multimodal elements in order to investigate the role of inscriptions in the appropriation process.

Radford (2000, 2002, 2012) reported on longitudinal studies involving students' group work with algebra and more specifically with patterns. This work involves methodological and theoretical developments that are interesting. Radford's research is based in a semiotic-cultural perspective on learning building on Vygotsky's view of signs as linked to and affecting our cognition. In Radford (2012), researchers took part in the process of designing the lesson material and students were organized in small groups. These sections were video recorded and student works were collected.

In his early work, Radford (2000) uses concepts from discourse analysis. He follows a three-step analysis of transcripts, a) valuing each utterance as equally important, b) contextualizing utterances, and c) including pauses and hesitations. This approach Radford (2000, p. 244) terms *situated discourse analysis*. The unit of analysis was conceived through a process of refining salient episodes through data managing by indexing and theorizing. Radford emphasizes the importance of natural language in the development of algebraic thinking and the use of algebraic symbols.

Radford, Demers, Guzmán, and Cerulli (2003) introduce the concept of *semiotic node*. This was a response to findings in many studies on the importance of gestures and artifacts in the production of graphs and algebraic expressions. Semiotic nodes are "pieces of the students' semiotic activity where action, gesture, and words work together to achieve knowledge objectification" (p. 56). The transcripts include description of gestures and the analytical tool of semiotic nodes was applied in the analysis. In Radford, Bardini, and Sabena (2007), the analysis was done in greater detail. A slow-motion, frame-by-frame, fine-grained video micro-analysis was carried out and complemented with a voice-analysis. The same kind of micro-analysis was carried out in Radford (2012), except for the voice analysis, where a multi-semiotic analysis (spoken words, written text, gestures, drawing, and symbols) was done.

Arzarello (2006) outlines a theoretical frame emphasizing the role of multimodality and embodiment in cognition. He argues for a multi-semiotic analysis of objectification processes and claims that the present semiotic frameworks cannot capture didactical processes in a satisfactory manner. Therefore, he introduced the idea of the *semiotic bundle*. In the semiotic bundle, which includes semiotic sets such as gestures, speech, written representations, as well as more formal systems, the distinctions between the sets are only made for analytical purposes while interpreted as a unitary system. The semiotic bundle is dynamic and can shift to include more or less semiotic sets as the event unfolds. The meaning of the mathematical object may not be the same in the different sets. Moreover, even if the transformation from one set to another is accomplished, the meaning the object had in the prior set may linger, and so it can take time before the concept is formalized. By looking at the data synchronically and diachronically, the genesis and evolution of the semiotic objectification process can be traced. The semiotic node introduced by Radford (2003) is similar to looking at the semiotic bundle synchronically.

Arzarello (2006) used the semiotic bundle to analyze the work of one group of five fifth-graders. Video recordings and student work were collected as part of a longitudinal research design. However, the episodes presented were chosen from a 30 minute session on problem solving. The selection process was not commented on, except by saying that four main episodes were chosen. The episodes were subjected to different analytical methods; (episode 1) synchronic analysis; (episode 2) diachronic analysis; (episode 3) synchronic + diachronic analysis; and (episode 4) diachronic analysis. The transcriptions include descriptions of gestures and pictures are presented in the analysis.

Roth and Thom (2009) looked at multimodality and learning from a phenomenological perspective. The aim of the study was to propose a new way of understanding mathematical concepts grounded in a case study. Data were collected in a second grade classroom during group work sessions in geometry. In addition, artifacts used and all work by the teacher and the students were photographed. One episode from a whole class session, lasting 69 seconds, which is called exemplary, was chosen for analysis. The episode is presented in the context of what happened before. The transcript includes details (length of pauses, pitch etc.). The episode is presented over 6 pages and several drawings depicting movements are part of the description. The authors argue that "conceptions can be understood as networks of experiences that indeterminately emerge from lived (rather than intellectual) reorganizations of embodied bodily experiences" (op. cit., p. 188).

The studies presented above are all conducted within the paradigm of interpretivism. They are ethnographic and researchers spend time observing, making field notes, and collecting students' work; the researchers are concerned with the context in which the events take place. The video and/or audio recordings are done in classrooms and are naturalistic in the sense that students are in their everyday environment engaging with mathematical activities. The studies also share a common focus on the multimodal aspects of learning, except Bjuland (2002) and Radford (2000).

METHODOLOGICAL CONSIDERATIONS

In spite of the commonalities of the studies presented, the transcripts look very different and include different features of interaction. Bezemer and Mavers (2011), investigating multimodal transcripts in research, point out that "transcripts should be judged in terms of the 'gains and losses' involved in remaking video data" (p. 204). The focus should not be on attempting to achieve representational accuracy, rather the approach should be transparent.

The studies use different analytical approaches to dialogue. Bjuland and Carlsen use the *dialogical approach* elaborated by Linell (1998), while Radford uses *situated discourse analysis*. Consequently, the process of analysis is different. Radford's first step is to look at each utterance in its own right and categorize it. As a second step, he contextualizes them. In contrast, sequentiality is central to the dialogical approach as each contribution in a dialogue gets its meaning from both prior and subsequent turns. Arzarello (2006) and Roth and Thom (2009) do not fully reveal their approach for analyzing dialogue.

An important aspect is the selection of salient episodes. Bjuland (2002) transcribed all verbalizations and then identified relevant episodes according to the analytical interest. Carlsen (2009) worked with video recordings. After several viewings, he chose 14 sessions which were roughly transcribed. Following this, relevant episodes were identified and transcribed in detail. From this sample salient episodes were chosen. Radford (2000) used *situated discourse analyses* as a first approach to the data set, which was transcribed in its entirety. The studies by Arzarello (2006) and Roth and Thom (2009) do not fully comment on the selection process.

These studies show that multimodality is an essential part of understanding how students learn mathematics. Thus, it becomes important for this branch of research to enter into a discussion on how to advance the use of multimodal methods of analysis.

CAPTURING LEARNING: SUGGESTED METHODS

The aim of this paper is to describe ways of doing video analysis that focus specifically on learning processes and which include attention to multimodality. The approach will be discussed in three sections: *video analysis*; *multimodal transcription*; *learning processes and video analysis*. Our discussion will be twofold as it a) provides arguments for the methodological choices and b) is practically oriented in that an excerpt of a multimodal transcript is included and analytical approaches are briefly exemplified.

Video analysis

The video analysis follows an interpretivist paradigm. The aim is to understand learning processes by closely following how participants engage in meaning making. Applying the notion of knowledge objectification through semiotic activity implies analyzing multimodal aspects of interaction. According to Knoblauch (2012), one has to apply two types of interpretation in order to preserve the essence of multimodal elements of interaction. The first is to interpret what is seen and heard as it appears from an everyday understanding and from the actors' point of view. The second level is the professional interpretation of the interaction.

The ethnographical aspect of this research is important in terms of the validity of interpretations. Observation of lessons, interviews with the teachers and the written materials collected improve the ability to interpret the situation. The validity of the interpretations will depend on the assumption that "people are existent and, that they have been conducting (acting) in ways that are open for reconstruction (capture) by video data" (Knoblauch, 2012, p. 73). This allows *subjective adequacy*, which means that there is a correspondence between what the researchers say and the statements by the participants. Psychological studies have shown that people often "see events similarly in terms of causal, behavioral, and thematic structures" (Derry et al., 2010, p. 7), which supports the validity of an *everyday* interpretation of interaction.

The empirical material in this study is considered to be *naturally occurring data*. We recognize that the presence of three cameras, two professionals operating them, and one to three researchers observing exert some influence on the situation. However, students today are familiar with cameras, and in consultations with the teachers after lessons they expressed that students behaved as usual.

In order to approach the complexity of the interaction in the groups, the discourse is separated into two main parts: dialogue and multimodal elements. However the two parts are interpreted as belonging to a unified system of communication and therefore seen as integral parts of meaning making. Two methodological concepts will be considered when analyzing the multimodal elements (Knoblauch, 2012, pp. 74–75): *sequentiality*, considering any action as motivated by prior actions and motivating future actions; *reflexivity*, actors do not only act but also indicate, frame or contextualize how their action is to be understood and how they have interpreted a prior action to which they are responding. These concepts correspond well with the dialogical approach which also emphasizes sequentiality.

The issue of multimodal transcription

In attempting to transcribe visual data of video recordings there is a challenge in doing adequate data reduction. The focus of our research is the interaction in the groups. Luckmann (2012, p. 32) argues that "the elements of the interaction which the analyst, based on his knowledge of social life, must assume were relevant to the participants in the original interaction, must be noted in the transcript". Knoblauch (2012, p. 75) argues that video analysis is a hermeneutic activity. "[T]he task set is not to only describe and explain non-verbal behavior". As a researcher one has to decide what knowledge is needed to make sense of a situation and to identify visible conducts constituting the situation. Therefore, multimodal transcribing is not only a preliminary stage to the analysis; the activity forms an essential part of the analysis.

The video material and the written works of students have been examined in order to understand the problem solving process through the dialogue and the semiotic actions that appear both by each individual student and as part of the joint group activities. Several multimodal elements of the interaction have been identified. These fall into three categories of use of mediating resources: *inscriptions* such as drawings, tables, texts, numbers, arithmetic, algebraic (including variable/s); *concretes* i.e. matchsticks; *gestures*

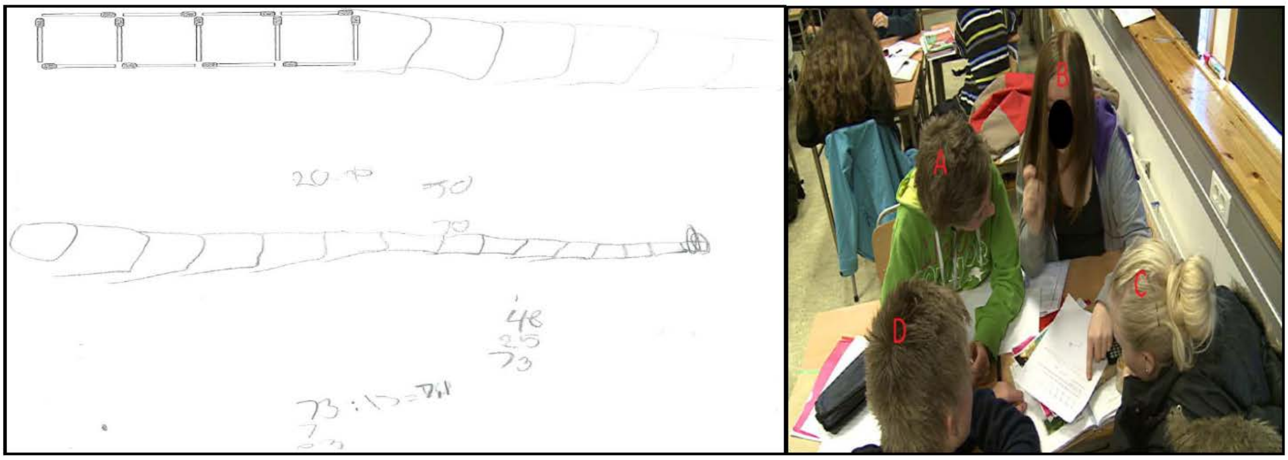

Figure 1: Representation of part of the multimodal transcription

such as pointing, tracing in air/figure/table, glance, rhythmic hand movement, raising hand.

The learning processes and video analysis

Derry and colleagues (2010) stress the importance of being systematic when selecting salient episodes. Schoenfeld (1985) parceled the dialogue according to the mode of reasoning (i.e. planning, exploration) as he expected strategic decisions to be located at the junctures between such episodes. In this study, the dialogue will also be parceled according to the mathematical strategy the students are working with. This is done in order to explore how the students' discourse on the problem evolves during the problem solving process and to reveal mechanisms which drives it. In light of these explorations, fragments of the text which show the first traceable step and its successors in the objectification process will be identified.

An excerpt from a multimodal transcription of a Norwegian group is presented below. A group of 8th grade students, Ben (A), Ann (B), Trish (C) and Sam (D), are given an algebra task (adapted from TIMSS 2007) involving matchsticks and patterns. The teacher hands out toothpicks as a material to use in order to solve the problem. Only Ann writes on the task paper. Marks indicating if the students are in the process of conjecturing (Cj) or convincing (Co) and also specifying the mathematical strategy used such as additive (A) or multiplicative (M) have been inserted into the transcript in order to show the analytical approaches to the text.

8 Trish: We can make them [*squares*] on the table. But should we just use these or? [*Trish shakes the can of toothpicks she is holding in her hand*].

9 Ann: But see, we get 7.1 [*Ann points to the division, 73 divided by 13, she has been working on*], then if you have taken ( ) then you get 7.1 squares. 1, 2, 3, 4, 5, 6, 7 [*Ann points at the squares in the task paper as she counts them and continues by pointing at imaginary squares until she reaches 7*]. So then you get less than sev...then we get, if we make 7 squares. Ok, 4.

The girls try to add a square to the figure using the toothpicks. They give it up quickly as they notice that the dimensions are different.

10 Trish: Ha..ha

11 Ann: You, this didn't work

12 Trish: We'll draw it.

13 Ann: [*She adds a square to the figure by drawing three sides in one motion, she then points at each square as she counts them*] 1, 2, 3, 4...[*adds another square in the same manner*], 5. [*starts counting the matchsticks making up the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14... 17, 18. Ok, but see...ah...I got a good idea...look [*Now she only counts the horizontal matchsticks*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12...[*adds more squares using the same motion*] 13, 14...15, 16...17, 18...19, 20 [*There are now 10 squares altogether*]. So if we take [*She now counts the squares*] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. When we have 20 rows we have [*writes 20 and then counts the vertical matchsticks silently*]...((then we have...then we have)) =

14 Sam: ((But what are we going to do with them...Ann?)).

15 Ann: = When we have 20 we have 50 pieces [*writes 50*]. Or, when we have 20, when

we have 20 such things... [*she points or taps repeatedly at the figure*].

16 Sam: It is those [*Sam holds up a toothpick*].

17 Ann: Yes, matchsticks, then we have 50 altogether [*points to the number written*], used 50 such matchsticks [*points back at the figure*] and we are going to use 73, right? =

18 Sam: Just make...

19 Ann: = So then...

20 Trish: ((really one more will be 53 and then 56))

21 Ben: ((We are going to use...))

22 Ann: No, if we have one more with 10 in it, then it becomes... =

23 Sam: ((Yes because it is four in one)).

24 Ann: = So, then we get 20 more and it becomes 70 [*writes 70*]. ((It is 1, 2, 3...so then we get 70... No, now there is too much here)) =

25 Ben: [looks at Sam and responds to his comment] ((No, it is 3, it is 4 in one and 3...1, 2, 3, 4, 5, 6, 7, 8, 9))

26 Ann: = I think I sort of lost count of it.

27 Trish: No, 70, and then you should have 1 thing more and then it becomes exactly 73.

28 Ann: Ah, but see, oh yes because 20...

29 Trish: It is really only three in each, it is only the first there is four in, and then there is only three in each the whole time [*points at the figure while she explains*].

30 Ann: But see...

31 Trish: If you do like that then...4 [*she holds her finger over the first square*]

32 Ann: 1, 2, 3. [*counts three matchsticks in the first square, then pushes away Trish's finger and starts counting in the pattern she has developed, horizontal matchsticks first and then the vertical ones*] Ok, 1, 2, 3, 4, 5, 6, 7, 8, 9 ( ) 18, 19. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20. 20. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. 11.

While Ann is counting, Ben and Sam start paying attention to something that is going on in the classroom which is not relevant for the mathematical discussion. When the teacher approaches the group, the boys attend to solving the task again.

33 Trish: [*traces the matchsticks in the squares using the same motion as Ann used earlier when drawing new squares*] Oh, you! 73 divided by 3 and then just add 1! [*she picks up her calculator*]

34 Ann: There you said one. [While Trish is working on the calculator, Ann traces first the four matchsticks in the first square and then the 3 matchsticks in each of the following squares. She is using the same motion as earlier when drawing the squares].

35 Trish: No. [*The teacher comes over to the group, but Trish only looks at the calculator while she speaks*] 73 divided ((by 3, plus 1, 25)).

36 Ann: [*Ann looks at the teacher*] ((divided by...3. Is that right?))

In turn (9) Ann suggests a solution to the task based on a multiplicative strategy. In order to make sense of the answer she found, she turns back to the task paper and applies an additive strategy.

The marks in the text indicate important events in the problem solving process. If we focus the attention on the objectification process, we see in (20) the first verbalization of the *3+3* pattern, which is discussed and developed by Sam (23), Ben (25) and Trish (29), and finally expressed as *4+3+3*. However, in (33) we see that Trish traces the matchsticks with the same motion used by Ann that appears early in the text (13), immediately before she expresses a new conjecture for how to solve the task (33). Ann is not taking part in the discussion of the *4+3+3* pattern but seems to drive it with the gestures and the drawing she is making.

CONCLUSION

The video recordings available of 16 groups working with the same task offer an opportunity to study the role of features of thinking in the objectification process. These features, as elaborated through the empirical materials and the theoretical perspective, have been identified as: *elements of reasoning* (sense-making, conjecturing, convincing, reflecting, generalizing), *mathematical strategies* (additive, multiplicative, equations, functional), *semiotic resources* (use of language, inscriptions, concretes, gestures) and indicators of the *culture of collaboration*.

The analytical methods described are developed in order to understand how these different features of thinking are incorporated in learning processes. The ambition is to shed light on a) what role mediating tools play as students decide on mathematical strategies, b) what features of the knowledge objectification process that can be discerned, and c) what are the differences, if any, between classrooms and cultures of work in the different countries.

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ENDNOTE

1. Defined in the project as when letters are introduced as variables in order to collect similar data in the four countries.

A case study of epistemic order in mathematics classroom discourse

Kenneth Ruthven and Riikka Hofmann

University of Cambridge, Cambridge, UK, kr18@cam.ac.uk

This exploratory study analyses the discursive frames through which ideas are developed and evaluated during one section of an early secondary lesson. The study employs a refined version of the classic IRF framework to analyse the interaction structure of classroom dialogue, linking this to semantic analysis of the ideological stance conveyed by participants' utterances and actions. The epistemic order which emerges emphasises the evaluation of ideas primarily in terms of whether they are understandable and make sense. The predominant discourse pattern is one in which the teacher exercises epistemic initiative but offers little overt epistemic appraisal. Occasionally, however, the teacher passes the epistemic initiative to a pupil and provides some form of supporting or concluding epistemic appraisal.

Keywords: Classroom dialogue, discourse analysis, school mathematics, IRF.

MOTIVATION FOR THE STUDY AND THEORETICAL FRAME

In this paper we report an analysis of how classroom dialogue functions to create what we will term an *epistemic order*. By this term we refer to the system of discursive frames within which ideas are developed and evaluated in the classroom. Recent theorisation of classroom discourse has distinguished two crucial dimensions, one concerned with *discourse structure* – the forms of talk and patterns of interaction in play – and the other concerned with *ideological stance* – the degree to which knowledge and ideas are taken as fixed and given as opposed to fluid and open (O'Connor & Michaels, 2007). Such theorisation has also challenged the assumption that these aspects are necessarily aligned; as fostered by the archetypical Initiation-Reply-Evaluation (IRE) structure of classroom recitation in which the opening Initiation move accords the teacher the *epistemic initiative* in posing the question, just as the closing Evaluation move makes the teacher the agent of *epistemic appraisal*.

However, within linguistic research, the limitations of the IRE template – even in representing the structure of conventional classroom dialogue – have long been known, and a broader and more flexible Initiation-Response-FollowUp (IRF) model has been preferred (Sinclair & Coulthard, 1992). Linguistic scholarship has also suggested modifications to the original IRF framework to better model the nuances of classroom dialogue (Coulthard & Brazil, 1992). In particular, the modified version of the IRF framework that we employ is designed to acknowledge the uncoupling of two aspects of the teacher's management of classroom dialogue which typically takes place as that dialogue moves away from an IRE pattern: management of the taking of turns by speakers in interaction slots, and of the substantive exchange of ideas through communicative acts. Thus, our adapted IRF approach introduces a distinction between the *interactional initiative* of launching an exchange, and the *epistemic initiative* of introducing the idea on which an exchange focuses. While one move often combines both types of initiative – as in the classic teacher I within IRE – this is not always so. In the transcript analysis which follows, for example, see E4 where the teacher simply initiates interaction by inviting a pupil to speak, followed by E5 where, by making a substantive contribution in response, that pupil initiates the idea to be discussed.

More recently, pedagogically motivated research on classroom dialogue has shown that triadic IRF interaction patterns continue to play an important part even in more enquiry-oriented classrooms, but fulfil a wider range of functions (Nassaji & Wells, 2000; Truxaw & DeFranco, 2008). This literature has identified markers of what is termed 'dialogic activity' in which classroom talk is more varied in its forms of interaction and more open to exploring differing perspectives.

Such markers include the extent to which questions are posed by pupils, the types of question posed by the teacher, and the kinds of teacher follow-up to pupil responses, including the extent to which responses are not explicitly evaluated and/or are taken up in further exchanges.

DESIGN OF THE STUDY AND ANALYTIC METHODS

In this paper we analyse one section of a video-recorded lesson. Our purpose is to employ an IRF system, adapted for analysis of epistemic order, to establish a baseline against which a later section of the same lesson will (in future) be compared. This later section is of particular interest because it appears to offer a strong example of 'dialogic activity' displaying a distinctive epistemic order. Our adapted system aims to underpin deeper analysis to provide a more rigorous basis for such judgements.

Coding in particular, and analysis more generally, were undertaken against a transcript of the classroom dialogue but involved referring also to the original video-recording. We employed an approach to transcription in which the emphasis was on capturing both the taking of speech turns and the development of substantive ideas (so, for example, excluding exchanges only concerned with classroom management, and omitting repetitions, stumbles or repairs in spoken expression which proved to have no analytic significance). This produces more accessible transcripts while ensuring that all analytic judgements are backed by the source video-record.

One of the reasons that we chose to update the classic IRF system is that it is very rigorously specified. Here, however, space permits us only to set out the essentials of the modifications we made to the original IRF analytic framework. First, a prefix is added to the coding of each move to indicate whether it was undertaken by teacher or pupil. Second, recognising the way in which interaction slots are sometimes linked in practice, the familiar repertoire of I, R and F slots was extended through the addition of two composite types: the R/I type (in which a Response consists of, or develops into, a further Initiation) and the F/I type (in which a Follow-up consists of, or develops into, a fresh Initiation). All moves are given a speaker prefix and slot code.

Next, to distinguish the interaction slot that a move occupies from the communicative acts that it accomplishes, a further code string indicates the character of such acts. Moves which convey some substantive idea about the operative topic (typically I or R moves) are coded either as a solicitation (s) or as a contribution (c), depending on whether they explicitly seek to elicit a response or not. Equally moves (typically F moves) receive a code if they provide an explicit evaluation indicating approval (a) or disapproval (d) of a prior contribution; and if they repeat, restate or revoice (all or a salient part of) a prior contribution (v). Thus in an archetypical IRE exchange the initiating teacher question would be coded tIs, the pupil reply pRc, and the teacher evaluation something like tFav or tFd. The purpose of refinement, however, is not just to enhance the information carried in the code string but to better model moves in dialogue. So, for example, a teacher move which simply nominates a pupil to speak would be coded tI; and the ensuing move in which the pupil poses a question pR/Is.

When italics are used this indicates a part of the code that has had to be inferred.

Finally, as well as recognising the basic interactional unit of the IRF *exchange*, we acknowledge two larger units. A *transaction* is a higher level unit consisting of one or more exchanges, grouped because there is linkage between these exchanges through uptake of ideas (often signalled by the presence of F/I moves). We refer to one particular type of transaction as a *duologue:* where interaction extends over more than one exchange and is restricted to the teacher and a single pupil. An *episode* is a still higher level unit consisting of one or more transactions forming a recognisable structural component of the lesson as marked out by participants and/or resources.

The lesson involved an experienced teacher with a class (aged 11/12) in their first year of secondary education in England. The lesson material came from a module on probability, making connections between mathematics and science (Ruthven & Hofmann, 2013). For reasons of space, we confine ourselves to five episodes making up a lesson section in which the whole class addressed a series of related questions. These questions appeared on two slides about the genetic model of the inheritance of the characteristic of attached/detached earlobes which had also supported a

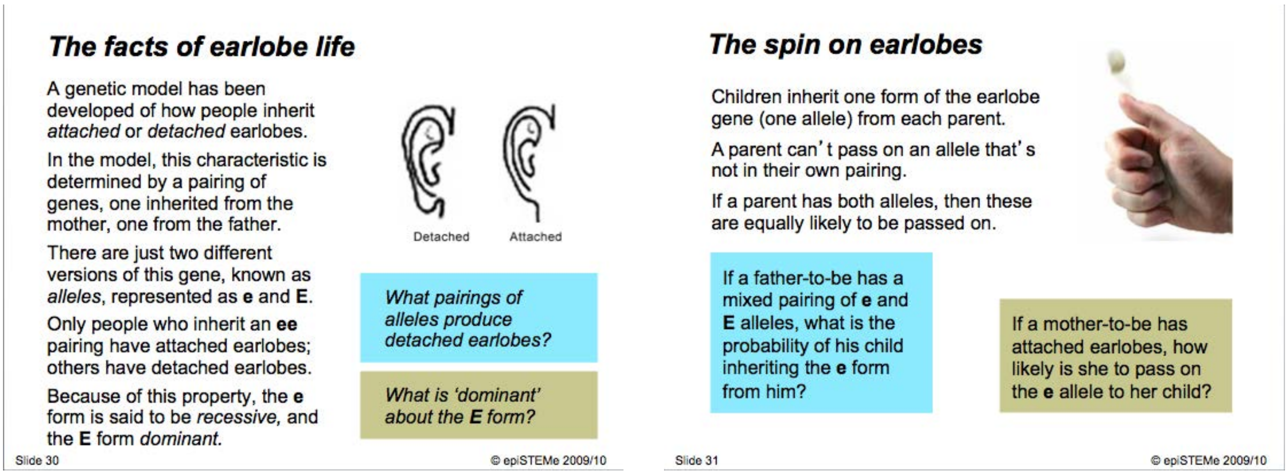

Figure 1: The slides supporting the section of the lesson under analysis

shorter introductory exposition by the teacher. These two slides are shown in Figure 1.

PRESENTATION AND ANALYSIS OF THE INDIVIDUAL EPISODES

Episode A

The first episode takes place when the teacher brings the whole class together after they have independently tackled the first question posed on Slide 30.

A1 T: Dan, what pairings have you got, because I think you look like you've finished? So for detached earlobes. tIs
A2 P: Two large ees.pRc
A3 T: Two large ees. *[Records on board]* tFv
A4 Ps: A big ee and a little ee. pRc
A5 T: A big ee and a little ee. *[Records on board]* tFv
A6 P: A small ee and a big ee. pRc
A7 T: A little ee and a big ee. *[Records on board]* tFv

Rather than repeating the scientific question about pairings posed on the slide, the teacher invokes it at one remove, asking a pupil to report the pairings that he had found (A1). The teacher follow-up (A3, A5, A7) to the pupil responses (A2, A4, A6) (which adequately answer the scientific question) repeats them in a neutral tone and records them on the board without offering any explicit evaluation.

A8 T: Everybody happy so far? tIs
A9 P [Hal]: No I don't get it. pRc
A10 T: What don't you understand, Hal? tF/Is
A11 P [Hal]: The big ees and the little ees. pRc
A12 T: What about the big ee and little ee don't you understand? tF/Is
A13 P [Hal]: How that represents anything. pRc

The transition to a new transaction is marked by the teacher's next solicitation (A8). She asks pupils whether they are "happy" with what has been presented (A8) (rather than, for example, whether it is correct). In response, Hal expresses unhappiness (A9) in terms of his "not getting it". This framing is taken up by the teacher in her subsequent solicitations (A10, A12) about what he "doesn't understand". This co-construction of the situation underpins the reflexive and diagnostic duologue which takes place, in which the teacher's probes serve to elicit Hal's thinking.

A14 T: Can anybody help him? Hal says he doesn't understand about the big ee and the little ee, and he doesn't understand what they represent. Can anybody help him? tIs
A15 Ps: *[Inaudible. Many pupils speaking over each other.]* pRc
A16 T: Not necessarily. No. No no no no. No no. Forget about X and Y. Forget about boys and girls. Just think earlobes please. tFd/Ic

A17	P:	A little ee and a big ee are detached, and a big ee and a little ee are detached, so [inaudible].	pRc
A18	Ps:	*[Inaudible. Many pupils speaking over each other.]*	pRc
A19	T:	All right. I'm not sure that Hal's getting the answer to his question. Hal, are you?	tFd/Is
A20	P [Hal]:	Yeah.	pRa

Opening the final transaction, the teacher invites pupils to help Hal (A14). However, this solicitation (A14) elicits a tangential line of thinking in response (A15; inferred from A16) which the teacher follows up with strong disapproval and a steer back towards the issue at hand (A16). This elicits further pupil responses (A17, A18). It is ambiguous whether the follow-up to these (A19) alludes to the breakdown of orderly talk or to the substance of the help being offered. The teacher solicits Hal's evaluation (A19), positioning him as the arbiter of whether his difficulties have been resolved. His positive response (A20) is allowed to conclude the episode.

In this episode, then, all solicitations are made by the teacher, and so epistemic initiative remains firmly with the teacher. However she exercises little overt epistemic appraisal. In this epistemic order, the teacher directs the unfolding of ideas which are to be evaluated by pupils according to whether they make sense to them. To this end, the teacher employs discursive strategies such as eliciting pupils' ideas and self-assessments, as well as avoiding evaluative follow-up to their responses. Where such follow-up does become evaluative, it relates to redirecting pupil contributions to the matter at hand or establishing their helpfulness to other pupils.

Episode B

B1	T:	Bet, you had a question.	tI
B2	P [Bet]:	Oh yeah. *[Referring to question on slide]* Like what is dominant about the ee then?	pR/Is
B3	T:	*[Reading question from slide]* What is dominant about the ee form? So if you've got a big ee, what is dominant? What are you going to see?	tR/Is
B4	P [Bet]:	If you, the little ee, you have to have two of them to have attached, but you only need one big ee and one small ee to have detached, so there's more ways you can have big ee than little ee.	pRc
B5	T:	Yes.	tFa

The next episode begins with the teacher inviting a pupil to speak (B1). Bet draws attention to the second question on Slide 30 by restating it (B2). Rather than responding by answering this question, the teacher restates it to create a fresh solicitation (B3). Through this discursive manoeuvre the teacher leads Bet to answer the question that she herself raised. When Bet responds with a broadly well conceived answer (B4) the teacher approves it (B5).

B6	P [Tia]:	Surely if you have a big ee then you're going to have detached earlobes.	pIs
B7	T:	Yes. Yes. So if you've got at least one big ee, then you are going to have detached earlobes.	tRav

The final exchange starts with a solicitation from another pupil seeking validation of a variant of the same answer (B6). The teacher approves and slightly refines it (B7).

In this episode, then, the opening solicitation for each transaction comes from a pupil. In both, the pupil puts forward her answer to the question under consideration in a rhetorical form which invites endorsement. The answer is duly approved (and in one case restated) by the teacher. Here, the discursive frame is one in which the teacher cedes epistemic initiative to the pupil but becomes the agent of epistemic appraisal.

Episode C

The next episode relates to the first question on Slide 31.

C1	T:	*[Reading question on slide]* If a father to be has a mixed pairing of ees, so a little ee and a big ee, what is the probability that the child will inherit the little ee. Tia?	tIs
C2	P [Tia]:	Surely it's quite low, because like	pR/Is
C3	T:	Can we put a figure on it?	tR/Is
C4	P [Tia]:	Zero.	pRc

C5	T:	So he's got one of each. He's got a big ee and a little ee. What is the probability that the baby will have a little ee, from their dad?	tF/Is
C6	P [Tia]:	Zero. I think it is.	pRc
C7	T:	Zero. You think it's impossible?	tFv/Is
C8	P [Tia]:	Surely if you have a big ee, somewhere... you're going to have detached?	pR/Is
C9	T:	Yes, but this question isn't about what sort of earlobes the child will have. It's about which of those two alleles the child will inherit.	tRc

The first transaction in the episode consists of a duologue between the teacher and Tia. It opens with the teacher reading the scientific question posed on the slide, restating some elements of it, and then nominating Tia to respond (C1). Tia's emergent answer to the question is ill conceived (C2, C4, C6) and is taken up by the teacher through a series of follow-up and solicitation moves which successively press for greater precision (C3), restate the question (C5), and draw attention to an implication (which would be incompatible with the question situation) (C7). In response to this last probe, Tia finally articulates the reasoning behind her answer (C8), allowing the teacher to pinpoint the misinterpretation of the question that underlies Tia's responses (C9). While the teacher makes no explicit evaluation of these responses, the direction and persistence of her questioning does imply dissatisfaction with them. This duologue, then, has a more dialectic quality.

C10	T:	So he's got one big ee and one little ee, the father. What is the probability that any baby he makes will inherit the little ee. Lea?	tIs
C11	P [Lea]:	[Inaudible] make it a half. [*Pause*] Yeah fifty per cent.	pRc
C12	T:	Lea says it is a half. Tia, you're now saying that makes sense.	tF/Is
C13	P [Tia]:	Yeah.	pRa
C14	T:	Could somebody just confirm why. Why does that make sense? Kit?	tF/Is
C15	P [Kit]:	It says up on the board, if a parent has both alleles, whatever, then there is equal chance, and if it goes up to one there's a half chance.	pRc
C16	T:	So it does indeed. [*Reading from slide*] Equally likely to be passed on. So that makes sense doesn't it. So the probability of a little ee is going to be a half.	tFva

The teacher then restates the question and nominates another pupil to answer (C10). Lea does so (C11). The teacher does not evaluate but, acknowledging a signal from Tia, refers Lea's answer to her (C12), eliciting Tia's agreement that it "makes sense" (C13). The teacher then solicits explanation of "why" from another pupil (C14). Kit's response refers to the key piece of information on the slide and pinpoints how it leads to the answer (C15). The teacher's concluding follow-up endorses this contribution, and reiterates the key point as the basis for the answer "making sense" (C16).

In the opening duologue of this episode, epistemic initiative remains firmly with the teacher, exercised through a series of questions probing the pupil's ideas. While the pupil also makes solicitations, these are by way of response, and in a rhetorical form which insists on a point and appeals for its endorsement. It is the absence of such endorsement from the teacher, accompanied by her probing of the point, that tacitly implies epistemic appraisal. This probing by the teacher leading to the diagnosis which concludes the transaction could, however, be viewed as compatible with the epistemic order enunciated in the ensuing transaction: that ideas should be appraised in terms of whether they make sense and are consistent with the institutionally approved knowledge available. In this transaction, while the teacher retains the epistemic initiative through a series of solicitations, she designates pupils as the primary agents of appraisal, only exercising such a role herself in the final move.

Episode D

The fourth episode comprises a short duologue on the second question on Slide 31.

D1	T:	[*Reading from slide*] If the mother to be has attached earlobes, so the mummy has attached earlobes, how likely is she to pass on a little ee? Tom.	tIs
D2	P [Tom]:	Certain.	pRc

D3 T: Certain. Hundred per cent. Why is that? tFv/tIs
D4 P [Tom]: Because if she's got attached earlobes, then she's got ee ee. pRc
D5 T: She's got two little ees. tFv

The opening teacher solicitation restates the question and nominates a pupil to respond (D1). Tom does so succinctly and correctly (D2). The teacher follows up by repeating and elaborating Tom's answer, and then solicits an explanation of it (D3). Tom highlights the key idea (D4) which the teacher follows up by refining it (D5).

Here, the teacher takes the initiative in posing questions but offers no explicit evaluation of pupil answers. Unlike previous episodes, the notion of appraisal by other pupils in terms of whether an idea makes sense is neither articulated nor enacted here. Indeed, by not subjecting the ideas put forward to such scrutiny, the teacher might be taken to be signalling her own approval of them, so employing a tacit form of appraisal within a more conventionally authoritative epistemic order.

Episode E

The final episode develops from public exploratory talk by two pupils which raises the question of whether both questions on the projected slide are intended to refer to the same situation (E1, E2), a suggestion rebuffed at this stage by the teacher (E3).

E1 P: But, if the mother to be and the father to be, like, are the same mother and father, and they both make, like pIs
E2 P: Yeah, does it matter? Is it like the same child, like, that they're talking about, or not? pR/Is
E3 T: I don't think it's a particular child. [*Pause*] tRc

However, after a short period during which the teacher consults teaching notes and pupils talk amongst themselves, the teacher gives Bet the floor (E4).

E4 T [*In response to indication from Bet*] Yes. tI
E5 P [Bet]: About the question that we've just said. The baby might not definitely have attached earlobes but it would definitely have a little ee because she has two little ees so she you'll definitely have one of them. But depending on what the father might have, detached ears he might have. pRc
E6 T: [*Intervening*] So if we actually join this mother and this father together to make a child. [*Gestures to Bet to speak*]tF/Is
E7 P [Bet]: It could have two little ees or one big ee and one little ee. So he's got one big ee. It's definitely going to have a little ee. pRc
E8 T: Definitely going to have a little ee. tFv
E9 P [Bet]: But it could get a big ee from the father, it could get a little ee. pIc

Bet starts to put forward her ideas (E5). The teacher intervenes by commencing a statement that explicitly restates the hybrid situation, inviting Bet to complete it (E6). Bet does so (E7), the teacher repeats a key phrase (E8), and Bet expands further (E9).

E10 T: [*Turning away from Bet towards class*] So which sort of earlobes is it more likely to have? [*Pause*] If these two parents get together which sort of earlobes is it more likely to have? Hyp, any thoughts? tIs
E11 P [Hyp]: Detached. pRc
E12 T: Bet was just saying that it's guaranteed to have one little ee but it could get a big ee, and I'm saying, what sort of earlobes is it most likely to have. Yes Jay. tF/Is
E13 P [Jay]: Half and half because, because of the father, because if you then get a big ee then it will be dominant, and so it'll be detached. pRc
E14 P: Yeah. pFa

In the final transaction of this episode, the teacher puts a new question to the class and nominates a pupil (E10). Hyp gives an incorrect answer (E11) which the teacher follows up without evaluation. Rather, she recapitulates part of Bet's earlier exposition, restates her own question, and accepts another pupil's bid to speak (E12). Jay's response provides both correct answer and supporting reasoning (E13), which appears to be approved by another pupil (E14). Again, by allowing the transaction to conclude in this way, the teacher might be taken to be tacitly signalling approval.

The teacher's rebuff to the idea proposed in the opening transaction, serves to reclaim the epistemic initiative in the face of an apparent digression. But, in the second transaction, she passes the initiative to another pupil who productively develops the idea. The teacher now changes her position, lending support to the idea, and eventually appropriating it in formulating the question to be pursued in the final transaction. The duologue is launched by Bet, punctuated by shorter contributions from the teacher which extend and revoice key points. In the final transaction, the teacher reclaims the initiative, and dialogue returns to interaction around teacher questions. The teacher makes no explicit evaluation of responses, but she follows up the incorrect answer to her first solicitation by restating the question. She then allows the episode to conclude without following up the sound answer to this second solicitation (or an apparent pupil endorsement of it). This pattern could be interpreted as one of tacit appraisal within a more conventionally authoritative epistemic order.

SYNTHESIS AND DISCUSSION

The epistemic order which emerges from these episodes is one in which initiative is generally exercised by the teacher. The predominant discourse structure involves a tIs>pRc>tF move sequence, where the tF component takes a tFv or tF/Is or tFv/Is form. It is notable that in such an interaction pattern, the teacher does not exercise any overt epistemic appraisal, at least not in the form of an explicit evaluation. Rather she probes the pupil response (C3, C5, C7, D3) or refers it to other pupils for approval and/or explanation (A8, C12, C14). The ideological stance guiding this approach is conveyed by the teacher's references to evaluating ideas in terms of whether they are understandable and make sense (A10, C10), and also whether they are consistent with institutionally accepted knowledge (C16). In particular, in the opening episode, this stance is articulated and enacted through the reflexive duologue concerning Hal's understanding (A8–13) and the ensuing transaction in which the teacher moderates the provision of help by other pupils (A14–21); although when she doubts whether the lines of explanation being put forward by pupils are appropriate, she does make an explicit evaluation (A16 and possibly A19). However, in later episodes (D, E) the referral of responses to other pupils falls away, leading to their ending as soon as a pupil has enunciated an institutionally accepted resolution of the issue of the matter under discussion. This could be interpreted in terms of the teacher exercising – and conveying to those alert to this scheme – tacit epistemic appraisal within a more conventionally authoritative epistemic order.

Beyond this predominant pattern are two occasions when the teacher passes epistemic initiative to a pupil (B1–7, E1–9). With few examples, all that be confidently said is that substantive development of ideas originates from a pI move. On the first occasion, the teacher exercises epistemic appraisal by concluding her exchange with each pupil with an explicit evaluation (B5, B7). On the second occasion, the teacher exercises a supportive, if tacit, epistemic appraisal through expanding on (E6) and echoing (E8) the ideas put forward by a pupil over a series of moves.

Both these discursive frames, then, grant pupils a degree of epistemic agency: over appraisal in the first type, and initiative in the second. In maintaining such frames, of course, the teacher exercises a more fundamental epistemic authority. Indeed, all five of the episodes conform to a pattern in which, once an institutionally accepted resolution of the topic has been enunciated, the teacher directs any further discussion towards elucidating that resolution. Such authority is apparent too on those occasions where the teacher judges it necessary to explicitly evaluate and steer pupils' exercise of agency. Such interventions show devolution of agency to pupils as conditional.

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Writing in mathematics lessons in Sweden

Cecilia Segerby

Malmö University, Malmö, Sweden, cecilia.segerby@mah.se

Previous research has shown benefits for both the students and the teachers in letting students write in different ways in mathematics. Consequently, communicating their understanding is emphasised in the current Swedish curriculum. In this paper, Swedish students' perceptions are examined of their writing during mathematics lessons and their assumptions about the purpose of keeping notes. The results come from a questionnaire answered by 136 randomly selected students in Years 3, 4 and 5 and show that writing is not extensively used during mathematics lessons with calculations being the dominant kind of writing. As well, half of the students considered their notes to be worthless.

Keywords: Mathematics education, primary school, communication, writing.

INTRODUCTION

Research has previously shown advantages from having students write more than numbers and symbols during mathematics lessons. This is because writing helps clarify and organize students' ideas, which then contributes to making sense of mathematics; in this way, students' thoughts become visible and provide opportunities for reflection (Freitag, 1997). In contrast to orally communicating ideas, writing allows students to develop a deeper understanding of concepts (Johanning, 2000). Consequently, writing contributes to documenting students' knowledge and experiences to others. Writing can also be an effective communicative tool as both students and teachers become aware of the student´s understanding, feelings and misconceptions about the content being learnt (Meaney, Trinick, & Fairhall, 2012).

There has been some research which has documented the kind of notes the students write during their mathematics education. Britton, Burgess, Martin, McLeod and Rosen (1975) examined the writing in mathematics of students between the ages of 11 to 18 years and found three categories: transactional, expressive and poetic. Transactional writing focuses on the final product as its purpose is to inform, advice, persuade and/or instruct. These authors found this to be the most common writing. Expressive writing is more personal and has been called "thinking aloud on paper", like a diary. Less than ten percent of the writing collected in Britton and colleagues (1975) research was of this kind. The last category, poetic writing, encourages imagination such as constructing your own exercises, drama and poetry and was about twenty percent of the collected writing.

Meaney, Trinick, & Fairhall (2012) examined the writing in mathematics lessons of students in Years 1 to 11. They divided what Britton and colleagues (1975) categorised as transactional writing into three different genres: description, explanation and justification. Descriptions were of mathematical situations or objects, such as definitions. Explanations showed how mathematical phenomena and events came to be, often through a series of steps showing the working out of a problem. Justifications involved providing information about why something is done and included reflections. In their study, Meaney and colleagues found that calculations were the students´ privileged of writing.

The current Swedish curriculum (Skolverket, 2011) emphasises developing students' communication skills in mathematics to support their understanding. However, there is limited research on the kinds of students' notes used in mathematics lessons and it cannot be found previous research in Sweden on students' opinion about their writing in mathematics.

The aim of this paper is to examine students' perceptions of the writing they do in Year 3 (when they are about nine years old), 4 (when they are about ten years old) and 5 (when they are about eleven years old) during their mathematical lessons. The research questions for the study reported in this paper are:

- What kinds of writing do Year 3, 4 and 5 students perceive that they write down during their mathematics lessons?
- What do these students consider to be the functions of that writing?

A questionnaire study was conducted with 136 students. The result of this study will be the baseline for an intervention study. Before discussing the methodology, next section discusses research connected to writing in mathematics education.

WRITING IN MATHEMATICS IN YEARS 3, 4 AND 5

Writing in mathematics is often considered as useful for developing students' vocabulary knowledge. For example, Lundberg and Sterner (2006) stress the fact that students should have possibilities to build up a dictionary in mathematics. These authors considered some vocabulary to be difficult to comprehend as there are terms (concepts and words) that only can be found in mathematics. Some words, such as odd, can have a different mathematical meaning to their meaning in natural language (Lee, 2006). Having different meanings for the same word can be confusing for students. This may be the case in Year 4 in Sweden as it is at this point that mathematical textbooks are considered to become more challenging with many new concepts and the quantity and complexity of the text increasing, thus putting higher demands on students' reading skills (Myndigheten för skolutveckling, 2008).

The importance of understanding the vocabulary in mathematics is also shown in Vilenius-Touhimaa, Aunola and Nurmi´s (2008) study with students in Year 4 and in Möllehed's (2001) study in Years 4 to 9 where a relationship appeared between students' mathematics problem solving performance and understanding the mathematical vocabulary in the exercises. Yet, according to Misono and Takeda's (2012) study in a fifth-grade class in Japan many of the students did not use mathematical terms and they found it difficult to write descriptions about mathematical operations. As a consequence these authors suggested that teachers need to train the students to describe each step using mathematical terms and not only let the students write down why they were able to get the correct answer by only using numbers.

In Sweden, Ebbelind and Segerby (2015) and Segerby (2014) found that few of the exercises in the textbook require the students to describe, explain and evaluate their understanding, so there are limited opportunities to develop this skill in the textbook. This is problematic since working in the textbook is the dominant practice in mathematics in Sweden (Johansson, 2006; Myndigheten för skolutveckling, 2008).

It has also been suggested that teachers should provide students with instructions about how to structure mathematical arguments, such as justifications and explanations, and to construct narratives, which support mathematical thinking (Meaney et al., 2012). In the research by Hensberry and Jacobbe (2012) with seven students, aged between 5 and 11, a problem solving model was used to structure students´ writing, which led to improvements in problem solving achievement.

Another positive aspect of keeping notes in mathematics was found in Johanning´s (2000) research about problem solving in groups. It indicated that writing helped students to find their mistakes and understand, remember and solve the problem better, when they first wrote in isolation before they met and discussed the problem in groups. However, this study was conducted among Years 7 and 8 students and it is not clear how relevant the results are for younger students.

There are several benefits of letting students write different kinds of texts in mathematics, but it is also essential that teachers explain the aim for the writing and for whom they write so they understand the purpose of doing that (Meaney et al., 2012; Morgan, 1998). In next section, the method for the reported study is described.

METHOD

To examine students' perceptions of their writing in mathematics, a quantitative study was conducted with 300 randomly selected students from Years 3, 4 and 5 (100 in each Year) throughout Sweden during spring 2012. A total of 136 students responded; 50 students from Year 3, 40 students from Year 4 and 46 from Year 5. When randomly selected samples are used, every unit in the target population has the same possibility to participate and it is reasonable to generalize the result. In this study, the selection of the students was

Britton et al.´s categories	Questions in the questionnaire
Transactional involving informing, such as calculations	1a, 1b, 1c, 1e, 1g, 3a, 3b, 3c, 3e, 3f, 4a, 4b, 4c, 4e
Expressive, such as evaluation	1d, 1f, 3d and 3e, 4d
Poetics, such as original mathematical exercises (stories)	2

Table 1: Categories connected to the questions in the questionnaire

done by the Swedish Tax Agency, who has information on all Swedish residents. By using this approach 136 different classroom contexts from across Sweden could be examined to reveal the context of culture involving writing in mathematics. Context of culture refers to what occurs outside language, such as the events and conditions of the world (Halliday & Matthiessen, 2004). For example, context of culture can involve how the mathematic education is designed in Sweden such as the reliance on the textbook and how that affects the teacher´s and the student´s roles.

As mentioned earlier, it is considered that mathematics becomes more difficult in Year 4 with texts becoming longer and many new concepts being introduced (Myndigheten för skolverket, 2008). This is why it was decided to investigate the writing in Years 3, 4 and 5 to see if there were differences in the writing.

Pre-testing the questions in the questionnaire is crucial to its success. Therefore, a pilot study was conducted involving 15 students in Year 3 to examine how the youngest students were likely to reply to the questions. The students answered the questionnaire individually before talking in groups of three or four students. This led to that question four was reformulated from "How does your understanding in mathematics being accessed?" to "How does your mathematics teacher find out about your understanding in mathematics?" With other questions, the numbers of alternative responses were expanded.

The questionnaire contains four questions and all of them, except for question 2, contain closed-response answers where the students could choose one or more alternatives. In question 2 the students could only choose one alternative since it involves students making decision of how often they write down mathematical exercises (stories) of their own. The alternatives in the questions contain limited amounts of texts in respect to the youngest students´ reading ability in this study.

Questions 1 and 2 of the questionnaire examine the first research question concerning what the students perceive that they write down during their mathematics lessons, while questions 3 and 4 examine the second research question involving what the students consider to be the functions of keeping these notes in mathematics.

In order to answer the research questions the categories by Britton and colleagues (1975), *transactional, expressive and poetic*, are used to structure the questions in the questionnaire to visualize the students´ different kinds of writing (see Table 1).

However, what differs between previous studies (Britton et al., 1975; Meaney et al., 2012) and this study is that is does not examine examples of students´ texts. Instead it focuses on students´ opinions and feelings about what they write down during mathematics lessons and what the purpose of keeping those notes are. This approach has previously not been used in Swedish mathematics education research.

In the next section each of the questions responses are presented and discussed.

RESULTS AND ANALYSIS

Question 1: "What do you write down during mathematics lessons?"

The results are presented in Table 2 and in the text that follows they are interpreted regarding the categories of transactional and expressive writing.

Transactional writing: The most common kind of writing, which the students perceived that they did, was writing calculations. This is a similar result to that of Britton and colleagues (1975) which was conducted 40 years ago in USA and suggests that writing in mathematics may have not developed much since then, at least from the perspective of how students refer to it.

What do you write down during mathematics lessons?	Year 3	Year 4	Year 5	Total
a) My operations (calculation)	100%	100%	100%	100%
b) Knowledge goals for the different areas in mathematics	10%	5%	4%	7%
c) Mathematical words with explanations	22%	5%	26%	18%
d) Thoughts about what I think is easy and/or difficult in mathematics (evaluation)	10%	2%	2%	5%
e) How to solve mathematical problems	30%	18%	37%	29%
f) Thoughts as preparation for group exercises in mathematics, such as problem solving exercises	12%	4%	9%	9%
g) Results from practical exercises	12%	11%	11%	12%

Table 2: Summary of results from question 1

The second most common kind of writing was keeping notes on different strategies connected to problem solving. Although this kind of mathematical writing has been connected to improve problem solving (Hensberry & Jacobbe, 2012; Johanning, 2000), only 29 percent of the students considered that they had been involved in producing this kind of writing.

Less than 20 percent of students had written definitions of mathematical words. In Year 4, only 5 percent of the students considered that they had written definitions in their mathematics lessons. Given that previous research has suggested that this is important for students in making sense of what they are learning (Lee, 2006; Lundberg & Sterner, 2006), there is some concern that so few students recognise this as part of the writing that they do in mathematics. As a correlation has been found between Year 4 students' knowledge of mathematical vocabulary and the problem solving performance (Möllehed, 2001; Vilenius-Touhimaa et al., 2008), this result suggests that an intervention study would be most beneficial for students in this Year level. The students, independently of the Year, considered that they rarely wrote about the knowledge goals that they were to achieve in different areas.

Expressive writing: Research has also shown benefits by letting the students write down thoughts as preparation for group exercises (Johanning, 2000), but very few students had done this in this study. Evaluation is another kind of expressive writing and as discussed in the part about the responses to question 4, evaluation in mathematics was said to be done by teachers. This would explain why students rarely wrote their own evaluations of their mathematics learning.

Question 2: "I create own exercises (stories) in mathematics"

This question is the only one in the questionnaire that examines the poetic writing. There is a summary of results in Table 3.

This table shows that in Year 3 approximately 70 percent of the students consider that they sometimes or often create exercises of their own but that number decreases to approximately 30 percent in both Years 4 and 5. The poetic writing seems to occur rather often in Year 3 but is barely used in Year 4 and 5. This kind of writing can contribute to reveal the students´ understanding and misconceptions about the content being taught (Meaney et al., 2012) so this

I create exercises (stories) in mathematics by myself	Never	Rarely	Sometimes	Often
Year 3	8%	26%	56%	10%
Year 4	29%	42%	22%	7%
Year 5	33%	33%	27%	7%
Total	23%	33%	36%	8%

Table 3: Summary of results from question 2

What do you use your notes in mathematics for?	Year 3	Year 4	Year 5	Totally
a) To correct my calculations	28%	38%	24%	29%
b) To practice and look at before tests	8%	15%	28%	17%
c) To look up explanations for different mathematical words	4%	7%	4%	5%
d) To follow my development in mathematics	20%	13%	9%	14%
e) To communicate with the teacher	4%	0%	2%	2%
f) As a help when I solve different exercises	24%	30%	19%	24%
g) Nothing (for my learning in mathematics)	48%	40%	52%	47%

Table 4: Summary of results from question 3

kind of exercises seems to be relevant regardless the students´ age.

Question 3: "What do you use your notes in mathematics for?"

Table 4 shows the results in relation to the students' responses to question 3 in the questionnaire. It is argued that the reported writing can be divided into transactional and expressive writing.

Transactional writing: The results show that approximately 30 percent of the students' used their notes in mathematics to correct their calculations. Notes as a help to solve different exercises were used by approximately 25 percent of them. Less than 20 percent used notes as a preparation before tests. In question 4 (see Table 5) students considered that tests were the most common method that teachers used to determine their understanding. Therefore to find that very few students considered their notes as helpful to study for these tests suggest that students are not seeing writing in mathematics valuable for their learning. As well, very few students considered that they used their notes to communicate with the teacher. Given that it has been found that writing as a communication tool between the teacher and the students can contribute to exposing the students misconceptions, feelings and understanding (Meaney et al., 2012), and both the transactional and the expressive writing are involved. This seems to be an area to focus on in an intervention study.

Expressive writing: It is also interesting to note that older students considered that they used their notes less for following their own development. This is related to responses to question 1 about determining whether what they were learning was difficult or easy. A higher percentage of students in Year 3 considered that they wrote about this in mathematics in comparison to students in Years 4 and 5.

Almost half of the students considered that their notes in mathematics lessons were not important for their understanding in mathematics. This suggests that the students are unclear about the aim of writing in mathematics. If students are unclear about the purpose of their writing, it may be that they do not perceive that they engage in different kinds of writing. The same might concern the use of explaining mathematical words. In question 1 approximately 20 percent considered that they wrote down explanation of mathematical words but according to the result of this question very few of the students used their notes as a dictionary. This might show that they know the words but it can also refer to students understanding of the usefulness in this kind of writing. Therefor it is essential that the aim for the writing is explicit

How does your mathematics teacher find out about your understanding in mathematics?	Year 3	Year 4	Year 5	Totally
a) Test and diagnoses	86%	90%	100%	90%
b) By the number of exercises I have done	28%	20%	17%	23%
c) Through the operations in my counting book/textbook	50%	38%	54%	48%
d) Through what I have written in my evaluation what I think is easy and difficult in mathematics	20%	18%	11%	16%
e) Other ways, such as homework	4%	5%	2%	4%

Table 5: Summary of results from question 4

articulated by the teacher (Meaney et al., 2012; Morgan, 1998).

Question 4: "How does your mathematics teacher find out about your understanding in mathematics?"

Transactional writing: The main sources for the teachers to evaluate the students' understandings are tests and diagnoses (90%). About 20% of the students also indicated that the teacher could work out how much they understood by counting the number of exercises they had done. Approximately 50% of students considered that the teachers looked at the calculations that they had done in their notebooks. The students' perceptions that the teachers focused on their calculations may contribute to them considering that writing mostly concerns doing calculations.

Expressive writing: Less than 20 percent of the students thought that their teachers used what they had written in their evaluations. This indicates that evaluation is not important for the students´ development in mathematics. However, letting the students evaluate their learning visualize their thoughts (Freitag, 1997) and thereby provide opportunities to make the teacher aware of the students´ understanding, misconceptions and feelings about the content being taught (Meaney et al., 2012).

CONCLUSION

According to the results of this study, it can be inferred that few classes in Years 3, 4 and 5 in Sweden use writing in mathematics extensively. Calculations are the dominant type of writing that appears, independently of the Year, and these notes are the main source for the teachers to evaluate understanding in mathematics.

However, in Year 3 it is more common for the students to use different kinds of writing connected to transactional, expressive and poetic functions, but that progressively decreases in Years 4 and 5. With older students the production and use of notes is less thought of as a way to follow their development in mathematics and to communicate with the teacher to expose misconceptions, feelings and understanding in mathematics. This is critical since Year 4 mathematics is considered to be more complex than in earlier Years and correlation between students' knowledge of mathematical vocabulary and problem solving performance has been found.

Further research is suggested where different kinds of writing activities are implemented into a Year 4 class in mathematics to examine how these can contribute to developing communication skills and thereby support the students' mathematical understanding. The purpose for the writing then needs to be explicitly explained for the students so the notes can become valuable for them and not, as approximately half of the students in the reported study say to think, worthless for their understanding in mathematics.

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Language use, mathematical visualizations, and children with language impairments

Elke Söbbeke

University of Paderborn, Paderborn, Germany, elke.soebbeke@math.upb.de

This article presents a first approach and theoretical foundation for a new research project. It focuses on the role of language in the process of communication about structures and relations in mathematical visualizations. Mathematical knowledge is abstract in most cases. Using means of visualization is indispensable for speaking with children about the abstract mathematical concepts. Against this background language is seen as an important tool in the construction of knowledge. But this raises the question, which features of language in children with specific language impairments (SLI) pose a challenge for verbalizing abstract structures. The interest in research relates to the question how students with SLI manage to communicate about the embodied structures in mathematical means of visualization.

Keywords: Means of visualization, communication, structures, language impairments.

EPISTEMOLOGICAL VIEW ON MATHEMATICAL KNOWLEDGE

For a better understanding of the focus on means of visualization in this paper, it is useful to think at first about the special character of mathematical knowledge and the special epistemological conditions involved in processes of learning mathematics.

Mathematics as a science of pattern, structures and relations deals with essentially abstract concepts. From the beginning of the learning of mathematics the young child is confronted with this challenge: Even dealing with elementary mathematical objects like numbers, operations and later the concept of the place value system the child is confronted in a first way with the abstract ideas of mathematical knowledge. Therefore the child has to develop awareness that mathematical concepts are not empirical objects. For example the child has to learn that numbers represent more than only an amount of objects. Rather, it is the theoretical relation between these objects that constitutes the mathematical concept of number.

> In the \`world of objects\`, \`0\`(zero or null) means \`no object\`: and in this world there is no principal difference between the removal of \`5 apples and 5 pears\` or of \`5 black and red chips\`. If, in the model with black and red chips, the same number of black and red chips is given to mean \`0\`, this theoretical relation has to be established \`by one's own and independent activities of thinking\` and, only in this way, a difference is constructed between the chips configurations, which symbolizes a number aspect, and the pears and apples, which belong to the world of things. (Steinbring, 2005, p. 20).

This example points out, that an *amount of ten chips* (five black ones and five grey ones) - that could be seen, counted and manipulated by the child - can also represent the *number zero* (see Figure 1). But this interpretation is only possible if the learning child does not only focus on the real objects, the concrete properties of the objects (the colour for example) but focuses on the *relations that are represented by the different coloured objects*. This particularity is an important basis of the understanding of numbers in general.

In his reflection about the character of numbers, Benacerraf (1983) points out:

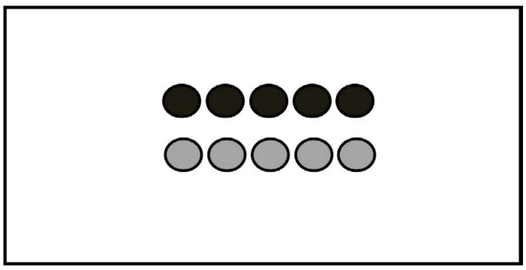

Figure 1: Number 'zero' as a relation between two amounts of five

> Therefore, numbers are not objects at all, because in giving the properties (that is, necessary and sufficient) of numbers you merely characterize an abstract structure - and the distinction lies in the fact that the elements of the structure have no properties other than those relating them to other elements of the same structure. (...) To be the number 3 is no more and no less than to be preceded by 2, 1, and possibly 0, and to be followed by 4, 5, and so forth. And to be the number 4 is no more and no less than to be preceded by 3, 2, 1, and possibly 0, and to be followed by (....). Any object can play the role of 3; that is, any object can be the third element in some progression. What is peculiar to 3 is that it defines that role - not by being a paradigm of any object which plays it, but by representing the relation that any third member of a progression bears to the rest of the progression. (Benacerraf, 1983, p. 291)

This means that even the numbers which often seem to be understood in everyday experience as an amount of concrete objects, or as objects with concrete properties have to be understood in the progression and relation to the preceding and following objects. More generally:

> So what matters, really, is not any condition on the objects (...) but rather in the relation under which they form a progression. To put the point differently - and that is the crux of the matter - that any recursive sequence whatever would do suggests that what is important is in not the individuality of each element but the structure which they jointly exhibit. This is an extremely striking feature. (Benacerraf, 1983, p. 290)

Mathematics is thus as shown above a science, whose concepts are used to describe and analyse abstract patterns and structures. But in contrast to this, in „non-mathematical" everyday experiences mathematics is often understood as a collection of rules, procedures and algorithms. Against this background, the mathematical signs and symbols are often used and (mis-)understood as important „anchor points", which have to be memorized and which are often confused with the underlying mathematical concept. However, the peculiarity about mathematics is - which Duval (2000, p. 61) describes as the "*paradoxical character of mathematical knowledge*" - that just these signs and symbols are not the mathematical concept in itself, they only refer to it, they represent it. Presmeg (2008) describes *signs* as *interpreted relationships* between a *representative* and an *object*.

> I shall take a sign to be the interpreted relationship between some representamen or signifier called the sign vehicle and an object that it represents or stands for in some way. In mathematics, the objects we talk about cannot be apprehended directly through the senses: for instance, "point", "line", and "plane" in Euclidean geometry refer to abstract entities that we can never see, strictly speaking, as in Sfard's (2000) virtual reality. We apprehend these objects, "see" them, and communicate with others about them, in a mediated way through their sign vehicles, which may be drawn or written by hand or through dynamic geometry software, labeled in conventional ways, moved and manipulated for multiple purposes. We work with these sign vehicles as though we were working with their objects. (...) It is this interpreted relationship between a sign vehicle and its object that constitutes the sign. (Presmeg, 2008, p. 3)

The result is a particular challenge for the development and insight of mathematical concepts: If mathematical signs are not the mathematical concept, but only a symbol of a relationship, and if mathematical concepts have to be understood as abstract relations, how is it possible to speak and to reflect about them at all - especially with young children in elementary school?

EPISTEMOLOGICAL VIEW ON MEANS OF VISUALIZATION

For these processes of thinking about abstract structures and teaching mathematics, the use of means of visualization is an important foundation to help young students in building adequate internal representations of mathematical ideas. The mathematical ideas, as

> theoretical ideas, are not things which could be conveyed as completed products. The mathematical subjects consist of relations between things and not in the objects and properties. Therefore, mathematical thinking (...) has to be visualized, in order to represent such relations. (Otte, 1983, p. 190, translated by author)

Goldin and Shteingold describe "the development of efficient (internal) systems of representation in students, that correspond coherently to, and interact well with, the (external) conventionally established systems of mathematics" as a fundamental aim for the process of mathematics teaching (Goldin & Shteingold, 2001, p. 3). But this aim implies a difficult and challenging task for the teacher: As different studies in mathematics education have shown, the intended way from external to internal representations is not straight, easy or clear (Söbbeke, 2005a, 2005b).

For a better understanding of this difficult requirement, it is helpful to consider the elementary learning processes in mathematics education in a more detailed way. For this purpose we will look at the use of means of visualization from different perspectives. In German mathematics lessons chips are a common means of visualization. In the following, the example chips illustrates in which way the interpretation of the material must be differentiated: an interpretation of the chips as concrete to an interpretation of the chips in a first systemic and relational way.

First Perspective: Especially in teaching processes with young learners the counting of concrete objects is an essential activity to develop a first concept of numbers. With this background, objects like "chips" have to be understood as an explanatory background for the understanding of the new number symbols and number words. The concrete materials seem to deliver an explanation for the new mathematical symbols (Steinbring, 2014). The material-based and concrete interpretation of each single object („one chip means '1'") is helpful and important.

2nd Perspective: The interpretation of the ten chips undergoes a first conversion / change if they are sorted in the place value system. A new interpretation of the material is necessary (see Figure 3).

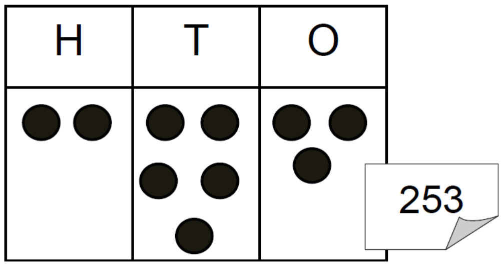

Figure 3: Ten chips in the place value table

The 10 coloured chips should be seen in their structural relationship to each other and to the whole system: Sorted in the place value table, each single chip is given a new meaning: It no longer represents „one", but also „10" and „100" (Söbbeke, 2005).

3rd Perspective: We continue the consideration of the 10 chips a bit further. The 10 chips are arranged now in a rectangular shape (Steinbring, 2014). The relationship of each single element to the overall structure is important. The learning child has to be aware of the "new" structure(-units): e.g. "twos" and "fives". This is an important modification and at the same time a more sophisticated approach to the material: for mathematics learning, the child has to see that the collection of these 10 chips does not only stand for the amount of ten, but also represents essential arithmetical ideas: e.g. "two times five" (2×5) or "five times two" (5×2). This view has to become even more differentiated, for example to see the possibility of a distributive decomposition of the "chips field" like "1×5+1×5=2×5" (see Figure 4).

These examples show that the first perspective on the material, using it to count objects ("one chip means '1' "), successively undergoes a major evolution. By putting the chips into the place value table or in the shape of the rectangle, the children can be encouraged to think about the first *systemic aspects* of the mean of visualization.

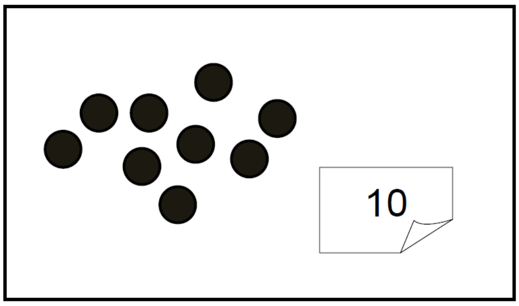

Figure 2: Counting ten chips to develop a first concept of numbers

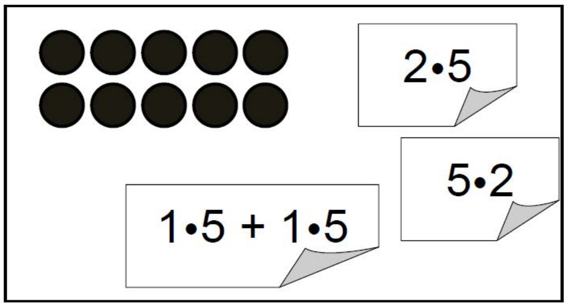

Figure 4: Ten chips structured in a rectangular shape

Against this background, it becomes clear that the children have to learn a new approach to the use and interpretation of means of visualizations (Söbbeke, 2005a): To build new mathematical knowledge it is necessary to disregard the concrete properties of the objects and to understand these objects as elements of the overall system. Therefore the child has to learn to examine the relations, structures and the theoretical ambivalence, which the means of visualization contain. The author's main interest in research belongs to this difficult and challenging requirement for the learning child (Söbbeke, 2005a, 2005b). For further research, it seems relevant to investigate, in how far it is possible for elementary learners not only to develop this systemic view on mathematical visualizations, but also to *communicate about* it.

THE ROLE OF LANGUAGE IN EXPLORING STRUCTURES IN MEANS OF VISUALIZATION

Language and the learning of mathematics

Language is important for the learning of mathematics in two ways: on the one hand language has a communicative function that serves the exchange among the children and with the teacher. On the other hand, it has a cognitive function that advances an increase of new mathematical knowledge (Maier & Schweiger, 1999).

The communicative function of language

The importance of language is highlighted in German mathematics curricula and the German "Bildungsstandards" (educational standards) by emphasizing the activities and processes of "communication" and "reasoning" as prominent skills for mathematics learning. The guiding principle on "patterns & structures" requests the children to describe mathematical relations and structures in general to all mathematical topics. Against this background the communicative exchange can be seen as a necessary component of a stimulating and challenging teaching and learning culture.

The cognitive function of language

Regarding different research traditions, today the learning of mathematics is no longer seen as a purely individual, mental process of the single child, but as a learning process in which the social interaction and communication with others is an essential basis for the development of new mathematical concepts. In this context, the sociologist Miller (2006, p. 200) distinguishes different types of knowledge (cumulative knowledge versus structural or fundamental knowledge) and justifies the importance of social discourse as an important factor for learning.

While cumulative knowledge can be developed quite individually on the basis of experiences by a subject on its own (e.g. a poem or foreign language vocabulary), structural or fundamental knowledge can only increase in the social debate and reasoning of learners with others (Miller, 2006, p. 200). As shown above, mathematical concepts are assigned to the structural knowledge. They arose historically and therefore they require cultural processes -processes of communication and interaction - to be developed and learned. Miller also highlights that new knowledge cannot be completely derived from existing knowledge, because new knowledge exceeds the acquired knowledge. That is, in processes of collective argumentation new findings, beliefs and concepts can be developed.

But children in elementary school are still on their way to become mathematically communicating people and thus „autonomous learners" of new structural knowledge. They still have to learn to derive new structural knowledge from their empirical experiences or activities with concrete materials or means of visualization (Steinbring, 2014). Finally it becomes clear, that the communicative function of language strongly supports and requires the cognitive function. This point shows the reference for the presented interest in research: The teacher has to *reveal an (seemingly) empirical world* to the children, in which they have to make *theoretically significant experiences* and in which they *learn to verbalize* them.

Mathematical communication with children with language impairments

At least in the German discussion about mathematics and language, it is quite popular to discuss primarily the influence of language impairments on performance in mathematics or to develop concrete actions (or aids) to support the learners (e.g., Donlan, 2007; Fazio, 1999; Jordan, Levine, & Huttenlocher, 1995). In contrast to that, the aim of the author's research interest is to better understand the theoretical and epistemological conditions of language use, and analyse the special features of language in the context of the interpretation of means of visualization. As shown above means of visualization deliver an important access to the theoretical and abstract "world" of mathematics.

The language plays a fundamental role to this access: Children learn to derive the theoretical and systemic aspects from their empirical experiences with the concrete materials *only by a special communication about the structures and relations in these means of visualization.* This aspect fits the author's main interest in research: Which kind of linguistic particularities affect the communication about mathematical means of visualization, especially the conceptual and systemic interpretation of the means of visualization? In the following, two essential aspects are described which seem to be relevant for a deeper understanding of the described research interest.

Symbol competence

Studies in cognitive sciences emphasize the importance of an „image and symbol competence" (DeLoache & Burns, 1994) for the cognitive development and learning of young children. They also show that this competence is a challenging skill, which has to be developed: "There are several reasons to suspect that recognition of a depicted object is not equivalent to understanding the nature of pictures or the relation between a picture and its referent" (DeLoache & Burns, 1994, p. 85).

Other studies point out that images are iconic symbols that are associated with a specific content. This "association" is based on structural similarities either on a concrete or abstract level (Elia, Gagatsis, & Demetriou, 2007). In processes of mathematical communication the children have to develop this understanding of symbols, and furthermore the competence to verbalize this. Several observations indicate that children with SLI show a delayed development of such a symbol understanding competence: Understanding signs which establish a relationship between signifier and signified is delayed. This delay can have an effect on the symbol competence in the context of interpretation and verbalisation of means of visualization in primary school education (Lorenz, 2005, p. 4).

Linguistic means to describe relations and generality

In the German discussion about language impairments and learning mathematics, different technical terms have been described, that affect the communication about mathematics: For example problems of seriality include difficulties to verbalize linguistic sequences, to differentiate and interpret them. An example is the major effects that arise in number words by interchanging the individual positions of the digits: 163, 631, 316, etc. (Lorenz, 2005). It is clear that difficulties in the context of seriality not only exchange the names of the numbers (factual knowledge) but can lead to problems for the *structural concept* of numbers.

It is important to understand the spatial relations in diagrams. The use of prepositions, which describe such spatial relationships, are important linguistic means for the development of a concept. In addition, the children have to verbalize relational concepts which combine different objects in a comparative sense and be aware of causal constructions (Nolte, 2000). These linguistic expressions are a key basis to describe first ideas of generalization; they are technical terms that need to be learned.

However, taking into account the epistemological perspective shown in the previous sections, *more* seems to be important than to examine the use of these *technical* terms. My current research aim is to develop a first theoretical framework to analyze the *children's linguistic means to express relations and generality*. The epistemological framework clarifies the special *content* of communication. But it does not clarify *in which way* the language use of children with SLI should be theoretically described. It is necessary to integrate certain theoretical concepts into the epistemological framework in order to describe the linguistic means of children with SLI adequately. A first approach to this could be the distinction of different levels and linguistic means by Akinwunmi (2012).

Mathematicians use algebraic expressions such as variables, terms or equations to express generality. But primary learners have no knowledge about variables to describe mathematical patterns and something universal. Akinwunmi (2012) examined processes of generalization. Through the analysis of clinical interviews with fourth graders, she reconstructed different types of verbal means, which the children (in this context without SLI) used to discover and describe mathematical patterns. Akinwunmi worked out five categories for generalization that could deliver a first access to the presented interest and further work: use of *a representative example*; use of *several examples*, development of *quasi-variables*, *conditionals* and *variables*. These categories embrace a range of linguistic expressions from relating to concrete objects and examples to describing generality.

CONCLUSION FOR RESEARCH

This paper is a first theoretical foundation of a research project. Based on an epistemologically oriented view of mathematical knowledge and the role of means of visualisation, the importance of communication about structures in mathematical representations had been worked out. In the following process of research a solid framework has to be developed. This work includes to integrate certain theoretical concepts (in the field of communication and language) into the epistemological framework in order to describe the linguistic means of children with SLI.

The communication with children with SLI about means of visualisation is of current interest. For mathematics teaching in Germany this requirement constitutes currently a special challenge. Because of a new law on inclusion, children with and without SLI have to be taught together. In order to develop for this kind of teaching not merely superficial recipes, it is important to investigate processes of challenging communication.

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Linguistic norms in mathematics lessons

Kerstin Tiedemann

University of Cologne, Cologne, Germany, kerstin.tiedemann@uni-koeln.de

The present paper reports on a qualitative study that focuses on linguistic norms in everyday mathematics lessons. The central question is what kind of norms students and teachers establish for their language use when talking about mathematics. The underlying data contains videos from two second grade classes. Two short examples from that corpus illustrate the empirical results and show two fundamentally different kinds of linguistic norms.

Keywords: Norm, form of language, function of language.

INTRODUCTION

In Germany, approximately one fifth of all students has a migration background. As a consequence, representatives from politics and science focus more and more on this special group. But international comparative studies like PISA 2000 or IGLU 2001, to mention the earliest, show that learners with a migration background are less successful in the German education system than other students of the same age. Another important result from those studies is that also the group of learners coming from families with a low socioeconomic status performs below average. What these students might have in common with those from migration families is that their language skills tend to be below average. Thus, with regard to both of these student groups, it is important to note that the proficiency in German academic language turned out to be more relevant than other background factors, such as multilinguism, socioeconomic status or immigrant status (see Prediger, Renk, Büchter, Gürsoy, & Benholz, 2013). For that reason, German mathematics educators pay increasing attention to the topic of language in mathematics classrooms.

Language is often seen from a normative point of view. Researchers use categories, such as those of everyday language and academic language (see Gogolin, 2009), and try to assess to what extent the language used in classrooms meets these categories. Having a slightly different focus, some other researchers use this helpful differentiation in order to describe everyday discourses in mathematics classrooms (see Riesbeck, 2010). Thus, although these categories can help us to describe the language which is used, they do not explain which rules or norms the interlocutors themselves establish when they talk about mathematics. One might expect that each learning group establishes its own norms for using language. But which norm could that be? What might be their point of orientation?

To answer these questions, my empirical analyses are focused on the negotiation of linguistic norms in everyday mathematics lessons. These norms are relevant in regard to mathematics learning because they can create or constrain learning opportunities for students. For that reason, the linguistic norms seem to be a good starting point for changes in mathematics classrooms.

LINGUISTIC NORMS

If a student talks about mathematics with the teacher or with other students in class, he will have to coordinate two levels in one utterance: First, there is the content level and, second, there is the language level. The student has to find an accepted and understandable way of expressing his mathematical idea. According to these two levels of an utterance, Sfard (2008) distinguishes between object-level rules and metadiscursive rules. Object-level rules are narratives about "regularities in the behavior of objects of the discourse" (p. 201). Sfard provides the example of the sum of angles in a polygon: "The sum of the angles in a polygon with n sides equals $(n-2)\times180°$". One can imagine that this narrative guides the content level of an utterance in a mathematics lesson. But, by that rule, nothing is said about when and how to say something about the sum of angles. This metadiscursive level is governed by another kind of rules: the metadiscursive rules. They offer orientation concerning the question of "when to do what and how do to it" (Bauersfeld, 1993, quoted in Sfard, 2000, p. 167). Sfard (2008, p. 201) clarifies that these rules "define patterns in the activity of the discursants trying to produce and substantiate object-level narratives."

Yackel, Cobb and Wood (1991), Yackel and Cobb (1996), and Voigt (1994) give further examples of metadiscursive rules which they name as norms. One of these examples is that of social norms. They are related to classroom interactions in general and are not necessarily unique to mathematics, e.g. teachers do not only expect their students to give answers, but also to explain and to justify them (Yackel & Cobb, 1996, p. 178). In contrast, sociomathematical norms are not object-level rules in Sfard's sense, but they have a component that focuses on the content level. For example, a sociomathematical norm guides the understanding of what counts as an acceptable mathematical explanation in a specific learning group (see Yackel & Cobb, 1996, p. 461). On closer inspection, one can see that these sociomathematical norms combine the content level and the language level. For example, in order to produce a good explanation you have to meet the established object level rules, but your utterance has to be appropriate to the recipient as well. To meet the latter requirement, you have to use a specific language. For the moment, I would like to push the focus even more on the language level than it has been done in the work of Yackel and Cobb or Voigt. As it is known the language is for some students a barrier to learning, it is necessary to understand the language use in detail. That is what my focus is on. One might expect that, similar to social or sociomathematical norms, teachers and students also negotiate norms for their use of language. This is what I refer to as *linguistic norms* in the following. Thereby, we have to keep in mind that these linguistic norms affect the content level and the development of mathematical learning processes because they determine the medium of learning, the language.

To supplement the preceding ideas, I would like to clarify the difference between rules and norms. Both terms refer to regularities that one can reconstruct in a given interaction. But, according to Sfard (2008, p. 204), not every rule is a norm. But to be a norm, a rule has to fulfill two conditions. First, it must be "widely enacted within that community." Second, it must be "endorsed by almost everybody." The first condition might be regarded as fulfilled if a norm is enacted in a situation where almost all the students are listening. We can think of an interaction between one student and the teacher in front of the board with everyone else listening and looking in their direction. Then, we can say that this norm is enacted in that specific learning group. In contrast, it is often impossible to decide whether the students really endorse the established rule, thus, to decide whether the second condition is fulfilled. But, as Sfard (2008) remarks, a norm does not necessarily have to be accepted by really all the members of a community to be considered as such. She stresses that in order to be a norm the rule has to be endorsed especially by those within the community who count as experts. Similarly Tatsis (2013) shows that a student might violate a norm to protect her or his face. Nevertheless, the considered rule might be a norm. To put it methodologically, the violation of a norm does not automatically mean that this rule was not a norm. What can help us in this regard is Sfard's (2008, p. 204) clue that a norm becomes explicit and most visible when it is violated. Then, this violation evokes "spontaneous attempts at correction". To sum up, a norm distinguishes itself from a rule by its obligation. While rules guide an interaction towards observable regularities, norms do the same, but are treated as a requirement by most of the interlocutors, in particular by the teacher.

LANGUAGE BETWEEN NEARNESS AND DISTANCE

Koch and Oesterreicher (1985) offer a distinction that has proved to be helpful in the context of linguistic norms. These authors differentiate the medium of language. Thus, an utterance can be phonic (oral) or graphic (written). For the context of this paper, I focus on phonic language in face-to-face interactions in everyday mathematics classrooms. Secondly, Koch and Osterreicher identify two conceptions that an utterance may have. It can be conceptually oral or conceptually written. This affects the question of communication strategies used. As an example, it is easy to recognize that we use different communication strategies when talking informally to our mother compared to when we are giving a lecture at the university. Conceptually oral language is often used when interlocutors are directly related and can refer to a given situation, e.g. an informal conversation with one's mother. This is what, consequently, gives a specific form to the language. For example, at any time, the interlocutors can ask questions of understanding, show emotions and influence the course of the interaction. For that reason, sentences may be short and even incomplete. Referring to the given situation, the speakers use deictic expressions and gestures. Thus, orality is characterized by interlocutors who spontaneously negotiate their roles and the course of their interaction. Koch and Oesterreicher call this a language of nearness. Examples are a conversation in the family (medially oral) or a chat among friends (medially written). In contrast, conceptually written language is used when the interlocutors are not necessarily in direct relation and processes of language production and language reception might be separated from each other. Thus, aspects of the situational and cultural context have to be made explicit. As a consequence, sentences are longer and more complex. The writer forms main clauses, but also subordinate clauses in order to express the relations he wants to inform about. And s/he uses more specific terms, e.g. mathematical terms, to be precise and explicit. Koch and Oesterreicher call this a language of distance. Examples are a text of law (medially written) or a scientific lecture (medially oral).

What we can learn from Koch and Oesterreicher's concept is that the language we use is not only characterized by its form, but also by its function. We have different purposes when talking to our mother or when giving a lecture at the university. The form of language changes according to these functions. To say it with Halliday (1994), we use different language registers in different situations. In the concept of registers, Halliday combines the aspects of form and function.

METHODOLOGICAL BASES

The goal of my research is to describe which kind of linguistic norms teachers and students establish for their language use in mathematics lessons. I have focused on primary classrooms and on conversations in which both students and teachers are involved. The reason for the latter decision is that the teacher plays a special role in the negotiation of (linguistic) norms. Because of his position in the social situation of a classroom, his norms might be accepted as such in almost every case. Thus, if the teacher introduces a norm and no student disagrees explicitly, we might assume that the rule is enacted in that learning group. By reconstructing linguistic norms in this way, we should keep in mind two issues. First, there might be other norms which are enacted only by a few students and which do not come to light in a class conversation at all. In this regard, Tatsis (2013, p. 1632) talks about "'minor' norms". But, for an individual, these norms can be as important as those confirmed by the teacher. Second, not all linguistic norms that are established in the classroom have to be desirable ones, some might even hinder mathematics learning (see Tatsis, 2013, p. 1629). My research goal is to describe types of linguistic norms which are established in the mathematics lessons and which are an important (although not the only and maybe not even the best) part of the learners' linguistic environment.

I filmed everyday mathematics lessons in two German classes of second grade for several weeks. The data contains both one unit about arithmetic (orientation on the hundred field) and one unit about geometry (characteristics of geometrical bodies). In all cases, I filmed the whole lesson and decided afterwards which scenes might be interesting with regard to my research aim. Those scenes were transcribed and became objects of analyses. In the process of selecting scenes, analysing them and describing the results, I revised the selection of scenes several times.

To get access to occurring negotiation processes in mathematics classrooms my analyses have to be of a reconstructive manner. Thus, they are analyses of interaction (see Cobb & Bauersfeld, 1995). This method refers to the interactional theory of learning, is based on ethnomethodological conversation analysis (see Sacks, 1996), and was devised by a working group directed by Bauersfeld. This method is especially suitable for my research focus because it is aimed at the thematic development of a given face-to-face interaction and allows to integrate both perspectives: the mathematical and the linguistic one.

EMPIRICAL EXAMPLES

To illustrate my results, I have selected two short scenes from the different learning groups. These examples allow reconstruct different types of linguistic norms, although they are very similar concerning the content. In both cases, the teacher and the students deal with the hundred field. According to the teachers who usually plan their mathematics lessons together, the goal of those lessons is the orientation on the hundred field. The students should find numbers quickly and describe their positions exactly. From other lessons, the students know already what rows and columns are. For that reason, both teachers start their lessons with a repetition.

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 1: Diagram of the hundred field

Example I: Ozan's number puzzle

At the beginning, the teacher, Ms. Kunz, and the students talk about rows and columns on the hundred field. They repeat that rows "go from left to right" and that columns "go from top to bottom". After that, Ms. Kunz starts to give number puzzles of the following form: "The number is in the third row and in the fifth column." When having given a puzzle, Ms. Kunz always asks one child to solve it. The following scene starts when the course of action changes. Now, one student is allowed to give a number puzzle on his/her own. It is Ozan, a young boy with a Turkish migration background.

47 Ozan: The number is in the fourth column and... in the third r-
48 Ms. Kunz: No, first the *row*.
49 Ozan Err... The number is in the third row and in the fourth column.
50 Ms. Kunz: Okay. Choose a child, Ozan.

If we assume that Ozan would have finished his number puzzle by saying "row", we can state that he has given a correct number puzzle (47). With respect to the content and the language, he has formed an appropriate sentence. However, he deviates from those examples that Ms. Kunz has given before. The difference is that he refers to the column first and then to the row, not vice versa. Ms. Kunz insists on her order and therefore on a specific sentence structure (48). The structure that she demands cannot be derived from general object-level rules or general meta-discursive rules. Instead, she starts to establish a new norm that only applies for this specific learning group. In his following turn, Ozan accepts the proposed norm. He restructures his number puzzle according to Ms. Kunz' example without change of content (49). He describes the position of the 24 again. Thus, in this example, it is evident that the linguistic norm which Ozan and Ms. Kunz establish does not focus on the mathematical content at all, but on the grammar. The interlocutors develop their own grammar for number puzzles which can be described as follows: When giving a number puzzle with regard to the hundred field, you have to refer to the row first and to the column thereafter. Ms. Kunz confirms that norm explicitly in the end: "Okay." (50) Finally, by asking Ozan to choose a child that may solve his puzzle, she leads back to the course of the discourse. The negotiation of the linguistic norm has come to an end.

In this process of negotiation, we can see that the teacher's language use is conceptually oral. Thus, Ms. Kunz forms an incomplete sentence in order to ask Ozan for a correction (48). That makes her utterance short and allows a rapid progression of the discourse. This kind of language use is typical for face-to-face interactions in our everyday life (see Koch & Oesterreicher, 1985). When looking at Ozan's language use in comparison, we can note that his utterances are rather conceptually written. He is asked to form a complete sentence with a given subject ("the number") and with two indications of location that should be linked by an "and". Thus, his sentence is rather complex and more explicit.

An interesting point about the reconstructed norm is its potential influence on the content level. On closer inspection, Ozan and Ms. Kunz establish a linguistic norm that might be relevant to mathematical learning processes. When you always name the row first, you allow an early and quick estimation of the size of the specified number. Thus, the row indicates the ten, the columns indicates the one. So when you know in which row the number is, you know approximately how big the number is. From that example, we can see that linguistic norms can create (or constrain) mathematical learning opportunities.

Example II: Zeynep's number puzzle

The second scene is taken from the corresponding lesson in the other learning group. The initial situation is quite similar. Ms. Yilmaz and her students talk about rows and columns and repeat that rows "go from left to right or from right to left" and that columns "go from top to bottom or from bottom to top". Thus, there is a small difference in defining the row and the column compared to Ms. Kunz' class. While Ms. Kunz and her students only mention one direction, Ms. Yilmaz draws attention to two possible directions in each case. After that, Ms. Yilmaz also starts to give number puzzles: "My number is in the sixth row and in the second column." Then, she always asks one child to solve the puzzle. In the following scene, Zeynep, a young girl with a Turkish migration background, is the first student who is allowed to give her own number puzzle.

16	Zeynep:	In the ninth colow-
		[In German, Zeynep says "Speile" which is a mixture of the two words "Spalte" (column) and "Zeile" (row). Accordingly, "colow" should be a mixture of the two words "colums" and "row".]
17	Ms. Yilmaz:	Err, either column or row.
18	Zeynep:	*Row*. Row nine and column four.
19	Ms. Yilmaz:	Yes, *good*. We can also say it like that, like Zeynep has done it. Who
20		can *say* to Zeynep now, which number is in the *fourth* column and in
21		the *ninth* row?

While giving a number puzzle to her class, Zeynep uses a word that does not exist in German (16). But it is easy to recognize its components. It is a hybrid of the German words for column and row. For that reason, one cannot decide whether Zeynep is talking about the ninth column or about the ninth row. Ms. Yilmaz interrupts her, shows a little bit of irritation by hesitating ("err") and asks Zeynep for a decision: column or row (17). In the following, Zeynep seems to be quite clear about that point and stresses her answer: "*Row.*" (18) Then, she gives her number puzzle in a new form: "Row nine and column four." In contrast to her first try (16), she does not use ordinal numbers any longer to appoint a specific column or row. Instead, she uses the numerals like names which do not necessarily have a mathematical meaning. Ms. Yilmaz accepts and values this new way of formulating a number puzzle and leads back to the course of interaction. She asks the class who can solve Zeynep's puzzle (19–21). In this short scene, Zeynep and Ms. Yilmaz establish a norm for the formulation of a number puzzle. From now on, it seems to be allowed to waive ordinal numbers and to choose from different linguistic possibilities. What remains the same through these different linguistic forms is the content. Zeynep and Ms. Yilmaz describe exactly the same position on the hundred field by their different sentences; they are both talking about the number 84. Thus, we can see that the linguistic norm which Zeynep and Ms. Yilmaz establish rather focuses on the function of language. They do not seem to be interested in a specific form of language, but in a precise description of a position.

In that process of negotiation, Zeynep's language is essentially characterized by orality. She uses mathematical terms (row, column), but does not form any complete sentence. Instead, her utterances are rather short and understandable in the context of that specific situation. In contrast, Ms. Yilmaz uses different registers. Parts of her language are conceptually oral, too. When pointing to Zeynep's word mistake, she forms an incomplete sentence, reduced to the necessary (17). Like in the scene before, that makes the utterance short and allows a rapid progression. But in the end, when Zeynep has given a correct number puzzle, Ms. Yilmaz offers a grammatically complete sentence (19–21). That way, Zeynep and all the other students who listen to the discourse can learn two different versions of number puzzles, one is conceptually oral (Zeynep) and the other one is conceptually written (Ms. Yilmaz). As both versions are valued, the learners are obviously free to choose one of them.

By negotiating the reconstructed norm, Zeynep and Ms. Yilmaz do not only touch a matter of language, but a matter of content, too. When a student who listens to the discourse recognizes that there are different ways of formulating a number puzzle, he may ask himself what the invariant through the different forms of representation might be and, thereby, he gets to a mathematically interesting question: What is necessary to describe a position on the hundred field clearly? Thus, one can see again that linguistic norms refer to the content level and cannot be separated from them.

CONCLUSION

The two examples do not only allow me to reconstruct two different linguistic norms, but two different types of such norms as well. Thus, they focus on different aspects of language. In both cases the interlocutors negotiate how one might talk about mathematics, in this specific case about a position on the hundred field. In the first example, the focus is on the form of language.

Thus, everyone is asked to always refer to the row first and to the column second. One could say that the interlocutors establish their own kind of grammar for number puzzles. In the second example, the focus is on the function of language. While the interlocutors obviously accept different linguistic forms of giving a number puzzle, they check whether an utterance fulfills its function: "We can also say it like that." (19)

When reconstructing these different types of linguistic norms, one can note that the belonging processes of negotiation are linked to different language registers. Thus, the negotiation of linguistic norms which are orientated to the structure or form of language rather need a conceptually written language (Example I). This assumption seems plausible because it is necessary that someone produces the whole linguistic form when you want to talk about it. Only then, it is clear what the topic of the discourse is. In situations like that, short clues or incomplete sentences might not be enough. In contrast, the negotiation of linguistic norms which are orientated to the function of language does not necessarily need a specific register (Example II). According to the situation, both a conceptually oral and a conceptually written language obviously might meet the given requirements. One can assume that the use of a certain register depends on the mathematical content. Thus, difficult mathematical contents possibly require a more written conception of language so that complex relations and processes can be described accurately, whereas rather simple contents might be expressed in a conceptually oral language, too.

Embedding the two empirical examples in the wider field of my data, it can be stated that it is not possible to assign one linguistic norm to one learning group. Instead, both types of norms can be observed in both classes from time to time. But when students and teachers negotiate how to speak about mathematics, there is a clear emphasis on one of the two types of norms. Obviously, it is not helpful or even possible to negotiate the form and the function of language at the same time. Furthermore, the differentiation between these two types of linguistic norms in mathematics classrooms can help us to be clear about the learning goals that we have as teachers and also to be clear about the perspective that we have as researchers. First, as a mathematics teacher, we cannot support the language development of our students in all respects at any time. Instead, we can decide where to put the focus on, so that we can offer situationally suitable exercises and materials to our students. While we work on some kind of grammar in one lesson, it might be more important to work on the function of language (e.g., a precise description, a convincing argumentation) in another lesson. And of course, there are lessons where our focus is not on the language at all. Second, as researchers, we possibly like to know more about the language development of learners in mathematics classrooms, about good materials and learning arrangements that help teachers to support their students' language development, but we should distinguish what the main focus is in each case. Depending on the linguistic norms which are established in a certain mathematics lesson at a certain time, the interlocutors have different needs of support and we might see fundamentally different ways of using language.

The two examples show that linguistic norms are related to the content level. By defining norms for their language use, interlocutors define their medium for teaching and learning. Thereby, they create or constrain learning opportunities. We can conclude that Sfard's (2008) distinction between object-level rules and meta-discursive rules is an analytic one. We can distinguish between the content and the language when we look at everyday interactions as researchers, but in the classroom, these levels always belong together and might interact with each other.

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Reconstruction of teachers' professional vision concerning important aspects of classroom interaction

Anna-Marietha Vogler

TU Dortmund University, Dortmund, Germany, anna.vogler@math.uni-dortmund.de

Participation in classroom interactions gives students the opportunity to learn subject-specific topics and to acquire discourse competences. Hence, there is a multitude of research concerning arrangements which enable learners to successfully participate. But what counts as an adequate contribution, enabling students to successfully participate in mathematical classroom interactions and, especially, what counts from the teachers' perspective? Based on the fact that the teachers' perspective on adequacy of considerations in interactions highly influences the teachers' acting and support in the classroom, the study INTERPASS investigates the teachers' perspective within video stimulated group discussions. One of the identified motives of the teachers' perspectives is presented in detail in the following.

Keywords: Group discussion, professional vision, participation in classroom interactions, documentary method.

INTRODUCTION

Acquisition of mathematical knowledge takes place in classroom interactions and through the opportunity to participate in the process of negotiation of meaning involved (Cobb & Yackel, 1998; Sfard, 2008). To be able to participate, content-specific comprehension and linguistic abilities of students as well as the teachers' efficiency in producing interactive scopes for participation are important.

On the one hand, based on a constructivist interactionist view, a lot of research has been done to empirically examine different aspects of influence on opportunities to participate by analysing classroom interactions. On the other hand, teacher education programmes have been created to establish eligible acting repertoires of teachers to foster interactive support. The so-called 'mistake-handling' concerning students' contributions thereby plays an important role in creating these programmes (Heinze, 2005). Two examples of professional demands on teachers on the basis of these acting repertoires are (1) to provide explicit feedback on the adequacy of students' contributions and (2) to form discursive competences. But, also ideas of students should be embedded in the interactive process in the mathematics classroom to foster participation. More or less successful teaching experiments often serve as examples of the implementation of professional demands into classroom interactions. This form of training is mostly based on the idea that one prototypical conceptual situation can be established in the classroom situation. Hence, most of the time the perspective of protagonists on such more or less supportive interactions was not conceptually considered in detail and realisation in every day classroom failed (Prediger, Quasthoff, Vogler, & Heller, 2015).

Furthermore, Sherin (2007) describes that the teachers' perspective and especially their conceptions and perceptions concerning linguistic and content-specific aspects of classroom interactions are strongly related to the way they give support. Thus, the teachers' expectations of students' contributions provide an insight into the teacher's supportive behaviour and their way of establishing opportunities to participate. But what counts as an adequate contribution for teachers in mathematics? What do teachers define as aspects of successful classroom interactions?

To answer these questions I will draw upon the constructs 'selective intention' and 'knowledge-based reasoning' developed by Sherin (2007) to investigate conceptions and perceptions of teachers empirically.

In this paper, I therefore analyse thematic motives of negotiation processes within video-stimulated group discussions of teachers. Here, Sherin's approach allows me to focus on recurrent motives of teachers concerning supportive or non-supportive interactions shown in the videos. What kind of linguistic, content-specific and interactive aspects of the shown classroom situations do they partake? Pursuant to the analysis, possible insights into the teachers' perspective on what counts as a successful interaction are delivered. Furthermore, corresponding to the presented empirical data I present one of the major negotiation themes and pick up the question, whether there are differences amongst the eligible repertoires of action between interactionally relieved or stressed teachers.

PARTICIPATION IN CLASSROOM INTERACTIONS

The theoretical principle for the following explanation is a constructivist view on learning as considered in cultural historical elaborations by Sfard (2008) and interactionist articles referring to approaches of Cobb and Bauersfeld (1995).

Participation in mathematics classroom interaction gives students the opportunity to learn linguistic, content-specific and interactive aspects of the discourse. However, to participate in these interactions puts great demands on students, because negotiation processes of mathematical meaning have their own alternating dynamic (Krummheuer 2011). To participate in classroom interaction, students have to interpret what topic is negotiated as well as how and when an adequate contribution can be presented. Thereby, mathematical meanings and linguistic demands of the discourse can be perceived and interpreted differently by all participants, students and teachers, depending on their definitions of the situation. Students' ability to comprehend the ongoing interactive process and its demands as well as the competence to produce matching contributions within the classroom interplay are important for participating and thus for learning. If students are not able to produce these matches, they can be excluded from the classroom discussion over time (Jablonka & Gellert, 2011). But who decides and controls what counts as an adequate contribution? Different studies have shown that in most classroom interactions the teacher is central to this controlling process (Lee, 2007). "Teachers' verbal utterances trigger, encourage, discourage, 'delete' students' verbal contributions and allocate evaluations accordingly" (Prediger et al., in press). These evaluations and feedbacks give accessory advice for expectations concerning content-related conceptions and conventions that are substantial for the specific discourse (O'Connor & Michaels, 1993). However, research on classroom interaction shows that mostly the 'rescuing' of an interactive fluency or a communicative order is more relevant than progress in the development of content-related aspects of the communication (for description of the characteristic funnel pattern: Bauersfeld, 1995). The phenomenon is justified by the assumption that teachers, stressed by the pressure to act in time in classroom interaction (henceforth: 'interactionally stressed teachers'), have to react on students' considerations in time and have to spontaneously manage the negotiation of meaning towards the content-related goals of learning and the communicative order within the polyadic interaction. The way in which teachers take up learners' contributions is crucial for their opportunity to participate in classroom interaction and to learn both mathematical and discursive competences.

TEACHERS' PROFESSIONAL VISIONS

A lot of classroom studies have described how the support of teachers enables students to participate, and what kind of support is particularly conducive for learning and participation. Some of the studies also give advice on how to fulfil certain conditions to help students to participate. Nevertheless, teachers' enacted strategies often do not match these demands in spite of different professional development programmes. An explanation of the mismatch between requested acting repertoires of teachers and everyday classroom interactions (of course) could be the pressure affecting teachers to act in time. Anyhow, there is a lack of comprehensive teachers' perspective regarding considerations about the improvement of classroom support. Relating to this, Sherin (2007) pointed out that every teacher has a professional vision, meaning the way she or he makes sense of issues happening in the classroom and that is shared in the professional community of teachers. This professional vision influences the way teacher act and also support in the classroom. For teacher training programmes it is necessary to conceptually comprise these perspectives of teachers and to improve the teacher trainings. But how can the teachers' professional visions towards the support in classroom interactions be observed?

To this end Sherin (2007) reconstructed the teachers' professional visions on classroom interactions in so-called 'video clubs' over time. She proposes to distinguish analytically between the process of 'selective attention' (of aspects from classroom interactions) and 'knowledge-based reasoning' wherein teachers link their perceptions with own experiences and knowledge. Especially the selective attention is taken into account in this paper to reconstruct recurrent motives of teachers in such discussions. These motives represent specific pattern of perception, evaluation, and interpretation. Hence, through these motives it should be possible to draw conclusions from the teachers' active repertoires of acting that are characteristic for them. Concerning the selective attention, Sherin (2007) observes that in the first meetings of the video clubs, when all participants were stimulated by videos for the first time, teachers exclusively focused on pedagogical aspects of the contributions of the other teacher. The students' acting was for the first time mentioned in the third meeting. Therefore, I will also pick up in details the process of knowledge-based reasoning in the exemplary analysis.

RESEARCH DESIGN AND EMPIRICAL DATA

The study presented in this paper is part of the larger project *INTERPASS*, an interdisciplinary study of linguists and mathematics educators, being led by Uta Quasthoff and Susanne Prediger at TU Dortmund University. The study combines a classroom video study and group discussions of mathematics and German teachers while they are relieved from any pressure of classroom action. Within the classroom video study 10x12 mathematics and language lessons (each 45–60 min.) in five grade five classes were recorded during the first inquiry. By means of comprehensive sequential analysis five sequences were selected for the group discussions.

The following three comparative categories were selected: 'lessons of mathematics and German', 'matches and divergences in micro-cultural practice' and 'German native speakers and speakers of German as a second language'. Also, only sequences showing emergences of subject-specific matches and divergences, i.e. sequences containing the formerly introduced structure and are therefore particularly substantial for the process of socialization in the discourse, were selected. The group discussions are based on video presentations of different sequences of interaction lasting an average of two minutes. To create the possibility of a detailed discussion based on the video data, the sequences were additionally transliterated and shown at the end of each unit of interaction via beamer. This paper focuses on group discussions with teachers. Four discussions with five to ten teachers, each lasting 1.5–2.5 hours, were recorded. The group discussions, based on this paradigmatic analysis, were held with German and mathematics teachers at different German secondary schools (Gymnasien). Each group only met once. After a short introductory round, the teachers were asked to observe a video of a short interactive sequence and to comment on it. This was the only impulse for the group discussions.

METHODICAL APPROACH FOR DATA ANALYSIS

The focus of analysis is exclusively limited to video clips from mathematics classrooms. In order to analyse the teachers' selective attention, parts of coherent negotiation processes concerning thematic motives of the teachers within the group discussion were identified. According to this, there is a short overview over all relevant categories of motives the teachers mentioned in their discussion. To select these parts of the group discussion, the methodical approach of the documentary method of Bohnsack (2009) is applied. The first step of this analysis is to organise the transcribed video material in interactional units, which are interpretatively described in categories. These categories are developed with respect to the content interpreted from the ongoing process of negotiation within the group discussion. Based on the sequential interpretation of 'turns' within these interactional units the negotiation of meaning, respectively the thematic development, is reconstructed. Therefore, (1) the categories describe the topics of all negotiated taken-as-shared themes, respectively all motives of the teachers. After the development of these *descriptive* (sub-) categories they are (2) summed up to the following seven main categories from the discussion of the teachers: participation aspects, acting and turn aspects, classroom management, teaching aspects, social aspects, aspects of professional identity, and subject specific aspects.

In the following sections, I will summarise aspects from the categorisation and present a detailed interpretation of a scene that is exemplary for two topics discussed, delivering first insight into teachers' motives.

ANALYSES OF TEACHERS' PROFESSIONAL VISIONS

A discussion between five teachers is analysed: two German teachers (Mrs. Nachbar, Mrs. Fuchs-Focke) and three mathematics teachers (Mr. Neumann, Mrs. Jacobi, and Mr. Klein) from different German Gymnasiums in urban areas. Background for the discussion in the following scene is the video "Explaining the procedure of rounding" from a mathematics classroom interaction on how to round 63 to 60 (see Prediger & Erath, 2014, for a more extensive transcript and analysis of the episode), wherein one teacher (Mr. Maler), a male student (Kostas) and a female student (Katja) in grade five interact. After the teacher had asked for a solution of rounding 63 to the nearest tens, the boy Kostas answered: "And then ... when you ... take away three and write down a zero, well ... you could do it now but actually it is wrong, you have to round down and wr ... write down the number closer to zero". Kostas describes the rounding rule based on the basic concept of geometrical representation of proximity and distance for a particular tenner on the number line by his answer. Thereby he marked that rounding is not only changing the last number to zero, but even more: one has to identify whether the last number is closer to the previous or subsequent tenner. But also Katja gives a solution: "You round down with zero, one, two, three, four and with five, six, seven, eight, nine, you, ... you round up". She termed the mathematical concept. While the teacher Mr. Maler does not evaluate Kostas' utterances positively, he comments on Katja's considerations with the phrase "Did everybody get that?"

In the group discussion presented below, the two mathematics teachers Mr. Neumann and Mr. Klein discuss the explanation of Kostas. Flashpoint for the discussion was the difference between the reaction of the teacher to Kostas' and Katja's contributions. Mr. Klein pointed out that Katja gave a "perfectly clear" answer.

1 Mr. Klein: Well, he did not1 name the rule. She has defined exactly...
2 What happens with each digit? When? This actually is the criterion.
3 When to round up or down he said eventually in the last sentence?
4 Closer number to zero. Well, he probably meant...
5 Because sixty-three is closer to sixty? Could you now...
6 This is highly interpreted. But...
7 Mr. Neumann: Yes sixty. Or seventy. Right?
8 Mr. Klein: Yes sixty or...
9 Mr. Neumann: The question is...
10 Why does he always have the same seventy...
11 Because sixty is the closer number.
12 Mr. Klein: That is a bit the... What is behind? Without being just the dull rule?
13 But that is difficult already! Also, to understand something at this
14 sound level. What he meant and...it was not really phrased clearly.
15 But...
16 Mr. Neumann: Now it is too... Because the five solution is not there.
17 That is important.
18 Mr. Klein: Well, anyways it was not clear.
19 Because he was not even counting the digits...
20 Mr. Neumann: Yes. Closer number to zero.
21 Mr. Klein: Logically... You can say...
22 Mr Neumann: It is certainly more general now he has to well count the
23 numbers, right?
24 Mr. Klein: Yes, exactly.
25 Mr. Neumann: That is already...
26 There is already an achievement.
27 Mr. Klein: Well, probably he tried to think it through!
28 Without possible understanding... Maybe ...
29 Mr. Neumann: But he has not phrased it by a rule, right?
30 Well, that is what is missing... and that is...
31 Mr. Klein: Exactly!
32 Mr. Neumann: But basically they hear....
33 And Kostas has not given it to him.
34 Mrs. Now it is named...
35 Fuchs-Focke: Wonderful.

ASPECTS OF THE RECONSTRUCTION OF THE INTERACTION PROCESS

Acting and turn aspects (with emphasis on students' concepts)

After 30 minutes of discussion, the teachers explicitly deal with the content of Kostas' solution for the first time. At this point of time, the presented sequence starts. Mr. Klein describes the central idea of the geometrical representation (proximity and distance on the mental number line) on which Kostas' answer is based for the first time (line 4–5). Mr. Klein and Mr. Neumann both highlight several positive aspects of Kostas' solution concerning his mental number line. These aspects are: creativity ("without being just the dull rule" – line 12), universal validity ("it is certainly general" – line 22) and cognitive performance of finding a solution ("There is already an achievement. Well, probably he tried to think it through!" – line 26–27). But also Katja's answer to the teacher's question is regarded to be sophisticated. Mrs. Fuchs-Focke gives a particularly positive evaluation of Katja's contribution in the last sentence (line 34–35), when she refers to Katja terming the rule discussed in the previous scene. Also Mr. Klein refers to this answer as "defined exactly" in his first sentence (line 1). The descriptive (sub-)category in this interactional unit can be summed up as 'rating of students' contributions within classroom discussion'. Thus, the category for the analysed sequence is marked as 'acting and turn aspects'.

Teaching aspects

Already in previous scenes, not being covered in this paper, the aspects of the teacher's acting concerning the didactical goal of the lesson are very prominent. The five teachers broadly discuss Mr. Maler's goal of the lesson and his methods to reach this goal. During the presented interaction several aspects of the students' utterances are named. The teacher Mr. Klein refers to the missing match between initiation of the teacher and Kostas' answer: Kostas does not phrase a rule (which is demanded by the teacher) (line 29–30). Also Mr. Neumann agrees with this negative evaluation of Kostas' answer (line 31). He names the high degree of the implicitness of Kostas' solution as reason for the teacher's evaluation (line 13–14). Although the student's contribution is evaluated positively towards aspects of subject-specific content, both teachers put the quality of the contents of the students' statements in another perspective as it is not matching the interpreted goals of Mr. Maler. In this case, the loudness within the classroom (as a additional context caused problem) (line 14), the absence of a solving strategy in case of the five (as a content regarding problem) (line 16) and the lack of comprehension are identified as a communicative problem of the student's statement, letting the rejection of Kostas' statement seem to be 'reasonable'. The subcategory that is found here is described by the phrase: 'description and rating of teachers' acting concerning didactical goals of the lesson'. Concerning the utterances of Kostas, the teachers share the opinion that it is legitimate to reject the solution because of a mismatch regarding the didactical goals of the teacher Mr. Maler. They agree that a rule is required to complete the lesson's goal Therefore, the category 'teaching aspects" can be summed up.

Acting and turn aspects (with emphasis on comparison of students' contributions)

Mr. Neumann's contributions at the end of the scene completed the process of comparing both students' utterances and the teacher's reaction to them (line 29–30 and 33). All three teachers agree on the crucial reason for Mr. Maler to reject Kostas' statement in the process of classroom interaction. The absence of a rule description is more important than the high quality of Kostas' described concept. They argue for the rejection of Kostas' contribution with his lack of linguistic standard. Mr. Klein, for example, points out that Kostas' solution is not phrased clearly (line 14). The teachers discuss that Katja's rule is the appropriate answer to the given task of justifying the solution 60. Furthermore, the way she presents it can be seen as a socially accepted practice of the rounding rule. In line with that, Kostas' solution is not accepted because his answer does not fit into the interactive fluency to reach the goal of the lesson. Therefore, the rejection of Kostas' solution is legitimated several times by the teachers referring to Mr. Maler's need to reach educational goals. The high degree of implicitness of the rejection Mr. Maler shows in the video is not mentioned within this short discussion. But while Mr. Neumann interprets Mr. Maler's rejection another time, he positively remarks that the teacher avoids face-threatening reactions towards Kostas. Besides the negotiation about the contributions of the students Katja and Kostas, also the comparison of both students' considerations result in the category for the analysed sequence as "acting and turning aspects". The interactive process is not mentioned in the presented sequence.

Reconstructed professional visions

The reconstructed aspects of the discussion process show that two major motives of teachers' selective attention can be identified: namely (1) keeping track of teachers' goals (Prediger et al., in press) and (2) rating of students' considerations. Concerning the first motive, the group-discussion teachers interpret easily that the favoured reaction of Mr. Maler is to give just positive evaluation on contributions that match lesson goals (for details, see Prediger & Erath, 2014). For these teachers the alignment of feedback with goals of lessons is a natural process of classroom interaction as well as legitimation for reaching educational goals. In comparison to the reactions given by the discussion group towards the different contributions of Katja and Kostas, it can be assumed that the saving of the fluency of the interactional process in classroom interplay is one of the major demands. Both Katja's and Kostas' utterances could be the starting point to develop a description for the rule for rounding. Nevertheless, the teacher only pays little attention to Kostas' solution, even though it is possible to evolve the rounding rule by his geometrical approach. The problem for students to achieve a high degree of discourse competence to manage the demands of an accepted participation in mathematical classroom is not mentioned within the discussion. These findings are confirmed by other scenes and through the clusters of recurrent combinations of (sub-) categories. Hence, aspects of legitimation are mostly motives that are mentioned while teachers discuss methodical aspects.

CONCLUSION

With this short insight into a complex group discussion it becomes evident that some non-supportive findings from classroom studies, like the implicitness of demands for presentation and content, are also judged as adequate from the teachers' perspective. However, there is a gap between the normative professional demands, which result from research on classroom interactions, and the professional visions of teachers (for details, see Prediger et al., in press). Especially motive (1) is of particular importance for teachers. While the teachers discuss pedagogical motives in detail, the content specific quality of students' utterances is picked up for the first time after 30 minutes of the discussion. This phenomenon could also be confirmed (with few exceptions) in other reconstructions of interaction units in different group discussions. These findings confirm results from Sherin (2007). Participants from Sherin's research as well as the teachers in our first video-stimulated group discussion of classroom interaction focus on the pedagogical behaviour of the videotaped teacher. In future, these qualitative findings shall be triangulated by quantitative analyses of the categories of the interaction units.

Despite the presented motives of the discussion group, it is remarkable that there is no stress on the interactive process and the negotiation of mathematical meaning. Considerations of the students' statements are only given in form of their matching with regard to the didactic ambitions of the teacher. From that perspective, the *interactionist* demand for support, enabling students to participate in classroom interactions, can be seen as contrary to the motives reconstructed here. Therefore, the process of negotiation of meaning, giving particular attention to learners' ideas, is opposed to the focus of keeping track of the teachers' goals. Also, the lack of Mr. Maler's acting to not provide explicit feedback on the adequacy of the contribution is not mentioned in our discussion group. Although the feedback is an important turn within the structure of the process of the negotiation of meaning, it seems to be adequate to the teachers that this turn is absent in the case of the interaction between Kostas and Mr. Maler.

Comparing the results from this illustrated analysis with reconstructions from research in classroom interactions, one can assume that also *interactionally relieved* teachers prioritise actions that are contradictory to normative professional demands. This is an unexpected result for research on group discussions. One consequence of this finding is to adjust approaches of teacher training programmes. Hence, the teachers' perspective on classroom interactions and supportive situations should be integrated into the process of forming professional demands and eligible acting repertoires. The inclusion of empirically based approaches concerning teachers' motives makes it thus possible to mention also meaningful motives of teachers instead of substituting them through the idea of one prototypical conceptual situation or new techniques.

ACKNOWLEDGEMENT

The research project INTERPASS ("Interactive procedures of establishing matches and divergences for linguistic and micro-cultural practices") is funded by the German ministry BMBF (grant 01JC1112) under the guidance of Susanne Prediger and Uta Quasthoff. I thank them, Kirstin Erath and Vivien Heller for the collaboration.

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ENDNOTE

1. In the transcribed sequence stressed words or appointments are coded in bold letters. All specialities of the spoken language (mistakes, grammar, etc.) are mentioned in the translation of the transcribed sequence.

Students’ language repertoires for prediction

David Wagner, Joseph Dicks and Paula Kristmanson

University of New Brunswick, Fredericton, Canada, dwagner@unb.ca

For communication about prediction (both relating to probability and to conjecture), language is by nature recursive – language is an indicator of meaning as well as a force that shapes meaning. We describe how this recursive nature of language impacted the choices we made in a cross-sectional longitudinal study aimed at gaining insight into children’s language repertoires relating to conjecture. We then use some of the data from the project to identify issues relating to interpreting data in such a context. Finally, we raise questions about implications for educators.

Keywords: Language, mathematics education, prediction, conjecture.

INTRODUCTION

The understanding of possibility, risk, and certainty, like the understanding of any mathematics, is mediated by language. Certain language repertoires are necessary to convey the ideas. At the same time, the language used to describe risk shapes the way people conceptualize it. This recursive nature of language compelled us to develop a research project to investigate children’s language repertoires in relation to conjecture. Having noted similarities in the language of conjecture and of prediction, we structured classroom and interview prompts to encourage students to make predictions and we talked with them about the meaning of the things they said. In this paper, we focus on research choices in relation to this endeavour. First, we describe choices we made to gain insight into children’s language repertoires. Second, we use some data from the project to identify issues relating to interpreting data in the characteristically mathematical contexts of conjecture and prediction.

Moving beyond our academic interest in mathematics education, we will argue that the issues we identify may be significant for understanding everyday experience. In particular, we will raise questions about the impact of mathematics class experiences with uncertainty. We will also raise questions about the impact of intertextuality between uniquely mathematical ways of communicating about conjecture and everyday ways of interacting about authority.

COMMUNICATING ABOUT UNCERTAINTY

The investigation of conjectures (hypotheses) is one of the most important mathematical processes. Much mathematics teaching focuses on enabling students to perform particular mathematical procedures, such as adding fractions, factoring polynomials, and calculating probability. These skills appear as standards in curriculum documents and frameworks (e.g., CCSSO, 2010) that are used by curriculum planners and teachers. Research and professional literature, including curricula (e.g., New Brunswick Department of Education, 2010) and curriculum frameworks, point to the necessity of students learning these intended outcomes through the exploration of mathematical problems.

When people explore a mathematical problem together, as with mathematical investigations in classrooms, it is necessary to have a way of suggesting an idea before knowing it is true. Rowland (2000) noted the centrality of such conjecture to mathematics, and coined this “space between what we believe and what we are willing to assert” (p. 142) as the Zone of Conjectural Neutrality (ZCN). Because of the recursive relationship between language and experience, the language resources available affect the possibilities for making conjectures.

Our theoretical perspective for this research draws on the work of Vygotsky and Wertsch related to the connections between thought and language, and, in particular, the central role that language as social interaction plays in the process of learning. Nevertheless, we have found it a challenge to avoid deficit framing because of the shaping force of one’s language repertoire. Deficit framing suggests that one's own way of speaking or thinking is superior by evaluating whether or not others have acquired the same skills. In the study of linguistic variation for numbers, the one area of mathematics register variation that has been documented significantly, Swetz (2009) pointed out how cultures have been rated on the extent of their number systems. In our research we are more interested in the potential for linguistic variation to open up opportunities to understand mathematics differently – for example, how does thinking of numbers as verbs (e.g., Lunney Borden, 2010) change one's conception of counting and arithmetic operations? In our case, how does linguistic variation express itself in relation to understanding probability? Because linguistic variation in mathematics (besides the area of number) has not been researched significantly, the discussion requires careful research to move forward.

Our focus in the article is on participants' repertoires for expressing modality. Modality refers to linguistic tools for expressing degrees of certainty, for example the use of modal verbs like *must* and *could*. "It must be six" is stronger, and thus has higher modality than "It could be six." Some modal verbs—e.g., 'can'—are ambiguous. "You can be excused from the table" indicates a degree of obligation; "You can finish the race" indicates ability; "I can help you" indicates inclination; and "It can be a six (because one of the remaining cards in the deck is a six)" indicates probability. When students hear the word *can*, what does it mean to them?

In an example of research that may appear deficit framed, Shaffer (2006) explained how deaf children with hearing parents did not develop what she called "The Theory of Mind" because of the absence of modality in their vocabulary. Theory of Mind relates to the understanding that different points of view are possible. Linguists Martin and Rose (2005) described the effects of modality (sometimes called 'modulation') this way: "it opens up a space for negotiation, in which different points of view can circulate around an issue" (p. 50) – a description that bears close resemblance to Rowland's ZCN. Shaffer reported that once the children developed vocabulary for modality (in American Sign Language) it became clear that these tools facilitated their quick development of this Theory of Mind.

We are especially interested in the way children use language to express modality in mathematics contexts (and beyond) because modality is important in conjecture (Rowland, 2000), to describe uncertainty, and to understand other points of view (Shaffer, 2006). In our research, we did not aim to look for holes in children's language repertoires. Rather, we focused on attending to the ways they talked about their understanding, to help us see a range of ways to talk about and understand conjecture and uncertainty.

METHODOLOGICAL CHOICES

The data for our cross-sectional longitudinal study comprise audio- and video-recordings from English-medium and French Immersion instructional contexts in an Anglophone region in Canada. Students worked in groups in class and were subsequently interviewed, extending the group work. At the end of the interviews we asked students about the meaning of words they used to describe degrees of certainty.

For each mathematical context we tried to avoid using specialized mathematical language ourselves. We know from second language acquisition literature that learners are generally good at noticing and, subsequently, using the language used in interactions with more able speakers (Long, 1996). We wanted to hear what language skills the children in our research used to communicate their ideas without setting them up with the specialist language to build on. As we struggled to construct problems without use of specialist language, we found that larger narrative contexts made this possible. Other strategies we considered became grammatically awkward. These narratives also made the problems accessible to very young children, perhaps partially because of the lack of specialty language, but mostly, we think, because they connect to children's experience. In addition to embedding our questions in a narrative context, we attempted to avoid specialized uncertainty language when we interviewed participants about their predictions in contexts based on uncertainty. We agreed it would be acceptable to use a language strategy after the participant did, but not before. This proved very difficult; indeed, in the interviews we often used words we intended to avoid, and sometimes used incorrect or awkward structures in attempts to avoid this. After completing most of the first year's classroom and interview interactions, we agreed amongst our team that we should be less paranoid about avoiding specialty language, but knew that this issue would plague interpretation of the data (from before and after our decision to loosen up.)

The first year's participants were in Grades 3, 6, and 9. We had them play a modified version of skunk, which is a game often used in the teaching of probability (e.g., Brutlag, 1994; Neller & Presser, 2004). We had them play in pairs so that they would be more likely to talk with each other about their ideas and strategies. We introduced the game with a narrative like this, varying slightly between contexts because we did not script the narrative: "I was picking strawberries in the forest. After a while, when my basket was quite full, a skunk wandered into the berry patch. I ran away so the skunk would not spray me. I lost the berries in my basket when I ran off." (This narrative also gave a reason for calling the game *skunk*.) Participants had a pile of beans (representing the berries), a cup (the basket), and a bowl (home). When the researcher rolled the die and called out the number, participants put that number of berries in their basket. A six represented the skunk. When it was rolled, everyone would lose the berries in their baskets. On the other hand, if they "went home" (dumping their beans into their bowl) before the appearance of the skunk, their berries were safe. We played seven rounds – one berry-picking expedition for each day of the week. We played the game with participants in their classrooms first. The following day we interviewed groups of students and played again but with six cards bearing the numbers one to six instead of the die. The interviewer would not replace the cards into the deck until the deck was completely played out, at which time it would be reshuffled. Thus the participants experienced the difference between independent and mutually exclusive events in probabilistic situations. During the game, the interviewer would ask the participants to say why they made their choices about when to "go home." After the game, the interviewer would ask participants about specific things they said, asking for clarification on meaning. The camera operator was helpful in this regard, acting as a second interviewer. She or he could make notes on what participants said, which was relatively difficult for the primary interviewer who was busy with the cards and interaction.

In the second year, participants from Grades 4, 7, and 10 (catching some of the same students as the previous year, one grade earlier) predicted the 50th car on different trains based on the first seven cars. The narrative context of this situation had the researcher tell a story about waiting with a friend for a train at a level crossing, and deciding to predict what kind of car the fiftieth car would be. Trains were then shown using presentation software, with an engine and the first six or seven cars, each labelled with their number. After students made their predictions about the 50th car, we had the train accelerate and then decelerate to settle on the 50th car. As with the game of skunk, we had students work in groups to draw out communication.

The sequences presented to students varied considerably to defy expectations of certain kinds of patterns. The cars were distinguishable by colour and shape – Yellow (Y) cars were rectangular boxcars, green (G) cars were tankers, and blue (B) cars were flatbeds carrying big triangles. Train 1 showed Y,G,B,Y,G,B,Y and continued with a pattern of threes (YGB). Train 2 showed Y,G,Y,Y,G,Y and continued with a pattern that increased the number of Ys before each G – i.e., Y,G,Y,Y,G,Y,Y,Y,G, etc. Of course, the initial seven cars could have suggested a pattern of threes (YGY) similar to the previous train – i.e., Y,G,Y,Y,G,Y,Y,G,Y, etc. For this train, we stopped the train at around the 25th car to let students reconsider their predictions. Of course, we invited students to tell us their reasoning whenever possible. Train 3 showed B,G,B,B,G,G, etc. and continued with B, B, B, G, G, G, etc. with increasing groups of B and G. The interviews on the following day started with Train 4 showing Y,B,G,Y,Y,B,G. It continued with groups of four (YBGY) – i.e., Y,B,G,Y,Y,B,G,Y etc. Train 5 started with Y,B,G,B,P,B,Y and continued with a random collection of cars, in which the colours started to misalign with the shapes and new kinds of cars appeared. As with Train 2, we stopped train 5 at around the 25th car so we could hear the students reconsider their predictions. In addition to the confounding randomness of the fifth train, there was no 50th car – it had only had 42 cars. As with skunk, we ended these interviews with questions about distinctions among various language choices we heard the students use.

For this paper, we focus on one interview with four Grade 6 students playing the game of skunk. However, we make some references to other data within the project to illuminate certain findings through comparison. This group of students was not identified by their teacher as exceptional in any way. The school is in an area that has relatively low socio-economic indicators. As noted above, these four students played skunk in class the day before, and subsequently one of our research team interviewed them – first playing skunk with cards instead of a die, and then asking them about some language meanings. We asked them to play skunk in pairs, and they somehow came to an implied agreement that the pairs were competing against each other.

LANGUAGE USED TO EXPRESS UNCERTAINTY

These four 11- and 12-year olds show considerable language repertoires, which we found to be the case for even the most mathematically and linguistically novice students in this project. We were the most careful about and attentive to modal verbs because of our earlier research and teaching work.

The modal verb *have* expresses high modality because it refers to events that must occur. The interviewer used it first (though trying to avoid doing so) in turn 111 and it was used again in turn 229 when Chris talked about the difference between playing skunk with cards and with the die: "It's easier this way because when the skunk first came you just don't have to worry." It wasn't used again until the interviewer asked questions about its meaning. Here is a short version of that discussion.

319 Interviewer: [Yesterday] I heard Terry say when you're working in your groups, "Do we both have to write this down?" So what's the difference between "it has to be the skunk" and "she has to write this down"? Is the "has to" the same? "This has to be the skunk." "She has to write it down." Do you notice a difference between them? ...
346 Terry: Do we both need to, like, do we both need to write it down?
347 Interviewer: No, but it's a proper use of the word. But is it the same as "this has to be the skunk"?
348 Terry: No.
349 Interviewer: No? Why not?
350 Terry: Because you know it has to be.
351 Dale: It absolutely has to be.
352 Interviewer: It absolutely has to be.
353 Terry: Yeah
354 Interviewer: But when asking "do you have to" it's not absolutely
355 Terry: No, yeah
356 Interviewer: Okay
357 Dale: Because the fire bell or something could ring or something and you all go outside and you don't have to write it down.
358 Interviewer: Don't have to write it down but if the fire bell rung this would still be the skunk.
359 Dale: It would still be the skunk.

We note that, to clarify meaning, the students introduce new vocabulary that was not part of the interview up to this point. Terry used the modal verb structure "need to" to emphasize the necessity of "have to." Dale introduced the adverb *absolutely* to further emphasize this sense. The students distinguished between instances of 'have to' depending on context.

We had a similar conversation about the modal verb *can* which had been used in its various forms, including *can't*, by the students in the interview. We started this part of the conversation by referencing Dale's writing in class earlier. When asked what is the most number of berries they could get in a day, Dale wrote, "You can get any number because it could just keep going." (This was with playing skunk with a die) The researcher referred to Dale saying in the interview that it is different with the cards because "we can't keep going." What follows is again a short version.

391 Interviewer: That *can't* – If you're wanting to go visit your friend, and your mother or father says that you can't go over to your friend's house, is it the same kind of *can't*?
392 Terry: No, that means you're not allowed. ...
395 Interviewer: You're not allowed to
396 Terry: Yeah.
397 Interviewer: Or how do you know it's not the kind of can't that Dale said? Where it just can't possibly happen? How can you tell the difference?
398 Terry: By the way she says it.
399 Chris: Yeah. ...
418 Interviewer: When you said earlier "you can't win", which one is that closest to? Remember, when you looked at your basket and you said, "Oh, we can't win." Is that like the "you're not allowed" or is it
419 Terry: It would be you can't
420 Leslie: You don't
421 Terry: Like you, it's impossible, like
422 Leslie: Yeah, it's impossible.
423 Terry: Well, it was because if you added it all up, the skunk
424 Dale: You'd only get, like, fifteen.

425 Terry: The skunk would have come.
426 Chris: Yeah, you'd only get fifteen so if the skunk is that...
430 Terry: Because the skunk was gone.
431 Interviewer: It would have been impossible.
432 Terry: Yeah, yeah
433 Interviewer: So if someone says *can't*, ... if I told you that you can't divide by zero in a lesson on dividing would you think that that means that you're not allowed to or that it is impossible to do?
434 Chris: That it is impossible.
435 Interviewer: Why would you think that?
436 Terry: Because you can't divide by zero.
437 Interviewer: Why can't you?
438 Terry: Because it is impossible.
439 Interviewer: How do you know?
440 Chris: Because you can't
441 Terry: Because you can't
442 Chris: If it is zero, you can't put it in any groups.

In this case, Terry introduced the adjective *impossible*, to clarify the meaning of *can't*. No one had used the word before this in the interview. As with "have to", the students distinguished among instances of *can* and *can't* based on context. During and after this interview, we wondered how students could make this distinction for instances in which they do not know a convincing logical argument for the assertion. With the example of division by zero, the students now knew that it is impossible, but how might they have thought about it the first time they heard their teacher say "you can't divide by zero"?

The students introduced three adverbs/adjectives that indicate degrees of probability into the interview. The word *probably* was first used by Leslie and not used again by others. When Leslie and Terry were considering whether or not to make the same choice about going home as the other group, Terry remarked, "One of [our groups] won't lose everything and the other would" (turn 204), and Leslie replied, "It is probably going to be us" (turn 205). The adverbs *absolutely* and *impossible* came up in the conversations about language choices when the students were trying to explain what the modal expressions meant, as noted above.

Other modal verbs used included *would*, which was first used (accidently) by the interviewer and used liberally later by the students, and *may* as in "you may be able to win" (Dale, turn 263). Another specialized linguistic form used by a student was the if-then statement, first used by Chris: "If it was two numbers then it would make a difference" (turn 39). This was in the discussion about the playing skunk with a die.

In addition to the relatively specialist terminology for modality (the modal verbs and adverbs), students expressed degrees of certainty in other ways. Terry introduced the expression "I think" in a conversation about playing skunk with the die. The researcher had asked if the number of berries they got would be different if the skunk came on a one instead of a six, to which Terry replied, "I think it would because we roll the one a lot" (turn 45). Terry introduced another expression to describe the differences between playing skunk with cards and with the die. In turn 236 Terry said, "You never know what is going to happen" (with the die). Terry also said "the odds are harder" (line 273) when the probability of success became lower. Dale was inventive too, and used the expression "I had a feeling" (turn 126) after "going home" to stay safe. This was in reply to the researcher asking, "Did you know that this was the skunk?"

Finally, the absence of any modal expressions is significant in the consideration of modality as well. The use of bald assertions can replace strong modal verbs or adverbs. Dale said, "the skunk is right there" (line 74) while pointing at the skunk card, as yet unrevealed but evidently the skunk by deduction. We might expect "the next one has to be the skunk" or "I am certain that the next one is the skunk" but the bald assertion serves the same function. Chris did the same on line 82 saying "it's there." In this interview (and others), there were many instances of this method for expressing certainty.

DISCUSSION

The four students in the interview described above demonstrated a wide repertoire of language for expressing degrees of certainty. Each of them used a range of expressions, and each of them introduced expressions that no one else had used before. Terry was the most talkative in the discussions about language meaning, but we caution that it would be unwarranted to make conclusions in comparison to the others on this basis. Many of the expressions introduced by the students came late in the interview, which tells us that if the interview had been shorter, we would not have known whether or not the students had these expressions in their repertoires. This serves as an exemplary caution against deficit-based assessments. Similarly, when one student said something, there was no need for the others to say it again or even speak about it unless they disagreed. Also, if students use an expression that has just been used by the teacher or tester, it may not be fully "acquired." We cannot assume someone does not possess certain language simply because they do not use it. However, we can claim that a student has an expression in their language repertoire if they introduce it. This is why we went to the lengths that we did for structuring our prompts carefully.

In addition to using (and introducing) specialist language, the students in the interview at times demonstrated ability to convey their meaning using very limited technical language. In particular, they could make their ideas clear when talking about the extremes of certainty – when events were impossible or certain. The more specialised language seemed to be relied upon either for describing events that were somewhere between impossible and certain, and for clarifying meaning on the extremes when pressed to do so. As noted above, Rowland (2000) introduced the idea of the zone of conjectural neutrality to describe language that specifies degrees of certainty, which is "in defiance of the cultural norm that the pupil is judged to be 'right' or 'wrong'" (p. 211). He claimed it to be helpful for a conjecturing atmosphere. We note that the same terminology is used to describe probability, and thus specialised modality language can defy situations in which predicted results may be between impossibility and certainty. We have only begun to consider the implications for pedagogy considering that the language is shared for both conjecture and probability spaces.

This brings us to discussion of the second research context, which was set up to be similar to but distinct from the game of skunk – a twist on the context. In both contexts, students were making predictions. What is the difference between a train and a pile of cards, both of which are sequences of physical objects? One difference is that the cards are shuffled and train cars are sequenced with some sort of intention. Nevertheless, our experience of real trains is that the sequence of cars seems to be quite random, or in groups (e.g., the boxcars first, followed by a bunch of tankers, followed by a few flatbeds, and finally the rest of the tankers). We have never seen trains with patterns similar to the ones introduced in our research – patterns like yellow boxcar, green tanker, blue flatbed, yellow boxcar, green tanker, blue flatbed, etc. A Grade 4 student in the second year of research involving the trains became increasingly frustrated with the rest of the class identifying what the 50th car would be. This student kept saying that it is impossible to know, while the class continued to ignore him. This student refused to make predictions.

This tension points to the presence of some sort of pedagogical contract in which students generally expect intention from their teachers. Even in the game of skunk, when the interviewers showed all the cards to the students and shuffled the cards directly in front of them, the students sometimes expected some kind of lesson – the appearance of a second skunk card, for instance. With the trains the phenomenon was more obvious; the students (with some exceptions, most notably the Grade 4 student noted above) assumed that the patterns would continue even though the researcher and teacher never said that these were patterns and the described context was one of a real-life train. The anger displayed by participants when they saw the fifth train (the random train) made clear to us the students' expectations for pattern. There is something about the transposition of a narrative into a mathematics classroom that changes it to a scenario in which everything should be predictable (and known by the teacher, or researcher).

In our research project, student predictions were based on both the probabilities inherent in the given scenarios and the students' second-guessing of teacher/research choices in constructing scenarios for pedagogic or other reasons. This raises questions about how students experience probability learning. Uncertainty in the mathematics classroom is experienced differently than outside the classroom. Furthermore we note that the language of conjecture shares language with probability, and so we wonder whether this ought to confound similarly our understanding of the way students experience proof and reasoning.

Finally, we turn our attention to implications beyond the classroom. Increasingly significant social phenomena, such as climate change, involve both calculations of risk, which are based on assumptions, and conjectures (hypotheses). The fact that risk calculation and conjecture share terminology may complicate communication about such social phenomena. Furthermore, both risk calculation and conjecture language about certainty is also used to express authority, as demonstrated in the above conversation about authority. When people in the public sphere who appear scientific make claims that sound authoritative, how are listeners to know whether these claims are warranted? It is incumbent upon mathematics teachers to be aware of these shades of meaning and the risk of ambiguity on such important social issues.

ACKNOWLEDGEMENT

This research was supported by the Social Sciences and Humanities Research Council of Canada, as part of a grant entitled "Students' language repertoires for investigating mathematics" (Principal Investigator: David Wagner).

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TWG09

Posters

A case study comparing the comments written by two students on their mathematics notebooks

Matías Arce, Laura Conejo and Tomás Ortega

University of Valladolid, Valladolid, Spain, arcesan@am.uva.es

The mathematics notebook of the student is a very usual tool in Spain, but, at secondary level, students are usually free to develop and use it as they want. Here, we put our attention on a particular aspect: the kinds of comments the students add in their notebooks when they are writing in them. As a first step, we have carried out a case study in which we compare the notebooks of two high school students with high grades. There are some similarities, like the large amount of comments, which link different concepts, but, also, important contrasts related to the role of the notebook.

Keywords: Notebooks, writing, comments, case study, high school.

RESEARCH QUESTIONS, FRAMEWORK AND METHOD

The mathematics notebook (MN), understood as the place in which students take their class notes, carry out and correct tasks and collect their mathematical work, is a very common tool in Spain. At secondary level, teachers usually do not indicate to their students how to develop it. We could see the MN as a tool with various use schemes among the students, but there is little research on this (e.g., Fried & Amit, 2003). In our research, we are looking for different profiles of elaboration and use of MN in students, in which writing plays an important role. Students usually transcribe in their MN the theoretical development of a topic (definitions, examples, theorems) and the solving of exercises. But students can add other written elements such as memories of previous concepts, clarifications or study aids. We call them *comments* and here we present an exploratory study focusing on them, with two research questions: What types of comments can we find in a student's MN? What are the relationships between them and how the students use their MN to study?

In the 80s and 90s a lot of studies about writing and its role in the learning of mathematics were developed. Some of them propose different classifications of students' writing in particular given tasks, according to the understanding and learning they reflect: in mathematics journals (Waywood, 1992) or in explanations of concepts (Shield & Galbraith, 1998). In our study, we do not have any particular task but we adopt these classifications as a basis to study what kinds of comments they add in their MN.

Four classes of high school students, chosen by availability, have participated in our research. The four have mathematics teachers with "traditional" methodology (theoretical exposition of contents, posing and solving of exercises). To start the study of the written comments, we have done a case study selecting two students of different classes with high grades in mathematics: a girl (S1) and a boy (S2). We have done photocopies of their MN to analyse the comments found in them (with the aid of the previous research) and an interview with each student about her/his use of the MN.

ANALYSIS AND FIRST RESULTS

In the poster we present and illustrate the similarities and differences between the kinds of comments we have found in the MN of these two students. Besides, we link them with the role of the MN and the way they use this tool to study mathematics, in order to extract the first results. Regarding the similarities, there are a high number of comments linking different concepts (or new concepts with prior knowledge) and justifications about the need to introduce some concepts or techniques or why they can apply a procedure. Comments of these types seem to be an indicator of good achievement in mathematics; they show that the students are beginning to understand the relational and logical nature of mathematics (Shield & Galbraith, 1998).

But there are key differences. The memories and clarifications about processes are more common in the MN of S1, and only in it can we find some recommendations about the resolution of exercises ("recipes" or good practices). We have found in S1's MN some signs (smiles or question marks) about her understanding (or not) of concepts and processes. In the interview, she says to base her study of the subject on the revision of theory and on the review and repetition of exercises made in her MN. These kinds of comments act as study aids for her, reinforcing an algorithmic aspect of mathematics (Shield & Galbraith, 1998) and a utilitarian stance towards knowledge (Waywood, 1992). In addition to these aids, she says she needs the visible correction of mistakes (using colours, marks or signs) and the order and cleanliness in her MN. It seems that S1 constructs her MN as a personal "textbook", emphasizing some clarifications and her difficulties to conduct her study throughout the reading of it.

There are almost no comments of those types in S2's MN, probably due to the different role of the MNs for him and his vision about mathematics. To prepare the test, instead of reviewing the exercises of his MN, S2 solves other exercises in which he has the solution to check. He seems to see mathematics as a constructive activity more than a gathering of contents submitted (Waywood, 1992). In the next steps, we have to extend the study to more students to learn more about the different types of comments and their functionality, in order to develop a classification of them.

ACKNOWLEDGEMENT

The first author has the support of an FPU Grant (AP12-2241).

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Talk about patterns in the mathematics classroom

Filip Roubicek

Academy of Sciences, Institute of Mathematics, Prague, Czech Republic, Filip.Roubicek@seznam.cz

The poster deals with the talk in the mathematics classroom which is focused on looking for relationships in a pattern. The communication of students is observed in the environment of geometrical patterns in a triangle grid and their transformation into arithmetic patterns or algebraic functions. It shows how pupils/students reason about relationships in these patterns and among these patterns, how they describe and express their generalizations in words or symbols.

Keywords: Talk, discourse, mathematics classroom, patterns.

DISCOURSE ON MATHEMATICAL TOPICS

The mathematical discourse among students is an attribute of thought-provoking classroom. Discourses are defined as different types of communication (and thus commognition) that draw some individuals together while excluding some others (Sfard, 2008). The classroom based on discourse approach brings a lot of situations of communication among students and among them with their teacher. They describe procedures of calculation or construction, verifying, justifying, deducing, generalizing, etc. Their talk reflects their mathematical experience, knowledge and thoughts. Under this perspective, the analysis of the students' talk is expected to enable to diagnose part of their mathematical understanding.

PATTERNS IN MATHEMATICS

Patterns represent the type of interesting and challenging tasks for students and the suitable environment to develop their mathematical thinking. "Generally, if we see a pattern in mathematics, we look for the relationship which will describe the pattern. Patterns are found in all aspects of mathematics, arithmetic, algebra, statistics and games." (Littler & Benson, 2005, p. 203) In the described case the environment of geometrical patterns in a triangle grid is transformed into arithmetic patterns or algebraic functions.

Talk among students is observed on the basis of modelling of geometrical patterns which are assembled from toothpicks in a regular triangular grid (geometrical figures are made of identical equilateral triangles). Series of these figures (see Figure 1) can be constructed by lengthening (the figure type is preserved but its shape is not) or by enlarging (the shape of the figure is preserved – similar figures). Elongation is characterized by a different quantitative change than construction of a sequence of similar figures.

At first students construct figures made of identical equilateral three-toothpick triangles. Then they state how many toothpicks the figures are made of, describe the arithmetical regularity in the sequence of figures and express the number of toothpicks needed for figure of length n.

Teacher: How many toothpicks are these parallelograms made of?

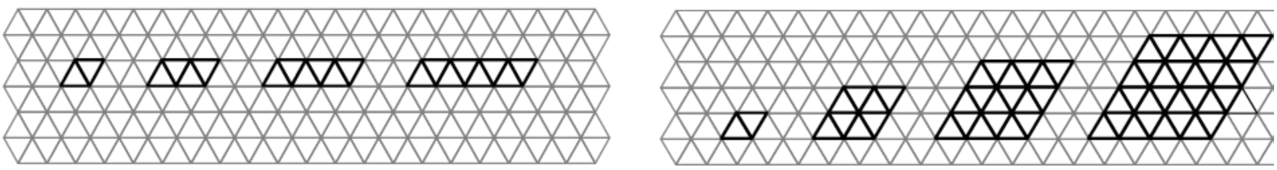

Figure 1: Two types of a geometrical pattern in the triangular grid

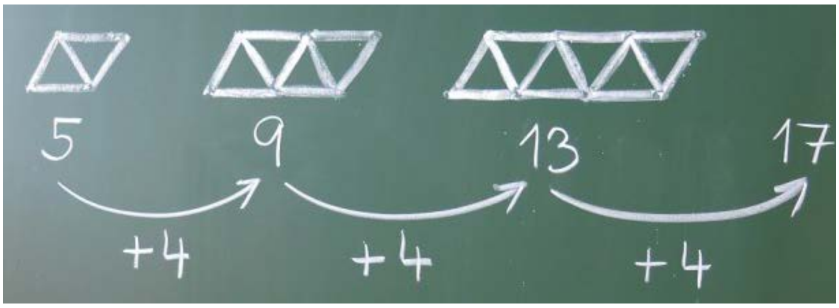

Figure 2: Transformation of a geometrical pattern into the arithmetic sequence

Student A: *She counts the number of toothpicks in the figure one by one.* The first one is made of five... the second out of nine...
Student B: ... and the third out of thirteen.
Teacher: How many toothpicks will you need to construct the next parallelogram?
Student A: I will add one, two, three, four... well, the next one will be made of seventeen.
Teacher: So, how will this go on?
Student B: Always plus 4... 21, 25, 29, 33...

The poster describes how students communicate their mathematical thoughts while looking for relationships in a pattern and express them through arithmetic rules or algebraic functions.

ACKNOWLEDGEMENT

The research has the support of GAČR 14-01417S and RWO 67985840.

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Testing and diagnostics of students' difficulties in CLIL teaching

Alena Šteflíčková

Charles University in Prague, Faculty of Education, Prague, Czech Republic, alena.steflickova@seznam.cz

The poster deals with CLIL and assessment and evaluation in CLIL. It summarises main positive and negative aspects of CLIL with regards to a Czech teacher and it concentrates on ways of diagnosing if a student has problems with language (English) or content subject (mathematics) while writing different types of tests. There are two main principles used – analyzing on the basis of the combining solutions in different tasks in set of tasks (alternative test) and analyzing of the procedure of a student. The poster introduces examples of several works of research and different alternative tests and it shows some possible ways how the test CLIL students. It also shows examples of diagnosing student's difficulties, mainly in language.

Keywords: CLIL, testing, assessment, alternative tests.

THEORETICAL FRAMEWORK

The current globalised world requires modern methods of teaching and learning. CLIL (Content and Language Integrated Learning) is one of these methods; it integrates learning of content subject and a language other than the language of instruction. Both of these subjects are taught at the same time and via each other. The method is in the spotlight of researchers and scholars who see it as one tool for better language education, also in mathematics education research.

CLIL has some positive aspects and also some problematic ones. One of these is assessment and evaluation. A key question is what should CLIL teachers assess and evaluate – content only (according to Hofmannová, Novotná, & Pípalová, 2004, this is the most common way in CLIL teaching), language only, both content and language separately or both content and language at the same time. Assessing and evaluating of both content and language at the same time can be difficult because sometimes it is not easy to distinguish if the difficulties of a student originated in language or content. Then there is one important question emerging: "How to integrate both parts?" (Novotná, 2011). A connected question is: If a student did not answer a question or answered the question incorrectly, what does it mean? Does it mean he/she did not understand the question or did not know the answer or was not able to produce it? A plain fact is that it is usually not possible to separate these items.

The poster aims at integrating math and English and the main objective of the poster is to introduce several types of alternative tests or tasks which were created and used in ongoing research regarding diagnosing where the difficulty of a student was.

METHODS

Assessment and evaluation is a part of CLIL which is not very often discussed. The ways of written assessment in CLIL are tests aimed at content or vocabulary, alternative tests, performance tests (students perform what they are asked for), portfolios and "can do" tables (there is a list of skills students should learn and the teacher ticks the ones the student has already learnt).

The examples of tests and tasks in my study are based on two basic principles. The first principle is inspired by Novotná (2011) who introduces an alternative test, which is a set of gradated tasks – some tasks are graduated in English and some are graduated in mathematics (gradated in difficulty). By comparing results in different tasks it is possible to make an educated guess where the student's difficulties are. The second principle is analyzing the procedure of a student's solution in order to understand his/her train of through.

CONCLUSION

Dealing with assessment and evaluation, if we choose to assess and evaluate content subject only, the tasks and tests would need to be in native language, because as my units of research showed the assignment in second or foreign language (L2) is one factor which influences the understanding. If we assess and evaluate in L2 then there is a question how to diagnose if the student's difficulty is in language or content subject. One possibility is to use an alternative test or to study the procedure of a student solution. To understand the procedure and train of through of a student is important for a teacher, so he/she is able to analyze knowledge of a student or a whole class and can use the information for other teaching. Further research could deal with other tests, comparing one test in different languages, as well as studying student's difficulties in oral testing or different school mathematical domains.

ACKNOWLEDGEMENT

This research was supported by GAUK 253543 (880213).

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Adaptivity challenges for relational scaffolding

Larissa Zwetzschler

University Duisburg-Essen, Essen, Germany, Larissa.Zwetzschler@uni-due.de

Scaffolding is a mean to support students' ability of expressing ideas in written language and to foster conceptual understanding. It is well known that scaffolds in words and chunks must be topic specific and adaptive to the individuals' steps of proximal development, but not what that means in detail, for example for percentages. The case study reported in this poster investigates which topic specific scaffolds are needed and reconstructs students' challenges with topic specific scaffolding as well as effects on their conceptions.

Keywords: Language, scaffolding, writing, percentages, adaptivity challenges.

THEORETICAL BACKGROUND

Scaffolding consists of chunks or parts of sentences, which students can use for their written texts (Smit, van Eerde, & Bakker 2012). Thereby the ability to express something in written language shall be developed. But students write texts topic specific challenges in percentages must also kept in mind. Parker and Leinhard (1995) resume backgrounds why percentages are so hard to learn; one reason is that connections between different elements (like part and whole) are often invisible in the language. These connections are part of a *relational understanding* that Skemp (1976)

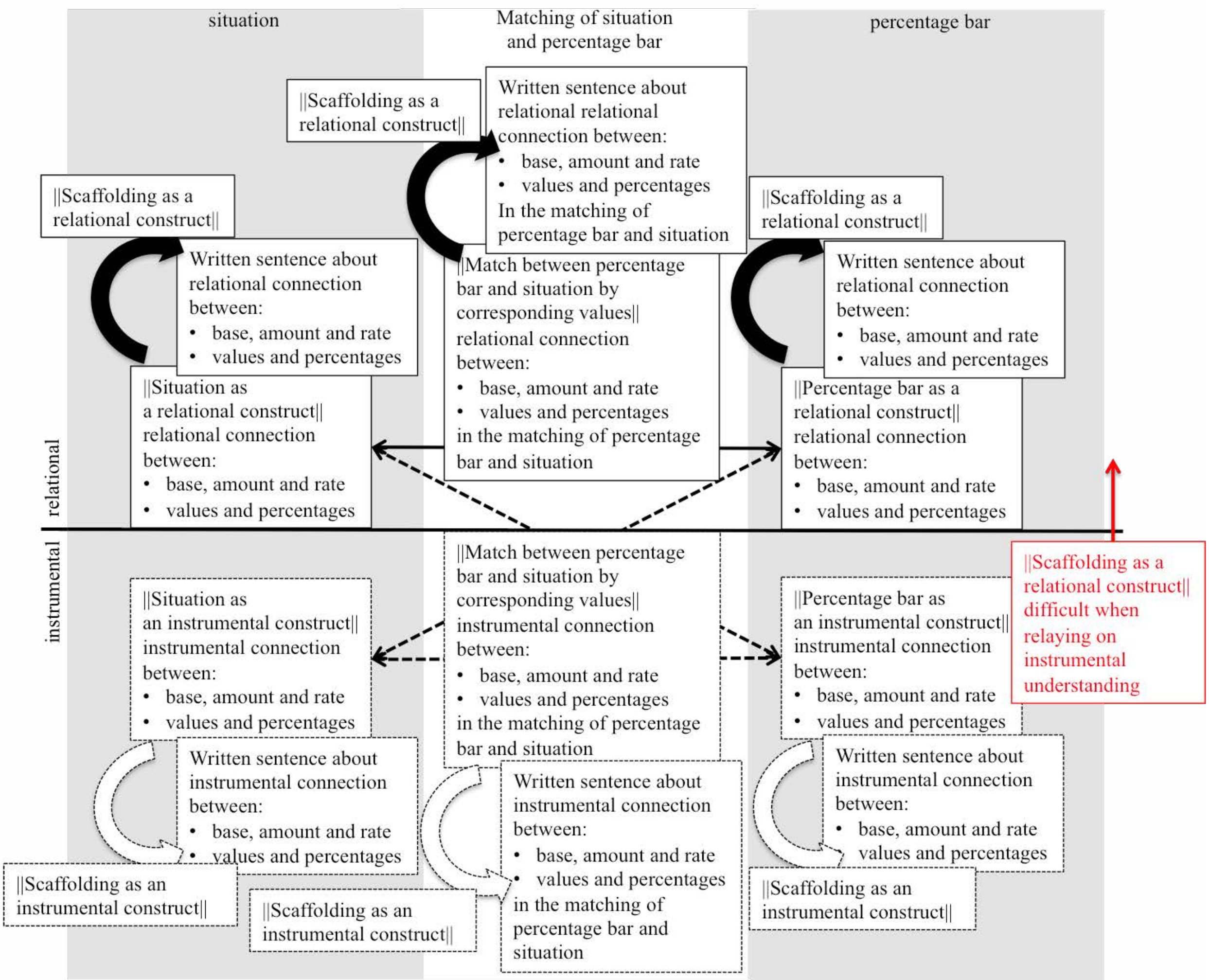

Figure 1: Adaptivity challenges for relational scaffolding

opposes to instrumental understanding. They are therefore chosen to be a core of a developed teaching learning arrangement in which students are asked to explain an error in a false advertisement (false rate). Their writing and understanding process is supported by scaffolding and percentage bars.

DATA GATHERING, ANALYSIS AND FIRST EMPIRICAL RESULTS

The learning arrangement was investigated by videotaped design experiments in laboratory settings (18 students of grades 7–8) and in 8 design experiments in classroom settings (with similar students), each lasting 35 to 50 minutes. Vergnaud's analytical model of concepts- and theorems-in-action is used for the data analysis, as "a fruitful and comprehensive framework for studying complex cognitive competences and activities and their development" (Vergnaud, 1996, p. 219).

The data analysis shows students difficulties to use relational scaffolds like 'because the base is... and of this the discount is reduced', if they had an instrumental understanding. A student says in those situations, that the scaffolding isn't helpful: "Because there, I – can't make the beginning of the sentences". Figure 1 shows the analysed interplay between an instrumental or relational understanding of the situation and the percentage bar, a matching of both and the consequences for instrumental or relational scaffolding. To sum up, relational scaffolding is needed to understand and express matching in percentages, but students can hardly make use of it when relying on an instrumental understanding.

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TWG10

Diversity and mathematics education

Introduction to the papers of TWG10: Diversity and mathematics education – social, cultural and political challenges

Lisa Björklund Boistrup[1], Tamsin Meaney[2], Mônica Mesquita [3] and Hauke Straehler-Pohl[4]

1 Stockholm University, Stockholm, Sweden, lisa.bjorklund@mnd.su.se
2 Bergen University College, Bergen, Norway
3 Universidade de Lisboa, Lisboa, Portugal
4 Freie Universität, Berlin, Germany

SCOPE AND FOCUS

Thematic working group 10 is interested in discussing diversity and mathematics education within the realms of the societal, the cultural, and the political. The working group was established at CERME7 in Rzeszów, Poland, in 2011 (Valero, Crafter, Gellert, & Gorgorió, 2011; see also Pais, Crafter, Straehler-Pohl, & Mesquita, 2013), but was an extension of the language diversity group which had been part of CERME since the first conference.

In the work of the group, mathematics education is assumed to refer to more than the encounter between an individual and a mathematical object and to wider contexts than exclusively classroom settings.

The group is specifically interested in discussing research that addresses how diversity affects possibilities in mathematics education. Diversity might be expressed in terms of gender, ethnicity, language, socio-economic status, disability, qualification, life opportunities, aspirations and career possibilities, etc. Contexts are diverse in terms of the variety of sites where mathematics education takes place, and the differences in the organization and structure of practice in such contexts—schools, homes, workplaces, after-school organisations etc. Contexts also include the political structures where policies are formed that draw on, make use of, or ignore mathematics education research. Diversity also occurs in relationship to who is doing the research and who is being researched, posing methodological issues of an ethical nature. These multiple diversities intersect, and in so doing pose challenges to intended and actual learning and teaching practices in their multiple forms.

ORGANISATION OF TWG10'S WORK

In the seminars during CERME9, papers were presented in a similar way to what had occurred in CERME7 and CERME8 in that the authors did not present their own paper. Instead each paper was presented by another author giving a neutral description of the main ideas from the perspectives adopted in the paper. The author(s) then had a few minutes to add to or comment on the presentation, with the possibility of pointing out or emphasising important aspects. The presenter prepared one question/comment for the author, which was discussed in connection to the presentation. In the end of each session, there was time for discussing the presented papers. These discussions firstly occurred in pairs or small groups and then were shared in the whole group. This was done to facilitate the contributions from as many participants as possible to the work of TWG10.

Although the papers in each session were grouped to facilitate discussions about similarities and connections between papers as well as tensions and contradictions, each session was not labelled as such in advance. One consequence of this procedure was that the efforts to thematize the contents of the research presented became a joint process within the group.

A poster session with 6 posters was held in which all presentors had 2 minutes to describe the content of their poster. The poster authors then positioned themselves next to their respective posters to engage in discussions with group members.

THE PAPERS DISCUSSED

In this section, we briefly describe the papers from TWG10 following the schedule from CERME9. We address some (dis)connections between each paper and the subsequent paper as examples of the diversity of papers in the group.

In the paper by *Albanese*, ethnomathematical dimensions are adopted as tools for analysing observations of teachers participating in workshops aiming at influencing their conceptions about the nature of mathematics. *Parra-Sánchez* similarly has an interest in ethnomathematics, but not as a tool adopted in empirical research in school, but as a broad focus of the study itself through a literature review. Parra-Sánchez problematizes the relationship between researchers and researched communities in the ethnomathematical field and proposes a more symmetrical approach. While this paper has an interest in power relations between the researcher and the researched, the next paper by *Hauge and colleagues* applies an approach where the gap between the researched (master students) and researcher (Hauge as main author) is closer than in much empirical research in a study on how mathematics related classroom discussions may enhance critical citizenship (drawing on Skovsmose).

Similar to Hauge and colleagues, the paper by *Kitchen and colleagues* critically scrutinises quantifications taking place in society, but with another topic in focus. Kitchen and colleagues focus on how teachers' assessment practices were largely influenced by the pressures to prepare students for success on the US state's standardized test. *Bagger* similarly presents research where assessment in mathematics was critically investigated. While Kitchen and colleagues pay attention to how official assessments affected teachers' practices, Bagger's interest is in effects on students, in terms of student positions which were construed drawing on Foucault. *Turvill,* similarly to Bagger, has an interest in mathematics education in relation to young students but with a theoretical object of a study on inequalities. Turvill examines number sense from the perspectives of cognitive psychology, situated cognition and Bourdieusian social psychology. In the paper by *Lembrer*, the children referred to are younger than in the case of Turvill, while also drawing on sociology, in a study on the relationship between socialisation and mathematics education in Swedish preschools.

Montecino and Valero present a theoretical analysis on texts, as does Lembrer. While Lembrer analysed official documents, Montecino and Valero adopt Foucault and Deleuze to explore how discourses in research literature are operating as part of the fabrication of the mathematics teacher as a subject and in the production of truths about them. *Pansell and Björklund Boistrup* also have an interest in the mathematics teacher but with data from communications within a collaborative teacher meeting where one teacher's justification of her professional decision making as part of a socio-political context is analysed and discussed. The teacher's decision making concerned, for example, calculations and this was an interest in the theoretical paper by *Kollosche*. Here, the focus is on connections between calculation and bureaucracy. Adopting a methodology following Nietzsche and Foucault Kollosche points out implications for mathematics education. Similar to Kollosche, the paper by *Dahl* is theoretical, although Dahl takes on a more structural perspective when adopting concepts from Bernstein suggesting how problem solving can be viewed in three different ways: as an ideology, a competence and an activity.

Dahl presents a foundation for a methodology for investigations of problem solving in mathematics education. *Norén and colleagues* also pay attention to methodological matters when revisiting their own research with a focus on methodologies for performing research while paying attention to diversity and equity issues, in this case in relation to newly arrived students. In *Radovic and colleagues* as well as in *Norén and colleagues* there is an engagement in the perspective of the students. Radovic and colleagues report on the intersection between mathematics identity and the peer positioning of high attainment girls in a particular mathematics' classroom in Chile. Also *Marks* has an interest in students' perspectives and this paper examines questionnaire and interview data to identify pupils' prevailing mindsets in primary mathematics. The findings, where a fixed-trait belief is dominating, are discussed in relation to mathematics education policy and practice in England.

Although having an interest in mathematics education in school, the paper by *Andrade-Molina and*

Valero takes on a more historical perspective than Marks when adopting cultural historical strategies (Foucault) to research the functioning of the school geometry curriculum, arguing that school geometry fabricates the scientific minds of the future. Similar to Andrade-Molina and Valero, *Helenius and colleagues* present a theoretically driven analysis. Drawing on Bernstein's ideas about vertical and horizontal discourse their paper raises issues about how the connection to the everyday in problem solving could reduce children's opportunities to learn mathematics.

Similar to Helenius and colleagues, the paper by *Albersmann and Rolka* concerns problem solving. Albersmann and Rolka do not critically examine the everyday context, but use problems with every day contexts when investigating parent-child cooperation in the course of a workshop. A quite different scope has the paper by *Black and colleagues* where the data derive from a mathematician. In this paper, the role of 'others' is explored in one woman's mathematical identity with the role of 'caring' as a cultural resource to identify as a mathematician. While Black and colleagues examine data related to the discipline of mathematics, *Mukhopadhyay and Greer* argue for the necessity of maintaining diversity in all its human forms, including mathematics and mathematics education. Central to this position is respect of the conception of mathematics and mathematics education as human activities, inextricably embedded in forms of life.

AN ELABORATION AND PROBLEMATISATION OF INCLUSIVENESS AND QUALITY IN MATHEMATICS EDUCATION

In TWG10, we agreed that matters discussed within the group were essential, not only for this group but for research in mathematics education in general. In relation to an interest in inclusiveness and quality in mathematics education research, discussions in the papers from TWG10 would be productive for elaborating and problematizing the research of the field, for example, concerning the development of research ethics, or finding productive ways of addressing the situatedness of any research process in mathematics education.

For TWG10, this concerns respect for diversity in a variety of ways, which also is constantly changing. Different demands for ethical responsibility were discussed, for example, a virtue of respect for participants/collaborators in research, such as teachers and students, when performing research with an emphasis on recognizing knowledge where it is situated; or, going further, the establishment of deepened connections between the researcher and the researched through allowing the researched a true stake in the collaborative development of the research project, thus sounding out political common ground. While TWG10 obviates a fixation of general demands for ethical responsibility, it formulates reflexive controversy as a requirement for mathematics education research that seeks to locate itself within the realms of the societal, the cultural, and the political. A consequence from such a view is an awareness of how the actions by any researcher within the field of mathematics education have political consequences.

In the work of TWG10, labels were discussed as "needing" not to being taken for granted, such as challenging the meaning of "success in mathematics education" or how a student "in need" may be construed. In discussions as well as in papers, the importance of investigating what lies between and behind labels was addressed.

Another theme within TWG10 was an interest to problematize and challenge mathematics education research done with the aim of identifying the teaching and learning practices that "work best". In the discussions within TWG10, the focus shifted from "what works" towards the question of rather „how what works looks like – and for whom". Furthermore, research within TWG10 addressed how the enforcement of accountability measures within many societies of today is not likely to promote any "deep" mathematical competence.

One such aspect was a discussion in which many of the issues within mathematics education were taken as not only being problems of scenarios of learning mathematics, but issues within the broader political context that still concern mathematics education. Consequently, while some of the papers explicitly established the relation between the local context, for example, a classroom, and the broader political context, discussions steadily established such relations where it was not an explicit focus of the respective papers.

In TWG10, we also identified tensions where perspectives within the group were not coherent. One such tension was a research interest in the subject of mathematics where mathematics education could be viewed as a gatekeeper where social order should be maintained. In other works from the group, the emphasis in the research rather was on how to invite all students into the discipline of mathematics. This included a discussion about mathematics itself in terms of how useful it may be for the individual, but also for societies as a whole. Different kinds of usefulness, at times not compatible, were discussed, such as mathematics as a problem-solving tool in life, or as a selection device (for example, to higher education).

Closely connected to this tension was a discussion in the group about whether change at all was possible, and in that case how. In some discussions, (mathematics) education was emphasized as a facilitator for changing the world we live in, whereas other discussions had a stronger emphasis on (mathematics) education as being structured by the world, with limited power to be a departure for a change in society. The dynamics of the discussions suggest a reflexive approach towards the relation between mathematics education and the societal structures in which it is embedded.

Similar tensions were discussed at CERME8 (Pais et al., 2013) and during CERME9 more topics were included in the discussions. The diversity of TWG10 is also possible to experience through a sound installation made by the group, the "cacophony" on link: https://www.dropbox.com/s/id10kp598jkc872/TWG10%20cacophony.m4a?dl=0

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TWG10

Research papers

Ethnomathemathical dimensions for analysing teachers' conceptions about mathematics

Veronica Albanese and Francisco Javier Perales

University of Granada, Faculty of Education, Granada, Spain, very_alba@hotmail.it, fperales@ugr.es

As Ethnomathematics involves deep epistemological implications about the nature of mathematics, we decided to hold a workshop for pre-service and in-service teachers, in order to influence their conceptions about the nature of mathematics. The aim of the research is to analyse the observations of these teachers related with the conceptions of the nature of mathematics after the proposed workshop. In this paper, we present different dimensions for analysing the teachers' conceptions about mathematics from the perspective of Ethnomathematics.

Keywords: Ethnomathematics, conceptions, teacher education, sociocultural perspective.

INTRODUCTION

Ethnomathematics is grounded in the anthropological studies on mathematics of indigenous peoples and aims to recognize and describe the ideas and practices of different cultural groups, indigenous communities as well as labour guilds (D'Ambrosio, 1985; Barton, 1996). Later, in the attempt to make sense of the confrontation of different perspectives of Ethnomathematics, its purpose was extended to address a broad view of mathematical knowledge in relation to its development in different cultures (D'Ambrosio, 2012). Ethnomathematics seeks to explain what and how different individuals show different interests, talents, skills and strategies to generate, organize, and share that knowledge. This sociocultural perspective is supported by the contributions of various disciplines which share a relativist position.

As the perspective of Ethnomathematics involves deep epistemological changes to move away from a positivist posture, many researchers propose seminars, courses and Ethnomathematics workshops for teacher education, with the common goal of raising awareness about the nature of mathematics as a social and cultural product and hence, directly or indirectly, influencing the participating teachers' epistemological conceptions.

In the United States of America, Presmeg (1998) organized courses where pre-service teachers became aware that mathematics is a cultural product, starting to affirm value and leverage in their future professional life concerning issues related to cultural diversity.

In Costa Rica, Gavarrete (2012) organized a course for training teachers who work in indigenous environments. She promoted relativistic conceptions of mathematics that integrated academic knowledge with traditional indigenous cultural knowledge and that was applied to various activities of the daily life of these communities.

Building on the work of Presmeg (1998) and Gavarrete (2012), we held a workshop for a group of pre-service and in-service teachers of the University of Buenos Aires. One of the workshop's objectives was to influence the teachers' conceptions about the nature of mathematics.

The aim of the present research is to illustrate the potential of different ethnomathematical dimensions for analysing the development of the participating teachers' conceptions concerning the nature of mathematics after the workshop. This allows us to point out the changes that the work-shop made. The dimensions themselves were defined from the point of view of the ethnomathematics program. We show how these dimensions were useful in order to systematize participants' conceptions.

We decided to work in Buenos Aires because the workshop was about the mathematics involved in an Argentinian craft and because the legislative documents of the educational reform of 2006 promote a model of teaching and learning where knowledge is conceived in relation to its development in cultures and its use in daily life (Albanese, Santillán, & Oliveras, 2014).

THEORETICAL BASES

The ethnomathematical dimensions

Based on a literature review, we propose three dimensions that we consider crucial for systematizing the perspective of Ethnomathematics.

The first dimension is the practical dimension: mathematics is *a tool that man develops to relate, understand, manage and eventually change his environment*. This dimension comes from the definition of D'Ambrosio (2008) of Ethnomathematics as a way (*tics*) of knowing (*mathema*) in the environment (*ethno*); in other words, mathematics is considered as a tool for the systematization and interpretation of reality such as identifying patterns that govern environmental elements and the relationship between them. Thus, mathematics is a way of knowing how to perform daily activities, acting in and controlling the surrounding context and eventually modifying it.

Secondly, we consider the social dimension: *mathematics is a consensual construction of a set of rules and norms within a group of people that decide to share it*. This dimension comes from Barton's (2008) proposal of interpreting mathematics as a system of meanings through which a group of people gives a sense of quantity, space and relationships (QRS-systems). These systems are built by communities that share the same vision of reality and agree on some common codes to communicate.

The third and last dimension is the cultural dimension: *different forms of mathematics exist in different cultures*. Ethnomathematics establishes a profound relationship between mathematics and culture. From a philosophical point of view, many of the researchers, like Barton (2012) and Knijnik (2012), insist on the importance of a relativistic perspective that allows the coexistence of different mathematics.

Teachers' conceptions

To delineate the notion of conception we consider the contributions of Ponte and Chapman (2006) and Pajares (1992). However, different to these authors, we decided not to distinguish between conception and belief, as we considered this distinction not clear enough, and as our interest was actually focussed on what the participants believe that mathematics is, i.e. their conceptions about mathematics. We understand conceptions as the underlying organizational structure, or conceptual substrate, of a knowledge; they are idiosyncratic, derived from experience or fantasy with a strong affective and evaluative component. They do not have an internal consistency, they are more inflexible and less dynamic than knowledge [1], they tend to be perpetuated despite contradictions; they are usually acquired through the process of cultural transmission.

METHODOLOGICAL ISSUES

The workshop

The workshop was about the mathematics and mathematical thinking involved in an Argentinian braid craft that we studied in some previous research (Albanese, Oliveras, & Perales, 2014; Albanese & Perales, 2014; Oliveras & Albanese, 2012). The aim of the workshop was to present mathematics from a different point of view and therefore to influence the participants' conceptions about mathematics.

The workshop was held by the main researcher as a part of a course of mathematical modelling of the degree of mathematics for secondary school teacher of the University of Buenos Aires. The participants in the workshop were twelve pre-service teachers, students of the course, and their two teachers.

The workshop employed a methodology based on direct experience where we showed and asked them to make braids, working in small groups as well as on the content that focused on a specific manual task.

The idea of the workshop was that the participants investigated, constructed and agreed on a creative representation of the making of a braid. Then the participants shared the different representations they made in small groups and we pointed out the difference and similarities between them and a craftsman's representation of the practice.

At the end of the workshop we invited the participants to answer five open questions about final observations on the work and its epistemological and educational implications. These questions are listed below.

1) What mathematical thought did you put into effect to make, represent and invent braids?

2) What implications about the nature of mathematics are involved in this activity?

3) Which aspects of Ethnomathematics have you experienced?

4) What aspects of the experience do you consider relevant in relation to the methodology of work?

5) What potential educational purposes can you see in this type of work?

The research
The research can be considered as an educational ethnography, that is, ethnography of small work groups or classes (Goetz & LeCompte, 1988). Ethnography was chosen for its affinity with the perspective of Ethnomathematics, as well as for the interest of the attitudes, opinions and beliefs of the people who we were researching.

We considered the oral and written observations of the participants as evidence of their conceptions about the nature of mathematics. We performed a qualitative analysis of the data based on content analysis with the help of the computer programme MAXQDA7 to systematize the information and to assign codes and categories.

The descriptive codes were obtained in a process combining both inductive and deductive procedures. The original idea for a drafted grid of categories came from a pre-analysis phase, in which we were first drafting and working with preliminary codes. Then we went back to the theory of Ethnomathematics and refined the original idea against the backdrop of to the theoretical bases described above. Thus, we finally obtained the three dimensions of Ethnomathematics in a deductive manner. These dimensions were then combined with the five items from the questions, that we posed to the participants. These questions were discussed according to their scientific relevance and validity for our research objectives by a group of expert ethnomathematics researchers. Fifteen potential categories resulted from the combination of the three dimensions and the five items. Finally, we inductively obtained the descriptive codes to fill these categories in confrontation with the empirical data. The development of the descriptive codes therefore follows an inductive-deductive cyclical process that produced eighteen descriptive codes, leaving one of the fifteen potential categories empty and allowing for a further differentiation of four of the categories within the practical dimension.

SOME RESULTS

Here we present the descriptive codes for the written observations, evidencing the empirical relation of teachers' conceptions about mathematics with the practical, social and cultural dimensions of Ethnomathematics. In Table 1 we summarize the descriptive codes obtained in the analysis, organized by item and category.

We will illustrate the descriptive power of the codes, by describing in detail the codes that came from the answers of the participants to the second item: *what implications about the nature of mathematics are involved in this activity?* These are summarized in the second column of the Table 1.

Items / Dimensions	1. Mathematics thinking	2. Nature of mathematics	3. Ethno-mathematics aspects	4. Methodology	5. Potentiality
Practical dimension	***→Model real situation***	***→Model of daily life*** ***→Patterns***	***→Experience, research*** ***→Mathematics of the practice***	***→Concrete experience*** ***→Motivator***	***→Daily life in school*** ***→Mathematics of reality***
Social dimension	***→Agreement***	***→Social construction***	***→Collective, consensus***	***→Group***	***→Social interactions***
Cultural dimension		***→Other mathematics***	***→Different mathematics***	***→Different point of view***	***→Thinking differently***

Table 1: Table of the codes distributed by item and by category

JO: This activity clearly shows the potential of mathematics as a modeler of daily events.

MAR: Mathematics arises as a need to systematize a concrete practice, while it allows people to imagine other possible braids.

These two participants underline the role of mathematics in creating models of daily life, these models give us the possibility to manage and control reality (like inventing new braids). We assigned them the code "model daily life" and we categorized them in the practical dimension.

VAN: Through this activity we could make a case analysis and generate patterns for modeling and generalization about braid weaves.

JE: Mathematics is (in part) to create representations of a part of reality and of its internal relations (or some of them), and from that model, to generate new knowledge on that portion of reality.

These two participants go a little further, they both recognize that mathematics is a tool to model reality but they also insist on identifying the internal relations and the patterns. So we coded these as "patterns" and again we considered these observations as referring to the practical dimension.

KA: Mathematics is a socially constructed science.

This participant makes explicit that mathematic is a social construction so this observation clearly belonged to the social dimension.

DI: Wondering: what is mathematics? Reflecting on transformations throughout history and open to new transformations.

This participant made evident his reflection about the nature of mathematics, and reflects in a historical way, seeing the transformation that mathematics has made over time. We associate this observation with the idea that, as mathematics is a historical product, it is a social product, so we considered it in the social dimension.

MAT: Mathematics is more than school mathematics or academic mathematics.

VE: The arbitrariness of the notations, the coexistence of some representations (over others) the relationship of that coexistence with the particular "reading" of each one using such representation.

These last two answers show respectively the existence of a different kind of mathematics with respect to the academic one, and the existence of different points of view of the same problem (with the consequent elaboration of different notation) and the different interpretation of the same representation depending on the viewer. Both these observations are related to the existence of different mathematics, that's why we categorize them in the cultural dimension.

The classification of conceptions according to the three dimensions allows us to summarize the results of the whole group of participants. Ten of the 13 participants who answered the question "*What implications about the nature of mathematics are involved in this activity?*" observed that the activity showed the importance of mathematics to *model everyday situations*, to represent reality and to recognize and control relationships and *patterns* to handle general cases (Practical category). Four participants manifested ideas related to the social category: mathematics is a *socially constructed* science, it is intrinsic to human activity, it is transformed throughout history and they recognized the role of institutions in indicating where there is mathematics. Finally, three referred to evidence of the cultural category: two of them suggested considering *something else* other than formal and classical mathematics and one insisted on the coexistence of different representations valuing the way of thinking of each one.

CONCLUSIONS

Our findings suggest that, in general, the codes of the social and cultural categories are much less present than the practical category, and we perceived that the social and cultural dimensions were always manifested in the participants' observations indirectly, the ideas were delineated but not fully developed while the ideas related to the practical dimension were richer and more developed in the teachers' conceptions about mathematics after the workshop.

The definition of three ethnomathematical dimensions, the practical, social and cultural dimensions was fundamental for carrying out the analysis. The definition of these dimensions provided the tools to interpret the teachers' observations. We think that the description of these dimensions is an important achievement of the work and its usefulness has been confirmed during the analysis of our data, as we could recognize almost all the participants' observations by assigning a category of conceptions.

The explanatory potential of similar operationalizations of dimensions of mathematics, could also be demonstrated by other studies, e.g. that of Albersmann (2015) about parents' utility-oriented beliefs about mathematics. Albersmann's study considers three dimensions that are related to the ones presented in this paper, which are conceptualized as 1) pragmatic relevance, 2) methodological relevance, and 3) cultural relevance.

In the studies of Presmeg (1998) and Gavarrete (2012), changes in teachers' conception have been recognized, but our work goes a little further because indicates the multiple directions of these changes.

Finally, this demonstrates how fruitful it was to link different philosophical contributions to Ethnomathematics by different authors, identifying and giving voice to their different positions. D'Ambrosio (1985, 2008) insists on the practical and moreover social dimensions, as he is a mathematical historian. Barton (1996, 2012) starts from the social dimension but he ends up focusing on the cultural dimension, because of its work with indigenous tribes. Our findings highlight the importance for ethnomathematical research to take into account both points of view.

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ENDNOTE

1. Here Knowledge indicates what a person knows.

Parent-child cooperation in mathematics learning: Insights into maths-experience days

Natascha Albersmann and Katrin Rolka

Ruhr University Bochum, Bochum, Germany, natascha.albersmann@rub.de

Considering the fact that the influence parents have on their children's mathematical development is unavoidable, it should be less the question whether parents should participate in their children's mathematical education, but more how their participation could be incorporated more consciously. This issue becomes even more crucial with regard to secondary school mathematics. In this paper, one approach is presented, namely a family math in which parents and their children get the opportunity to deal with mathematics in problem-based discovery learning environments together as a team. The focus of this paper lies in the investigation and characterisation of parent-child cooperation in the course of the first workshop of the project, as well as factors promoting such cooperation or hindering it.

Keywords: Parental involvement, secondary school, cooperation, problem-solving.

INTRODUCTION

The critical role that parents play in their children's educational development in general and specifically in mathematics is undeniable (Desforges & Abouchaar, 2003). Parental beliefs about mathematics, its teaching and learning, emotions regarding mathematics or parents' mathematical competencies, to name just a few factors, come together to diverse influences on their children's mathematical education. What is more, parental participation in mathematical learning activities seems to decrease with their children getting older. In this context, Eccles and Harold (1993) mention, among other reasons, a change in parents' views of their competences to help their children with schoolwork as they enter higher school grades. In particular, with the switch from primary into secondary school, in Germany at the beginning of the 5th grade, mathematical contents start to become more and more unfamiliar for parents. Whereas they could rely on being an omniscient answerer or the conveyer of knowledge in earlier school years, parents ultimately need to shift roles in order to further support their children regarding mathematics. Besides mathematical contents, parents may also feel that current methods of teaching mathematics may differ a lot from those they experienced in their school years (Eccles & Harold, 1993).

All in all, the conditions for parental support in their children's mathematical education change fundamentally in the course of secondary school years. Therewith, new challenges but at the same time new possibilities for parental involvement in their children´s mathematical learning emerge.

INVOLVING PARENTS IN MATHEMATICAL EDUCATION

Initiatives for parental integration in their children's mathematical education often consist of one-way transmissions of information and materials from schools or other educational institutions. However, drawing on the social constructivist view, learning is not a simple process of transferring information, but involves an active social interaction. The emphasis thereby lies on learning as an ongoing process in activity-based learning situations with meaningful purpose (Vygotsky, 1978). In this context, findings from a project called MAPPS (Math for Parent Partners) indicate that giving parents opportunities to actively construct their own understanding of mathematical concepts provides an indisputable foundation for their work with their children. Moreover, research results of this project imply that the nature of interactions that parents developed with their children might be even more relevant than their ability to help them with specific content (Civil, Guevara, & Allexsaht-Snider, 2002). These findings emphasise that it is not only necessary to introduce parents to current mathematical concepts in an active way, but also to foster parent-child interactions in mathematical contexts.

Since in secondary school mathematics parents' knowledge advantages decline, the foundation for parent-child interactions in context of mathematical problems change. Following the idea of a "community of learners", one goal is that parents and children work together with everyone serving as resources to the others, with varying roles according to their understanding of the problem at hand (Rogoff, Matusov, & White, 1996). This concept goes hand in hand with a cooperative learning approach that constitutes a foundation from which parent-child interactions can be fostered in the context of secondary school mathematics. In this sense, we understand parent-child cooperation as one main feature to describe the nature of parent-child interactions in mathematical learning situations.

Guided by the concept of dialogic learning, conversations in a cooperative learning approach should follow an egalitarian dialogue where different contributions are taken into consideration according to the validity of their reasoning instead of the position of power held by those who make the contributions (Flecha, 2000). Thus, in order to foster parent-child cooperation, it is essential to create a learning situation in which parents will not automatically act as the conveyer of knowledge, but rather become a learning partner for their children. This speaks for a concept with open-ended and problem-based discovery learning situations (Bruner, 1961) in which parents and their children get involved with a mathematical problem in a self-directed way (Hmelo-Silver, 2004).

Considering the demand to integrate parents more actively in their children's mathematical secondary school education, the questions arise: How does the cooperation of parents and their children in the context of a discovery learning situation work out? Which facets promote a parent-child-cooperation, which not?

METHODOLOGY

The project and its participants

One possibility to investigate forms and conditions of parent-child cooperation in an open-ended and problem based discovery learning situation is provided by a family math project, called maths-experience-days. This project started as a pilot project in the school year 2012/13 with parents and their children from a 5th grade (10–11 years old) of a German higher-level secondary school, a so-called gymnasium. The project workshops lasted three to four hours and took place in the school building on Saturdays, hence outside of schooldays. The participation was on a voluntary basis.

In the first workshop, which builds the basis of the results presented and discussed, twenty-two families participated with both parents appearing in five cases. Hence, a total number of twenty-seven parents attended the first workshop, among these seventeen mothers and ten fathers. The parents' ages varied between 36 to 52 years old. Due to incomplete data sources only the data of twenty-five could be taken into account in the research presented.

The families participating in the project live in rather high socio-economic conditions. With respect to parental educational backgrounds, it can be noted that every parent has a school leaving-qualification. All in all, seventeen parents achieved the general qualification for university entrance, of which fifteen of those parents obtain a university degree and two did an apprenticeship. Moreover, seven parents achieved a subject-related entrance qualification. Two of those parents finished a polytechnic degree and five an apprenticeship. Only one parent achieved an intermediate school-leaving certificate who did an apprenticeship as well. The occupational fields are diverse and range, for example, from business management, law to medicine, natural sciences or social professions. Each and every parent has a job at the time of the survey.

The first station of the discovery journey – content and structure

According to the above-mentioned general conditions for the learning environment, a didactical reduction of the mathematical content seems to be reasonable. This concerns the complexity of the problem presented to the parents and their children at the beginning of the workshop (see Figure 1).

„A friend of mine, let's call him Harry, used to be a paper boy when we were a bit younger. His job was to deliver a free regional newspaper once a week. He loved working in the fresh air and meeting all the people living in his delivery district. However, there

„A friend of mine, let's call him Harry, used to be a paper boy when we were a bit younger. His job was to deliver a free regional newspaper once a week. He loved working in the fresh air and meeting all the people living in his delivery district. However, there was one thing, he struggled with. His tour took far too much time.“

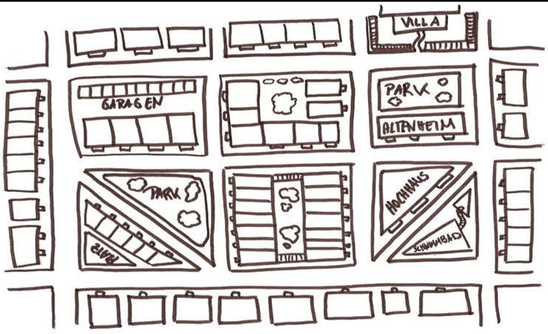

Figure 1: Problem and Harry's delivery district – working material

was one thing, he struggled with. His tour took far too much time.“

Due to the requirements for autonomous learning and performance of the parent-child teams, the role of the workshop coordinator rather consists in encouraging cooperation by scaffolding strategies like recruitment, direction maintenance and frustration control than providing scaffolding methods with stronger content-related guidance (Wood, Bruner, & Ross, 1976). In the need of content-wise support, parent-child teams can draw on different hint cards according to their progress in problem-solving. Those hint cards are constructed on the basis of scaffolding methods like reducing degrees of freedom by simplifying the task as well as accentuating critical and relevant features or, at last, demonstrate the solution itself.

The phase of the problem investigation is structured through the cooperative learning method "think-pair-share" (Sharan, 1994), which particularly shall subserve parent-child cooperation. First, parents and children think about possible approaches to solve the paperboy's problem for themselves and, then in pairs (parents and children) exchange their idea is, carrying them forward together. At last, the ideas are collected and discussed in the whole group.

Data sources

We have several data sources at our disposal that play a role for the analysis and hence the presentation of the results. While considering multiple data sources, we draw on an approach that is characterized by triangulation of data (Schoenfeld, 2008).

Before the first workshop, and again after the last workshop, parents are asked to answer a questionnaire containing both open and closed items. Besides personal information like age, gender, school education, vocational training, and occupation, the questionnaire consists of parental beliefs about mathematics considering both cognitive and affective aspects as well as their personal reasons for participation.

Likewise, data concerning the process of working on the problem in the course of each workshop is collected. Here, reflective journals are used which have become an accepted method for qualitative researchers to gain insights into their participants' thinking (Mewborn, 2013). On the one hand, parents and children document their mathematical considerations during the problem investigation. On the other hand, the reflective journals are used by parents and children at the end of each workshop. In order to encourage reflection, guiding questions are displayed which address their experiences, difficulties that occurred as well as strategies to overcome them and new insights that parents and children gained during the workshop.

Besides parents' and children's journal entries after the first workshop, the results presented in this paper are based on field notes taken during and immediately after the workshop by the coordinator.

Analysis

The experiences of parents and children during the first workshop indicate diverse aspects for an analysis, all of which cannot be taken into account. In this paper the focus lies on the analysis of data referring to parent-child cooperation during the problem-solving process.

On the one hand, parents explicitly referred to forms of parent-child interactions throughout the entire reflective journal from which the nature of their cooperation could directly be derived. On the other hand, some parents did not mention their children or the work with their children at all. Thus, cooperation patterns were implicit. In order to interpret those journal entries and moreover validate interpretations, they were matched with other data sources like the corresponding journal entries of their children and field notes taken by the coordinator of the workshop.

The analytical framework for the open data sources is informed by the principles of qualitative content analysis (Mayring, 2010). First, hints on parent-child cooperation were independently selected from the reflective journals by the authors and significant themes identified with the goal to induce a categorisation of parents' cooperative behavior. Second, interpretations were compared between the authors and discrepancies were discussed. A category formulation was determined and checked in a final sifting through reflective journals. Finally, in due consideration of all data sources, parents were assigned to one of the inductively developed categories.

RESULTS AND DISCUSSION

To understand the whole picture, some facts need to be pointed out before the discussion of results. First, the sample consists of parents with a high socio-economic status and therefor is of limited representativeness. Second, they participated in the family math project voluntarily. Consequently, the sample is selective by its nature. However, taking into account all of the conditions, the data still allows some interesting insights.

The findings show that parents can be assigned to one of two categories: *parents as learning partners* and *parents as learners*. The two notions of parental roles are similar to those used by (Civil, 2001), although the conditions of parental integration are different from the MAPPS project with regard to the direct involvement of children in our project. While the first category *parents as learning partners* could be observed for thirteen parents, six parents were assigned to the category *parents as learners*. Another six parents do not give any information about the parent-child cooperation and could not be coded.

In the following, we first briefly describe the meaning of the two categories and use excerpts from parents' and children's journals to exemplify our shared understanding of the categories. In the course of exemplifying the categorisation, some aspects with a positive effect on parent-child cooperation, but also some with a negative effect are pointed out. We indicate the authors of the answers by M for mother, F for father, and C for child, combined with a respective number.

Parents as learning partners

In the category – parents as learning partners, the parents are open to cooperation with their children and accept the children as "equal partners" (M20) in the discussion, like it is intended by the above-mentioned dialogic learning approach. Here, parents often emphasise the positive experience of this cooperation. In what follows, we give an example of a statement on the parent-child cooperation experienced as positive:

[The process of solving the problem was] constructive, through cooperation with my child. [...] and we perfectly complemented each other. [...] My child approached the problem with more enthusiasm and curiosity than me. That made me happy. (M8)

Additionally, field notes give a closer insight in the kind of cooperation between mother and child. During the problem-solving process the mother took a step back, gave her daughter room and let her explain her approaches and insights first. Here and there, the mother summarised partial results and asked some crucial questions. Thereby the mother-child team got further in their problem-solving process and at the end reached an overall solution. It moreover is interesting that the daughter shares this positive experience of working together with her mother, as she expresses:

> We solved it [the problem] in teamwork. That was cool. (C8)

In this context, one identifiable factor for the positive experiences regarding parent-child cooperation is that the mathematical problem is not only unknown for the children but the parents as well. Moreover, since the problem is open-ended, both, parents and their children could contribute to the problem solution in their own ways. They could even approach it on an equal footing. However that was not the case for every parent-child team. In a few cases the fact that the problem was an unknown riddle for both, parents and children, did not work in favor of parent-child cooperation, but against it. Those cases are discussed below in the part of *parents as learners*.

Other parents emphasise the good parent-child cooperation during the workshop while confirming the stressful relationship to mathematics learning situations in context of school mathematics at home, like for example:

[...]. I especially do find the cooperation between child and parents very refreshing. Away from the mathematics-homework-problems to a "voluntary" learning and approach to mathematics. [...] There was no "school pressure" as probably usual. It simply was fun and had a good result. (F11)

The child as well remarks the positive parent-child-cooperation but furthermore values the mathematical context away from numbers and calculations like it is excessively stressed in school.

I liked it because it was something new and in maths with Miss H we always do calculations. (C11)

Indeed, arithmetic commonly builds a focal point in school mathematics. With that, this comment points out that an open-ended and problem-based discovery learning environment is a fruitful opportunity to gain positive new experiences with and diverse insights into mathematics, which are often missed out in school contexts.

Other parental journal entries clearly show that a cooperation does work, when parents do not take over a dominating role, a role of being master of knowledge, but become a learning partner for their children, sometimes on an equal footing. This aspect is for example reflected by the following parental expression:

At times the cooperation was good, when we came closer to the solution and respected each other as equal "providers of ideas / partners". There also were moments when my child lost interest [...] when the solution of the problem at times receded into the distance. (M20)

One main problem becomes apparent with this quote. Sometimes children get set back by occurring difficulties in the mathematical problem-solving. In this case parents should support their children more actively in order to overcome the difficulties together and refocus. Such situations in the workshops constitute an opportunity to impart methods and techniques to parents for supporting their children in a critical mathematical learning situation.

Parents as learners

In the category – parents as learners, it becomes obvious that parents are keen to learn mathematics for themselves. They are fascinated by the mathematical problem at hand and take the opportunity of getting fully absorbed in the task. Being completely engaged in their own problem-solving process, they lose sight of their children and do not enter into cooperation with them.

One mother describes in detail her work on the mathematical problem, exclusively using the first person. For example, she comments on her thinking process, occurring difficulties while working on the problem, and finally concludes with the following statement:

> I would have liked to cooperate more closely with my daughter. (M7)

Not only the field notes point in a similar direction, but also the child comments on the workshop exclusively using the first person. This further strengthens the impression that no cooperation took place.

Other responses give a closer look into forms of cooperation between parents and children. One father points out, that he would have liked to accomplish the problem solution on his own and hence is not content with *his* work. He solely refers to his child in one line with the coordinator and the other participants when asking for exchange as he got stuck in his own problem-solving process:

> Thinking into an impasse, solved it via exchange (with child, coordinator other participants). (F22)

However, the field notes confirm the impression of the father deepened into his own problem-solving process and completely forgot his child over it. The child himself tried to get some glimpses at his father's problem-solving process, asking for insights, but was on his own for the rest of the problem-solving process. This impression is again strengthened by the child using only the first person in his journal entries, not referring to his father at all.

The following example, where the child together with father and mother took part in the workshop, is another illustration for the category of *parents as learners* where a competitive thought is decisive for a lack in parent-child cooperation. While the father describes his point of view in a concise way, the mother is more detailed and discusses her despair in the problem-solving process, but also her ambition to overcome her difficulties on her own. The problem-solving process of this family can be best described as competitive, which can be confirmed by the following mother' statement.

> What my husband is able to accomplish, I need to accomplish as well. (M24)

Nevertheless, she clearly expresses her wish for more cooperation in her further remarks.

> [I would have liked] that we cooperated more closely – instead of everyone for himself. (M24)

The child strengthens the impression of competition and the lack of cooperation by writing the following:

> Nasty that dad always was the first one to find out everything. (C24)

Taking the competitive nature of some parent-child teams into account a different approach is needed in order to strengthen parent-child cooperation. One promising approach could be to let parents and children work on a mathematical problem that directs them to work together as a team in order to compete against other parent-child teams. Such a context could fuel the competitive nature of some parents and children, but at the same time strengthen their team building.

CONCLUSION AND OUTLOOK

The discovery learning situation of the paperboy's problem, presented here, offers parents and their children a possibility to investigate a mathematical problem on an equal footing. They have the opportunity to experience mathematics in an open, assessment-free setting together as a team and therewith could gain positive feelings of social relatedness, as well as mathematical competence. Sometimes such experiences between parents and their children were more intense, sometimes less.

Searching for parental attributes that may be linked to the form of parents working with their children in context of the math-experience-days, namely *parents as learners* or *parents as learning partners,* no direct relation to parental gender, age, educational qualification or the occupational field could be found. That does not mean that there is not any connection to those attributes, but that they are not explicitly related to the parent-child-cooperation, rather multifaceted in its nature or interwoven with other factors. Those factors could for example be parents´ emotional dispositions towards mathematics or parental motivations to do mathematics. In some cases it already became obvious that parents´ desire to investigate a mathematical problem on their own, their ambition to solve the problem for themselves is so dominant that the risk exists that they forget to cooperate with their children. Such factors influencing parents' ways to work with their children need to be investigated further in future research on the family math project.

Furthermore, it would be interesting to investigate more closely the impact of such a family math project on mathematical learning situations at home. As it seems, general mathematical learning situations at home, mostly in context of homework, are often accompanied by tension and stress. Several parents mention this issue in their answers being completely aware of the project's special status. In this context, the here presented results first and foremost indicate that an open-ended, problem-based discovery learning situation represents a fruitful context for children working together with their parents on a mathematical problem. In any case, the risk that parents overtake the role of the conveyer of knowledge is minimised by the nature of the presented learning environment. When, moreover, the task for parent-child cooperation is explicitly defined by activities like investigating the problem, comparing approaches and therewith generating ideas or strategies without pointing to an existing solution, the pressure to find an overall solution could be reduced even more. Hence, one promising approach, in order to foster parental involvement in their children's mathematical learning, could be to integrate such open-ended, problem-based discovery learning situations as special parent-child homework.

However, the nature of parent-child cooperation in those problem-solving contexts needs to be investigated further. It is clearly not enough to examine only the course of one workshop, with one type of mathematical problem at hand, in order to get a holistic picture of parent-child cooperation and promoting or hindering factors. Consequently, it needs more and diverse family math workshops to investigate the conditions and dynamics of parent-child cooperation in open-ended, problem-based discovery learning environments. That will follow in the main survey.

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The sightless eyes of reason: Scientific objectivism and school geometry

Melissa Andrade-Molina and Paola Valero

Aalborg University, Faculty of Engineering and Science, Aalborg, Denmark, melissa@learning.aau.dk, paola@learning.aau.dk

There is a gap between the aims of school geometry in terms of the teaching of spatial abilities to young children, and the dominance of a school geometry rooted in Euclid's axioms and abstractions. Such gap is not to be explained in terms of a "misimplementation" of the curricular intentions. Rather, the gap evidences elements of the power effects of school geometry on children's subjectivities. We problematize both the truths circulating in school geometry discourses and the effects on children's subjectivities, by adopting cultural historical strategies to research the functioning of school geometry curriculum. We argue that school geometry fabricates the scientific minds of the future by educating students to see not with the eyes of their bodies, but with the eyes of reason and logic.

Keywords: Objectivity, subjectivity, school geometry, power effects.

INTRODUCTION

Nowadays it is argued that spatial visualization ability plays a key role in shaping the successful scientific minds of the future, probably as important as verbal and mathematical thinking (Newcombe, 2010). This ability becomes critical when developing expertise in STEM domains (science, technology, engineering, and mathematics). It is believed that including spatial ability as a criterion for identifying talented youth would help recruiting many adolescents with potential for studying STEM fields, but who are currently being missed (Wai, Lubinski, & Benbow, 2009). There is a research trend claiming the importance of providing spatial education for young children because "increasing access to a preschool "spatial education" constitutes a safe bet for fuelling school readiness and igniting long-term performance gains in STEM-related fields" (Verdine, Golinkoff, Hirsh-Pasek, & Newcombe, 2014, p. 20).

But why has spatial visualization become so important? One possible reason is how research has found that visualization is central in the conceptualization processes of scientific discoveries. For example, the use of spatial reasoning is implicated in several physics discoveries such as Galileo's laws of motion, Faraday's electromagnetic field theory, and even Einstein's theory of relativity (Kozhevnikov, Motes, & Hegarty, 2007). Furthermore, the discovery of the structure of DNA was "centrally about fitting a three-dimensional spatial model to existing flat images of the molecule" (Newcombe, 2010, p. 29).

It is postulated that "spatial thinking can be taught; [... and that] it is possible with appropriately structured programs and curricula" (National Research Council [NRC], 2006, p. 109). Several studies on school geometry deal with the development of spatial abilities in students, from introducing diverse activities with building blocks to changing the entire school geometry curriculum. Despite the recognition that spatial ability is a key element to science and that it can be taught, it is also highlighted that the teaching of this ability is largely ignored in formal school settings (Clements & Sarama, 2011).

It is the contention of this paper that there is a gap between the aims of school geometry in terms of the teaching of spatial abilities to young children, and the dominance of a school geometry rooted in Euclid's axioms and abstractions, which prompts to a flat and abstract world. Our assumption is that such a gap is not to be explained in terms of a "misimplementation" of the curricular intentions. Rather, the gap evidences elements of the power effects of school geometry on children's subjectivities. Adopting cultural historical strategies to investigate this contention in the constitution of school geometry, the paper deploys an argument in three movements. Firstly, we examine how notions of space move between discussions of perception and formalization. The formalization of the language of Euclidean geometry, through scientific objectivism, provides ways of building a scientific self. Secondly, we explore how such objectivation entails the subjectivation that takes place through the discourses of school geometry. By examining the curricular materials of Chile, we exemplify the existing gap between such discourses and the expressions of the ideal student. Finally, we problematize both the truths, in terms of Foucault, circulating in school geometry discourse and the effects on children's subjectivities. We evidence how the school practice of knowing geometry has effects on how students understand and train themselves to see space.

ANALYTICAL STRATEGY

There are many truths that circulate in mathematics education research, and such truths constitute unproblematized understandings of the practices of mathematics education. The type of research deployed in this paper assumes that mathematics education practices are political because they govern subjectivities in both productive and constraining ways (Valero & García, 2014). Evidencing the subject effects of a series of practices and discourses, such as school geometry, is a contribution in understanding how the school mathematics curriculum fabricates the subjectivities of children through educational processes. This is important since mathematics education is not only a process of knowledge objectivation, but also a process of subjectivation or of becoming within culture.

More concretely, this approximation is inspired by the work of Michel Foucault. Our strategy is composed by some concept-tools we borrow from Michel Foucault (subjectivity, discourse, truths). We bring this concept-tools to help us reasoning about the problem of the apparent gap between the aims of school geometry in terms of the teaching of spatial abilities and the dominance of a school geometry rooted in Euclid's axioms and postulates. Our empirical materials consist of students' textbooks, curricular guidelines for teachers and geometry maps of learning progress, all of them produced by the Chilean Ministry of Education (MINEDUC).

Since discourses are produced by the interaction of different spheres of social life and are shaped by statements and their related truths (Foucault, 1972), to understand how school geometry is operating it is also necessary to study how geometrical knowledge has been shaped. Here we delineate elements of such study connecting geometry with cultural historical studies of science (e.g., Daston & Galison, 2007). The discussion of objectivation/subjectivation in science and geometry invites us to approach the school geometry curriculum as practices that govern subjectivities through the enunciation of the ideal student. Hence, problematizing the naturalised truths that circulate in school geometry discourse and the statements about the aims of school geometry will help us to elucidate the effects of geometrical knowledge objectification on the self.

THE OBJECTIVATION OF SPACE

Mathematics lives in a world of abstractions, axioms and formulas. There is a perfect and ideal world within mathematics, every calculation applied correctly should work impeccably, even if is not about a real object. For de Freitas (2013), mathematical objects are taken to be entirely free from spatio-temporal conditions. Hence, if mathematics is universal and has no context it is possible to understand it as a blind sight, without inference, interpretation or intelligence (Daston & Galison, 2007).

However, it is believed that mathematics can describe the world we live in. But if geometry can describe what we are able to see, our surroundings, why has it become a blind sight? According to Boi (2004), the anatomy of the eye entails light on a curved retina, therefore our visual system deploys a projective geometry rather than Euclidean metrics. An experiment conducted by Blumenfeld (Hardy, Rand, & Rittler, 1951) demonstrated that phenomenological visual judgments do not satisfy all Euclidean properties, he revealed that physical configurations do not coincide with Euclidean geometry (Suppes, 1977)[1]. Likewise, Burgin (1987) claims that the conception of Euclidean geometry's space was based on technique rather than on visual evidence. It was based on axiomatic, which deployed an idealized world of ideal shapes, such as triangles, squares, platonic solids and so on. It is an objective knowledge.

For Daston and Galison (2007)[2], objectivity in science was not a matter of viewing nature as it really was, but as it should be to be studied – nature as an ideal nature –. The result of objectivism was an annulation of the self by the self, it was "the suppression of individuality, including images of all kind, from sensations of red to geometrical intuitions" (Daston & Galison, 2007, p. 46). Images, within science, were 'left behind' because it was the only way to break the mental world of individual subjectivity.

If objectivity in science aspires to a knowledge that bears no trace of the knower, how is geometry supressing individuality? For Daston and Galison (2007), objectivity becomes an 'epistemic virtue' when abstractions are able to transform subjective representations into objective concepts. For example, Tazzioli (2003) shows that Mario Pieri, an Italian mathematician, introduced the axioms and methods of projective geometry without any reference to intuition (neither to monocular vision). He was close to cut the link between geometry and empiricism. Hilbert's work, inspired by Pieri, was the masterpiece that led to a geometry based on logic, axioms and theorems (*règles*), a form of geometry in which intuition and experience do not have a strong role. This led to a space that can only be reached by mathematics, very distant to the one we are able to see and interact with.

But space is a product of concrete practices and attempts to representing them; it is not abstract at all. However, when it comes to the knowledge that traditionally has dealt with space —geometry—, then it becomes the realm of abstraction. It becomes an objectified space. For instance, according to Lefebvre (1991) space can be understood in three forms: space as perceived, as conceived and as lived. The first form takes space as a physical form, as real space, a space that is generated and used. The second form, the space of knowledge (*savoir*) and logic, takes space as an instrumental space. Space becomes a mental construct, an imagined space. The third one, space of knowing (*connaissance*), sees space as produced and modified over time and through its use. It is a space that is real-and-also imaginary. Geometrical space is a space of *savoir,* within Lefebvre's forms; it is an idealized, imagined and constructed space.

In the same fashion, school geometry is leading us to see space as an instrumental space, a mathematical space of *savoir*. Space becomes Euclidean, Cartesian, and flat. It is very distant to the one we can perceive. Ray (1991) stresses that school mathematics has been based on the axioms of Euclidean geometry because they provide an internally consistent system, evident to the eye. But this objective geometrical space deployed by Euclidean geometry is limited, in terms of Lefebvre's forms of space.

BLINDING THE CHILD

In school, geometrical knowledge tends to be constructed outside the body. It is a fixed knowledge that has to be learnt by students. School geometry is based on abstractions, very distant from our perception of 'daily space', as if we had a body without the sense of sight. In other words, dealing effectively with school geometry tends to fabricate a sightless body. Then, learners are not really prompted to use their senses to learn or interact with geometrical knowledge.

For example, Chilean mathematics curricular guidelines claim that mathematics was developed to solve diverse challenges of mankind and of mathematics itself, within history and culture. Therefore, school mathematics must provide and facilitate the understanding of the real world we live in (Ministry of Education of Chile [MINEDUC], 2010). In this sense, school mathematics should supply students with tools to interact with the world they are able to see. An analysis of official documents of the Chilean Ministry of Education (MINEDUC) shows that its claims are built around certain statements, in a Foucaultian sense, that delineate the features of an ideal child, such as:

> School mathematics curriculum is aimed to provide students with the basic knowledge of the field of mathematic, and, at the same time, that students develop a logical thinking, deduction skills, accuracy, abilities to formulate and to solve problems and abilities of modelling situations (MINEDUC, 2010, p. 3, our translation).
>
> To learn mathematics enriches the understanding of the reality, facilitates the selection of strategies to solve problems and contributes to an autonomous and own thinking (MINEDUC, 2010, p. 3, our translation).

School geometry becomes a valued and useful knowledge, a set of tools that will help students to fulfil any situation of the real world. For Valero, García, Camelo, Mancera and Romero (2012), school mathematics "inserts subjects into the forms of thinking and acting needed for people to become the ideal cosmopolitan citizen" (p. 4). In this case, an ideal student should be

When students have achieve this level, they solve activities as the following:

- To calculate, by using geometrical and analytical methods, the area between four circumferences of radius a, which centers are the vertices of a square of length 2a.
- To determine the number of diagonals of an n-sided polygon, by using combinatorial techniques.
- To prove by contradiction, that a line is tangent to a circle when the radius drawn to the point of contact is perpendicular to the tangent at that point.
- To prove, by using congruence axioms, that the exterior angle of a triangle is grater that either of the remote interior angles.
- To construct a regular pentagon using a compass and straightedge or a software, and to justify the construction.
- To deduce the vector equation of a line from its Parametric and Cartesian form.
- To formulate and to verify which geometrical shapes satisfy that: the sum of the areas of the shapes built up on the sides of a right triangle is equal to the area of the shape built up on the hypotenuse.

Figure 1: Seventh level of the map of progress (MINEDUC, 2010, p. 18, our translation)

a logical thinker and a problem solver. The student should be capable of modelling real life situations by only using mathematical knowledge.

Furthermore, the Chilean Ministry of Education (MINEDUC) established a map of progress with seven levels that students have to achieve along school geometry in compulsory education. An ideal student should perform successfully in activities where he/she must be able to solve problems by only using geometrical axioms and theorems (see Figure 1), which leads to a notion of space in terms of the formal system of Euclidean geometry. What it takes to deal successfully with these tasks is far from spatial visualization. It seems that this ability is something that has to be developed by the student itself.

The mismatch between the expectations and the description of abilities could be related to the fact that Chilean school geometry is based on Euclidean, Cartesian and vectorial geometries, necessary to cope with other school subjects, such as physics.

> The world we live in is three-dimensional [...]. It is aimed that students are placed in a real three-dimensional context, providing new tools to make spatial and flat representations, such as the vector model. This model constitutes one of the basic foundations of physics and mathematics. (Ministry of Education of Chile [MINEDUC], 2004, p. 68, our translation).

According to this quote, it becomes important to link "reality" with school mathematics. By doing so, it is assumed that students will be able to use geometrical tools to solve everyday-life problems. The way to achieve this link is by introducing three-dimensionality to students. Within school, the "world we live in" becomes vectorial. Consequently, the expressed desire to link spatial thinking and the real world seems to blur, and the only important part left are the perennial mathematical abstractions. This generates a new type of space, the space of school, which has been 'chopped' and has been restricted.

As an example, the MINEDUC (2004) enhance certain types of activities where students "emphasize relations between Cartesian and vector equations within geometrical shapes" (MINEDUC, 2004, p. 68, our translation) than other type of activities where students could develope spatial skills. Space for school is in terms of XYZ, a space than can be modelled by school mathematics. Chilean school geometry is based on a flat geometry, mainly Euclidean. But in Euclidean geometry the studies are on objects situated in the void; objects that are not real (Kvasz, 1998). A possible question to pose is if Euclid of Alexandria was living in an abstract world? The easy answer is that of course not he was, and, moreover, he started his studies by analyzing his surroundings. He developed a theory known as Euclid's optics. Which is a theory of vision and of intuition (Suppes, 1977).

According to de Freitas (2013), logic and axiomatic relations in mathematics tend to erase the temporal and ontological. As a result, school mathematics is an untouchable knowledge that becomes universal, decontextualized and, therefore, without culture or the possibility to influence in it (Valero & García, 2014). It is an unalterable truth, installing mathematics as the science of pure logical structures and negating all connections between mathematics and the real world we live in (Kollosche, 2014).

> The concept of space to be reconstructed in the students' understanding is that of a rational, referential space with fixed points in two or three dimensions. It is assumed that the conceptual development of the child will lead to an internal and abstract representation which will contribute to making a decontextualized child, freed from the practical capacities of acting with objects in space, particularly of those spaces where everyday life occurs (Valero et al., 2012, p. 7).

However, school keeps making this link between what we are able to perceive and the abstract space of mathematics. We claim that there is a gap here to explore.

THE SIGHTLESS EYES OF REASON

We deployed a discourse analysis of official curricular materials of the Chilean Ministry of Education. This analysis, built from Foucault, has pointed to the existence of statements circulating about an ideal student. In this existing discourse, it is believed that by connecting school geometry and reality students will become problem solvers, logical thinkers, 'reality modellers' and so on. The existing discourse requires that students perceive themselves as agents who are able to change the world, but also as agents who are responsible for their own learning (Foucault, 2009). More precisely, school geometry deals with power-knowledge relations [3], by promoting the fabrication of a certain type of subject, a scientific trained child.

But, how is school geometry discourse operating on students? Here the discussion is not about the contents of school geometry itself rather it is on how school geometry is operating in the fabrication of children's subjectivity. In other words, it is on the power effects of school geometry in fabricating forms of being in the world. In this sense, human beings become subjects through the objectifying effects of scientific knowledge, a knowledge that is also objective (Foucault, 1982). And, at the same time, the practice of knowing generates effects in the form of knowing and in the subjects who know (Daston & Galison, 2007). Therefore, students must train themselves to become part of a practice. An example of this self-training is illustrated in the following activity proposed in the 6th level of the progress map of Chilean school geometry (MINEDUC, 2010, p. 17).

> A young girl is observing a pit formed by two concentric circumferences, 1 m. and 1.2 m. of radius respectively, and 3.5 m. of depth. Using this information, she is able to make a model by drawing a rectangle ABCD in a coordinate system XYZ. It is asked to the girl to determine the rectangle's vertices and to argue on which axis the rectangle must rotate to obtain a three-dimensional representation of the pit. Finally it is asked to the girl to calculate the pit capacity, in litters.

Clearly the ideal student must forget about his/her senses; must train him/herself to be able to model real life situations using geometrical deductions; must be able to think space in terms of XYZ. It is not necessary to use spatial visualization ability to solve this activity because is useless. Is it relevant to mention that the

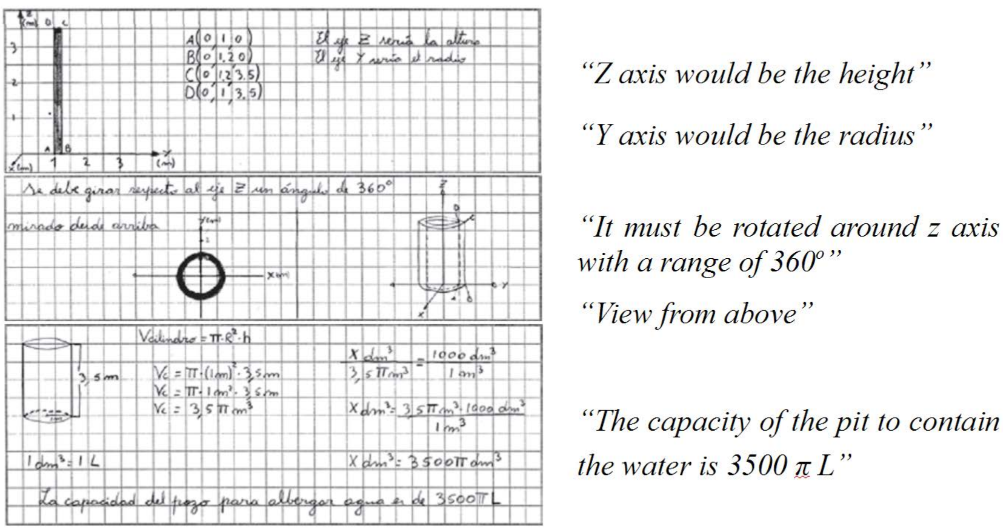

Figure 2: Example of a expected answer of a student

girl was observing the pit? Would it make any difference if the problem had not been contextualized to a real life situation? Why is it relevant to use a coordinate system to calculate the volume of the pit?

This evidences a gap between the aims of school geometry on curricular materials of Chile, in terms of the teaching of spatial abilities, and the notion of space that school geometry promotes, which is rooted in Euclid's axioms and abstractions. Reasoning with the objectification of space in geometry, to shape a scientific self implies to suppress the self, which means to cut all links between perception and geometrical formalizations. This suppression leads to perceive space and geometrical knowledge as decontextualized, and as universal and timeless.

Nonetheless, space outside the school is not universal neither timeless. What if this notion of space changes? Sanjorge (2003) argues that the space in which the subject is constructed has already changed; consequently the subject itself has been changed. There are virtual movements, where there is no orientation, not right or left, it is a post-Cartesian space, which is nonlinear. It is a "spherical space, where up and down are not positions in the world but situations of the viewer" (Sanjorge, 2003, p. 5, our translation). This is a notion of space unfolded by technology, a virtual world that is subjectifying children to perceive no orientation, where everything is reachable by a 'click'. It is a space opposed to Cartesian movements within 'school space' and the different spaces are not related at all.

At the end, the interplay between power and mathematics education is on how the school mathematics curriculum generates cultural and historical subjects (Valero & García, 2014). Then, school geometry becomes a technology of the self and the others, by regulating children's conduct, and by developing 'cultural thesis' (Popkewitz, 2008) about an ideal student who is able to see with sightless eyes. This generates systems of reason in which forms of life and subjectivity are made possible, organized and constrained. Therefore, school geometry has power effects on how students understand and train themselves to see space. Such subjectification pursues to fabricate the scientific minds of the future by educating students to see not with the eyes of their bodies, but with the eyes of reason and logic.

ACKNOWLEDGEMENT

This research is funded by the National Commission for Scientific and Technological Research, CONICYT PAI/INDUSTRIA 79090016, in Chile and Aalborg University in Denmark. This research also makes part of the NordForsk Center of Excellence "JustEd". We would like to thank the members of the Science and Mathematics Education Research Group (SMERG) at Aalborg University for their comments to previous drafts of this paper.

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ENDNOTES

1. Blumenfeld performed a series of experiments with parallel and equidistant alleys. One of the tasks in the experiment was to arrange two rows of point sources of lights as straight and parallel to each other as possible. The lights were placed on either side of a plane. The results revealed that physical configurations do not coincide with Euclidean geometry. In Euclidean geometry, parallel lines are equidistant along any mutual perpendicular. However, in the experiment, the resulting lines diverged, they were not parallel at all. He concluded that Euclidean geometry does not apply to our visual space.

2. Daston and Galison (2007) analysed images on scientific atlases to study its history, emergence and development.

3. This power is not to be understood in terms of domination of the self; it is not an imposition to train students' sight. This power understands 'the other' as a person who acts on his/her own; depending on the freedom of the subject (Foucault, 1982). Likewise, school geometry discourses are not a form of impositions; they are produced because we reproduce them through language. They are an 'action upon action' (Foucault, 1982).

Pressures and positions of need during the Swedish third-grade National Test in Mathematics

Anette Bagger

Umeå University, Department of Science and Mathematics Education, Umeå, Sweden, anette.bagger@umu.se

This paper presents and discusses parts of a large-scale [1] ethnographical and longitudinal study that has followed the process of implementing the National Test in Mathematics for third graders (Ntm3) in Sweden during its first three years (2010, 2011, and 2012). Pupil talk from 2011 about pressure and what is at stake was used to construe three positions of need that might characterize pupils during the National Test in Mathematics in their third school year: the position of shame, the position of unfamiliarity, and the position of stress. How these might be handled in educational practices is discussed briefly.

Keywords: National test, pressure, mathematics, third grade, position of need.

INTRODUCTION

The National Test in Mathematics for third graders (hereafter called Ntm3) in Sweden was introduced in 2010 (Utbildningsdepartementet, 2012). Since this test in its current form is a new phenomenon, research is scarce. Levlin (2014) has investigated how reading and langugage skills in the second school year predict achievement on the National Test in Mathematics for third graders, test anxiety and its connection to acievement have also been examined (Nyroos, Bagger, Silfver, & Sjöberg, 2012), and the tests have been studied as a technology of disciplinary power (Foucault, 1980) exercised over children (Sjöberg, Silfver, & Bagger, 2015). Other research on tests for children of similar ages has shown that national tests can contribute to difficulties in mathematics (Sjöberg, 2006). There are ways of handling test anxiety, coping with stress, and learning test-taking skills (Sena, Lowe, & Lee, 2007), but these require identifying needs and pupils in need. The very concept of a pupil who needs support in mathematics is used and understood in a variety of ways in research, contributing to challenges in communicating and building knowledge about needs and support for these pupils (Bagger & Roos, 2014). The concept of *need* is in this paper understood as a position that a pupil might move in and out of, depending on the situation and the individual or the environmental prerequisites (Silfver, Sjöberg, & Bagger, 2013). This paper looks further into some of the discursive preresiquites for developing the position of 'being a pupil in need'. The approach is to investigate pupils' perspectives from an ethnographic point of view. Hereby I strive to adopt an insider's perspective in the research by revealing the participants' perceived experiences and, through this, building an emic narrative. The pupils are then the source for the emerging story about being a pupil in need. Three focus areas guide this investigation in exploring how pupils are positioned and position themselves in terms of possible needs: (1) pupils' talk about the experience of pressure, (2) whether and how stakes are expressed by the pupils describing negative pressure, and (3) pupils' talk about the position of being in need. This paper represents a pilot study that elaborates on the methodology before it is applied in a larger sample of pupils at the same three schools. Larger numbers will make it possible to discuss issues of diversity and equity and to take socioeconomic factors into account.

NATIONAL TESTS IN SWEDEN

Educational reforms in which assessment and comparison of pupils have played key roles have become more common internationally in the last decades (Clarke, Madaus, Horn, & Ramos, 2000). One purpose of the testing is to identify pupils at risk of not achieving the educational targets (Wyatt-Smith & Castleton, 2005). Media often publicize the results, reporting how well schools perform in relation to each other and whether more or fewer pupils have succeeded. Thus, the tests sort and categorize not only pupils but also schools. This can be understood as part of an international trend in neoliberal educational policy, whereby schools are governed and govern through the test (Hudson, 2011). Parents' choice of schools might then be described as a consumer's choice of education, where the child's knowledge is perceived as a product (Lange & Meaney, 2014). In this way, the discourse surrounding testing activates power relations between schools, teachers, and pupils.

Ntm3 has the purpose of both evaluating the pupil's level of knowledge and evaluating education at large. It also aims to support a just evaluation of pupils' knowledge by securing the equality in teachers' assessments (Björklund Boistrup, & Skytt, 2011). The third-grade National Test in Mathematics is taken by all pupils with few exceptions and requires approximately one month to administer. It consists of several subtests connected to different (but not all) parts of the curriculum. In addition, there initially is a self-evaluation section and a co-operation subtest. The test is administered and corrected by the classroom teacher. Afterwards, the schools' test results are saved in databases (SIRIS, Swedish Ministry of Education).

THE PUPIL IN NEED DURING TESTS

Pupils might experience standardized tests as a pressure on them, but whether the pressure is perceived as positive or negative varies. Negative pressure can increase failure rates and affect the learning progression because it threatens pupils' self-worth (Putwain, Connors, Woods, & Nicholson, 2012). Putwain and colleagues (2012) describe how stakes relate to negative pressure:

> Higher stakes tests increase the threats to self-worth, unfamiliarity increases the uncertainty of being able to demonstrate competence and low competence beliefs increase the likelihood of failure, all of which result in the appraisal of tests as threatening rather than challenging. (p. 300)

That is, if pressure is negative, the pupil might fear or imagine that something is at stake, and this could contribute to pupils "in general" being transformed into pupils in educational needs. A need is not only constituted of what the individual wishes and asks for but is also a matter of necessity and depends on the situation at hand (McLeod, 2011). The need for support during tests can be explained as an occurring agency need that for some individuals remains also on other occasions and at other times but may also be caused by a dispositional educational need. An occurring need may therefore be temporary or persistent and in some cases both. Teachers are good at detecting achievement but are less skilled at detecting what the pupil thinks about him- or herself (Urhahne, Chao, Florineth, Luttenberger, & Paechter, 2011); thus, it is crucial that research seek to clarify pupils' understandings of themselves. Research about how pupils position themselves in testing situations contributes to this end, since it may reveal some of the social dimensions of testing from the pupil's perspective and positions of need that are foreseen. Giving pupils support and equal chances of passing the test is especially urgent because reports indicate that the school system is segregating pupils, which might be seen in test scores and grades (see, e.g., Swedish Agency for Education, Swedish School Inspectorate, 2012). Drawing on Atweh, Graven, Secada, and Valero (2011), I understand the issue of providing high quality mathematics and equity as an ethical imperative that schools must address.

METHODOLOGY

Ethnographical data were produced before, during, and after the national test. Contextual facts were collected before the test regarding the municipalities, schools, and classes.

Data collection and selection

In order to access the pupils' firsthand experiences about the test, video-stimulated recall dialogues (VSRD) were conducted with pupils individually and video recorded (see Silfver, Sjöberg, & Bagger, 2013). The technique used in interviewing the children was inspired by Morgan (2007). Data from these interviews have been analysed, whilst other data from the project constitute a contextual background for understanding and interpreting the interviews. The pupils with the highest and the lowest test scores at three schools in one municipality (2011) were selected. Choosing these pupils presumably brought forward some of the critical issues regarding mathematics testing. The selection makes it is possible to investigate preresiquites for needs regardless of achievements, gender, background, or skills. The schools ranged in size from medium to large, and their locations varied from inner-city to suburban and from lower class, immigrant-majority to native Swedish-speaking, middle-class areas. The selection of schools allows for the discussion of social justice and diversity, in addition to discourses in the area of test taking in both policy and practice.

Year 2011	Outer-city school		Inner-city school		Non-city school	
Pupil	Sofie	Ali	Ellen	Emmanuel	Anna	Sara
Passed subtests	All	None	All	None	All	None
Score (max 101)	87	55	89	19	97	75

Table 1: Pupils with the highest and the lowest test scores from three classes (2011) in three schools, from various socioeconomic settings (the names are fictional)

Analytical process

Data were examined using Transana, a computer software for encoding and analysing video and audio recordings. Analysis was performed in three steps, according to guidelines from Heath, Hindmarsh, and Luff (2010) and is displayed in Table 2. Clips were organized, after repeated reviews, into collections with open codes. Patterns, such as similarities and contradictions, were searched for between collections. Positions of need were construed as categories to summarize the patterns found. This is presented in the results. Examples are given of what was said.

Negative pressure was identified in the talk when pupils implied that they had felt uncomfortable and experienced disturbing thoughts about the test, math, and themselves. *Positive pressure* was identified when the talk was influenced by positive feelings, expectations about the test, math, and themselves. Whilst searching for things at stake, I applied Putwain and colleagues (2012) definition of *stakes* as

> the real or imagined consequences of testing students' academic credentials (test scores or grades), educational access (for example to a particular school) and educational progression/ability setting, and also for teachers professional status (e.g. from school league tables). (p. 291)

Instead of *teachers' professional status*, I searched for *pupils' status*.

Positioning and discourse as analytical lenses

Drawing on Foucault, one can understand discourse as constructing and constructed by individuals and society through systems of representation. Through these systems, knowledge, truth, and power are produced (Hall, 2001). An individual's ability to speak is regulated by discourse in determining how, when, and from what position people may speak (Davies & Harré, 2001). These positions depend on the discursive context, the participants themselves, and the activity at hand (Lofors-Nyblom, 2009). This means that the discourses in the test situation will govern possible positions of need and the ways that the test can be talked about. An important theoretical distinction is that a position of need is not the same thing as one pupil's *being* in one fixed position of need. The concept is instead a construction of possible positions of need that a pupil may take on or be given in the test situation. One pupil's talk might in this way contribute to several positions of need, depending on the position from which that pupil is talking. It is not the subject (pupil) who talks but the discourse that talks through the subject (Foucault, 1983).

RESULTS

Positions of need construed from pupils' talk about pressure and stakes are presented in the following paragraphs. Positions of need were expressed only in regard to negative pressure. Positive pressure, along with what could be described as positions of 'the good test taker,' is also displayed. In pupil talk about positive pressure, no needs were apparent; neither was any talk about stakes.

Pressure as positive and the understanding of responsibility

Sofie and Ellen were both high achievers who talked about the test as interesting and the nervousness as stimulating. Ellen emphasized that loneliness and being on one's own contributed to the feeling of positive pressure. These two girls were emphatic about positive aspects of the test. Ellen said, 'The teachers want to see what we can do, so that we can achieve our goals.

Selection of pupils	Review 1	Review 2	Review 3: Analysis
Highest and lowest performing in class	Talk about pressure and test consequences	Talk about negative pressure and stakes involved	Positions of need during tests

Table 2: Process of analysis

I think it's really important—it's important to achieve goals in school, and you can do that if you take the test.' Ellen had great trust that the actual accomplishment of the test would help her reach her goals. And Sofie was very clear about where the responsibly for her development lay:

> Because this test is for—I think it's for you to see what you already know and what you need to practise. And what the teachers will ... if they need to give someone more instruction and if they should be clearer on something ... So it's really the teachers that get to know if they did well or not.

Despite this outlook and the fact that Sofie had the highest score in her group, she referred to skilled boys in order to position herself as talented:

> Yes, I think I had the fewest mistakes in the class, because there are two other boys that have other math books since they are so good ... [and] I had one mistake less than them.

Lack of curiosity regarding the test

Sara did not express pressure in either direction but thought that the math on the test was difficult. She implied that she might lose knowledge that is important to managing in life—for instance, 'if you have to go somewhere, or buy things in a store.' She also said that it could be fun to take the test because she was allowed to draw, something she liked to do. And it was fun to write the test because it was something unusual. Although she had the lowest scores, Sara passed all the tests. Her school's area was a white, middle-class neighborhood, and almost all its pupils spoke only Swedish.

Pressure as negative and what is at stake

In the talk about negative pressure, three different positions of need were construed: the position of stress, the position of shame, and the position of unfamiliarity. The talk also contains signs of things perceived as at stake. Of the six pupils, four talked about the test as pressuring them in negative ways and mentioned things being at stake.

The position of stress

Both high achievers (Anna) and low achievers (Emmanuel and Ali) reported having experienced anxiety and stress or disturbing thoughts in connection with the test. These emotions were felt in the body, whilst only the high achiever Anna described stressful thoughts. Emmanuel said, 'I was nervous; my hands shook. It felt like they would write the wrong answer.' Anna talked about the fear of not being able to retrieve her knowledge whilst working on the test. Even though she was sure that she was good at math because adults had told her so, she was insecure about whether her knowledge could really be trusted:

> But later on, I had a little panic ... so that I could barely ... I was still nervous, so I panicked because I was writing it. But exactly when you needed it [the knowledge] most and you needed ... to write the answer ... to remember the answer—then it flew out of your mind, and when you don't need it, it comes back. I got more than a little irritated.

Anna also vividly described how stressed she felt during the test and how she tried to handle these feelings. She was worried about not finishing in time and about making mistakes; several times she indicated that she was afraid of not being permitted to continue to the fourth grade. Her social belonging was at stake: 'If you didn't get the right answer, maybe then you would have to retake third grade. It felt like that.' At the same time, she understood that this was not the case: 'I know that it's not like that, but I just got that feeling, and it also felt like I would get a heart attack a little.' The feeling of potentially loosing her social context grew whilst Anna was waiting for the results: 'Waiting for the results felt like waiting to know whether I had to retake third grade or could move on to grade four.' The stress may also have involved knowing that the test would be difficult and that the situation would demand concentration and intense thinking. Ali commented on this: 'It's hard, and you have to think really, really hard, and that might be tough.' He also spoke of finishing as a relief from pressure: 'Finally, I'm done, and now I'm free.'

The position of disappointment

Ali implied that knowledge might be lost if a pupil did not pass the test: he assumed that taking the test *in itself* generated mathematics learning. Along with this lost knowledge, his dignity would be at stake if he did not learn more, along with a potentially loss of social status:

> I think you can embarrass yourself sometimes if you don't know it [math] ... also, if we answer questions ... and they say how much is one times

> one or ten times ten, and when you don't know, then you might embarrass yourself.

He also mentioned another possible outcome of not succeeding in the test—namely, feelings of sadness: 'If you don't pass, you'll be sad and feel sorry that it didn't go well.'

The position of unfamiliarity

The fear of not knowing mainly involved feelings about scores. Both Ali and Anna said, 'What if I make a mistake?' Anna also worried about being able to finish in time; that is, she did not know whether she had enough time. Emmanuel spoke of uncertainty about his skill and his possible results. 'Will I pass this? (Interpreted as: I'm afraid I won't.)' This fear may have arisen partly because the test took place in an unknown situation. Anna said, 'Beforehand, I didn't know what to expect, what the test would look like, and I wasn't prepared.' This concern could be connected to Anna's fear of being unable to retrieve knowledge and putting her educational progress at stake.

DISCUSSION

The analysis of how pupils talked about pressure in order to study discursive preresiquites for positions of need in the test situation seems to have yielded rewarding results, since the pupils talk about pressure and stakes seem to give valuable information to possible educational needs in the test situation. To perform the same analysis in a larger sample of pupils might reveal still other positions of need, beyond the ones discussed here. Clearly, issues of power and governance were involved in the practice of test taking for the pupils in this article. They talked about the risk of losing social belonging, pride, and knowledge. Some critical gendered issues, as well as matters of diversity and social justice, are revealed, and it will be interesting to examine these further in a larger study. One conclusion from this limited sample, is that the test leads to a comparison and sorting of pupils. Pupils start to relate their achievements to those of others in order to understand their placement within the competitive discourse, which is more apparent in the multilingual schools. Comparisons were highlighted more often by the pupils in the multilingual schools, regardless of which kind of pressure they expressed. This implies that pupils on these schools are participating in a competitive discourse regarding scores which position them as having the potential of 'not winning' or 'loosing'. In the suburban school in a middle-class and mainly Swedish-speaking area, neither of the pupils compared themselves with others. Instead, they talked about their knowledge as the focal point in the testing situation. Those pupils are in this way participants in a testing and learning discourse where their knowledge is at stake. Anna is a key informant since she had much to say about her experiences. It is an important methodological reflection that the boys who reported negative pressure and stakes did not manage the Swedish language very well, going to the multilingual schools. It could be that their test performance was hindered because of this problem, which might cause negative pressure. It might also be that if they had been interviewed by someone who could speak their language, they could have provided much more information.

The results suggest that the position of unfamiliarity could both lead to uncertainty about one's ability and derive from uncertainty about one's ability. One educational need here is to cultivate knowledge about skills and the test, before the test. Knowledge about skills could decrease the risk of test anxiety (Ahmed, Minnaert, Kuyper, & Van der Weerf, 2012), but that would require that the teacher learn whether pupils lack these insights and have these feelings. The test situation enhances the focus on scores, which might lead to a heightened risk that teachers will not notice how pupils think or feel about this issue (see Urhahne et al., 2011). This also applies to the position of shame. Talking about these feelings would most probably positively affect both the achievement in the testing situation and the overall learning situation. Tests can actually create good opportunities to talk about feelings of putting things at stake, not remembering, not knowing, and embarrassment. This might lead to a balancing of the test situation so that pupils' needs come to the fore—not only the needs of the test. Clarifying the purpose of the test, how it will be used, and how it is understood by the teacher could perhaps lead to a leveling of inequalities for the pupils.

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ENDNOTE

1. The project "What Does Testing Do to Pupils?" was led by Gunnar Sjöberg and Eva Silfver and financed by the Swedish Research Council.

Mathematics as caring: The role of 'others' in a mathematical identity

Laura Black[1], Yvette Solomon[2] and Darinka Radovic[1]

1 University of Manchester, Manchester, UK, laura.black@manchester.ac.uk

2 Manchester Metropolitan University, Manchester, UK

In this paper, we explore the role of 'others' in one woman's mathematical identity. We draw on sociocultural theory to analyse identity in terms of the relationship between self and others: that is, as mutually shaped in interaction with others through the enactment of an 'identity in practice'. Our analysis focuses on the role of 'caring' as a particular form of this relationship, suggesting that our respondent uses this as a cultural resource to identify as a mathematician. We argue that caring supports her access to central mathematical spaces and, consequently, her potential to change the nature of mathematics itself.

Keywords: Gender, caring, identity, others.

BACKGROUND

Mathematics is perceived by many people as abstract, masculine, irrelevant, boring, uncreative and difficult (Mendick, 2006; Schoenfeld, 1992). Such negative views of mathematics appear to be particularly common amongst certain social groups – e.g. girls, low achievers, low SES (Forgasz & Rivera, 2012). Girls, for example, criticize school mathematics as repetitive and rote, and show a "quest for understanding" (Boaler, 2002) which places both the relevance and use of mathematics as central. A considerable body of research has suggested that such negative attitudes are symptomatic of a wider process through which girls are excluded from high status mathematics as a white, male, privileged social practice. Walkerdine's (1989) now classic study highlighted how from a very young age girls enter into mathematics though enacting domestic/caring roles in play with their mothers. But once at school, this "domestic mathematics" was discounted and de-valued against "formal-school mathematics" which subsequently positioned girls as merely hard working in contrast to boys' perceived natural and effortless ability. In school, the girls in Walkerdine's study continued to find social status within the classroom through exercising domestic/caring roles (such as being the class helper), but were positioned as lacking in school mathematics, foretelling restricted access and participation in mathematics later in life. Walkerdine thus highlighted how the domestic/caring roles that girls are socialised into ultimately served to exclude them from higher status forms of mathematics valued by school.

In this paper, we explore this account of 'caring' and its connection to girls' access to mathematics further. Whilst we recognise that many students' experiences of mathematics are shaped by its position as a gatekeeper to powerful and privileged domains, we argue that the processes by which individuals are included/excluded from mathematics are complex and that in certain circumstances, individuals may 'succeed against the odds'. Thus much of our previous work has provided examples of successful women (from non-privileged backgrounds) who have persisted with mathematics, examining the complex and contradictory structural conditions which have afforded their success (Solomon, 2012; Black, 2013, 2010; Solomon, Radovic, & Black, submitted). With this in mind, we hypothesise that 'caring' (both for others and by others) whilst still a result of marginalisation more generally, can potentially act as a cultural resource for individuals and, in the right circumstances, sustain a positive mathematical identity. We explore this hypothesis here by drawing on one of our previous examples – Roz – a professional 'academic' mathematician who one of us (Yvette) interviewed on a number of occasions over a period of 5 years. This period included an interview with Roz as a PhD student (Solomon, 2012) and as a post-doctoral researcher (Solomon, Radovic, & Black, submitted). We draw primarily on Holland, Lachicotte, Skinner and Cain's (1998) theory of figured worlds, and the ways in which individuals draw on cultural resources and models in 'identity in practice', using this as a lens for understanding the role of caring and its relevance in Roz's history of engagement with mathematics.

Theorising identity and 'caring for/by others'

Holland and colleagues (1998) argue that identities (rather than identity) are relatively durable entities which are drawn on and constructed in the stories we tell ourselves about who we are. Thus, they are multiple in nature and connected to the various practices we engage with. This is elaborated by Holland and Lachicotte (2007) who suggest that our perception of our own self is rooted in how we are perceived by others as we engage in practice:

> We actively internalize a sense of our own behavior as compared to the behavior of others acting in related roles and positions. We develop an inner sense of the collective regard that society is likely to have for our performances. Then, we craft our own way of being in roles and positions in relationship to this "generalized other," the collective sense that we gradually develop from those who evaluate us (Holland & Lachicotte, 2007, p. 107).

Elsewhere we have argued (Black et al., 2010; Black & Williams, 2013; Solomon, Radovic, & Black, submitted) that 'the internalisation of our own behaviour' which Holland and Lachicotte outline here takes place through reflection – in reflection we construct a story about who we are and thus our subjectivity (our own behaviour in practice) becomes crystallised as an identity (I am a certain kind of person). Crucially then, the stories we construct are mediated and shaped by our subjectivities 'in practice' to the extent that we cannot simply decide to adopt a particular identity on a discursive whim.

We find the idea of cultural models useful in this respect since they lie on the boundaries between 'practice' and the narrated self (Gee, 2010; Holland et al., 1998). Holland and Skinner (1987) describe cultural models as shared implicit knowledge about types and ways of talking about them, which become crystallised so that the knowledge/story/assumptions that they embody remains implicit. Gee (2010) provides an example in questioning 'Is the Pope a bachelor?' – which illustrates that although the Pope might be included in the formal definition of the term bachelor (a male who is unmarried), he does not typically fit with the implicit story or assumptions which have become crystallised through our use of this term. Therefore, we can argue that cultural models are culturally and socially shared and, to a large degree, shape what is possible and what is not. In Holland and colleagues' (1998) account of the US college dating scene, we see cultural models relating to various figures (jocks, nerds and so on), and these determine a story about who can date which kind of girl. Applying this to our concern with girls and women in mathematics raises the issue of what cultural models and figures intersect with self-positioning in mathematics, and the related issue of what kind of engagement with mathematics these models support. Furthermore, given Walkerdine's (1989) conclusions regarding the role of caring for others (in domestic roles) in excluding girls from mathematics, we are interested in how this particular cultural model might be used by someone who is female and yet apparently successful in mathematics. In this paper, therefore, we address the following questions: What is the role of 'caring for/by others' in a developing mathematics identity? How does the individual draw on a 'caring self-other relationship' in their self-authoring as a mathematician? How does 'caring for/by others' relate to the construction of mathematics itself?

METHODOLOGY

Roz began her university career at the age of 44, taking undergraduate, masters and PhD degrees in mathematics. At 52, she is now engaged in post-doctoral study at the same highly prestigious university where she undertook her PhD. The interview, which was fully transcribed, took place in a university cafe, and was loosely focused on her ongoing mathematics career, and her account of the past, present and future, including the impact of being a mother on her career. We see the interview as a place which encourages reflection and, therefore, supports identity formation of the kind we have described above. It provides the space for Roz to reflect and thereby, co-construct with the interviewer a story about who she is and what identities she positions herself towards. In our analysis, we focus on the role of caring and others in her account of herself as a mathematician, and in particular the part they play in her vision of the mathematical world and her place in it.

ANALYSIS

The role of caring for/by others in a developing mathematical identity

Roz's talk is populated by references to the role of caring for/by others in her decision to become a mathematician, and her on-going identification of what kind of mathematician she is. She especially identifies her relationships with her mother and father as crucial. Specifically, Roz locates the origins of her desire to become a mathematician in terms of a 'genetic' alignment with her father:

> ... when we were tiny he would bring home the scrap paper with all the technical drawings on for us to use the other side of – it was a purple one. (?) I was just fascinated with the drawings themselves, and the precision with which they designed even the (inaudible) in the lid, and how things worked – I just loved it. And I was just gazing at these and how things worked and whatever ... so it was always in there I think. ... I mean part of it's intrinsic isn't it, genetic – I'm much more like my father than my mother. So the things that he liked – I like. I don't feel let down by him in the way I feel let down by my mother. So there's all those things going on, but there's also sort of the natural being drawn to those pictures rather than turning it over and scribbling on the other side ... so that was already intrinsic. So I think it was already in there to a large degree.

Here we suggest that Roz is re-constructing a particular type of relationship with her father based around shared interests and enjoyment which, she believes, has resourced her mathematical identity. This relationship is presented in direct opposition to that with her mother with whom she feels 'let down by'. Elsewhere in the interview, she explains this more negative relationship in further detail:

> She was a social worker, so she was working shifts in a home for children who'd been taken away from their parents. [..] So there was no regular routine at home. I was the eldest so I was expected to look after my sisters. [..] And I think ... it might be the particulars of her job, but she never had time for us. She did loads of stuff for us – washing, ironing, cooking and everything ... but never anything with us. [..] So I think it really was important to me to stay home with my boys. Because she was always saying 'Not now Roz, not now Roz' but there was never a time when it was okay to talk to her. And if you can't share the trivial things, you can't share the important things – cos you don't establish a relationships. So ... yeah. So that's why I say that.

Whilst she recognises that in some ways she was cared for by her mother in that 'she did loads of stuff for us', the contrast she constructs between this and the 'with us' type of relationship she had with her father is notable. Roz clearly dis-aligns herself from this more distant relationship with her mother – for her, it was 'regrettable' that her mother worked when her children were young, and she emphasises her own choice to be a 'stay at home' mother, placing value on 'being there for them':

> I was 20 years full time at home with my boys, I was. [...] but at the same time I could be there for them, and they weren't left to squabble it out, fight it out between themselves in the school holidays like we had to.

We suggest that Roz's account of how she became interested in mathematics and how she came to identify with it is shaped by her re-telling of these significant relationships with caring others. In this re-telling she invokes a type of being cared for which involves spending time together and sharing interests. This is manifest in the relationship she had with her father whom she identifies as a critical 'other' in explaining the origins of her love of mathematics. But this is also not a simple causal relationship of positive parental influence = positive mathematical identity since Roz also frames her father's influence in terms of what it is not i.e. the negative, more distant relationship with her mother who she was 'let down by'. We suggest that these two ways of 'being cared for' ('spending time with' versus 'doing stuff for') have shaped her trajectory in becoming a female mathematics academic at different moments of her life. First, by resourcing her original interest in mathematics through shared interests with her father, and then, by motivating her late entry to academia as a mature student which emerged from her decision be a 'stay at home' mother.

The role of a 'caring self-other relationship' in being a mathematician

In a previous paper (Solomon, Radovic, & Black, submitted), we noted how Roz constructed a narrative of her present and future self as a mathematician which entailed negotiating the contradiction of being feminine but doing 'male' mathematics. We suggested that this contradiction led her to enact her mathematics activity in a particular way, hybridising femininity (through dress, investing in relationships with other staff members and organising social activities) with her interest in the 'theoretical stuff' which she sees as belonging to the 'masculine', systematising domain. These findings relate to the focus of this paper, since the 'caring-other' relationships we have discussed appear to be crucial in enabling Roz to negotiate this contradiction. For instance, in the following extract, Roz discusses how her previous experience as a stay at home mother (caring for others) has affected her current role as a mathematician:

I	Being a mother and being a mathematician, do they ... sort of go together?
R	I think it's added to me as a person. ... [..] And I think that part of coming to maths as a mature student has given me things to hook it onto. [...] And I can understand it more quickly because I can relate it to things I already know. And I think that being a mother has helped me to be the kind of person who will help the other students and who they can relate to and that kind of thing, and that persists. .... and I think that being a mum has helped me with that.

Here, we can see that being able to 'relate to others' is not only valued by Roz but also that she sees it as helping her to perform her role as an academic, connecting to her earlier emphasis on the form of 'caring for others' as sharing interests. The fact that she connects this to 'being a mum' re-iterates the particular kind of parental, caring relationship she values – one which now enables her to act in a certain way as an academic mathematician:

> But when I'm here [in her department] I'm... you know okay I can be very firm ... you know if need [to talk to] people with ice behind my eyes I can do that and get them to do what I need ... but I can also be thoughtful ... and people say 'Oh I didn't think of that' – and it's the most obvious thing, like you know just being extra careful to talk to the new person, or you know every day 'How are you getting on?' – this kind of thing.

The role of 'caring for/by others' in the construction of mathematics itself

The particular form of 'caring for/by others' that Roz values not only appears to resource her identity as an academic mathematician in the context of her university department, but is also evident in her construction of what mathematics itself is. Roz's research expertise lies in the area of complexity theory and statistical modelling to address real world problems such as finding the most effective treatment in healthcare and structuring networks for food security decision support. A persistent theme in her story is an emphasis on mathematics as a means to 'describe the world' and make it a better place (see Solomon, 2012; Solomon, Radovic, & Black, submitted), and in describing this mathematical activity she sees herself as 'helping people' – she uses the term 'the people element' frequently. This suggests that the 'caring-other' relationship which Roz invokes in describing both her historic love of mathematics and her current identity as an academic mathematician is also part of her construction of her mathematical activity itself.

But crucially, Roz is at pains to point out that she is not succumbing to a dominant (gendered) discourse about mathematics which creates a division between those who do applied, 'people friendly' mathematics/statistics and those who do 'real' theoretical mathematics (see Mendick, Moreau, & Hollingworth, 2008). This is apparent when she discusses the kind of work she does in her current role:

> ... the thing is it's such an outward facing role – the ability to engage with people is you know the thing that's really really important. [..] I started picking through some papers to get ready for the interview [for her current role], I was kind of like ambivalent about the job. And then I saw this 'algebraic statistics' and there was ... it was talking about ideals and this kind of stuff, and I was thinking 'It's real maths, I don't feel like I've done that for ages, oh I really want this job now.' And just suddenly I thought ... Yeah real maths! Oh ... it's a dichotomy isn't it, because this is the really pure stuff, that really is not meant to ... I mean the model was [for helping] people, but this is the really pure stuff, and it's just kind of playing with toys, and I think 'Ooh ...' [..] But also we needed to prove some stuff so ... with its coherence... And I need to get into Bayesian statistics which I've not done before, and I need to go into

> ... yeah apparently you can represent a Bayesian network as a set of polynomials, in which case the algebraic geometry comes in and you can prove stuff about it.

Here we suggest that whilst Roz recognises that a dichotomy between applied and pure mathematics exists in people's perceptions of mathematics, her insistence that she can do both in one role indicates that she feels they are not mutually exclusive – i.e. her current 'identity in practice' involves doing mathematics which is both 'outward facing' and 'the really pure stuff'. In this way, Roz does not merely engage with mathematics to enact a feminine 'caring for others' role but, rather, envisages herself as changing the context of mathematics itself so that the traditional gendered and excluding binary (male – pure mathematics vs female – applied mathematics) no longer exists. The following extract highlights the energy and motive that is provided by this perception that she is changing mathematics:

> I sat through a series of lectures that my now boss gave,... and the lecture that I absolutely adored had a proof with three lemmas in it you know and .... 'I really am a mathematician – I just love this stuff' – my heart was racing, I just enjoyed it that much, you know. [...] And um ... when I was at [her undergraduate university] ... it was actually to do with coding and cryptography. But then you were calculating in rings and we were doing some stuff like that, and we just did a little bit of algebra ... and even then I was kind of like 'Oh I wish I could do more of this, I wish I could do more of this' you know. So ... and this is the precisely the sort of thing I'm going into now.

DISCUSSION

In this paper, we have highlighted how a particular form of the self-other relationship – 'caring for/by others' – can be used as a cultural resource in narrating a positive mathematical identity. Our analysis of Roz's story has shown how she uses this cultural model to account for the kind of person she has become and to self-author as a particular kind of academic mathematician. We have also argued that her use of this cultural model has mediated a particular view of what mathematics is and what mathematics communities should be like. So, rather than subscribing to a gendered binary division of mathematics (female-applied vs male-pure) such as that reported by Mendick, Moreau and Hollingworth (2008) in their study of undergraduate students' perceptions, Roz sees herself as enacting both 'people friendly' and 'real maths' in her one role and, thus changing the world of mathematics itself.

The role of the 'caring for/by others' relationship in this process is worth commenting on. We recognise that 'caring' as a construct is heavily loaded with connotations that relate to unequal gendered divisions of labour in society which for many women are a source of marginalisation. As Walkerdine (1989) has suggested, 'caring' plays a particular role in marginalising girls away from school mathematics at a young age. Here, we suggest that given the right circumstances, caring for others AND being cared for by others can be used to good end. Rather than being a source of dominance/manipulation of women which reinforces their marginalisation, caring can be used to develop a positive mathematical identity and potentially a tool for change. However, our reference to the 'right circumstances' here is pivotal – Roz's story must be seen in context as one of 'success against the odds'. In taking up her role as a mathematician in a prestigious university, she has to some extent already overcome the powerful structural conditions which make becoming a mathematician particularly difficult for women like her. Now, Roz's ability to use 'caring for/by others' as a cultural resource and thereby construct a way of doing mathematics that combines 'helping others' with 'real maths' is a result of her new found location in a profession where relating to others now holds value and where this form of capital is not perceived as readily available to many of her male colleagues.

The implications of this argument are significant. Archer, DeWitt, Osborne, Dillon, Willis and Wong (2013) have recently suggested that in order to raise girls' aspirations in relation to STEM subjects (including mathematics), interventions are needed which make a future in STEM more acceptable to girls. They note that many (predominantly working class) girls focus on practical subjects and do not see science in this light. Their findings denote particular forms of capital which include caring and domestic competences which are not seen as part of science:

> The data from the girls without science aspirations reinforce the importance of capital in that their stated aspirations are clearly rooted within particular forms of social and cultural capital

> (family contacts, everyday experiences of e.g. babysitting/childcare, fashion and sport). The absence of science capital within their daily lives renders science aspirations less conceivable (and achievable), not only reducing their opportunities for developing a practical 'feel' for science but also of being able to see science as a 'thinkable' performance of femininity (p. 186)

Thus Archer, de Witt and Willis (2014) suggest, among other measures, that

> In practice, our study indicates that "one size fits all" approaches to increasing science participation (whether generally or for "girls" or "boys") are unlikely to have much effect on improving post-16 participation. Approaches that are sensitive to young people's gendered and classed identities would seem more appropriate for delivering STEM enrichment and/or interventions aimed at increasing participation. (2014, p. 24)

While this appears to be a reasonable suggestion, it is less than clear what exactly this would entail, and an emphasis on 'broader forms of engagement', while laudable, might not address the binaries which Archer and colleagues (2013) themselves have noted, and which may simply position girls and women in particular (lower status) areas of STEM. Archer et al add that there is also a need to change cultures:

> there is also a need to ensure that the cultures operating within post-16 science (in colleges, universities and workplaces) are indeed equitable and do not alienate or disadvantaged 'non-traditional' participants. (2013, p. 189)

Again, however, it is less than clear how this might be brought about or what it in fact entails. True equity means enabling access to powerful knowledge and deepening engagement rather than simply widening participation; our findings suggest that this requires a fundamental change in the high status contexts of STEM to create new practices which critically engage with and ultimately transcend the binaries we have outlined. We argue that Roz's approach to mathematics can ultimately challenge the structures which define its status as a powerful gatekeeper in maintaining wider social divisions of class and gender.

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The three faces of problem solving

Jonas Dahl

Malmö University, Malmö, Sweden, jonas.dahl@malmo.se

The point of departure in this paper is my previous research in which I analysed how the idea of problem solving is recontextualised into the mathematics curriculum for upper secondary school in Sweden and how this increases the risk for excluding lower SES students from future power. I discuss how this research could be followed up through a suggestion of how problem solving could be viewed in three different ways: as an ideology, a competence and an activity. Bernstein's pedagogic device and dichotomy of vertical and horizontal discourses are crucial in this suggestion. By seeing problem solving as an activity, and connect this to what Bernstein labels the evaluative field, I thereby tie the whole pedagogic device together by taking an overall view of problem solving as global policy-speak.

Keywords: Pedagogic device, evaluative field, vertical and horizontal discourse, social equity, problem solving.

INTRODUCTION

> Certain aspects of our way of life, certain kinds of knowledge, certain attitudes and values are regarded as so important that their transmission to the next generation is not left to chance in our society but is entrusted to specially-trained professionals (teachers) in elaborate and expensive institutions (schools) [...] Different schools may make different priorities, but all teachers and all schools make selections of some kind from the culture (Lawton, 2011, pp. 6–7).

This selection is not only about what kinds of knowledge, attitudes, and values that should be transmitted, but also to whom. Since this selection is often made on socio-economical grounds, school reproduces social inequity (Bernstein, 2000). Instead of 'school being for all' (see, e.g., Skolverket, 2012b) there is instead a risk of 'school being for some' (Dahl, 2014).

Furthermore, education should not only be the transmission of tools to interpret the world, but also the means to change the world (Atweh & Brady, 2009). Bernstein (2000) suggests that what schools should provide students with for them to become citizens with a stake in society is a sense of, what he calls enhancement, inclusion and participation. According to Wheelahan (2007) this has to do with the form of knowledge that students are given access to:

> Unless students have access to the generative principles of disciplinary knowledge, they are not able to transcend the particular context. Students need to know how these complex bodies of knowledge fit together if they are to decide *what* knowledge is relevant for what *particular* purpose, and if they are to have the capacity to transcend the present to imagine the future. (p. 10, italics in original)

In the mathematics curriculum for upper secondary school in Sweden (year 10–12) problem solving is emphasised as a main competence that students should develop (Skolverket, 2012a). This emphasis, both on competences and problem solving, is common to other curricula and frameworks, for instance *Curriculum and Evaluation Standards for school mathematics* (NCTM, 1989), *Principles and Standards for School Mathematics* (NCTM, 2000) and *Adding It Up* (Kilpatrick, Swafford, & Findell, 2001) in the USA, the KOM-project in Denmark (Niss, 2007) and PISA (Programme for International Student Assessment) (OECD, 2012). Sweden, in following this international trend without questioning the roles of competences and problem solving can be seen as adapting to, what Ball (2013) calls the global policy-speak (Dahl, 2014). Global policy-speak "produces, or is the effect of, a cutting-off of research in mathematics education from its political responsibility and consequences and from its philosophical and political roots" (Gellert, Barbé, & Espinoza, 2013, p. 303).

I have in previous research (Dahl, 2014) investigated how the need for problem solvers is expressed in the Swedish mathematics curriculum, where Sweden is seen as an exemplar of a wider trend in mathematics education. I have also suggested that, with the new curriculum in Sweden, launched in 2011, there is a greater risk of reproducing inequity through school compared to the older curriculum; mainly due to the greater division between vocational education and academic preparatory education that is more solidly build in to the system with the curriculum (Dahl, 2014). These different types of programmes attract students with different social background with vocational education attracting to a greater extent from lower SES groups (Broady & Börjesson, 2005).

The curriculum is here defined as the official documents that address teachers and in Sweden these are the same for all upper secondary schools and includes the national tests since one purpose with these tests is to concretise the curriculum (Skolverket, 2012b). Together, these documents inform teachers and students of what should be taught/learned, in some sense how it should be taught, but also what should be assessed and how. Questions that arise from this are about what really happens in schools and in the classrooms in regard to competences, and especially problem solving competence. Is there a difference, as the curriculum suggests that there should be, between different educational tracks? If so, how does this intended difference affect students' foregrounds?

Before moving in to the theoretical background and framework, I need to describe what I mean with *the problem-solving citizen*, a term I introduced in my previous research. The problem-solving citizen is a citizen who is flexible, employable and one who gives his/her "best in responsible freedom" (Skolverket, 2012b, p. 4). Drawing on OECD in defining the problem-solving citizen, I add that he/she also has to be(come) an "intelligent consumer" (OECD, 2006, p. 72). With this view on the citizen, which is the dominant view in the curriculum, the citizen is one with duties to the country rather than rights (Dahl, 2014). As a contrast, with Bernstein's view on the citizen, the rights of a citizen are emphasised.

THEORETICAL BACKGROUND AND FRAMEWORK

Bernstein (2000) described the pedagogic device as illustrating how education acted as a filter for ensuring that class distinctions were reproduced. The pedagogic device consists of three sets of interdependent rules: *the distributive rules* regulate the power relationship by distributing different forms of knowledge to social groups; *the recontextualising rules* regulate the formation of pedagogic discourse; *the evaluative rules* constitute pedagogic practices that are realised in instructional and regulative texts (Bernstein, 2000). In relation to the problem-solving citizen, the distributive rules can be considered as controlling how the discussions of politicians, education bureaucrats and educational researchers are relayed to those formulating the curriculum. The recontextualising rules control how the distributed knowledge about problem solving is incorporated into the curriculum and national tests which in turn control the ways that educators in schools and the wider education sector come to discuss problem solving. The evaluative rules control how teachers teach problem solving in classrooms. Within this field, the regulative discourse is dominant over the instructional discourse (Bernstein, 2000). The regulative discourse "refer[s] to the forms that hierarchical relations take in the pedagogic relation and to expectations about conduct, character and manner" (p. 13). The instructional discourse "refer[s] to selection, sequence, pacing and criteria of knowledge" (p. 13).

From a Bernsteinian perspective, the "generative principles of disciplinary knowledge" mentioned above in the quote from Wheelahan, could be equated with knowledge forms in a *vertical discourse*, which is context-free or generalisable, in this case, from the world of mathematics. The opposite is forms of knowledge in a *horizontal discourse* which includes mundane or common sense knowledge. According to Bernstein (2000), this latter form of knowledge is context-dependent and not easily used outside the given context, thus not generalisable. The difference between knowledge from the horizontal and vertical discourse is not a matter of abstract or concrete; rather, it is a matter of reference to a specific material base, that is, to a context outside mathematics (Bernstein, 2000). For instance, in school mathematics, a task or problem that is situated in a context outside mathematics can be regarded as within a horizontal discourse. These contexts could be domestic areas, such as best-buy strategies, or vocational settings, but also other school subjects, such as economics or chemistry. Knowledge from the vertical discourse is context-free and can be realised in mathematics tasks that make no reference to the outside world, such as find the greatest area of a triangle with a given perimeter. Due to their context-independency, tasks such as this are generalisable to different contexts or can give access and insights to the esoteric world of mathematics. From the point of view of social equality, access to the vertical discourse is crucial.

In her small study, Lubienski (2000) raises questions about NCTM's (1989) suggestion that problem solving should be a means for developing other mathematical skills or competences. Drawing on Bernstein, Lubienski (2000) states that "contextualised mathematics problems seem to align nicely with lower-class students' preferred ways of thinking" (p. 457) and "[o]ne might expect that if lower SES students tend to have a contextualised orientation to ideas, they would benefit from contextualized problems" (p. 467). However, she concludes that this is not the case. Lubienski draws the conclusion that the lower SES students in her study were not able to transcend the context of the problems and had greater difficulties learning from solving contextualised tasks than their higher SES peers. Her conclusion was that by putting mathematical problem-solving tasks in a context, some students, particularly students from a lower socio-economical background, are hindered in their mathematics learning.

Similarly, when it comes to assessing through problem solving tasks, Cooper and Dunne (1998) found that sometimes students, in solving contextualised tasks draw too much on the real world and, at other times, too little. They further suggest that there is a relationship to students' socio-economic background wherein students from lower socio-economic backgrounds struggle more often to bring in the appropriate knowledge form.

Bernstein (2000) states, "to make specialised knowledge more accessible to the young, segments of *horizontal discourse* are recontextualised and inserted in the contents of school subjects" (p. 169). Following Lubienski (2000) this is problematic when it comes to teaching or learning through problem solving. Following Cooper and Dunne (1998) it is also problematic when it comes to assessing problem-solving competence. This means that problem solving is problematic both when it is seen as a means for developing other mathematical competences and as an end in itself, both views highlighted in the mathematics curriculum in Sweden (Skolverket, 2012a). As Bernstein suggests, segments of the horizontal discourse can be brought into school mathematics as hooks, but "using horizontal discourses other than as a 'hook' to entice pupils into vertical discourses is to destroy what is distinctive about pedagogic communication" (Whitty, Rowe, & Aggleton, 1994). From a categorisation of school mathematics tasks in a textbook series with tracks for different ability levels, Dowling (2005) concluded that students who are labelled low-ability, mainly from working class are denied access to, what Dowling calls the esoteric domain. This is another way of saying that the students are denied "access to the generative principles of disciplinary knowledge" (Wheelahan, 2007, p. 10).

Thus, although the purpose of using contextualised problems is to develop mathematical awareness and competences, it may have the opposite affect, especially if students from low socio-economic backgrounds are the students given these tasks more often, as suggested by for instance Dowling (2005), Lubienski (2000) and in my analysis of the mathematics curriculum in Sweden (Dahl, 2014). Contextualised problems, including those based on the students' own experiences, can be used to hook students and promote mathematics, but can be problematic if, in solving them, students never leave the horizontal discourse and enter the vertical discourse.

FitzSimons (2008) suggests "these complementary discourses, vertical and horizontal, need to converge in formal education settings in order to enable richer forms of knowledge construction by learners" (p. 3). In order to see and understand the boundaries that exist between them, students need access to both discourses. Access to the vertical discourse may be denied to certain groups of students. In particular, low achievers from low socio-economical areas (Dowling, 2005; Lubienski, 2000; Wheelahan, 2007) may be restricted access into the vertical discourse. In order to gain equal outcomes and ensure that social differences are not reproduced, *all* groups of students need access to the vertical discourse in order to "transcend the present to imagine the future" (Wheelahan, 2007, p. 10). Therefore, a way of ensuring equality becomes

<table>
<tr><td>Problem Solving</td><td colspan="2">Ideology</td><td colspan="2">Competence</td><td colspan="2">Activity</td></tr>
<tr><td rowspan="2">Pedagogic device</td><td colspan="2">DF</td><td colspan="2">RF</td><td colspan="2">EF</td></tr>
<tr><td></td><td colspan="2">ORF</td><td colspan="2">PRF</td><td></td></tr>
</table>

Table 1

a question of ensuring equal access to the vertical discourse. My analysis of the mathematics curriculum in Sweden (Dahl, 2014) suggests that there is a risk this is not the case. Instead there is a risk that lower SES students are hindered from access to the vertical discourse.

Sjöstedt (2013) draws the same conclusion, but adds Bernstein's ideas about visible pedagogy (Bernstein, 2003). In viewing pedagogic practice as a cultural relay, Bernstein (2003) discusses some criterial rules about "criteria which the acquirer is expected to take over and to apply to his/her own practices" (Bernstein, 2003, p. 64). Visible pedagogy has to do with whether or not these rules are explicit in the pedagogic practice. If the criterial rules are visible, students see the connection between pedagogy and evaluation; that is, they know what is expected of them. The syllabus in Sweden is divided into core content (what should be dealt with in the classroom) and competences (what should be assessed and graded). If there is a difference in what happens in the classroom and what is assessed, then there is a risk that the pedagogy becomes invisible, meaning that the students will not know what is expected of them. According to Bernstein (2003), this is problematic for lower SES students. For a more equitable outcome, there is, besides being given all access to the vertical discourse, also a need for the pedagogy to be visible.

THE CONNECTION BETWEEN THEORY AND METHOD

The term problem solving is ambiguous and my suggestion is that problem solving can be seen to have three different faces: 1. Problem solving as an activity, that is, when someone, for instance a student, solves a problem. 2. Problem solving as a competence, that is, an interpretation and a description of problem solving in, for instance the curriculum, and 3. Problem solving as an ideology, that is, how problem solving is a part of the global policy-speak. I further suggest that these three faces can be connected to the evaluative field (EF), the recontextualising field (RF) and the distributive field (DF) respectively.

Bernstein (2000) distinguishes "between an *official recontextualising field* (ORF) created and dominated by the state and its selected agents and ministries, and a *pedagogic recontextualising field* (PRF). The latter consists of pedagogues in schools and colleges, and departments of education, specialised journals, private research foundations" (p. 33, italics in original). In Table 1, I add this division of the recontextualising field but also suggest that the ORF is the bridge from the DF to the RF and the PRF is the bridge from the RF to the EF.

Furthermore, within the recontextualising field, Bernstein (2000) distinguishes between two different discourses: the instructional discourse (ID) and the regulative discourse (RD). RD is dominant over ID and "refer[s] to the forms that hierarchical relations take in the pedagogic relation and to expectations about conduct, character and manner" (p. 13). Those students become and act as genuine problem solvers is seen as the main goal of mathematics education, the end in itself.

To summarize and put things in different words: Problem solving as an ideology is when politics, following a global policy-speak, talks about and uncritically assume that problem solvers is something all need to become. Within the RF this is transformed into competences and goals of the school. In the classroom (the EF) problem is an activity, used both to develop other mathematical competences but also for assessment, that is, to assess if the students "have the competence", if they have become problem-solving citizens. That the RD is dominant over the ID means that the actual goal of problem solving as an activity in the classroom is to make the students behave as problem solvers (problem solving as a competence) because it is assumed what the country needs (problem solving as an ideology).

In my previous research, it is the operation of the distributive and recontextualising rules that are in focus around the construction of the problem-solving citizen and this project was an analysis of the curriculum, including the national tests. In order to see the whole picture, an investigation of what happens in schools and in the classroom needs to be added. By moving the analysis into the classroom would mean an investigation situated within the evaluative field which would cover the whole pedagogic device because "[t]his is what the device is about. Evaluation condenses the meaning of the whole device" (Bernstein, 2000, p. 36). The issue on the level of evaluative rules (the classroom) could be transformed to be an issue about how the instructional discourse (about problem solving) supports or hinders access to the regulative discourse (the problem-solving citizen). My suggestion is that this is an investigation of problem solving seen as an activity but also that it connects back to the other views on problem solving.

INITIAL THOUGHTS FOR A METHOD

To investigate the evaluative field there is not only a need to investigate *how* problem solving is reused as an activity. There is also a need to investigate *why* from the teachers' points of views. And also what the effects could be from the students' points of views. This would give indications of the influence that the global policy-speak (problem solving as an ideology) and the curriculum (problem solving as a competence) have on teachers and students and will hence tie the whole pedagogic device together.

The investigation should have as it aims to find differences and similarities between vocational programmes and academic preparatory programmes. Further, it should be based on access to the vertical and horizontal discourses. That is, how do teachers within the different types of programmes interpret problem solving as an activity?

It is also important to address another dichotomy that I used in the analysis of the curriculum (Dahl, 2014): if problem solving is seen as a means for other competences or areas or if it is seen as an end in itself. This categorisation is important because of the issues that arise when problem solving is seen as a means for learning mathematics highlighted by, for instance, Lubienski (2000) and Dowling (2005). Together with the why question, this dichotomy is also important in order to investigate how the instructional discourse supports or hinders access to the regulative discourse.

Finally, the visibility of the pedagogy is important to take into consideration. With an assumed risk of more recontextualisation for the lower SES students there is also a greater risk of the pedagogy being invisible. The risk of the lower SES students "looses" is then doubled.

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Critical reflections on temperature change

Kjellrun Hiis Hauge, Maria Astad Sørngård, Tor Inge Vethe, Terje André Bringeland, Andreas Austlid Hagen and Marius Sætren Sumstad

Bergen University College, Faculty of Education, Bergen, Norway, khh@hib.no

Advanced mathematics often plays an important role in risk evaluation, for example in relation to climate change. How can mathematics related classroom discussions enhance critical citizenship when societal issues are accompanied by advanced mathematics? In a master course in mathematics didactics, a figure from IPCC was discussed. The figure shows developments of the average global temperature changes. In this paper, we analyse the classroom discussion in accordance with Skovsmose's six reflection steps, with focus on societal aspects. The students reflect critically on the data used to support the figure, on characteristics of the models, on the models' significance in society and on their own participation in the classroom discussion. We discuss the results in a broader educational and societal context.

Keywords: Critical mathematics education, climate change, reflective knowing.

INTRODUCTION

Mathematics often plays an important role in risk evaluation as support for policy making, for example in relation to climate change. However, risk issues are often associated with complexity, uncertainty and conflicting stakes, which implies that mathematics supporting political decisions is advanced, but also associated with uncertainty (Funtowicz & Ravetz, 1993). A consequence of this uncertainty is that mathematics based arguments can support opposite political stances.

This is in line with Skovsmose's claim that mathematics has a formatting power in society in that it influences how reality is perceived (Skovsmose, 1992). He further argues that the ability to recognize this formatting power, and reflect on it, is an essential democratic competence in order to balance the experts' influence on politics and society. He refers to this ability as reflective knowing. The role of offering critique to experts and politicians on societal issues can be denoted as critical citizenship (Jablonka, 2003). Mathematics and science educators offer perspectives on what critique may signify in relation to the issue of climate change in education. Barwell and Suurtamm (2010) argue that the mathematising of climate change makes human activity invisible and call for more visibility in the modelling process, for example through information on model assumptions to help evaluating whether the model fulfils its purpose. They suggest that mathematics education has a responsibility in facilitating reflective knowing on the role of mathematising climate change and call for research within the area. Barwell (2013) further supports these arguments by showing how a particular philosophy of science matches ideas from critical mathematics education, as for example the formatting power of mathematics, reflective knowing and critical citizenship. Hansen (2010) discusses what critical democratic competence might be in relation to predicted sea levels as an effect of global warming. As Barwell and Suurtam, she links this to the modelling process and underlying assumptions, and her emphasis is on mathematical modelling as a classroom activity to prepare students for future critical engagement in such issues.

All these three papers are theoretical papers and include the idea of mathematics education as a preparation for critical citizenship. Although they suggest various student activities for enhancing reflective knowing, they do not suggest how this reflective knowing may be expressed when facing claims developed through advanced mathematics of climate science. Looking to science education, Erstad and Klevenberg (2011) describe a classroom study where the student task was to explore topics in "An inconvenient truth" with Al Gore. The aim of the task was to learn about aspects on science in society. From searching the internet, the students discovered that certain controversies on climate change were linked to disagreements between scientists. The paper thus addresses complexity in climate change through expert disagreements.

The issue of climate change may be too complex for students to develop alternative mathematical approaches that can pinpoint socio-political consequences of mathematized information. A crucial area of research would therefore be on what kind of critical reflection non-experts can provide on advanced mathematized information that is useful for critical citizenship.

In our study, students of a master course on mathematics education discuss a figure (Figure 1) taken from a report by the Intergovernmental Panel on Climate Change (IPCC, 2013) as an introduction to critical mathematics education. The authors of this paper are five of the master students and the lecturer. The IPCC graph presents different modelled courses of average global temperature changes. The mathematics supporting the graph is far too advanced for the students and the lecturer to grasp. Still, can reflecting on the graph enhance mathematical understanding that is useful for critical citizenship?

The research question of our study is: 1) What kinds of reflection related to critical citizenship are expressed during the classroom discussion on the IPCC graph? In the following, we describe the data and present Skovsmose's (1992) six reflection steps, which we apply to our data analysis. In the analysis part, we follow the course of the classroom discussion and offer examples of expressed reflections. Finally, we discuss the significance of our findings to critical citizenship.

DATA AND METHOD

As an introduction to critical mathematics education in a master course, the nine students and the lecturer (Kjellrun) discussed a graph produced by IPCC on predicted temperature changes (see Figure 1). The discussion was followed by a lecture on selected literature from critical mathematics education, where links between the initial discussion and the literature were collectively drawn. The lecturer chose this figure because of its potential for generating reflections about mathematics and its use: (1) its political relevance and dispute, (2) it expresses uncertainty through statistical spread, different pathways of future emissions and implicitly through dissenting model results, (3) the social construct of "global average surface temperature" and (4) the problem of measuring temperature in space and time. Her intended role was to keep the discussion as student driven as possible, but at the same time ensuring that student reflections took place. Before the discussion, the lecturer invited the students to write an academic paper on an analysis of their discussion together with her. The analytical tool to be applied was part of the curriculum. Five of the students accepted the offer and are co-authors of this paper.

The classroom discussion is analysed by using Skovsmose's (1992) six reflection steps, of which five are later picked up by Gellert, Jablonka and Keitel (2001) and referred to as five reflection levels. The steps can be summarised as reflections related to the following questions: (i) Are the calculations right? (ii) Was the right algorithm used? (iii) Is the mathematical approach reliable? (iv) Could the problem be solved without formal mathematics? (v) How does the mathematical approach affect the specific context of the problem? (vi) Could we have reflected on this in another way? The six reflection steps refer to what Skovsmose calls steps towards reflective knowing and which is essential for critical citizenship. They represent steps of students' reflections while engaged in problem solving.

The first pair of steps refers to reflective thinking within pure mathematics, the next within applied mathematics and the latter within a broader, societal, understanding of mathematics (Skovsmose, 1992). As these reflection steps are developed for situations where students reflect on their own problem solving, in contrast to our case, we have adjusted the reflection steps for our purpose. We have chosen (i) and (ii) to include our reflections on the mathematics expressed in the graph, which are developed by others. Since the underlying mathematics is hidden for us, we have concentrated on our understanding of the graph contents, for example: Have we understood the graph correctly? Have we understood underlying algorithms? We consider such reflections as a required basis for reflective knowing and critical citizenship and are therefore relevant to include in our analysis.

The two classroom discussions on the graph and on applying the reflection steps on the graph are audiotaped. Additional data include notes and the lecturer's PowerPoint presentation. We have applied the

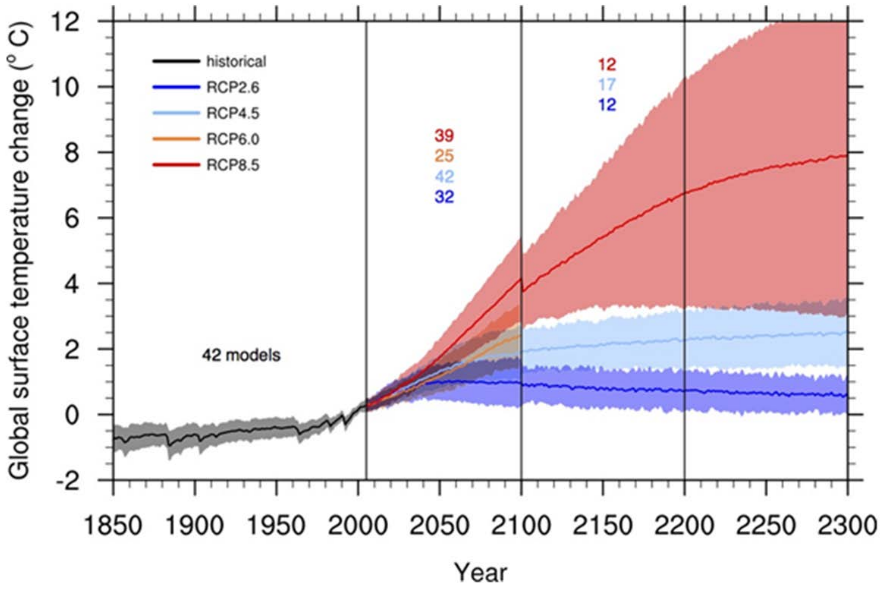

Figure 1: Temperature change. The graph shows modelled developments of the global annual mean temperature change (IPCC, 2013, p. 89). The change is calculated relative to the period 1986–2005. Four future scenarios are presented with mean (the coloured trajectories) and 90% confidence interval (the coloured shadings). The colours represent different possible emission levels of greenhouse gases (RCP – representative concentration pathways), which will depend on today's and future global climate politics.

authors' names in the excerpts. The other names are anonymised. The audiotaped discussions were transcribed and coded by the students in accordance with the six reflection steps. The coding turned out to be challenging as the steps were not straight forward to apply. This was discussed during meetings.

ANALYSIS

In the following, we present some excerpts from the classroom discussion, which we analyse in accordance with Skovsmose's six reflection steps. The beginning of the discussion was dominated by the students making sense of the figure (Figure 1): the general features, the significance of the colours, the lines etc. Repeated words are deleted in the excerpts. The first excerpt takes place early in the discussion.

Tor Inge: Where does the limit go for where the measurements fall within?

Maria: It depends on which scenario is taken into account. If they have calculated with an RCP value of 8.5, and that resulted in. Those who achieved the lowest values at the lowest one, right? And then they calculated with the same model with a value of 2.6. And that could have resulted in one of the lowest values there. While some with a high value could have ended down there. What you can see here is that there's a lot of variation here. That here it seems that they disagree much more.

Tor Inge: More uncertain?

Maria: Yes, while at the blue, they quite agree all the way, in a way. They are more certain.

Tor Inge seems to misunderstand what the colours represent, as "limit" may refer to dividing mathematical model *output* into colour categories. Maria responds by explaining that the colouring depends on the *input* of the model, "which scenario is taken into account", determined by specific RCP values (see Figure 1 for an explanation). Further on, she offers a reason on why the shaded areas vary in magnitude: "they disagree much more." When Tor Inge asks "More uncertain?", he is probably asking whether Maria refers to the spread in the graph, or uncertainty.

When Tor Inge asks his first question, he is probably seeking to understand the graph, which can be categorized as Skovsmose's reflection step (i). Indirectly, Maria expresses that the figure is based on several models that do not produce the same results. The students do not discuss further what this disagreement implies, for example how and why the models are different, the reliability of the various models or whether such disagreements can explain why there are climate change deniers. This means that the excerpt has onsets to Skovsmose's reflection step on reliability (step iii) and on implications for society (step v).

The next excerpt is taken from a part where the students discuss natural temperature fluctuations, exemplified by ice ages. Tor Inge has expressed that he would have liked to know more about the temperature back in time.

> Tor Inge: Yeah, I would have gone further back in time, and seen. We talk about they only are natural fluctuations, which also have taken place before, right? And how high were the peak temperatures longer ago, before the measurements?

It is likely that Tor Inge imagines that knowing "the peak temperatures longer ago" would have strengthened his confidence in that the temperature development after year 2000 is extraordinary. Since Tor Inge is addressing the reliability of a conclusion drawn from the graph, we categorize his utterance to be a critical reflection in line with Skovsmose's step (iii). However, the reflection can also be denoted as step (v), since the utterance may suggest that a different mathematical approach could have affected his perception of climate change.

In the following, Kjellrun draws the students' attention to a specific feature in the graph.

> Kjellrun: [...] But if we look at year 2100, what is happening there? [pause 16 sec.]
> Anne: [inaudible] is a shift?
> Kjellrun: Yes, why is that?
> Anne: At least the red one.
> Kjellrun: Yes, at least the red one, that's very distinct.
> Anne: There are fewer models, you know.
> [...]
> Tor Inge: I'm thinking that the most critical until 2010 do not continue further in the models, [Kjellrun: Yeah, well that's true.], so that the curve isn't as steep when it continues.

The combination of Anne pointing to that "There are fewer models" and Tor Inge stating that "the most critical" models "do not continue" after the drop, suggests that Tor Inge reflects on how the prediction results depend on which models are used. We assume that Tor Inge means 2100 and not 2010, and we interpret "critical" models in this case to be the models predicting greater and more dramatic temperature changes. Anne's statement that "There are fewer models" can be denoted as reflection step (i), as she seems to be in the process of making sense of the shifts in the graph. Tor Inge's remarks can be categorised as questioning whether the right algorithm was used in selecting models (step ii), whether the approach is reliable (step iii) and whether the selection of models affects the perception of the severity of the problem (step v).

Later, Kjellrun asks about the relevance of graphs like the one we had discussed:

> Kjellrun: Has it had an impact? Have the climate panels' - I guess we can call it warnings?
> Terje: Maybe routines and control measures have become stricter?
> Anne: I'm thinking about the discussion he had with Vesterålen and Lofoten. The loss of oil. This model may influence whether in fact there will be oil production there, because you can see what oil production causes.

Terje and Anne (and other students) suggest various impacts of the graph, or maybe IPCC graphs in general, and thereby argue that such graphs do have an impact on society and has certain potentials. Anne is probably referring to the Minister at the time ("he") of the Ministry of Petroleum and Energy in Norway, who was in favour of oil exploration in the Vesterålen and Lofoten area. In the continuation of this excerpt, the students also reflect on how IPCC has had limited influence, for example in regard to the Kyoto Protocol. Since these reflections link the graph to societal aspects we characterise them as Skovsmose's reflection step (v), although the students declare that mathematics makes a difference rather than discuss how the mathematics influence ways of understanding.

Towards the end a student started reflecting on the discussion:

> Andreas: Well, I thought that we were trying to find shortcomings or flaws in all statements, on the graphs you have shown

us. That we have been very, well, very. Don't know if we've been negative, or ...

Theodor: Are receptive to flaws in the models. [Andreas: Yeah. Yeah.]

Tor Inge: We had that starting point all the time. I'm thinking about that the headline for the lecture is "in a critical". [Andreas: Yeah.] I imagine this [Theodor: Yeah.] goes into that direction.

Andreas: There have been few positive arguments for the things that have, the statements, on the graph. There have been few arguments on that they are right or that it is correct, and, on, of the researchers, of the. Yeah. ...

Kjellrun: So, we have, in a way, been looking for shortcomings or flaws? Or how it was expressed? But *are* there shortcomings about what the climate panel does? So, is our critique completely unreasonable?

Anne: No, but you know, it's. When saying that there's a 95% probability, for example. That there's a great divergence here. That maybe that's why you become so critical?

Marius: The critique has shed light. That, the critique, by closer inspection, then this critique will strengthen, you know, the findings, or what I should call it. Not the findings, the prognoses. That if we investigate further, we'll read more what was done. Then we maybe: how often do they measure the temperature, right? You know, we find: "Yeah! They do the measurements quite often, which means that they have quite accurate prognoses". Then this would strengthen this graph. That, when we take a critical stance, it can also strengthen our impression. It's hard to explain. That they, what we read here is. We believe it, maybe, [Kjellrun: Uhum.] that when being critical, you can. Difficult to explain.

Several of the students express unease about having been critical during the discussion. For example, Andreas says "we tried to find shortcomings or flaws" and expresses a concern for having "been negative". When Tor Inge suggests that the topic of the lecture may have influenced the critical stance, he too indicates that they might have been too critical. When Anne points to the "great divergence" she may be suggesting that the lack of certainty to be an explanatory factor, although her reference to "95%" is unclear. She is probably referring to the 90% confidence intervals in the figure, or she might be referring to an IPCC statement that was presented early in the discussion about human influence causing global warming. While Andreas, Theodor, Tor Inge and Anne seem to associate "critical" with negative criticism, Marius suggests that "critique will strengthen [...] the prognoses". He indicates that critique can also increase confidence in a graph. Since the students are suggesting that they could have reflected in different ways, the whole excerpt can be categorised as Skovsmose's reflection step (vi).

DISCUSSION

The analysis shows that the students reflect in accordance with most of Skovsmose's six reflection steps, taking our modifications of the steps into account. The students help each other interpret the graph, they question whether the graph supports IPCC's claim on the present situation being extraordinary and they respond to the information about the graph being constructed by a number of models and which produce diverging results. They thereby reflect on the relevance and sufficiency of the mathematical approach, and they further reflect on how the approach affects perceptions of climate change. Some students show resistance to their own critical reflections, expressing them to be negative, while one of the students defends them as *critique* of the graph rather than *criticism*. The analysis suggests that reflecting together on the graph, helping students to understand its mathematical contents, was vital for the later reflections linked to the reliability and the implications of the graph. This supports Skovsmose's (1992) reasoning that reflective thinking about the mathematics itself and its application is necessary for reflective knowing.

Skovsmose (1992) further promotes reflective knowing as essential for critical citizenship. Although the students demonstrate reflective knowing, key questions remain. Do our findings show that these reflections are significant for critical citizenship? We argue that in several ways they do. We experienced that the graph was challenging to interpret, requiring insights in statistical concepts and mathematical modelling. Yet, the discussion demonstrates that although we did not have the expertise to understand the mathematics behind the graph, the students are capable of reflecting on how the mathematics relate to its context. A key capability for critical citizenship is being able to question whether certain data are included, model assumptions and whether certain perspectives are taken into account (Barwell, 2013; Barwell & Suurtamm, 2010; Hansen, 2010; Funtowicz & Ravetz, 1993), of which all were demonstrated in the discussion. On the other hand, the students had only onsets to expressing a connection between uncertainty aspects and the role of this uncertainty in society and policy making. The students connect uncertainty in the temperature time series to society's perception of climate change, but they do not explicitly reflect on the significance of the discrepancies between models.

This brings us to another question: What is the role of the lecturer/teacher in promoting reflective knowing? Should the lecturer have been more active in making links between the mathematics and society? She did bring the students' attention to some of the features of the graph, as for example the shift. She also asked about societal impacts of the graph. Still, she could have taken a step further and invited the students to specifically reflect on the formatting power of mathematics, as for example the significance of the shift for how climate change is perceived, how society can cope with the uncertainty they had addressed and the difference between critique and criticism in this context. This might have led to other crucial aspects of critical citizenship, as for example uncertainty in science and why there are disagreements about "facts" and how we should cope with this uncertainty (Funtowicz & Ravetz, 1993). We thus conclude that Barwell's (2013) call for addressing the formatting power of mathematics in climate change was partly achieved in the classroom discussion, but that it still had a potential that was not fulfilled.

With some amendments, Skovsmose's reflection steps showed to be useful as an analytical tool for our study. Still, they are quite broad and do not guide a lecturer or a researcher on what specific attributes of the involved mathematics that can be connected with societal aspects and how. In our case, the time series, the predictions, the spread, the number of models and the drop were significant mathematical features for the discussion. We therefore recommend the development of a framework that captures attributes of various mathematical models in society, of which students/citizens would benefit from gaining insights in. Jablonka's (2003) paper on mathematical literacy shows a potential for such a framework as she characterises a handful of mathematical model categories accompanied with their limitations.

Taken together, the classroom discussion demonstrates that students are capable of reflecting on mathematical information although the underlying mathematics is too advanced to grasp. There are various attributes to the involved mathematics which non-experts, like us, could critically reflect on. We find such reflections useful for critical citizenship because it shows that non-experts can contribute by posing crucial questions about mathematics in a given context. As a final remark, we would like to add that discussions on IPCC graphs can also facilitate learning about climate change and that there are several other ways to discuss climate change in classrooms which complement our approach. We welcome research on related issues in classrooms, spanning both primary and secondary school levels.

ACKNOWLEDGEMENT

We would like to thank our fellow students for letting us site them. We are also grateful to IPCC for granting a permission to reprint the figure.

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Mathematical exclusion with the every day

Ola Helenius[1], Maria L. Johansson[2], Troels Lange[3], Tamsin Meaney[3] and Anna Wernberg[4]

1 Gothenburg University, NCM, Gothenburg, Sweden, ola.helenius@ncm.gu.se

2 Luleå Technical University, Luleå, Sweden

3 Bergen University College, Bergen, Norway

4 Malmö University, Malmö, Sweden

Problem solving can involve using mathematics to solve everyday problems. In this study, we examine an interaction between a teacher and a class of six-year olds in Sweden around an open-ended problem, from an everyday context. Using Bernstein's ideas about vertical and horizontal discourse, a mixture of everyday and mathematics understandings is identified in the interaction. This mixture seems to result in confusion for both the teacher and the children over what should be the focus. This paper raises issues about how the connection to the everyday in problem solving could reduce children's opportunities to learn mathematics.

Keywords: Real-life problems, Bernstein, vertical discourse, horizontal discourse, young children.

INTRODUCTION

Problem solving is often considered to provide a purpose for students to learn mathematics (Dahl, 2014). For example, van Oers (2001) discussed how Freudenthal (1973) promoted the use of real-life contexts in his realistic mathematics education – "the realism of mathematics then is seen in the applicability of self-invented mathematics in a meaningful problem, and for many people this seems to mean a real-life problem" (p. 64). Nevertheless, van Oers (2001) queried whether it was possible for higher levels of mathematics to arise from real-life problems:

> Despite the enormous innovation this view could produce in the content and activities of the mathematics classrooms, it entails a serious danger by focusing too exclusively on the real life quality of the contexts from which the mathematical thinking originates. (p. 64)

In high school classrooms, research has shown that the use of contexts can result in some students being excluded from mathematical learning opportunities (Meaney & Lange, 2013). This could be because they have difficulties recognising the mathematics in a problem solving task when it is posed in an everyday setting (Zevenbergen & Lerman, 2001). Students also can be uncertain about whether they are expected to ignore their everyday experiences (Gellert & Jablonka, 2009). Similarly, Boaler (1994) raised two related issues. The first is that students do not know how much or how little of the everyday they should use. The second is that students may not realise that the rules of the mathematics classroom require them to suspend their knowledge of reality to make sense of the mathematics classroom reality. Her research found that girls gained poorer marks on problems of which they had had real world experiences.

Although it has been suggested that using contexts could support working class students' learning of mathematics (Lubienski, 2000), previous research has shown that these groups of students are more likely to draw on their everyday experiences than those with middle class backgrounds (Cooper & Dunne, 1998; Gellert & Straehler-Pohl, 2011). For example, Cooper and Dunne (1998) found that "working class children are almost twice as likely as service class children to refer only to their everyday knowledge in answering our enquiry" (p. 128).

As is the case with the research already described, most research on the confusion caused from drawing on everyday knowledge to solve mathematics problems has been done with high school students. Little research has investigated how young children make sense of mathematical problems set in everyday contexts. Therefore, our research question is: how do young children solve mathematical problems situated in their everyday knowledge?

The research is situated in a Swedish preschool class with mostly six-year-olds. This class is considered a bridge between preschool and school and as such is the first place children have contact with formal, school knowledge and ways of working.

THEORETICAL FRAMEWORK

In order to respond to the research question, we follow Gellert and Straehler-Pohl's (2011) and Dahl's (2014) lead in using the concepts of vertical and horizontal discourse developed by Basil Bernstein. Over several decades, Bernstein developed a systematic sociology of education which included the development of many different ideas. One of these was the distinction between what he labelled the horizontal and vertical discourses:

> A vertical discourse takes the form of a coherent, explicit, and systematically principled structure ...
>
> A horizontal discourse entails a set of strategies which are local, segmentally organised, context specific and dependent, for maximising encounters with persons and habitats. (Bernstein, 1999, p. 159)

Horizontal discourse is vital for solving specific issues relevant to the solver. Consequently, it is often related to everyday understanding, gained through practical experiences. However, the knowledge gained through the horizontal discourse is not easily transferable to other situations because of how it is organised and its strong connection to a specific context (Bennett & Maton, 2010). Vertical discourse is considered generalisable to a range of situations. In reviewing research on these concepts, Knipping, Straehler-Pohl and Reid (2012) suggested that vertical discourse is often equated with the knowledge learnt in schools.

Bernstein (1996) stated "to make specialised knowledge more accessible to the young, segments of the *horizontal discourse* are recontextualised and inserted in the contents of school subjects" (p. 169). Therefore, the distinction between vertical and horizontal discourse seems useful because although school mathematics problems are often situated within everyday contexts, they require generalisable knowledge to be solved. In order to make use of these concepts, they need to be operationalised. Before describing the operationalisation, we provide background to the collected data.

DATA COLLECTION AND ANALYSIS

In the first half of 2013, video recordings were made in one preschool class on four different occasions. Our wider research aim is to investigate what mathematics is or could be in preschools and our video recordings in this preschool class were to be compared with the preschool data. In order to get comparable material, we specifically asked to video problem solving sessions. In the problem solving situations, the children worked in pairs, groups or by themselves, with a sharing session at the end. We had permission from parents for 12 children to be filmed and the films to shared, but not all children were present at each session. The class had 2 teachers who shared being filmed and working with the children whose parents did not give permission for them to be filmed.

The lesson examined in this paper was typical in the format of all lessons, by beginning with a warm up activity, in this case around pairs of numbers that added to ten. Then a problem was posed to the children, who were given time to solve it, this time, individually. The problem in this lesson was about ten children in a small preschool class who needed to be distributed to three different activities, woodwork, baking and painting. The teacher stated that there are no wrong or right answers and that it might be possible to distribute them evenly or it could be that one group had more children or another group had no children at all. The children were given paper on which to record their groups but were told that they could record them in any way that they liked. While the children worked, the teacher moved around the class, talking with each child. At the end of the session, the teacher had the children fold their papers and sit down in a horseshoe. She told the children that they must explain why they have distributed the class in the way that they did. With more or less help from the teacher, each child constructed a story about their distributions.

We discuss three children's interactions with the problem, using photos and transcript extracts. In order to identify whether the teacher and/or the children

Discourse	Characteristics	Examples
Horizontal	Context specific	Filippa's reason for only splitting the ten children into two groups was that only two teachers would be needed. In her class there were two teachers, suggesting she drew on her own experiences.
	Segmentally organised knowledge	In the presentation session, the teacher queried the children about their distributions. The children did not know that they needed to have a story for their distributions until they did their presentations. The solving of the problem and the telling of their stories were separated both in time and in space and so can be considered as taking place in different sites and thus were segmentally organised.
	Maximising encounters between persons and habitats	When Nicolas presented a paper, on which there was written, 5, 5, 5, the teacher worked hard with the other children to provide a story that allowed for this, so Nicolas could be considered successful. The emotional demands of not having a child fail became the issue to be resolved.
Vertical	Coherent, explicit and systematically principled structure	Lova uses a series of different strategies (using her fingers, making marks on the paper) to determine different combinations of 3 numbers which added up to 10. Although she does not systematically list all of the possible answers, her actions suggest that she is aware of the underlying principles which allows her to determine appropriate solutions.

Table 1: Characteristics of vertical and horizontal discourses

used vertical or horizontal discourses, we looked for particular characteristics as identified in Table 1.

THREE CHILDREN'S PROBLEM SOLVING

We have chosen to discuss the problem solving of Nicolas, Lova and Filippa as they illustrate a range of responses as well as showing different problem solving methods.

Nicolas

In the warm-up activity about ten-friends, Nicolas was

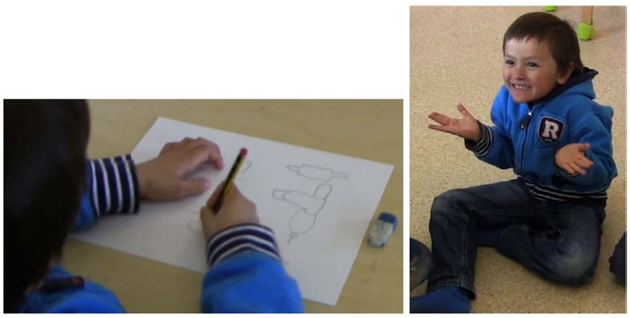

Figure 1: Drawing distribution and showing uncertainty in the presentation

given a card with 9 on it. Teo had 1 and soon located Nicolas. Teo ensured that they are acknowledged by the teacher as having the correct answer. During the individual work, Nicolas sat next to Teo and followed him in spending time copying the symbols (rolling pin, hammer, paint brush) for the three groups. He then wrote 5, backwards, against each of the symbols (see Figure 1). When the teacher called the children together, Nicolas was the fourth child to show his solution. The following exchange accompanied, Nicolas turning over his paper. When the teacher asked in the first turn, how many he had he shrugged his shoulders (see Figure 1) to indicate that he was uncertain.

Teacher: Nicolas, can you show? You've got five, is it fives? Five and five and five in each group. How many children is that? I see five here and five and five. How many children is that? (Nicolas kan du visa? Du har gjort fem, är det femmor? Fem och fem och fem i varje grupp. Hur många barn blir det? Jag får se här fem och fem och fem. Hur många barn blir det?)

A child: Twenty? (Tjugo?)

Children: Fifteen. (Femton.)

Teacher: Fifteen children, how many children were there? (Femton barn, hur många barn var det?)

Nicolas: Ten. (Tio.)

Teacher: Ten, how many children too much? Did you mix in a few more children from another preschool class? You thought fifteen children were better so you could share or? (Tio, hur många barn för mycket? Blandade du in lite barn från en annan förskoleklass? Du tyckte femton barn var bättre så man kunde dela eller?)

One child: It may be that five new children have started in that preschool class. (Det kan ju va så att fem nya barn har börjat på den förskoleklassen.)

Teacher: That I did not know of. So it became five in each group. I'll take it. Thank you very much. (Som jag inte visste om. Så blev det fem i var grupp. Jag tar den. Tack så mycket.)

The discussion initially focused on the numbers, both on how many Nicolas used and how many more this was than in the task. At this point, the conversation seemed to be within the vertical discourse, as the teacher is sequencing the knowledge in a cohesive, structured manner. However, Nicolas' body language (see Figure 1) indicated that he had become aware that his response was incorrect, perhaps because the teacher asked him number questions that she had not asked earlier presenters. The teacher then shifted direction and provided a possible story so that Nicolas' numbers could be considered appropriate, perhaps because she had earlier indicated that there were no right or wrong answers. By producing a story about Nicolas' numbers, the teacher moved back to the context-specific-ness of the horizontal discourse and restored the personal relationships. The difference between 10 and 15 disappeared from the focus and was replaced by a discussion about how 15 was a better number to share and how this number of children could come about. Nicolas made only one contribution to the discussion. It is therefore unclear whether he had understood either the vertical discourse about the difference between 10 and 15 or the horizontal discourse about the ease of splitting the class into 3 groups. The teacher's shifting between the two discourses seems to provide him with no opportunities to develop his problem solving skills either in a context-specific or more generalised format.

Filippa

In the introductory warm-up activity, Filippa had a card with 5 on it. It was not until all the other pairs were formed that Filippa realised that Hugo did not have a partner and therefore their numbers (5 and 5) must equal 10.

In giving the instructions about working individually, the teacher specifically mentioned Filippa. Filippa seemed to have taken the request to heart in that she covered her work with her arms so that Lova could not

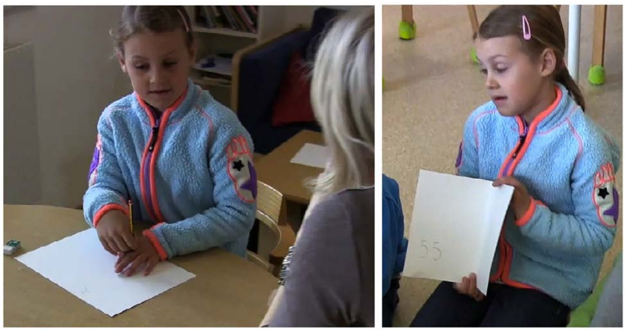

Figure 2: Filippa's problem solving

see what she did. After Lova was moved by the teacher, Filippa looked at Teo using his fingers to work out a solution. It was not possible to hear, the teacher's comments to Filippa as she moved around the room but the consequence of it was that Filippa erased the beginning of a symbol for the first group on her paper. Filippa's presentation was the third presentation. Her paper showed two 5s.

Teacher: Filippa, can you show what you have done? Okay, you've written five and five. Five in two groups, it is. Which group was it that there was no one in? Baking, woodwork or painting? (Filippa, kan du visa vad du har gjort? Okej du har skrivit fem och fem. Fem i två grupper blev det. Vilken grupp var det som det inte blev någon i. Baka, snickra eller måla?)

Filippa: Woodwork. (Snickra.)

Teacher: Woodwork there was no group of, no children at all, but why not? Do you remember, how you thought? (Snickra blev det ingen grupp i, inga barn alls i, men varför inte det? Kommer du på det, hur du tänkte?)

Filippa: Because they will do it another time. (För dom skulle göra det en annan gång.)

Teacher: They would do woodwork another time instead so then it became an equal number of children in each group. (De skulle snickra en annan gång istället så då blev det lika många barn i varje grupp.)

Filippa: Otherwise they thought that it became a little too bustling with everything. (Annars tycke de att det blev lite för stimmigt med allting.)

Teacher: Yes it could be, they would have had to be many teachers. Thank you so much. (Ja det kan ju bli. De hade ju fått va många fröknar. Tack så mycket.)

Filippa: Although they had only two teachers. (Fast de hade bara två fröknar.)
Teacher: Two teachers and then there was one teacher in each group. Brilliantly solved. (Två fröknar och då blev det en fröken i var grupp. Strålande löst.)

As with Nicolas, the teacher presented the drawing and immediately asked which group did not have any children in it. From this question, both the teacher and Filippa built up the story about why there were only two groups. Filippa was complemented by the teacher as having a brilliant solution. In developing the story, it seemed that Filippa drew on her own experiences of only having two teachers in the class and finding moving between too many activities busy and noisy. At no time, did the teacher bring up mathematical understandings. Whereas Nicolas' unexpected answer resulted in the teacher moving into the vertical discourse, the interaction with Filippa remained firmly situated in the horizontal discourse. Unlike the girls in Boaler's (1994) study, Filippa did not seem confused by the familiarity of the context, rather the teacher emphasised that she wanted the context-specific information.

Lova

In the warm-up activity, Lova was the first to stand up and try to find the pair number for her 2. However, it was her partner, Svante, who told the teacher about their pair. As the teacher described the problem of sharing ten children in the three groups, Lova could be seen using her fingers to work out possible solutions (see Figure 3). Before she collected her paper, she shared her solution with Svante who also used his fingers to find a solution.

In working on the problem, Lova moved from using her fingers to putting tally marks next to the symbols for each of the groups (see Figure 3). She seemed to recognise that there was more than one solution. After she has added one round of tally marks, she counted them before adding the next round. Lova's actions suggested that her interest was in the vertical discourse surrounding the principles connected to adding three numbers together to make 10. When sitting in the horseshoe, Lova was one of the few children who the teacher asked to present her response (see Figure 3), although in this case, the teacher indicated that Lova had a proposal, not a solution.

Teacher: Lova, what proposal do you have? Oh, okay what is there? Can you tell me? (Lova vad har du för förslag. Oj okej vad står där? Kan du berätta för mig?)
Lova: Three, five and one. Three, five and two. (Tre, fem och ett. Tre fem och två.)
Teacher: Let me see, three five and two, okay. It is, let's see here. (Jag får se, tre fem och två okej. Det är, ska vi se här.)
Lova: Three in one group, five in another. (Tre i en grupp fem i en.)
Teacher: But where are the three, is it in the baking group? (Men var är de tre, är det i bak gruppen.)
Lova: Yes. (Ja.)
Teacher: And then it's five in the woodworking group and two in the painting group. (Och så är det fem i snickargruppen och två i målargruppen.)
Lova: Because the woodworking group, it's many more who like to do woodwork, less who like to paint and in between who like baking. (För att snickargruppen, det är mycket mer som tycker om att snickra, mindre som tycker om att måla och mittemellan som tycker om att baka.)

Figure 3: Lova's problem solving

Teacher: So they could choose for themselves in that class, okay. (Så dom fick välja själva i den klassen, okej.)

In presenting her solution, Lova seemed to focus on the numbers, suggesting that she wanted to stay within the vertical discourse. However, the teacher shifted her to the horizontal discourse by asking about which group had three children in it. Lova happily participated by providing details about why she had split the ten children.

DISCUSSION AND CONCLUSION

Previously, van Oers (2001) highlighted how too much attention on children's everyday experiences could be detrimental to their learning formal mathematics knowledge. In this study, we have examined examples from one lesson in a Swedish preschool class which suggests that the issue is more complex.

To acknowledge the preschool class as a bridge between the informal learning in preschools and the formal learning in schools, teaching activities could invite children to participate in both vertical and horizontal discourses. Certainly, the problem solving in this lesson provided a context with plenty of opportunities to work within both the horizontal and vertical discourses. However, the interactions within the three cases suggest that the two discourses, metaphorically speaking, are conflated.

In the case of Nicolas, the teacher made an initial attempt to work in the vertical discourse by discussing the relationship between 5, 10 and 15. In a few, very complex and fast moves a horizontal discourse was created in which the question "what numbers are good for creating three groups" became legitimate. Like the initial vertical discourse discussion, the new horizontal discourse "story" from Nicolas' perspective seemed to make little sense. In this exchange, Nicolas' opportunities to extend his understanding of either the horizontal or the vertical discourse seemed to disappear.

Filippa herself made use of the horizontal discourse to make a connection between the story and the mathematical context. The teacher had opportunities to introduce potential mathematical obstacles to Filippa's story by suggesting that there could be as many teachers as groups. This would have indicated that in mathematics, it is possible to make such assumptions. However, the discussion stopped and was kept within the horizontal component where Filippa seemed most comfortable. Possibilities for learning about the vertical discourse of formal mathematics never eventuated.

Finally, Lova presented her answer in such a way that it invited a discussion about numbers within the vertical discourse. The teacher subtly hinted at this by remarking that 2, 5 and 3 "is okay", but then shifted the discussion to the horizontal discourse. Thereby Lova, who of the three children, showed the most interest in the principles behind the different combinations of numbers that added to ten, lost an opportunity to extend her understanding.

The three cases reveal a group of children, which although not homogeneous, had in common the experience of not being provided with possibilities to connect their experiences to formal mathematics knowledge. The children were not confused over the knowledge needed to solve the problem. They all understood that numbers were expected. Instead confusion may have occurred when the teacher brought the solutions into the horizontal discourse of everyday knowledge, but without extending the children's social concerns, such as having sufficient teachers. If similar lessons continue to highlight the everyday knowledge, even when children such as Lova show interest in the mathematics, then there is a risk of mathematical exclusion.

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Student assessment in an era of accountability

Richard Kitchen, Sarah Ridder and Joe Bolz

University of Denver, Morgridge College of Education, Denver, USA, richard.kitchen@du.edu

In this research paper, the assessment practices of mathematics teachers at an urban high school in the USA who primarily taught racial and ethnic minority students and low-income students are described. We found that the teachers' assessment practices were largely influenced by the pressures to prepare students for success on the state's standardized test. For instance, teachers regularly used the language found on "the test" to classify students by their performance (i.e., Unsatisfactory), was regularly used by the teachers to label students, and was used in both assessment design and assigning student grades. Moreover, student performance on the test influenced how teachers viewed students, and consequently, how they viewed them in the assessment process. This is problematic given the long history in the USA of low-income, diverse students being denied access to challenging mathematics instruction.

Keywords: Assessment, diversity, urban education.

INTRODUCTION

The purpose of this research study is to examine how the assessment practices of mathematics teachers at a highly diverse urban high school in the United States of America (USA) were influenced by their students' performances on the state test ("the test"). At a time when large-scale assessments such as No Child Left Behind mandated tests (NCLB, 2001) [1] have been dominant in the USA, more research is needed to understand the impact of these tests on teachers' practices. It is important to understand how the test influences teachers' assessment practices because assessment is at the heart of what Ball and Forzani (2007) refer to as the "instructional dynamic."

The five teachers who participated in this study taught at Chavez High School [2] during the 2013–14 school year, a school situated in an urban school district in the state of Colorado. Chavez High served slightly more than 2,000 students in grades 9–12 and had a highly diverse [3] student population. In 2013–14, the Free Reduced Lunch rate at the school was 75%; 64% of the Chavez High student body was Hispanic, 15% was African American, 13% was White, 4% was Asian American, and 1% was Native American or Native Hawaiian [4]. At the time this study was conducted, the standards-based Transitional Colorado Assessment Program (TCAP) assessment in mathematics was being administered to all grade 3–10 students in the state in the spring. The research question that we address in this paper is: How did the Colorado state mandated test in mathematics influence the assessment practices of mathematics teachers at Chavez High? Before we offer our research findings and address our research question, we provide reviews of the relevant research literature in classroom assessment and research pertaining to the mathematics education of low-income and diverse students. The research methodology used in this study is also described and we conclude with a brief discussion related to our research findings.

CLASSROOM ASSESSMENT

Classroom assessments are used to inform teachers, students and parents about student knowledge and understanding of mathematical concepts, processes and skills (Wiggins, 1993). Two ways of viewing assessment are assessment as an evaluative process focused on "an accounting of what is" (Webb, 1992, p. 663) and grade assignment (Kulm, 2013), and assessment as "the process of gathering evidence about a student's knowledge of, ability to use, and disposition toward mathematics…" (National Council of Teachers of Mathematics, 1989, p. 3). The former, summative assessment focuses on what students know at a given time (Guskey & Bailey, 2001). The latter, formative assessment, differs from summative assessment in that the focus is not just on summarizing students' learning, but on using information derived from assessments to inform instruction (Black & Wiliam, 2009). After more than a decade in which large-scale summative assessments such as No Child Left Behind mandated tests (NCLB, 2001) have dominated the educational landscape in the U.S., we agree with Kulm (2013) that it is time to "reclaim the true meaning and purpose of enlightened mathematics assessment" (p. 4). Such assessment places a premium on using assessment to support teachers and the development of their instruction for the betterment of their students. For instance, Shepard (2000) called for assessments in which students actively make meaning of mathematical concepts by building on their previous knowledge and experiences and making connections to previous knowledge and new understandings.

With the introduction of the No Child Left Behind (NCLB) legislation and the testing that accompanies it (NCLB, 2001), educators and researchers have expressed a number of concerns about the impact that the introduction of high stakes, large-scale testing in the United States have had on teaching and learning (Nichols, Glass, & Berliner, 2006). For instance, students are reduced to test performers and "teachers find themselves using students to protect or help themselves… The marketplace mentality expands its reach" (Gergen & Dixon-Román, 2014, p. 8). "Teaching to the test" not only limits the content that is taught to what is tested, but also promotes superficial student learning that tends to be more skill focused and furthers student alienation toward school and learning (Nichols, Glass, & Berliner, 2006). Furthermore, instruction may be more targeted to those students who are close to meeting proficiency on the test and less instructional attention is paid to those who are far above or below meeting proficiency (Harlen, 2007). In a nutshell, high-stakes tests have had many negative consequences such as categorizing students as numbers to be compared to their peers, while limiting instruction to the detriment of students and their educational opportunities (Linn, 2000; Messick, 1995a; 1995b). Rather than holding schools and teachers accountable simply for student achievement on high-stakes tests, a range of student assessment information should be collected that will also inform teachers about how to improve their instruction (Harlen, 2007).

MATHEMATICS FOR LOW-INCOME AND DIVERSE STUDENTS

Globally, mathematics has served the historic role of sorting and stratifying students by race, ethnicity, and gender (Gerdes, 1988; Jurdak, 2014). In the U.S., white and Asian middle class and upper-middle class students have been privileged to have greater access to challenging mathematics curriculum and instruction (DiME, 2007; Tate, 1995). Educational opportunities and access to such opportunities are influenced by where one lives, what Tate (2008) refers to as the "geography of opportunity" (p. 397). Massey (2009) contended that advantages and disadvantages procured from an individual's socioeconomic status (SES) are both reinforced and compounded by geographic concentration. For instance, students from low-income communities attend schools in which pupil expenditures compare unfavorably to pupil expenditures in schools located in wealthy communities and achieve at lower levels than their wealthy counterparts (Payne & Biddle, 1999). Hogrebe and Tate (2012) found that algebra performance is also influenced by where students live; the SES of local communities is significantly related to students' performance in algebra. Brynes and Miller (2007) found that SES has direct effects on mathematics achievement and indirect effects on both the opportunities students have to enroll in advanced mathematics classes in high school and on their propensity to take advantage of learning opportunities in mathematics.

In addition to poverty and SES, student access to a challenging standards-based mathematics education is influenced by race, ethnicity, and English language proficiency (DiME, 2007; Gutiérrez, 2008; Martin, 2013). A role that mathematics has played historically is to sort and stratify students by race, ethnicity, and gender (DiME, 2007; Gerdes, 1988). Specifically, white and Asian middle class and upper-middle class students have been privileged to have greater access to challenging mathematics curriculum and instruction (DiME, 2007; Tate, 1995). Schools that enroll large numbers of African American students often have disproportionally high numbers of remedial classes in mathematics in which instruction is focused on rote-learning and strategies that are intended to help students be successful on standardized tests (Davis & Martin, 2008; Lattimore, 2005). In response to NCLB (2001) and the demands to increase test scores, Davis and Martin (2008) argue that the preponderance of skills based instruction "[negatively] shape the lives of poor African American students in more significant ways than middle-class or affluent students" (p. 18). An extensive research base demonstrates that low academic expectations and lower pupil expenditures have historically been the norm for schools that serve students from low-income communities and racial and ethnic minority students (Ferguson, 1998; Knapp

& Woolverton, 1995). Given this research, it is not difficult to surmise that millions of low-income students of color are being denied access to instruction in which students regularly engage in mathematical reasoning and discourse to solve complex tasks (Davis & Martin, 2008; Kitchen, Burr, & Castellón, 2010; Téllez, Moschkovich, & Civil, 2011; Valero & Meaney, 2014).

RESEARCH METHODOLOGY

Beginning in the fall semester of 2013, we visited Chavez High 2–4 times a month for the duration of the 2013–14 school year. A school visit included two classroom observations of one of the five participating mathematics teachers on consecutive days in fall 2013 and again in spring 2014. A classroom observation consisted of videotaping the participating teacher teach a mathematics lesson as well as videotaping a group or groups of students who had provided consent to participate in the study. Every attempt was made to videotape in a manner that minimized interference in the mathematics lesson (e.g., the video camera was placed in a location in the classroom such as the back of the room so as not to block students' view of their teacher, the whiteboard, and any other instructional resources used by the teacher). An interview was conducted with each participating teacher immediately following the first or second observation. Interviews with individual teachers were 30–45 minutes in length. Two focus group interviews were also conducted in spring 2014. Four of the five participating teachers participated in the initial focus group interview and all five of the participating teachers attended the second focus group interview. An interview was conducted with the Chavez High principal in the spring of 2014 as well.

For the purposes of this study, the data analysed were all the interviews conducted that included the individual interviews conducted with participating teachers, the two focus group interviews and the interview with the Chavez High principal. The interview transcripts were analysed using interpretive methods (Erickson, 1986; Maxwell, 2005). Each interview was read as a whole, followed by a period of open coding to allow for the emergence of themes, and themes were then compared across interviews conducted. After a set of themes were obtained from the dataset, we searched for commonalities and differences across interviews conducted (Miles, Huberman, & Saldaña, 2013). We also sought both confirming and disconfirming evidence by searching for supportive and non-supportive evidence (Erickson, 1986; Miles, Huberman, & Saldaña, 2013).

The five mathematics teachers at Chavez High who participated in this study were Ms. H, Ms. K, Ms. S, Mr. T, and Ms. V. All five teachers were chosen by the school's administration as among the best mathematics teachers at the school and were recommended for inclusion in this study. Ms. S is Hispanic and was the only teacher of color in the group of participating teachers.

HOW "THE TEST" INFLUENCED ASSESSMENT PRACTICES

In this study, we investigated how the Colorado state mandated test in mathematics influenced the assessment practices of the mathematics teachers at Chavez High. It became clear from the outset of this study that the teachers' assessment practices we report here were largely the result of administrative mandates to improve student achievement at the school. For example, teachers were frustrated that students were often allowed to retake assessments on which they had not performed well. Ms. H discussed a concern shared by her colleagues about how some students approached assessments; "I'm sick and tired of hearing that before I hand out the test, 'there is a re-test, right?'" (Ms. H, Faculty Focus Group Interview, May 19th, 2014). Teachers explained that they were under pressure from the school's administration to do everything they could to avoid failing students in their classes. Teachers knew that if 20% or more of their students failed a class, they would be called in for a meeting with a Chavez administrator (Faculty Focus Group Interview, May 19th, 2014). Thus, students were often allowed to take an assessment more than once to improve their grade. Many of the assessment practices that the teachers pursued, and discussed below such as "deployment" and Exit Tickets had been mandated by the Chavez administration.

Our first finding was that because of the intense focus at Chavez High on students' performance on the TCAP ("the test"), it was common for teachers and administrators to refer to students as Advanced Proficient or just Advanced, Proficient, Partially Proficient ("Bubble Students") or as Unsatisfactory (or just as "Unsats"). This language reflected not only how students had performed on the test, but had also become language that teachers had adopted to design assessments and even assign students grades. The "Bubble Students" and "Unsats" were specifically earmarked for additional mathematics instruction as a means to change their status to "Proficient students." We also found the "Bubble Student" or "Unsat" label persisted; it was well documented who these students were and teachers were well aware of who was a Bubble Student and who was an Unsat student. "Bubble students" were specifically targeted for supplemental instruction since these students were within reach of achieving Proficient status on the test.

Though students were classified as a "Bubble Student" or as an "Unsat" based on performance on the previous year's test, teachers also used these categories to label student performance on summative and formative assessments that they used in their classrooms. For instance, mathematics teachers at Chavez High School engaged in what was referred to as "deployment." Students who were evaluated as Partially Proficient or Unsatisfactory on a unit test were "deployed" to receive supplementary instruction on the mathematics unit just completed. After the 1–2 day deployment, students were administered a post-test. When discussing the impact of one such deployment, Ms. K reported that "77% of the Unsats moved up to Partially Proficient" (Ms. K, April 17, 2014) on one of the re-tests administered.

Teachers also discussed aspects of classroom assessments they designed using the language from the test. For instance, parts of unit tests used during deployments included "Unsat" items, or questions that generally required less of students (e.g., recall of mathematical vocabulary). The "Unsat" portion of an assessment was meant to provide "access points" for students (Ms. S, Focus Group Interview, April 17, 2014). By this, Ms. S meant that the content of the "Unsat" portion should be more elementary, focusing for example, on vocabulary items in geometry. Teachers explained that the majority of the test items on summative assessments and even formative assessments such as Exit Tickets were "Unsat" items (e.g., 70–80% of the items), while the remainder of the items on the assessment were more advanced. Mr. T explained that he generally included only "the naked" or skills based tasks on his Exit Tickets (Mr. T interview, 2-20-14). The content of mathematics assessments was clearly influenced by the content teachers perceived would be included on the test. Ms. S, for instance, discussed modifying the curriculum she used because, in her opinion, it did not align well with the TCAP.

Finally, mathematics teachers at Chavez High used the test language categories to assign grades to their students. For instance, Ms. S used the classification language used on the test as part of her grading system, she did not assign grades based upon percentages: "I don't have any numbers in my grade book. It's a Partial, Unsat, or Advanced" (Ms. S interview, 10-24-13). For Ms. S, grades were assigned based upon how each student was performing, in a holistic manner, relative to language aligned with the test: "Unsat is they have some knowledge of some of the math that we did, so that's about a D. Partial Proficient is C-ish. And Proficient is about a B because you're doing what the standards are asking you to do. For students to earn an A, they have to take the math they've been doing and apply it to new problems that hasn't been taught to them." (Ms. S interview, 10-24-13).

In summary, we learned that the TCAP, Colorado's standards-based test profoundly impacted teachers' assessment practices, including their grading practices. Because of the strong focus by the Chavez High administration on student achievement on the test, and their goal of increasing student test scores, much of how teachers classified students and designed instruction was driven by the need to improve student achievement. Perhaps more importantly, student performance on the test greatly influenced how teachers viewed students, and consequently, how they viewed them in the assessment process. The language that was used on the test to classify students by their performance on it (i.e., Advanced Proficient or Unsatisfactory) was used as a means to label students and had become the taken-for-granted language teachers had adopted in many of their daily practices.

DISCUSSION

Few studies exist in the U.S. that examine how a state's large-scale test impacts high school mathematics teachers' assessment practices, specifically at a secondary school that serves primarily low-income and diverse students. An important contribution to the research literature is our finding that the test influenced the mathematics teachers at Chavez High to frequently use assessments in a summative manner and that they used their assessment results to categorize students vis-à-vis student performance categories used on the TCAP. Moreover, the mathematics teachers were largely concerned with their students, mostly low-income, students of color, deriving correct answers to skills based assessment tasks, rather than engaging them in solving rich mathematical tasks. This is problematic given the long history in the U.S. of low-income, diverse students being denied access to challenging mathematics instruction (Davis & Martin, 2008; Kitchen, DePree, Celedón-Pattichis, & Brinkerhoff, 2007). The intense focus on preparing students for the test, specifically students who have historically been denied access to a strong mathematics education program in the U.S., is continuing the destructive legacy of prioritizing low-level mathematics instruction for students from marginalized and oppressed communities (Davis & Martin, 2008; Lattimore, 2005). Finally, attaching a label such as "Unsat" to any student is unjust, particularly so when we consider that most Chavez High students were from historically marginalized and oppressed communities.

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ENDNOTES

1. The No Child Left Behind Act of 2001 was passed by the U.S. Congress and requires states to develop assessments to be administered at the state-level. To receive federal funding, states must give these assessments to all students at select grade levels <http://en.wikipedia.org/wiki/No_Child_Left_Behind_Act>.

2. "Chavez High" is a pseudonym.

3. "Diverse students" refers to students who are members of a racial or ethnic minority group and is synonymous with "students of color," a phrase commonly used in the U.S.

4. The ethnic/racial categories reported here are the categories used by the school district in which Chavez is located.

School mathematics and bureaucracy

David Kollosche

Universität Potsdam, Potsdam, Germany, david.kollosche@uni-potsdam.de

This essay focuses on the connections between calculation and bureaucracy and points out implications for mathematics education. A genealogic analysis which methodologically follows Nietzsche and Foucault is used to show these connections firstly on a historical and secondly on the level of styles of thought. In the process, Weber's theory of bureaucratic administration, Sybille Krämer's history of formalisation and Foucault's analysis of the modern episteme will be considered. This study shows that mathematics – even in its unapplied and theoretical form – and bureaucracy share a common style of thought. Consequently, school mathematics can be understood as an institution which trains and examines a bureaucratic style of thought – an understanding supported by Paul Dowling's sociology of mathematics education.

Keywords: Calculation, formalisation, bureaucracy, Foucault, genealogy.

FROM SKOVSMOSE'S QUESTION TO THE METHOD OF GENEALOGY

In mathematics education research, calculation tasks, i.e. tasks which can be solved following prescribed rules for the manipulation of terms of numbers and variables, are often considered to be too 'bureaucratic' and to be over-represented in the mathematics classrooms. Usually, this claim serves to promote innovative approaches in mathematics education, but within this study it has not been possible to find research with such an object of an academic study itself. Ole Skovsmose (2005) also refers to the dominance of calculation tasks to distinguish his *Critical Mathematics Education.* He tries to explain the social function of the vast amount of calculation tasks each student has to solve during his school career. He refuses some common explanations and eventually asks:

> Could it be that 'normal' students in fact learn 'something', although not strictly speaking mathematics, and that this 'something' serves an important social function? If we look back again at the 10,000 commandments [which students have to solve during their school careers], what do they look like? Certainly, not like any of those tasks with which applied mathematics occupies itself, tasks in which creativity is needed to construct a model of a selected piece of reality. Nor do they look like anything a working mathematician is doing. However, they might have some similarities with those routine tasks, which are found everywhere in production and administration. An accountant has to do sums day after day. A laboratory assistant has to do a series of routine tasks in a careful way. [...] All such jobs do not invite creative ways of using numbers and figures. Instead things have to be handled carefully and correctly in a pre-described way. Could it be that the school mathematics tradition is a well functioning preparation for a majority of students who come to serve in such job-functions? (Skovsmose, 2005, pp. 11–12)

At a first glance, it seems surprising that Skovsmose does not provide any answers to his questions although he might be considered one of the founders of socially concerned research in mathematics education. But a closer look shows that he does not consider mathematics itself but only its application and teaching as socially critical. For Skovsmose, mathematics education is "undetermined", "without essence"; it can "serve a grand variety of social, political, and economic functions and interests" (Skovsmose, 2011, p. 2). Indeed, only under this assumption can he reasonably promote his emancipatory pedagogy. Then again, this very assumption hinders him from socially analysing non-applied calculation. So what if this assumption is misleading, if it is hindering us from discussing the social functions of what might belong to the essence of mathematics education: pure but nevertheless socially relevant mathematics, e.g. calculation? The question would then not only be how calculation *is* *applied*, but how it *can be applied*, which perspectives it allows and which ones it denies.

As calculation is no matter of applications and teaching alone, but one of mathematics itself, the usual methods of Critical Mathematics Education are not promising for a comparison of calculation and bureaucracy. Instead, this study uses the method of *genealogy* as introduced by Friedrich Nietzsche and continued by Michel Foucault. Genealogy is a method for "examining the historical origins of present day philosophical concepts, ideas and discourses along with the institutions that sprang from them" (Lightbody, 2010, p. 1). Unlike other historical approaches, it questions the ethical, metaphysical and epistemological values which implicitly underlie our knowledge and practices. It assumes that our ideas and practices are not born perfect, but have evolved in a struggle against other ideas and practices. Therefore, genealogy "is not the erecting of foundations: on the contrary, it disturbs what was previously considered immobile; it fragments what was thought unified" (Foucault, 1971/1984, § 3). By confronting the taken-for-granted with its fragile genesis, one gains the possibility of looking at it from an outsider's perspective and recognising the ideas it was directed against, the interests it served or the complicity into which it seduced us (Saar, 2007, p. 15). As every genealogy involves the danger of shaking our own convictions, it cannot be performed in any 'objective' way. Consequently, the purpose of genealogy is not to constitute any kind of 'true' history (although it might contribute to such an endeavour), but to help us understand our contemporary ideas and practices. Especially, it allows us to gain a critical distance from the values operating in mathematics education and to analyse calculation inside and outside school as a morally ambiguous socio-cultural phenomenon. Martin Saar states:

> Genealogy is more qualified than any other form of critique to grasp phenomena such as imperfect liberty, complicity with authority and subtle heteronomy, for it illuminates the conditions of the possibility of life forms in which heteronomy stabilises and power effects mentalities. (Saar, 2007, pp. 15–16; my translation)

The following analysis will first consider the genesis of calculation and bureaucracy in order to trace similarities in development. It will then match the styles of thought represented by calculation and bureaucracy in order to substantiate the claim that both share a common style of thought. The genealogic questions to ask are: *At what times, under which circumstances, to satisfy which needs and to serve whose interest did calculation and bureaucracy develop? What are the values underlying calculation and bureaucracy? What can the history of both tell us about commonalities between them?*

TOWARDS A GENEALOGY OF CALCULATION AND BUREAUCRACY

Unfortunately, neither a genealogy of calculation nor one of bureaucracy exists. Therefore, this study will try to sketch a genealogic approach from the theory that exists on the history and theory of calculation and bureaucracy. Contemporary theories of bureaucracy are still based on the sociology of Max Weber (Anter et al., 2010). Apart from that, his approach towards bureaucracy is particularly valuable for this study as it has genealogic features (Saar, 2007, p. 296). Weber describes an 'ideal type' of bureaucracy which is well summarised by Robert K. Merton:

> A formal, rationally organized social structure involves clearly defined patterns of activity in which, ideally, every series of actions is functionally related to the purposes of the organization. In such an organization there is integrated a series of offices, of hierarchized statuses, in which inhere a number of obligations and privileges closely defined by limited and specific rules. Each of these offices contains an area of imputed competence and responsibility. Authority, the power of control which derives from an acknowledged status, inheres in the office and not in the particular person who performs the official role. Official action ordinarily occurs within the framework of pre existing rules of the organization. The system of prescribed relations between the various offices involves a considerable degree of formality and clearly defined social distance between the occupants of these positions. (Merton, 1949, p. 151)

Although Weber considers the rise of bureaucracy a modern phenomenon and contrasts it with patrimonial forms of administration, which were typical for pre-modern monarchies, he acknowledges that some of its social and economic preconditions (such as the economic need for an effective, professional and centralised administration or the development of monetary economy) existed before, leading to historical forms of administrations with bureaucratic traits. He explicitly mentions the New Kingdom of Egypt, the late Roman Principate and the absolute monarchies of early modern Europe (Weber, 1922/1972, pp. 556, 560).

It is striking that our records of the development of calculation date back to the very same places and eras. In Sybille Krämer's unique history of formalisation (1988), which builds on the work of Jacob Klein (1936/1992), the outstanding contributors to that development of calculation from purely arithmetic to algebraic forms are the Egyptians at the beginning of the New Empire in the 16th cent. BC, Diophantus at the time of the late Roman Principate and Vieta when monarchy began its change towards absolutism. In spite of their enormous contributions to philosophy, the (decentrally administered) ancient Greeks considered calculation unworthy of a scientific discussion. It took only 200 years and the uprising of strict philosophical logic to have many contributions of the Pythagoreans, mathematicians said to be influenced by the Orient, excluded from the corpus of mathematics, most notably from Euclid's *Elements* (Krämer, 1988).

In a collection of application tasks and solutions, the Egyptians documented their mathematical techniques which include fractions, written methods for multiplication and division, applications of the Pythagorean theorem, solving quadratic equations and calculating areas and volumes. The Egyptian 'aha-calculus' is the earliest record of the use of variables; it documents the transfer of algorithms from numbers to signs. Different from our use today, the Egyptian variable could only stand for a specific, yet unknown number. It could be used in expressions such as $4 + h = 15$, but not in expressions which describe relations of values such as $y = 4x - 1$ or $a + b = b + a$ (although we have to keep in mind that these expressions could only be recorded verbally as our formalistic writing of mathematics developed only during the last few centuries). The variable was always connected to a certain number, it was its placeholder; and initially this was the only reason to treat it as a number (Krämer, 1988). Greek algebra separated values from their contexts of application and linked them to geometry. While expressions such as $a + b = b + a$ could now be interpreted as an apposition of line segments, the use of algebra and variables was constrained by the necessity of its geometrical interpretation. Five centuries after Euclid, Diophantus emerged as the *enfant terrible* of classical mathematics: he added lengths and areas, thought of triangles as triples of numbers, introduced symbols for operations and facilitated a formal notation for terms and equations. Nevertheless, he still considered variables the mere placeholder of a fixed number. Thus, he was unable to present universal algorithms and had to document his techniques in examples of tasks and solutions (Krämer, 1988; Klein, 1936/1992). This did not change until Vieta developed his algebra. Vieta was the first to consider variable as autonomous entities, independent from any number(s) it might represent and defined only by its rules of calculation:

> Algebra is no longer calculation with unknown numbers. Instead, it can be conceived as a calculation with characters, i.e. with 'undetermined' symbols which can represent all possible numbers that – substituted into a given equation – form a right expression [...] This is how the mathematical formula came into the world. (Krämer, 1988, p. 61; my translation)

A COMMON STYLE OF THOUGHT

The joint development of calculation and bureaucracy merely indicates a connection between both. The further analysis will show that calculation and bureaucracy do not only share a common style of 'bureaucratic thought', but that this style of thought is exemplary and prototypical for the modern thought since the 17th century. To this end, considerations about the characteristics of symbols in modern thought will lay the basis for analysing the role of bureaucratic thought in contemporary society.

Again, the genealogic analysis lays its focus on historical events of change and conflict. In this case, the biggest changes can be spotted around 1600, when Vieta's algebra became influential and the rise of bureaucracy allowed absolutism to develop. In his *Order of Things* (1966/1970), Michel Foucault identified a strong change in the episteme, i.e. the way people perceive and make sense of the world, in the years around 1600. Until the end of the Renaissance, thought was dominated by the principle of resemblance – a relationship considered to be unbreakable. Signs were "thought to have been placed upon things so that men might be able to uncover their secrets, their nature or their virtues" (p. 59). Signs resembled the represented, literature resembled truth, variables resembled numbers and money was made of valuable materials. In contrast to that, signs gain their independence in the 17th century. Suddenly, they are considered arbitrary constructs and require legitimisation. Consequently, science begins to discuss the criteria for the significance of symbols, leading to the appreciation of calculation and the evolution of formal logic. From then on, symbols are interrelated by their order (*taxonomy*, connected to logic) and by their measure (*mathesis*, connected to calculation) (pp. 71–76).

In his history of Algebra, Klein argues that while "in Greek science, concepts are formed in continual dependence on 'natural,' prescientific experience, from which the scientific concept is 'abstracted'", in modern science "nothing but the internal connection of all the concepts, their mutual relatedness, their subordination to the total edifice of science, determines for each of them a *univocal* sense". Klein recognised that "the nature of the modification which the mathematical science of the sixteenth and seventeenth century brings about [...] is *exemplary* for the total design of human knowledge in later times" (Klein, 1936/1992, pp. 120–121). From that perspective, modern calculation is not only an example of the new episteme, since it uses autonomous symbols; it is also a condition of the possibility of the modern episteme, for it constitutes a method to interrelate autonomous symbols. Accordingly, Klein points out that the new form of calculation is not a mere "device" of science but predefines the forms, e.g. the possibilities and restrictions, of scientific understanding (pp. 3–4).

The modern episteme is a prerequisite of bureaucracy, too, for it builds on the dissolution of the resemblance and the installation of symbolic practices. Bureaucracy has the purpose to provide predictable and equitable, i.e. non-arbitrary, forms of administration. For that reason, administrative acts are bound by "a consistent system of abstract rules which have normally been intentionally established" (Weber, 1921/1947, p. 330) instead of resembling any natural, traditional or divine law. Within this system of rules, officials act in a "spirit of formalistic impersonality", "without hatred or passion", "without affection or enthusiasm"; "everyone in the same empirical situation" has to be treated equally and the official is not allowed any "personal considerations" (p. 340). According to this, obligations, administrative means and authority are linked to positions, which are abstract symbols within the system of rules and do not resemble any natural person; positions are only 'held' by persons (p. 330). Bureaucracy follows the principle of impersonality by separating the official and the client from the human, by ignoring their individuality: their hope, fear, anger, gratitude, concern and doubt.

Calculation embodies a similar style of thought. Firstly, calculation is used for non-arbitrary, i.e. 'objective' predictions. The German *berechenbar* means 'calculable' as well as 'predictable'. Secondly, calculation works along a system of abstract rules that are culturally established and that the individual has to conform to. Thirdly, this system of rules demands 'formalistic impersonality' as calculation operates by its rules alone. This formalism disregards any 'personal considerations' of the calculating individual just as it disregards those of the official. But on top of this, the variables of each calculation also have to be manipulated 'impersonally', i.e. without any regard for what they might stand for. Every situation is only perceived in the boundaries of the pre-defined cases, i.e. cases that rules (for calculation or administration) exist for. It is this separation of sign and represented, of case and individual, of variable and number in the modern episteme that allows both bureaucracy and calculation as known today. Calculation is not a mere tool of bureaucratic administration, but it is in itself a technique for the "de-humanised" (Weber, 1922/1972, p. 563) processing of situations. Therefore, calculation is not an "undetermined" technique that can "serve a grand variety of social, political, and economic functions and interests" (Skovsmose, 2011, p. 2); it is a technique which resembles a style of thought that is: bureaucratic.

CALCULATION AND BUREAUCRACY IN THE CLASSROOM

Merton acknowledges that bureaucracy has to exert "a constant pressure upon the official to be methodical, prudent disciplined"; it must attain "an unusual degree of conformity with prescribed patterns of action" in order to fulfil its purpose (Merton, 1949, p. 154). Accordingly, Weber states that the bureaucratic style of thought requires "specialised training" (Weber, 1922/1972, p. 552). He explains the uprising of general education in modern times with the need for preparatory training and selection. Mathematics education in particular has historically developed alongside the cultivation of bureaucracy, incorporating calculation which has been shown to share a common style of thought. As Skovsmose points out, students have to solve a large amount of calculation tasks during their school career (Skovsmose, 2005). Solving these tasks is a prescribed activity with abstract symbols, following prescribed rules. Compared to other tasks used in school, calculation tasks specifically cannot be solved *without* a bureaucratic style of thought: There is usually no other valued solution to a calculation task than the development or application of a rule-bound and impersonal algorithm. The experience of these ever-repeating challenges causes the student to adapt. On the one hand, she may be able and willing to cultivate a bureaucratic style of thought. This would allow her to perform well (at least as long as mathematics education incorporates calculation to a large extent) and experience herself as a successful learner. On the other hand, she may be either unable or unwilling to cultivate a bureaucratic style of thought. This would leave her to ever-repeating failure in calculation tasks. In the case of such a trauma, the only adaptation securing the student's dignity is to escape from the humiliating situations. As a physical escape is not tolerated, it has to be performed mentally: those students 'learn' that mathematics is 'nothing for them'. Ideal-typically, this organisation of the mathematics classroom results in the production of either accomplices or avoiders of mathematics, ensuring that mathematical rule goes unquestioned and thereby contributing to the domination of humans by mathematics which Roland Fischer (1984) has warned against.

In his *Sociology of Mathematics Education* (1998), Paul Dowling describes "myths" about mathematics which are spread in the mathematics classroom. He is especially interested in so called 'real world problems' in which problems formulated in situations of the real world are interpreted and solved by calculation. While no specific applications of calculation will be discussed here, Dowling's analysis helps to understand who calculation is *per se* positioned in the mathematics classroom. The *myth of reference* is a mechanism which makes students think that calculation is a universal tool capable of solving any real world problem. Dowling states and exemplifies that many 'real world problems' build on situations which would not be solved mathematically in everyday life (Dowling, 1998, pp. 4–7). In addition to that, the mathematics classroom does not usually present any 'real world problems' that cannot be solved mathematically. Therefore, school mathematics provides experiences which foster the belief that mathematics can be reasonably applied to solve *any* problem of the real world. To the extent to which these 'real world problems' are based on or result in calculation, 'real world problems' function as a mechanism to install calculation as an omnipotent means of perceiving and handling our world.

Dowling's *myth of participation* refers to a mechanism which fosters the belief that students will need mathematics to succeed in their everyday life outside school (Dowling, 1998, pp. 7–11). 'Real world problems' are often student-oriented, i.e. they build on situations that are close to the experiences of students, although the mathematics involved would usually not be used to solve such problems in the real world. Their latent message is: *Look at these examples from your everyday life and see how calculation is needed to manage them!* That is how school mathematics provides experiences which foster the belief that the students need mathematics and especially calculation to cope with their everyday life outside school.

Both mechanisms bear the possibility of intensifying the experiences students have with calculation tasks. On the one hand, those succeeding in calculation may be happy to master the seemingly omnipotent and even privately relevant, de-humanised, rule-bound approach towards our world. On the other hand, those failing in calculation may explain their failure with their own incompetence rather than with insufficiencies of the calculation method as the latter is believed to be omnipotent. Nevertheless, they may believe that calculation is important for their life outside school. In the end, they might come to think that they lack the ability to handle the mathematics necessary for a fulfilled life and feel compelled to lay their trust in mathematical experts. Thus, a function (although not an intended goal) of school mathematics would be not only to separate the capable and willing from the unable and unwilling, but also to make the latter appreciate their subordination.

As bureaucracy and calculation share a common style of thought, performance in calculation indicates whether or not students are suited for administrative positions: whether or not they can reduce situations to cases and calculations, and whether or not they can handle these cases calculations according to imposed rules disregarding their personal thoughts and feelings; whether or not they can separate from themselves an administrative or calculative processor of rules. But school mathematics also educates in the sense that it produces situations in which students cultivate their relationships towards calculation. Whether or not, or rather: how far this experience affects the student's relationship towards bureaucracy, is hard to tell. It seems at least natural that the estrangement from calculation to some extent coheres with an estrangement from any practices sharing a similar style of thought, especially from bureaucratic ones. Therefore, school mathematics can be considered an institution which (alongside other functions) identifies and trains a calculatory-bureaucratic elite and teaches the rest to subordinate to the calculatory-bureaucratic administration of our society. And although this explanation deserves further theoretical and empirical elaboration, it can already serve as a first answer to Skovsmose's questions, contribute to the socio-philosophical discussion about the essence of mathematics, question the educational objectives which mathematics educators assign to school mathematics, explain why mathematics is such a polarising subject and shed light on anxiety and joy, motivation and estrangement in mathematics classrooms. Within the research community especially, it would also add a new dimension to socially concerned research in mathematics education and call for ways to deal with this social function of school mathematics.

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Socialisation and mathematics education in Swedish preschools

Dorota Lembrér

Malmö University, Faculty of Education and Society, Malmö, Sweden, dorota.lembrer@mah.se

This theoretical paper investigates the relationship between socialisation and mathematics education in Swedish preschools. Socialisation is considered to be the process by which children construct their own childhoods and experiences as a preparation for adulthood. Mathematics education as defined by the curriculum outlines what learning possibilities preschools and the adults working in them should provide to children. The production and reproduction of cultural knowledge as components of socialisation are connected to the global issue of early year's education and schoolification. I suggest that it is important to discuss how learning of content and subjectivities is a key feature in an investigation of the relationship between socialisations and mathematics education in the Swedish preschool curriculum.

Keywords: Learning, mathematics education, preschool, socialisation.

INTRODUCTION

In the research on mathematics education, there is a perception that young children need to have strong mathematical understandings when they begin school (Duncan et al., 2007). However, the implementation of programmes to provide this has led to concerns about the schoolification of preschool. Schoolification is described as:

> an emphasis on the acquisition of specific pre-academic skills and knowledge transfer by the adult rather than a focus on broad developmental goals such as socio-emotional well-being and the gaining of understanding and knowledge by the child through direct experience and experimentation. The push-down of grade one materials, specific learning standards and the traditional primary school model of didactic instruction to pre-kindergarten and kindergarten in some US states has heightened concern about the possible schoolification of ECEC [early childhood education and care]. (Doherty, 2007, pp. 7–8).

Increased references to mathematics in the revised Swedish preschool curriculum (Skolverket, 2011) suggest that perceptions of its value as being beneficial to society have increased. In this paper, I contribute to a discussion of the role of mathematics education in early childhood studies by reflecting on its relationship to learning and socialisation. I link socialisation processes and mathematics education with two components: the reproduction of culture from one generation to another; and the recognition of young children in preschool as knowledgeable and active participants in today's society. Learning is a vital component of this investigation because it highlights the connection between content and subjectivities. Therefore, my aim is to offer a theoretical contribution regarding the understanding of socialisation processes and what learning possibilities preschools should provide.

By placing a particular emphasis on preschool as the site of socialisation processes, I argue that there is a need for a discussion on what grounds and in what ways are children modified by the institution of preschool (Kampmann, 2004). Preschool as an institution is a place where children's social context and experiences are formed (Ebrahim, 2011), but with a specific focus on learning and development. Consequently it can be said that society considers preschools to be the necessary institutions for strengthening children's social competence and general ability to develop their childhood so that they can live in a modern world both independently and as part of a democratic society. However this raises the question, how does increasing the importance of mathematics education affect children's socialisation and learning?

SOCIALISATION

Socialisation occurs from living within a society and taking part in activities with others (Thorne, 1987). Given that in 2013, 77 per cent of children aged between one and three years and 94 percent of children aged between four to five years attended Swedish preschools[1], it can be stated that children are socialised within the institution of preschool as this is where they participate in society:

> By interacting with playmates in organized play groups and nursery schools, children produce the first in a series of peer cultures in which childhood knowledge and practices gradually are transformed into the knowledge and skills necessary to participate in the adult world. (Corsaro, 1992, p. 162)

Walzer and Miller (2007) stressed that within culturally diverse societies, educational opportunities provide an understanding of some of the meanings to be found in that society. Socialisation provides an educational structure which children learn to recognize and interpret (Trondman, 2013).

Socialisation as replication

Socialisation for young children, including that which occurs in preschools, has been equated with a process or a journey towards adultness. This journey contributes to children's gaining of knowledge of their own and others' roles in society, in order to reproduce society's key institutions (Lee, 2001). James, Jenks and Proud (1998) stressed that socialisation includes a transmission of culture from one generation to another, in order to ensure that societies sustain themselves over time. As a consequence of the process of change and alteration, the child is seen as developing socially, so that they become the adults that society needs.

Curriculum as a body of knowledge is an example of institutionalisation of the norms and values seen as important by a society, but also as an example of adults determining what skills, norms and values children need to become acceptable adults. The mathematical goals in Swedish preschool curriculum represent some of these institutionalised norms and values. The goals related to mathematics require preschools to provide opportunities for children to:

> develop their understanding of space, shapes, location and direction, and the basic properties of sets, quantity, order and number concepts, also for measurement, time and change,
>
> develop their ability to use mathematics to investigate, reflect over and test different solutions to problems raised by themselves and others,
>
> develop their ability to distinguish, express, examine and use mathematical concepts and their interrelationships,
>
> develop their mathematical skill in putting forward and following reasoning (Skolverket, 2011, p. 10)

In these goals, what mathematics is has been already determined. Inclusion in the curriculum means that they take on the aura of being the valuable cultural knowledge which should be transmitted to children so that society can be sustained over time. However, if the process of socialisation is for children to gain valued knowledge about the subject, as part of the reproduction of society's key institutions (Lee, 2001), then children may need to recognise the activity as mathematics.

Yet, a focus on the future may lead to unwelcome consequences. Sarama and Clements (2004) argued that such a focus can limit a child's own hunger for knowledge and their willingness to engage in mathematical activities. As well, it may be that the inclusion of more mathematical goals in the revised Swedish preschool curriculum (Skolverket, 2011) could restrict teachers' possibilities in planning activities which value what children already know and can do. Concerns have been raised that the focus on mathematical knowledge needed for school learning is a form of schoolification regarding the effect on the kind of socialisation that preschool children receive (Alcock & Haggerty, 2013; Gunnarsdottir, 2014; Sofou & Tsafos, 2010).

Socialisation as creation

An alternative view of socialisation is that it can be considered as a process in which children co-create new cultural norms and values together with others (Thorne, 1987). For Ebrahim (2011), socialisation is the process by which people, who inhabit a society, create it. From this perspective, children need to be considered as knowledgeable, active participants in

1 For more information see http://www.scb.se/en_/

the construction of their childhood and experiences (James et al., 1998). This would include producing norms and values connected to the societies of their childhoods.

In the Swedish preschool curriculum (Skolverket, 2011), the preschool is expected to provide opportunities for children to engage with more general goals. Although many general goals also suggest that adults determine the necessary knowledge and skills for young children to know, some position children as having possibilities to create rather than just replicate cultural norms and values. These include:

> Each child should have the opportunity of forming their own opinion and making choices in the light of their personal circumstances (p. 4)
>
> Children should also have the opportunity to explore on their own issues in greater depth and to search for their own answers and solutions (p. 5)

In these goals, children are situated as persons with their own rights, interests and experiences who can influence the acquisition of the necessary skills to perform as functioning members of their society.

Having different emphases in the goals (creating versus recreating societal values and norms) could restrict teachers' possibilities in planning activities (Lembrér & Meaney, 2014). The focus of the goals for mathematics education on replicating cultural knowledge, including valuing certain aspects of mathematical knowledge, may mean that teachers do not consider children as needing opportunities, for example, to form their own opinions and make choices about mathematics. This could be an example of schoolification where the kind of socialisation that preschool children receive is restricted to ensuring that they become the kind of mathematicians needed for school learning.

Replicating and creating through socialisation

Socialisation as a process of creating/recreating society and the transmission of culture is connected to perceptions of what young children are capable of doing. For example, Lee (2001) highlighted how a young child's age affected adults' perceptions of them having rights to have opinions and desires, as often children are considered too young to be worth listening to. Discussions such as these make it difficult to recognise children as fully human or people in their own rights (James & Prout, 1997). Consequently, researchers have discussed the necessity of making such a distinction. For example, Thorne (1987) discussed the adult/child dualism as being socially constructed and therefore possible to change.

Rather than seeing creating and replicating cultural norms and values as being in opposition, it has been argued that children's own knowledge can be a starting point for initiating social interaction in play and promoting construction of subject knowledge (Edo, Planas, & Badillo, 2009). However, this requires a delicate balance between production and reproduction of societal norms and values, a sense of responsibility for the future of the society while at the same time allowing them to create own values, knowledge and even cultural understanding.

LEARNING

Although curriculum goals frame the operationalisation of mathematics education in preschools, they alone do not determine the socialisation that children gain from participating in activities. Instead, learning, not as a cognitive activity done by individuals, but as that done within societies and in particular in societal institutions such as preschools, needs to be considered in relationship to socialisation. Radford (2008) stated, "learning does not consist in constructing or reconstructing a piece of knowledge. It is a matter of actively and imaginatively endowing the conceptual objects that the student finds in his/her culture with meaning" (p. 223).

Often preschool children's early learning is described as essential for further learning processes. Children acquire the understanding, skills and awareness of different mathematical concepts, developed in the course of their own experiences (for example, Brenner, 1998), through the process of learning and the reproduction of norms and values (Lee, 2001). Studies of how children learn mathematics together with their peers, family, environment and in culture, indicate that interactions around mathematical activities are of importance (for example Carruthers, 2006).

However, within discussions of socialisation, learning is conceived as being about learning knowledge or skills, either already found within a society as valued norms and values or newly created within the process of learning. Such discussions fail to recognise that learning cultural norms and values results in children (and adults) learning to become someone, in other words – learning subjectivities. Such a view of learning positions children as human and contribute to an understanding that mathematics learning is socially constructed, not merely reproduced, so that children can explore and thus produce new forms.

Osberg and Biesta (2008) describe learning as something that can occur anywhere at any time but that education was about learning about taking on a particular subjectivity such as being a responsible member of society. Radford (2008) also saw learning as being more than simply learning about ideas. He stated that learning is "not just about knowing something but also about becoming someone" (p. 215). The role of the curriculum in shaping the kind of person that evolves from participating in activities in preschool.

> This 'shaping of subjectivity' is generally understood to be achieved through the curriculum (and the pedagogy 'supporting' the curriculum). With the concept of 'education' the notion of curriculum therefore acquires a very specific meaning. It becomes a course by means of which the subjectivity of those being educated is directed in some way. (Osberg & Biesta, 2008, p. 314)

From this perspective, Biesta (2007) considered much of what occurs in institutional settings, such as preschools, to be socialisation, as for him socialisation is the "insertion of 'newcomers' into existing cultural and socio-political settings" (p. 26). On the other hand, education as defined by Kant is the self-education needed to achieve rational autonomy in order to become fully human. Biesta (2007) argued that this view of education was also a form of socialisation because it set up what the end product of self-education had to be, that of rational autonomy. Education of this kind results in individuals taking on the attributes of existing members of a society but without a recognition of the role of the community in this process (Radford, 2008). Socialisation, then, must be considered as a not just learning cultural norms and values but living those norms and values, as with becoming a rational human being.

However, this suggests that those who did not have or did not gain the appropriate attributes could not be considered human (Biesta, 2007). Thus, young children can be categorised as being non-human as they do not have the necessary desired rationality. As research has come to highlight the importance of mathematics in early years (Ginsburg & Amit, 2008; Sarama & Clements, 2009), this category of being non-human could be extended from not being rational to not being mathematical sufficient or "at-risk" from beginning school without the mathematical knowledge deemed necessary by adults (Meaney, 2014).

In contrast, Biesta (2007) postulated that education should be deemed as preparation for an uncertain future, where freedom "needs to be realised again and again" (Biesta, 2007, p. 32).

In a report for the Swedish National Agency of Education (Skolverket), Johansson (2011) stressed that the Swedish preschool curriculum clearly states that children's own experiences should be actively drawn upon in preschool contexts because it is a source for knowledge and learning. Activities are to be based on a creative form of play, with opportunities for other kinds of expressions. This can be seen in the following example of goals from the curriculum (Skolverket, 2011):

> Learning should be based, not only on the interaction between adults and children, but also on what children learn from each other (p. 6)
>
> The preschool should promote play, creativity and enjoyment of learning, as well as focus on and strengthen the child's interest in learning and capturing new experiences, knowledge and skills (p. 9)
>
> Take account of children's eagerness, desire and enjoyment to learn, as well as strengthen confidence in their own ability (p. 11)

In these goals, although children are positioned as having a major role in their own learning, their developing subjectivities are only apparent in regard to the kind of learner they should be encouraged to be. Similar points can be made about a lack of awareness of the subjectivities that preschool children learn while engaged in mathematical activities.

As noted earlier when children's socialisation is discussed in terms of learning mathematics, knowledge and skills are considered crucial. However, what young children are capable of doing mathematically is determined by the opportunities provided for them to engage in activities. When preschool children are often actively engaged in mathematical activities (see, for example, study by Lange, Meaney, Riesbeck, & Wernberg, 2012), they can construct new knowledge as well as making sense of existing knowledge in a process of cultural reproduction. Becoming aware of the knowledge they are learning can be seen as an active process of meaning making based on understanding and interpretation. In an earlier study on measurement (see Lembrér, 2013), a group of preschool children drew a map. As they became aware of the importance of different measurement attributes like the length of the boat and the height of the train, they utilised the knowledge that they already had to gain culturally valued knowledge that extended their measurement understanding. Consequently, learning is a process in which socialisation is bound together through mathematical activities. The subjectivities that the curriculum suggests should be available to children through participating in mathematics activities are not explicit. Still the goals in the curriculum indicate that preschool children's possibilities for learning certain subjectivities is to be reproducers of existing norms and values.

SCHOOLIFICATION AND SOCIALISATION

While engaged in activities based on the Swedish preschool curriculum (Skolverket, 2011), such as those involving mathematics, children learn to reproduce societal norms and values so as to become the adults society expects as a result of socialisation. This is likely to occur even when children are recognised in the curriculum as active participants, rather than passive learners. Thus, learning is associated with socialisation.

Societal views and perceptions of children and childhood influence what is stated in the curriculum. For example, research, such as that reviewed in Clements and Sarama's (2007), indicates that the development of young children's mathematical ideas and skills has come to be regarded as one of the core purposes of preschools in some countries. What is included in the curriculum in turn influences how activities are implemented and children then are socialised. In regard to the revised preschool curriculum (Skolverket, 2011), Lembrér and Meaney (2014) indicate that schoolification through the increased emphasis on mathematical goals is likely to be affecting preschool education in Sweden and this may have an impact on the subjectivities available to children. The results of their study suggest that there are societal expectations about children's need to acquire the skills to perform as members of their society. This contributes to a tension between schoolification, where expectations about what children can learn shifts from school to preschool, and traditional foundations of preschools as institutional practice in Sweden, which has focused on children learning through play.

However, not only the curriculum determines the available possibilities for children concerning subjectivities. These possibilities are further determined by an interplay, including how the curriculum is interpreted and implemented by teachers, but also how children respond to the activities developed from it. Furthermore, dialogues with peers and adults can contribute to children gaining awareness of their own ability regarding mathematical knowledge, by inventing new meanings for improving their mathematical knowledge and for widening their awareness of possible subjectivities that are available (Lembrér & Meaney, submitted). Thus, when engaged in open-ended activities, children can encounter mathematical concepts/knowledge which can contribute to them asking for new knowledge, interests and ideas. This realisation of freedom can lead to children being better prepared to face the uncertain future that present-day adults are unable to predict (Biesta, 2007). This implies children developing the ability to be an agent for their peers, teachers and/or other adults in preschools. However, a tightening of what is acceptable as mathematical knowledge and skills and a restriction of the way that young children interact with it as a result of schoolification may lead to children being exposed to a narrowed range of potential subjectivities. Rather than being prepared for an uncertain future, children learn how to become the adults of today. Society would waste the potentials of imagining a tomorrow that could be different than today.

CONCLUSION

In this paper, I examine the connection between socialisation and mathematics education in Swedish preschools. Based on the curriculum, adults in preschools mediate children's learning by creating mathematical activities or environments with an expectation that valued norms and values are passed on to the children. However, the curriculum does not situate this delivery of cultural norms and values as a passive process for children. Rather children are proposed to be seen as active participants. However, learning cannot be considered to be a mere passing of norms and values, whether or not this is done as an active or as a passive activity. This is because learning something also results in learning how to become someone. It is therefore possible to conclude that an individual is socialised when she or he has learnt to think and feel according to society's expectations.

Socialisation processes may vary depending on the institutional setting and educational discourses. As almost all Swedish children attending preschool, this institution plays an important role in children's lives. Children become organised by institutions' norms and values which have profound effects on the socialisation process.

The impact of mathematics education as stated in the preschool curriculum is dependent on the interpretation of the goals and guidelines of the curriculum and hence on how children are seen by teachers and working teams in preschools. In this endeavour, it is relevant to consider the tension between schoolification and traditional foundations of preschools as institutional practice in Sweden which saw children as having a wider set of possibilities of subjectivities open to them. When the possibilities for the norms and values transmitted through mathematics narrows the types of activities that adults in preschools feel able to offer children, then there will also be a limiting of the type of subjectivities made available to children to adopt. Therefore, mathematics education has a major impact on early childhood education in Sweden. When schoolification affects how preschool teachers implement mathematical learning situations, children's possible subjectivities are narrowed by moving from broad developmental goals to learning mathematics.

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Primary pupils' perceptions of mathematical ability

Rachel Marks

University of Brighton, School of Education, Brighton, UK, r.g.marks@brighton.ac.uk

There is increasing evidence that holding a growth-mindset in mathematics, and hence a belief in the capacity for change, pays dividends in terms of mathematical engagement and attainment. However, much mathematics education policy and practice in England is embedded in fixed-trait theories; a belief that some people can do mathematics and some people cannot. Drawing on a wider mixed-methods study involving 284 pupils and 13 teachers in two primary schools, this research used attitudinal questionnaire and interview data to identify pupils' prevailing mindsets in primary mathematics. Pupils were found to hold predominantly fixed-trait theories strongly grounded in a biological discourse. The potential implications of these perceptions are examined.

Keywords: Primary mathematics, ability, growth-mindset, fixed-mindset.

INTRODUCTION AND CONTEXT

Ability-grouping has long been proposed as one answer to concerns over standards in school mathematics. With a controversial history and concerns over equality and diversity, ability-grouping is often central to educational debates. As I write this, England is witnessing ferocious political and media discussion as to whether setting – between-class ability grouping for individual subjects – should be made compulsory in secondary education with its use mandated through the inspection system. Whilst the literature on ability-grouping in secondary mathematics is quite extensive, the literature for primary mathematics is more limited. However, the wider study this report draws on suggests that primary ability-grouping practices and their impacts essentially mirror the literature in secondary mathematics [1], and there is evidence that the use of ability-grouping is currently increasing in primary schools (Hallam & Parsons, 2013).

Underlying ability-grouping practices must be some notion(s) of "ability", yet defining this concept is far from straightforward. Mathematical ability is a pervasive discourse within the English education system. Selection practices, such as ability-grouping, are commonplace, being grounded in 'common sense' fixed-ability thinking where individual potential is thought to be immutable and easily determined (Marks, 2013). The concept of individual boundaries, neatly described as ability, is entrenched in social attitude. A belief in the 'correctness' of this underlies many educational debates. This was vividly illustrated during a television debate on secondary school selection:

> It seems to me that 1000 kids in a comprehensive, sooner or later, the ones who are good at maths will have to be told "you are good at maths" and the ones who aren't, will have to be told "you are not good at maths" and you should be doing your darnedest to break the barriers but you should be learning as a young person that there are limits to what you can do. (Richard D. North, The Big Questions, BBC1, 26.07.2009)

This paper examines how pupils' come to understand themselves and others within this discourse and practices. The paper asks: How do pupils construct themselves as mathematicians within the prevailing discourse of ability in primary mathematics? Understanding this is fundamental in that beliefs about the capacity for change are known to impact on mathematical engagement and attainment (Dweck, 2000). This paper extends previous work, being grounded in primary mathematics, and hence allowing us to examine belief development within the earliest stages of schooling.

THEORY: THE CONCEPT OF ABILITY IN PRIMARY MATHEMATICS

Adopting a socio-cultural approach, learning is taken as a process of participation and enculturation (Kirshner, 2002). In developing an understanding of how pupils construct themselves as mathematicians within the prevailing discourse of ability, mathematics education is taken to extend beyond the classroom, incorporating the wider cultural contexts in which pupils participate. Learning is seen as identity development in which pupils negotiate between themselves and the social context in which "a culturally and personally located social schema" may be "transacted, redefined ... resisted and, like discourse, called upon when the moment is opportune" (Carr, 2001, p. 527). In doing school mathematics, pupils can adopt and adapt available learner and/or mathematical identities and thus become enculturated to the mathematical world. However, this identity 'choice' is both constrained and constraining, with some identities excluding learners from mathematics. It should also be highlighted that, whilst research on affective issues generates considerable interest, exploring pupils' beliefs is complex (Hannula, 2011).

Ability is a difficult concept that lacks solid definition and is conceptually challenging (Howe, 1997), with these complexities debated in earlier CERME papers (e.g., Brandl, 2011). Despite this complexity, the term ability is in widespread use in education, usually going unquestioned. The dominant view of ability in schools – and perhaps particularly in mathematics – is as a fixed determinant of pupils' future attainment, relatively impervious to change. Through a long history, such beliefs have become elevated to the status of truths through the simple stories they tell and the appeal to a "basic human need to stratify society" (Kulik & Kulik, 1982, p. 619). Within this research, ability is conceived of as an aspect of identity rather than an individual attribute. Ability is co-constructed through discourse within social practices and pupils construct their ability identity in relation to those around them.

METHODS

This research formed part of a wider study into ability in primary mathematics. The aspects of the study presented here sought to ascertain primary pupils' perceptions of their mathematical ability and elicit their views as to what mathematical ability is and the stability of their constructs. The wider study was a longitudinal mixed-methods study conducted over one academic year in two primary schools – Avenue Primary and Parkview Primary – in Greater London. [2] Both schools had similar Contextual Value Added scores (used to measure academic improvement) but employed different degrees of ability-grouping, allowing for a range of experiences. All pupils in Year 4 (ages 8–9) and Year 6 (ages 10–11, the final year of primary schooling in England) were included in the sample for the quantitative aspects of the study, totalling 284 pupils. From this sample, a sub-sample of 24 focal-pupils representing a range of attainment was selected for the in-depth qualitative aspects.

Data collection

Attitudinal questionnaires were conducted with all pupils (n=284) as pre- and post-tests (testing construct stability) in October 2007 and July 2008. This instrument consisted of four sub-scales: motivational orientation, beliefs about the causes of success, perceived ability and enjoyment of mathematics. Reportage is limited to perceived ability. Given widely acknowledged difficulties in measuring affective characteristics, with no 'best test' (Kline, 1990), the instrument used was developed, with permission, from earlier work by goal-theory researchers (Nicholls et al., 1990). This instrument has been widely used, particularly in mathematics education, and favourably acknowledged in reviews. Perceived ability was presented as a one-item scale asking pupils to indicate their perceived standing in mathematics related to their peers. The original instrument had high test-retest reliability (0.83). Pupils were presented with a horizontal line labelled from 'best in maths' to 'worst in maths'. Pupils marked the line to indicate how good they were at mathematics, with piloting interviews suggested that this was easy for the pupils to understand. Questionnaires were administered, following training, by class teachers in lesson time with administration observed in two classes. Data were collated for analysis in SPSS.

Individual and group interviews with the sub-sample of 24 focal pupils were used to "examine individuals' thoughts, feelings, and experiences, which are not easily observed" (Moore, Lapan, & Quartaroli, 2012, p. 251). Individual pupil interviews were semi-structured using tasks adapted from Personal Construct Interviewing techniques. Focal-group interviews were semi-structured with a schedule and tasks – including a discussion of mathematical work – developed from earlier work by the researcher. Follow-up group interviews were used as a form of participant validation developing conversation around themes emerging from early analysis. All pupil interviews were conducted away from the classroom and developed as conversations with themes explored as brought up by the pupils. All interviews were audio-recorded, transcribed and filed with task outcomes.

Data analysis

Following the first administration of the perceived ability scale, the distribution of the data was graphed, then descriptive statistics, Kolmogorov-Smirnov tests of normality, and z-scores for skewness and kurtosis calculated to establish the characteristics of the data and appropriate statistical analysis tests. The first administration produced a distribution that did not differ significantly from a normal distribution, D(219)=0.06, $p=0.08$, allowing the use of parametric tests. The data for the post-test did differ significantly from a normal distribution but it was considered that the overall distribution was near normal and that the parametric tests were robust enough for them to be used on the untransformed data.

Transcripts from interviews were imported into a single NVivo project allowing for consistent coding and analysis. Interview analysis was conducted using constructivist grounded theory, the more theory-driven approach developed by Charmaz (2000) in response to criticisms of grounded theory as narrowly empiricist and atheoretical. In this approach, analytical categories (codes) were derived from reading the data alongside existing theoretical analyses. Codes were structured into trees prior to axial-coding. Mirroring Hamilton's (2002) secondary-school work on ability constructions, analysis was split into internal beliefs and external references. From the corpus of data appended to each theme, extracts, often critical incidences, were selected which best illustrated the area under discussion.

Reliability and validity were key considerations throughout the data collection, analysis and presentation processes. Where possible, established instruments with known reliability and validity were used; in all cases instruments were extensively piloted. Across the analysis, quantitative and qualitative data were linked using methodological triangulation (Denzin, 1997) where data types were compared to determine if there was convergence, difference, or some combination. Participant validation and inter-researcher scrutiny of coding use and application provided a proxy for the validity of the themes drawn from the data (Kurasaki, 2000).

RESULTS: PUPILS' PERCEPTIONS OF MATHEMATICAL ABILITY

The data suggest pupils tend to perceive ability as an internal construct, determined biologically and relatively impervious to change. Self-perceptions of ability appear to remain fairly stable. There is a tendency towards positive self-perceptions but this is accompanied by a long tail of pupils holding weak self-perceptions.

Primary pupils' perceptions of ability: Questionnaire analysis

The self-perception of ability scores for the post-test covered the range of available scores from 0–100 with a median of 68.5. These are illustrated in the boxplot in Figure 1. These data are significantly non-normal, D(239)=0.09, $p<0.0001$, being negatively skewed ($Z_{skewness}$=–4.73). A median of 68.5 suggests a tendency towards more positive self-perceptions. However, there is a long tail of weak self-perceptions with outliers representing pupils holding very low perceptions. Change in ability-perception scores were calculated (post- minus pre-test scores) (M=3.2, SE=1.4, sd=20.3) with scores ranging from -44.0 to +96.0. On average, there was a small increase in self-perceptions between the pre- (M=61.4, SE=1.6, sd=23.0) and the post-test (M=64.6, SE=1.5, sd=20.9). This difference was not significant $t(396)=-1.46$, $p=0.15$. Overall, pupils' perceptions remained fairly stable over the year.

Primary pupils' perceptions of ability: Interview analysis

Of note, when conducting the interviews – both individual and group – no pupil struggled to place themselves or peers on the perceived ability line. As Howe (1997, p. 2) suggests is the case with the wider population where "people today have so little hesitation about ranking individuals as being more or less intelligent", many pupils appeared enthusiastic in positioning their peers. Pupils regularly talked about ranking and had no difficulty in labelling how 'good' or 'bad' they were or of categorising other pupils into a dichotomy of the 'top' and the 'others'. Whilst all avail-

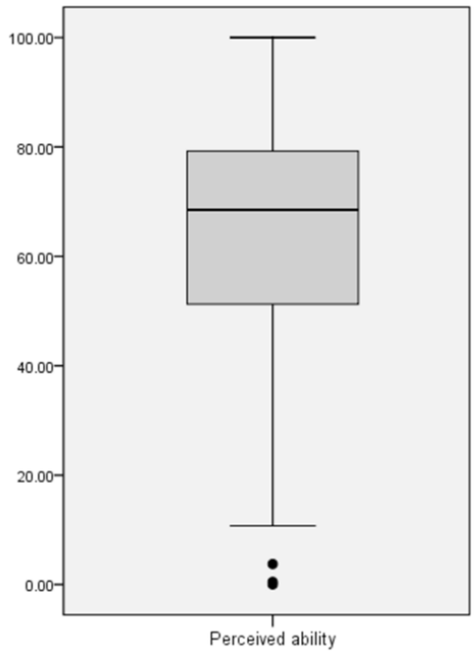

Figure 1: Pupils' perceived ability – full dataset

able evidence points towards a continuum of attainment (i.e., there are no distinct high or low attaining groups) and shows that learning trajectories are not fixed (Brown et al., 2008), the mathematical identities dominating pupils' talk seemed to be predicated on a notion that some are, and some are not, mathematical.

Using Hamilton's (2002) internal beliefs and external references dichotomy for coding, there was a discernible bias in where pupils perceived mathematical ability to be located. Across the data set, pupils made 121 references to ability being an internal construct (biologically determined) and 52 references to mathematical ability being driven by external factors (such as age and experience). 81% of pupils' references at Avenue Primary, which employed strong ability-grouping, located mathematical ability internally, compared with 54% of the total references made by pupils at Parkview Primary which employed weak ability-grouping structures.

Many pupils constructed mathematical ability, mirroring societal conceptions, as something real and located within the individual rather than being an aspect of a person's developing and changeable identity. When asked what caused high or low mathematical ability pupils tended to recourse to natural variation and neurological and genetic differences. Of all pupils experiencing setting for mathematics, pupils in top sets made 80% of these links suggesting they held a stronger belief than pupils in lower groups in a biological explanation for individual differences in mathematical ability. Pupils expressed a belief that differences in mathematical ability were apparent from birth "because you are born with an ability" (Victoria, Avenue Primary, Year 4, Top Set). When asked, individually, what made someone good, or otherwise, at mathematics, pupils repeatedly talked about those who were good at mathematics as being born to be good at mathematics and vice versa:

Wynne: Their brain's bigger. And they're cleverer and better [...] I don't know, it just happens. They were born like that. They were born clever.
Zackary: Some people are just not born clever.
Yolanda: Some people are really good at maths and some people aren't that good at maths. Probably it sometimes runs in the family.
(Avenue Primary pupils, Year 4, Bottom Set)

Talking about ability differences was a natural discourse to the pupils and strong links emerged between this and the ability-grouping practices they experienced:

Uma: Cause it's like the erm, ability of what you can do, so there's like a high, there's like a top maths group, then a middle maths group then a bottom maths group
Victoria: And then you know which one is which
Uma: Because if you are like in one big maths group and you're all different abilities then there might be something too hard for like the people that need to do easy questions, and the people that need to do it hard, it would be too easy for those people
(Avenue Primary, Year 4, Top Set)

In interview, the pupils introduced the language of ability themselves; in the extract above the pupils brought the terminology in at the beginning of the interview in response to being asked to describe what happens in their mathematics lessons. For these pupils, ability, and the practices of ability, are important constituents of what mathematics is. The extract, in common with many pupil interviews, carries an unquestioned assumption that there are different types of people in terms of ability levels and these can be clearly demarcated into groups. Based on this belief in clear groups, pupils voiced an acceptance that "some people are more clever than other people" (Catherine, Parkview Primary, Year 6, Top Set). People are seen as different, with, as in society more generally, ability providing a simple explanation for individual success and failure. Pupils accepted that this was right without question. These beliefs may be influenced by ability practices:

> Natalie: Well some people are just, you know, cleverer than other children, that's what decided our groups in year 3 and it hasn't changed.
> (Avenue Primary, Year 6, Top Set)

Pupils construct an explanation that fits what they see. They hold the belief that individual differences lead directly to group placement and that group placement has not changed as differences are innate and unchangeable. Holding a fixed-ability belief appears to be self-perpetuating with pupils viewing mathematical ability as an internal force that drives, and limits, what they can do. External factors are seen as relatively inconsequential to outcomes with a belief that individuals can only take their attainment to a maximum level determined by internal limits.

Given that limits to attainment appeared to feature strongly in pupils' constructs, I asked pupils if they felt they could improve upon their current position. The responses across schools, ability-groups and year-groups were consistent and stark:

> Zackary: I think I would not move. I think I would normally stay in the same place. I don't think there's anything I could do to make myself better.
> (Avenue Primary, Year 4, Bottom Set)
> Megan: I think I could move a few centimetres further up the line, not far.
> (Avenue Primary, Year 6, Top Set)
> Peter: Just about here, not a huge way, well because you can only do so much can't you, it's quite hard.
> (Avenue Primary, Year 6, Bottom Set)

Most pupils suggested limited room for improvement. They positioned themselves within a hierarchy seen as normal and accepted the place they, their teachers and others gave them, believing they simply did not 'have' something that others did which might have allowed them to achieve more highly in mathematics. Peter's statement was not made as a question, but as an acceptance coupled with an assumed shared understanding with myself as the interviewer. Other pupils made similar comments. Whilst there were some positive statements from pupils who felt that some improvement could be made through teaching and learning in mathematics, this was tempered by the consistent underpinning theme of immutable limits:

> Researcher: Could anything help you to improve?
> Uma: Yes, if we had something like, Mr Iverson, if he explained it out a couple of times and actually came up to me in the lesson and talked it through then I would understand it a bit better.
> Researcher: Could that make you move up higher?
> Uma: No, because I have some trouble on a lot of sums with carrying over. I'm way past there in history though, but not in maths, there's this bit [≈ the top 20% of the perceived ability line] I can't get.
> (Avenue Primary, Year 4, Top Set)

Although Uma suggests that intervention from her teacher could lead to improvement, she does not see this as having an impact on her ability, which she constructs as fixed and internal. She talked about a part she would never be able to attain, even with teaching, suggesting a belief in upper boundaries. Extending this, pupils suggested that effort cannot overcome predestined limits:

> Natalie: I don't think all children can do really well in maths though
> Megan: Even if they tried really hard, even if they tried really hard
> Natalie: If they tried really hard their best might not be a 5A, but if you have lots of ability and you tried your best then you would do very well in maths. So not all children can do well. [...] If you're determined you might be better but I don't think all children, I don't think, all children can't be, well they could be okay at maths but not really brilliant, because...
> Megan: Well you could have people who had lots of ability but they just weren't trying hard enough so they were considered to be not as good but then when they try

> hard they are really good, but they have to have lots of ability.
> (Avenue Primary, Year 6, Top Set)

Natalie and Megan suggest that you can have ability and not use it but that you cannot move beyond innate ability limits; effort alone is not enough to achieve success. Such persistent fixed-ability beliefs hold implications for mathematics education.

IMPLICATIONS FOR MATHEMATICS EDUCATION

This paper is significant for those working in mathematics education, illustrating how the pupils were, as Boaler (2000) suggests, not only learning mathematics, but also learning to be a mathematician. Understanding these pupils' constructs has important implications for future research, as these foundations cannot be ignored when looking at any intervention aimed at increasing engagement or attainment.

From early on in their mathematics careers, pupils are engaged in producing understandings of mathematical-ability that are likely to be carried forward into and beyond secondary mathematics. These productions are strong in Year 4 and particularly salient in Year 6, mirroring the "evolving sense of ability identity" found in Hamilton's (2002, p. 601) secondary school study. Pupils' models of ability portray a stable concept with little plasticity. These models can be complex, drawing on multiple ways of thinking including internal and external references. However, the overriding view of mathematical-ability is as an innate, biologically determined quantity, residing within individuals in specific quantities, with limited possibility for change. Few pupils suggested they could move in terms of their mathematical attainment and those that believed they could move placed boundaries on this.

It is perhaps not surprising that pupils are holding entity-theories of mathematical ability given the clear and consistent messages to this effect they receive from teachers, parents, and the media. Previous writing has suggested that primary teachers are engaged in reproducing their own relationships with mathematics in their language and practices (Hodgen & Marks, 2009). As such it is imperative that spaces and opportunities are provided for primary teachers to engage with their perceptions of mathematical ability and disrupt 'common-sense' practices which may set-up, or perpetuate, the limits pupils impose through their constructions.

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ENDNOTES

1. See Marks (2012). The researcher would like to acknowledge the willing participation of the schools, teachers and pupils without whom this research would not have been possible. The study was funded by a studentship from the Economic and Social Research Council (award number: PTA-031-2006-00387).

2. All names in this paper are pseudonyms.

Statements and discourses about the mathematics teacher. The research subjectivation

Alex Montecino and Paola Valero

Aalborg University, Faculty of Engineering and Science, Aalborg, Denmark, montecino@learning.aau.dk

The research fabricates an image of mathematics teachers, which shape our knowledge and truths about teachers. This image sustains the development of different discursive formations. The image's configuration is entangled with spatio-temporal conditions, which are shaped by diverse social, cultural and political contexts. In this work, we are studying discourses —about the mathematics teacher that are circulating in the research— from some theoretical toolbox of Foucault (1980) and Deleuze (1994) and from the methodological toolbox of Pais and Valero (2012). We are seeking to explore how discourses are operating in the fabrication of the mathematics teacher as a subject and in the production of truths about them.

Keywords: Statement, discourses, subjectivity, dispositive, mathematics teacher.

INTRODUCTION

The research on mathematics education, focused on the mathematics teacher, has been a growing subfield of research since the 1990s. The research has shaped the constitution of diverse discursive formations around teachers. Also, from the research, certain images, truths and corresponding systems of reasoning have emerged to encourage us to think about the mathematics teacher from pedagogical topics that mainly emphasize teachers' knowledge (of mathematics, of mathematics for teaching, of pedagogy, etc.), competencies and skills. A cognitive perspective has dominated most research. The impact of socio-cultural theories in the understanding of teachers has added different views, such as identity formation and communities of practice.

The diverse studies about the mathematics teacher have attempted to set an understanding, a problematizing and a reasoning about the mathematics teacher. These studies have established ways of thinking and ways of understanding the mathematics teacher, which are configured from the confluence and the convergence of ideas, notions, interests and reasoning. Since, the discourses circulating within mathematics teacher research are shaping what is accepted or rejected and what is considered true or false about mathematics teachers.

This paper opens up a discussion that will help us to understand the configuration of different discourses that circulate about mathematics teachers and how these discourses operate the construction of knowledge, of truths about teachers. We are seeking to show how discourses are operating in the fabrication of the mathematics teacher as a subject and how is configured an ideal image of them, which produce truths, knowledge and an ideal to follow or to aspire. Hence, in this paper we develop a discursive analysis focused to understand how the research is shaping an ideal image of subject, the mathematics teacher.

To construe the mathematics teacher from the ideal image displayed in the literature will help us to position the teacher as an historical and political product, which is produced through games of power. In other words, this will help to conceive the teacher as a person subjected to diverse technologies (Foucault, 1980, 1997) and dispositive of control (Foucault, 1980), where thinking about the mathematics teacher is a dynamic idea, a theoretical construct established from diverse practices. This study is built upon three premises: (a) intentions, needs, and desires configure the conditions to set mechanisms of power, truths and discourses; (b) power produces knowledge, that is, the power is both the object and instrument of knowledge,

> "What makes power hold good, what makes it accepted is simply the fact that it doesn't only weigh on us as a force that says no, but that it traverses and produces things, it induces pleasure, forms knowledge, produces discourse' (Foucault, 1980, p. 119);

(c) the mathematics teacher is an historical and political product. The idea of the mathematics teacher — as construct — is in constant development, and this development depends on spatio-temporal conditions.

STUDYING THE RESEARCH DISCOURSES ON THE MATHEMATICS TEACHER

We consider the discourses as a group of statements with regularities in its use and enunciation (Foucault, 1980). Therefore, the discourses are not understood in terms of "a particular instance of language use —a piece of text, an utterance or linguistic performance— but describing rules, divisions and systems of a particular body of knowledge" (Arribas-Ayllon & Walkerdine, 2008, p. 99). The discourses are generated in spatio-temporal conditions. Moreover, the production of discourse is controlled, is selected and is redistributed by a number of procedures (Foucault, 1971). In every society the

> "production of discourse is [...] controlled, selected, organized and redistributed according to a certain number of procedures, whose role is to avert its power and its dangers, to cope with chance events" (Foucault, 1972, p. 216).

The discourses are established as truth through diverse dispositive of control, where regularity in the use of certain statements leads to the configuration of certain discourses, which are accepted as true and naturalised. Therefore, these discourses are not questioned and are accepted. For this work, we keep in mind that the discourses are composed of statements.

> "We shall call discourse a group of statements in so far as they belong to the same discursive formation [...discourse] is made up of a limited number of statements for which a group of conditions of existence can be defined. Discourse in this sense is not an ideal, timeless form [...] it is, from beginning to end, historical – a fragment of history [...] posing its own limits, its divisions, its transformations, the specific modes of its temporality" (Foucault, 1972, p. 117)

And, the prohibitions that are surrounding the discourse reveal its link with the desired and with power (Foucault, 1971).

Thus, we consider the discourses that are configured around mathematics teachers as a technological power, which "determine[s] the conduct of individuals and submit them to certain ends or domination" (Foucault, 1997, p. 18). Mathematics teachers are subjected to discursive formations, since, the discourses that circulate are shaping the 'must be' of teachers. This 'must be' is leading to ideas and images, which generates ways of subjectivity. Within the discourses that circulate in the research about the mathematics teacher, it is possible to see resonances and recurrences in the statements that constitute these discourses. These resonances emerge from predominant rationalities, through the entanglement between the desired and ideas and ways of thinking. The predominant rationalities lead to some statements becoming the discourses that are repeated again and again. And these discourses are shaping ways of being for teachers from what is desired by society.

All research is developed given certain intention, assumptions, ideals, and rules or notions, which are part of games of power that are configured within a social, political, and ideological contexts. The research shapes practices and knowledge through discursive formations that are set in the diversity of studies. Moreover, the research establishes networks, where it is valid or possible to enunciate determinate things about the mathematics teacher. These enunciations are the product of predominant rationalities, which lead to ways of understanding and thinking about the teacher. The subject is fabricated and configured within these networks; thus, the mathematics teacher as a subject and as an ideal image will be the result of discursive formations produced by the research.

This study is focused on the discourses that circulate about the mathematics teacher within the mathematics education research. We seek to understand two questions: (a) how discourses are configured? In other words: What were the necessary conditions to establish these discourses and their prevalence through time? And why has a particular discourse been established (predominate discourse) about the mathematics teacher and not others? (b) What are the implications of those discourses. In other words: How are discourses operating in the fabrication of the mathematics teacher and how is an ideal image of them configured, which produce truths, knowledge and rationalities, to produce an understanding about them.

ANALYTICAL STRATEGY

In our strategy we deploy concepts from the theoretical toolbox of Foucault (1971, 1972, 1980, 1997) and Deleuze (1994); our focus is mainly on discursive formations and its resonances. Hence, we will deploy a discourse analysis, which will allow us to detangle the statements, revealing the possible conditions of power effects by studying the discourses that circulate in the research. Valero (2014) argues that the mathematics education research creates language for naming study objects and ways of thinking about these objects. This language is composed of discourses, which are configuring new discursive formations and/or strengthening the diverse discursive formations that are circulating.

Therefore, in this study, we perform a Foucault-inspired discourse analysis, which seeks to ascertain the regularities and systematicities that lead to discursive formations, where the diverse statements form a rhizomatic field affecting the desired subjects within mathematics. The role of the analysis is to reveal the convergence of a complex network of discursive practices and to allow us to study the constitution and configuration of ideas or notions within diverse games of power.

With this discourse analysis enables us to view the mathematics teacher as a discursive formation, namely, as a subject immersed in discursive practices and configured from such practices. Within discourses, the images of the mathematics teacher are configured, establishing ways and networks to think about teachers, to formulate ideas about them and to validate statements.

To perform our discourse analysis, we consider —as empirical material— the 17th volume of the Journal of Mathematics Teacher Education (Journal of Mathematics Teacher Education [JMTE], 2014), composed of 6 issues; each issue has 3 or 4 papers with an introduction written by the editor. This journal was selected because it is one of the most important sources of mathematics teacher research. Moreover, the journal publishes research about mathematics teachers from diverse topics and theoretical frameworks. This analysis of recent documents is a part of historicizing of the mathematics teacher configuration. With this analysis, we are seeking to establish the shaping of the current image of the mathematics teacher from our historicity.

The focus is on statements that circulate in research and its resonances. These resonances lead to discourse formations about the mathematics teacher. When analysing the research, we took only the enunciations; we did not evaluate the researcher of the study. In our analysis as the focus is on the regularities or resonances of statements and not the people that formulated it, which led us to lose the notion of author. Rather we have quoted the journal pages because the journal provides us with the empirical material that evidences how the mathematics teacher is thought of.

We propose, in a Foucaultian sense, that discourses are generated by a spatio-temporal rationality and not by some particular people. Authors reveal the convergence of a complex network of discursive practices; hence, the discourses are not established because a person formulated them, rather because we reproduce them through discourse, "the function of an author is to characterize the existence, circulation, and operation of certain discourses within a society" (Foucault, 1977, p. 124).

Studying the diverse statements formulated in the research, it is possible to see regularities or resonances in the different arguments deployed or in the antecedents or in the conclusions of the studies. This study has allowed us build a rhizomatic web of statements that circulate, leading us to formulate categories of statements that emerge from ways of thinking and reasoning about mathematics teachers. The resonances are shaping to desired images of the mathematics teacher and lead to conditions that configure the teacher as a subject. From the diverse resonances that emerge in the research it is possible to establish two categories, these categories are clustering the discourses that emerge and circulate in research. In both categories, the mathematics teacher is configured as a discursive formation, synthesizing in him or her that which is desired and feared. The categories present the "must be" of the mathematics teacher and shapes truths about an ideal subject.

In short, these two categories emerged from: first, we selected our samples of text, the investigations reported in 17th volume of the JMTE. Subsequently, we have identified, how in our empirical material is shaping conditions of possibility, which are configuring enunciates about the mathematics teacher. But, we focused on enunciates that had resonances and recurrences. On the one hand, these resonances and recurrences have systematicity in its use and configuration; on the other hand, these resonances and recurrences show continuities and discontinuities in statements and its ideas, notions, intentions, social interest, among others. Finally, we have clustered the diverse enunciates in two categories —through an interpretative work.

TRUTHS AND KNOWLEDGE ABOUT THE MATHEMATICS TEACHER

"Truth is a discursive construction and different regimes of knowledge determine what is true and false" (Jørgensen & Phillips, 2002, p. 13). Truth is founded on systems of reason that characterize community and society; this system of reason sustains the production of knowledge and lead to the fabrication of a particular teacher subjected within a system of beliefs and ways of thinking. For example, currently it is possible to see a predominate reasoning, which prioritizes the calculation and standardization of everything (e.g., JMTE, 2014, pp. 5–36, 429–461), where the concept of the mathematics teacher forms part of this reasoning and in turn contributes to its construction. This reasoning is based in the objectivity of knowledge that has developed around mathematics teachers. More precisely, "objectivity and subjectivity are expressions of a particular historical predicament, not merely a rephrasing of some eternal complementarity between a mind and the world." (Daston & Galison, 2007, p. 379)

Moreover, from a Foucaultian approach, knowledge is conceived as a set of assumptions; these are based on the theoretical and personal experiences that emerge within a network and engage in interplay of different practices. Hence, knowledge is understood as an event, not as a universal structure, unique, absolute, or unbiased. The "knowledge is always a certain strategic relation in which man is placed. This strategic relation is what will define the effect of knowledge" (Foucault, 1970, p. 14). Knowledge is produced within different discursive practices; it cannot be conceived without a particular discursive practice and a discursive practice is defined by the knowledge itself (Foucault, 1972). In short, knowledge is composed of a series of continuities, events and discursive formations established by diverse configurations of power. Knowledge is partial and fickle in relation to its historical-political context. Therefore, far from preventing knowledge, power produces it (Foucault, 1980).

Truth and knowledge that emerge from the discourses that circulate in the research shape ideal images of mathematics teachers and configuring a subject. This ideal image is used as a framework to think about the teacher, to speak about him or her, to recognize the teacher socially, and to understand his or her practices, training, and work. Moreover, this ideal image is a product of the detangling of diverse games of power, dispositive regimes of knowledge, discursive formations, and rationality. Through this detangling, we can formulate the two statements category, where it is possible to see statements, such as:

> Many PPTs [prospective primary teachers] wanted to continue taking another mathematics course because they wanted to improve their mathematics knowledge and skills not only for themselves but also for the sake of their future students (JMTE, 2014, p. 356)

The statements that circulate about the mathematics teacher are composed of enunciations that evidence "well-intentioned" principles about the mathematics teacher. For example, it is possible to find enunciations, such as, "[the teacher should] provide students opportunity to clarify and communicate their thinking" (JMTE, 2014, p. 483). Showing the desired and feared ideas about mathematics teachers; therefore, these principles help to configure an ideal image, knowledge, practices and discourses, and moreover, help to configure the subjectivity of the mathematics teacher.

TEACHERS' KNOWLEDGE AND THEIR PASTORAL CALL

The job of the elementary school mathematics teacher (i.e., teaching), is generally regarded as a complex and demanding practice that requires a mixture of both theoretical and practical knowledge, rehearsed skills and deep understanding of children (White, Jaworski, Agudelo-Valderrama, & Gooya, 2013). In addition, the practices of mathematics teachers are configured within a network of practices and discourses, which fabricate the rational, objective, and universal subject to become the modern cosmopolitan citizen (Valero & García, 2014). Therefore, the mathematics teacher becomes an important agent for governing others, since, currently, governance is required to shape particular types of subjects. In other words, the mathematics teacher has an important role in developing and fabrication of the modern subject. In the 19th century, the narrative connected progress to economic superiority, and citizens began to develop an intelligible mathematical competence; by the end of the 20th century emerged the connection between people's mathematical qualifications and social progress (Valero, 2013); Changes in demands for skills have profound implications for the competencies which teachers need to acquire to effectively teach 21st century skills to their students.

Through an analysis of empirical research materials on the mathematics teacher, it is possible to observe how the regularities in the statements that circulate in the research are shaping ideal images of mathematics teachers. Moreover, these ideal images are setting the "must be" of the mathematics teacher; hence, these ideal images regulate the understanding of "a good teacher," and also define the knowledge, skills and qualities that the teacher should have to reflect that ideal. The regularities observed are clustered into two categories: The first category considers statements where the mathematics teacher is reduced to the knowledge and the skills that he or she has, for example, mathematical knowledge and pedagogical knowledge, among others. The second category considers statements where the mathematics teacher is thought of as a useful tool for the governance of others, to conduct others, and to conduct oneself, for example, the teacher is responsible for the fabrication of a particular subject, a rational and logical student.

Examples of these two categories of statements that emerged in the discourses of the research on the mathematics teacher are revealed below:

The first category: The mathematics teacher is reduced to his or her knowledge and skills

> They found that teachers' lack of content knowledge interfered with their judgements and that there was a mismatch between their perceptions of students' difficulties and the actual difficulties demonstrated by their students. (JMTE, 2014, p. 405)

> [Teachers need to] develop professional knowledge in support of their practice. (JMTE, 2014, p. 455)

> There is a strong correlation between the teacher's knowledge of mathematics and successful classroom practice. (JMTE, 2014, p. 373)

The second category: The mathematics teacher as agent for governing.

> Mathematics teachers play a unique role as experts who provide opportunities for students to engage in the practices of the mathematics community. (JMTE, 2014, p. 105)

> ... promoting reform, considered by many to be a major responsibility of prospective teacher preparation (JMTE, 2014, p. 295)

It is possible to enunciate that both categories emerged from the statements that circulate in studies, which are based on an ideal of perfection (ideal teacher, ideal situations, ideal practices, among others) and on the mathematics teachers' "must be" configured from an ideal image. This image is shaping from diverse forces that are part of the society and at the same time this image promote what the society desired with the mathematics teacher.

More specifically, the first category responds to objective knowledge and the importance that was given to mathematical knowledge in modern society. Mathematical knowledge is privileged knowledge and is related to progress and the societal development. The second category argues that discursive formations are favouring the fabrication and conduction of the subject toward an ideal, located in an epistemology of that which is desired, therefore, the mathematics teacher is thought of as a dispositive. Hence, mathematics teaching is thought of as a profession that has a pastoral call. For example, the mathematics teacher is believed responsible to promulgate the ideas and the ideals that mold the desired citizen by impelling to his/her students toward what is desired, aspired for, and accepted within society.

In some statements it is possible to see both categories imbricated, as for example:

> Teachers rely on established beliefs to choose pedagogical content and curriculum guidelines [...]; and teachers reflect their beliefs in their teaching, thus shaping their students' beliefs [...]. (JMTE, 2014, p. 305)

> They [effective teachers] ensure that the lesson content has a strong mathematical focus and contains opportunities for students to think, reason, communicate, reflect upon and critique mathematics. (JMTE, 2014, p. 299)

These two examples show that mathematical knowledge is not questioned; the knowledge is considered an important and sacred truth. This stance toward knowledge influences how students understand the world, which favours a type of rationality and subject.

CONCLUSIONS

The discourses reveal our historic, ideological and political framework. Because the statements express desired ideas about the subjects involved in education and the role of school mathematics to imagine a better world, they also express some truths circulating in diverse teaching practices of school mathematics and its learning (Valero & García, 2014). Moreover, we can understand the discourses as forces acting on the subject, since these discourses are promoting practices, rationalities, thoughts, assumptions and knowledge, among others, from a spatio-temporal context, all which favour the fabrication of a particular subject. Therefore, we can understand discourses about the mathematics teacher as a dispositive power.

From the review of the pedagogical research on mathematics teachers, it is possible to see that it seeks to homogenize mathematics teachers in a context where differences and particularity are predominant. Moreover, diverse research promotes a denaturing and an abstraction of mathematics teachers, ignoring aspects of the complexity of teaching, for example, about the particulars of teaching as work, the internal dynamics of the work place, etc. The denaturation and objectivity in the research promotes a rationality, which favours the thought of the mathematics teacher as a neutral and perfect subject, an established ideal image. A large number of researchers have sought to compare the mathematics teacher (subject) with this ideal image, favouring the establishment of an epistemology of the deficit knowledge. Consequently, many researchers have focused on exploring the skills and knowledge of mathematics teachers, emphasizing their deficiencies and their negatives aspects. For example:

> ...the teachers' knowledge of functional thinking was below the level expected for teaching middle-school algebra. This provides further evidence of teachers' inadequate understanding of mathematics for teaching (JMTE, 2014, p. 418)

This can generate a paradox because the ideal is defining the mathematics teacher from an unattainable idea, but when the teacher approximates this ideal, the ideal is redefined and increases the distance between the ideal and the subject (the mathematics teacher). In addition, this ideal is constantly reformulated, according to the research and the new demands of the society.

Hence, the diverse affirmations that were found in researched enunciations can establish a framework and a network for imagining the subject of the mathematics teacher, establishing statements about good practices, appropriation and management of specific knowledge, and its importance for society. For example, it is arguable that "mathematical knowledge for teaching stresses the importance of using mathematical knowledge to bring about pedagogically useful mathematical understanding" (JMTE, 2014, p. 229). Consequently, the intention of the mathematics education research is not only to show, question or analyse what mathematics teachers doing. Mathematics education research configures a disposition that shapes the subjectivity of the mathematics teacher, where the knowledge has great value and the mathematics teacher is thought of as an agent for governing others.

We can see that the statements established have been strongly influenced by objective ideas, the favouring of certain practices, and discursive formations seeking abstraction and generalization. Moreover, thinking of the mathematics teacher from a social perspective, we can see diverse discourses that establish knowledge of a useful subject that seeks to form part of the desired truth and conduct other toward the desired truth. The discourses and knowledge established from the research are subjected to the study of the mathematics teacher under a particular logic that defines the real; this logic is developing from an epistemology of ideas both desired and feared. In addition, the discourses determine what is true or false about mathematics teachers, which helps establish a particular ideal image of the teacher and a particular subject. In the words of Deleuze (1994), the subject does not pre-exist, the subject isn't reproducing the repeats that are part of the world. But rather the subject is produced by the multiples games of the real and these games are validating practices and knowledge.

ACKNOWLEDGEMENT

We would like to thank the members of the Science and Mathematics Education Research Group at Aalborg University for their comments to previous drafts of this paper. This research also makes part of the NordForsk Center of Excellence "JustEd".

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Cultural responsiveness and its role in humanizing mathematics education

Swapna Mukhopadhyay and Brian Greer

Portland State University, Graduate School of Education, Portland, USA, swapna@pdx.edu

In opposition to neoliberal forces that are furthering homogenization of mathematics education worldwide as part of globalization, we argue for the necessity of maintaining diversity in all its human forms, including in mathematics and mathematics education. Central to this position is respect is the conception of mathematics and mathematics education as human activities, inextricably embedded in forms of life.

Keywords: Diversity, cultural responsiveness, ethnomathematics.

INTRODUCTION: GEOPOLITICAL BACKGROUND

Among the most salient aspects of education in the United States and many other parts of the world are: privatization and corporatization of public education, with associated profiteering by IT and publishing corporations; homogenization that ignores all forms of diversity; as well as excessive and irrational use of high-stakes standardized tests (Apple, 2000; Picciano & Spring, 2013; Spring, 2008).

Aspects that bear particularly on mathematics education include:

— homogenization of the mathematics curriculum, reinforced by international testing

— pervasive rhetoric about the necessity of high levels of formal mathematics education, typically phrased as essential for economic competitiveness in the global marketplace

— unwarranted weight afforded to performance on tests of mathematics as a gatekeeper to educational and economic opportunities

— continuing perception of mathematics as acultural, and academic mathematics as a purely European achievement

In this paper, we present a counterposition.

MATHEMATICS AND MATHEMATICS EDUCATION AS HUMAN ACTIVITIES

We take it as axiomatic that mathematics itself, and mathematics education, are human activities, embedded in historical, cultural, social, and political contexts. Accordingly, we argue for mathematics education that valorizes diversity in all its forms, stemming from diversity in forms of life, that we term "culturally responsive mathematics education" (CRME). In opposition to the corporate goal of (mathematics) education as a means to increase human capital, we would redirect it towards increasing human capability, which Sen (1997, p. 35) defined as "the ability of human beings to lead lives they have reason to value and to enhance the substantive choices they have". In short, we argue for humanization of mathematics education.

Mathematics as cultural construction

> [...] mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context (Hersh, 1997, p. xi).

Particularly since the 1980s, the position expressed by Hersh has been articulated among teachers, scholars, and researchers who critique the sociopolitical systems of mathematics education. Central to the concerns we raise in this paper is the Ethnomathematics movement, essentially launched when Ubiratan D'Ambrosio gave a plenary talk at the International Congress on Mathematical Education in Adelaide in 1984 on "Socio-Cultural Bases for Mathematical Education", and the conference included an extra day on the theme "Mathematics Education and Society". Shortly afterwards, in another key work, Bishop (1988) framed mathematics education as enculturation and identified counting, measuring, locating, playing, designing, and explaining as mathematical activities found in essentially all cultures.

One definition is that:

> Ethnomathematics is the mathematics practiced by cultural groups, such as urban and rural communities, groups of workers, professional classes, children in a given age group, indigenous societies and so many other groups that are identified by the objectives and traditions common to these groups (D'Ambrosio, 2006, p. 1).

Thus, although much of the work in Ethnomathematics analyses mathematical aspects of practices in non-industrial societies, it also applies to cultural groups in the industrialized world. For example, carpenters in many cultures operate very efficiently, when measuring or designing, with decimal fractions or binary fractions such as $\frac{3}{4}$ and $^5/_{16}$ – they never need to calculate something like $^3/_7 + ^4/_{11}$. From this perspective, it must be considered a major weakness in mathematics education as typically practiced in many parts of the world that there is a lack of connection between what happens in schools and the lived cultural and sociopolitical experience of students, their families, and their communities, and, for most people, their future lives.

Emphatically, the statement just made in no way negates society's need for a cadre of people with mathematical expertise to provide the benefits that come from technological advances (and indeed to advance mathematics as a discipline that continues to evolve (Hersh, 2006)). What seems absurd to us is pervasive rhetoric (in which many in our own field join) about the need for *all* students to learn substantial amounts of technical mathematics, ignoring the most obvious fact about any society, namely that it relies on people filling a diversity of roles. Thus, instead of a soundbite such as "algebra for all", we would suggest a less catchy, but more real, slogan like "a great deal of algebra for a few, a lot of algebra for a larger number, and the opportunity to learn useful algebra for everyone" together with the understanding that lack of performance in formal algebra should not be a barrier to lives and careers for which it is not necessary. We also suggest that the opportunities for Internet-based learning, including courses delivered by top mathematicians, are ideally suited for the nurturing of the next generation of mathematically gifted students, who are typically self-motivated.

While the study of mathematical practices among cultural groups may be defined as the essence of Ethnomathematics, from its inception another focus has been the construction of a counter-narrative to the Eurocentric – indeed, arguably, racist (Raju, 2007) – account of the history of the development of academic mathematics, as addressed in the collection of key papers edited by Powell and Frankenstein (1997). Historians have documented the contributions of many cultures, including Indian, Chinese, and Arab, to the development of academic mathematics in Europe (Joseph, 1992; Raju, 2007). In other parts of the world, very sophisticated and elaborate systems of mathematics, astronomy, navigation, engineering, and science, were developed across millennia. D'Ambrosio (1985) pointed out that colonialism grew in a symbiotic relationship with modern science, in particular mathematics and technology, and Bishop (1990) characterized mathematics as a tool of imperialism. Mathematics was, and remains, a powerful way to convey the supposed intellectual superiority of Europeans and cultural groups deriving from them. Accordingly, the rewriting of history of mathematics is essential, not just as a matter of truth and justice, but also because the continuing belief in the intellectual inferiority of non-White people as doers of mathematics, deeply and unconsciously rooted among the colonizers and internalized by the colonized, is an obstacle to the construction of the identity of a non-White person as a doer of mathematics, for example an African or Asian immigrant in a European school.

Humanizing mathematics education

Mathematics education, like mathematics, is a human activity – indeed, even more so, given the centrality of interpersonal relationships in learning/teaching. An adequate analysis of the relationship between mathematics-as-discipline and mathematics-as-school-subject is beyond the scope of this paper; we simply state our convictions that mathematics education is not reducible to the immaculate transmission of a structured body of knowledge from experts to learners, and that mathematicians may be necessary, but are certainly not sufficient, when it comes to framing mathematics education.

Paulo Freire's observation that "education is politics" (Freire & Macedo, 1987, p. 47) applies specifically to mathematics education in many ways, with far-reaching consequences. A very important case is the extent to which so many aspects of modern life are governed by mathematical models that are often invisible to, and almost always beyond the control of, most people – what Skovsmose (2005, p. 86) termed "mathematics in action". We have argued elsewhere (Greer & Mukhopadhyay, 2012) that mathematics education predominantly fails to prepare students to become citizens with a critical disposition to understand, and agency to disrupt, misapplied mathematical models, for example in relation to economics. We would argue that this failure in mathematics education serves to protect political systems from critique. By contrast, Gutstein (2012), operating in the spirit of Freire, has shown how mathematics can become a weapon in the struggle for social justice by teaching students how it can be a tool for analyzing and then acting upon, issues of importance in their sociopolitical reality (reading and writing the world, in Freirean terms).

To raise another political theme in relation to mathematics education, there is pervasive rhetoric across the world to the effect that advanced levels of mathematics (and science) education are essential for all students for a given country's economic survival in a globally competitive world. Indeed, within the United States of America, this rhetoric is increasingly couched in terms of threats to national security. Contrast this nationalistic stance with D'Ambrosio's (2010) passionate plea that mathematicians and mathematics educators should collectively be seeking solutions to the crises facing humanity.

CULTURALLY RESPONSIVE MATHEMATICS EDUCATION

In the previous section, we argued for a conceptualization of mathematics as culturally constructed and for humanizing mathematics education, including recognition of its political roles. In this section, we consider, as an important aspect of humanization, making mathematics education culturally responsive. This theme plays out, with variations, all over the world, of which we mention just a few.

As movements of populations increase, for many reasons, children find themselves in complex intercultural life situations. By way of example, Ali (2012) presents a detailed account of a young Pakistani immigrant in Barcelona constructing her mathematical identity and planning her career in a multicultural city, in the context of several languages (Punjabi, Urdu, Catalan, Spanish, English) and having experienced greatly contrasting styles of mathematical instruction in Pakistan and Saudi Arabia before coming to a school in Barcelona in Grade 10.

As a more general example, with the ending of colonialism in its original form in many countries, liberated peoples face the issue of reconciling their cultural identity with the need to be economically competitive. Arguably the clearest example is South Africa (Graven, 2014; Vithal, Adler, & Keitel, 2005). In many other parts of the world, such as South America, Australia, and New Zealand, campaigns for the rights of Indigenous peoples include work on both mathematics education and on Ethnomathematics (Ferreira, 2015), not without many conflicts and dilemmas (Greer, 2013). A very striking manifestation of tension was manifest in a report (Atweh & Clarkson, 2001, pp. 86–87) of interchanges at a conference at which Clements (1995, p. 3) stated that "Over the past 20 years I have often had cause to reflect that it is Western educators who were responsible not only for getting their own mathematics teacher education equation wrong, but also for passing on their errors to education systems around the world". Yet, at the same conference, the president of the African Mathematical Union (Kuku, 1995, p. 407) "warned against the overemphasis on culturally oriented curriculum for developing countries that act against their ability to progress and compete in an increasingly globalized world" (Atweh & Clarkson, 2001, p. 87).

In India, activist academics have been striving to create curriculum and textbooks for elementary mathematics that "address diverse children's knowledge through a (re)humanizing pedagogy of empathy, despite the constraints of a large bureaucratic and increasingly neo-liberal state system" (Rampal, 2015). In the same context, Subramanian raises ethical issues involved in designing and developing a uniform curriculum for an educational system of such a size and with such diversity of languages and cultures.

In the United States of America, since the passing of the legislation "No Child Left Behind" in 2001, intensive use of standardized tests, combined with disaggregation of test scores by ethnicity, has led to considerable attention being focused on attempts to reduce the differences in test scores among ethnic groups, in particular to raise the test scores of Black, Latino/a, and Native American students. (The usual terminology for such efforts is "closing the achievement gaps" which is problematic for a number of reasons, including connotations of deficit models, the positioning of white students' achievements as the norm, and concerns about the nature of the tests that yield the scores.)

Attention to these aspects has been concentrated by rapid demographic changes – at the time of writing, the US Department of Educational Statistics has just projected that in the coming school year, the proportion of school students that are White (by the classification structure used) will be less than 50%. The teaching population, on the other hand, remains predominantly White, being 83% in 2007.

Against this background, the concept of culturally responsive teaching has made considerable progress within the United States in the last twenty years (Gay, 2010; Ladson-Billings, 1995; Villegas & Lucas, 2002). Contributing to its emergence, movements within critical education that have been foundational include Multicultural Education and Critical Race Theory. The term "Culturally Responsive Mathematics Education" (CRME) originated in a 2004 conference in Washington which we helped to organize, leading eventually to an edited volume (Greer, Mukhopadhyay, Powell, & Nelson-Barber, 2009).

Given that central to all education is the relationship between the student and the teacher, then the cultural and mathematical identities of both students and teachers are of paramount importance, as is the degree to which they do or do not align, particularly in terms of culture, ethnicity, and class. In the context of the United States of America, Gay (2009, p. 189) posed many key questions: "How can middle-class monolingual European-American math[ematics] teachers work better with students who are predominantly of color, attend schools in poor urban communities, and are often multilingual?". Similar questions apply in all educational settings in which there are class, ethnic, and cultural differences between students and their teachers. Gay (2009, p. 194) also stressed that teachers-in-training should examine questions such as "What is it about the way math has been socially constructed that is exclusive, rather than inclusive to culturally, racially, ethnically, and socially diverse students?". She also pointed to the fact that:

> [many students] find it difficult to see the relevance of many math concepts, principles, and operations to real life, when they are perpetually presented as decontextualized formulas and abstractions. Teachers need to be taught how to humanize mathematics, and to place these reconstructions into the lived realities of different racial, cultural, social, and ethnic groups. (p. 195, emphasis added)

There is a conception that an educational system full of human beings, with all the complexity that implies, can be treated as a black-box model controlled by crude external levers of standardized testing designed to expose schools to market forces that will automatically improve education. This conception has its epicentre in the United States of America, but the seismic effects are widespread. Quite apart from the absurdity of such a position, and the lack of any evidence that it is viable, in the context of the current paper, we point out that standardized, mass-administered tests by their nature cannot take account of the diversity of students' lives (Miller-Jones & Greer, 2009).

CONCLUDING THOUGHTS: HONORING DIVERSITY IN MATHEMATICS EDUCATION

We argue for mutual respect for the Other, through:

> [...] celebration of diversity in all its human forms, specifically in relation to mathematics and mathematics education: culture, ethnicity, gender, forms of life, worldviews, cognition, language, value systems, perceptions of what mathematics education is for (Greer, Mukhopadhyay, & Roth, 2012, p. 1).

One way to frame the argument that we are seeking to advance in this paper is to consider interactions between three families of mathematical activities, namely mathematics-as-discipline (MD), mathematics-as-school-subject (MS), and mathematics-within-culture (MC). The simplistic notion held by some mathematicians that the role of MS is simply to pass on the rudimentary contents of MD to another generation, primarily as groundwork for the reproduction of their own species, needs to be confronted because it exerts unwarranted influence on mathematics education, in our view. Rather, we believe that MC should play a central role in constructing mathematical schooling, a position aligned with the concept of "funds of knowledge" that is based on a simple premise, that "people are competent, they have knowledge, and their life experiences have given them this knowledge" (Gonzalez, Moll, & Amanti, 2005, pp. ix-x).

As argued in this paper, a clear implication of this viewpoint is that diversity in all its forms, of which diversity of forms of life may be considered the bedrock, must be fundamental to mathematical schooling. Skovsmose (2012) draws attention to the variety of sites for learning mathematics, the variety of forms of mathematics in action, and the variety of educational possibilities. With reference to the first, he points out how research discourse in our field is dominated by an unexamined stereotype of "the prototype mathematics classroom" that ignores the reality of many classrooms around the world. The second type of variety that he mentions is an effective response to the question "But isn't mathematics the same everywhere?" which is not tenable if the boundary of mathematics is drawn to include its applications (and, as Raju (2007) has pointed out is negated by the predominant reliance of Western mathematics on two-valued logic, a cultural choice). And the third is central to the argument in this paper.

While we have not enough space within this paper to address linguistic diversity, it is of the greatest importance, not simply in its own right in a world increasingly under the sway of English, but also in relation to issues in multilingual classrooms and the importance of language in framing and communicating thinking, including in mathematics.

Epistemological pluralism is another central issue, even from the perspective of mathematics-as-discipline. Pinxten, van Dooren, and Harvey (1997, pp. 174–5), citing the fundamental different epistemology of the Navajo, in particular in relation to space, point out that diversity is essential in order for evolutionary selective processes to operate in the further development of MD. They comment that "Through a systematic superimposition of the world view and thought system of the West on traditional non-Western systems of thought and action all over the world, a tremendous uniformization is taking hold... The risks we take on a worldwide scale, and the impoverishment we witness is – evolutionarily speaking – quite frightening" (p. 174).

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Newly arrived students in mathematics classrooms in Sweden

Eva Norén, Jöran Petersson, Cecilia Sträng and Petra Svensson

Stockholm University, Stockholm, Sweden, eva.noren@mnd.su.se

In this paper, we discuss how newly arrived students experience, and perform in, school mathematics. There is little research on immigrant students' initial time in Swedish school, and it is methodologically underdeveloped. Our own research will be revisited, and we give an account of the methodologies we have developed. We look for analytical tools using both qualitatively as well as quantitatively, to interpret classroom interaction, social practises, individual performance and achievement. Our attention to diversity and equity issues includes avoiding deficit discourses explaining both success and failure in school mathematics, in relation to backgrounds, language and culture.

Keywords: Newly arrived, mathematics education, methodology, foreground.

INTRODUCTION

Sweden today is a multilingual and multicultural society. Some suburban schools in the largest cities, Stockholm, Gothenburg and Malmoe have schools with up to 98 % of students speaking first languages other than Swedish. 44 % of the total amounts of immigrated students, 7–19 years of age (Statistics Sweden, 2014), are from outside Europe. Several of these countries suffer from political instability and some of them are developing countries. Newly arrived student numbers increase as the worlds' hot spots spread. The term *newly arrived students* is used in Swedish education policy and academic discourse. It defines students arriving from abroad during the time of primary or secondary school as newly arrived (Utbildningsdepartementet, 2013). In Sweden, students can be regarded as newly arrived up to four years from arrival. Resources are tied to that time span within the educational system, and offers optional teaching in both mother tongue and Swedish as a second language (Utbildningsdepartementet, 2013). The students cannot yet speak Swedish, and are entitled to a special introduction to school. The present years Arabic is the most common mother tongue spoken after Swedish (Skolverket, 2013/14).

There is little research on immigrant students' initial time in a Swedish school, and there are no studies comparing Sweden with other countries. Bunar (2010) mapped and reported research on newly arrived students in Sweden. He found the Swedish research is "scarce and theoretically and methodologically underdeveloped" (p. 6). It is impossible to draw certain conclusions and the Swedish research does not support understanding of learning conditions for newly arrived students. Bunar also states there are, to a high degree, local variations and a lack of a common educational policy, that might lead to arbitrariness and uneven quality in education. This paper is a response to Bunar's call for more research on newly arrived students in Sweden and the purpose is to discuss the question:

How do newly arrived students experience, and perform in, school mathematics in Sweden? To do this we look at data from our own research retrospectively. An underlying aim is to pay attention to diversity and equity issues in mathematics education, and to develop theoretical and methodological tools for research on newly arrived students in Swedish mathematics classrooms.

In the following section, a brief review of research on newly arrived students in Sweden is given, and thereafter our own earlier research will be summarised. The methodologies of our presented work will be indicated and then some final remarks for future research will be made.

BACKGROUND

Studies on newly arrived students that have been undertaken in Sweden are divided into mainly three areas: 1) A social and individual perspective that include mostly identity formations among adolescents, the power relations in the society, and how students integration are affected by these factors. 2) An institutional perspective that focus on transitions and on measurements of the importance of migration age for students' achievement. 3) A pedagogical perspective that focus on second language development and learning in a second language (Bunar, 2010). The studies come from a variety of academic disciplines.

Newly arrived students contribute to characterize the Swedish school with an increased diversity of languages, ethnicities, cultures, religions and nationalities. Their introductory time in school is crucial to their continued performance in the Swedish school. Introductory teaching differs greatly between different schools and we know very little about what works (OECD, 2006; Nilsson & Axelsson, 2013). Education for newly arrived students is commonly arranged in introductory classes, providing a basis in the Swedish language for later transition to mainstream classes. In order to deal with language requirements of academic school work in the language of instruction (Swedish) it is necessary to develop what Cummins (2008) refers to as Cognitive Academic Language Proficiency - CALP. That means to be able to use language in context-reduced situations such as advanced school mathematics. In a second language environment it may take about seven years to fully develop CALP skills, in the second language. Hakuta (1986) suggests that CALP skills are transferable across different languages.

Criticism from both researchers and authorities inform us that students are retained in introductory classes for too long. In fact, it is the individual student's progress that determines when s/he will be integrated into a mainstream "regular" class. The rules on how the assessment of this progress is done and by whom is unclear. And as researchers have pointed out this assessment could be disastrous for the newly arrived students' motivation to study if there are delays in their transition to mainstream classes (OECD, 2006). In their recent study, Nilsson and Axelsson (2013) analysed the social and pedagogical resources in the contexts of introductory classes and how newly arrived students experienced the time in and transition between introductory and mainstream classes. The students had arrived during the last years of lower secondary school. The result points to a tendency of allocating responsibility for newly arrived students' education solely to the introductory classes or the individual student. Nilsson and Axelsson (2013) argue that this is an insufficient praxis, and that pedagogical and social provisions also have to be made in the mainstream system in order to fulfil inclusive and educational aims.

In an introductory class where students have different backgrounds, different educational background and knowledge of the Swedish language, there are often also various forms of trauma in the picture. One important factor for integration in school is that teachers take into account the interactions and, above all, to create a supportive network around the individual student (Rodell-Olgac, 1999).

EARLIER AND ONGOING STUDIES IN SWEDISH MATHEMATICS EDUCATIONAL CONTEXTS: DRAWING FROM OUR OWN RESEARCH

Norén (2007, 2010) investigated bilingually instructed mathematics classrooms where teachers and students used Arabic and Swedish. Some of the students in her study were newly arrived and attended introductory classes except for mathematics and physical education classes. According to Norén the newly arrived students, benefited largely from the bilingual education. Knowing and using several languages opened up an opportunity for mathematics development. The bilingual teaching of mathematics favoured learning the Swedish language as Swedish mathematics text books was used and Swedish was focused for mathematical concepts.

An example from the study (Norén, 2007) was centered on Nada, a 15 year-old girl originally from Iraq, Arabic as mother tongue. According to Nada, from interviews and participating observations, when she as a newcomer arrived into Swedish school, she was immediately placed in an introductory class where the focus was on learning the Swedish language at the expense of continued learning in mathematics. Nada was self-conscious about priorities in the introductory class, saying:

> Then we worked almost nothing with math. It was just numbers (arithmetic) and plus and minus.

> No texts. We invested in Swedish. Nothing in the maths. I have never got a passed grade in mathematics, I didn't know any maths. (Nada, from interview in eighth grade)

Her account is a reflection of a strong influential discourse "Swedish only", to learn Swedish first and fast. It is an everyday opinion that it would enable her to quickly start studying mathematics in her second language Swedish (Sjögren, 1996; Runfors, 1993). When Nada started eighth grade, she was offered, and said yes to participate in the bilingual mathematics education program. In a later interview Nada said:

> I have learned more (mathematics and Swedish) /…/ Arabic makes it easier and possible to learn more /…/ Language is an important issue. /…/ When I started the bilingual program with other Arabic speaking students, when they got grades like pass and higher, I thought I could get grades as well. (Nada, from interview in ninth grade)

Heading towards the end of ninth grade Nada started to identify herself as an engaged student working hard to pass the examinations in mathematics. That was also the expectations from her mathematics teacher. In the end of the ninth school year Nada got grades in mathematics good enough to get into the upper secondary school program she wanted.

In another example Norén (2010) informs us of a group of three newly arrived students from Iraq, 15, 15 and 16 years old. When working with text problems in mathematics, they had access to dictionaries. The context in the text problems was arranged around a fishing trip. Although students with help of the dictionaries, and a voluminous workload, managed to translate the information word for word the context was shrouded in mystery. With the help of the bilingual mathematics teacher who clarified the cultural contexts, the students solved the tasks. It is not common with fishing trips in Iraq, the teacher explained, and it is not that the translations of words and sentences from Swedish into Arabic help students solve text data in mathematics. By working with text data, students in addition to working with mathematics also learn some Swedish, and they learn something about the Swedish culture.

In an ongoing study Petersson (2012, 2013) investigated newly arrived immigrant students' achievement on various mathematical tasks on a test in mathematics. The tasks are formulated as to not cause language obstacles for second language learners, and to involve algebra, statistics, proportional reasoning and negative numbers. The performances of the newly arrived students, defined as those who immigrated during school years 8–9 are compared with students who immigrated during school year 1–7, and Swedish native students. One important result is how the achievement for the different student categories was distributed over different topics in mathematics. On most tasks newly arrived and early arrived immigrant students achieved similarly and on a level of about two thirds of the level of the native students. For tasks on advanced school mathematics, such as algebra and negative numbers, the newly immigrated students performed as native students or better while the earlier immigrants performed significantly lower than both native and newly immigrated students. Petersson concludes that there is a need for research on how different student categories perform in specific topics in mathematics. That is, how achievements and various solution strategies are distributed among newly and early arrived students. It is not sufficient to draw conclusions that immigrant students in general perform lower than native students or that immigrant students perform lower in mathematics the later they immigrated, like, for example, Böhlmark (2008) does. An assumption, by Petersson, is that early arrived students have had most of their mathematics instruction in their second language Swedish, while newly arrived immigrant students have had instruction and textbooks mainly in their first language. That newly arrived students in average perform better on some tasks and native Swedish students perform better on other tasks may have to do with the fact that teaching of mathematics in countries over the world put emphasis on different mathematical topics.

Svensson (2014) examined how immigrant students experienced their possibilities to learn mathematics. This was done to broaden and critique common explanations given concerning immigrant students' unsuccessfulness with mathematics. In their narratives in which, the fifteen years old students positioned themselves in discourses that contributed to poorer performance in school mathematics than Swedish native students. The students were often confronted with not belonging to the normal i.e. Swedish students and describing their experiences of alienation and a sense of injustice. The students seemed to create themselves as individuals with limited room for manoeuver in the present and in the future as they identified themselves as belonging to a problematic group of students. One of the students in the study, Khaled from Afghanistan, had been going to a introductory class for as long as three years, only the last year of compulsory school (ninth grade) was in a regular mainstream class. He expressed the need of knowing Swedish to be able to manage mathematics instruction and the text books in Swedish, when he was talking to his fellow students in a focus group interview:

> Yes, but like this, I cannot so good language, I do not understand certain questions in math when they say it, I read it several times but I do not get it anyway, because I can not handle it. You have lived long time here, and know good Swedish so you can do it better than me. (Jo, men en del typ jag kan inte så bra språket, jag fattar inte vissa frågor på matematik när de säger det, jag läser det flera gånger men jag fattar det ändå inte för jag inte klarar det. Ni har bott länge här, och kan bra svenska så ni klarar det bättre än mig.)

Further on he says:

> It is not so easy to learn so quickly the Swedish and then change it to a maths language, to then I do not understand so much at math, some questions you do not understand so well / ... / like full sentences / ... / when I read it gets, it becomes quite strange to me so I do not understand so I skip it. (Det är inte så lätt att lära sig så snabbt svenskan och sen ändra det till mattespråket, till sen man fattar inte så mycket på matte, vissa frågor förstår man inte så bra /.../ typ hela meningar /.../ när jag läser det blir det, det blir helt konstigt för mig så förstår jag inte så hoppar över.)

Khaled tells about his experiences of being taught mathematics in Afghanistan. He does not believe that mathematics in Swedish schools is the same as in Afghanistan. He managed mathematics quite well there but not in Sweden. He says it may be because of language issues, and "I am mostly used to that, the Afghan maths one can say, what I did there". He also states that his father cannot help him with his mathematics homework because the father knows Afghan mathematics, not Swedish mathematics.

In a recently started study Norén and Sträng (in preparation) investigate how newly arrived immigrant students get access to mathematics teaching when starting school in Sweden. They will also investigate how the newcomers experience their education in Swedish mathematics classrooms. Sträng does participant observations during mathematics lessons in introductory classes. Norén interviews students, who just have been transferred to mainstream classes. Also their parents are interviewed as well as their teachers. The study is part of mapping mathematics education for newly arrived students in a Stockholm suburban municipality. There are yet no results reported from this study.

METHODOLOGICAL ISSUES

As this paper is a response to Bunar's (2010) call for more research on newly arrived students, and to develop theoretical and methodological tools, we give account of the methodologies we have developed in our research respectively. As we have concerns for equity in mathematics education we have to problematize how our research is carried out, in terms of "normality", Swedishness, and newly arrived students who not speak yet Swedish. It is about power relations as well as communication difficulties. To do so we use discourse theory in line with Foucault (2008). In classroom research, students' experience, performance and achievement analysis require a variation of research methods. So far we have used both qualitative and quantitative methods. Both may use document analysis and interviews while they typically differ in the use of classroom observations and statistical analysis of the outcome. So what are the methodological concerns when the focus is on how newly arrived immigrant students' experience, and perform in, school mathematics in Sweden?

NEWLY ARRIVED IMMIGRANT STUDENTS' EXPERIENCES OF MATHEMATICS CLASSROOMS

Part of our research is mainly qualitative, using participant observations (Hammersley & Atkinson, 2007), individual interviews, and focus group interviews (Kvale, 1997). The research is inspired by critical mathematics education research. Skovsmose (1994, 2005) has had a major impact as we are using his theoretical construct *foreground* (Norén, 2010; Svensson, 2012, 2014). According to Skovsmose (2005), a foreground represents a student' interpretation of his/her learning opportunities and life choices in relation to what the student finds acceptable in the current socio-political context, but also what a student in question perceives as available for him/her. Skovsmose (2005) writes:

> Intentions do not spring to life from nothing. They are grounded in a landscape of pre-intentions or dispositions, and I divide these into "background" and "foreground". The background of a person can be interpreted as the socially constructed network of relationships belonging to the history of the social group to which the person belongs. When one tries to understand an individual's intentions, one often refers to his or her background. But equally important is the persons' foreground. By this, I refer to those opportunities that the social situation makes to the social group to which the person belongs. Opportunities are not to be understood as sociological facts but as collectively or individually interpreted opportunities (p. 89).

Using *foreground* as an analytical tool, we have noted that students' success in school mathematics associates with the opportunities they have to positions formed in classroom practices. These practices are affected by public discourses on immigrant students and mathematics that occur on a societal level. Our focus is on the opportunities made available for the students to get involved in school mathematics, and in the long run, the hopes for the future that they will persieve through mathematics classroom practices. In line with Rodell-Olgac (1999) we argue for schools and teachers to take into account the social interactions in classrooms, and to create a supportive network around the individual newly arrived student. The contrast is what has been shown in earlier research: A focus on solely language aspects might position newly arrived students as "problems", and the answer to that will be the remedy "Swedishness" with a quest to homogenize (Sjögren, 1997; Runfors, 2003; Norén, 2010; Svensson, 2014). Students will be viewed and treated at the basis of what they lack, and what knowledge they don't have, in relation to how Swedish students the same ages are viewed and treated. The Swedish students will be the role model of what is "normal" while students whose mother tongue is not Swedish will be regarded as deviant with "weak language skills" (in Swedish). We name these aspects deficit discourses. In our research we look for other types of discourses.

Another part of our research is to compare how newly and earlier arrived students perform in different topics in mathematics. Seen as a sample for educational statistics purpose, immigrants are in many aspects a heterogeneous group. As discussed earlier they are a diverse group with respect to how recent they immigrated, their socio-economic situation, and their earlier experiences of schooling.

NEWLY ARRIVED IMMIGRANT STUDENTS' PERFORMANCE IN SCHOOL MATHEMATICS

Newly arrived students are rare. About 9 % of the students are born abroad; about 4 % are newly arrived the first school year and about 1.6 % of the 16-year-old students. The difference in percentage is mainly due to the longer stay of the older students and not due to age of immigration. This is a challenge when studying achievement data of newly arrived immigrants, since the sample must be large enough for conducting quantitative research. One example is Heesch (2000) who re-used TIMSS 1997 data for a study on Norwegian immigrants. In their study, there were few immigrants and the authors often failed to reach statistically significant results despite sometimes large differences in achievement. Secondary data may also have other limitations. For example test questions may be under secrecy and the sample may not be purposeful for the research question.

Another factor to consider is the socio-economic situation of newly arrived students. Some of them come as refugees, some with less schooling than others. For this reason there is residential area segregation due to economic resources of individual families. In larger cities with many schools, this causes school segregation since most students are enrolled in a nearby school. This indirectly has consequences for students' achievements. Schools with a high proportion of first and second-generation immigrants, also have a slightly lower ratio of qualified teachers and slightly higher ratio of turnover of employed teachers (Skolverket, 2004, p. 57). Hansson (2012) found a positive correlation between student achievement and whole class instruction in mathematics. She also found a correlation between high proportions of students' individual work and students with immigrant background or low socioeconomic status. This means that the students, who can be seen as in more need of teacher support, get less teacher support. Hansson interpreted this fact as existence of segregation in Swedish mathematics classrooms.

A challenge for quantitative studies is that random samples of entire classes needed for collecting a large enough sample of, say, newly arrived immigrants, and would be unmanageably large. An alternative to random samples is purposive sampling when designing a student sample (Cohen & Manion, 1994). Petersson (2012, 2013) used a purposive sample with the aim of decreasing the sample size needed as he selected schools with an over average proportion of first generation immigrants. There are challenges also with purposive sampling since the samples can be biased. There is a risk of comparing students from a similar background instead of comparing students that represents a national random sample.

Petersson (2012, 2013) handled the possible bias by comparing the achievements on the compulsory national test of the purposive sample with a national random sample. The latter sample was the same as was used for the evaluation of the national test.

FINAL REMARKS

This paper is a response to Bunar's (2010) call for more research on newly arrived students. Our focus on a particular school subject is important, as mathematics education has limited space in earlier research on newly arrived students. Newly arrived students must learn the new language while also having to acquire knowledge in other school subjects. For this to be possible, second language instruction has to be related to teaching in the school's all subjects. The effects of the positive impact mother tongue has for newly arrived students must not to be forgotten.

Our aim in this paper was to pay attention to diversity and equity issues in mathematics education includes avoiding deficit discourses explaining both success and failure in school mathematics, in relation to backgrounds, language and cultural issues. As shown in this paper, we look for analytical tools, qualitatively as well as quantitatively, to interpret classroom interaction, social practises, individual performance and achievement. Multiculturalism and multilingualism do not have to be constructed as obstacles to learning mathematics.

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Justifications for mathematics teaching: A case study of a mathematics teacher in collegial collaboration

Anna Pansell and Lisa Björklund Boistrup

Stockholm University, Department of Mathematics and Science Education, Stockholm, Sweden, anna.pansell@mnd.su.se

The broad interest of this paper lies in how a mathematics teacher, Mary, justifies her professional decision making. The reported study draws on aspects of a PhD project and analyses Mary's communications within a collaborative teacher meeting focused on the teaching of mathematics to grade five students. The analysis, drawing on social semiotics, highlighted the significance of artefacts, such as multiplication tests, in Mary's articulated decision making. We also give account for what is addressed in a teacher's justifications and how the teacher relates to her students in the justifications. Finally, we discuss the wider social and political context in which the teacher is working.

Keywords: Mathematics teacher, mathematics teaching, teacher collaboration, case study, social semiotics.

INTRODUCTION

This paper is about a primary school teacher specialised in mathematics and science, Mary (pseudonym). She has about fifteen years of teaching experience. Together with Mary there are three teachers teaching about 100 fifth graders and they meet alternate weeks to discuss the mathematics they will be teaching. This study focuses on one of these meetings when the group discussed the assessment of multiplication tables and the use of textbooks to differentiate teaching. We followed Marys' communication through the meeting with an interest in how she justifies her professional decision making.

It is a challenge to discuss and understand why a mathematics teacher teaches the way she does. An easy response could be that the curriculum says so; another answer might be that the text book says so. From our perspective it is, in such a discussion, essential to understand that teachers' actions are undertaken within a particular social and political context. One example of this is how for the last twenty years, Sweden has had a decentralised curriculum that has forced autonomy on teachers. This forced autonomy, which has moved teachers to the centre stage of curriculum enactment (Skott, 2004), raises the demands on mathematics teachers to make informed decisions in the very complex set of actions that constitutes mathematics teaching. As a basis for teacher decisions there are different aspects of justifications possible to construe. The aim in this paper is, from a perspective close to a teacher's, to investigate her justifications when she discusses mathematics teaching with her colleagues. We also discuss the wider social and political context in which the teacher works, while focusing on artefacts as a part of the school system. We pose the following research questions:

1) What role do artefacts play in Mary's justifications in the teacher meeting?

2) What circumstances, contents and ideas are addressed in Marys' justifications?

3) What relational aspects concerning herself as a teacher and her students are a part of Mary's justifications?

LITERATURE REVIEW

When studying teachers in their social settings, one possibility is to examine how they collaborate with colleagues. Even though teachers are autonomous they need to work in close relationship with colleagues, parents and students (Hargreaves, 1994). Findings from research about teacher collaboration indicate that collaboration within a culture where teachers engage in mathematics teaching together, have a positive impact on both teaching practices and student achievements (e.g., Honingh & Hooge, 2014). Research situated in teacher collaboration is often used to study teacher development programs, e.g. professional learning communities (Riveros, 2012), communities of inquiry (Goodchild, 2014) or learning studies, which are common in Sweden today (e.g., Kullberg, Runesson, & Mårtensson, 2014).

Teacher collaboration research is rarely situated in a naturalistic setting, with a collaboration that would have taken place whether the researcher was there or not. A review of CERME proceedings 6, 7 and 8 reveal only a few papers on teacher collaboration (e.g., Spencer & Edwards, 2011) and none situated in an everyday situation. If research is situated in a development program it could be difficult to distinguish if the positive effects come from the collaboration or if it is from being in a development program. Here we can see a need for more studies on teacher's everyday life, also teacher collaboration.

Studies concerning mathematics teachers and their practices are diverse. In literature relevant for this study we can see that different countries have different school-cultures. Cross-nationally, students receive different numbers of mathematics lessons of different length, but within countries there is considerable consistency in how mathematics teaching is executed (Andrews & Sayers, 2005). This influence of tradition and school culture on mathematics teaching is, for example, described in the way teachers engage with and use curriculum materials (e.g., Remillard, 2005). As a part of a school tradition, mathematics curricular materials such as the mathematics text-book are said to have a prominent role and bear traditional forms of discourse (e.g., Johansson, 2006; Herbel-Eisenmann, 2007). Teachers deal with a complexity including the impact of culture, tradition, curricular materials and the competing objects and motives created by teachers and students (e.g., Skott, 2001).

METHODOLOGY 1: ANALYTICAL PROCESS

We have divided the methodology into two parts. Here we describe the interplay between preliminary data analysis and theoretical considerations which led to the analytical framework that we finally exploited. In another part, below, we describe more about the research design and data collections.

In order to acquaint ourselves with the data (transcripts from teacher meetings), we undertook several constant comparison readings, inspired by Glaser & Strauss (1967), with the purpose of identifying recurrent themes focused on a teacher's professional justifications. Our interest was directed only towards those justifications that were possible to observe in Marys' utterances and not those that Mary may have kept to herself. In the initial readings three major themes emerged. One related to artefacts as part of the school system, for example multiplication tables and the role they played in the teachers' discussion. Another was "what" Mary focused on in her justifications and a third concerned interpersonal aspects such as how Mary related to her students in her justifications.

In other words, during our initial analyses we identified themes with a conceptual similarity to those of social semiotics (Halliday, 2004; Van Leeuwen, 2005, Morgan, 2006). Consequently, refocusing our analyses around functions from this theory, provided a basis for new rounds of more focused readings.

ANALYTICAL CONCEPTS – SOCIAL SEMIOTICS

There are three social semiotic meta-functions that suited our preliminary analysis: textual, ideational and interpersonal. In this paper we adopt the meta-functions mainly according to a multimodal approach (Van Leeuwen, 2005; see also Björklund Boistrup & Selander, 2009). The meta-functions are not independent, they constitute each other.

The *textual* meta-function is, in this paper, understood according to the roles different communicative resources (such as artefacts, speech, voice, gestures and the like) play in communication, (Van Leeuwen, 2005). Communicative resources then constitute texts in a broader sense than only taking language into account. In this paper our interest is focused on what roles artefacts, for example textbooks, play in Mary's communication.

The *ideational* meta-function is used to reflect the explicit content of the communication under scrutiny (Herbel-Eisenmann & Otten, 2011). In this paper the ideational meta-function is used to discern what Mary focuses on and addresses in her observable justifications.

The *interpersonal* meta-function can be used to describe interactions, roles and relations between people "who are the participants in the interaction /.../ what relationships do they have to each other and to subject matter" (Morgan, 2006, p. 229). In this paper this meta-function is mainly adopted to construe how Mary relates to the students in her justifications.

METHODOLOGY 2: RESEARCH DESIGN AND DATA COLLECTION

Trying to understand a case, in this case when one teacher is studied, within the social setting and investigate it with depth is how case studies are used (Hammersley & Gomm, 2009). Consequently, we do not seek to offer material for generalisation but to provide a description of a part of one teacher's reality, and the result will widen our experience of mathematics teachers and these kinds of situations which could be seen as an alternative to generalisability in case study research (Donmoyer, 2009).

In this paper the collaboration between Mary and her colleagues served as the social setting where communications on mathematics teaching could be observed. We followed teacher meetings when Mary and her colleagues discussed different problems and possibilities concerning their mathematics teaching. Here we concentrate on one of the teacher meetings, although data on eight other meetings served as a background, and helped our understanding of this particular meeting. In order to capture mathematics teachers' collaborative discussions, audio recordings were made and transcribed directly in the software Videograph [1]. The same software was used to assign codes in terms of the research interest.

ANALYSIS AND FINDINGS

We present two excerpts, followed by our summary analysis, from the teacher meeting between Mary and her colleagues: Tomas, educated in advanced mathematics, but not in education; Peter, an experienced primary school mathematics teacher like Mary; and Sara, a primary school mathematics teacher a little bit less experienced than Mary and a new teacher at the school. The described excerpts serve as to illustrate the various features identified from the teacher meeting. The analysis draws on the three meta-functions which also are reflected in the research questions.

Episode 1: Discussion of a multiplication test

In the beginning of the teacher meeting a new teacher, Sara, was invited to describe what she had done during the previous week. Tomas, the official leader of the group, stated that one of Sara's teaching actions has been "really good" and referred the group to a times tables test comprising items such as 6 x 7, (the "really good" refers to the teacher having the test and not to the result of the test). In the following we present an episode from the meeting. To make the analysis as transparent as possible, some comments have been added to the excerpt.

1	Sara:	Yes, and then I did the multiplication test in the afternoon
/.../		[The teacher group talk about the result of the test which was not satisfactory]
2	Peter:	How many did you do? Five?
3	Sara:	Five minutes
4	Peter:	Hundred exercises?
5	Tomas:	How many exercises?
6	Sara:	Hundred and twenty exercises
/.../		[They talk about the test and other topics for about ten minutes]
7	Tomas:	Er, but then we have one or two in one class that thinks maths is like the plague and really hard, and that always feels like, and that has its grounds in this multiplication and then it was the worst anxiety attack and tears fell and it, it is really tough.
8	Mary:	Arr! [Said with a voice construed as compassionate]
/.../		[About five minutes of discussion on other topics]
9	Mary:	I am sitting here thinking about your test, 120 exercises in 5 minute. Is that reasonable is it a lot or little or is it...
10	Sofia:	That is reasonable! I had students that did it in three minutes
11	Mary:	Ah, well then
12	Sofia:	Two and a half minutes, if you know them it is there...
13	Mary:	Then it is...
14	Peter:	Yes
15	Sofia:	When you see the problem you know that it is, you don't have to.
16	Mary:	You don't have to figure something out no that is good...

17 Tomas: I can agree, if it was a long ago since last time, you could get seven, eight minutes the first time.
18 Mary: Yes
19 Tomas: But the thing you test is if they have automated it if they know...
20 Peter: Hmm
21 Tomas: Not that they will sit there and think five times five let's see that is five that is ten that is fifteen that is twenty, twenty five...
22 Mary: No, that's not, you don't have time for that...
23 Tomas: They should know that five times five is twenty five.
24 Mary: Hmm

Analysis of Episode 1

We view the test as an artefact and thus part of the multimodal "text" (Van Leeuwen, 2005). In this *textual* analysis we adopted social semiotics as "the way people use semiotic 'resources' both to produce communicative artefacts and events and to interpret them [...] in the context of specific social situations and practices" (ibid, p. xi). Our analysis revealed that, for Mary, the test played the role of reflecting the students' knowledge of automated multiplication skills. As the teachers discussed the test they talked about how the test was organised (line 2–6 and 9–14), and what the students need to know (line 12, 15–16, 19–24). We can also see the test having the role of a tradition keeper. Peter, who asked the question about how Sara did the test (line 2), almost said the answer together with Sara. This idea of having about 100 exercises in five minutes' was only questioned by Mary, for example, in line 9. No one offered an alternative or questioned the organisation of it. This way of doing this test seemed to be taken for granted. Drawing on Björklund Boistrup (2015), we argue that artefacts like this test, and how it is executed, are part of the assessment tradition within Swedish mathematics education. Looking at old Swedish text books, similar tests are found in teacher guides, so it can be inferred that this is part of a long term Swedish mathematics teaching tradition. When Mary tried to challenge the role of this test, she had not only her colleagues to argue against, she had a whole testing tradition to speak against.

We adopted the *ideational* meta-function (Herbel-Eisenmann & Otten, 2011) to analyse "what is being talked about or the specific content of the interaction". In this case, the content matter of Mary's justifications was knowledge concerning the automated recall of the multiplication tables. We construed this from the episode when Mary (line 16) said that it was good when the students do not have to calculate to know how much a simple multiplication is. In line 22 she agreed with Tomas that the test situation did not give time for calculation, and this was interpreted as Mary viewing it to be important when testing automated knowledge. Mary also agreed with Tomas in line 24 that the students should know five times five. Our analysis identified that Mary justified the test with the argument that the automated knowledge of the times tables is important mathematical knowledge. This was also present when the teachers talked, in another episode not discussed here, where Tomas said "they understand how you should calculate the more difficult exercises but since they don't know the times table it turns out wrong any way". On this occasion, Mary agreed with an "mm". This notion of 'important knowledge' was one of justification for Mary's support of the test.

We also identified an *ideational* aspect from Mary's justifications in an episode before this one, where Mary displayed resistance to the test. She then justified her questions and proposals drawing on the matter of how to organise classroom work in mathematics teaching. She stated that testing multiplication tables could take too much time from her lessons, saying "I do that in third and fourth grade but after that I don't want to use time from my lessons on this". Here we construed a hesitation in Mary's communication about the importance of doing the test.

Looking at *interpersonal* aspects in Mary's justifications, we were inspired by Morgan's (2006) description of the interpersonal meta-function as bringing forth relational aspects as well as "meanings, including the possibilities for emotional experiences" (p. 224). We then focused on how Mary's justifications in this episode concerned her relations with and emotions towards her students. Beginning at line 9, we identified a conclusion in Mary's question; that this test might not be for all students. A care for students is construed in Mary's justifications for resisting the test. Throughout the meeting Mary expressed a care for "low performing" students in relation to the test, for example her response to Tomas statement in line 7. Our understanding from this analysis is that the justification for Mary's resistance to the test came from this care for the students.

Episode 2: Discussion of mathematics textbooks

The next episode is when the teachers discuss text books. In this case the discussion concerns what text books to have available for the students.

1	Mary:	Yes, I will probably have Tom, he is on chapter four now, he will finish it soon...
2	Tomas:	Yes, then we have the book for grade six
3	Mary:	But then we have it so I don't have to order...
/.../		[A few minutes discussion on what books to order and not]
4	Tomas:	/.../ Peter and I agreed on, what we see as a wise thing. Those who work with more advanced mathematics so to say than grade five, they need two books. One for grade five and one for grade six /.../ they will do a test on this section to show, I can do this. If you showed that you can you can work in the grade six book but if you can't, then you need the teaching for grade five /.../ if it is too easy, you can show me that you can
5	Mary:	That's right
6	Tomas:	/.../ if they show us what they can do we should not hold them back.
7	Mary:	No

Analysis of episode 2

In our analysis we construed the role of the text book as a differentiating artefact. Mary agreed with Thomas, when he argued that if the students have a text book relevant for the "own level" they will have mathematics teaching suitable for their knowledge (line 1–2 and 4–7). This idea of using the text book as the solution for differentiated teaching says something about the position the text book holds.

Again we adopted the ideational meta-function to help us identify the idea that "high achieving" children need differentiated mathematics teaching. We construed that the teachers wanted to achieve a teaching suitable for these students through offering them more advanced text books. The teachers all agreed on the textbook as differentiator, as we can see when Tomas and Mary discussed this in lines 1–7. Tomas and Peter also had a special solution in line 4, concerning how to organise who get access to the more advanced text book and who does not. Mary justified her support for the idea when she expressed the need for one of these text books in her own classroom (line 1–3) and when she agreed with Tomas' idea in line 5. This is also construed from line 7 when Mary agreed that the students should not be "held back".

Looking at the interpersonal aspects of Mary's justifications, we can see how Mary agreed with Tomas in line 5, when he stated that the students could be trusted to take the responsibility to show what they can do before they got a text book different from the rest of the class. Here we construed that Mary related to her students with trust.

Another interpersonal aspect is construed when Mary, in line 1, made sure that there was a book for one of her students who would need it very soon. Here we identified that Mary related to her student with care, but, when compared to the case of multiplication tests, this time the care was for the high achieving students and the purpose was to challenge, not to be careful with.

CONCLUSIONS

In these two episodes, two artefacts were identified as indicators of Mary's professional justifications: the multiplication test and the text book. In the analysis, we construed the multiplication test as playing two roles, a reflector of students' skills as well as a tradition keeper, while the text book played the role of a differentiator.

Ideational aspects were identified in Mary's justifications. When she argued for and against the multiplication test, we identified in our analysis how she argued that automated recall skills of the multiplication table is important mathematical knowledge. We also identified how she wanted to allocate time for mathematics teaching in the sense of a communicative practice and that this test was taking too much of teaching time. Discussing the mathematics text book Mary justified the need for different books with the experience that her students would need a certain book and with the idea that "high achieving" children need a differentiated mathematics teaching.

We also read in our analysis how Mary related to her students in her arguments. In both these discussions Mary related to different students with care when she expressed how the students were in some kind of need. In the first episode she addressed "low achieving" students and the fear she had that they would suffer from failing at the test. In the second episode she addressed the "high achieving" students, whom she wanted to provide with the "right" kind of teaching. Mary also related to her students with trust when she agreed that they could be trusted to show that they could do the exercises in the ordinary text book before leaving it for a more advanced one.

To sum up, we have seen a variety of roles, ideas, experiences and relations in Mary's justifications. The result from this small study makes it clear to us that a mathematics teacher has a very complex broader context to take into account. The roles of the artefacts, such as tradition keepers and time savers, say something of the strong position they have. Discussing them, Mary also dealt with content-related issues and her students, in terms both of what teaching they needed and what they needed emotionally. All this was visible when Mary justified her views on mathematics teaching, negotiating with her colleagues.

DISCUSSION

The three meta-functions that underpinned the analytical process facilitated a diverse description and understanding of the data. They provided ideas to view the collaborative communication in different ways with and different aspects emerged.

In the case of the multiplication test as artefact, we described Mary's support of the test mainly in relation to an ideational aspect, the important knowledge, while her resistance appeared mainly in a relational aspect, the care for "low performing" students. This contradiction came close to what Skott (2001) describes as competing objects and motives. Something interesting here is that the importance of this knowledge wins over the care for the students, since Mary both questioned the test and seemed convinced that since they were testing automated recall it was okay. In the case of the text book as a differentiator there was support both from ideational and relational aspects which made the teacher group unanimous.

Looking at both the text book and the multiplication test with an interest in the socio-political context in which a mathematics classroom is immersed, we can see tradition (Remillard, 2005; Björklund Boistrup, 2015; Herbel-Eisenmann, 2007) shine through. The test has a role as a tradition keeper since it is a part of the school tradition. The discussion of the text book also offers an interesting perspective, being focused on differentiated teaching. It still shows the strong position the text book holds when the teacher group unanimous justifies this idea. There is no doubt that the relation between Mary and curricular materials (text books) is very complex, and it would be interesting to see deeper analysis of the role of curricular materials in relation to tradition connected to teachers' justifications.

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ENDNOTE

1. http://www.dervideograph.de/enhtmStart.html

Dialogues in ethnomathematics

Aldo Ivan Parra-Sánchez

Aalborg University, Denmark, Aalborg, aldo@learning.aau.dk

The relationship between researchers and researched communities in the ethnomathematical field has been conceived and developed broadly and deserves a problematization. A revision of previous ethnomathematical research is presented, finding how theoretical and methodological assumptions are related with leading notions of ethnomathematics, as dialogue, communication, respect and otherness.

Keywords: Ethnomathematics, ethnography, symmetry.

INTRO

Ethnomathematics emerged in the 80's within Mathematics Education, as a program of research in the History and Philosophy of Mathematics that seeks to understand mathematical knowing and doing throughout the history of humanity. It does so by studying "mathematics practiced by groups, such as urban and rural communities, groups of workers, professional classes, children in a given age group, indigenous societies, and so many other groups identified by objectives and common traditions" (D'Ambrosio, 2006, p. 1). Nowadays, ethnomathematics is a well-established field of research with a worldwide community of practitioners, most of them exploring implications in Education, using anthropological perspectives.

The field was based since their origins on a critical attitude that promotes the emancipation and equality of discriminated groups (Powell & Frankenstein, 1997). As two of the main leaders of the field assert:

> Ethnomathematics fits into reflection about de-colonization and the search for real possibilities of access for the subordinated, the marginalized, and the outcast, or excluded. The most promising strategy for education in societies that are in transition from subordination to autonomy is to restore dignity to their individuals, recognizing and respecting their roots (D'Ambrosio, 2006, p. 30).
>
> In other words, ethnomathematical studies may broaden the (intercultural) understanding of what are mathematics, of what are mathematical ideas and activities. There cannot be a sole, unified view of mathematics. For a monolithic and dominant view there is no basis. At the same time, for the other extreme, a cultural relativism concerning mathematics, there is also no ground: intercultural intelligibility seems possible (Gerdes, 2001).

As ethnomathematical research pursues the aim of valorising new understandings and recognitions, there is a strong need for the development of ways to relate and interact with communities. Therefore communication arises as an important issue. Some researchers consider it explicitly, with notions such as dialogue (Vergani, 2007), respect (Powell & Frankenstein, 2006) or mutuality (Adam, Alangui, & Barton, 2010) while others deal with it in a tacit way. Indeed, all of them make methodological decisions about the participation of the community and how that participation will be considered and registered in an academic diffusion of the research.

In an early attempt, S. Ferreira (1994) proposed ethnography as the proper and natural methodology for the growing discipline. To subscribe to that proposition has implications on how to conduct the relationship between research, researched and researcher. However, with the development of the field, several other methodological approaches have been attempted, including culturally responsive action and participatory action research (Mukhopadhyay & Roth, 2012). In addition, several ethnographical approaches were developed in the field of ethnomathematics.

This paper focuses on ethnomathematical research developed in, or with, living communities. The main goal is to problematize how that type of research has been devising the roles of communities and researchers, and contrast those enacted roles with well-known aims of ethnomathematics. To do so, a theoretical positioning is presented, introducing the concept of symmetry; then, an analytical strategy is proposed for a review of previous ethnomathematical studies, looking for some regularity within theoretical or methodological decisions. Some theoretical remarks are presented at the end.

THEORETICAL FRAMEWORK

As many practitioners adhere, ethnomathematics involves a program of research in the history and epistemology of mathematics (D'Ambrosio, 2006). For this analysis, epistemology of mathematics is not considered as a regular thing that exists in a void, but as a system of practices working in concrete places, where actual subjects make assertions. In that sense, for this approach, ethnomathematics has to cover not only an examination of the knowledge and practices that count as mathematics, but also encompass the procedures and instances of enunciation that allow to typify knowledge as mathematical.

This approach intends to give new insights to the recurrent discussion about the "ethnomathematical paradox" pointed by Millroy (1992). In that dilemma, it is considered problematic to use the "lenses of a mathematical observer" to analyse practices that are not originally based on Western mathematics. Important critiques and counter-critiques to ethnomathematics have debated that issue (Knijnik, 1996, pp. 84, 75; Pais, 2013) expressing concerns about the constraints and contingencies of the academic observation. Nonetheless, the debate could be reduced to a claim for reflexivity: "check your lenses", "research your own research" or "be aware that your standpoint" that does not alter the relation between observer and observed. However, building a broader vision of mathematics requires going beyond reflexivity, especially when living communities are researched. Such move towards a broader series of concerns is what I would call the problem of *symmetry*.

The problematic issue with ethnomathematics might not only be the use of mathematical lenses by the outsider researcher, but also to assume that insiders do not have their own ways to observe and analyse practices. Or even more, to assume that it is not possible to establish interchanges or interactions between those approaches. It becomes central to problematize how the involved subjects in the ethnomathematical field (both scholars and practitioners) position themselves with respect to each other trough the practices of research, recognizing or neglecting degrees of legitimacy that, therefore, allow or constrain the presence of multiple voices in the research. I am arguing that, instead of limiting the problem to reflexivity, ethnomathematics research can be problematized around the issue of *symmetry*. Symmetry as a problem of research in ethnomathematics is the study of how the participation of both researchers and practitioners have been conceived and expressed in research. In simple words, the problem of symmetry is the recognition that "the other has his own lenses" and that "researched people can research too".

Anthropology and sociology have explored several alternatives in the fieldwork to face similar problems, with approaches like collaborative ethnography, participatory action research, autoethnography, public ethnography and others (Denzin & Lincoln, 2011). All of them respond in different ways to the challenge that Woolgar identifies in the work of James Clifford, about a "dispersal of ethnographic authority in the sense that both researchers and natives be recognized as active creators (authors) of cultural representations" (Woolgar, 1988, p. 25).

A classic assumption of ethnographic fieldwork is encapsulated in the phrase "being there, writing here", demarcating a clear division between places, subjects and roles. In this paper I endorse a non-colonial posture about power and knowledge that questions such assumption, acknowledging the rights and the capacities to theorize that communities and subjects have. Elsewhere I have pointed:

> [Individuals and people] do not only have various types of knowledge, but also have the capability to disseminate them, broaden them and contrast them with the knowledge of others. Indeed, people and individuals have the power to define how, when and where their knowledge can circulate. (Parra, 2011)

Symmetry is related with the notion of infra-reflexivity that Bruno Latour proposed for studies in Sociology of Scientific Knowledge, trying to achieve "equal status for those who explain and those who are explained"

(Latour, 1988, p. 175) through an engaged interaction in each stage of the research process. I am aware that the word symmetry can raise many doubts because it can be easily associated with a supposedly friendly ideal horizontality that is often not the case. If we consider the complex power relationships involved in the very act of research, "equal status" has to be understood as a way to go, instead of a place to arrive. Notwithstanding that, I propose the notion of symmetry to stress the presence of practitioners as actors with voice and agency. Additionally, symmetry can subsume notions like dialogue, respect and mutuality, which are used by ethnomathematicians with very diverse meanings, when they explore possibilities of partnership, collaboration and reciprocity between scholars and practitioners in the ethnomathematical observations and representations.

In this paper, I assume that a singular piece of research does not express only the personal intentions of its direct developers [1], but also a shared set of truths and beliefs, that includes rules to follow, practices to perform, words to say and ideas to organize and explain within the research field. That set comprises what can be accepted in a specific space-time setting as a research experience.

Consequently, to scrutinize the understandings about symmetry within a particular ethnomathematical research project, it is necessary to observe this project as a whole, This includes an analysis of how symmetry is addressed in the methodology, in the theoretical approaches invoked, and also in the process of drawing conclusions. Moreover, whenever possible, subsequent activities of researchers and communities after the publication of the experience also should be observed because they allow to study the positioning of each agent about symmetry. Procedures are described in the next section.

ANALYTICAL STRATEGY

The empirical material for this paper will be a selection of ethnomathematical research projects that were reported in PhD theses or peer-review journal papers. The selected reports vary broadly in the date, language and place in which they were developed and in the places that their results were published in order to reinforce the fact that the main focus is to study assumptions that circulate within the field and not on particular researchers or research centers.

The goal is to trace some features, present or not, that allow to establish at what extent the interplay with the researched communities has been developed. For that reason it is important to elucidate the comprehensions that previous ethnomathematical research developed about symmetry, when they faced the problem. For analytical purposes, clusters of questions have been defined following a simple division in: inputs, actions and outputs. Research stages are distributed in the three clusters.

Inputs: this cluster covers justifications, antecedents, goals and theoretical positioning. It is important to observe the motivation to undertake the research; whether the community demanded the research beforehand, as part of their own interests and concerns, or whether the motives follow mainly the researchers' interest and how they found an agreement, if any. How the researchers have met the community or group, as well as if the goals and the objectives for the research explicitly mention some benefits to the studied community, even if it is in a long-term, is illustrative of the motives. It is further intended to trace symmetry into the theoretical framework exposed by the researchers, searching if they have used explicitly a definition of dialogue, mutuality, reciprocity or respect, or in some way made theoretical considerations about their interaction with the people being researched.

Actions: concerns about methodological and analytical strategies are treated in this cluster; In order to asses in which conditions the interaction was conducted, it is relevant to analyze the applied procedures to collect data, for example, if there was an immersion or a sequence of visits or for how long interaction was undertaken. In which roles were the members of the community and practitioners involved? In what stages of the research process did they contribute? These questions attempt to grasp the type of communication that researchers establish with the researched people. The attention provided to collaboration, reciprocity and partnership, even through any kind of feedback or reverting action, will be examined also in the ways that the selected projects analyzed and discussed their data.

Outputs: this cluster comprehends the research outcomes, including not only the conclusions and recommendations that the researchers can draw in a published text. Issues of intellectual property, applicability and uses of the research within the group can be considered. Long-term or short-term experiences can develop different kinds of partnership and dialogue in the aftermath of a published research.

PRELIMINARY FINDINGS

For the scope of this conference paper, I will only present the first results of literature review, covering mainly the first two clusters of questions, leaving for further study considerations about the continuity of the relationship between researchers and groups after a publication of the research.

For the first cluster, motivations for an ethnomathematical research can be inquired, suggesting that collaboration and partnership are strongly connected to the preexistence of a sociopolitical project providing a frame to the research, as we can see in experiences like (Knijnik, 1996, 2007), invited to participate in the Educational Literacy Project of the Landless movement in Brazil, or (Gerdes, 2007, pp. 227–256) commissioned by the government of Mozambique to educate mathematics teachers. Also the work of Meaney, Trinick, and Fairhall (2012) is embedded in the educational process of the Maorian people in New Zealand. In those three cases, researchers had in common the necessity to articulate their theoretical interests within a non-academic process that surrounds the research and demands some sort of result from the researchers. This does not imply that researchers were employees or militants of those processes but that they have to interact with them, and that a part of the projects' theoretical concerns were related with those non-academic processes.

Cases like those contrast with other experiences conducted for the pure sake of a theoretical interest, like the PhD project "Interpretacion matematica situada de una práctica artesanal" carried out by Albertí Palmer (2007), who worked in Indonesia, seeking to develop a method to identify mathematics in a practice, founding on the identification of mathematics in the Torajan architectural ornamentation, or a project carried out by Rohrer (2010), who interviewed sculptors in Mozambique, trying to solve questions like "Which mathematical means and tools do Makonde artists use and which mathematical knowledge can be revealed in their practice?"(Rohrer, 2010, p. 15) "Is it possible to find golden sections within their final sculptures, even though there might be no prior knowledge of this term as such?" (Rohrer, 2010, p. 149). Those questions where part of her doctoral thesis, which attempts to give resources for a theorization of ethnomathematics as an interdisciplinary theory.

Motivations in the projects of Albertí Palmer and Rohrer appear to act completely outside of the interests of the researched groups because they are devoted to register some specific practice or notion present within the community that can be identified by the researcher as mathematical, aiming to expand the idea of mathematics, from a culturally-free practice formalized by western rationality, to a human and contextualized one. However, such expansions happen inside a western academic context in the spirit of "Nothing better to show a new view of this Science [Mathematics] than observe how other societies are building it" (S. Ferreira, 1994, p. 94, my translation).

Moving to the theoretical positioning, S. Ferreira commented first about the relationship between researchers and researched groups, when he contributed to theorize ethnomathematics using modelling: "I can not dismiss ethnography, ethnology, validation and, mainly, *the action of return the model´s outcomes*" (S. Ferreira, 1991, italics added). Some years after he proposed a cycle based on those ideas, conceiving ethnomathematics as a pedagogical model, where the fieldwork involves an action over the community.

> In my view, *every ethnographic research has to have, by necessity, a return of their results* to the communities who are being researched. This proposal of a return is one of the *indispensable* actions in the process. It is up to the community to decide whether accept it or not. (S. Ferreira, 2004, italics added, my translation)

By addressing educational possibilities of ethnomathematics, authors like Lübeck and Rodrigues (2013), Vergani (2007) and Oliveira (2013) use explicitly the term "dialogue", invoking Paulo Freire`s ideas to guide the encounter between different people in a respectful way. Mutuality becomes central to ethnomathematical research with Alangui's (Adam et al., 2010) work:

> Mutual interrogation is the process of setting up two systems of knowledge in parallel to each other in order to illuminate their similarities and differences, and to explore the potential of enhancing and transforming each other. In the

> context of ethnomathematical research, mutual interrogation is a process facilitated by the ethnomathematician – the researcher (p. 11).

Mutual interrogation locates ethnomathematician as an intercultural speaker that gathers different worldviews, transmitting questions and answers among practitioners of different traditions of knowledge. This approach goes beyond classic ethnography, because the parties involved become researchers as well, blurring the borders among "there" and "here".

In the second cluster of questions, the amount of time invested in the fieldwork with the communities is considered a key issue to unfold questions around symmetry. Usually, when a research for a PhD thesis adopts ethnography as a methodology, the common option is to give a short full immersion of a couple of months to collect data, like the studies by Rohrer (2010, as discussed above) or Millroy (1992). In those cases the researcher did not know the members of the community before she entered the field and she had to elucidate which member could provide the best information. Then she tried to establish as soon as possible a communication with those special members. Inferences are built using only one data collection, and other resources come from previous academic literature.

It is less common in a Ph.D. thesis to make a sequence of visits along an extended period, like (Knijnik, 1996) or (Silva, 2013), who worked regularly with their studied groups before they accomplished their doctoral studies. This allowed them to interact with communities in other ways, to participate in community's special events, and to know a wider range of members of the community, obtaining contradictory and complementary data in each visit. These two experiences have transcended the scope of a PhD study and developed further books and papers

Far from classic ethnographical approaches, continuous long-lasting experiences made possible a deeper interaction between communities and researcher; projects like Urban Boundaries in Portugal (Mesquita, 2010), MACIMISER in Micronesia (Dawson, 2013), or the already mentioned works of Paulus Gerdes in Mozambique, as well Tony Trinick and colleagues in New Zealand. Those cases permanently collect data, create small projects framed within a bigger research process, release multiple publications and occasionally create re-elaborations of previous findings as a result of the continuity in the communication between researchers and researched. In these projects, community members can hardly be considered "informants" in a simplistic sense. They become another researcher in special stages of the experience, discussing and assessing questions, procedures or methodologies with the academic researchers. However, there is only one reported case (Caicedo et al., 2009) in which a group participated in the design and implementation of all stages of the research, even in the search for funding. This preliminary result does not deny the existence of some kind of reciprocity or collaboration in brief ethnographic experiences, but certainly confirms a basic idea: dialogue requires time, persistence and patience to be developed.

The third cluster of questions demands to seek for tensions outside texts, like the research impact. As announced before, few considerations resulting from the literature review are presented here. Even though it is a very uncertain question to identify what kind of outcomes can be considered as a contribution from the perspective of the researched group, collaboration can be traced in the multiplicity of products resulting in a research that can be used by community members: textbooks, booklets, portfolios, videos, and other resources.

Research in doctoral theses is often driven by its intention to answer a theoretical question. Very seldom does it create the opportunity to report implications of the research with the community, or at least to discuss the published thesis. The educational models produced in doctoral theses leave the possibility to be implemented to the community and remain on a level of proposals. This is consistent with the approach of Sebastiani quoted above.

Long-term research has to deal with several challenges, like the teamwork continuity, the funds for the research and the perception of the community about the presence of the researcher. Interactions and dialogues are developed between people defending their interests and immersed in power relationships. Questions about the social relevance of the research can appear after some years of continuous work: members of the community ask for the benefits that they can or cannot receive for their participation. Additionally, researchers in long lasting experiences become engaged with the community not only for the matter of research, but also with their social struggles. The relationship transcends the academic instances, gaining a personal dimension. Summarizing, these experiences involve ethical and political dimensions, not only theoretical or mathematical ones.

OUTRO

This literature review has suggested the classification of ethnomathematical research in two groups, concerning virtues like dialogue, collaboration and associate concepts. This distinction is related with, but not determined by, time. It is obvious that a clear cut about what is a "short" amount of time in research cannot be provided. Such classification does not intend to formulate any criteria to predict, to control or to evaluate ethnomathematical research. The intention is to pay attention to the existence of one dimension usually overlooked in the field, that I have referred to as symmetry. This dimension expresses concerns about the consistence and effectiveness of our own actions towards a wider conceptualization of mathematics.

Ethnomathematicians who work with communities, should be aware of the risk to betray their own goals, turning a thing that supposed to be a dialogue into a communication in only one way. To elude colonialism is not an easy task, because it is still present in our conceptualizations and beliefs about what can count as research. Classic ethnography exemplifies those beliefs, determining and distributing roles in the research process; as Woolgar states, ethnographers who stand for a division between image and reality try to maintain the exoticism of the reality observed and the epistemological superiority of the observer:

> Not only is the tribesman different, the implication is that this difference entails the subject's inferior access to reliable procedures for observation and report. In an almost paradoxical way, the more exotic the native, the more we can depend on the accuracy of the ethnographer`s report (Woolgar, 1988).

Of course, researcher's intentions cannot be simplified as "bad" or "negative". Most of the times, ethnomathematicians express their sincere respect of community heritage and act consistently with that. However, the less time invested in the fieldwork research, the more the researchers are tempted to do classic ethnography; despite researchers claiming their good intentions, not seldom the idea of dialogue in ethnography is in fact a dialogue between ideas interplaying inside the researcher´s mind. It is important to realize that institutional frameworks prompt research by imposing measurements of time to assess the quality and the impact of an investigation, and accordingly to that, to provide financial support. This economical rationality of completion rates promotes certain specific styles of research.

To conclude, it is not easy to imagine how a dialogue around mathematics could be established with a group. However, there is some research that can give indications. For instance, ethnomathematicians could include practitioner´s way of thinking, and give them more control in the research process, as a way to lose pre-eminence in the report, accepting other voices and other interests.

ACKNOWLEDGEMENT

This research is funded by Departamento Administrativo de Ciencias, Tecnología e Innovación (COLCIENCIAS) and Aalborg University in Denmark. This research also makes part of the NordForsk Center of Excellence "JustEd".

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ENDNOTE

1. Of course the individuals are not helpless operators in that set, they can strength it with their actions, change some components, or eventually they can deflate its importance as paradigm, accordingly to the awareness and consciousness that each individual has about the ways that set operates. Studies like the intended here, try to provide such awareness.

The intersection of girls' mathematics and peer group positionings in a mathematics' classroom

Darinka Radovic[1], Laura Black[1], Christian Salas[2] and Julian Williams[1]

1 University of Manchester, Manchester, UK, darinka.radovic@postgrad.manchester.ac.uk

2 Bangor University, Bangor, UK

This research paper reports the intersection between mathematics identity and the peer positioning of high attainment girls in a particular mathematics' classroom in Chile. Following an ethnographic approach, we explore how this intersection mediates girls' perceptions of what mathematics is and how it is supposed to be done/practised. The main finding is that different female 'peer' group processes and identities appeared to mediate different mathematical positional identities. The discourse models of maturity/immaturity, compliance/resistance and effort/effortlessness appear to be crucial in mediating peer processes – these are used to define membership and identity of the peer group (including relations with boys) and also mediate girls' positioning in mathematics.

Keywords: Mathematics identities, peer relations, gender and 'maturity'.

BACKGROUND

The study of gender equity in mathematics education has been an important focus of research for 40 years. Several studies have tried to understand how girls have been persistently "counted out of maths" (Walkerdine, 1998), by lower mathematics grades (e.g., Hyde, Fennema, & Lamon, 1990) and lower representation in post compulsory mathematics courses and careers (e.g., Mendick, 2005) compared to their male counterparts. Recent studies have shown that these differences are not homogeneous across countries (Guiso, Monte, Sapienza, & Zingales, 2008), and national heterogeneity suggests cultural constructions of these differences. One concept that has been increasingly used in exploring this cultural diversity is that of identity (e.g., Solomon, 2008). For example, the concept of mathematical identity has helped explain how women in some countries (e.g. the UK) come to see themselves as not belonging in mathematical (or STEM) contexts, despite their high performance in the subject (Solomon, 2012).

Alongside concerns regarding girls conflicted relationship with mathematics, there has also been a significant amount of research trying to explain girls' high achievement in most subjects (including mathematics) when compared to that of boys. This research has focused on masculine subjectivities in schooling, suggesting the existence of a "laddish" culture of (particularly working class) boys rebelling against the school in contrast to a feminized culture of compliance, and effort more apparent amongst girls (Jackson, 2003). However, simple dichotomies (e.g. rebellious boys/compliant girls) have been recently challenged by evidence showing that girls often aspire to effortless achievement (Jackson, 2006)I found this book a fascinating and engaging read...It provides a useful analysis and exploration of the classed and gendered 'anti-school' ethic in place presently within many schools, and it will provide a meaningful analysis for academics, policymakers and practitioners and anyone with an interest in gender, education and young people.\" Fin Cullen, Goldsmiths College, Review in Gender and Education \"I would [therefore] urge everyone concerned with what is happening in schools to read this book, with its fascinating data and nuanced arguments.\" Heather Mendick, London Metropolitan University – Review in British Journal of Educational Studies This innovative book looks at how and why girls and boys adopt 'laddish' behaviours in schools. It examines the ways in which students negotiate pressures to be popular and 'cool' in school alongside pressures to perform academically. It also deals with the fears of academic and social failure that influence pupils' school lives and experiences. Drawing extensively on the voices of students in secondary schools, it explores key questions about laddish behaviours, such as: Are girls becoming more laddish and if so, which girls? Do boys and girls have distinctive versions of laddishness? What motivates laddish behaviours? What are the consequences of laddish behaviours for pupils? What are the implications for teachers and schools? The author weaves together key contemporary theories and research on masculinities and femininities with social psychological theories and research on academic motives and goals, in order to understand the complexities of girls' and boys' behaviours. This topical book is key reading for students, academics and researchers in education, sociology and psychology, as well as school teachers and education policy makers." (Jackson & Jackson, 2006, and furthermore, that the apparent girls' compliant behavior may also be a way of resisting conformity by becoming invisible rather than disruptive (Fisher, 2014).

With these arguments in mind, this paper aims to explore further notions of diversity in the construction of girls' mathematical identities in classroom cultures. Particularly, we are interested in understanding how high attainment girls' positionings in the peer group and in the mathematical practices intersect, and how this intersection mediates their mathematical identities in the classroom.

THEORETICAL BACKGROUND

In order to account for the relation between learning and identity, this study takes a sociocultural perspective on identity. This perspective considers that both learning and identity develop in moment-by-moment engagement with others in, and through, sociocultural practices (Holland, Lachicotte, Skinner, & Cain, 1998). In these practices, individuals are positioned, and position themselves in relation to others, thus, constituting *positional identities* (Holland et al., 1998). As people move between practices, these identifications are multiple and therefore a single identity is never crystalized. A consequence of such processes is that identifying with contradictory practices can prove challenging and sometimes even generate contradictions in or crises of identity (Black & Williams, 2013).

The prime cultural practice that needs to be considered when researching mathematical positional identities is the *mathematical* practice itself, and how notions of competence or 'smartness' are culturally constructed therein (Gresalfi, Martin, Hand, & Greeno, 2008). These notions constitute social norms and hence obligations against which students will be positioned by others, and will position themselves, for example, by resisting or complying with these obligations (Cobb, Gresalfi, & Hodge, 2009).

A second practice that has received attention by researchers is that of students' positioning in their peer group culture. Studies that have explored how the school culture relates with the peer culture, have found that bringing these two (sometimes opposite) cultures together in a developing sense of self, can be challenging for individuals. For example, in a recent study, Francis and colleagues (2010) found that for high achieving students, notions of high achievement were in conflict with social popularity with peers, requiring a particular kind of "identity work". This identity work was extremely challenging for girls, who had to negotiate demands for being both attractive/desirable (feminine) and clever/smart (masculine) (Skelton, Francis, & Read, 2010). This is particularly important in mathematics, as this subject is often seen as geeky/non-popular and male (e.g., Mendick, 2005b), so being mathematically able can act as a double marked category for girls (Damarind, 2000).

Considering the existing evidence, it appears necessary to explore further the complexity of cultures in the classroom and avoid essentialising specific cultural forms as male or female, both in the mathematics and in the peer cultures. Hence, this study explores how girls' peer group relationships and values relate to their mathematical identity in the classroom.

METHODOLOGY

Context of the classroom study

This study was purposefully situated in a school with average attainment, medium socioeconomic status (SES) and in a traditional working class area in Santiago, Chile. It was a private subsidized school that received government funding and charged parents a tuition fee. As some studies have found (Mizala & Torche, 2012), these characteristics generate a highly homegeneic population in terms of SES. In addition, it was considered a demanding school where students were asked to reach a 70% mark for a passing grade. Students that did not reach this score were held back a year.

The study focused on one case study in a year 7 mathematics classroom of 39 students. Not unusually for Chilean classrooms (see Ramirez, 2007), the students' scores varied between failing and excellent. In addition, because of the hold-back policy, the classroom was also diverse in terms of students' age, with a range between 13 and 16 years. These characteristics made this classroom a very typical one in this country for this type of population.

The mathematics' teacher [who we will call Ms. Paula] had 2 and half years of experience in teaching mathematics. She came from a general teacher-training course (non-mathematics specific) in a non-traditional University, with relatively limited experience and specific preparation in mathematics. However, the Head of Department and her students characterized Ms. Paula as a "very good teacher".

Similarly to many Chilean classrooms (e.g., Martinic & Vergara, 2007), Ms Paula's lessons followed a very consistent routine with three main parts: (1) whole class introduction of the day's topics; (2) students' working on exercises; and (3) a closure. In this particular classroom, during the closure part "reward points" were distributed among students according to whether they completed the activities. As also described in previous Chilean studies (e.g., Radovic & Preiss, 2010), most of the talking was done by the teacher in the whole-class parts, with students' role mainly focusing on answering the teacher's questions. Students had relatively more space for independent work and peer talk during exercises, where they could choose to work collaboratively.

Data collection and data analysis

This study used an ethnographic approach to access the classroom culture and individual student's and group's mathematics identities. Following Cobb and colleagues framework (2001), instruments were designed to capture both a social perspective (shared meanings in the classroom's culture) and a psychological perspective (individual student positionings in this culture), the latter particularly focused on girls. One of the authors [DR] visited the school regularly during the second semester of the school year, collecting data from individual and group interviews with students, conversations with the teacher and observations. Different interview techniques and schedules were used to allow students to talk about their classroom, lessons, teacher and peers.

A *Mathematical Groups* interview was concerned with the students' perceptions of the different mathematical groups in the classroom and their own place in these groups (self and other's positioning). It was used to explore what constructs of competence were working in the classroom and which students were consistently positioned as 'competent' or 'non-competent'. Sixteen students (7 boys and 9 girls) were sampled to represent different levels of attainment, and interviewed individually for about 5 to 10 minutes. They were asked to group their classmates by answering the following question: *are there groups of students that show a similar relationship with mathematics? Build as many groups as you want considering that students in a group should be similar within and different from other groups.* After grouping, students were asked to talk about what criteria they used to group their classmates.

A *Natural Peer Group* interview followed Cairns, Xie and Leung's approach (1998), approach that allows mapping the classroom's peer group structure in terms of 'naturally occurring' (i.e. informal, out-of-school, or non-academically shaped) social clusters. Students were purposely sampled to represent different peer groups that were observed or mentioned in previous interviews until reaching saturation (i.e. no new groups appearing). In total, 12 students were interviewed (5 boys and 7 girls). They were asked to group their classmates by answering the following question: *are there people in this classroom who hang around together a lot?* In a follow-up interview, meanings associated with each group were explored. Interviews again lasted between 5 to 10 minutes.

This paper's analysis is focused on 5 high attainment girls (between A- and A+) that were recognized as belonging to two big social peer groups, each of which included boys from the classroom, but in significantly different ways. These girls were also interviewed more in depth about their relationship with mathematics. In a semi-structured narrative interview that lasted about half an hour, they were asked to talk about their mathematics story and to describe how they felt, how they behaved, and how others saw them when doing mathematics. For the preliminary analysis that follows, the five girls participation in the classroom as observed during observations and their interviews were explored looking for communalities within the peer group and differences between the groups.

RESULTS: INTERSECTIONS OF GIRLS' MATHEMATICAL AND PEER GROUP IDENTITIES

When the students were asked to group their classmates in *terms of their relationships with mathematics*, they tended to consider oppositions in *achievement* (high v/s low achievers) and *effort* (high effort v/s effortless) as significant categories for distinction. Thus, three main groups were mentioned in this regard: Effortless High Achievers [two boys and three girls], Effortless ('*lazy*') Low Achievers [four boys, of whom 3 were held back], and High Effort students with various levels of attainment.

In terms of naturally occurring peer clusters, the analysis that follows is centered on the girls that belonged to two of the six peer groups named by students '*Adolescent*' and '*Normal*'. While the '*Adolescent girls*' were said to behave as if they were older teenagers (e.g. who 'wear make up and go out partying with boys'), the '*Normal girls*' were described as either immature or more appropriate to their age, and so positioned in contrast to the '*Adolescent*'.

Adolescent/Effortless girls

The intersection between the mathematical and the natural peer group revealed that the conformation of the '*Adolescent group*' mixed students from two contrasting mathematics groups: three effortless high achieving girls and four effortless low-achieving boys. Their classmates described them as the group that acted as if they were older, thought they were the 'coolest', showed off about what they owned (clothes/technology, i.e. making the link with higher economical capital), and appeared less concerned about school. Some classmates mentioned that the girls in this group were too worried about their appearance, wearing too short skirts, make up, and always hugging and kissing each other. Some of them, they said, even had boyfriends.

When talking about themselves, students from the '*Adolescent group*' said they were the most mature group in the class. They said other groups of students (especially the '*Normal group*', see below) played like children, ran about during breaktimes and were loud and euphoric. In contrast, the girls in the adolescent group said they hung out with boys and talked about their different interests: for example, Justin Bieber. They also claimed to balance their school life with other social obligations, like going out and partying. According to one of these girls, this was an important aspect of growing up.

> New things will come in life, people can't be all the time worrying about school. It's not that school goes to second priority, but that new things come to life (...) like more friends, new things, I'm starting to go out (partying) and those things are taking more of my time, and that's it... less time...

This notion of maturity was further supported by these girls' alignment with the 'effortless low achievers' boys' in the group who had been held back a year and who, in consequence, were one year older than the rest of the class.

During mathematics' lessons, these girls were on task but somehow also seemed disengaged from the classroom activities. During whole class teaching it was common to hear them making jokes (e.g., giving related funny examples from the telly) and commenting between themselves (in group) on what the teacher was saying, but without sharing this commentary with the whole class. They were never seen offering an answer to the teacher's question, or asking or commenting spontaneously to the whole class. Thus, they gave an appearance of not engaging but did so without being openly disruptive as such. During independent work, in contrast to other students, they remained sitting at their own desks, working with their partners. These girls were fast in their work, finishing ahead of time and, therefore, having the opportunity to chat about other things with their friends. Also, in contrast to other girls in the class (e.g. the girls from the '*normal group*' see below), they rarely engaged with the teacher while working; instead, they sat back and waited until someone came and checked their work.

During individual interviews, these girls reported that mathematics for them was an active, collaborative, and fast activity, where they were able to use their skills ("I'm good with numbers", "I'm good with finding the solution"). Their perceived competence allowed them to enjoy and value Ms. P's pedagogic approach as it allowed them to be mathematically active while remaining independent:

> I like maths more than other subjects because it is not always the same. Exercises change; there are always new things to do and new ways to do them. It is not like other subjects that are always write,

> read, write (...) Ms. P gives some time for doing the sheet and this time is like a little free time because you can.... I mean, not a free time in a bad sense, but you can work with your desk partner, or talk about the exercises with your group.

What they seem to accomplish in the classroom was the tenacity to maintain their academic performance as high achievers, but, at the same time, also preserve their status in the peer group.

> I try to do the exercises fast, so I can think in maths for a moment and then I can return to the "normal life" (...) that's what I like to do... to do my work and then relax...

These data suggests a relationship between peer group membership and mathematical positioning, where each one appeared to be resourcing the other. Their effortless identity as a high achiever in mathematics meant they could complete the work quickly – this then allowed them to accomplish their social 'obligations', which were vital to maintain peer group membership and cohesion (e.g., chat between them and with the boys during the mathematics lessons). At the same time, performing their social identity as 'mature girls' involved achieving a good balance between social peer-group maintenance work and academic work (i.e., not working too much, and not being seen to engage with the classroom process, with the teacher, or with their immature inferiors) which at the same time resourced their 'effortless' mathematical identities. This relationship allowed these successful girls to construct a positive relationship with the subject and with their teacher's pedagogic practice, while affording social independence and maturity as it aligned with the held back boys of their group.

The Normal/Non Effortless girls

The Normal Group presented a rather different intersection between mathematical and peer group identities. This group was comprised of six girls and three boys. Other students in the class described the girls in this group as 'the quiet ones'; the group that never got told off by teachers. Girls from the adolescent group also commented that these girls were childish, contrasting them with their own social identity as 'mature'.

Two of the girls within this group named themselves as the '*Normal Group*' (hence the adopted name) and, in opposing themselves to the '*Adolescent Group*', they said that they enjoyed childhood as appropriate to their age, not wanting to grow up too fast: as one of them said, "we live what we are supposed to live".

During observations, the high achievers girls of this group appeared as described by their peers and in contrast to the adolescents: always on task and actively participating in classroom activities. During whole class teaching, these girls consistently offered answers to the teacher's questions. As with the adolescent girls, the normal girls finished their work fast. However, instead of waiting for the rest of their classmates to finish, they would seek help from the teacher in order to check their answers. Furthermore, the relative compliance of these girls with the teacher provided a degree of freedom in their behaviour: during independent work, they were frequently standing and walking around the room. Although this was supposed to be against the rules and proscribed as disruptive, they were never reprimanded.

The Normal girls portrayed different ideas of what learning mathematics was when compared with the views of the Adolescent girls. Although they mentioned being relatively good at mathematics, they did not claim to value the independence of working alone to solve problems and felt that sometimes doing mathematics became too repetitive. For them, learning mathematics involved applying a method taught by the teacher and then waiting for the teacher to offer a new method if the first one was inadequate or too challenging.

> I would like the teacher to teach us techniques that weren't difficult so we could understand them in the moment and then we could move quickly, like the magic formula, easy procedures to understand (...) But in mathematics the teacher is like too serious and she explains everything as it is very complicated...

This version of what mathematics was (or was supposed to be) appears to relate to how they saw their peer group role in the classroom. The Normal girls talked about maturity in order to differentiate themselves from their peers (particularly the boys in their group), but for them this term appeared to hold a different meaning. For them, maturity involves being responsible in doing their work and helping their (immature) male classmates:

> Yes, because when we start working on exercises, I look around and all the boys are just playing... some of them have it done but most of them just play (...) I think it is just that women are more professional, more mature, they keep focus on things (...)
>
> I think we are 4 girls and we will have to be with the boys and we will have to help the boys to focus, we will have to be responsible, because I want to get a good grade...

These extracts again show how these girls' peer and mathematical identities are related. Being 'normal', acting according to their age, feeds into their version of how mathematics should be: a set of methods that the teacher has to transmit to them. How are their peer group social relations then related to their mathematical positioning/identity? One can hypothesise that this peer group is constituted as being 'child like' in their peer activities, but also in its compliant/dependent relation to adults – in this case the teacher – in contrast to the adolescent girls whose 'maturity' demands a degree of independence. Mathematically, this is associated with following rules and techniques, which should be simple and straightforward. Even the social relation to their male peers is not yet adolescent, and positions these males as in need of their (parent-like) surveillance/monitoring, reflecting their own compliance with the authority of their teacher.

CONCLUSIONS AND DISCUSSION

This study offers further evidence for the idea that mathematical identities are intertwined with the students' peer culture (Gholson & Martin, 2014). More specifically, this study contributes to the literature by describing a dialectical relationship where peer group membership and identities inter-relate with their engagement in classroom activity and mediate how girls come to see what mathematics is, and how it is suppose to be done. In turn, emerging mathematical identities in the classroom may play some role in maintaining peer group membership.

These preliminary results suggest that two cultural models can arise in this dialectical relationship: (i) effort/compliance with authority, and (ii) maturity. In the mathematics education literature, the notion of effort has been identified as crucial in defining mathematics as a male domain where essential ability involves effortless high achievement (Walkerdine, 1998). Evidence from this study challenges such an argument by suggesting that girls can also position themselves as effortless high achievers as was noticed by Jackson (2006). However, it is important to point out that these girls are not particularly high achievers when compared with performance of students in the private school system in Chile and did not mention intentions of continuing studying mathematics after compulsory schooling: it seams that this effortless identity is still counting them out of mathematics, or it will in the future.

This study also suggests that girls' relationships with mathematics is also influenced by the meanings they attach to the process of growing up, particularly to their notions of maturity, which is a central theme in peer group membership (including relations with boys). On the one hand, the adolescent girls use maturity as a way of addressing their need to position themselves outside the school academic culture, a process which is further aided by their alignment with the oldest boys: Despite the boys' being lower achievers – they gave status to the girls who wished to use maturity to differentiate themselves and their group from the rest of the class. On the other hand, the normal girls associated maturity with compliance with school culture – not just being compliant but even taking responsibility for the poorly behaved boys as another means of performing their (apparent) compliance. These different notions of maturity not only signify peer group membership but also how to approach mathematics, since for these girls their maturity was representative of the school culture (i.e. doing maths quickly, easily to allow for other activities v/s following rules/methods and verifying such procedures to perform well).

ACKNOWLEDGEMENT

The work of the first author was funded by the National Commission for Scientific and Technological Research in Chile (CONICYT).

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Number sense as a sorting mechanism in primary mathematics education

Rebecca Turvill

Brunel University, London, UK, rebecca.turvill@brunel.ac.uk

This paper aims to explore the way mathematics education systematically disadvantages particular groups of children, beginning in their early education. I focus on the concept of number sense to illustrate how it acts as a gatekeeper to wider mathematical learning and subsequently life opportunities. By examining number sense from the perspectives of cognitive psychology, situated cognition and Bourdieusian social psychology, I demonstrate inequalities in how young children develop number sense in primary school. I suggest that it is important to consider these different perspectives to reveal the dominance of cognitive theories on practice in primary schools. I propose that a Bourdieusian analysis of number sense reveals how number sense works to sort children and ultimately reproduce social divisions.

Keywords: Number sense, social justice, habitus, Bourdieu.

INTRODUCTION

Mathematics education systematically disadvantages some people (Zevenbergen, 2001). Crawford & Cribb (2013) report that high attainment in mathematics leads to higher earnings potential for individuals; with mathematics attainment at age ten being linked to individual earnings at age thirty. Similarly low numeracy along with low literacy is found to be a genuine barrier to employment (Wolf, 1999). We need to view such evidence alongside the knowledge that children from lower socio-economic backgrounds generally achieve lower outcomes in mathematics than their more advantaged peers (e.g., Nunes, Bryant, Sylva, & Barros, 2009). Thus there is a cycle of disadvantage, where children who are born into homes of low socio-economic status are likely to be failed by mathematics education and have limited opportunity to improve their life chances. In this paper I argue that an individual's 'habitus' (Bourdieu, 1989) causes misrecognition of number sense skills as mathematical ability. Such attribution of success to individual both ignores the systematic advantages and disadvantages for particular groups of children as they navigate the education system and actively creates further advantages and disadvantages for those groups.

Situated cognition has sought to challenge such outcomes through offering an alternative approach to teaching mathematics (e.g., Boaler, 2001). This is based on wider literature which suggests that mathematics in 'real-life' is not well supported by the mathematics taught in school (Carraher, Carraher, & Schliemann, 1985; Lave, 1998). However, I argue that situated cognition does not go far enough to explore the systematic failures children experience in schools. By placing a particular emphasis on primary schooling as the site of rapid development of number sense, I argue we need to broaden our view of number sense and that a Bourdieusian framework is useful in this context. Such a perspective allows us to explore how children are being sorted by their 'ability' in number sense in this very early stage of education, which becomes a self-fulfilling prophecy of how well children will achieve in mathematics throughout their education.

The purpose of this paper is therefore to offer a theoretical contribution regarding the examination of this development of number sense. Before outlining the contribution of the different theoretical perspectives, I begin with a short discussion of the construct of number sense.

NUMBER SENSE

Number sense is a term widely used in educational literature, curriculum documents and other academic fields such as cognitive psychology. In terms of education, as Greeno (1991) comments, although the concept is not defined well, we recognise it when we see it. Largely, the child's "fluidity and flexibility with numbers" (Gersten & Chard, 1999) is useful as an educational definition of number sense in primary education, although this is a much boarder definition than that usually employed in cognitive psychology. As I work through the theoretical positions of cognitive psychology, situated cognition and Bourdieusian social theory I demonstrate the aspects of number sense each focusses upon. I use this to illustrate the utility of a Bourdieusian analysis to highlight the structural inequalities that the concept of number sense reproduces through the education system.

COGNITIVE INFLUENCES ON NUMBER SENSE

Cognitive psychology offers a conception of number sense based solely on enumeration of numerosity. Within this definition mechanisms underlying numerosity discrimination continue to be explored. In general two systems for processing numerosity are proposed: subitising and approximate number recognition. The assumption I read from this literature is that these systems are taken to be innate, building on specific neural pathways. Xu & Spelke (2000) suggest that number sense develops spontaneously and have demonstrated how children as young as six months can discriminate between sets of two and three. Halberda & Feigenson (2008) demonstrate that this numerical discrimination improves rapidly throughout infancy and then continues to develop until approximately the age of thirty, suggesting the presence of a developmental mechanism. Finally, Feigenson, Libertus, & Halberda (2013) demonstrated a correlation between the accuracy of enumeration and individual mathematical ability as recorded in school attainment data, which was found to be reliable even when applied retrospectively across school records. Together, such data suggests an innate developmental mechanism which allows improvement in number discrimination and that this skill may be linked to formal achievement in mathematics. Such a perspective is influential on the development of primary school mathematics curricular (e.g., DfE, 2013) which replicate these proposed hierarchical mechanisms; early counting practices in particular are heavily influenced by work on the way children 'develop' counting skills (Wynn, 1990). There is suggestion however, that a curriculum based on developmental stages is outdated (Goswami, 1991), yet the pervasiveness of such a model remains within the education system.

I argue that a view where an innate capacity underpins mathematical learning can lead to a view that individuals have a particular 'mathematical ability'. It can also lead to a view that certain people will be able to learn mathematics better or worse than others. This is to some extent a mis-use of the cognitive literature, and indeed educational neuroscience is working to address such a misconception through work such as that by Howard-Jones (2007) which looks to examine how neurological evidence can be used to enhance the learning experience, rather than restrict it. What is important when reflecting on the dominance of such cognitive theories on the learning of number sense, is not simply how well the definition describes the concept, but to interrogate the effect of the concept on practice (Henriques, 1984). Thus, despite greater understanding of flexibility of neural pathways in learning, a fixed ability mechanism is common in mathematics learning (e.g., Marks, 2014). We attribute individual differences in mathematical attainment to differences in individual abilities and this becomes a self-fulfilling prophecy.

It is simple to see how such a situation operates in practice. For example, although there is no definitive list of skills, number sense is recognised when it is demonstrated (Greeno, 1991) – or perhaps when a 'lack' of it is shown. When children appear not to have mastered a particular aspect of this body of knowledge, say number bonds, this is identified as a 'gap' in their knowledge. Such children are given further time and additional practise to acquire this skill, contrary to the evidence that such an approach may not in fact allow them to develop the skill being targeted (Denvir & Brown, 1986). Thus children without the necessary number sense experience a curriculum of practice and repetition (e.g., Gersten & Chard, 1999). In contrast to this, children who experience success are given more advanced work, further extending their attainment. The achievement gap in mathematics is known to widen through primary schooling and such a model for teaching and learning begins to demonstrate a possible mechanism for this.

A further difficulty with operating in this paradigm to try and 'fix' number sense is that children are likely to be labelled. Those who need further practise of the basic number sense skill are then considered to be 'low-ability'; they do not have the required skill. Those who do have it are considered better. Not only then do the 'low-ability children get a narrowed curriculum, there is a high risk that they view themselves as unable to learn mathematics and have the associated difficulties of such a label, for example low self-esteem and low motivation. This aspect of identity will be referred to through the work of situated cognition.

Whilst cognitive psychology continues to provide us with increasingly sophisticated techniques to investigate and understand the neural basis of learning, over-emphasis on this paradigm may lead to a narrowed curriculum, particularly for those children who experience mathematical difficulties.

SITUATED COGNITION

Situated cognition offers an interesting lens through which to view number sense due to the discrepancies observed in formal and informal arithmetic. For example, Lave (1988) demonstrates how adults were able to use mathematical information in informal settings such as the supermarket to make 'best-buy' calculations and purchases. However, when the same scenario was presented as a 'pencil-and-paper' calculation, much like a school setting, the individuals often failed. As Greeno (1991) proposes, part of the concept of number sense is the ability to make connections and draw across different number concepts. In Lave's (1988) study this was largely attributed to the formal setting causing the individual to turn to the use of an algorithm which was then not successfully remembered or executed. I offer this as an interesting perspective on number sense as the presentation of a calculation seems able to affect the way an individual engages with it.

Situated cognition addresses this by proposing that mathematics as it is 'traditionally' taught in schools does not represent the mathematics used in real-life. McIntosh, Reys, & Reys (1992) state how as students increase their technical procedures in mathematics, they may in fact narrow their range of available strategies. If informal strategies are more generally called upon in 'real-life' mathematics then it is easy to see how procedural teaching limits the way mathematics is of use to people beyond school. The work of Boaler (2001) has shown improved attainment through working on 'real-life' problems in context. Not only did the children report that such problems made the mathematics they learned more relevant to their wider lives, the young people achieved more highly in formal testing than other students with higher prior attainment. Another important outcome was a reduction in the way girls were 'excluded' from mathematics.

The focus of Boaler's (2001) work is on the high school context. Whilst this work is important in understanding older pupils' engagement with mathematics, the concepts it addresses are ones I argue build on number sense. Children often report much greater dissatisfaction with mathematics at secondary school where they fail to see the relevance of the topics to their lives. This is not usually the case with primary school mathematics as the concepts involved are in fact largely 'number sense' and such learning is generally regarded as important by young people (e.g., National Audit Office, 2008). To solve more challenging problems requires prior learning; aspects of number sense need to be automatic to allow mental effort to be focussed on the more complex aspects of problem solving. If some calculations are not automatic, or at least fluent, then focussing on specific calculations becomes time consuming and distract from the wider task in hand. Situated cognition identifies this dilemma, where people cannot connect calculations presented in a particular format to one in which they are able to solve such a calculation, they cannot demonstrate the "flexibility" expected of number sense (Gersten & Chard, 1999). This again returns us to a cognitive paradigm of rehearsal and repetition of skills before moving onto more complex problem solving; it is just such a requirement to have fluency with basic calculation that leads to a narrowing of the mathematics curriculum for children struggling to demonstrate these skills.

Askew (2008) argues that it is not simply learning which is situated in mathematics, but that social identities in classrooms are also situated and that normalised routines of the classroom lead to maintenance of inequalities. Including social identity as an element of the situated nature of mathematics learning does broaden the context and it is important to examine the way children develop such identities in relation to mathematics learning. Number sense is well placed to allow such an analysis. Number sense as a cognitive domain where we make sense of situations and different contexts (Greeno, 1991) is likely to be influenced by how we feel in the situation, previous experiences and the response of others both to us and to our ideas – in short our mathematical identity.

Why, then, do children experience the same mathematical environment in such different ways? Situated cognition poses this as a challenging question. In order to begin to address this, I propose that a Bourdieusian perspective allows us to examine wider societal factors influencing the learning of mathematics, the cultural influences of the school environment and the school within its broader environment. We can appreciate that some children have already been excluded from learning mathematics by assessment of their number sense early in their schooling. I now turn to Bourdieu to develop this broader perspective in the role of number sense development.

BOURDIEUSIAN PERSPECTIVE

Viewing number sense with a Bourdieusian lens allows us to see this number sense not only as a body of facts, but as a way in which we interact with mathematical information. Bourdieu's method offers a set of conceptual tools through which to examine practice, namely field, habitus and capital. These concepts can be used to examine the way number sense is constructed demonstrating how it excludes particular groups of children.

Field

A field is defined by Bourdieu as "a universe in which the producers' characteristics are defined by their positions in relations of production" (Bourdieu, 1993, p. 51) and is considered a site of competing interests (Green, 2013); and as such 'primary mathematics education' is conceptualised as a field in this sense. Mathematics education is often used as a field in such analyses as it has a particular language, specific expectations and outcomes (Jorgensen, Gates, & Roper, 2013). Proposing primary mathematics education as a field allows the range of external and internal influences to be analysed. The nature of the British education system which separates largely primary and secondary schooling – at age 11 – further exacerbates the hierarchical nature of mathematics education. My suggestion is that primary school mathematics is largely seen as a gatekeeper to secondary school mathematics, and within this, number sense is a gatekeeper to the wider range of skills that allow one to access the wider curriculum to make success in primary school a possibility. Therefore, by conceptualising primary school mathematics as a field using Bourdieu's terms appreciate the expectations, barriers and opportunities presented by the system can be appreciated.

By conceptualising primary mathematics as a field, we can look beyond the classroom as the only site in which such skills are developed and thus have greater explanatory scope and power than the work of situated cognition. Children clearly develop much mathematical knowledge and understanding beyond the classroom, arriving at school with highly varied experiences. Whilst situated cognition begins to address the need to make mathematics learning relevant to 'real-life' it is not able to address the diverse contexts from which children are learning mathematics. In particular it is not able to explore the dissonance that children from lower socio-economic backgrounds experience between their home learning environments and that of the school. For example, the work of Street, Baker and Tomlin (2008) showed how children may have rich mathematical experiences at home, through activities like pigeon racing, but that the knowledge and experience that this provided the children was not reflected in the school environment. The Bourdieusian framework permits the competition between the recognised and the un-recognised styles of mathematics to become apparent.

Habitus

Habitus is a term used by Bourdieu to define the taken-for-granted way in which we act; a person's natural manner or disposition (Bourdieu, 1989). This is more than a set of preferences as habitus is operationalised by class, gender and race (e.g., Reay, 1998). Habitus is shaped throughout one's early socialisation and is well established by the time an individual enters education. Habitus is stable and durable, yet can be shaped, for example through education. Thus we can learn to act and operate in new ways, but our early socialisation never leaves us. Bourdieu expresses the way that early socialisation has such a powerful influence through concepts such as 'manners' or ways of acting in particular situations (Bourdieu, 1984). This is exemplified through the confidence that is acquired through 'growing up around' something, knowing 'intuitively' how to act and the confidence this brings, rather than having to learn it and being painfully aware that it is quite possible to make the wrong choice.

Considering number sense in such a framework highlights the inequalities that exist in the education system. Children who experience a home environment which is rich in number experiences – with parents who are confident and highlight numerical experiences – are likely to enter school with a greater awareness and confidence with numbers themselves. They have an apparently 'innate' confidence with numbers, which is mis-recognised as an 'innate' ability. Those children who do not have such opportunities appear as less able. Number sense is particularly suited to this explanation where it is seen broadly as the connected way of thinking about numbers, as linking different aspects of number, rather than solely elements of mathematics such as number bonds or multiplication facts. The way it privileges children becomes clear. Those children, who are used to working within the context of such number sense, used to talking about numbers in a confident way, will demonstrate 'ability' in school. Those who are not used to the language of mathematics (Walkerdine, 1989) are mis-recognised as 'low-ability' and thus excluded.

Habitus, is at home in the "field it inhabits" (Bourdieu, 1992, p. 128) thus the habitus of the middle classes, which aligns well with the expectations of the education system positions them to excel. From this Bourdieu argues that achievement becomes self perpetuating preserving the hegemony of dominant classes. Middle class discourses are largely in line with school discourses and the increased ability of such families to prepare their children for school allows them to take every opportunity offered by the education system. It is the available resources of the middle classes in Bourdieu's opinion which can readily be converted into academic achievement; such resources are considered to be capital as outlined below.

Thus, habitus can offer itself as a tool to examine the development of number sense. How does a child's habitus impact learning through development of their mathematical identity? The final tool to which I now wish to turn is cultural capital as a way to further explore how habitus can advantage and disadvantage different groups of children.

Capital

Capital is the outcome of the combination of habitus and field. Bourdieu defines different capitals of which cultural, economic and social capital are of relevance here. As stated, children bring varied experiences of learning mathematics to school with them and particular cultural experiences offer educational advantages to a child. It is the ability of those from higher socioeconomic backgrounds to utilise their economic capital to offer increased cultural capital which offers the systematic advantage such children experience in school.

What Bourdieu argues is very important in this transition from economic to cultural capital is the misrecognition of the conversion; that a child with such cultural experiences is better placed to learn mathematics is not recognised in the education system. Schools attribute being well placed to learn as being more 'able'. We know who is well placed to learn from our assessment of the children in school. Such educational assessment whilst pertaining to assess a child's 'ability' and to shape subsequent schooling, is in fact measuring wider socialisation. Habitus is exchanged for a culturally-valued outcome in the specific field, in this case achievement in mathematics, and the individuals increase their cultural capital.

CONCLUSION

A Bourdieusian analysis allows a wider conception of number sense – with number sense acting as a sorting device itself for participation in mathematics learning more generally. Thus children 'with' a habitus which fits well with the mathematical expectations of the education system are mistakenly considered able, whilst those with a poorly-aligned habitus are not. The consequence for this latter group, predominantly those from lower socio-economic backgrounds, is a message about their own 'ability' to learn mathematics and an altered curriculum which is repetitive and narrow – in order for them to 'fill gaps' in their learning. Such an approach, whilst well meant and standard practice in primary schools, is a form of 'symbolic violence' (Bourdieu, 1986) enacted upon children from lower socio-economic backgrounds. Number sense, mistakenly viewed as individual ability, shapes children's mathematical futures and potentially their life chances, from their earliest engagement with mathematical learning.

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TWG10

Posters

Teacher training through research in ethnomathematics

Veronica Albanese[1] and María Elena Gavarrete[2]

1 Universidad de Granada, Facultad de Educación, Granada, Spain, very_alba@hotmail.it

2 Universidad Nacional, Heredia, Costa Rica, marielgavarrete@gmail.com

Two experiences in primary school teacher education are reported. Both of them are based on Ethnomathematics understood as a tool to reflect about the ways of doing mathematics of two cultural groups. Geometric figures are built in different way by the two cultural group analyzed: a group of folk dancers of Argentina and an indigenous people of Costa Rica.

Keywords: Ethnomathematics, teacher education, cultural practice, geometric figure.

INTRODUCTION

We report two experiences performed with primary school teachers respectively in Argentina and Costa Rica, where it was proposed to use Ethnomathematics as a tool to promote professional teacher development through a process of reflective inquiry about the ways of doing and understanding mathematics (ethnomathematics) of a cultural sign of the cultural groups involved: the guild of Argentine folk dancers and the Cabecares, an indigenous group of Costa Rica.

METHODOLOGY

Both works consist in organizing a course for the training of primary school teachers, where we propose the development of Ethnomathematical Microproject (Oliveras, 1996). These include the identification and characterization of ethnomathematics in a cultural sign of the group chosen, to develop contextualized teaching activities to be presented in the classroom of primary education. The objective was to engage teachers in a research about the ethnomathematics embedded in the cultural practices in each context and to identify its relationship with academic-formal mathematics.

In both countries we organized a course for teacher training, which promotes the research on the context through micro ethnographies realized by the participants of the courses. The data collected consists of the reports of the Micro-projects and the audiovisual recordings of the course session.

SOME RESULTS

The cultural signs studied were the folk dance *chacarera* (Argentina) and the traditional conical House of the *cabécar* culture (Costa Rica). We highlight here the participants' observations about the ethnomathematics found in the cultural signs and about the relations and differences with academic and formal mathematics.

In the dance *chacarera* the participants focus on the geometric figures that outline the choreography and they observed that the circumference is associated to the movement of steps that include a turn of the body on itself for folk dancers, while in school mathematics, it is defined by an equidistance of all points in relation to a center. In fact the same steps in the figure called *avance retroceso*, which does not involve a rotation of the body on itself, are represented by a diamond, while the figure called *giro*, where a rotation is involved, is represented by a circular line; this is the evidence that the teachers provide for the hypothesis that dancers resemble the circular line to the rotation of the dancer's body by a round shape without corners. Thus we conjecture that the circumference of the guild of the dancers is conceived as a regular polygon that tends to have no angles (Albanese and Perales, in press).

The *cabécar* traditional conical house is called *Ju-Tsini* and is itself a system of cosmological representation where elements of the cultural heritage and particular geometric concepts of the group –a specific logic and a particular way of localizing- join together, since, in this ethnic group, the physical world serves as a system to represent the mythical world (Gavarrete, 2012). The participants identify the construction of the *Jú-tsiní* as a powerful and contextualized example to work with pupils concepts related to the solid corps, as symmetry (axis of symmetry and homologous points). The circle is drawn using a center pole, this means that here the conception of circle is associated to the distance from the central point. The space is represented by the union of a cone and an inverted cone, whose circular bases overlap and therefore have the same axial center, resulting in the model *Nopatkuö* which describes the three cosmic levels of the legendary tradition of the ethnic group. This specular model represents the dual opposition of elements. The principle of complementary opposition is equivalent in school mathematics to that of symmetry even if in this culture it has an added value of cosmological meanings.

CONCLUSIONS

In both countries, the research experience impacted the teachers training, as they reflected on the universality of mathematical knowledge and teaching applications, promoting teacher creativity to facilitate the developing of the mathematical curriculum in connection with the sociocultural environment.

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Power-relations in participatory action research project in mathematics education

Lisa Björklund Boistrup[1,2] and Joakim Samuelsson[1]

1 Linköping University, Linköping, Sweden

2 Stockholm University, Stockholm, Sweden

The purpose of the study presented in this poster is to examine characteristics of power relations between researchers and teachers in an action research project. We position the study within a critical and social approach and pay attention to inequities that may concern power relations between researchers and teachers in an action research project (Skovsmose & Borba, 2004). We draw on Gellert (2008) who argues that if "professional development of mathematics teachers is considered a collective affair, then the concepts used to describe the teachers' actions and cognitions should reflect this perspective" (p. 94). We also draw on Atweh (2004) when we problematise "the process of research itself and critiques it in terms of power relationships between the participants" (p. 194).

Keywords: Mathematics classroom, action research, power relations.

ANALYTICAL FRAMEWORK: PARTICIPATORY ACTION RESEARCH

In our analysis we adopt a description by Atweh (2004), where participatory action research is characterised as:

— a *social activity*. This research always recognizes the broader institutional context as part of the process, for example the municipalities or colleagues etc.

— a *participatory* action research. This research engages teachers and researchers to investigate their own knowledge and actions. A consequence is that people can only carry out action research 'on' themselves. It is not research done 'on' others. Emphasised here are the conditions for the research in which teachers should be given the opportunities to fully participate in research.

— research which involves *collaboration* where teachers and researchers engage in research together. Everyone in the project strives towards developing her/his own professional competence with the support of each other. The collaboration may also include cooperation with people who are linked to, but not directly involved in, the research project.

— research which is *emancipatory*. Participatory action research may afford teachers to see and analyse the mechanisms that put limitations to their work as mathematics teachers.

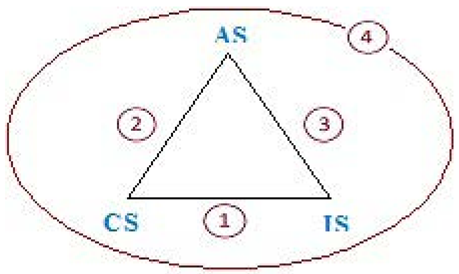
— research which is *critical*. Included here is that the research itself seeks to challenge a mathematics teaching that do not provide equal opportunities for all students to learn mathematics.

— research which is *reflexive* in that it goes in two directions. The participants in the research investigate their practice, *and* they also aim to change it.

METHODS AND FINDINGS

This is a case study where we reanalysed data from an action research project with a specific interest in power relations between teachers and researchers. The data consist of participants' logs. We also analysed audio recordings from three of the meetings in the project as well as notes from these meetings.

In the action research project, a model by Skovsmose and Borba (2004) was adopted with an addition by Björklund Boistrup and Norén (2013), see Figure 1.



CS: Current situation
IS: Imagined situation
AS: Arranged situation
1: Pedagogical imagination
2: Practical organisation
3: Explorative reasoning
4: Scrutinising the institutional context

Figure 1: (Skovsmose & Borbra, 2004) plus (4) from (Björklund Boistrup & Norén 2013)

This model frames our account of the findings that is presented on the poster while we also use the mentioned characteristics by Atweh (2004) as analytical concepts. One finding we elaborate on is how teachers went from acting as observing participants towards participatory researchers during the project, while the responsible researcher went from research leader towards a facilitating participatory researcher. Simultaneously there were constraints identified in the institutional context which affected the affordances for participants to take on new roles.

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Mathematics in agriculture and vocational education for agricultures

Karolina Muhrman

Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden, karolina.muhrman@nbg.nu

This paper presents some of the results from a research study about mathematics in agriculture and agricultural education. The purpose of the study is to investigate the role of mathematics as a professional knowledge, and how to organize vocational secondary school education so that students receive the math knowledge and skills they need for their future profession. The study is done from a curricula theoretical perspective with concepts from Bernstein (2000). The results indicate that mathematical skills are essential for a professional farmer, but according to those interviewed, many agricultural students have deficient mathematical skills for their future profession after their education.

Keywords: Mathematics, workplace, agriculture, vocational education, Bernstein.

INTRODUCTION

In Sweden it has in recent years been pointed out from various professions that vocational students do not have the desired or adequate mathematical skills for the professional life after their education. Based on that I found it interesting to study the reasons for vocational students to learn math, which math skills the professionals deem most important, how this is reflected in the school's math education and also how the students' mathematical knowledge affects their future career possibilities within their profession.

THEORETICAL FRAMEWORK

The study is done from a curricula theoretical perspective with concepts from Bernstein (2000). Bernstein's pedagogical codes, classification and framing and his theory of discourses is used to analyse how the curriculum is realized in mathematics education and how this may affect students' professional career.

METHOD

To answer the questions, I have in the winter of 2012 and the spring of 2013, done interviews with 15 professional farmers, 13 vocational teachers in agriculture, 11 mathematical teachers who teach agriculture students and 40 secondary vocational agriculture students. The students are interviewed in focus groups. The teachers and students come from eight different schools with vocational education in Sweden.

RESULTS

The results indicate that mathematics is an essential professional skill for farmers because they use mathematics all the time. Farming profession is today very advanced and as a farmer you do not only have use of practical skills. The farmers that I interviewed gave examples of 54 different job tasks that require skills of mathematics. Many of the farmers also said they do not want to hire someone who doesn´t have sufficient skills of mathematics since miscalculations can mean costly mistakes. The farmers claimed that they did not need any advanced skills but that they must have very good basic mathematical knowledge, since there is often advanced applications of the mathematics (cf. Fooreman & Steen, 1995). It is mainly the areas of percent, geometry and statistics that farmers need to have knowledge about. Farmers must also be very good at mental calculation, rough calculation and plausibility assessments.

The interviews revealed that many students did not really seem to understand the importance of learning mathematics and they claimed it to be boring. Students' lack of motivation to learn mathematics results in deficient mathematical skills to handle the calculations appearing in the profession. Lack of mathematical skills also results in that many students do not pass the professional certificates available in agriculture. To learn the required mathematics of the farming profession, many of those that were interviewed advocated that mathematics- and vocational classes should be largely integrated, and that even the professional farmers should be responsible for teaching mathematics. Students working with an integrated mathematics education say that it gives them more motivation, they learn better, and understand and remember more, when it's related to their reality. Using concepts from Bernstein, an integrated code was requested with a more a context-bound and horizontally organized mathematics teaching. But most of the schools in the study has what Bernstein calls a strong classification, where the school mathematics is clearly separated from other subjects and from work outside of school. At these schools the teachers have no or little cooperation with each other. Hierarchies and power relations described by Bernstein between different categories of teachers, were found to be prevalent. In order to enable an integrated code both the teachers and the school management must agree. An easy way to create a collaboration between vocational- and mathematics teachers, and thereby increase an integrated mathematics education, seemed to be to let these categories of teachers have a shared office space.

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Discussing teaching/learning methods in a complex educational context

Alessandro Spagnuolo and Michele Canducci

University of Ferrara, Department of Mathematics and Computer Science, Ferrara, Italy, spglsn@unife.it

This paper describes the state of research concerning an international instruction collaboration between Italy and Tanzania. During the last five years of the project it has been possible to detect some of the educational aspects that might have a negative impact especially regarding mathematics. Starting from this analysis, it was elaborated a peer education proposal as a possible didactic methodology suitable to be used in the school context. Research was made and the last results show that the aforementioned methodology could be particularly useful in this specific educational environment.

Key words: African secondary education, peer education, CLIL, complexity.

INTRODUCTION

Since 2010 the Akap association (Italy), has brought together a group of researchers in mathematics education from the University of Bari, Bologna and Ferrara and teachers at Daudi Secondary School (Tanzania). Each year a meeting among these teams is organized in order to discuss and experience learning opportunities for Tanzanian pupils which respects the many interrelated factors operating within the school context.

THEORETICAL FRAMEWORK

Mathematics is recognized as important by governments all over the world. Why is it so important to teach it? Many reasons are provided and debated by influential people, both in mathematics education but also in government organisation. In many countries so-called "underdeveloped", is still frequently the opinion that school systems must teach content that in the past were decisive factors in the industrialisation of "developed" countries, without thinking on the historical, social and cultural reasons behind their inclusion. Instead it has been suggested that the goal for "under-developed" countries is not to get where "developed" countries are today, but to develop and build a educative model that is appropriate for the social organization, based on ethical values which seek the common good (D'Ambrosio, 1997).

1) The data analysis published by the Tanzanian Ministry of Education and Vocational Training [MoEVT] at the end of the first "Secondary Education Development Programme" (June 2010) showed a progressive decrease in the number of students promoted in the two main national examinations: the PSLE (Primary School Leaving Examination) and the CSEE (Certificate for Secondary Education Examination).

2) Our investigation focuses on promoting quality in the education offered within a particular context in Tanzania, characterized by specific social, cultural and political factors. Through studying the MoEVT official documents and a collection of testimonials from Tanzanian teachers and students, we became aware of the complexity[1] of the issues regarding teaching in an environment so different from the European one. Some of the issues that the teachers had to contend with were: overcrowding classes; lack of textbooks and teaching aids; English as the language of instruction instead of Swahili mother tongue; poor teacher recruitment program. Consequently, the teams developed a peer education proposal for this specific context. According to Cooperative

1 We thought that the theory of complex systems in social sciences could allow us to manage the multiple dynamics between the components of the project, through a dialectic that goes continuously from the global to the specific. The main idea is that acting on a component means acting on the whole system, with a retroactive view

Learning Theory (Johnson & Johnson, 1980), there are many reasons to investigate this methodology: improvement in all students' academic performance; increased motivation to study, due to the development of psychological well-being and a positive climate into the classroom.

MAIN RESEARCH RESULTS

The latest results show that a peer education program could be particularly useful in this school. The mathematics workshops carried out in 2014 confirm the usefulness of a teaching approach based on social mediation for Daudi Secondary School. The division of the students into groups, the organization of specific roles assigned to each member and the use of appropriate working schedules seem to overcome the problem of the high number of students in each class and to involve each pupil in all of the planned activities. The possibility of discussion within the groups has led the students to overcome language barriers related to the use of English and to develop skills in critical reasoning and understanding mathematics.

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No reason to believe in numbers: Using a commercial for making ethical filtration visible

Hauke Straehler-Pohl

Freie Universität, Berlin, Germany, h.straehler-pohl@fu-berlin.de

This poster illustrates how an exercise of ideology critique can employ a commercial for making ethical filtration visible in the context of Critical Mathematics Education.

Keywords: Critical mathematics education, ideology critique, ethical filtration

CONTEXTUALIZATION IN CRITICAL MATHEMATICS EDUCATION

We are living in a mathematised world. More and more aspects of social live are penetrated by mathematics-based technologies. As Davis and Hersh (1986, p. 17) make us aware, "we should observe these developments critically, as they could do damage to all of us". This accounts *a fortiori*, as the trend of mathematisation is accompanied by a simultaneous trend of demathematisation: while mathematics becomes more and more pervasive in all spheres of life, it vanishes more and more from the surface and becomes more and more invisible (Jablonka & Gellert, 2007). The emergence of Critical Mathematics Education (CME, e.g., Skovsmose, 1992) has brought critical attention to the role of mathematics in our world. The demand for critical awareness has in the meantime grown beyond the boundaries of CME and has reached the mainstream of Mathematics Education (ME). That school mathematics *must* contribute to critical citizenship has become common sense within the field of ME (Pais, 2012). As it appears, what has been lost on the way is the care for the *limits* of mathematics for handling socially critical issues – the juxtaposition of potentially more relevant non-mathematical strategies and their sincere consideration (Skovsmose, 1992). It appears that the problematizing of socially critical issues within the mathematics classroom tends to structurally prioritize strategies that make use of mathematics. Skovsmose (2008) has described and de Freitas (2008, p. 87) has further analysed this effect as *ethical filtration*: "the agent [...] reduces the complex ethical and political situation to a set of abstract parameters which are then combined into a simplifying model primarily used for prediction. According to Skovsmose (2008), ethical filtration is built into the practice of mathematics in action". She complements that "mathematics education *actually* teaches students to enact ethical erasure so as to successfully generate solutions that seem unambiguous" (p. 88). A truly CME makes this ethical filtration visible and – where necessary – promotes support to put mathematics back to its refined place.

CONTENT OF THE POSTER

The poster suggests an activity that reverses the process of ethical filtration by carrying out an ideology-critique that problematizes the improper use of numbers – improper in this case is not meant as a use of "wrong" numbers, but that the use of numbers itself is a dead-end for the social issue under question. The poster illustrates how the philosophy of Slavoj Žižek (e.g., 2008) can serve as a means to deconstruct the improper instrumentalisation of numbers for a rankly capitalist interpellation. In a first step, the poster illustrates how cultural capitalism simultaneously installs the contradictory imperatives of a) acting sustainable and b) increasing consumption, and hereby releases an unconscious feeling of guilt within the interpellated subject. Then the poster illuminates how the commercial instrumentalises this unconscious feeling of guilt in order to be able to promise liberation from this guilt - through consumption. In a third step the poster illuminates the role of numbers in the construction of the commercial's message. It shows, how the success of the commercial is dependent on the effect that the viewer believes in the fantasy that the numbers are indeed relevant and yet simultaneously maintains a critical distance to the numbers – as taking them as actual facts that really have a meaning would lead to disintegration of the entire commercial's appeal. In a fifth step the poster proposes "over-identification" (Žižek, 2008, p. 29) as a pedagogic strategy that treats the numbers and numerical relations that are employed in the commercial as if they were *actually real.* This leads to a reversal of ethical filtration, as the initial ambiguity of the complex social problem is brought back into light and further reveals the cynicism that is implicated in the advertising campaign. Finally, over-identification is proposed as a pedagogic strategy that allows for bringing social and political issues into the mathematics classroom, without stepping into the trap of ethical filtration.

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Mathematics diagnosis and support: Sensitivity of pre-service teachers to social disparities

Vivica Zweidar

University of Bremen, Bremen, Germany, vivica@math.uni-bremen.de

Mathematical diagnosis and support require a high level of professional expertise from teachers. However, profound subject knowledge and the awareness of pupils' processes of learning mathematical concepts and strategies are not sufficient for appropriate diagnosis and support. Raising teachers' awareness of hidden mechanisms of discrimination within the practice of diagnosis and support could be a valuable contribution in the struggle for more educational equity.

Keywords: Diagnosis, support, Numeracy Project, pre-service teachers, sensitivity.

THEORETICAL BACKGROUND

In the organisation of pupil-centred teaching and learning in school, diagnosis and support play a significant role. Diagnosis means for teachers that they need to recognize pupils' current knowledge and proficiencies (Wartha & Schulz, 2012), so that they can appropriately foster learning. This fostering by a teacher can be labelled 'support'. In the mathematics classroom, teachers are faced with the challenge of using mathematically and didactically appropriate methods of diagnosis and support, while at the same time bearing in mind the heterogeneity of the pupil body. This contribution focuses on teachers' reflectiveness using mathematical diagnosis and support materials from New Zealand's "Numeracy Project (NP)" in a German classroom.

In a quantitative study by Irwin and Irwin (2005), children from lower socioeconomic backgrounds were shown to benefit less from diagnosis and support with NP materials compared to children who were ranked more highly on a socioeconomic scale. My contribution focuses on how sensitive prospective teachers are to both subject-specific and generic aspects of social heterogeneity. Generic perspectives in particular, such as language or contexts in problem statements, may appear insignificant for mathematics teaching at first glance. They can, however, become relevant in mathematical diagnosis and support (e.g., Cooper & Dunne, 2000; Jablonka & Gellert, 2010), and lead to discrimination against certain pupil groups, or at least put the adequacy of diagnosis and support tools for them into question.

The overall hypothesis I adopt is that attitudes and behaviours of teachers are crucial for effective, pupil-oriented education. Against this background, not only teachers' subject-related levels of proficiency concerning tools of diagnosis and support are important, but also their awareness of potential discrimination. This discrimination within diagnosis and sup-

critical, reflecting dimension (relating to the NP materials)	high awareness of socioeconomic and social bias	SOCIOCRITICAL SENSITIVITY	MULTI-PERSPECTIVE SENSITIVITY
	no/low awareness of socioeconomic and social bias	NO/LOW SENSITIVITY	SUBJECT-ANALYTICAL SENSITIVITY
		no/low awareness of the NP material's design and goals	high awareness of the NP material's design and goals
		subject-specific (mathematical, mathematic educational, pedagogical) dimension	

Table 1

port can take place by treating unequal individuals identically as well as by treating equal individuals differently (Gomolla & Radtke, 2009).

MY RESEARCH

For the purpose of my research I conducted five guided interviews with pre-service teachers from the University of Bremen who diagnosed and taught pupils in a German school with translated NP materials.

The analysis of the interviews revealed that most pre-service teachers showed mathematics education-related sensitivity to a certain extent and at the same time almost no awareness of socioeconomic and social bias in the NP material. They were not even aware that language plays a significant role in the NP and that language can be a filter to allow access to certain practices in the mathematics classroom exclusively to particular groups of pupils. All interviewed pre-service teachers had to be placed in the 'sensitivity-matrix', as shown above, in the field of 'no/low sensitivity' or at best in the field of 'subject-analytical sensitivity'.

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TWG11

Comparative studies in mathematics education

Introduction to the papers of TWG11: Comparative studies in mathematics education

Paul Andrews[1], Eva Jablonka[2], Arne Jakobsen[3] and Constantinos Xenofontos[4]

1 Stockholm University, Stockholm, Sweden, paul.andrews@mnd.su.se

2 King's College London, London, UK, eva.jablonka@kcl.ac.uk

3 University of Stavanger, Stavanger, Norway

4 University of Nicosia, Nicosia, Cyprus, xenofontos.c@unic.ac.cy

As with earlier CERMEs, TWG11 adopted an eclectic perspective in its interpretation of comparison as referring to any study that documents, analyses, contrasts or juxtaposes cross-cultural or cross-contextual similarities and differences across all aspects and levels of mathematics education. In this way, the TWG aimed to encourage critical but supportive discussion that would enable colleagues to:

- share findings and outcomes of empirical studies that adopt a comparative approach;
- outline the delineation of comparative and non-comparative research;
- develop and refine research methodologies specific to comparative studies;
- explore the interaction of macro-level survey studies and micro-level case studies;
- understand how various theoretical approaches and conceptual frameworks shape the goals and the design of comparative research;
- understand how comparative studies can inform teaching and learning practices;
- understand the role of culture in the construction of mathematics teaching and learning

A recurrent but very productive aspect of this working group has been the relatively small number of paper presentations. This year thirteen papers created space not only for colleagues to share their research in some detail but allowed participants to engage in lengthy and inclusive discussions on the nature of comparative mathematics education research and the means by which it can be meaningfully and rigorously undertaken. The papers, along with two posters, were themed in various ways, highlighting significant substantive and methodological variation. However, two theoretical papers provided an interesting and provocative introduction and conclusion to the sequence of papers. The introduction being Jablonka's "rationales for comparative classroom studies in Mathematics education" and the conclusion being Clarke's examination of "the role of comparison in the construction and deconstruction of boundaries". Between these bookends five themes were examined by means of papers and posters authored by colleagues, due to their nationality or current professional location, representing Australia, Austria, China, Cyprus, England, Finland, Germany, Ghana, Greece, Hungary, Italy, Japan, Kosovo, Poland, South Africa, Spain and Sweden.

Firstly, three papers offered differently conceptualised perspectives on how the intended curriculum is reified in official documents and school textbooks, particularly in contexts where the latter are regulated by the curriculum authorities. These were Vula and colleagues' analysis of the treatment of fractions in Kosovan and Albanian mathematics textbooks, Xenofontos and Papadopoulos' study of the ways in which the history of mathematics is incorporated into the lower secondary textbooks of Cyprus and Greece and Gosztonyi's comparison of the ways in which the "new math" permeated Hungarian and French mathematics education discourses.

Secondly, two papers and two posters confirmed that mathematics and the manner of its assessment is a cultural construction. The papers were Peng and colleagues' comparison of Swedish and Chinese teachers' perspectives on what constitute good national mathematics test tasks, Branchetti and colleagues' longitudinal analysis of the Italian national standardized mathematics tests. The posters, which were also discussed, were Lemmo's comparison of paper and pencil and computer based mathematics assessments and Bauer and colleagues' provocative appeal for teachers to unite in addressing the expectations of PISA.

Thirdly, two papers, investigating different aspects of mathematics affect, highlighted the paucity of research undertaken in African contexts and the frequent inappropriateness for use in those contexts of tools developed in the West. These were Bofah's demonstration of a reciprocal determinism between students' mathematics self-concept and achievement in five culturally diverse African contexts and Joubert's use of social media to examine the perceived causes of mathematics problems in England and South Africa.

Fourthly, two papers framed a discussion on the processes of mathematics. These were Saeki and colleagues' comparison of how Japanese and Australian students responded to a mathematical modelling intervention and Sajka and Rosiek's use of eye tracking technology to examine differences between different ability groups' approaches to mathematical problem solving.

Finally, two papers offered different methodological perspectives on the use of classroom video-recordings in comparative mathematics education research. These were Andrews and colleagues' exploitation of a foundational number sense framework to analyse learning opportunities for grade one students in Poland and Russia and Hommel and Clarke's analysis of how teachers, in four different cultural contexts, use questions to encourage students to reflect on their learning of mathematics.

In sum, the papers presented to the group reflected not only cultural diversity but also methodological pluralism, allowing a number of commonalities to emerge. Most presented studies aspired to data flexibility, with few being constrained by a priori theoretical assertions. Classroom processes have both form and function and a comparative study of one may transform the other. In comparative research the responsibility on the researcher to define adequately one's concepts and constructs is of great significance, particularly with respect to large scale tests of educational achievement. Lastly, in undertaking comparative research one should be mindful of the possibility of misapplying a set of culturally informed values.

TWG11

Research papers

Developing foundational number sense: Number line examples from Poland and Russia

Paul Andrews[1], Judy Sayers[1] and Gosia Marschall[2]

1 Stockholm University, Stockholm, Sweden, paul.andrews@mnd.su.se

2 Comberton Village College, Cambridge, UK

For a variety of reasons, children start school with differing number-related skills, leading to differences in later mathematics achievement. Such differences prompt the question, what number-related experiences are necessary if the first year of school is to prepare children appropriately for their learning of mathematics? In this paper, we discuss the development of an eight dimensional framework, foundational number sense (FoNS), that characterises those learning experiences. We then demonstrate the framework's analytical efficacy by evaluating episodes from two sequences of lessons, one Polish and one Russian, focused on the use of the number line. The results show that the FoNS framework is cross-culturally sensitive, simply operationalised and analytically powerful.

Keywords: Foundational number sense, grade one mathematics, Poland, Russia, number line.

INTRODUCTION

Evidence internationally shows that the depth of a child's number sense predicts later mathematical success (Aubrey, Dahl, & Godfrey, 2006; Aunola, Leskinen, Lerkkanen, & Nurmi, 2004). For example, basic counting competence has predicted later successes in, for example, Canada, England, Finland, Flanders, Taiwan and the USA respectively (LeFevre et al., 2006; Aubrey & Godfrey, 2003; Aunola et al., 2004; Desoete, Stock, Schepens, Baeyens, & Roeyers, 2009; Yang & Li, 2008; Jordan, Kaplan, Locuniak, & Ramineni, 2007). Also, under-developed number sense leads to later mathematical failure (Jordan, Kaplan, Ramineni, & Locuniak, 2009; Gersten, Jordan, & Flojo, 2005; Malofeeva, Day, Saco, Young, & Ciancio, 2004). Thus, understanding number sense and how it can be promoted seems a sensible goal.

However, despite its importance, number sense has been poorly defined (Griffin, 2004, not least because mathematics educators and psychologists work with different definitions (Berch, 2005). Indeed, "no two researchers have defined number sense in precisely the same fashion" (Gersten et al., 2005, p. 296). Over the last two years we have been working on overcoming this definitional impasse. At the same time, we have tried to develop a classroom-focused analytical framework that is simple to operationalise and sensitive to different cultural practices. In this paper, we summarise our progress before evaluating the framework, foundational number sense (FoNS), against case study lessons from Poland and Russia.

EARLIER WORK ON FOUNDATIONAL NUMBER SENSE

As indicated above, this paper draws on earlier work, including a paper presented at CERME8 (Back, Sayers, & Andrews, 2014). Since then our understanding of number sense in general and foundational number sense in particular has developed further. For example, our initial reading identified two broad conceptions of number sense. Today we argue for three, including the earlier two. The first, preverbal number sense, refers to those number insights innate to all humans and comprises an understanding of small quantities in ways that allow for comparison (Ivrendi, 2011; Lipton & Spelke, 2005). For example, young babies can discern 1:2 but not 2:3 ratios (Feigenson, Dehaene, & Spelke, 2004). The second, applied number sense, concerns those number-related competences that make mathematics sensible for all learners and prepares them for an adult world (McIntosh, Reys, & Reys, 1992). It underpins many curricular specifications and much of the material written on number sense (See, for example, Anghileri, 2000). Finally, the primary focus of this paper, is foundational number sense (FoNS). This comprises those understandings that require instruction and typically arise during the first year of school (Ivrendi, 2011; Jordan & Levine, 2009). Unlike preverbal number sense, it is something children acquire rather than possess. Unlike applied number sense, its focus is not a world beyond school but later arithmetical and mathematical competence.

When developing the FoNS framework, our intention was not to construct an extensive list of characteristic learning outcomes but a small set of simple to operationalise components amenable to cross-cultural application. Our view was that extensive lists of number sense components and typically comprising around thirty components (Berch, 2005), would be unwieldy. Consequently, we exploited the constant comparison approach of the grounded theorists. Articles and book chapters typically addressing grade one students' acquisition of number-related competence were identified. These were read and broad FoNS-related categories identified. With each new category, previous articles were re-examined for evidence of the new. This approach, placed, for example, *rote counting to five* and *rote counting to ten*, two narrow categories discussed by Howell and Kemp (2005), within a broad category of systematic counting. Among the works examined in this process were (Aubrey & Godfrey, 2003; Aunola et al., 2004; Berch, 2005; Booth & Siegler, 2006; Clarke & Shinn, 2004; Dehaene, 2001; Desoete et al., 2009; Gersten et al., 2005; Griffin, 2004; Howell & Kemp, 2005; Hunting, 2003; Ivrendi, 2011; Jordan et al., 2007; Jordan & Levine, 2009; Lembke & Foegen, 2009; LeFevre et al., 2006; Lipton & Spelke, 2005; Malofeeva et al., 2004; Noël, 2005; Yang & Li, 2008).

In this paper, we summarise these eight FoNS components before showing how they play out in two post-Soviet educational contexts. This is the third case study pilot of the emergent FoNS framework, undertaken to ensure its viability for a large scale international study. The first case study examined two teachers, one in each of England and Hungary, working with grade one children on number sequences (Back et al., 2014). The analyses, based on an earlier seven component FoNS framework, indicated not only that the original framework's categories were sensitive to culturally different classroom traditions but also that the ways in which the categories combined resonated with earlier studies' showing high levels of didactical sophistication in Hungary and, in relative terms, low levels in England. The second case study, involving two teachers, one in each of Hungary and Sweden, focused on the ways in which children were encouraged to acquire the skills of conceptual subitising (Sayers, Andrews, & Björklund Boistrup, 2014). In this case, the findings, based on the revised eight component framework, again showed a sensitivity to cultural context and highlighted well how different approaches to the same topic yield different FoNS-related outcomes, pertaining again to different levels of didactical sophistication. This paper reflects a third, and final, pilot evaluation of the framework. Before presenting the analyses, however, we present a summary of the eight components, which derived from the literature review described above. To avoid repetition and save space, each component is summarised independently of the literature on which it is based. The components of foundational number sense are:

Number recognition: Children recognise number symbols and know their vocabulary and meaning. They can identify a particular number symbol from a collection of number symbols and name a number when shown that symbol;

Systematic counting: Children are able to count systematically and understand ordinality. They count to twenty and back, or count upwards and backwards from arbitrary starting points, knowing that each number occupies a fixed position in the sequence of all numbers.

Awareness of the relationship between number and quantity: Children understand the correspondence between a number's name and the quantity it represents, and that the last number in a count represents the total number of objects, its cardinality.

Quantity discrimination: Children understand magnitude and can compare different magnitudes. They deploy language like *bigger than* or *smaller than* and understand that eight represents a quantity that is bigger than six but smaller than ten.

An understanding of different representations of number: Children understand that numbers can be represented differently, including the number line, different partitions, various manipulatives and fingers.

Estimation: Children can estimate, whether it be the size of a set or an object. Estimation involves moving between representations of number; for example, placing a number on an empty number line.

Simple arithmetic competence: Children perform simple arithmetical operations, which Jordan and Levine (2009) describe as the transformation of small sets through addition and subtraction.

Awareness of number patterns: Children extend and are able to identify a missing number in a simple.

Importantly, the eight FoNS components, while distinct, are not unrelated. This is because number sense "relies on many links among mathematical relationships, mathematical principles..., and mathematical procedures" (Gersten et al., 2005, p. 297), links that help avoid situations where children can count but not know that five is bigger than three.

METHODS AND RESULTS

The data examined in this study derived serendipitously from video-based teacher professional development programmes. However, both sets of lessons exploited the number line with grade one children and, therefore, proved amenable to a topic-based FoNS-related analysis. Both teachers, construed locally as effective, were video-recorded in ways that would optimise capturing their actions and utterances. Lessons, with transcripts, were scrutinised by at least two of the three authors with the intention of identifying episodes suitable for demonstrating a range of FoNS-related opportunities. In the following we present three episodes from each teacher's lessons as examples of the ways in which they worked with the number line.

The Russian episodes

The Russian teacher, Olga, began by sketching a horizontal line across the board, telling her class that this was a number line before asking what was missing. Over the next two or three minutes, four volunteers, with appropriate commentary from Olga, completed the number line as follows. The first drew a small arrow at the right hand end of the line to signify that numbers go from left to right also extend indefinitely. The second drew a small flag near to the line's left hand end to represent the start or zero point. The third, using what looked like a postcard, added regular intervals, as shown in Figure 1. The fourth added the integers correctly.

Figure 1: Marking the number line with an arbitrary unit

Commentary: In this first episode can be discerned several FoNS components. The use of the number line reflected an expectation that students would engage with different representations of number. The manner in which the line was constructed, using a repeated measure, implicitly addresses the relationship between a number and the quantity it represents, while the process of numbering the line, including the emphasis on the placement of a zero, highlight both number recognition and systematic counting.

The lesson now progressed to the class using this new number line. A girl came to the board and was told to show three. This she did by pointing to the flag (zero) with the index finger of her left hand and three with her right. Next, she was asked to show five, which she did in the same way. This was followed by Olga asking the girl to show how she would get from three to five, which she did by counting on two units.

Commentary: Within this episode, which was no more than two minutes in length, can be seen at evidence of least five FoNS components. The manner in which three, and other numbers, was demonstrated highlighted not only an identification of the symbol but also how three's position reflected a measure of units, essentially arbitrary, along an axis. In other words, it reflected the relationship between number and quantity. The task included an expectation that learners would count systematically, work with a particular representation of number, and engage with simple arithmetical operations.

In related fashion, the next task involved starting with seven and subtracting three. While the girl concerned initially struggled to stretch her arms sufficiently to reach seven, as shown in Figure 2, she seemed confident with the mathematics. On Olga's invitation, she counted out to seven, while keeping her left hand forefinger at the flag (zero). Next, she was invited to count back three spaces, which she did. Olga asked the class what the girl's action signified and was told that she had subtracted three from seven to get four.

Figure 2: A demonstration of seven on the number line

Commentary: Within this third episode, which lasted less than a minute, can also be discerned at least five FoNS components. The manner in which seven was demonstrated highlighted not only an identification of the number symbol but also the relationship between number and quantity. The task included an expectation that learners would count systematically, work with a particular representation of number, and engage with simple arithmetical operations.

The Polish episodes

Maria began her lesson by inviting each child randomly to the front to receive a sticker placed on his or her chest bearing a number, with Maria beginning by giving herself zero. She then asked the class to arrange itself in numerical order in a line down the middle of the room. Once this was done, she asked the class to return to their seats before asking them to repeat the task as quickly as possible. On this occasion, with great excitement, the class arranged itself within a few seconds.

Commentary: With respect to the FoNS components, Maria's actions were commensurate with her encouraging her students to recognise numbers and, essentially, count systematically. It also highlighted the extent to which the units of the number line are arbitrary and the use of the number line as a representation of number.

Later in the lesson Maria presented a number line with units but no numbers. She asked what should be placed at the end and was told zero, before being told that this should be followed by one, two and so on. At this point she asked her students to complete the number line on their sheet, as shown in Figure 3. Next she asked what would happen if the first marked point had been two and not one. This initiated a discussion on the importance of each unit being a representation of the same value with the consequence that the line would now show even numbers, 0, 2, 4, 6 and so on. Finally, in response to her asking what would happen if the first number had been three, it was agreed that the sequence would go 0, 3, 6, 9, 12, ... with each successive number being found by counting on three.

Commentary: Within this episode could be discerned five FoNS categories. At the most obvious level, Maria was attending to number recognition, different representations of number and systematic counting. Also, the introduction of the multiples focused attention on number patterns and, in the counting on of threes, simple arithmetic.

Later, Maria sketched a number line from zero to fifteen on the board. She marked a point at five, and asked her students to do the same. She then wrote 5 + 7 = before showing how the sum can be counted out, as in Figure 4. Following this she asked her students to do the same on their sheets. The students were

Figure 3: Completing the number line

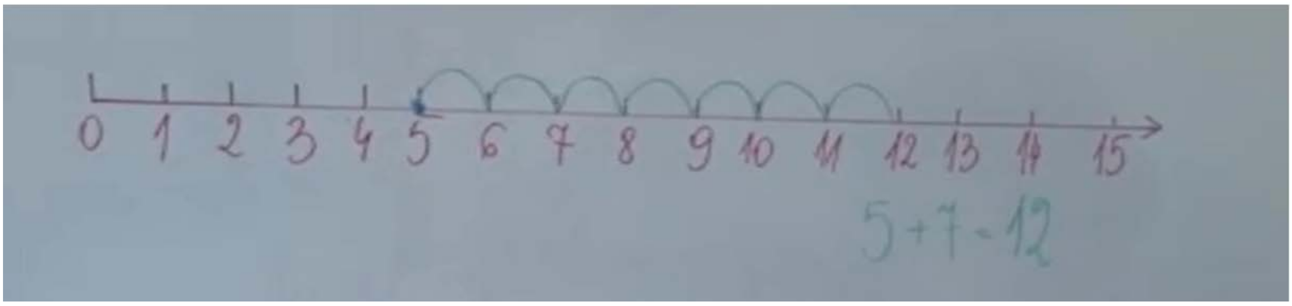

Figure 4: Using the number line to undertake simple arithmetic

then invited to repeat the process for 6 + 6 = , 14 − 9 = , 5 + 6 − 4 = . After each one, Maria repeated the process on the board, with instructions provided by different students. She counted very slowly and deliberately as she marked off each number on the way.

Commentary: In this final episode, in addition to the already familiar number recognition, systematic counting and number representations, Maria was attending to simple arithmetical operations and how they can be modelled on the number line. Interestingly, she did not just focus on addition but included subtraction and, with the third problem, both operations.

DISCUSSION

The analyses above, summarised in Table 1, indicate both similarities and differences in the ways in which FoNS was addressed. In respect of similarities, both teachers addressed, over the course of the analysed episodes, five categories. Both encouraged, throughout their respective excerpts, students' recognition of number alongside, systematic counting and awareness of different representations of number. Both also, but not to the same degree, focused attention on simple arithmetic operations. Overall, and bearing in mind the number line focus, such findings were of little surprise and, from the perspective of validating the framework, helpful – an analytical framework that failed to identify the expected would be of little use.

While it is always important to acknowledge similarities, differences are frequently more enlightening. On the one hand, Olga emphasised, through her insisting that students make a bodily link between a number and zero, the connection between number and quantity. On the other hand, through her discussion of multiples of two and three, Maria was seen to use the number line to support children's engagement with simple sequences. However, it is our view that such findings reflect not insignificant qualitative differences in the teachers' emphases. Olga's exploitation of the bodily link avoided too early a shift to working with numbers as abstract entities. This seems a more significant didactical decision when compared to Maria's emphasis on sequences, albeit a key category of FoNS in its own right. Such qualitative differences show that the FoNS framework has the propensity for highlighting, in much the way that generic learning outcomes exploited in other studies have shown, both simple analyses based on the frequencies of particular events and more sophisticated analyses based on the interactions of those events (see Andrews & Sayers 2013).

In this paper, we have shown how opportunities for students to acquire FoNS played out in two post-Soviet classrooms. This is of particular interest in the light

	Olga's episodes			Maria's episodes		
Number recognition	X	X	X	X	X	X
Systematic counting	X	X	X	X	X	X
Relationships between numbers and quantities	X	X	X			
Quantity discrimination						
Different representations of number	X	X	X	X	X	X
Estimation						
Simple arithmetical operations		X	X			X
Number patterns and missing numbers					X	

Table 1: the distribution of the categories across the episodes

of recent evidence that the rise of the free market had had markedly different impacts on student achievement on international tests (Bodovski, Kotok, & Henck, 2014). In this respect, Poland's PISA mathematics scores, reflecting students' real-world application of mathematical knowledge at age 15, have risen from significantly below to significantly above the international mean at a time when Russia's have remained largely static, constantly significantly below the international mean. These scores are interesting when set against Russia's grade 8 TIMSS scores, assessments of students' technical competence, has been consistently above the international mean. With respect to TIMSS grade 4, Russia has been consistently one of the higher achieving nations, while Poland has remained significantly below the international mean. In other words, and putting the case crudely, if such tests tell us anything it is that Polish students are increasingly competent on real-world mathematical tasks located in text and requiring a degree of interpretation, which Russian students are not, while Russian students are strong on mathematics tasks located in world of technical mathematics, which Polish students are not. In light of this, what do our limited analyses have to say?

Firstly, acknowledging the limited sample presented here, Olga's teaching seemed more focused on the structural properties of number and mathematics than Maria's. Not once did Olga make any number line-related reference to a world outside the classroom. Her efforts were focused constantly on a world contained solely within mathematics. Maria, on the other hand, although not reported above for lack of space, made frequent use of different representations of the number line drawn from the real world. For example, at different times she made reference to several thermometers, each of which presented a different scale and starting value, different tailors' measuring tapes, a carpenter's retractable tape, a measuring jug, various skeletons of fish and snakes showing, in particular, the regular spread of ribs. Each one elicited a brief discussion as to its purpose and relationship to the number line. On a separate occasion she based a counting activity on the use of a representation of a hotel lift travelling between the many floors of a very tall hotel. Thus, these differing emphases, essentially unrelated to the mathematics being taught, may have profound implications for students' successes on international tests of achievement. Olga's teaching seems unrelated to PISA expectations and Maria's to TIMSS. Such matters clearly require further examination.

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Reciprocal determinism between students' mathematics self-concept and achievement in an African context

Emmanuel Adu-tutu Bofah

University of Helsinki, Department of Teacher Education, Helsinki, Finland, emmanuel.bofah@hotmail.com

The study tests the theoretical and methodological models of the direct feedback loop in which mathematics self-concept and achievement are specified as both causes and effects of each other using the TIMSS–2011 cross-sectional data set. The participants were students in grades 8–9 from five African countries participating in TIMSS–2011 (N=38,806, M_{Age} =15.42, SD =1.37). Using nonrecursive structural equation models, the author examined the reciprocal-effects model indicating that achievement has an effect on self-concept (skill-development model) and that mathematics self-concept has an effect on achievement (self-enhancement model). There was support for the skill-development, self-enhancement as well as direct feedback loop models. Discussion centres on the theoretical, methodological, and practical implications of the results.

Keywords: Mathematics self-concept, achievement, reciprocal determinism, TIMSS.

INTRODUCTION

Cross-national comparative studies such as the TIMSS and PISA have recently gained considerable attention. Research on TIMSS and other large-scale surveys has consistently shown strong relationship between achievement and self-beliefs across nations (Chiu & Klassen, 2010; Huang, 2012; Marsh et al., 2013; Parker, Marsh, Ciarrochi, Marshall, & Abduljabbar, 2014). Studies reporting different levels of self-belief (e.g., self-concept) across nations normally report with strong theoretical backing, a common trend of lower self-concept among Asian countries when compared to other nations (e.g., Wilkins, 2004).

Culturally, the causal relation between affect (e.g., self-concept) and achievement has been demonstrated to be valid cross-culturally, but more typically in the Western hemisphere (e.g., Seaton, Parker, Marsh, Craven, & Yeung, 2014). Moreover, other studies have shown that self-belief (e.g., self-concept) may operate differently across cultures (Chiu & Klassen, 2010; Markus & Kitayama, 1991) due to the fact that the self is highly influenced by *the frame of reference effect*—social comparison, causal attribution, and reflected appraisals from significant others (Bong & Skaalvik, 2003).

In the present study, the nature of the relationship between students' mathematics self-concept and achievement was investigated using a non-recursive structural equation models in the five African countries that participated in TIMSS–2011. The bidirectional cause–effect between affect and achievement is of practical importance because many affective enhancement programs as well as educational policy statements throughout the world are based on the fact that an improvement in affects (e.g., self-concept) will lead to better academic achievement. The full intent is to test the reciprocal relationship between affect and achievement. The study is also based on the fact that indigenous research and theorising which integrate cross-cultural perspectives are crucial to the establishment of more useful and universal theories (e.g., van de Vijver & Leung, 2000). As Chiu and Klassen (2010) put it: "Cultural differences in self-beliefs can challenge the foundations of current theories and provide new ways of looking at the self" (p. 2). Furthermore, there is a paucity of cross-cultural studies on domain-specific self-concept and achievement in the African context. The data for the present study is from TIMSS–2011, which provides a comparable open data source for these analyses.

The causal determinism of affects and achievement

Self-belief (e.g., self-concept) theories have developed into several branches (for a review, see Wang & Lin, 2008). One such branch concerns the differing relationships between self-belief and achievement (Calsyn & Kenny, 1977; M-S. Chiu, 2012). Within this context are competing hypothesis such as the *self-enhancement model*, which posits that self-belief is a determinant of academic achievement (Chiu & Klassen, 2010; Marsh & O'Mara, 2008), and the *skill-development model*, which sees self-belief as simply a reflection of performance (Chen, Yeh, Hwang, & Lin, 2013; Chiu & Klassen, 2010; Ma & Xu, 2004; Wang & Lin, 2008). Moreover, a more realistic and logical compromise between the self-enhancement and skill-development models is the reciprocal-effects model (Guay, Marsh, & Boivin, 2003; Hannula, Bofah, Tuohilampi, & Metsämuuronen, 2014; Parker et al., 2014; Seaton et al., 2014), which posits that prior self-belief influences subsequent achievement and prior achievement influences subsequent self-belief (for an in-depth review, see Guay et al., 2003).

Cross-culturally, reciprocal effect models have been found to exist. For instance, evidence has been found in Germany (Marsh & Köller, 2004), Finland (Hannula et al., 2014), Canada (Guay et al., 2003), Australia (G. Marks, McMillan, & Hillman, 2001), United States (Marsh & O'Mara, 2008), and many OECD countries (Williams & Williams, 2010) but not in the African context. However, on the relationship between affect and achievement, some studies have indicated affect as the strongest predictor of achievement (Marsh & O'Mara, 2008; Morony, Kleitman, Lee, & Stankov, 2013), whereas others have indicated achievement to be the strongest predictor of affect (e.g., Hannula et al., 2014; Ma & Xu, 2004). Other studies have found no causal relationships (e.g., Williams & Williams, 2010). Taken together, no firm conclusions can be drawn about the causal ordering of affect and academic achievement cross-culturally.

Assumptions in cross-sectional models in causal determinism

Modeling reciprocal effects with cross-sectional data sets is uncommon in mathematics-related affect studies; rather the normal approach uses a longitudinal data set. The arguments lie in the methodological challenges, assumptions and theoretical considerations associated with cross-sectional data to examine the causal ordering (Kline, 2011; Ma & Jiangming, 2004; Wong & Law, 1999). Structural equation models that are used to estimate the reciprocal relationships involving cross-sectional data are known as non-recursive models. Kaplan, Harik, and Hotchkiss (2001) indicated that data from cross-sectional designs only give a "snapshot" of an ongoing dynamic process; as such estimations of reciprocal causal effects with cross-sectional data sets require the assumption of *equilibrium*. Kline (2011 p. 108) summarized that: "any changes in the system underlying a presumed feedback relation have already manifested their effects and that the system is in a steady state. That is, the values of the estimates of the direct effects that make up the feedback loop do not depend on the particular time point of data collection". For an in-depth discussion, see Heise (1975) and Kaplan and colleagues (2001). With respect to the present study, the argument is that during the eight or nine years that these students have been in school and engaged in mathematics learning, their mathematics mastery levels and self-concept have reached a point of equilibrium where each student has formed a realistic view of their perception with regards to performing a given task (See Williams & Williams, 2010, for similar arguments).

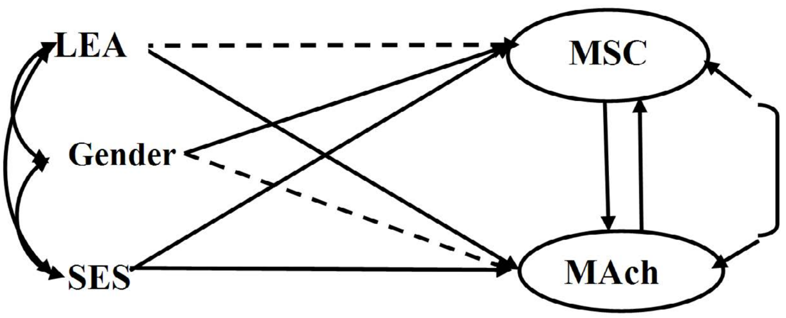

Figure 1: Reciprocal causal effects between maths self-concept (MSC) and achievement (MAch) with a direct feedback loop based on a cross-sectional design. The assumptions are shown with dotted lines. These indicate effects that were fixed to zero a priori. SES = socioeconomic status, LEA = Students' long-term educational aspirations. To avoid cluttering, only paths are shown

METHODS

The hypothesized model for the study is depicted in Figure 1. Each model was analyzed separately for each country. The result of a confirmatory factor analysis (CFA) verified a clear factor structure after incorporating the *method effect* associated with combining both negative and positive items in a survey (for similar argument see M-S. Chiu, 2012; Marsh et al., 2013). Moreover, in a multigroup CFA, measurement invariance (e.g., factorial invariance) of the MSC construct across the five educational/cultural groups was supported. Due to restrictions of space, these results are not presented here.

The students' long-term educational aspirations (LEA), gender, and socioeconomic status (SES)—measures of their home educational resources—are the exogenous variables—variables that no explanation is offered by the model; that is, no directional path point to them. In order for a non-recursive model to be identified, some parameters need to be specified a priori as such gender and LEA were assumed to have no direct influence on MAch and MSC respectively. The theoretical basis for fixing the effect of gender on achievement to zero is based on Deaux and Major's (1987) interactive approach model of gender differences which recognizes the importance of cognitive and cultural influences on gender roles, suggesting that more immediate factors, such as social and cultural patterns of discrimination, shape gender-related belief (see also gender stereotype threat: Spencer, Steele, & Quinn, 1999).

Arguing similarly as in Williams and Williams (2010), for all things being equal, the notion of males reporting higher mathematics achievement reflects a similar notion in mathematics self-concept (e.g., Nagy, Trautwein, Baumert, Köller, & Garrett, 2006). The assumption about SES is based on numerous studies that have established a firm relation between SES and achievement, and self-belief (Chiu & Klassen, 2010; Howie, 2013; Williams & Williams, 2010). The level of educational resources at home has been found to relate to students' achievement even after controlling for parental education and other factors (e.g., Teachman, 1987). However, in developing countries where large numbers of students have no access to basic educational resources, home educational resources are likely to be a more important component of socioeconomic disparity in education (Marks, Cresswell, & Ainley, 2006).

Concerning LEA, documented evidence (e.g., Gil-Flores, Padilla-Carmona, & Suárez-Ortega, 2011) suggests higher educational aspirations to be associated with higher achievement, and vice versa. With no documented evidence on the relationship between LEA and MSC, our argument is that students' mathematics self-concept mediates the relationship between LEA and MAch. Moreover, the error terms associated with the unexplained variance in mathematics self-concept and achievement were allowed to covary since they influence each other (Wong & Law, 1999; Kline, 2011).

Model evaluation and estimation criteria

The models tested in the present study were assessed by Mplus 7.2. The estimator used was the Mplus Robust Maximum Likelihood Estimates (MLR) with standard errors and tests-of-fit that are robust to non-normality and non-independence of observations (Muthén & Muthén, 1998–2012). The Mplus feature of full information maximum likelihood (FIML) was used to impute missing data. To ascertain the model fit, emphasis was placed on the comparative fit index (CFI), the root-mean-square error of approximation (RMSEA) as well as the chi-square test statistic (for informative purposes only because of it sensitivity to large sample size). The CFI is normed along a 0-to-1 continuum with values greater than .95 reflecting excellent fits to the data, and for the RMSEA, values less than .05 are indicative of a "close fit"(West, Taylor, & Wu, 2012). Due to the complex design of the TIMSS survey, the Mplus complex survey design option to account for the clustered design and to adjust standard errors was used. Students' class was used as the clustering variable, and students' sampling weights were also taken into account (*weighting variable supplied with the data*). The fit indices shown in Table 1 indicate the model fits well in all countries.

Data Source

Data were obtained from 38,806 (M_{Age} = 15.42, SD = 1.37) students who participated in TIMSS–2011 in five African countries (see Table 1). For detailed TIMSS sampling and method procedures, see Martin and Mullis (2012).

The maths self-concept scale

Maths self-concept (MSC) was measured through five items on a scale with a 4-point Likert response format: Agree a lot (1), Agree a little (2), Disagree a little (3), Disagree a lot (4). Item scales were reverse-coded to indicate higher values represent a more positive self-concept and vice versa. The items on the MSC are the following: 1) I usually do well in mathematics, 2) I learn things quickly in mathematics, 3) I am good at working out difficult mathematics problems, 4) Mathematics is more difficult for me than for many of my classmates [reverse coded], and 5) Mathematics is not one of my strengths [reverse coded]. The reliability (composite reliability) (Raykov, 2012) of the MSC ranged from acceptable in Botswana (.645) and Tunisia (.624), to low in South Africa (.557), Morocco (.526), and Ghana (.513). The maths self-concept scale was treated as a latent variable to account for measurement error. The reliability of the MSC was incorporated into the measurement model by fixing the variance of the error term to [(1- reliability)*variance]. This approach is discussed in (Heise, 1975), and a similar procedure was used in (Williams & Williams, 2010).

Mathematics achievement

TIMSS–2011 reported students' MAch in terms of five plausible values—random numbers drawn from the distribution of scores that could be reasonably assigned to each individual (Martin & Mullis, 2012).The use of plausible values has been discussed at length in the TIMSS–2011 methods and procedures (Martin, & Mullis, 2012). The composite reliability of the MAch score ranged from .969 in Botswana, .966 in South Africa and Tunisia, and .957 in Morocco, to .946 in Ghana. Mathematics achievement was treated as a latent variable and was given the same format and procedure (i.e. on the variance and reliability) as that described earlier for MSC.

Instrumental variables

Socioeconomic Status (SES). The SES scale was derived from students' reported home educational resources based on their responses concerning three home resources: Number of books in the home, Highest level of education of either parent, Number of home study supports: Own room, Internet connection. (See Foy, Arora, & Stanco, 2013 supplementary 3 for more details on the process of determining SES.)

Long-term educational aspirations (LEA). The LEA scale was a self-report asking participants to indicate the highest level of education they expected to attain on a scale ranging from (1) "Lower secondary education" to (6) "University program - Master/Doctorate." (See Foy et al., 2013 supplementary 2 for specific nationally define classifications).

Gender. The gender measure was based on students' responses to the questionnaire coded as 1 = *girl* and 2 = *boy*.

RESULTS

Overall, the results shown in Tables 1 and 2 are congruent with similar models estimated by other researchers (e.g., Williams & Williams, 2010). This lends credence to the validity of the results and supports the hypotheses. As indicated in Table 2, the causal relationship between MAch and MSC is supported in Tunisia. Furthermore, in the remaining four countries one or both of the bidirectional relationships failed. In Ghana and Botswana, only the effect of MSC on MAch was statistically significant, validating the self-enhancement model. In Morocco, the effect of MAch on MSC was statistically significant supporting the skill-development model. In South Africa, neither the self-enhancement nor skill-development model was supported. These findings indicate that the countries vary in their causal relationships between MSC and MAch.

The effect of SES on MAch was statistically significant and positive in all but two nations (Ghana and Botswana). Similarly, the relationship between SES and MSC was less consistent across the countries. A statistically significant relation between SES and MSC was found in all but two of the countries (South

Country	χ^2	*df*	S	CFI	RMSEA	Sch	Stu	Grade
Ghana	537.451	57	1.580	.991	.034	161	7,323	8
Botswana	654.534	57	1.105	.989	.044	150	5,400	9
South Africa	931.099	57	1.764	.990	.036	285	11,969	9
Morocco	642.498	57	1.359	.990	.034	279	8,986	8
Tunisia	430.827	57	1.181	.994	.036	207	5,128	8

Table 1: Measures of model fit and sample size by country. χ^2 = chi-square; *df* = degrees of freedom ratio; s = Mplus scaling correction Factor; CFI = comparative fit index; RMSEA = root-mean-square error of approximation; Sch= Schools; Stu = students (unweighted sample)

	Achievement estimates			Self-concept estimates		
Country	MSC	SES	LEA	MAch	Gender	SES
Ghana	38.519***	-2.158	2.744***	0.003	0.089***	0.071***
Botswana	-13.009***	-0.359	3.474***	0.000	0.096***	0.036*
South Africa	11.045	4.458***	3.847***	0.000	0.069***	0.012
Morocco	0.898	6.842***	1.726***	0.016***	0.098***	-0.057**
Tunisia	41.140***	2.757***	0.685***	0.015***	0.035**	-0.022

Table 2: Path Estimates by country. The absolute size of these metric estimates varies considerably because of the differences in the scales of Maths self-concept (MSC) and Maths Achievement (MAch). The effect of MSC, SES, and LEA on MAch is shown under the heading Achievement estimates. The effects of MAch, gender, and SES are shown under the heading Self-concept estimates. Socioeconomic Status (SES); Long-term educational aspirations (LEA). ***p <.001, **p <.01, *p <.05.

Africa and Tunisia). This indicates that the relationship between SES, MSC and MAch is dependent on the national context (William & William, 2010).

With regard to the effect of gender, all parameter estimates were positive and statistically significant. This indicates that males reported higher levels of MSC in all five countries. The effects of LEA on MAch were statistically significant for all the countries as well. The coefficients indicate that higher LEA predicts higher MAch in all countries.

DISCUSSION

The reciprocal determinism of MSC and MAch was validated in one of the five countries. This supports the cross-cultural dimension of the reciprocal models (Guay et al., 2003; Marsh & O'Mara, 2008; Seaton et al., 2014). Nevertheless, the analysis indicated that reciprocal determinism is dependent on the national context specific, thus supporting Williams and colleagues (2010). In South Africa, neither the reciprocal relation nor the self-enhancement model or the skill-development models were supported. Ghana and Botswana supported the self-enhancement model (Calsyn & Kenny, 1977; Chiu & Klassen, 2010; Marsh & O'Mara, 2008), whereas Morocco supported the skill-development model (Hannula et al., 2014; Ma & Xu, 2004). Moreover, the effect of students' long-term educational aspirations and gender on their MAch and MSC, respectively, also shows evidence of cross-cultural generalization, since they are all positive and statistically significant. The findings are also consistent with higher reported MSC for males (e.g., Nagy et al., 2006) and higher MAch for students' with higher long-term educational aspirations (e.g., Marsh et al., 2013). It also supports our assumptions behind the use of gender and LEA as *instrumental variables*.

The effects of SES on MAch and MSC were less consistent, but were evident in more than half of the countries. The relationships between SES, MSC, and MAch are consistent with other studies (Chiu & Klassen, 2010; Howie, 2013; Marks et al., 2006; Teachman, 1987; Williams & Williams, 2010). In countries where a relation was found between SES and MAch, governments can institute financial support schemes for low-income families in the light of our results. Moreover, upgrading schools and increasing funding for schools in low-income areas could help bridge the gap between low and high achievers (Marks et al., 2006).

The present research is one of the few cross-cultural studies on causal relationships between the MSC and MAch in an African context, and provides important new evidence regarding the generalizability of the uni- and bidirectional relationship between MSC and MAch.

A limitation of the study is the assumption behind using cross-sectional data when modeling a reciprocal analysis. For instance, the required assumption of *equilibrium*. The problem is that there is no statistical measure to evaluate the equilibrium assumption with a cross-sectional data set; *it must be argued substantively* (Kline, 2011, p. 108). Moreover, others have argued that the equilibrium assumption does not justify using cross-sectional models for bidirectional determinism, because cross-sectional models are miss-specified due to the fact that they do not take time lags into account (Gollob & Reichardt, 1987; Wong & Law, 1999). However, others have argued for the importance of cross-sectional data to test reciprocal models because in most situations cross-lagged effects are virtually impossible to obtain (Wong & Law, 1999). As we have seen, "causal attribution is not an automatic process; useful causal conclusions are the product of careful thought, high-quality data, and sound data analysis" (Rogosa, 1979, p. 301).

The findings of this study clearly challenge some of the foundations of current theories on self-belief, and provide new ways of looking at the self (Chiu & Klassen, 2010). Reciprocal determinism was found in some countries and was non-existence in others. The data shows the degree of cross-cultural variations on the reciprocal determinism between affect and achievement. The author could not provide sound cultural theory to explain these phenomena. Moreover, it is important to bear in mind that the analysis may not represent the dynamics of the feedback loop between math self-concept and achievement because the findings represent a static view of an ongoing dynamic system and may vary based on when the system is observed as it moves toward equilibrium (Kaplan et al., 2001).

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A longitudinal analysis of the Italian national standardized mathematics tests

Laura Branchetti[1], Federica Ferretti[2], Alice Lemmo[1], Andrea Maffia[3], Francesca Martignone[4], Mariagiulia Matteucci[2] and Stefania Mignani[2]

1 University of Palermo, Palermo, Italy, laura.branchetti@unipa.it, alice.lemmo@unipa.it

2 University of Bologna, Bologna, Italy, federica.ferretti6@unibo.it, mariagiulia.matteucci@unibo.it, stefania.mignani@unibo.it

3 University of Modena and Reggio Emilia, Modena, Italy, andrea.maffia1988@gmail.com

4 University of Eastern Piedmont, Vercelli, francesca.martignone@unipmn.it

This paper presents a longitudinal analysis of the outcomes of the Italian national standardized mathematics tests. By intertwining quantitative and qualitative methods, we selected and analysed a set of linked questions among the tests administered to the same cohort of students first in grade 6 and then in grade 8. In particular, we focus on poor knowledge students and we argue an example of the analysis of two linked questions about graphical representation of fractions. The comparison between the two questions allows us to interpret some difficulties of students and to expect possible future behaviours.

Keywords: Standardized mathematics tests, longitudinal analysis, fractions, qualitative and quantitative analysis.

INTRODUCTION

This paper presents an "in progress" research developed within the *Ideas for the research* project, funded by the Italian national institute for the educational evaluation of instruction (INVALSI). This project asks for new qualitative interpretative tools with the aim to integrate the statistical ones with them. These tools can be used by teachers to point out some groups of questions (in different grades) that could identify *poor knowledge students* (see the next section for a definition of this set) in specific mathematics contents. Our study focuses on a longitudinal analysis of mathematics standardizes tests outcomes collected by INVALSI in 2011 and 2013 respectively in grade 6 and 8 (therefore we analyse the same cohort of students who faced the INVALSI tests in grade 6 and 8). The aim of this analysis is to identify questions in which poor knowledge students have difficulties and, observing their behaviours, to describe possible causes of these difficulties. Such causes are going to be the object of our future analysis in classroom activities.

We carried out an analysis that intertwined qualitative and quantitative interpretative tools in a longitudinal study: in particular, we analysed items of the INVALSI tests focusing both on the mathematical contents involved and on the solution strategies developed by students to face the tasks (qualitative analysis), and using the information given by a statistical analysis of the national sample results (quantitative analysis). We expect this study to be an useful tool for teachers because the outcomes of the research could help teachers in identifying students' difficulties in longitudinally linked items. These information could be used to implement educational activities aimed to avoid the persistence of wrong behaviours of students, similar to the ones observed in the research, and future failures in mathematics.

In the next phase of our research some of the items, selected from the INVALSI tests and analysed in the first part of our study, will be administered again during classroom activities in order to give empirical support to the analysis presented in this paper.

THEORETICAL LENSES

The INVALSI test is designed by Italian teachers selected by INVALSI according to their experience and education. INVALSI framework is based on to National Standards Ministero dell'Istruzione, Università e Ricerca, 2012). In our study, we take into account INVALSI framework but we integrate it with specific theoretical tools taken from the literature in mathematics education.

The items are presented in a written form with multiple choice questions, open questions, true or false and closed. As the topics are many and the test is faced by a large amount of students, we conjecture that the most frequent difficulties described in literature appear in students' answers. For this reason, we create a list of difficulties found in literature for each topic. For instance, in this paper, we analyse two questions about fractions. Our theoretical framework on educational studies about fraction refers to the research carried out in the last thirty years and summarized in the Encyclopaedia of Mathematics Education (2014). In particular, Demetra Pitta-Pantazi describe some recurring misconceptions, for example:

Students often do not interpret fractions as numbers but view fractions as two numbers with a line between them. When adding fractions, they often add the numerators and denominators or are unable to order fractions from smaller to larger (e.g., Behr et al., 1992). (Encyclopaedia, 2014, pp. 470–476)

We use also the study of Fandiño Pinilla (2007) who offers a wide review about difficulties in fractions domain. These researches gave us the tools to identify and to interpret students' answers to the INVALSI test items. In this paper, we analyse students' answers identifying some common difficulties and errors linked to the concept of fraction: (M1 – Divide in non-equivalent parts: count-and-match misconception) the "Epistemic" misconception of fraction as a part but non-equivalent to the others; (M2 – Answer d – n instead of the total d) the mistake in identifying the fraction numerator and denominator when students analyse some coloured parts in a grid of equivalent parts of a figure and considering the coloured part as numerator (n) and only the rest (d–n) as denominator, not the total amount of parts (d); (M3 – Divide in equivalent parts and take some parts) Student implicitly suggests that "some" cannot be "all"; it becomes easily an obstacle for the fractions equivalent to the unit. The INVALSI items are given to students in written form, but words are often mixed with images, tables, graphs and other representations. For this reason, some students' difficulties could be related to semiotic representations management. Indeed, we need a theoretical tool of analysis to figure out possible obstacles arising from this semiotic richness. We use semiotic approach proposed by Duval: "there are always many possible semiotic representations of the same object. The higher process of thought, and especially mathematical activity, are based on this plural semiotic object representation" (Duval, 1993; 2008). Even if a richness of semiotic representations is necessary to conceptualize mathematical objects, in the very first steps of learning or for poor knowledge students coordinating many registers and connecting representations can be an obstacle. Different students errors in INVALSI tests can be interpret as a failure in verbal-graphic conversion, in the decoding of the text or in expressing a right solution in another register. In the examples argue below the students mistakes could be caused by a wrong transformation of the fraction verbal representation in the right graphic one or vice versa (S1) or in a incorrect transformation of the graphic results in the fractions' register (S2).

Our research question is: what kind of information, emerging from the analysis (both quantitative and qualitative) of the 8th grade items, are useful for the interpretation of the outcomes of the analysis of the 6th grade questions?

We conjecture that a longitudinal analysis of the answers of the same cohort of students, carried out through the comparison between the two data sets from different years, could be useful to better interpret the 6th grade outcomes in terms of the analysis of 8th grade items. Unfortunately, by means the analysis of the statistical data, we cannot follow the specific test outcomes for each student in different years; for this reason we need a criterion to link questions from different grade tests. In particular we decided to select questions that deal with longitudinal topics, identified in the National Standards - Indicazioni Nazionali (2012), and which can be solved using the same *scheme*, namely the same sequence of actions, controls, operational invariants and so on (Vergnaud, 2009).

METHODOLOGY

A longitudinal analysis of the tests, administered to the same cohort of students in different years has been performed integrating qualitative methods with quantitative ones. Starting from 2013 INVALSI test for grade 8 students, we consider 2011 test for sixth graders.

As concerns the qualitative analysis we analysed all the items of the INVALSI test for grade 8 and grade 6 focusing on some longitudinal contents: e.g., fraction and decimal representations of rational numbers and equivalent polygons. In this paper, we analyse items that deals with the fractions of a square area.

The INVALSI team verifies the consistency of the whole test by Classical Test Theory tools (Cronbach alpha, point-biserial correlation coefficient). Then using an Item Response Theory approach (Van der Linden & Hambleton, 1997) the estimation of item characteristic parameters is carried on.

In our work we use some INVALSI test results to investigate the behaviour of the items. First we apply a latent class analysis [1] to classify students and to judge the item characteristics. This statistical method gives a classification of students in a fixed number of groups characterized by different levels of performance. The classification is based on the estimated probability of correct answer of each item. Chosen the best number of groups, it is possible to interpret them (for example, the group with the worst performance, the group with the best performance and so on) and to investigate the probability of a correct answer for all items within each group. In this way items that show a particular response behaviour could be identified. Most items included in the test are unordered categorical (nominal) multiple-choice items. The statistical analysis and specifically the latent class analysis were conducted on dichotomous items, i.e. correct/incorrect response. The analysis of the national sample (i.e., the students of the same cohort in grade 6 in 2011 and in grade 8 in 2013) showed the presence of groups/classes of students with a lower probability of correct answers in comparison with the results of the whole of students' answers to all the items of the INVALSI test. Analysing the results obtained by the group with the lower performance, we selected some items in which the 'weaker' group had the lowest probability of correct answer compared to all the students who faced the same test. In our data we identify five groups/classes [2] which consist of students with a similar response probability for the same items. We analyse the results on a national sample of about 28000 grade 8 students.

Class 5 (about the 23% of the students) is the class with the lowest outcome probabilities so we consider it as the set of *poor knowledge students*. Comparing class performances item by item, a set of questions in which only class 5 has low performances can be observed (Figure 1). In particular, we gather the items in which the success probability for class 5 is less than half of outcome probabilities for the other classes. Item D25 (Figure 1) has the maximum ratio, equal to 0.46, indeed the probability of class 5 right answer is 14% instead of the other classes whose outcome probabilities are a range from 30% to 81%. This item seemed to be a good candidate to study the possible difficulties face by poor knowledge students in INVALSI test. It was identified through the interweaving of qualitative and quantitative analysis: this is meaningful both as regards the contents and schemes involved (it concerns

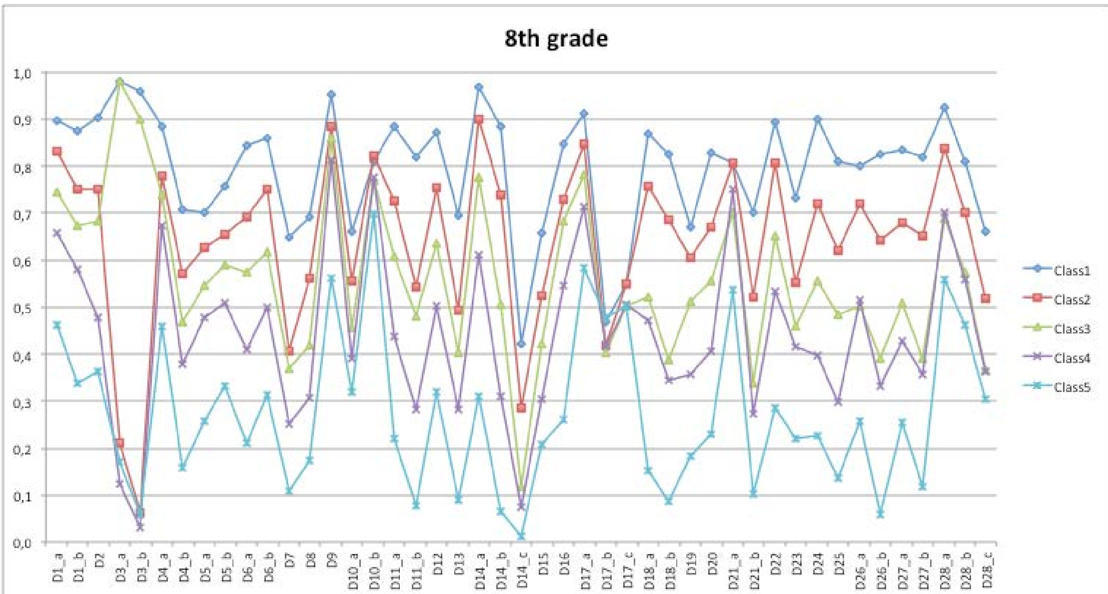


Figure 1: Classes' outcomes probability of correct answer in 8 grade test

the identification of relationships between areas of polygons), and the results on the national sample.

AN EXAMPLE OF ITEMS ANALYSIS

Using Latent Class analysis on data from grade 8 test of 2013, we select a set of questions with low performances in particular for poor knowledge students. To give an example of our analysis we consider question D25 (Figure 2).

This is a multiple choice question with four options and only one right answer. The correct answer is Option D. Option A considers the number of all pieces as denominator, so it could be chosen by students who do not take in account the different areas of the pieces (M1); Option B may be selected considering only half of the Tangram maybe because of a wrong calculation of the area of the triangle not dividing by 2; Option C is selected by students that correctly divide in equivalent parts the square, but take 1 as numerator and the difference between total and 1 (16−1=15) as denominator (M2).

Looking at data, 42% of the students answer correctly and few students choose Option B and C (respectively 8.0% and 11.3%). A relatively high percentage of students, about 35%, chooses Option A.

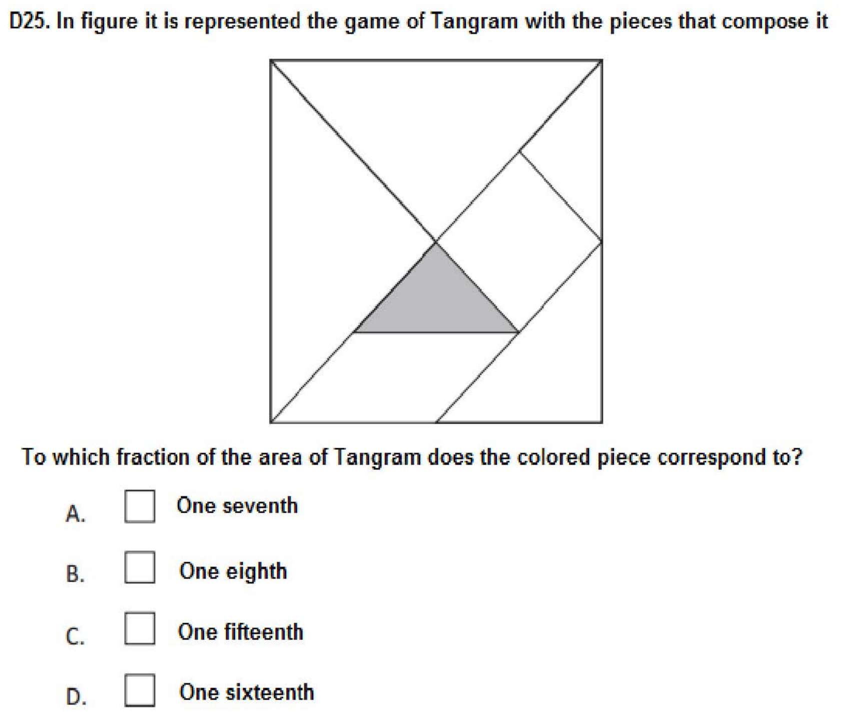


Figure 2: Question D25 from the grade 8 INVALSI test administered in 2013 [2]

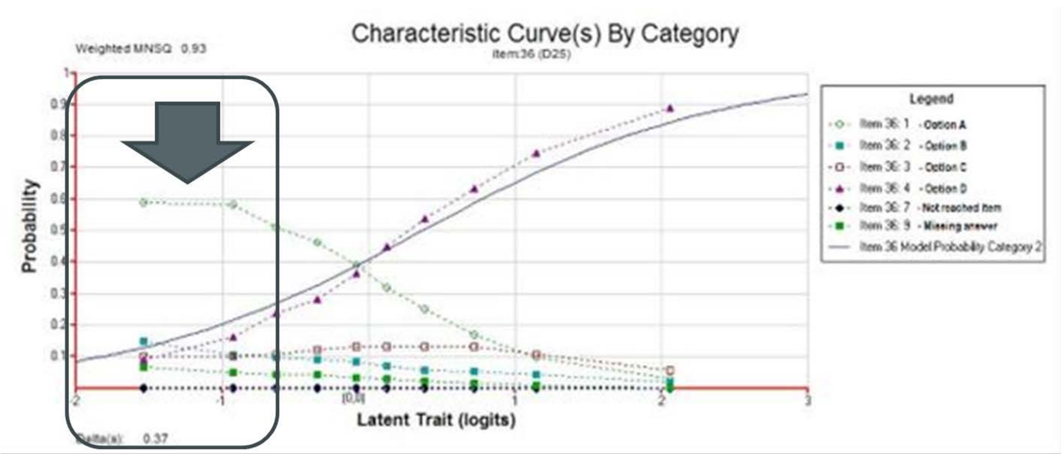
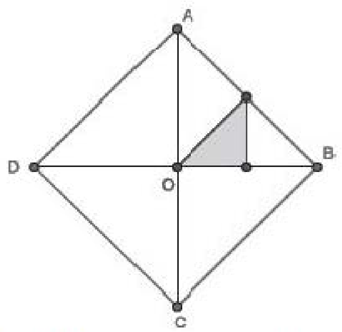


Figure 3: Characteristic Curve of question D25 from the grade 8 INVALSI test, 2013. The probability of options choice is shown on the vertical axis, while the corresponding ability estimate is shown on the horizontal axis.

In the Figure 3 on the abscissa of the graph we can read the measure of the ability of the students in the test (Latent Trait), and on ordinate there are the probabilities of options choice. Each option choice is represented by a different Characteristic Curves (see the Legend in Figure 3). Analysing Characteristic Curve (Figure 3), we can notice that students with lowest performances, among which we expect to find poor knowledge students, have 60% of probability to choose Option A and less than 20% of probability to choose the others (see the framed region indicated by the arrow in the Figure 3). We link the question of grade 8 test, D25, to question D2 of grade 6 (Figures 4–5).

There's a very good relation between the two questions in terms of statistic trends. Moreover the task is similar and the strategies developed to answer can be compared. We have to stress that the two questions, even if similar, are different. The grade 6 question asks to cover while the grade 8 one to find a fraction, but in both the cases the operation to carry out the solution is the same: to compose the square using triangles equivalent to the coloured one. In item D2, the percentage of correct answers is 55.3%; since the question is not a multiple choice one, the Characteristic Curve gives information only in terms of "right-wrong" answer. In this case, students belonging to the lowest performances in the graphic have a probability about 10% to provide a correct answer, with more than 70% of failure. The percentages given by national results didn't help us to understand which kind of errors students made because, in open questions, we can know only how many students gave the right answers (and how many students gave the wrong ones). In order to identify the possible wrong answers, we collected a sub-sample of 74 tests from the national survey of 2011 and we analyse the students' answers. Obtained data are summarized in Table 1.

D2 In the square ABCD the middle points of side AB and segment OB are connected



How many triangles, as the colored one, can we exactly cover the surface of the squ ABCD with?

Figure 4: Question D2 from the grade 6 INVALSI test administered in 2011 [3]

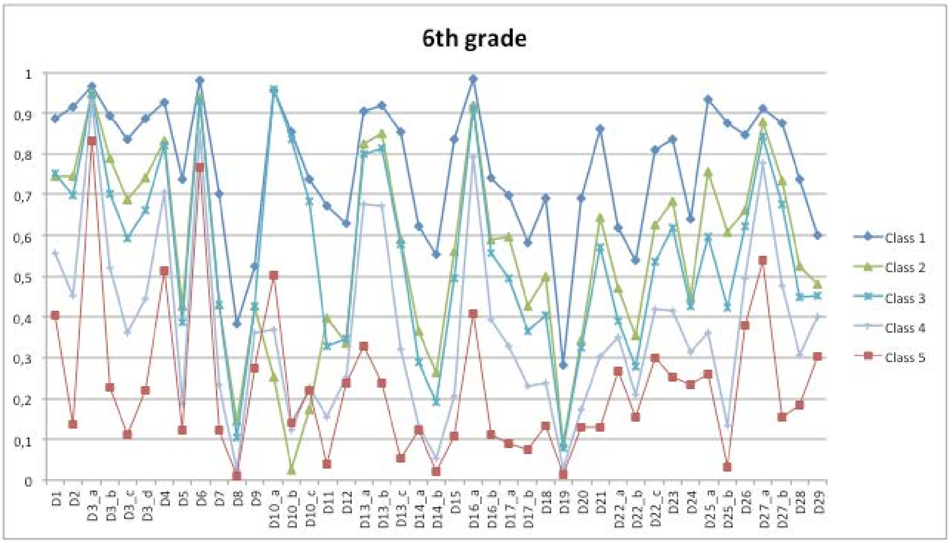


Figure 5: Classes' outcomes probability of correct answer in 6 grade test

Answer	16	12	8	4	3	other	missing
Percentage	52.7%	10.8%	5.4%	6.8%	4.1%	6.8%	13.4%

Table 1: 74 students' response to D2: answers (16 is the right answer, the others are wrong)

The percentage of right answers of this sub-sample is coherent with the national sample. Analysing wrong answers in Table 1, it is possible to identify some results that can be the outcome of the same students' solution schemes captured by the options of the D25 in 2013: therefore we conjecture a link between these difficulties observed in grade 8 and grade 6 results. As a matter of fact, students who answer 12 maybe look at triangle AOB as composed by three triangles, even if they are not equivalent (M1), and repeat this procedure in all the quarters of the square. This attitude could be linked with students who answer 3. They probably consider the three triangles but do not extend the procedure to the others. Another interpretation of this phenomenon is that students could individuate the right number of triangles composing AOB, but do not consider in the calculation the grey one and count 3 triangles or 12 in the case of extending procedure for the whole square (M2). Students that answer 8 probably do not consider the grey part and count only the white pieces of AOB, then multiply by four (M2+M1). Who answers 4 maybe does not reach the instruction and refers to AOB instead of the grey triangle or just focus on AOB's pieces (S3). As a matter of facts, same codes are present in both questions' analysis. This fact allows us/makes us able to link difficulties in the two questions longitudinally. In particular, students who answer 12 in D2 can behave as students who choose Option A or C. For instance, M1 refers to students who consider non-equivalent pieces and count 12 triangles in D2 (or 3 in the case in which do not extend the procedure to the whole square) and "one seventh" in D25. Otherwise, if we consider the second interpretation (M2), it is possible to link this attitude with the choice of Option C i.e. who does not consider the coloured triangle in counting. From the semiotic point of view, in both questions students have to recognize verbal, graphic and symbolic representations of fraction and to switch from one to the others. Indeed, one of the students' difficulties could be found in conversion between registers (S1–S2).

An interesting difference we have to take in account is that the multiple choice question obliges the student to coordinate graphical register with the verbal one, because using written verbal register to represent fractions is quite unused in Italian schools for the solutions. Probably some wrong answers are due to a combination of these difficulties. It will be possible to validate this conjecture only in the second part of the project, that will be carried out through interviews and classroom experiments – on the qualitative side – and correlating sets of data from the same sub-sample of students in the two grades. Comparing the percentage of correct answers, we notice that from grade 6 to grade 8 it decreases of 13 percentage point (from 55% to 42%), thus there is a significant group of students that is able to give a correct answer to D2 question but not to D25, even if the operations required to give the right answer are the same. Indeed, there is a peculiar feature of D25 which make this question more difficult and the main difference can be identified in the presence of a direct reference to fractions.

CONCLUSION AND FUTURE PRESPECTIVE

In this paper, we show an example of longitudinally linked items (these can be seen as paradigmatic examples of questions) that can identify poor knowledge students in specific tasks. From the methodological point of view we selected these questions both using quantitative and qualitative methods intertwined. In particular statistical analysis on a big amount of data allowed us to focus our attention on a group of students, and starting from their performances we could select a set of items of interest for our aim. The opportunity to use data about the same cohort of student in different grades has been exploited linking the selected questions from 2013 test to questions of the 2011 one. This connection is achieved through a qualitative study focuses on the epistemological, cognitive and educational aspects: i.e. we analyse the concepts involved, the representations and schemes (Vergnaud, 2009) developed by students in order to face that the task, and the possible difficulties arising from conversion among different semiotic representations (Duval, 2003).

The comparison of the linked questions outcomes strengthens the a priori analysis of possible students' difficulties facing these items. We identified the same coded difficulties and the percentages in outcomes are similar. The qualitative analysis of the grade 6 item allows us to formulate hypothesis for the selection of options in grade 8 item; it is reasonable to assume that grade 6 question outcomes can have a predictive power for the grade 8 ones, but only the classroom experiment can confirm it. In particular, as concerns the analysed questions, we can assume that the students who have difficulties in dividing a figure in equivalent pieces in grade 6 will tend to consider non-equivalent parts in graphical representation of fractions in grade 8. Similarly, the students who do not consider all the parts in a figure (in our example, the grey part) will be inclined to convert graphical representation of fraction in symbolic or verbal ones with a wrong denominator. Further interviews can confirm if students that have a difficulty in one of the questions have the same difficulties in the other. These interviews could be integrated with a correlation analysis of outcomes within already collected data.

The next phase of the project is going to involve some schools to identify students with peculiar difficulties on the selected items and to analyse their behaviours. We can show how our analysis points out some groups of questions (in different grades) that could identify poor knowledge students in specific mathematics contents. We would like to share with the teachers involved both our theoretical tools and the analysis of the data. It is important for us to understand if the tools produced are usable by the teachers.

These qualitative analysis of the items could be also useful for the teachers who design the test items because it shows some elements that can discriminate in the assessment of poor knowledge students.

From a statistical point of view, future developments of this work could include the implementation of a multilevel latent class analysis (Vermunt, 2003; 2008). In fact, traditional latent class analysis assumes that observations are independent while a hierarchical structure could be present (e.g., students nested within classes or schools). The multilevel latent class analysis could account for the nested structure of the data by allowing latent class intercepts to vary across groups (level 2 units) and by investigating if and how the groups affect the level 1 latent classes. In our study, this analysis would allow to examine how the probability of belonging to a particular performance group could vary across level 2 units (classes or schools). Finally, covariates could be introduced at level 1 and level 2 in order to predict the probability of belonging to a certain latent class.

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ENDNOTES

1. We implemented a Random Effects Latent Class Analysis (random LCA); this R package fits latent class models, may include a random effect.

2. In our analysis the classification in five classes gave a good division of the group of students according their performance in the test.

3. Translation from Italian realized by authors.

The role of comparison in the construction and deconstruction of boundaries

David Clarke

University of Melbourne, Melbourne, Australia, d.clarke@unimelb.edu.au

This paper addresses comparative research from the perspective of boundary crossing and argues that all research is intrinsically comparative and, as such, continually engages in the useful and productive activity of constructing and reconstructing boundaries. Recognition of the significance of acts of comparison in both boundary crossing and boundary construction foregrounds comparison as a key tool in the essential act of boundary deconstruction. International comparative research in mathematics education provides the examples illustrative of the points being made. It is incumbent upon researchers in mathematics education to consider what boundaries they invoke in their comparisons and to examine critically the form of boundary crossing implicit in their particular comparison event.

Keywords: Comparative research, boundary construction, boundary crossing.

COMPARATIVE RESEARCH AND BOUNDARIES

It is the assertion of this paper that it is the business of research to continually engage in the useful and productive activity of constructing boundaries. It is also true that some of the least useful and most harmful boundaries are also products of research. I would further argue that there is a fundamental redundancy to the expression "Comparative Research," since comparison is implicit in all research. Nonetheless, this paper will continue to employ the expression "Comparative Research" to refer to those research designs for which the focus is on specific, differentiated objects, communities or systems about which an act of comparison is to be undertaken. By contrast, a longitudinal study of evolving practice in a single classroom would not conventionally be thought of as a comparative study, yet the comparison between current and recent practice is continual in such a design. At the heart of this paper is the argument that acts of research comparison necessarily construct boundaries that distinguish between the objects, groups, communities, settings or systems that are compared. These boundaries are important. Without them, our acts of comparison are futile. As a consequence, boundary construction is an inevitable entailment of all research activity.

RESEARCH AS COMPARISON: THE RIGHT TO COMPARE

A paper presented at the previous CERME conference, posited the Validity-Comparability Compromise as a central consideration in cross-cultural research in mathematics education (Clarke, 2013a). In this earlier paper, commensurability was interpreted as the right to compare. And the central assertion was that this right to compare cannot be assumed, but is contingent on our capacity to legitimise both the act of comparison and the categories through which this act is performed. This earlier paper identified key considerations affecting the conduct and utility of international comparative research. It posited as central to research design the dual imperatives of validity and comparability and argued that these imperatives are inevitably in tension. The paper identified, illustrated and discussed those tensions, utilising very specific examples from current international comparative research. It was argued that any value that might be derived from international comparisons of curricula or classroom practice is critically contingent on how the research design addresses the dual priorities of validity and comparability. It was further argued that since these priorities act against each other, researchers undertaking international comparative research must find a satisfactory balance between these competing obligations.

Since it can be argued that all research involves acts of comparison (or "comparison events" from the perspective of Verran (2011)), any examination of the principles and contingencies which frame international comparative research will have implications for research in general. Those research designs currently associated with Comparative Research provide rich and fertile ground in which to speculate about the association between boundary construction and the acts of research comparison. The previous paper (Clarke, 2013a) questioned the assumptions we might make about our right to compare and attempted to foreground the decisions and obligations confronting a researcher undertaking particular types of comparison. This paper examines the nature of the boundaries constructed through our acts of comparison, the status that might be accorded to those boundaries, and our responsibilities as researchers to acknowledge our role in boundary construction. Further, making a virtue of necessity, I argue that sensitivity to the entailments of our comparative acts can assist us in the deconstruction of those boundaries created by our research. Such deconstruction would then better equip us to celebrate the useful work performed by those boundaries, while sensitising us to possible dangers, such as unwarranted extrapolation or generalisation, reification, segregation, stagnation or sanctification.

COMPARATIVE RESEARCH AS BOUNDARY CROSSING

If all research involves comparison, and all comparisons invoke or create boundaries, then my further proposition is that all research, and Comparative Research in particular, involves acts of boundary crossing.

It is useful at this point to consider the proliferation of boundary-related terms pervading educational literature at the moment: boundary crossing, boundary object, boundary interactions, boundary practices, and boundary zones (see Akkerman & Bakker (2011) for a useful discussion). Underlying all these terms is an inevitable uncertainty about what the term "boundary" actually refers to; inevitable, because its use and referent will vary from study to study. Boundaries are constructions, built of language through discourse. However, we respond to boundaries in different ways. Sometimes the boundary appears as a natural feature, like a river, separating one habitat from another; sometimes, as an artefact, like a wall, constructed to enclose or to separate; and, sometimes, as the principles by which the members of a club or society are distinguished from non-members [1]. Given such variation, the nature of boundary crossing itself must take different forms. The remainder of this paper addresses possible different approaches to boundary crossing and attempts to illustrate its points with examples relevant to mathematics education. The lead question, to structure this discussion, is "How do you cross a boundary?" Sometimes, the answer to this question can provide significant insight into the nature of the particular boundary. Each method of boundary crossing comes with its own caveat.

METHODS OF BOUNDARY CROSSING

One way to cross a boundary is to abolish it

> The insertion of cultural artifacts into human actions was revolutionary in that the basic unit of analysis now overcame the split between the Cartesian individual and the untouchable societal structure (Engeström, 2001, p. 134).

In this instance, the boundary between the individual and the physical world was abolished as a matter of theoretical dictate. In the field of research, the redefinition of metrics can significantly reconstruct boundaries. As a case in point, between the 2000 and 2003 administrations of PISA, Australia moved from "low equity" to "high equity" status without apparent change in practice, but through "slight variation in the way 'equity' was measured in PISA" (Gorur, 2014). In such cases, boundaries are re-drawn without additional evidence and a school system may cross from one grouping to another as a matter of legislation, rather than any change in either practice or outcome. Political examples of such boundary crossing by proclamation are extremely common.

Every act of boundary crossing can be associated with at least one potential danger, represented in this paper as a caveat.

CAVEAT: the abolition of boundaries can deny the recognition of diversity.

Each abolished boundary assigns an integrity or connectedness to otherwise distinguished entities (students, teachers, school systems, or task types) as members of a unified aggregate that conceals diversity. These concealed diversities may disempower the communities now integrated and may deny the researcher both explanatory alternatives and possibilities for advocacy of action. A particularly obvious example is the national aggregation of student achievement scores across category distinctions of ethnicity or socio-economic status that, once dissolved, no longer offer avenues for researcher comparison, explanation, advocacy or political action (e.g., Berliner, 2001; see also Clarke, 2003).

Another way to cross a boundary is to demolish it

The distinction between abolition and demolition for me is one of theoretical dictate vs empirical demonstration. Theory or simply accepted wisdom (entrenched belief) may treat a boundary as well-established, in that it distinguishes two categories of occurrence or situational domains that are conceptually distinct in a useful way. However, if empirical evidence pertinent to the characteristics held to distinguish the bounded domains is not consistent with the posited difference, then the boundary must be considered demolished (or at least destabilized) on evidential grounds. This destabilizing of boundaries can be highly productive. The lack of evidence of difference, where difference might be expected, should lead us to interrogate the original assumptions on which that difference was posited.

As a case in point, PISA student achievement performance is commonly invoked as providing evidence of curricular or pedagogical difference. The inability of PISA scores to distinguish between Korea and Finland, therefore demolishes a putative boundary that would have those two school systems in distinct domains, while calling into question the evidential value of similar distinctions that appeared to reify expected boundaries between other school systems (Clarke, 2013b). To labor this point: the distinction between high-achieving and low-achieving countries, as identified through either PISA or TIMSS performance, has been interpreted as indicating associated pedagogical differences that should inform educational change in less successful school systems. Research in classrooms in Korea and Finland problematize any simplistic clustering of Korean and Finnish school systems as pedagogically similar. Similarity seems limited to the comparable (measurably equivalent) success of their students on international tests of student achievement. The comparability of Korea and Finland in one respect usefully destabilizes generalized assumptions of difference with respect to the compared systems, but also (because of differences known from other measures/studies) suggests that the dissolution of boundaries is highly specific and cannot be simplistically generalized. It does, however, suggest the particularly useful question: "For what educational attributes might Korea and Finland be considered to reside in the same domain?"

CAVEAT: The demolition of a boundary can create a misleading appearance of homogeneity. Boundaries are situated constructions of prescribed conceptual tenure.

Yet another way to cross a boundary is to build a bridge

What is the work of a bridge? A bridge conveys individuals, groups, ideas or artefacts between domains. It does not interact with the boundary, but passes over it. When attempts are made to measure a construct like "Civic Principles" with sub-constructs such as equity, freedom, and social cohesion (IEA Civic Education Study, Shultz & Sibberns, 2004), the assumption that the construct can be defined in a commensurable fashion across two school systems builds a bridge between those school systems. Emergent empirical differences reify the boundary without necessarily interacting with or interrogating it. In the same way, assumptions of curricular comparability with respect to mathematics, such that mathematical performance is commensurable across various political and cultural boundaries, then generate differences, which retrospectively consolidate the boundary and the respective domains being compared.

We know from comparative analyses of mathematics curricula that different school systems do not organize their mathematics content in the same way (see Figure 1). Figures 1 and 2 show the results of comparative analyses of the Australian, Chinese and Finnish national mathematics curricula for the years of compulsory schooling. The categories employed in both analyses are adapted from the work of Porter and his colleagues (Porter & Smithson, 2001; and see Xu, Kang, & Clarke, 2011).

As can be seen from Figure 1, both the content and its sequencing differ significantly between the three countries. Perhaps, as importantly, the types of mathematical performances (levels of cognitive demand) specified in the three curricula also differ significantly (see Figure 2). Figures 1 and 2 demonstrate profound differences in not only the nature of the mathematics considered essential but in the types of

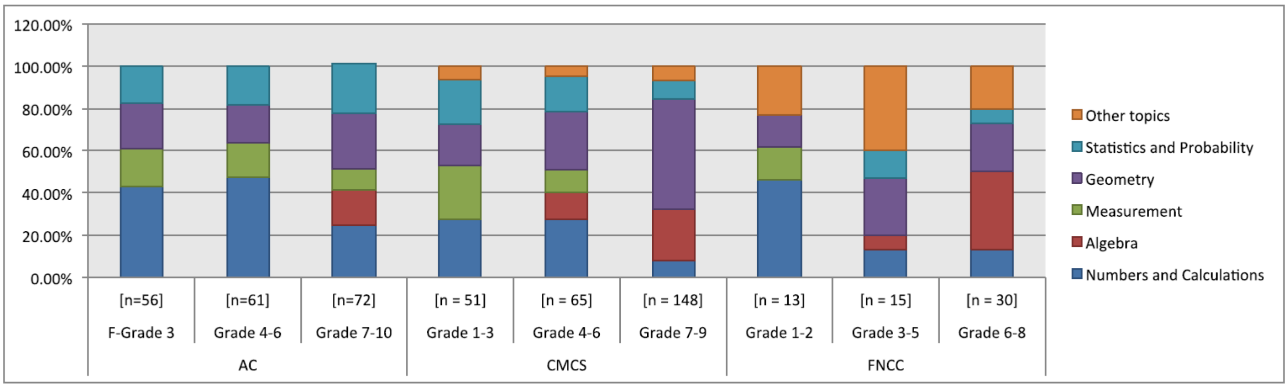

Figure 1: Comparison of Australian, Chinese and Finnish Mathematics Curricula by Content Category

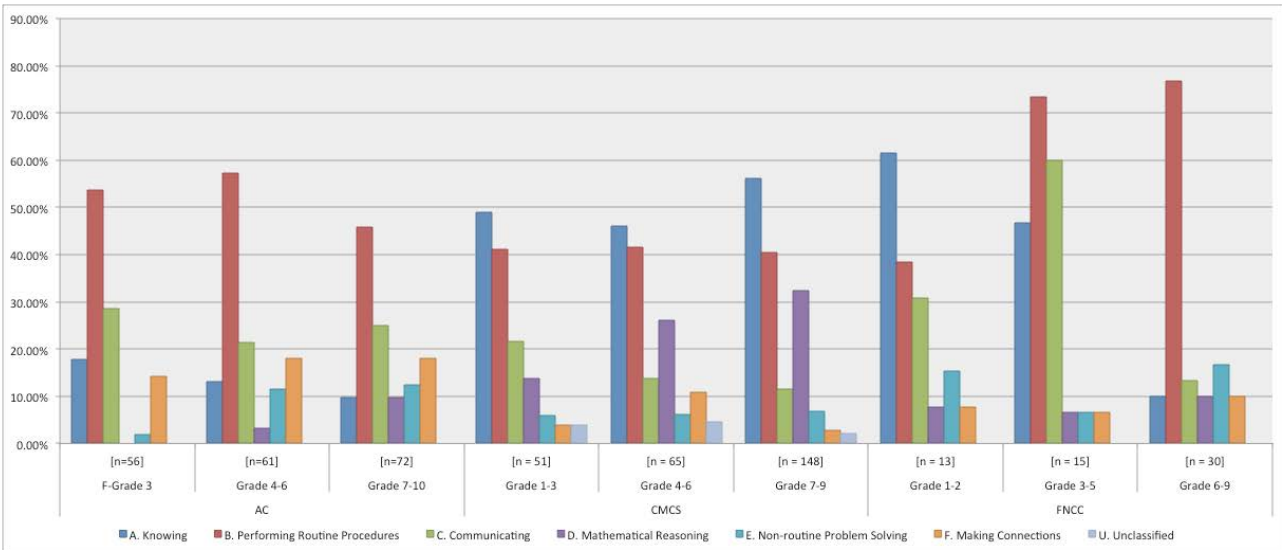

Figure 2: Comparison of Australian, Chinese and Finnish Mathematics Curricula by Performance Type (Cognitive Demand)

student performances promoted in relation to this content.

The measurement of student mathematical achievement on international tests such as PISA or TIMSS constructs a bridge between the mathematics curricula in the participating countries that affords comparison with respect to the performances attributable to students benefiting from the various curricula. The institution of international testing provides the bridge for this form of border crossing and reifies through the international acclamation of its findings the boundaries its acts of comparison have simultaneously surmounted and invoked.

CAVEAT: Bridges can institutionalize both difference and the defining boundary, differentiating what is being connected.

Bridges such as international testing regimes, university ranking schemes, and school comparison metrics (see, for example, the Australian *My School* website: http://www.myschool.edu.au) institutionalize the measures they document and the differences arising from their comparison. The paradox of simultaneously differentiating what is being connected through the act of comparison is at the core of the activity of research comparison. As was discussed in the previous CERME paper (Clarke, 2013a), we must scrutinize the legitimacy of the act of comparison because its consequences can consolidate the boundary it appears to transcend: both constructing and concealing difference (Clarke, 2013b).

A fourth way is to find objects to which the boundary is permeable

A truly impermeable boundary would prevent all possibility of comparison. Another way to say the same thing is that there would be no objects pertaining to one domain that had meaning within the other domain and nothing, therefore, that could serve as the basis for comparison. In one form of contemporary boundary-speak, there would be no possibility of a "boundary object" (Star & Griesemer, 1989). I do not propose to make continued use of the term "boundary object" in this paper. The term is sufficiently widely used to have become ill-defined or at least variously defined and my metaphor of permeability will serve my purpose without the added burden of any possible misunderstandings of another multi-faceted term.

To provide a contemporary context for this form of boundary crossing, I would like to situate the discussion around the acronym "STEM." The acronym STEM has achieved recent popularity and is simultaneously invoked to affirm a perceived distinction between STEM and non-STEM disciplines, the implicit integrity of a knowledge domain designated by STEM, and the independent integrity of the constituent elements of STEM, that is, Science, Technology, Engineering and Mathematics [2]. In fact, it seems that STEM is invoked in educational discussion as frequently to distinguish its constituent knowledge domains as it is to affirm their connectedness. This simultaneous invocation of integration and distinction makes STEM an ideal site to employ "boundary crossing" as an interrogative tool.

We have become so accustomed to the subject grouping for which STEM is the acronym, that it is difficult to recognize that STEM could be the name for a fairly monumental category error. One approach is to consider the nature of the truth claims characteristic of each discipline and the authorities to which these might appeal:

Science – empirical consistency

Technology – tool utility

Engineering – built viability

Mathematics – logical coherence

This approach demonstrates just how fundamental are the differences between STEM disciplines. If STEM, as representative of some unitary aggregate or assemblage, is to be of value in educational (or other) settings, then we need a mechanism to enable boundary crossing between the STEM disciplines. In this fourth approach to boundary crossing, we examine those constructs to which the boundary walls of the STEM disciplines seem most permeable. What we need to identify are constructs that demonstrably do explanatory or at least classificatory work in more than one domain within STEM.

How permeable are the disciplinary boundaries within STEM? And to which constructs are they permeable? Here are four contenders [3]:

Discourse – reasonable speech

Artefacts – constructed objects

Reasoning – purposeful thought

Evidence – objects of justification

Take "Evidence" as a construct having currency in each of the STEM disciplines. What qualifies as evidence in the domain of mathematics may be differently conceived than in science. Yet the function of evidence remains arguably the same in each domain: the validation of truth claims. Research seeking to compare phenomena across the STEM disciplines can do useful work by addressing how constructs such as Evidence are employed. How are these constructs transformed in their passage between STEM cells? Do we find conservation of function accompanied by transformation of form? Are the differences between STEM and non-STEM domains with respect to either the function or form of Evidence so disjoint that our act of comparison, legitimized by the existence of a construct (Evidence) to which the STEM to non-STEM boundary is permeable, retrospectively amplifies the STEM non-STEM distinction and consolidates that same boundary?

CAVEAT: How are these objects transformed in their passage through the boundary? Does conservation of function but transformation of form maintain object identity and consequently comparability?

We start to hear echoes of the Validity-Comparability Compromise (Clarke, 2013a), as the empirically-driven opacity (impermeability) of the boundary undermines the legitimacy of the very comparison that is rendering it more opaque. I suggest that the status of our "boundary object" as "boundary object" is critically dependent on the balance between sufficient similarity to support comparison and sufficient difference to sustain the boundary.

A fifth way to cross a boundary is to federalize the collective of bounded regions into a structured unity

STEM also provides an example amenable to our fifth method of boundary crossing. If we consider STEM to be a confederation of states subject to the same legislative and constitutional principles, but independently organized for many practical purposes, then boundary crossing is achieved through the identification or articulation of those constitutional (and constituting) principles. Not only does this approach constitute a form of boundary crossing by transcending intercellular STEM boundaries, but it also holds the capacity to regulate the process of boundary crossing by legislating which responsibilities are shared and which are the specific province of each domain. For example, is Evidence universally invoked, but Proof restricted to the domain of Mathematics? The mechanism whereby such principles of intellectual trafficking are laid down will reflect the relative agency and voice given to the constituent entities in the federated states of STEM. Dominance of any particular voice in determining the principles of exchange (e.g., the standards for evidence-based practice) would constitute an act of colonization.

CAVEAT: Federation is a commendable aspiration provided it does not become colonization. Who speaks for each bounded region?

Again, we find echoes of the concerns expressed by Clarke (2013a), elaborating the proposition: "Comparison must not be unilateral" (Stengers, 2011). In the context of international comparative research, any construct employed for the comparison of the classroom practices of Confucian-Heritage-Cultures and "Western" cultures must be sanctioned by each conglomerate community as legitimately typifying some shared aspect of both. Such shared aspects cannot be identified unilaterally (Clarke, 2013b).

A sixth way to cross a boundary is to accept responsibility for its construction (and deconstruction)

Each act of comparison simultaneously achieves the researcher's creation of the domains that are the subject of comparison and the boundary by which the domains are defined and distinguished. Each research report solicits the reader's complicity in these acts of construction and distinction. As already discussed, the event of comparison may be predicated on a presumption of difference that provided the warrant for comparison, but the consequences of the comparative activity may provide evidence that could either consolidate or destabilise the boundary on which the legitimacy of the comparison was predicated.

From this perspective, boundaries must be seen as fragile entities, ephemeral, continually changing and immensely useful. Eugene Ionescu once stated, "Only the ephemeral is of lasting value" (see Rothenburg, 1993). Whatever ideological commitments we might all feel to inclusivity, our practice as researchers acts to divide, to create boundaries. We do this most visibly in Comparative Research, where acts of comparison are foregrounded, as are the domains across which we compare. As has been argued, these acts of comparison have the inevitable outcome of constructing boundaries. Our obligation as researchers is to acknowledge this activity and engage simultaneously in both the construction and the deconstruction of these boundaries. In this way, by accepting our role in boundary construction, we position ourselves across (on both sides of) the boundary, not only able to make comparison but also to examine the implications of that comparison for the boundary it presumes. This examination will require the deconstruction of the boundary, providing insight into its utility, its fluidity and what I have called its conceptual tenure.

SUMMATIVE DISCUSSION

Boundaries are constructions, built of language through discourse. They are inevitably purposeful and can be both useful and affirming. They must also be fluid, in the sense that they must always be subject to contention, to destabilisation, and, consequently, open to deconstruction and reconstruction.

In this paper, I have foregrounded the role of comparison (in Comparative Research and in research in general) in creating and crossing boundaries. Viewing the various events of Comparative Research from the perspective of boundary crossing sensitizes us to the role research plays in creating boundaries and to the implications for our research of both the possible nature of these boundaries and of the process of boundary crossing that is also intrinsic to our research activity.

International comparative research in mathematics education can both create and destabilize boundaries in ways that enhance or impede our ability to benefit from the practices of mathematics classrooms and school systems elsewhere. The boundaries we construct should clarify our understandings, not impede their application. Equally, our destabilisation of existing boundaries should result from our demonstration that some boundaries do no useful work, but rather inhibit our consideration of alternative ways to conceptualise our discipline, our pedagogy, and even our research.

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ENDNOTES

1. It is important to note the dangers of specific instantiations of a more generalized construct: a river, a wall and a set of principles can all serve as boundaries; but a river can connect, a wall can protect and a set of principles can empower. The illuminating illustration can also mislead and we need to be sensitive to entailments of such examples that are no part of the boundary function for which they were invoked.

2. In fact, it is arguable that STEM encompasses many more knowledge domains than simply Science, Technology, Engineering and Mathematics; each of which (as we have seen for mathematics) cannot be treated as consensually defined. Bioinformatics provides a useful example of a STEM discipline that is both additional to and a hybridization of the 'primary' STEM domains.

3. More detailed discussion can be found in (Clarke, 2014).

The 'New Math' reform and pedagogical flows in Hungarian and French mathematics education

Katalin Gosztonyi

University of Szeged, Bolyai Institute, Szeged, Hungary
University Paris Diderot, Laboratoire de Didactique André Revuz, Paris, France, katalin.gosztonyi@gmail.com

In this paper, I present my PhD research (in progress) focusing on the mathematical reform movements of Hungary and France in the 1960s and '70s and their origins: especially how much they were influenced by the international "New Math" movement, and by the mathematical and mathematics education traditions of these countries. I consider different aspects of the reform: curricula, textbooks, teaching practices. I look for their general characteristics, in the sense of the Pedagogical Flow approach (Schmidt et al., 1996), the leading principles behind these characteristics and the historical-cultural origins of these principles. In this paper, I present elements of this research, focusing on the example of the Pythagorean theorem.

Keywords: Pedagogical flow, New Math, Pythagorean theorem, textbook analysis, comparative research.

INTRODUCTION

Although the reform introduced by Tamás Varga during the 1960s and '70s is generally recognized by the Hungarian mathematics education community as a key moment in the history of Hungarian mathematics education, having an important influence and keeping its values until today, its detailed historical or didactical analysis is lacking [1]. In my research, I consider Varga's work in its international context, comparing to the French "Mathématiques Modernes" reform of the same period: one of the most influential reforms during the international "New Math" movement, and also one of the well described ones, thanks to French historians of mathematics education (e.g., d'Enfert & Kahn, 2011). I attempt to describe general characteristics of these reforms, looking for coherent aspects between their different elements. Beyond common characteristics of the two countries' reforms, issued from the international discourse, I attempt to underline differences, and to show how they ensue from cultural traditions of the countries in question. I focus especially on mathematicians' conceptions about the nature of mathematics and its teaching.

I apply the term "pedagogical flow" in accordance with (Schmidt et al., 1996), where the international research group preparing the TIMSS studies introduces it to describe general characteristics of the mathematics educational system of a country which are present in different elements of mathematics education (as in curricula, textbooks, teaching practices). However, the cultural background of these characteristics, even if supposed, is hardly explored in (Schmidt et al., 1996); and the general model used by this research, the model SMSO which presents interrelations between the different levels of an educational system in detail, does not take into account the cultural, political, social or scientific background. Another model, the levels of codetermination of Chevallard (2002) takes these aspects into account, but integrates them into one hierarchical and linear system, and supposes that broader institutions like society or the scientific community of a country determine the lower levels of the system, such as the teaching of mathematics, for example.

In my research, I take into account complex interrelations between different elements (e.g., political, social, scientific and cultural) of the historical context, and the characteristics of the reform in each country. I focus more in detail on the epistemological background of mathematics and its teaching, expressed by a community of mathematicians in each studied country (communities which were particularly influential in the "New Math" period, as it is showed in the historical part of the study). By analysing written documents of the reforms: curricula, textbooks, teachers' handbooks and also the teaching practices suggested by these documents, I try to show how the conceptions expressed by mathematicians about mathematics and its teaching appear in the didactical characteristics of the reforms. In this paper, I focus on the example of the geometry curricula and, in particular, the Pythagorean theorem.

The "New Math" reform period is particularly suitable for this kind of research for several reasons. First, the "New Math" reforms are profound reforms, transforming curricula, resources and attended teaching practices in a coherent way, in accordance to some leading principles [2]. Secondly, exactly because of the profound changes, characteristics of the reform and the underlying principles are often explained in detail to inform the teachers and the society. Thirdly, the wide international discourses of the period make the comparison easier.

Finally, even though the "New Math" period is already history, it is not very far-away, and has its influence until today. Comparison of the present research to some more modern studies about "pedagogical flows" (e.g., Schmidt et al., 1996, for France; Andrews & Hatch, 2001, for Hungary) allows us to suppose a certain continuity. In this sense, the research on the "New Math" reform of Hungary and France can contribute to a better understanding of the historical, cultural and epistemological background of pedagogical flow in these countries.

METHODOLOGY

The research consists of three major parts: a historical, an epistemological and a didactical part. The first part of the research, concerning the history of mathematics education, is based on existing historical studies about France and about the international discourse of the "New Math" period. Concerning Hungary, general works on the history of pedagogy, original official sources, as well as written and oral memories of Varga's colleagues are used.

For the second, epistemological part, writings of mathematicians influencing the reforms are analysed: publications and lectures about mathematics education, mathematics popularisation books and correspondences. I look for characteristics of these mathematicians' conception about mathematics, and their main principles about its teaching.

The third, main part of the research is based on analytical tools provided by French theoretical frameworks of mathematics education research. After a general analysis of the content and of the structure of the curricula, three chapters are chosen from the first 8 grades in Hungary, and from the first 9 grades in France (primary and middle-school in each case). The analysis contains 1) an analysis of the place and role of the chosen chapter in the curriculum, based on the "ecological approach" of the Anthropological Theory of Didactics (Artaud, 1997); 2) a structural, rhetorical and linguistic analysis of the textbooks and teacher's handbooks; 3) an analysis of the teaching practices suggested by these resources, based on the Theory of Didactical Situations of Brousseau (1998).

THE HISTORICAL CONTEXT

During the 1960's and 70's, the international "New Math" reform movement, starting from the US and from some countries in Western Europe, influenced mathematics education in many countries of the world. France was one of the leading countries in this movement. International and French research studies underline the role of the technological competition of the Cold War, of mathematicians' efforts to integrate elements of modern mathematics, of the psychological discourses (first of all around Piaget), of the development of the educational systems and of society in the "New Math" reforms (e.g., d'Enfert & Kahn, 2011; Kilpatrick, 2012). Similar processes can be shown concerning Hungary.

The French reform called "Mathématiques Modernes" was introduced in 1969 for secondary, and in 1970 for primary education, following the work of a national committee led by the mathematician Lichnerowicz, but also vivid debates in teachers' associations and different, mostly short term experimentations. A modification of the reform took place in 1977.

In the same period in Hungary, a reform project was led by Tamás Varga, inspired by experiments of different countries but also by some Hungarian mathematicians and psychologists, and based on a long experimentation process since 1963 (Varga, 1975). This project was selected by a ministerial committee as basis of the reform of mathematics education, and the new official curriculum was introduced in 1978.

MATHEMATICIANS' DISCOURSES ON MATHEMATICS EDUCATION

In the "New Math" period, mathematicians participated actively in influencing mathematics education. In France, several mathematicians, often members of, or near to the Bourbaki group, expressed their opinions (e.g., Dieudonné, Choquet or Lichnerowicz, the leader of the committee preparing the reform). They emphasize the importance of modern, unified formal language, abstraction, structures and the axiomatic-deductive method in mathematics education. According to them, structures of modern mathematics correspond perfectly to the structures of human thinking; therefore they suggest that students should be introduced as quickly as possible to the use of this language and methodology (see, e.g., Piaget et al., 1955) [3].

In Hungary, mathematicians also took an active role in the reform movement of the period in question. In my present research, I focus on a group of first-rate Hungarian mathematicians who were interested in education since the 1940's and had important influence on the later reforms: first of all L. Kalmár, R. Péter, A. Rényi, L. Surányi, but also Hungarian thinkers living abroad: G. Pólya and I. Lakatos.

The analysis of their diverse writings (Gosztonyi, 2012) shows that these Hungarian mathematicians' image of mathematics is in deep contradiction with the one represented by the Bourbaki-school. They see mathematics as a constantly developing and changing creation of the human mind, and this development is guided by series of problems. According to them, the source of mathematics is intuition and experience; mathematical activity is basically dialogical and teaching mathematics is a joint activity of the students and of the teacher, where the teacher acts as an aid in the students' rediscovery of mathematics. Excessive formalism is discouraged; formal language being also seen as a result of a development. Mathematics is presented as a creative activity closely related to playing and to the arts.

DIDACTICAL ANALYSIS OF THE REFORMS

The content and the structure of the curricula

Concerning the curricula, both reforms aim to introduce new chapters from modern mathematics (such as set theory, logic, topology etc.), and to present mathematics as an integrated science (not "counting and measuring" as before the reform). But the way of realising this, the structure of the curricula is very different in the two countries; while the French curriculum is strictly hierarchic and linear, based on set theory, the Hungarian curriculum contains five big topics which are present in parallel during all the curriculum and interact with each other in a dialectic way: 1) sets and logic 2) arithmetic and algebra 3) relations, functions and series 4) geometry and measure and 5) combinatorics, probability and statistics.

In the followings I briefly present the case of the geometry curricula and that of the Pythagorean theorem.

In the French geometry curriculum of 1969 and 1970, an important break is marked between the lower grades (until 7th grade) and the last two years of the middle-school (8th and 9th grade). In the lower grades, geometry has minor importance, and is not recognised as 'veritable mathematics': the related chapters of the curriculum are named "observations of physical objects" and "practical exercises". The curriculum underlines that the study of 'veritable geometry' starts from the 8th grade, as an example of axiomatic thinking. Axioms and notions have to be introduced via physical observations, but once they are admitted, they have to be clearly distinguished from the physical word and every further theorem has to be deduced by formal demonstrations.

The study of geometry follows the axiomatic construction of real numbers, and is based on this last notion. Classical synthetic geometry is completely eliminated: the main aim of this geometry curriculum (in accordance with Bourbaki's construction of mathematics, where geometry is not an autonomous domain but part of topology) is not to study geometrical figures but to construct an algebraic tool to describe first the affine, then the Euclidian plane and space. Principal notions are projections, vectors, frames, transformations etc.

In this French curriculum, the Pythagorean Theorem is of limited importance: it is integrated in a bigger chapter about the Euclidian plane, as an algebraic consequence of a property of the orthogonal projection, and contributes to the construction of the notion of an orthogonal frame [4].

The Hungarian curriculum links geometry to other domains of modern mathematics (e.g., to set theory; to functions by transformations treated as movements; to combinatorics by discrete geometry), but not in a hierarchic, rather in a dialectic way. The visual nature of geometry plays an important role: geometry offers intuitive examples to treat problems of the other above mentioned domains. Although coordinate-systems are introduced, the studied geometry is mainly synthetic and concerns figures and their properties, transformations and symmetries.

The curriculum emphasises continuity; physical world experiences are present until the end of the middle-school, in a dynamic relation with argumentations and proofs on ideal figures. There is no completely axiomatic geometry in the Hungarian curriculum.

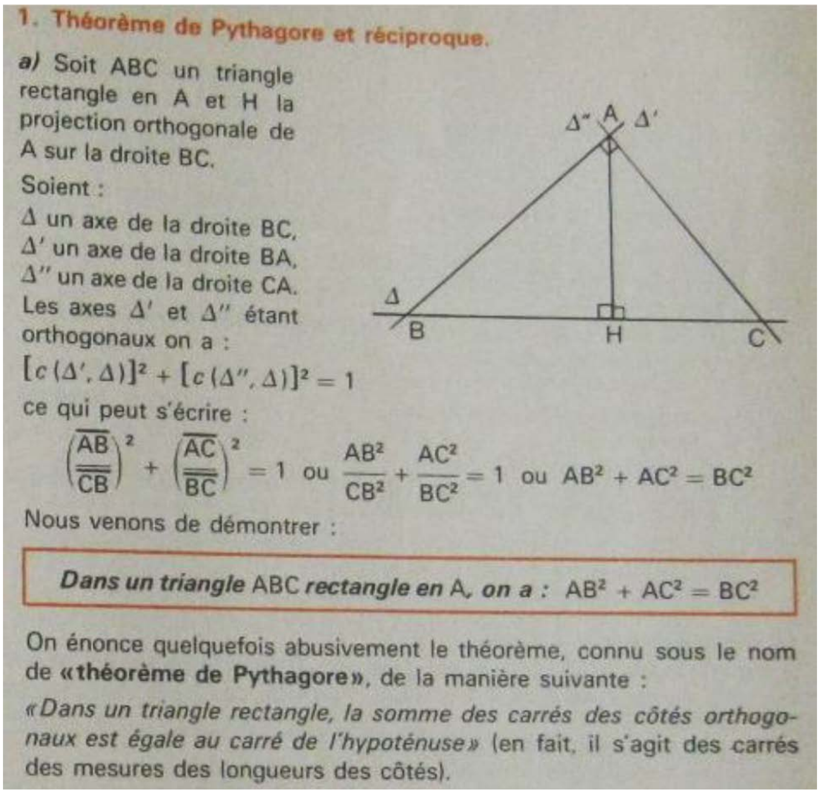
The Pythagorean theorem plays a significant role in this curriculum, not only as an important and useful property of right-angled triangles, but also as one of the first theorems, where students can discover the significance of proving.

The textbooks, teacher's handbooks and the attained teaching practices

In France, there is a great diversity of textbooks and of related handbooks, but some general tendencies can be observed. In Hungary in the period in question, there is only one obligatory series of textbooks. Here I present some structural, rhetorical and linguistic characteristics of middle-school textbooks, their suggestions about teaching practices; treating the example of the Pythagorean theorem in detail.

French middle-school textbooks, according to the curriculum, emphasise the initiation of students in the precise use of mathematical language. The first two years' books give also some natural language examples and describe some physical experiments; the second two years' books contain mainly formal mathematical discourse in an axiomatic-deductive form, followed by some "exercises" at the end of each chapter.

Figure 1 illustrates a typical treatment of the Pythagorean theorem in one of the French textbooks of the period. The demonstration is purely algebraic, requires developed formal and theoretical knowledge. The figure is only an illustration and what it represents is not really a triangle, rather three lines projected on each other. The textbooks, in accordance with the curricula, present the Pythagorean theorem as an element of a big theoretical system, constructing an algebraic tool to describe the plane and the space.



1. Théorème de Pythagore et réciproque.

a) Soit ABC un triangle rectangle en A et H la projection orthogonale de A sur la droite BC.

Soient :

Δ un axe de la droite BC,
Δ' un axe de la droite BA,
Δ'' un axe de la droite CA.

Les axes Δ' et Δ'' étant orthogonaux on a :

$[c(\Delta',\Delta)]^2 + [c(\Delta'',\Delta)]^2 = 1$

ce qui peut s'écrire :

$$\left(\frac{\overline{AB}}{\overline{CB}}\right)^2 + \left(\frac{\overline{AC}}{\overline{BC}}\right)^2 = 1 \text{ ou } \frac{AB^2}{CB^2} + \frac{AC^2}{BC^2} = 1 \text{ ou } AB^2 + AC^2 = BC^2$$

Nous venons de démontrer :

Dans un triangle* ABC *rectangle en* A, *on a : $AB^2 + AC^2 = BC^2$

On énonce quelquefois abusivement le théorème, connu sous le nom de **«théorème de Pythagore»**, de la manière suivante :

«Dans un triangle rectangle, la somme des carrés des côtés orthogonaux est égale au carré de l'hypoténuse» (en fait, il s'agit des carrés des mesures des longueurs des côtés).

Figure 1: (Fauverge, Jeanmot & Rieu, 1976, p. 163) Handbook for the 9th grade

This purely mathematic, deductive treatment corresponds to a lecture form in education: a direct transmission of institutional knowledge by teachers, and passive understanding by students (as we can understand from several textbooks' introduction). Although the introductions of the textbooks emphasise the importance of modern pedagogical methods and students' activity, they give only some general pedagogical indications and little help to concrete pedagogical practice. From the point of view of Brousseau's theory, these textbooks offer little *adidactical potentiality*: occasions to situations when students would engage in the construction of their own mathematical knowledge.[5]

Hungarian middle-school textbooks of the period are different from the presented French ones in several aspects. As we can see in the example presented below, the books contain a number of non-mathematical illustrations and didactical signs (the STOP sign means for example, as it explained in the introduction, that the reader should stop and think about the asked question). They also contain, in every grade, fictive dialogues of students to introduce new knowledge. The related teacher's handbook proposes to provoke similar discussions in the class. The dialogues are guided by a series of problems.

In the case of the Pythagorean theorem (see Figure 2), the first problem concerns the length of a rope stretched across the classroom, so that a student with a given height could stand under it. Students first estimate the result, and then solve the problem by experiment and measurements. The second problem is similar, but instead of the classroom, it concerns the bigger sports hall where students can't perform real experiments. The question is whether the difference between the length of the cord and that of the hall is bigger or smaller than in the case of the first problem. They first try to solve the problem by modelling and measuring, but the approximate result obtained this way isn't precise enough to answer the original question. Then they look for another method to solve the problem, "only with the help of calculations".

At that point, the handbook suggests finding a relation between the sides of a right-angled triangle, and introduces the figure of a classical geometrical proof of the Pythagorean theorem. Even the proof of the theorem is problematised, interrupted by questions and by discussions of students studying the figure. Finally, the theorem is applied to solve the original problem, as well as other problems.

The teaching practice suggested by the teacher's handbook and illustrated in the dialogues of textbooks is a kind of "guided discovery" process: students are guided through a series of problems, while continuing a dialogue between each other and with the teacher about the problems. Intuition, visuality and experiences play important role in this discovery process.

From the point of view of the theory of Brousseau, it is difficult to determine whether this work of students can be called *adidactic*: they rarely work autonomously, without the teacher's intervention (which is a necessary condition of the classical notion of an adidactical situation); nevertheless they take important responsibility in the process of constructing mathematical knowledge. So, the "guided discovery" teaching practice can be interpreted as involving an adidactical character of student's work, even if it doesn't correspond exactly to the classical notion of adidacticity.

CONCLUSION

The different presented aspects of mathematics education: the content and the structure of the curricula, the form of textbooks, and the attended teaching practices show great coherence both in the case of Hungary and of France. This observed coherence allows us to talk about "pedagogical flows" in the sense of (Schmidt et al., 1996).

Some common characteristics of the two reforms can be observed which may take their origin from the international discourses of the New Math period (like the ambition to present mathematics as a new, coherent subject; the emphasis on 'mathematical thinking'; new topics introduced like set theory or logic; the use of manipulative tools, especially in primary school). But there are also some important differences between the two countries, and the analysis of the mathematicians' principles let us suppose that they can be traced back to some mathematical traditions of these countries.

In France, such characteristics are the focus on big theoretical systems and on the strict hierarchical structure of mathematics, the emphasis on the axiomatic-deductive method and on the formal language

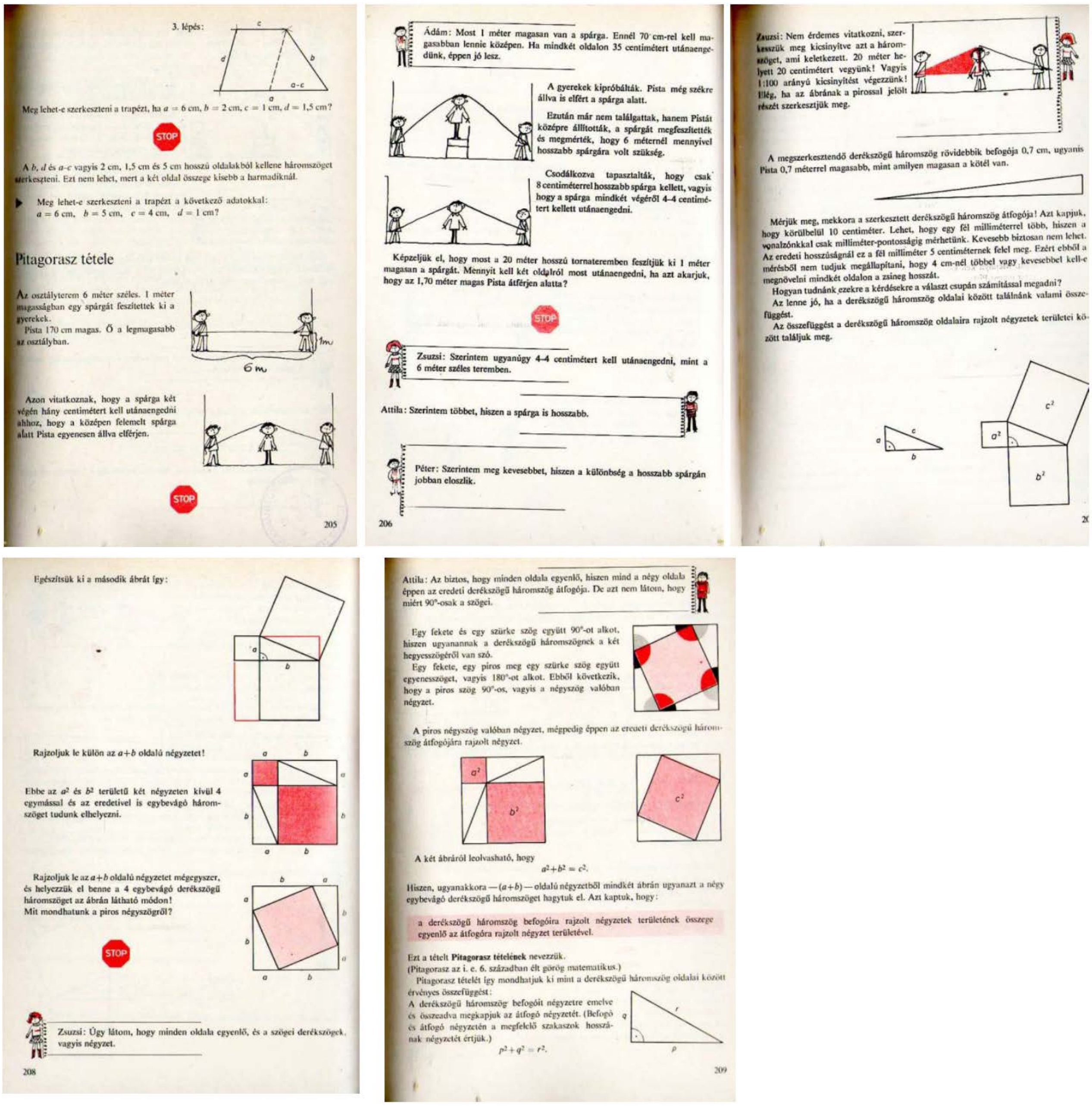


Figure 2: (Kovács, Sz. Földvári & Szeredi, 1980 pp. 205–209.) Handbook for the 7th grade

of mathematics. A principal aim of mathematics education is to initiate students into the knowledge and the methodology established by 'modern mathematics', considered as an ideal of human thinking. A tension can be observed between mathematical and pedagogical ambitions: although teacher's handbooks suggest to middle-school teachers to use some modern methods of active pedagogy, they offer little concrete suggestions to their realisation; the textbooks mostly correspond to the lecture form as a typical teaching practice.

In Hungary, the emphasis is more on the natural development of students' mathematical thinking and problem solving skills. The curriculum content is diversified and different topics interact dialectically, presenting the developmental, rather than the hierarchical nature of mathematics. Varga's curriculum is very careful with introducing formal language, and relies on empirical knowledge and on manipulative tools even on higher levels of mathematics education. A typical teaching form is the dialogue between the teacher and the students while they participate in a common discovery process based on series of problems.

To summarize, the "New Math" reforms of the two countries represent two, almost paradigmatic cases of mathematics education, related to different mathematical traditions and different epistemologies: a "bourbakist" view in the French case and a "heuristic" or "lakatosian" view in the Hungarian case.

The "New Math" is already history, and this research couldn't even attempt to have access to practices in ordinary classes. Curricula and resources changed in important ways in both of the countries since the 1970s, following social debates and didactical research among other factors (actually, the changes seems to be even more important in the French than in the Hungarian case). But, as I mentioned above, a comparison with results of other research works based on current classroom observations, confirms that several observed characteristics remain present both in French and in Hungarian mathematics education (see, e.g., Schmidt et al., 1996; Andrews & Hatch, 2001). The analysis of the "New Math" reforms may complete these existing observations, provide basis for further ones, and contribute to understanding the complex interrelations within a country's pedagogical flow more profoundly.

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ENDNOTES

1. Short commemorations with his colleagues are accessible, e.g., in the proceedings of the yearly organised 'Varga Tamás Days' (http://mathdid.elte.hu/html/vtn.html). In English see for example, Szendrei (2007).

2. This research takes into account the *intended* and *potentially implemented* curricula, described in the official documents and in the textbooks and teacher's handbooks. I don't consider the *implemented* curricula, the practice of ordinary teachers in the period, which can be fundamentally different from the intended practices.

3. At the same time, the teachers' association, also influential in the debates around the reform, and convinced by the importance of reforming the content of the curricula, emphasises also the use of modern, active pedagogical methods.

4. The next French curriculum, of 1977, doesn't ask the complete axiomatic construction of real numbers or geometry in the middle-school any more, and emphasises the practice of proof rather than axiomatisation. The mathematical organisation of the geometry curriculum remains similar to the preceding one, however the curriculum provides broader liberty in the organisation of textbooks and in the practice of teachers.

5. This contradiction between pedagogical ambitions and their realisation can be interpreted in the con-

text of the debates mentioned in note 3. The observed tension between mathematical and pedagogical ambitions probably contributes to the emergence of French didactical researches during the 1970's. The textbooks related to the new reform of 1977 follow the development of the debates: although some of them remain similar to the preceding ones, new textbooks appear with more developed pedagogical suggestions, e.g. in problem solving, and with a more classical treatment of geometry, among other things.

Reflection and questioning in classrooms in different cultural settings

Mandy Hommel[1] *and David Clarke*[2]

1 TU Dresden, Faculty of Economics, Chair of Business Education and Management Training, Dresden, Germany, mandy.hommel@tu-dresden.de

2 University of Melbourne, Melbourne, Australia, d.clarke@unimelb.edu.au

Reflection in classroom learning leads to a deeper understanding and helps to connect knowledge with application situations. Socially initiated reflection can be observed as a lesson event embedded in Review, Elaboration, and Summarization. Questions constitute a primary catalyst for stimulating reflection, particularly in classroom settings. This study[1] *investigates reflection events and related questioning behaviour of students and teachers by undertaking a comparative analysis of video data from the Learner's Perspective Study (LPS; Clarke, Keitel, & Shimizu 2006) in classrooms in Australia, Germany, Japan, and the USA.*

Keywords: Reflection, questions, patterns of question sequences.

INTRODUCTION

Already Dewey has pointed out, that reflective thought "... alone is truly educative in value..." (1910, p. 2). The importance of reflection in facilitating and shaping learning processes is broadly accepted. Reflection creates the conditions for the utilisation of new information in several types of action situations. One key function of reflection is the connection, integration or synthesis of existing knowledge. Without reflection, an individual's newly constructed concepts might remain abstract and isolated. If there is no connection to prior knowledge and to ways of action, new knowledge is useless, lacking either conceptual foundation or the connection to contexts in which it might be employed.

Processes of reflection in classrooms are frequently initiated by questions (White, 1995). Questions typically communicate a specific purpose related to identified content and context, and are usually intended to elicit an answer. But additionally, a question also conveys a more general indirect request (Searle, 1969). This indirect request implies: Think about it! That is, reflection can be triggered through the use of questions. An individual's response to the attempted initiation of reflection depends on situational factors and individual conditions, such as prior experiences and knowledge. With the help of questions, teachers can invite students to follow and even participate in the teacher's externalised way of thinking and thereby model both reasoning and reflection (Walsh & Sattes, 2011, p. 69), approaches to solving a problem, and the generation of insight by the elaboration of information. Nevertheless, recognising the individual character of reflection as a cognitive process (function), it has to be supposed, that students' respond differently to attempts to stimulate reflection in classroom learning situations. Classrooms all over the world are embedded in different cultural settings and it has been shown that teaching and learning are influenced profoundly by culture (Clarke, Emanuelsson, Jablonka, & Mok, 2006; Stigler & Hiebert, 1999, p. 87). It is certainly possible that cultural influences might have an effect upon both questioning and reflection, leading to similarities and differences in the performance of both in classrooms.

To get more insight into reflection in classroom learning, this investigation focused on observable indicators and patterns for such processes in classrooms. In addition to the consideration of questions as initiators of reflection, observable phenomena related to conducting reflection in classrooms (methodical forms like question-answer processes between stu-

1 The project was supported by DFG (German Research Foundation, HO 5092/1-1).

dents and/or between teacher and students, a mind-map, or a task situated in a real-world context) have to be documented. Importantly, reflection phenomena must be studied in classrooms in different countries in order to know about the role of reflection in classroom learning in different cultural settings. That is, in undertaking cross-cultural comparison of reflection as a socially-initiated process in classrooms, it is important to attend to both the form and the function of reflection in the different cultural settings.

THEORY OF REFLECTION AND QUESTIONING

Referring to White (1995), reflection can be thought of as initiated by questions and this may occur both as self-interrogation by an individual as well as by means of a socially performed process. Skovsmose (2006, p. 327) emphasized the importance of questions as facilitator and initiator of reflection. In classroom learning these questions could be raised by teachers, but also by students themselves. The assumption is that questions can be employed as the first observable indicators of the occurrence of reflection. The use and the effects of questions in learning have been investigated in many studies. Previous research has shown that most questions in classrooms are asked by teachers (Wragg & Brown, 2001). "It is normal for students not to ask questions" (Dillon, 1988, p. 12). To change this state, it would be necessary to get more into a "habit of reflection" (Costa & Kallick, 2000, pp. 60 ff.; Walsh & Sattes, 2011). To get into such a habit of reflection would constitute a significant change not only for students but also for teachers and schooling in many school systems and cultural settings (Costa & Kallick, 2000). Thought-provoking teacher questions as well as student-generated questions can be utilised to initiate reflection. Because students may not be accustomed to generating questions, it could be necessary to scaffold them with the help of question stems (King, 1992; Hommel, 2012). Such guided student-generated questioning supports to higher level questioning (King, 1992). Extensive analyses by Clarke and his co-workers of video records of large numbers of mathematics lessons in China and Korea revealed a complete absence of student-initiated questions (Clarke, Xu, & Wan, 2013). Such a uniform absence suggests a well-established history of pedagogic practice wholly reliant on the teacher as the source of all classroom questions. It is possible that the contemporary dissatisfaction among Chinese and Korean educators with the capacity of school graduates for innovation and novel problem solving may be a consequence of less well-developed habits of inquiry and reflection. Certainly, recent reforms in curriculum and pedagogy in both China and Korea seem directed towards more interactive and interrogative modes of classroom participation by students. Teacher questions promoting reflection, together with the opportunity for students to replicate such questioning in classroom interaction, may provide the means to realise not only contemporary Chinese and Korean educational aspirations, but also the aspirations of communities where student-initiated questions already occur, but are not promoted to best effect.

The concept reflection can be differentiated in content-oriented reflection and self-reflection. Beside self-reflection, Lengnink (2005) refers to different forms of content-oriented reflection: reflection of situation, reflection of sense, model-oriented and context-oriented reflection (p. 247). The focus of this study is socially-enacted, content-oriented reflection in classroom learning processes. That is, reflection as it is associated with the actual learning of content, application possibilities, and the further use of the learned content in the student's participation in various communities and contexts (Skovsmose, 2006, p. 328). Based upon the outline above, reflection can be defined as the process of further meaning-making and the deepening of an individual's understanding of their existing knowledge, drawing connections to other experiences and prior knowledge, as voluntary, conscious, systematic; embedded in social context, requiring attitudes of willingness, openness for novelty, interest, and the acceptance of responsibility for learning and for the outcomes of learning. Defined in this way, reflection in classroom learning is part of "student content engagement" (Mullis, Martin, Foy, & Arora, 2012 [TIMSS 2011]). The students' embodied "in-the-moment cognitive interaction with instructional content" (Mullis et al., 2012, p. 358) contains, beside other processes, precisely this "reflection in classroom learning." Without reflection, new information could remain disconnected from prior knowledge and ways of action.

RESEARCH QUESTION

Is there empirical evidence for the assumption that reflection processes in classroom learning are associated with questions? Are there commonalities and differences of observable reflection phenomena in classrooms in different cultural settings? It seems reasonable to expect that the variable "culture" is also influencing the occurrence and nature of reflection. For this reason, this study analysed selected classrooms of different countries to obtain an indication of this kind of influence.

METHOD

This investigation is based upon video data drawn from the international comparative Learner's Perspective Study (LPS). The LPS data set comprises lesson data from eighth grade mathematics classrooms in different countries (Clarke, Keitel, & Shimizu, 2006; Clarke, Emanuelsson, Jablonka, & Mok, 2006). The investigation reported here accessed data from classrooms in Australia, Germany, Japan, and the USA. All LPS teachers were recruited on the basis of their competence as judged by local criteria. Twelve lessons were selected. Three consecutive lessons for one teacher from each country were analysed[2]. This sample provided both the opportunity to gauge consistency of practice for each teacher across the three lessons and a sufficient database to facilitate comparison of practice between the classrooms situated in the four countries. Selection of teachers for this analysis was based on the existence of a coherent three-lesson sequence addressing an identifiable sub-topic within the lesson sequence recorded.

For the empirical identification and analysis of *reflection* in classroom learning, it has to be considered first, when and in which form reflection could occur. Reflection events can assumed to be embedded in different "lesson events" (Clarke, Emanuelsson, Jablonka, & Mok, 2006). The following stratified forms of reflection are assumed to be observable, in the order of their occurrence within the course of a lesson: Review, Elaboration, and Summarization. With '*Review*,' Mesiti and Clarke (2006) describe one of the dominant components in the beginning of the lesson as reflection activities related to the content of prior lessons and also to the prior knowledge of students (p. 51). These activities could be whole class activities, involving either review of previous lesson content in form of teacher led discussions, or question-answer situations for repetition of prior knowledge, or the comparison of student solutions to homework tasks. '*Elaboration*' is a form of reflection, which can occur at the end of a lesson, but also during the whole lesson. It implies a deep processing, while a systematization and abstraction relating new concepts and existing knowledge takes place. During this further processing, facts and concepts will be clarified and corrected. The third form, '*Summarization*', normally occurs at the end of a topic or a lesson. In Japanese classrooms, this lesson event is known as "Matome" (Shimizu, 2006). The core functions of Matome are highlighting and summarising the main point in the lesson, promoting students' reflection on what they have done, setting the context for introducing a new mathematical concept or term based on previous experiences, and making connections between the current topic and previous one (Shimizu, 2006, p. 141). The authors' experience of contemporary classrooms suggested the pessimistic hypothesis that Review and Summarization occur most frequently in classroom learning and that Elaboration, as the most desirable form of reflection on the basis of the depth of processing, would be the least frequent.

Following the assumed association between questions and reflection, questions need to be coded. To create an objective, comparable, and reproducible taxonomy, amenable to low inference empirical application, it should be useful to anchor the categories of *questions* to cognitive processes. The "Taxonomy for learning, teaching, and assessing" (Anderson & Krathwohl, 2001) offers a suitable frame. The developed category system consists of Rote (or Recall) questions (remember), Comprehension Questions (understanding), and Elaborative Questions (elaboration in the sense of apply, analyse, evaluate, and create) asked by teacher or students. Rote questions could be further distinguished into Single answer questions and Remembering questions. Single answer questions could be (theoretically) answered with 'one' word (e.g. yes, no) or the recipient is requested to name something. These kind of questions merely require a single (not a simple) and short answer. Single answer questions could be differentiated further into "organizational" (coded SAO) and "learning content" (SAC) regarding the content focus of the questions. Remembering Questions (REC) require an answer more than one word. In this case, the requested answer

2 Overview about the learning content: Australian lessons: area concept, area of a triangle, area of a rectangle; German lessons: binomial formulae; Japanese lessons: equations; U.S. lessons: positive and negative exponents, prime factorization. For details see (Shimizu, Kaur, Huang, & Clarke, 2010; Mesiti & Clarke, 2010).

exceeds only a single-word answer and consists of naming a concept, recalling a procedure or definition. Comprehension Questions (COQ) refer to eliciting a meaningful understanding of facts, concepts and procedures and thereby leading to broader learning than rote questions do. Students' need to have already understood the concepts and procedures the question is addressing in order to answer this kind of question. Elaborative questions (ELQ) are associated with a more deep and intensive form of processing than the previous question types. They refer, for example, to applying a procedure, analysing a relationship or explanation, or to evaluating or creating something. Irrelevant Questions do not have a conceptual connection either to the topic content nor the actual learning task. We decided to consider only questions which really required an answer. Utterances like rhetorical questions (e.g., "Isn't this great?", "We said that's a prime, right?"), in which the person who is asking the question does not actually expect a reply, explicit requests to students to do something, or questions which are answered immediately by the speaker himself, are not targeted.

The development of the coding system followed two steps. First, a deductive approach, based on a literature review and second, an inductive approach, within the coding progress, in discourse with other researchers. The validity of the coding system was proofed by means of intra-coder reliability (Kendall's Tau $\tau = .926$, $\alpha \leq .01$) and inter-coder reliability ($\tau = .875$). Reliability could be further improved by generating a detailed coding handbook with question examples as indicators for the different categories. For the purpose of this study the reliability was sufficient.

The investigation relies upon video-based observation. Of particular interest are observable reflection phenomena and the assumed association with questioning behaviour in classrooms. The unit of analysis for this investigation consists of "lesson events" (Clarke, 2003). "A Lesson Event is intended to connote a form of classroom interaction occurring within a lesson, but at a level of social complexity greater than just a statement or action taken by an individual" (Clarke, 2003, p. 10). Some regularity and recurrence are necessary to label a phenomenon as a lesson event. Reflection as *form* is assumed to be an identifiable recurrent phenomenon in the classroom. Lesson events involving reflection are related to a specific topic, task, problem or/and situation. As an individual process, reflection has the *function* of deepening understanding and elaboration. Whether a student responds effectively to the offer provided by a reflection initiation depends on several situational and individual factors (like emotion, motivation, prior experiences and knowledge). Individual reflection can occur within the socially-performed instructional form of teacher-orchestrated reflection, but is, of its nature, individual and regrettably mostly non-observable during the lesson. Beside, being an individual process in each student's mind, reflection as a socially-performed instructional form occurs with sufficient regularity to be defined as a particular type of lesson event. Observable indicators for these events of further meaning-making, deepening understanding, and drawing connections are also associated with questioning. Questions and the following question-answer sequences can give insights into externalized processes of reflection.

For analysing the lessons, the following forms of research data were used: classroom videos providing different perspectives (teacher, students), videos and transcripts of the post-lesson interviews with several students, lesson tables providing an overview about time, progress, content, and social-interaction form within the lesson; transcripts (original language and transferred into English). The various forms of data were analysed with respect to the research questions, using the forms of reflection and question categories identified from the research literature. Within the coding process, all questions occurring within the selected lessons were coded and the lesson events involving reflection (in the form of review, elaboration, and summarization) were identified and the associated question behaviour identified and documented. The data analyses included quantitative and qualitative procedures.

RESULTS

Over the 12 lessons, 43 reflection events were identified (Table 1).

All reflection events belong to a specific task, content, relationship or procedure and were initiated by a first question. The reflection events showed 21 phenomena of review, 18 elaborations, and 4 events of summarization. The highest amount of reflection initiating questions (21) belonged to *Review* in the beginning of a lesson. Most of these (18) were rote questions (ROQ)

	Review	Elaboration	Summarization
Australia	7	3	0
Germany	3	4	0
Japan	3	3	3
USA	8	8	1
Sum	21	18	4

Table 1: Reflection events in Australian, German, Japanese, and USA-lessons

(Table 2). Events in this category of reflection events were only initiated by teachers, for example:

J1-L03 So, let's take a look and try to remember what we did last time, and go over it before we go on. Um, do you all remember the equations, those we talked about in class yesterday, um, let's share them with the class. What kind of equations we had? (REC)

Student questions were all at the elaborative level and consisted of three questions:

A1-L12 By that couldn't a rectangle be a special kind of square? (EAN)

G1-L05 And what do you need ... what's the practical application? (EAP)

Elaboration events in sum were observed 18 times during the lesson (initiated 3 times by student questions and 15 times by teacher questions). Elaboration did not occur at the end but during the lesson. For the observed elaboration events, initiating teacher questions were found at every level of question. So, it could be suggested that the level of the first initiating question was not crucial for the elaboration progress itself. Rather the subsequent questions-answer process determined the progress of stimulating and scaffolding reflection in the classroom.

The most frequent instances of Summarization were observed in the Japanese Lessons, for example:

US1-L05 Let's - quickly guys, let's quickly do this. What would be, then, a good way to go ahead and sum up then what exactly is a composite? (REC)

Other than in the Japanese classroom, there was only one event of Summarization, which occurred at the end of one of the USA lessons. Summarizations during the lesson that might have been assumed to take place at the end of one topic were not observed. Summarization events were exclusively initiated by teacher questions at the level of rote questions. The distribution (Table 2) could indicate some support for the researchers' initial hypothesis of a theoretically-based hierarchy of reflection, with summarization associated with the lowest (least sophisticated) form of reflection, below review, and with elaboration as the deepest (most sophisticated) form of reflection. The observed "reflection events" provide support for the assumption, that reflection processes are associated with questions. From a practical perspective,

Question Level	Review	Elaboration	Summarization
SAO	1	0	0
SAC	6	2	3
REC	11	0	1
COC	1	8	0
EAP	0	3	0
EAN	1	5	0
EEV	1	0	0
ECR	0	0	0

Table 2: Distribution of initiating questions for reflection events

questions provided the primary observable indicators of reflection.

Within the qualitative analysis of the reflection phenomena in the Australian classroom, intensive recap activities were observed in the beginning of the lesson. But the Australian classroom did not show recurrent reflection activities at the end of a lesson. Rather, the Australian teacher used the specific characteristics of a situation within the lesson, for example a student question, to foster reflection.

The first German lesson (new content: binomials) started without the typical recap activities for German lessons as reported by Stigler and Hiebert (1999, p. 81). Further, the new content was practiced in group work, followed by comparing solutions. Group work was the predominant social-interaction form in the three German lessons. In fact, group work provided the context in all three German lessons for the majority of student questions. Compared to the Australian, Japanese and USA-classrooms, the most student questions were observed in the German classroom.

In contrast with the German classroom, the low frequency of student questions in the Japanese classroom is remarkable. Student utterances were rarely self-initiated. Similarly, Kawanaka and Stigler describe a proportion of 90 per cent teacher spoken words compared to 88 per cent in the USA and 76 per cent in Germany (1999, p. 261). Mostly students only spoke in response to a teacher request.

The three USA lessons started with a few written tasks for the students to solve. A remarkable characteristic was the high frequency of teacher questions as well as question sequences. To understand the process, we added a qualitative analysis of question sequences. Question sequences related to a concrete situation provide a deeper insight into the process. In this analysis, question sequences were defined as three or more subsequent (but not instantaneous) questions belonging to an interaction about a specific content in a delimitated situation.

Observing reflection behaviour in the lessons, three particular patterns of question sequences (Table 3) were identified: sequences of questions belonging to an equal level of cognitive dimensions, sequences of alternating question levels. A third possibility, the funnel pattern (Bauersfeld, 1980): sequences of questions getting a more narrow range of required cognitive dimensions in progress, was not explicitly found in the data. Instead, we found the opposite: sequences of questions leading from a single answer level to an elaborative level (inverted funnel pattern).

CONCLUSION

The initial assumption of this study was that reflection in classrooms is facilitated by questions. The data support the association between questions and observable reflection processes. However, a student's reflection process can only be observed once it is externalized in some way. This form of externalization can frequently be associated with the occurrence of a question. The association between questions and reflection does not allow the conclusion that the externalized question is actually the first incident of an individual reflection. The question could reflect a student's sudden idea or cognizance, the result of a preceding internal (unobservable) process.

This investigation analysed reflection in classrooms in different cultural settings. But the observed phenomena also reflect the specific instructional behaviours of the teachers. Observed differences between the four different teachers might suggest cultural differences but cannot be generalized to represent cultural patterns. The different forms: Review, Elaboration, and Summarization were not observed in equal measures in the twelve classrooms. The reflection events of Review in Australian, Japanese and USA-classrooms, the Elaboration in the German and

	Review	Elaboration	Summarization
equal level	13 (5 AU, 3 GE, 3 JA)	3 (1 GE, 2 JA)	3 (2 JA, 1 US)
alternating levels	7 (2 AU, 1 GE, 4 US)	10 (2 AU, 8 US)	0
inverted funnel pattern	1 (US)	1 (GE)	0
	21	14	3

Table 3: Reflection events in Australian, German, Japanese, and USA-lessons

USA-classrooms, as well as the Summarization of the Japanese classrooms could be employed to increase reflection for supporting students' learning in classrooms all over the world.

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Why look into mathematics classrooms? Rationales for comparative classroom studies in mathematics education

Eva Jablonka

King's College London, London, UK, eva.jablonka@kcl.ac.uk

Larger comparative studies of mathematics classrooms became most prominent as an appendix of international achievement studies. With the advance of video technology and the potentials it offers for researchers, however, comparative classroom observation studies became attractive and feasible also for smaller scale and low budget projects. This paper intends to provide a basis for discussing rationales for comparative studies of mathematics classrooms. It is suggested that the affiliation of classroom observation with school inspection on the one hand, and ethnographic research on the other hand, lingers on in comparative classroom research. The paper provides a narrative, illustrated by examples, that exposes this tension between evaluation and documentation inherent in the field.

Keywords: Classroom research, mathematics education, international and comparative education, teacher evaluation.

INTRODUCTION

There is no shared set of specific goals or methodologies that would justify characterising 'comparative studies of mathematics classrooms' as a sub-field, other than reference to the classroom as a focus of interest and 'comparative' as a research strategy. One might contemplate what constitutes a classroom as a unit; for example, the fact that all inside happen to gather in the same 'room' [1] in the presence of a teacher, or that there is a common practice in which all engage? The latter is not the case for individualised instruction or when students work on-line and access different sites, even if they are in the same room. Further, one can look at classrooms as micro-cultures, as reflecting school culture and schemes of work, as sites for teachers enacting broader curriculum traditions, as sites for transmitting norms and values of larger communities, or as sites for differential distribution of curricular knowledge, amongst others. The 'unit of comparison' [2] will be different in each case. Not all of these interpretations are featured as comparative studies in the mathematics education literature. These are usually comparisons across (groups of) countries, regions or districts. While comparative studies of mathematics classroom practice are driven by diverse interests and methodologies, all assume that classrooms are an obvious site to look at.

The range of comparative studies of mathematics classrooms presented in this working group at previous CERME conferences highlights well the diversity of goals pursued [3]. The intention of this paper is neither to provide a comprehensive review of comparative classroom research nor to present its history in terms of research traditions, but to offer a narrative that exposes the tension between documentation and evaluation inherent in the field.

SCRUTINISING THE QUALITY OF EDUCATION

There is a long tradition of carrying out comparative classroom observations that seek to identify how pedagogy and curriculum relate to students' learning, with the aim of scrutinising the quality of pedagogic practice. In 1891–92, Joseph Mayer Rice took stenographic notes from lesson observations in classrooms in primary and grammar schools in 36 United States cities (East Coast and Mid West). In addition to lesson observations, he talked to teachers, parents and staff in education authorities, visited teacher education institutions, collected student productions and also tested year-3 pupils in arithmetic, amongst others. He classified schools into 'classes' of excellence, ranging from a mechanical 'antiquated' drill-and-practice to a 'scientific' approach. Further, he argued that the differences between the two higher 'classes' of practices can be brought out more effectively by a method of comparison. Accordingly, he summarised commonalities and differences that accounted for his ranking: In the schools of the two 'higher classes', teachers aimed at children's development in all faculties, took into account their 'developmental stage' and made the work interesting. In addition, he observed an 'excellent spirit' amongst the teachers, who treated the children kindly, were enthusiastic and constantly strived to increase their professional strength. The schools in the highest class differed in their practice of integrating school subjects in an 'attempt being made to teach the subjects in their natural relation to each other' (Rice, 1893, p. 222). Further, he notes a strong focus on having children express their ideas in written form in all subjects as a distinguishing feature of the highest class. He also reports that the supervision in these districts takes the form of guidance, instruction and inspiration rather than inspection.

Notably, Rice took the school as his unit of analysis and occasionally included the local policy context as well as the economic conditions of the schools' district in his report. For example, he observed the 'poorest' teaching as regards 'methods and tone' (Rice, 1893, p. 131) in a primary school in one of the poorest neighbourhoods in Boston. [4] This points to another feature of his report, namely the assemblage of impressions on curriculum and authority relations from classroom observations across a range of school subjects.

The report is explicitly evaluative and critical, with clear preferences for a 'progressive' curriculum and pedagogy as outlined in the introduction (notably the labelling of the preferred pedagogic practice as 'scientific'). However, the principles for compiling his notes and the collected student work remain largely implicit. The writing style is journalistic and often scathing, which might be partly motivated by Rice's trip being financed by the magazine *The Forum* that first published part of his report as a series of articles and later financed a second study trip. His overall approach to data generation could be classified as ethnography, while his goal clearly is critical evaluation. Hence he engaged in two different activities.

Around 120 years later, comparative classroom studies produce large data sets due to an increased level of methodological differentiation, such as refined multi-camera video observation, complex scaling and other statistical techniques for constructing reliability and validity of a range of measurements, inclusion of a variety of context variables, systematic procedures for 'coding' segments of classroom video footage, often embedded in research processes characterised by a division of labour in academic work (e.g., coders/report writers). These large studies are affiliated with educational psychology and its tradition of measurement. On the other hand, there are many smaller projects, mostly within a tradition of ethnographic classroom research. The distinction between the activities of evaluation and documentation, however, does not necessarily specialise studies into these two forms. The picture looks more complex.

COMPARING AND MEASURING THE QUALITY OF TEACHING

Comparing and measuring the quality of teaching can be seen as an elaboration of strategies that have been used in many places for assessing teachers during classroom visits, including judgment of the level of teachers' subject-related knowledge and appropriateness of pedagogic strategies, in addition to curriculum coverage and classroom management. Instead of a holistic evaluation based on largely implicit performance criteria (practiced in many places by school inspectors), coding of lessons is based on developing a list of relevant aspects and identifying specific performances that relate to these aspects. Presence/ absence or proportion of lesson time spent on different activities, are then used as a basis for numerical measures, but for lack of theory also expert ratings for each aspect are conducted ('high-inference' coding). The latter leaves the criteria for the 'marks' on each aspect implicit. Scales based on ratings of a range of aspects of teacher performance are increasingly employed in the USA for formative teacher assessment, evaluation of curriculum policy and professional development (Hill et al., 2012). Sapire and Sorto (2012 used adaptations of such a scale (the Mathematical Quality of Instruction - MQI score) for comparing teaching quality in Botswana and South Africa, where teaching appeared much less complex in terms of pedagogical techniques and use of resources than in the USA, which renders the application of the measure questionable. They complemented the measure with curriculum coverage and other codes. In this context, with reference to (Knight & Sabot, 1990), Carnoy (2012) uses the term 'natural experiment' for the situation where national social conditions are similar but policies and outcomes differ. Even though measures of instructional quality originate in the idea that students' scores on mathematics tests are not an appropriate measure of the quality of teaching and hence classroom teaching practice needs to be looked at, correlations with some measures of student achievement are still often incorporated in studies that use such measures and are sometimes used as an argument for their validity. Comparison and evaluation become intertwined.

Studies that aim at comparing instructional effectiveness of different teaching approaches by means of randomised controlled trials have not (yet) been associated with comparative studies in mathematics education. These studies reflect a comeback of experimentalism, asserting it a position of scientific superiority for identifying 'what works'. They occasionally include classroom observation in order to check the fidelity of the teachers' dispending of the intervention (treatment), or to complement measurement of gain scores with scores from classroom observations (e.g., Clements, Sarama, Spitler, Lange, & Wolfe, 2011). Classroom observation in these studies is asserted an auxiliary role, as the classroom is only relevant in relation to the statistical regularity the black box produces as its achievement outcomes. Hence these studies do not qualify as comparative studies of mathematics classrooms.

RELATING INTERNATIONAL ACHIEVEMENT RANKINGS TO TEACHING APPROACHES

The international comparative measurement of average outcomes, which may include affective co-productions (e.g. attitudes) in addition to academic achievement, preceded international studies that included mathematics classroom observations. As a result, in many classroom observation studies, the nominal unit of comparison is countries and administrative units, or larger units defined by shared cultural traditions (such as the 'Confucian Heritage Culture' or CHC), with the aim of identifying characteristics of representative mathematics teacher practice based on random samples of lessons. This is motivated by an interest in high achieving countries (e.g., Hiebert et al., 2003). Comparisons of teacher practices in countries at the bottom end of the achievement rankings rarely attract wider than local interest.

In reflecting upon experiences with the video surveys from the Third International Mathematics and Science Study (TIMSS) and its follow-up study (TIMSS-R), Stigler, Gallimore and Hiebert (2000, p. 87) noted that "the most obvious reason to study classrooms across cultures is that the effectiveness of schooling, as measured by academic achievement, differs across cultures (e.g., Peak, 1996)". A similar observation by the Under Secretary of State for Education recently prompted the invitation of 60 Shanghai teachers to give workshops for teachers in England. Stigler and colleagues (2000), however, also mention the illuminating effect of making the familiar look unfamiliar when confronted with other cultural practices as a rationale for the TIMSS video studies, which does not suggest replanting of teaching practice. They describe the TIMSS video surveys as 'integrating' the tradition of ethnographic classroom research with a survey tradition of schools and classrooms that aims at allowing generalization to a wider population.

As 'comparing' only rests on generating analytical categories to find commonalities and specialising these for describing differences or variations, there is no restriction to the level of detail to which this strategy can be applied; neither is there any restriction to what could be usefully looked at in mathematics classrooms. Lesson structure, based on low-inference codes for time spent on a range of activities, has been a focus of many studies. In doing so, lesson structure ('script', 'pattern') is taken as both an unconscious routine (e.g., Hiebert et al., 2003) or as an outcome of a more conscious act of teachers' planning (e.g., Leung, 1995; see also the discussion in Clarke et al., 2007). Generally, it is not much looked at how instruction is constrained by traditions of classroom management other than through interpreting differences in student behaviour in terms of culturally typical inclinations, e.g., of Chinese students being obedient, a form of analysis, which Jablonka (2013) sees affiliated with cultural essentialism. Wong (2004) is critical towards attempts of interpreting differences in terms of culturally typical behaviour of Chinese students.

As the achievement measures are reported in the form country/region averages, relating classroom teacher practice directly to these measures can only be suggestive. In the first TIMSS study, the classroom samples for the achievement test and the video samples did not overlap in Japan and US, but only in Germany. Klieme and Bos (2000) found a differential item function on the Japanese test results in the achievement test for the type of mathematical tasks they observed in the lessons. For the German videos, relations between teaching practice and achievement were explored through high-inference ratings of some aspects of teaching quality with not much significant outcome. Clarke and colleagues (2007) observed more variability in lesson structure than reported in the TIMSS video study, and see the location of the lesson in a topic sequence as a key influence on a lesson's structure. The TIMSS video studies have not only been criticized for taking single lessons as analytical units, but also for not exploiting the potential of their rich qualitative data sets (Stigler et al., 2000; Andrews, 2007; Clarke et al., 2007). Other studies reported more consistent patterns in lesson structure in some places, as for example in China (Beijing) and also included a wider range of analytical categories than the TIMSS (Leung, 1995). In a random sample of lessons from Finland and Iceland (20 each), Savola (2010) found that the Finnish lessons followed a 'conventional' review-lesson-practice script, whereas more than half of the Icelandic lessons exhibited versions of individualized learning. When looking at the range of codes employed for characterising lesson structure, a major challenge appears to be the apparent embeddedness of instructional and regulative discourse [5]; hence categories for coding tend to include 'mixes' of privileged teacher-student relations and mathematical knowledge structures.

Relating characteristics of teaching practices to national achievement levels did not work for the TIMSS-R video studies either. The official executive TIMSS report (National Center for Education Statistics, 2003) is not explicitly appreciative of any practice, but the mentioning of the achievement levels of the countries implicitly suggests a ranking of some aspects identified in high achieving regions: "Results from the 1999 study of eighth-grade mathematics teaching among seven countries revealed that, among the relatively high-achieving countries, a variety of methods were employed rather than a single, shared approach to the teaching of mathematics." (p. 11). The Pythagoras Study (Klieme, Pauli, & Reusser, 2009) set out to further investigate relations between characteristics of teaching practice and student achievement as well as motivation in 20 classrooms from Germany and Switzerland each, covering a broad range of achievement levels. They focussed on particular topics (the Pythagoras Theorem and algebra 'word problems'), including some briefing for the teachers about how to approach these. This study, then, departs from the ethnographic tradition mentioned by Stigler and colleagues (2000) and moves towards an intervention and evaluation study. It included a broad range of methodological questions and produced a range of publications with detailed analyses. These studies are affiliated with educational psychology usually associated with measurement and with a conception of the curriculum as socially and culturally neutral content delivered in different ways by teachers.

As to the PISA, there is a limit in identifying the schools and classrooms that were chosen for further research. Hence, the use of the available contextual school data for comparative classroom studies is restricted. There is anecdotal evidence, however, that some complementing video studies are planned by the OECD. All attempts at relating regularities of teacher practice to a country's average achievement are further hampered by severe methodological problems in the production of achievement rankings, including sampling problems. Also, different international achievement measures privilege different forms of mathematical knowledge (Wu, 2010), which are balanced differently in national curricula (Cai & Howson, 2013).

This does not derogate the value of looking at larger samples of classroom videos. Most of the reports, however, are ignorant of the socio-political contexts, in which these classrooms are situated. Not much is said, for example, about practices of streaming and selection based on educational credentials, school fees, accountability regimes or working conditions of teachers. The highly competitive nature of the college entrance test in China or the amount of private tutoring in some places (cf. Bray & Kwo, 2014) are rarely mentioned in reports. In addition, classrooms are featured as culturally and socially homogeneous conglomerates, and one is left in the dark about the student intake and neighbourhood of the schools.

EXPLORING REGULARITIES IN MATHEMATICS CLASSROOMS WITHIN AND ACROSS DIFFERENT CURRICULUM TRADITIONS

A range of comparative observation studies are more in line with an ethnographic tradition or interpretive sociology (e.g., Kaiser, 2002; Knipping, 2003). Rather than analysing random samples of classrooms, typicality is achieved through selecting classrooms of experienced teachers judged locally as competent (Clarke, 2006). The practices of these teachers and students can then be seen as representing a range of pedagogic discourses in different curriculum traditions. Identification of regularities and exposure of similarities and differences facilitates interrogation of assumptions and so opens up possibilities for theoretical differentiation and practical innovation that might not otherwise be recognised. Exploring regularities of classroom practice can also be aided by use of statistical techniques (e.g., Andrews, 2009).

The Learner's Perspective Study (LPS) (e.g., Clarke, Keitel, & Shimizu, 2006a) views classroom practice as co-constructed by teachers and students and so included student video-stimulated reconstructive interviews and a camera focussing on changing groups of students (during ten lessons), in addition to a teacher camera and a whole-class view, which affords considering multiple points of view. This is a reminder of the insight that different techniques for producing video data reflect different analytical gazes (cf. Hall, 2000). The variety of analyses conducted by different research groups and teams composed of insiders and outsiders is reported in a series of volumes (Clarke et al., 2006a; Clarke, Emanuelsson, Jablonka, & Mok, 2006b; Shimizu, Kaur, Huang, & Clarke, 2010; Kaur, Anthony, Ohtani, & Clarke, 2013) and other publications. This demonstrates the productivity of a departure from the search for 'best practice' and shows how a substantial data set can be used for iteratively developing languages of descriptions ('theories') as well as for complementary analyses. The advantage of publishing reports that contain transcripts, student work and examples of teaching material is that the audience can engage in their own interpretations.

DEPARTURES AND POSSIBILITIES

While classroom practice also produces valued learning outcomes not captured by the examination system, curricular choices might appear most aligned with external examination specifications or with criteria derived from other policies (e.g., school inspection regimes). In systems with strong regulation there is very limited space for teachers to act upon curriculum. Weaker regulation affords more diversity in classroom practice and more adaption to apparent needs of different categories of students. Hence it would seem reasonable to analyse classroom practice with reference to a level where teachers and schools make deliberate decisions about curriculum, which clearly differs in different socio-political contexts. This strategy departs from attempting to uncover taken-for-granted lesson scripts, patterns or rituals.

Cross-subject studies with the same students or the same teacher would allow to create differentiated accounts of regulative principles of instruction and behaviour in relation to teacher authority and mathematical knowledge structures. Comparison is then a methodological strategy for creating substantial variation in the empirical data as a starting point for developing a language of description (see Gellert & Jablonka, 2009; Knipping, Reid, Gellert, & Jablonka, 2008). Studies with a broader conception of curriculum that illuminate issues of power, identity and subjectivity are still almost absent amongst comparative studies of mathematics classrooms.

Irrespective of the substantial amount of work produced by comparative mathematics classroom studies, those that employ quantifications, in particular for establishing descriptive causality ('findings') between characteristics of teacher practice and some 'student outcomes', still (or again) earn more scientific respectability, despite their often antiquated mechanistic conception of curriculum.

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ENDNOTES

1. Including, for example, open-air arrangements found in rural areas in periphery countries or in refugee camps.

2. The distinction between unit of analysis and unit of comparison has been discussed in the working group on comparative studies in mathematics at CERME 7.

3. These are published in the CERME proceedings. For lack of space for an extensive bibliography, the examples mentioned in this paper will remain limited.

4. It needs to be pointed out that Rice did not comment on the racist nature of some student writing assignment, which he collected from a grammar school.

5. The notion of 'embeddedness' of these two discourses is taken from Bernstein (e.g. 2000).

The perceived causes of the (assumed) mathematics problems in England and South Africa: A social media experiment

Marie Joubert

African Institute of Mathematical Sciences, Cape Town, Republic of South Africa, mariervjoubert@gmail.com

This paper presents an exploratory study, using a methodology based on the idea of building knowledge through the use of social media such as blogs, discussion forums and twitter. The study builds on work carried out in the UK, collecting and synthesising reports related to mathematics education, and from this developing research on the perceived causes of the (assumed) problems with mathematics education in the UK. It develops the research to take account of the South African context, and investigates the same question in this context. The results are compared, and findings include the fact that in both countries 'negative attitudes' towards mathematics are seen as a major cause. The paper concludes by reflecting on the methodology adopted.

Keywords: Negative attitudes, poor teaching, nature of mathematics, social media.

INTRODUCTION

This paper presents an exploratory study, which draws on, and extends, my previous work on the use of social media (e.g., twitter, blogs, wikis) in research (e.g., Joubert & Wishart, 2012; Joubert, 2011). I constructed a blog in which I collected and synthesised over 50 'official' reports broadly about mathematics education, published in the UK after January 2011. The blog can be found at https://mathsreports.wordpress.com. The reports were commissioned by a range of governmental and non-governmental organisations, such as, for example, the Advisory Committee for Mathematics Education (ACME), whose membership includes researchers and practitioners in mathematics education and which aims to advise policy, the Royal Statistical Society and the Nuffield Foundation. Naturally the reports can be seen to reflect the interests of these organisations. Many, but not all, of the reports make generalised statements without reference to empirical research, raising questions about where these statements came from and what the status of the reports is; there is much more discussion about the reports on the blog.

I used social media such as twitter and online discussion forums to invite the mathematics education community and others (anyone who was interested and was willing) to help me understand more about these reports, by making comments on the blog. For example, what did they aim to achieve? Why were they commissioned? Whose interests did they serve? What is their role in shaping – or reflecting - a collective mindset?

On moving to South Africa in mid-2014, I aimed to carry this work forward but within the South African context. The work is at a very early stage but it is nevertheless perhaps interesting to compare initial results with the findings emerging from the UK. Of course the context is very different and one might expect there to be some differences; the value of any findings is under review.

The research has two distinct focus areas: the use of social media in research and the reported state of mathematics education. In this paper, I focus on one aspect of the latter, (the perceived causes of the perceived problems in mathematics education) and comment on the former.

BACKGROUND: PROBLEMS WITH MATHEMATICS EDUCATION?

In both countries there seems to be a belief that mathematics education is problematic, in terms of, for example, low levels of achievement on international benchmarking tests and low take-up of mathematics at post-compulsory levels. The extent to which these problems actually exist or could be said to be similar is of course debateable, but for the purposes of this paper the so-called problem is assumed. Perhaps the most compelling symptom of the (assumed) problem is that in both countries employers and higher education claim that the mathematical knowledge of school leavers falls short of what they require (ACME, 2011a; British Academy, 2012; The Centre for Development and Enterprise, 2013).

It could be argued that the commissioning of the UK reports is in itself a response to the assumed problem and a number of reports claim that a problem exists (ACME, 2011a; Harris, 2012; Vorderman, Porkess, Budd, Dunne, & Rahman-hart, 2011).

In South Africa, education generally is troubled (Spaull, 2013) but it seems that mathematics is particularly problematic (Howie, 2003; Human Sciences Research Council, 2011; Mji & Makgato, 2006) with, for example, South Africa at the bottom of the league table in mathematics in the 2008 international benchmarking tests, TIMSS. Even the highest achievers in South Africa performed less well than the average students in top performing countries (Human Sciences Research Council, 2011).

A second, related, problem in both countries is a reported shortage of young people with appropriate skills in mathematics and so-called STEM subjects. (ACME, 2011a; British Academy, 2012; Southern African Catholic Bishops' Conference, 2012).

The problems, it seems, are similar although the extent of the problem is perhaps not. The focus of this paper, however, is on the causes of the problems rather than the differences in the extent of the problems. If the causes are well understood, then governments might be in a better position to target interventions; and if the causes are similar in both countries, then it could be that recommendations or successful interventions in England might be applicable in South Africa and vice-versa.

THE CAUSES OF THE PROBLEMS

Many of the reports collected on the blog suggest the causes of the problems, which fall into three broad areas: young people's (and adults') mathematical activity; the curriculum, qualifications and assessment; and teachers, teaching and schools.

In terms of young people's mathematical activity, negative attitudes are widely seen as a cause for the problems. The term 'negative attitudes' includes a general dislike of mathematics and a cultural acceptability of 'I can't do maths'. For example, the Royal Society mentions 'a distinct general lack of warmth towards science and maths' (2011, p. 55). It is further reported that not being good at mathematics is generally seen as acceptable and leads to low levels of post-compulsory participation (ACME, 2011b; Harris, 2012; National Numeracy, 2012; Vorderman et al., 2011). Harris sums up the claim, stating that:

> It seems that the UK has a culture where being less skilled in mathematics and numeracy is perceived as acceptable and not uncommon.... learners do not see the need to increase their mathematical skill. (p. 11)

The argument seems to be that because it is acceptable to be bad at mathematics, young people think that they do not need to study mathematics.

The curriculum is sometimes seen as a cause of problems in (mathematics) education. In 2011 the then secretary of state for education announced a review of the national curriculum, stating that 'The previous curriculum failed to prepare us for the future. We must change course.' (Department for Education, 2011). This announcement seems to imply that the Government believed that a different (better?) curriculum would drive standards up.

The ACME report, *Mathematical Needs of Learners*, stated that 'it is not necessarily mathematics itself that is problematic, but rather the nature of the curriculum and the teaching methods and assessment regimes'(ACME, 2011b, p. 8). It went on to say that 'the current curriculum is seen as being fragmented' (p. 6). As described by ACME, the curriculum is linear and lists topics and skills which should be taught and tested in order. This approach can mean that, for students, learning in mathematics is 'fragmented and incoherent' (p. 17).

Assessment is widely mentioned in the reports and there seems to be general agreement that it needs to be improved. However, assessment is not frequently cited as a cause of the mathematics problems. Where it is given as a cause, this is in terms first of the content of some assessments and second the 'assessment regime'.

In terms of the content, there appears to be a tension between promoting mathematical thinking within, for example, extended investigative projects or problems solving, and the ways in which mathematics is assessed. The claim is that examination questions tend to be routine and familiar, and the only mathematical thinking that is assessed is that which is easy to test, and reliability of assessment tends to be highly valued possibly to the detriment of validity. (ACME, 2011b; Norris, 2012; Vorderman et al., 2011). As ACME says:

> GCSE and A-level examinations are dominated by routine procedures and familiar applications. There is strong agreement among teachers, educationalists and Ofsted inspectors that unless all aspects are assessed they will not be given significant teaching time and resource in schools and colleges. (2011b, p. 6)

A bigger concern than the content of the assessments seems to be the 'assessment regime'. Many reports suggest that assessment takes place within a 'culture of performativity' (Norris, 2012, p. 11) driven by the annual publication of league tables and that operating within this culture maybe at the expense of the best mathematical experiences for pupils (ACME, 2011b; Norris, 2012; Ofsted, 2012; Vorderman et al., 2011).

It appears that there are two main causes of the mathematics problems which are directly related to teachers and teaching. First, there is a shortage of specialist mathematics teachers (Parliamentary Office of Science and Technology, 2013; Royal Society, 2011; Vorderman et al., 2011). Second, teacher knowledge is sometimes limited; there is much within the reports to imply that poor or inadequate teaching is the cause of much of the mathematics problem although this point is not always made explicit.

Where teaching is given as a reason for the problems in mathematics, it is argued that the approaches which are often privileged in classrooms are over-simplified and mechanistic and that students do not benefit mathematically from these (ACME, 2011a, 2011b; Norris, 2012; Ofsted, 2012; Vorderman et al., 2011). A number of reports claim that there is tendency to 'teach to the test' avoiding innovative teaching approaches, and using methods that are unlikely to lead to anything more than superficial learning (ACME, 2011a, 2011b, 2012; Harris, 2012; NFER, 2013; Royal Society, 2011). It seems that increased accountability (performance tables and inspections) has a role to play in this (Ofsted, 2012).

It is difficult to find South African reports equivalent to those cited above. However, the South African research literature suggests a range of causes for the problems. For example, Spaull (2013) identifies factors such as teacher education, parents' education and speaking English at home as being strongly associated with mathematics performance. Howie (2005) explored the relationships between background variables and mathematics achievement at the school level and the classroom level. She found that the variables at the combined school and classroom level most strongly associated with mathematics performance were the community where the school was located, size of classes, attitudes, beliefs and commitment of the teachers including dedication towards lesson preparation and the workload of the teacher. In a separate earlier study (Howie, 2003), she also explored the relationship between language proficiency and mathematical achievement and found that students who spoke English or Afrikaans at home, and those coming from classrooms where these languages were mainly used, overall gained higher scores in mathematics. In this same study, she also found that there was an association between achievement in mathematics and socio-economic status of the student, their 'self-concept' in terms of finding mathematics difficult and their perception of the importance of mathematics (their own perception as well as those of their mothers and friends).

Research by Mji and Makgato (2006) investigated the causes of poor achievement in mathematics and science by seeking the views of teachers and learners in a small number of schools. They found direct influences such as teaching strategies, content knowledge and understanding (of teachers), motivation and interest (of both teachers and learners) and non-completion of the syllabus; and indirect influences such as parental role and language.

Whereas research such as Spaull's and Howie's, and the reports produced in the UK, can be seen to take a top-down approach, the approach used by both Mji and Makgato and within this research is perhaps more 'bottom-up' or grass-roots. Building a comprehensive understanding may require both.

METHODS

As discussed above, this research draws on previous experience of using social media and web 2.0 within research, from which the principle of quick and easy data collection was derived. A very short online questionnaire was devised, which begins by stating 'Too many of our young people leave school with mathematical understanding which falls short of the needs of employers and higher education' and then asking 'Why?'.

Five possible causes were tested in the UK. These were derived from the UK reports and discussion points made on the online forums mostly by teachers and parents: assessment focuses on the wrong things; the curriculum is not fit for purpose; societal attitudes towards mathematics are negative; teaching is not good enough; and 'mathematics is hard'.

The same questions were used for the South African audience, and to take account of the South African context two further causes were included. The first, 'Too many learners opt for maths literacy' because of obvious concern amongst the mathematics education community about it (see for example Southern African Catholic Bishops' Conference, 2012). Mathematics literacy was introduced in South Africa 2008 as an alternative to pure mathematics and was designed to provide functional mathematical learning to address the needs of the workplace. However, it is generally seen as an easier version of mathematics rather than an effective preparation for the workplace, as explained by Spaull (News 24, 2014). The second additional cause, 'The language of learning and teaching (LoLT) is not the home language', takes into account the findings by Spaull, Howie and Mji and Makgato, see above.

Respondents were asked to rate each of the causes in terms of a) a key reason b) having some influence and c) not a reason. They were also provided with an opportunity to add 'your longer comment' in recognition of the fact that these five or seven 'causes' are somewhat over-simplified and the list is almost certainly not exhaustive. It also allows respondents to disagree with the 'causes'. Respondents were also asked for their main interest in mathematics education, to allow for an analysis of the responses of different groups (if needed). Further this provides some sort of understanding of the demographic profile of the respondents.

FINDINGS

There were 101 and 62 responses to the questionnaire in the UK and South Africa respectively. Figure 1, below, gives the breakdown of the respondents' main interest in mathematics education in the two countries. This has been included to demonstrate that in both countries responses come from a mixed demographic, with the majority of respondents being teachers.

In both countries, negative societal attitudes are seen as a major cause of the problems in mathematics; in both countries fewer than 10% of the respondents gave this as 'not a reason' and in both almost 50% gave it as a key reason.

For assessment, similar proportions in both countries (about 20%) stated this was not a reason. There is a relatively big difference, however, in whether it is a

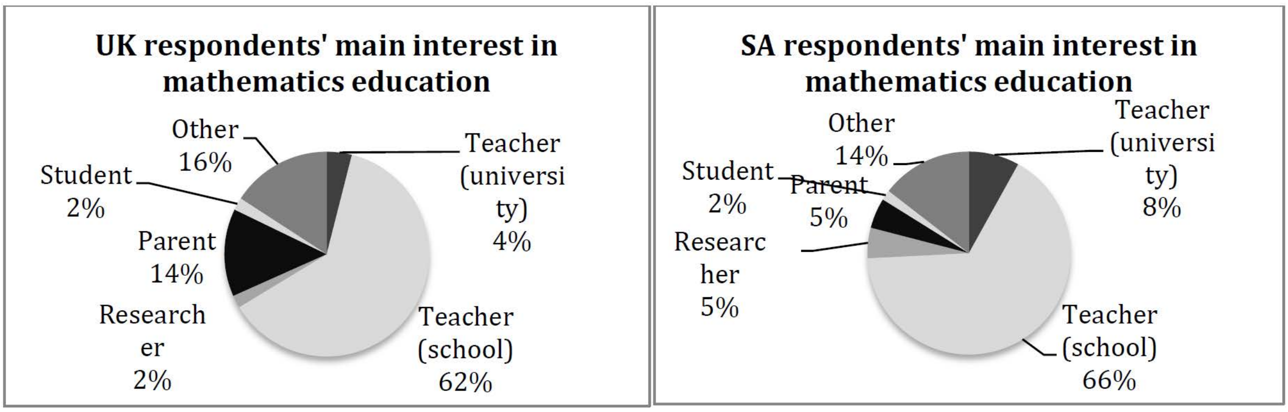

Figure 1: Main interest in mathematics education

key reason; in the UK the figure is over 30% while in South Africa it is under 15%.

In terms of the curriculum, the biggest difference in the two countries is in the numbers stating that it is not a reason, with under 20% in the UK making this choice in comparison to just over 40% in South Africa.

In the UK, under 20% gave 'teaching is not good enough' as a key reason, in contrast to 50% for South Africa. Just under 30% in the UK stated that they did not think it is a reason, whereas fewer than 10% said this in South Africa.

Finally, for 'maths is hard', fewer than 10% chose 'key reason' in the UK, whereas almost 30% made this choice in South Africa. In both countries about 40% chose 'has some influence' but the figures for 'not a reason' are 50% and just over 30% in the UK and South Africa respectively.

For South Africa, for mathematics literacy, just under a third of the respondents chose key reason (28%), just over a third chose 'has some influence' and about a third (32%) chose 'not a reason'. For the language of learning and teaching, however, almost 50% stated that this is a key reason with under 20% stating that it is not a reason.

Almost all respondents also provided longer answers sometimes adding further causes, such as a lack of commitment by the students in both countries (twelve comments). A number (seven) of responses from the UK mention accountability, 'excessive' monitoring, a target-driven approach and the need to see constant progress. In contrast, South African responses give reasons such as lack of accountability, unsafe schools and large classes. In both countries home life is given as a reason. Five UK respondents made comments to this effect, suggesting, for example, that parents do not support the students or that they 'have rotten home lives'. Six comments from the South African survey suggest that home life and parents cause problems in mathematics learning. Of these, four suggest that parents do not support their children in their school work, sometimes explaining how difficult it is for parents:

> 'Majority of our learners have no support at home and therefore have no culture of learning. Parents are too busy trying to make a living and do not have time to check up on their children's progress or lack thereof. The learners only do the Mathematics during the Mathematics period and do not consolidate/practice/do homework, at home.'

Two further comments explain that children are hungry and lack the basics in life.

DISCUSSION

The sample for this research was an opportunity sample of 101 in the UK and 62 in South Africa, very many of whom are teachers. Some of these people were invited directly to complete the survey but most responded to invitations on social media which implies at least that they had access to the Internet and excludes those without Internet access. It probably cannot be seen as a representative sample and I do not claim that the results reported above hold true for the whole population. However, the findings of this study do seem to indicate that there are differences in the two countries with respect to the perceived causes of the (assumed) problems with mathematics education. This is hardly surprising as the contexts are so different. What is more interesting, perhaps, is what the differences and similarities are.

In terms of the findings, in both countries the reason 'negative societal attitudes' was selected as a key reason or as having some influence by very many respondents, with almost half the respondents selecting it as a key reason. This is the only reason where there is strong similarity in the responses in the two countries. Of course, 'negative societal attitudes' over-simplifies the issue, which has many facets and could be seen as related to 'maths is hard' in complex ways. However, the point is that a) this is seen as the 'biggest' cause by the samples in each country and b) that there is agreement between the two respondents from the two countries. In the UK, this cause is well recognised and attempts are made to tackle it (e.g., setting up the charity 'National Numeracy'). The extent to which these attempts are successful is difficult to gauge, but it could be that South Africa might learn something from the UK's example.

A key difference between the responses from the two countries is in terms of teaching. For the South African respondents this cause was selected as a key reason or as having some influence by as many people as selected negative societal attitudes, whereas the figure for the UK is considerably lower. The biggest difference, however, is in the numbers selecting it as a key reason. It seems that the standard of teaching is a major concern in South Africa. One major factor contributing to the poor teaching, across all subject areas, could be the lack of accountability reported above (Spaull, 2014). However, in the UK accountability does not seem to work well, also reported above as a cause of the problems. It is not clear how or whether either country could learn from the other.

There is also a difference in the responses from the two countries related to the cause 'maths is hard' where the proportion of South Africans selecting this as a key reason is over double the proportion for the UK. There is no obvious explanation for this although it may be related to the language of learning and teaching.

Finally, for South African respondents the fact the language of learning and teaching is not the mother tongue appears to be a major cause of the problems. Whereas this is probably the case across all subject areas, in mathematics it is possibly more acute as many of the African languages do not have words for mathematical concepts (e.g., there is no word for 'parabola' in Zulu).

CONCLUDING COMMENTS

The mathematics education landscape is highly complex in the UK as is perhaps evidenced by the large numbers of reports produced on the topic. It seems clear that there is no 'quick fix' (such as a review of the national curriculum) for the problems although there does appear to be some general agreement about what needs fixing (attitudes mainly).The landscape in South Africa, however, is perhaps even more complex, and it is less clear where to put efforts in; negative attitudes, poor teaching and LoLT are all seen as major causes.

It would be unrealistic to expect research of this kind to capture the complexity of the situation in either country but the research does, perhaps, begin to reveal something of the causes of the problems. Learning from this research, further work in the area will pay more attention to the school environment, policy and accountability.

The research approach is experimental and I conclude with a review of its success. To some extent the blog can be seen as successful as it has had over 11 500 views (at January 2015) from all over the world. However, the number of comments, and the level of debate, is perhaps disappointingly low. The comments did not help me answer the questions posed at the start of this paper, but in a sense that is not relevant to this paper which has a different focus. The online forum activity generated active discussion which was useful in informing the questionnaire design. The micro-blogging (twitter) attracted some visitors to the blog and the questionnaire.

As remarked above, the respondents and commenters were essentially a self-selected opportunity sample, mostly anonymous (although some provided email addresses). These were people who volunteered to contribute, and, for the questionnaire, many of them gave substantial responses to 'your longer answer'. In my view, the data collected is interesting and worthwhile and justifies the approach adopted. However, I would like to explore further about how to use web 2.0 and social media approaches more effectively.

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Swedish and Chinese teachers' views on what constitutes a good mathematical test task: A pilot study

Aihui Peng[1], Håkan Sollervall[1,2], Erika Stadler[3], Yueqiang Shang[4] and Li Ma[1]

1 Linnaeus University, Faculty of Technology, Kalmar, Sweden, aihuipeng@gmail.com
2 Malmö University, Malmö, Sweden
3 Stockholm University, Stockholm, Sweden
4 Southwest University, Beibei, China

Mathematical tasks in tests are central to students' learning. Research shows that there is a significant gap between mathematical tasks in national tests and teacher-made tests. In this pilot study, we examine nine Swedish and nine Chinese teachers' views on what constitutes of a good mathematical test task at the lower secondary school level. E-Mail Interviewing is conducted by presenting seven mathematical tasks from national tests in Sweden and China, respectively. The preliminary results show that Swedish and Chinese teachers hold some common views on the characteristics of good mathematical test tasks, but they also show different views on some mathematical tasks with high level of abstraction. Implications of the results and the methodology informed by the pilot study are discussed.

Keywords: Abstraction, assessment, mathematical task, pilot study, teacher belief.

INTRODUCTION

This study is motivated by the observations of a significant gap between tasks in teacher-made tests and national tests. Palm, Boesen, and Lithner (2011) investigated the mathematical reasoning required to solve the tasks in the Swedish national tests and a random selection of Swedish teacher-made tests, and found that most of the tasks in the teacher-made tests did not require the students to produce new reasoning and considering the intrinsic mathematical properties involved in the tasks, findings that were contrasted with the national tests that included a large proportion of tasks for which memorization of facts and procedures were not sufficient. Senk, Beckman, and Thompson (1997) examined the kinds of assessment tasks used in 19 mathematics classrooms in the United States, and found that, in general, teachers selected low-level abstract tasks that did not reflect the aims of reform curricula. Tasks in teacher-made tests generally were low-level, were stated without reference to a realistic context, involved very little reasoning, and were almost never open-ended (ibid). Chen (2013) investigated a group of Chinese teachers' views of the tests made by them, and found that the tasks in the teacher-made tests were mostly adapted from previous tasks in different levels of tests, and a small portion of them were simplified from tasks in mathematical competitions, in general, tasks in Chinese teacher-made tests were predictable and therefore did not require students to produce new reasoning. Researchers state the urgency to investigate the potential different factors that result in the gap between teacher-made tests with mostly routine tasks and national tests with a larger proportion of creative tasks, so as to gain further insights into necessary and sufficient conditions under which teacher-made tests can be improved (Palm, Boesen, & Lithner, 2011). Our study presented here aims to make such a contribution.

Assessment has become a central issue in the discussion of ways to improve mathematics education. Teacher-made tests are an important constituent of assessment. Mathematical test tasks are central to students' learning because what students learn is largely assessed by the tasks they are given. The characteristics of mathematical test tasks in teacher-made tests are influenced by different factors including teachers' knowledge and beliefs, as well as the content and textbook of the course (Senk, Beckman, & Thompson, 1997), but the mathematics teachers' beliefs are so instrumental in shaping their final decisions concerning what tasks to include in a test (Nathan & Koedinger, 2010). Accordingly, teachers' beliefs on what constitutes a good mathematical test task to a great extent determine the characteristics of mathematical tasks in teacher-made tests. In this study, we investigate this issue by comparing Swedish and Chinese teachers' views. We focus on Swedish and Chinese teachers because Sweden and China run totally different educational systems and the assessment of mathematics education in both countries are differently practiced by that, one has a long traditionally high-stakes assessment culture whereas the other has not. This leads us to consider whether teachers' views on mathematical test tasks may be different or common across the two educational systems. We specifically address the following questions: What constitutes a good mathematical test task from a teacher perspective? What are the commonalities and differences between Swedish and Chinese teachers' views on the characteristics of a good mathematical test task?

Our current study is a pilot study, which forms the foundation of a larger study that aims to develop a framework for examining good mathematical test tasks. A pilot study is defined by Wiersma and Jurs (2008) as:

> A study conducted prior to the major research study that in some way is a small-scale model of the major study: conducted for the purpose of gaining additional information by which the major study can be improved – for example, an exploratory use of the measurement instrument with a small group for the purpose of refining the instrument. (p. 427)

It is widely accepted that pilot studies precede the main study and form an important component of the research design. Therefore, we have a dual purpose of the current study, one is to address the research questions stated above, and the other is to examine the methodology of the study.

LITERATURE REVIEW

In this section we give a brief literature review on the nature of mathematical tasks and the attributes of a good mathematical test task, which helps to identify the potential contribution of the current study.

The nature of mathematical tasks

A mathematical task is defined as a set of problems or a single complex problem that focuses students' attention on a particular mathematical idea (Stein, Grover, & Henningsen, 1996). According to the different purposes, researchers classified mathematical tasks into different types. For instance, Yeo (2007) classified them into mathematically-rich tasks, such as analytical tasks and synthesis tasks that can provide students with opportunities to learn new mathematics and to develop mathematical processes such as problem solving strategies, analytical thinking, metacognition and creativity; and non-mathematically-rich tasks, such as procedural tasks that can only provide students with practice of procedures. Sullivan, Clarke and Clarke (2013) classified three types of tasks, namely, purposeful representational tasks, contextualised tasks and content-specific open-ended tasks, and highlight the importance of providing a wide range of types of tasks in a variety of different sequences:

> ... allow students opportunity to have a sense of control by allowing them to make decisions, are interesting to the students, incorporate a rationale for them to engage, provide some challenge, reduce the risk of failure, and for which success provides the motivation for further engagement. (p. 10)

In reference to Sullivan, Clarke and Clarke's work, Foster (2013) claimed that the most powerful task type presented was the content-specific open-ended one, which accommodates tasks that are "accessible by students, able to be used readily by teachers, foster a range of mathematical actions, and contribute to some of the important goals of learning mathematics" because "such rich tasks would seem to offer students deep opportunities for learning mathematics". As we can see, the different types of tasks have explicitly or implicitly implied some features of a good mathematical task. However, a good mathematical task might be not necessarily a good mathematical test task. Below we address what makes a good mathematical test task shown in the literature.

Attributes of a good mathematical test task

Our literature survey shows that there are only a few studies that focus on the attributes of a good mathematical test task. These studies can be classified two types: one is from a researcher perspective and the other from a participant perspective. For the former, Kin (2010) suggested three characteristics that a mathematical test task should have, firstly, the correctness and rigour (mathematically sound, correct, and accurate); secondly, must be set within the scope of the stipulated syllabus; thirdly, possess a respectable amount of beauty (in the form of neat formula, symmetry of the situation, beautiful link with a few topics, or ingenuity of ideas). For the latter, Kontorovich (2011) argued that in the case of competition problems there is a "built-in kernel" which has the potential to be appreciated by the intended solvers. And therefore he investigated 22 adult participants' perception of an interesting mathematical problem in mathematical competition and identified four characteristics of those interesting problems, which are as follows: consisting of meaningful mathematics beyond the problem, the misleading problem situation image, A-typicality and novelty of problem's formulation, and the dilemma on the problem's "wrapper". Building on the previous studies, our study aims to contribute to better understand the attributes of a good mathematical test task from a teacher perspective.

METHODOLOGY

Participants

The participants consisted of 9 Swedish and 9 Chinese teachers, from two Swedish schools and two Chinese schools and teaching mathematics in grades 7–9. Most of them hold a Bachelor's degree in Mathematics Education. They participated in the study through our contact of a liaison person who was one of their colleagues. Their participations were voluntary. They were informed about the purpose of the study, that all information they provided would be kept confidential, and that only the collective results would be shared.

Material and its translation

To avoid our study resulting in general claims by just asking the teachers "what constitutes a good mathematical test task", we adopted a task-based interview that attempted to elicit specific and detailed responses about their views on characteristics of good mathematical test tasks. Mathematical tasks, selected from national tests in Sweden and China, served as a foundation of the current study. Based on researchers' views on good mathematical tasks in national tests, and having considered both the coverage of different mathematics topics and the representativeness (or typicality) of tasks in each country, fifty tasks were firstly identified. Next, to ensure tasks from each country could be adapted to the other country, we discussed further with mathematics educators from both countries about each task and finalized seven mathematical tasks from each country.

In a cross-national study, the equivalence of the two language versions of the instruments is an important issue (Andrews & Diego-Mantecón, 2014). In our study, the mathematical tasks presented to teachers were the main research instruments. In order to produce equivalent information in the different language versions of the tasks, both the Swedish and Chinese tasks were first translated sentence by sentence from their original language to English. Only the names of persons in the tasks were changed to those, which were familiar to teachers in their culture. An English version of the fourteen mathematical tasks was formed and tested by oral interview with two Swedish colleagues and a Chinese colleague. In the end, the Chinese teachers were presented with the original mathematical tasks from China and a Chinese version of the mathematical tasks from Sweden. The Swedish teachers were presented with English versions of the mathematical tasks from both China and Sweden. During this process, the first author who is proficient in Chinese, English, and basic Swedish, checked whether the two versions of the instruments were equivalent.

Data collection

As a pilot study, we choose E-Mail Interviewing as a main method to collect data. We chose this method as it cost considerably less to administer than telephone or face-to face interviews, and it also allowed us to invite participation of geographically dispersed samples from both Sweden and China (Meho, 2006). Furthermore, the participants also showed their preference to use E-Mail Interviewing because it saved time for them compared with arranging face-to face interviews, and also allowed them to allocate enough time to read and think about the mathematical tasks as an important precondition to answer the interview questions in this study.

Through the contact of the liaison persons, every teacher in his/her school got a copy of the task-based interview questionnaire consisting of the fourteen mathematical tasks in the order of tasks from their own country and followed by those from the other country. In the beginning of the questionnaire, there were instructions about the interview where they were specifically informed to consider the tasks from the perspective of assessment, and after every mathematical task, there were specific interview questions "do you think this is a good mathematical test task? If so, clarify the reason; if not, clarify the reason". After having finished the interview questions, the liaison persons collected all the questionnaires and sent back to us by emails.

Data analysis

Our data coding was inspired by grounded theory (Corbin & Strauss, 2008) to understand the differences and commonalities of the teachers' views on good mathematical test tasks. The data was coded in the original languages, Swedish and Chinese, respectively. The coded data was translated into English. Finally, the data was scrutinized by two steps: 1) identify the good mathematical test tasks recognized by teachers, and 2) identify recurring themes on how the teachers viewed the selected mathematical tasks. Disagreements were resolved through discussion between the researchers.

RESULTS AND DISCUSSION

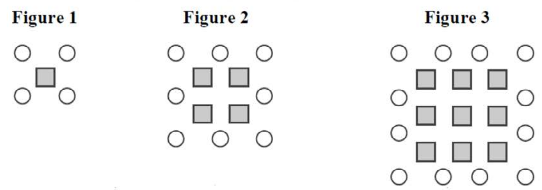
Good mathematical test tasks as identified by Swedish and Chinese teachers

The results show that teachers differ in their views on the 14 mathematical tasks. For some tasks, more teachers regard them as good mathematical test tasks, but for some tasks, only a few regard them as good mathematical tasks. The task that was identified by the highest number of teachers as a good mathematical test task was one of the Swedish tasks (S6 – Figure 1). This task is about looking for patterns, inductive reasoning and generalisation, which is regarded as "A good mathematics test task because it provides a figure with a clear visual representation, with subtasks ranging from simple to complex, and it also has strong regularities and suitable level of difficulty (CT2)."

The task that the lowest number of teachers identified as a good mathematical test task is one of the Chinese tasks (C7). C7 is a task with a similar knowledge background as S6, but it is regarded as "too difficulty for students".

C7: As shown by the figures (Figure 2), in the rectangular paper ABCD, $AB = \sqrt{6}$, $BC = \sqrt{10}$. Firstly, fold the paper to overlap point B and point D, where O_1 is the intersection of the crease with BD and denote by D_1 the middle point of O_1D. Secondly, fold the paper to overlap points B and D_1, where O_2 is the intersection of the crease with BD and denote by D_2 the middle point of O_2D. Thirdly, fold the paper to overlap points B and D_2, where O_3 is the intersection of the crease with BD, and so on. Assume the intersection of n^{th} crease with BD is O_n. Then BO_1 = ___, BO_n = ___.

S6: In a fruit plantation there are some mango trees (■) surrounded by orange trees (○) as shown in the figures.



a) How many mango trees and how many orange trees would there be in figure 5?
b) How many mango trees and how many orange trees are there in figure n?
c) In figure 2 there are twice as many orange trees as mango trees. Investigate in what figure there are twice as many mango trees as orange trees.

Figure 1

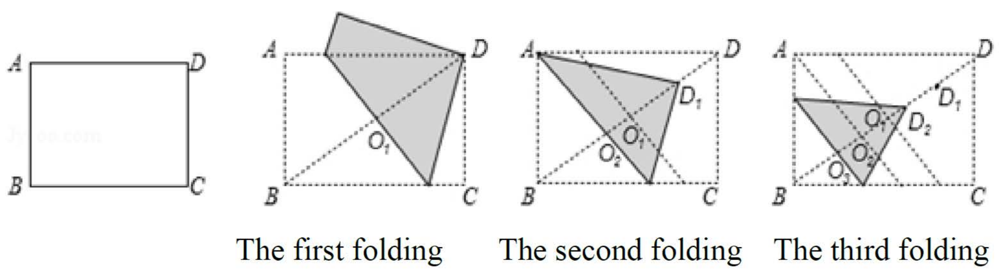


Figure 2

In generally, the Swedish tasks were more appreciated by the teachers than the tasks from China. Table 1 shows the number of teachers from each country (#CT = number of Chinese teachers, #ST = number of Swedish teachers) that considered the specific task (C1-C7: Chinese tasks, S1-S7: Swedish tasks) as a good mathematical test task.

Commonalities of Swedish and Chinese teachers' views on characteristics of good mathematical test tasks

Two recurring themes regarding the characteristics of a good mathematical test tasks commonly held by both Swedish and Chinese teachers emerge in our analysis. These are having connections with real life, and being accessible to average students.

Our results show that both Swedish and Chinese teachers highly recognized real life context as a characteristic of a good mathematical task. With an increasing emphasis being placed on the applications of mathematics in real-life situations in school mathematics curriculum in most countries for various motivations, for instance, real life tasks are often familiar enough and imaginable to students and can therefore serve as a stepping-stone for thinking about important mathematical concepts (Yeo, 2007), as the enactors of curriculum, it is understandable that teachers also commonly recognize the value of tasks has collections with real life. Below are quotations from both Swedish and Chinese teachers:

ST3: The pupils can relate to it when they go for a trip.

CT4: It is a good task because it has close connection with everyday life application, which reflects the idea that mathematics comes from real life and has the function of serving for real life.

A basic function of test is to assess students' learning. Therefore, the accessibility to average students is an important indicator for mathematical test tasks, which is commonly shared by Swedish and Chinese teachers. Below are some selected comments from the teachers.

ST6: They (the Swedish tasks) all have a reality in which students can, without having lots of theoretical knowledge of mathematics, see a "picture" of the task and solve the task and think about the answer reasonably.

CT8: The task is not very hard, can be solved by average students.

Differences of Swedish and Chinese teachers' views on characteristics of good mathematical test tasks

Two recurring themes regarding the differences of Swedish and Chinese teachers' views on characteristics of good mathematical test tasks emerge in our analysis, which is, students' mathematical competency is emphasised by the Chinese teachers but is not visible in the Swedish teachers' responses, and that Swedish and Chinese teachers have different views on some mathematical tasks with high level of abstraction.

Different from the Swedish teachers, when arguing the characteristic of good mathematical tasks, the Chinese teachers always refer to students' mathematical competency that is expected to have in order to solve the tasks successfully. Niss (2003) defines that mathematical competency means the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. And he categorizes eight different types of mathematical competencies. In our study, the Chinese teachers continuously mention different mathematical competencies such as computational competency, imaginative competency, observational competency, inductive competency, etc., which is not visible in the Swedish

Task	C1	C2	C3	C4	C5	C6	C7	S1	S2	S3	S4	S5	S6	S7
#CT	6	5	5	5	8	2	1	7	6	4	6	5	7	5
#ST	4	5	1	4	0	1	1	6	4	5	5	5	7	7
Total	10	10	6	9	8	3	2	13	10	9	11	10	14	12

Table 1: Number of teachers considering the task as a good mathematical test task

teachers' responses. We interpret this difference by referring to mathematics education in China, where there is a long heavy tradition on emphasizing the two basics, basic knowledge and basics skills, which forms the foundation for any further study in mathematics and students' mathematical competency development. And this tradition is deeply rooted in Chinese teachers' beliefs in their mathematics teaching practice, therefore it. Below are the Chinese teachers' responses on their perspectives on the Swedish and Chinese tasks by phrasing some mathematical competencies.

CT6: The Chinese test tasks aim for cultivating students' problem solving competency, which is a little bit far from reality. And most of the Swedish tasks relate to every life, which could have the realistic significance for students' development.

CT2: The Swedish tasks are simpler, but they focus on the application of mathematics, which is close to real life. And the Chinese tasks examine students' synthesis competency.

For some mathematical tasks with high level of abstraction, the study shows that Swedish and Chinese teachers have different views. As seen from Table 1, there are gaps between Swedish and Chinese teachers' views on two tasks (C3 and C5), which Swedish teachers do not regard as good mathematical tasks, but Chinese teachers regard as good ones. Below are some comments from the teachers.

ST5: The Chinese task has a much higher level of abstraction and students need to have much more theoretical mathematics.

CT6: This task reflects the category idea; such tasks are very helpful to cultivate students to form a mind of thinking comprehensively.

As Kin (2010) suggests that a good item must be set within the scope of the stipulated syllabus, Swedish and Chinese teachers differ in their views on mathematical tasks with high level of abstraction could be explained from this perspective. Mathematics curriculum standards (or syllabus) in Sweden and China are different with different requirements for students, the mathematical knowledge and competency covered in those mathematical tasks with high level of abstraction might be covered in curriculum standards in China but not in Sweden, therefore, Chinese teachers value them but Swedish teachers do not. Another interpretation for this difference might be connected to the different cultural profiles in mathematics education. Grønmo (2013) stated that the Nordic and English-speaking countries has the profile of applied mathematics that can be referred to as mathematics applied to solving problems in everyday life, whereas the East Asian and East European countries has the profile of pure mathematics that is an abstract world with well-defined symbols and rules. Obviously, mathematics education in Sweden has the applied mathematics profile and whereas in China it labelled as pure mathematics, therefore, mathematical tasks with high level of abstraction are appreciated by Chinese teachers but not Swedish teachers.

CONCLUSIONS

Our study has contributed to our better understanding on characteristics of good mathematical test tasks from a teacher perspective. To be a good mathematical task in tests, both the features of the tasks and the expected difficulties of the tasks for students need to be considered. Through the comparison between the Swedish and Chinese teachers, we uncover some common beliefs they hold and their different views on characteristics of good mathematical test tasks, which provide helpful insights for our reflections on mathematics education in Sweden and China.

Furthermore, the pilot study presented in this paper illustrates that there are several potential flaws in the instruments and in the procedures. Firstly, we would like to address the task selection. Surprisingly, although the researchers consider the tasks used in the interviews as good mathematical test tasks, not all of them are recognized as good ones by the teachers. Therefore, a pre-interview with schoolteachers might be helpful to ascertain whether the tasks are good. Secondly, we address the task organization. In the interview, tasks are labelled by their sources either from Sweden or China, which may produce a priori prejudice for teachers when making their judgements, as we see from teachers' comments on their view on differences of mathematical tasks in Sweden and China. This could be avoided if we remove the information on the exact sources of the mathematical tasks. Thirdly, the E-Mail interviewing doesn't allow for further prompting and clarification of responses, for such a reason we lose the opportunity to achieve deeper understanding of teachers' valuable insights. In all, the feedback obtained from this pilot study enables us to make revisions aimed at overcoming these identified weaknesses and improve on the design for our main study. Additionally, data analyses suggest that the research findings from the main study would be fruitful and likely to make valuable contributions to the knowledge in both characteristics of good mathematical test tasks and teachers' different perspectives across educational system.

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Examining the heart of the dual modelling cycle: Japanese and Australian students advance this approach

Akihiko Saeki[1], Akio Matszaki[2], Takashi Kawakami[3] and Janeen Lamb[4]

1 Naruto University of Education, Naruto, Japan, asaeki@naruto-u.ac.jp
2 Saitama university, Saitama, Japan, makio@mail.saitama-u.ac.jp
3 Nishikyushu University, Kanzaki, Japan, kawakamita@nishikyu-u.ac.jp
4 Australian Catholic University, Brisbane, Australia, janeen.lamb@acu.edu.au

The aim of this study is to compare how Japanese and Australian teachers utilise opportunities to promote students' switching between mathematical modelling cycles based on the dual modelling cycle framework (DMCF). This study found that teachers need to change how they assist students when transitioning from one modelling cycle to another not only based on differing levels of student ability, but to account for differences between countries as well. The Japanese students had more sophisticated visualisation skills than the Australian students when working with the geometric structure of an ordinary helix on the side face of a cylinder. However, both groups of students benefited from use of the DMCF to develop their understanding of the mathematical problem as they moved between modelling cycles.

Keywords: International comparisons, dual modelling cycle framework (DMCF), ordinary helix, modelling teaching, mathematics education.

INTRODUCTION

Problem solving in the traditional mathematical classroom has tended to be an individual task. Students work in isolation as they go about problem solving. When teaching problem solving skills, teachers have relied on the work of Polya (1945). His approach included the hermeneutic of solving a similar, simpler problem which provided each student with an approach that they could use to find a solution for their problem. In more recent years there has been a realisation that to effectively function in society students need to develop the skills of being more flexible and creative problem solvers. Mathematical modelling provides an opportunity to develop these skills, as it is designed for group work that promotes collaborative interactions. This approach is usually set in the context of real world problems where information is incomplete or ambiguous, promoting questioning and the posing of conjectures (Brown & Walter, 2005). As a result it allows for multiple solution paths, permitting discussions around the *best* solution rather than *the* solution. In these situations research indicates that modellers' attempts to find a solution usually results in their shuttling between the real and mathematical worlds (e.g., Stillman, 1996; Stillman & Galbraith, 1998; Borromeo Ferri, 2007; Matsuzaki, 2007, 2011). According to Busse and Kaiser (2003), modellers construct their own *subjective figurative context* from the modelling task, and the modellers' perception of the task context can affect their modelling progress. When the modellers' modelling processes have stalled, evidence suggests that students move from their initial modelling task to a similar and simpler modelling task where some traction is considered possible. In this paper, we explore a theoretical extension to this approach to mathematical modelling, as limited research exists on how to facilitate the teaching of mathematical modelling when responding to a diversity of modeller abilities.

Saeki and Matsuzaki (2013) proposed a new theoretical modelling framework called the *dual modelling cycle framework (DMCF)* (see Figure 1). This DMCF re-conceptualises the modelling cycle explicated by Blum and Leiß (2007). In the case of solving an initial task located on the first modelling cycle, one modelling cycle is enough if modelling is proceeding successfully. If problem solving is unsuccessful or the

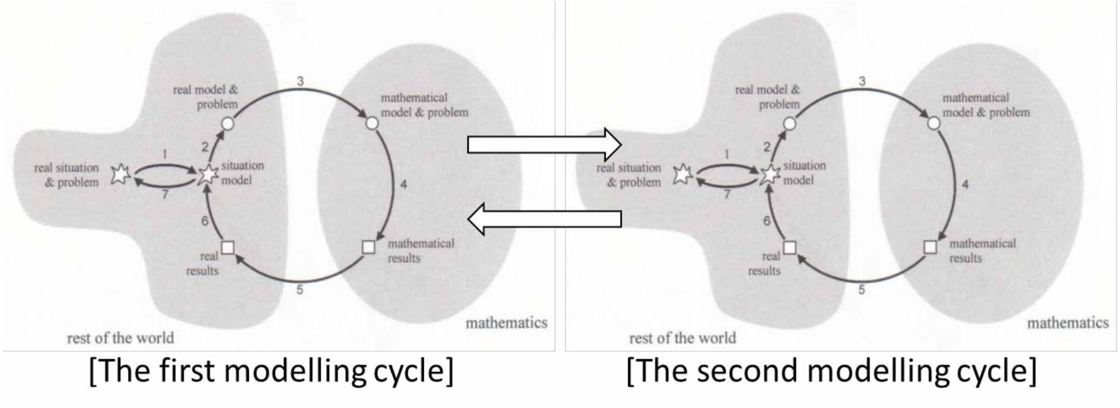


Figure 1: Dual modelling cycle diagram (Saeki & Matsuzaki, 2013, p. 94)

modeller does not know enough to solve the task, the modeller can be assisted by trying to solve a similar task as was proposed by Polya (1945) in his earlier work. One rationale for using two separate modelling cycles when changing from an initial modelling task (TASK1) to a similar and simpler modelling task (TASK2) is that there may be cases when doing so leads to more success.

Research by Saeki and Matsuzaki (2013) has identified that to support successful outcomes for all modellers using the DMCF, the most important point is for teachers to support switching between the first modelling cycle and the second modelling cycle by providing a similar and simpler task. Matsuzaki and Saeki (2013) implemented experimental modelling lessons for undergraduate students in Japan and identified three stages: transition from the first modelling cycle to the second modelling cycle, modelling within the second modelling cycle, and transition from the second modelling cycle back to the first modelling cycle. Kawakami, Saeki and Matsuzaki (2012, 2014) implemented DMCF-based modelling lessons with Year 5 elementary school students in Japan and classified six types of students' responses. They also described modelling lessons in terms of a first trial of two tasks, one in each modelling cycle; a second trial of TASK1 based on TASK2, and a final trial of TASK1 through classroom discussion. DMCF-based modelling lessons were also implemented with Year 6 students in Australia (Lamb, Kawakami, Saeki, & Matsuzaki, 2014), permitting international comparisons. The aim of this paper is to compare how teachers can assist students in switching between modelling cycles while supporting a diverse range of student capabilities within two different countries.

CHARACTERISTICS OF MATHEMATICS LESSONS IN JAPAN AND AUSTRALIA

In this paper, we explain the differences in mathematics teaching in Japan and Australia in order to make international comparisons between the two modelling lessons. Some of these differences are based on work by Mok and Kaur (2006), where characteristics of mathematics lessons are explained with a focus on the '*learning task*'.

The teaching strategy used by Japanese teachers is one that supports each student's level of ability. Teachers lead lessons by considering the needs of each student and providing a variety of activities to suit. Furthermore, many Japanese teachers have adopted problem-discovery oriented teaching methods based on Yamamoto's (2007) work, which outlines three stages when detailing such methods: (1) initial learning activities, (2) discovery of a problem that must be solved, and (3) solution of the problem. A characteristic of this method is to emphasize the children's change of awareness. Consequently this style of lesson can be challenging for the teacher. Thus this view of mathematics teaching matches the modelling teaching practice described above (Kawakami et al., 2012, 2014).

On the other hand, the teaching strategy adopted by Australian teachers relies on mathematical tasks based in daily-life contexts where students make links to their daily life activities (Mok & Kaur, 2006). Supporting this Australian teaching strategy, Stillman, Brown, Faragher, Geiger and Galbraith (2013) analyzed the goal of mathematics by analyzing textbooks and curricula in secondary classrooms in Australia from a socio-cultural perspective. This led to three findings: (1) textbooks were used as a foundation for teaching materials, (2) teaching materials were based

Figure 2: Oil tank image URL<http://blog.goo.ne.jp/kobeooi/e/b021c971381154725 fc3ee4a3d645aa8> [18 Mar 2014] (Note: Picture reversed in class)

in contexts that enhanced students' understanding of the world, and (3) assisted the development of a critical disposition towards the surrounding world that requires decisions to be made. Thus teaching tasks emphasized daily-life contexts that evoked a need for decision making. With this in mind we were conscious of the need for context based problem-solving in Australian schools and we found data for this perspective (Lamb et al., 2014).

EXAMINING THE HEART OF THE DUAL MODELLING CYCLE FRAMEWORK

We developed DMCF-based teaching material for elementary school students to assist them in understanding the geometric structure of an ordinary helix on a side face of a cylinder. The students were initially provided with a picture of oil tanks with differing diameters (see Figure 2). The students were then provided with an *Oil tank task* (TASK1) and a *Toilet paper tube task* (TASK2), displayed in Figure 3.

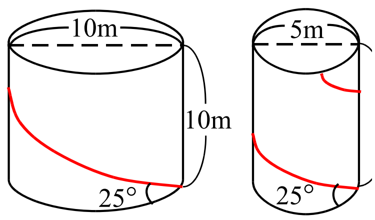
In our earlier research using the same task as above, we found that modellers who could not solve TASK1 were able to advance their modelling of this task by modelling a similar but simpler task, TASK2 (Kawakami et al., 2012; Lamb et al., 2014). Students who could solve TASK1 but were encouraged to engage with TASK2 developed a more advanced understanding of TASK1. By actively switching between TASK1 and TASK2, most students were able to solve TASK1 (see Kawakami et al., 2012; Lamb et al., 2014 for details). These tasks helped students understand the geometric structure of an ordinary helix on the side face of a cylinder. This structure is important because it forms the foundational knowledge necessary to solve the oil tank task in higher grades (using either the Pythagorean theorem or trigonometric ratio). Student investigation of the 3D model leads to understanding the rectangle model and the parallelogram model (see Figure 4).

The DMCF aims to deepen students' mathematical understanding by switching between two modelling cycles, as indicated in Figure 5. There are two kinds of switching that lead to in-depth engagement in the tasks.

The first is a teacher's intentional switching to facilitate student understanding. It is therefore very important for teachers to design an approach to switching through the use of teaching material before implementing the lesson. Teacher's intentional switching is done twice. The first instance of switching is the transition from the first modelling cycle to the second modelling cycle. In this transition, the teacher used the toilet paper tube to present an opportunity for students to work with a concrete object. The second instance of switching uses feedback from the second modelling cycle to return to the first modelling cycle. In this transition, it is necessary for all students to recognize that they have returned to TASK1. Therefore

***Oil tank task* (TASK1)**

There are several types of oil tanks. Their heights are equal but their lengths of diameters are different. Is the length of the spiral stairs on these oil tanks equal or not? As conditions, angles to go up spiral banisters are all the same.



***Toilet paper tube task* (TASK2)**

It is impossible to open along the actual spiral stair of the oil tank. We can use a toilet paper as a similar shape to an oil tank as it can be opened along its slit to show the 2D shape. Con

Figure 3: Teaching material based on DMCF

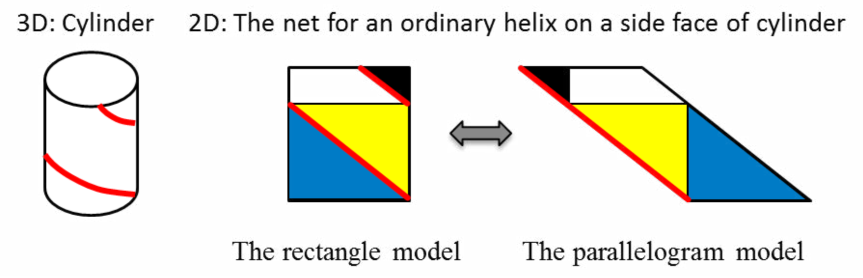


Figure 4: The geometric structure of an ordinary helix on the side face of a cylinder

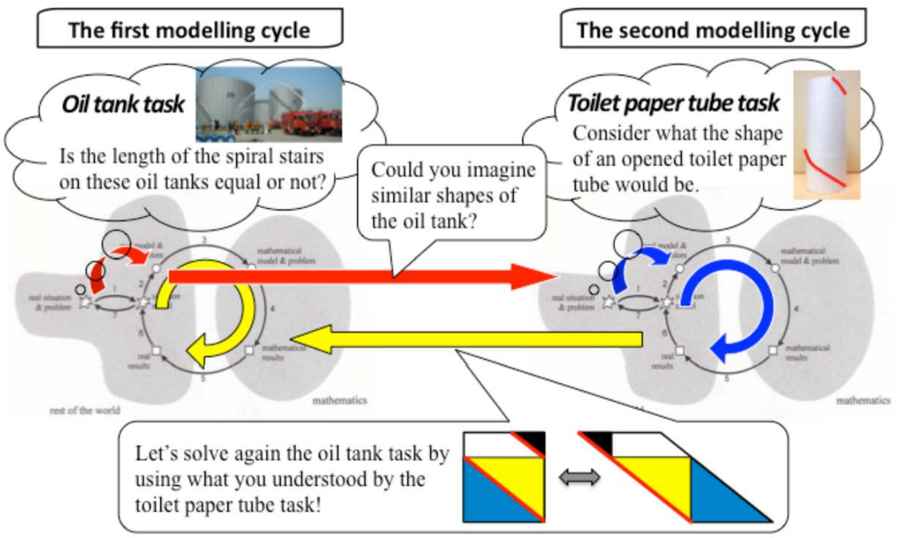


Figure 5: Dual modelling cycle diagram of switching between two modelling cycles

we offered the information of TASK1 to students again and let them predict whether the spiral stairs would be the same or not. It is necessary for the teacher to prepare methods of switching that accommodate differing levels of student ability. In the case of the Australian school, significant student difficulties necessitated a substantial change in the switching methods used (see description on pages 7 and 8).

The second is students' intentional switching to solve TASK1. This switching is important as it provides an opportunity for students to develop ideas by themselves. Hence, teachers have to prepare a range of alternative approaches to stimulate the transition between modelling cycles that correspond to differing student needs. It is important to note that one of the problems in this process is that some students lose track of which modelling cycle they are in. When this is the case it is necessary for teachers to guide students in understanding their position in the modelling sequence and the correct direction they need to take to move to TASK2 or back to TASK1.

THE MODELLING LESSONS IN JAPAN AND AUSTRALIA

Case of Japan

The Japanese experimental class consisted of three 45-minute lessons (see Kawakami et al., 2012). The class included 33 Year 5 students (aged 10 or 11) from a Japanese private elementary school.

Showing the Oil tank task

At the beginning of the lessons, the teacher showed photographs of two oil tanks and asked the students if the length of the spiral stair was equal or not (see the oil tank photograph, Figure 2). In order to simplify the *Oil tank task* (Figure 3), the teacher asked which part of the spiral stair should be measured, its banister or its steps. Through discussion with the students they agreed to measure the length of banister at the side of the oil tank. Then the teacher showed 3D models of the oil tanks displayed in Figure 3 and asked students what they could do to solve the *Oil tank task*. The students responded by producing 2D drawings.

Seventeen students (52%) were able to draw the mathematically correct 2D rectangle models of the oil tanks from the 3D models, representing the spiral staircase as straight lines on their models (see Figure 6). The remaining students were not able to draw the mathematically correct models, as the representation of the staircase was not connected on their models (see Figure 7). Some students rounded their paper to check whether the staircase would be connected or not. However they did not make the full link to the mathematical structure of the spiral shown in Figure 4.

Teacher's intentional switching (1):

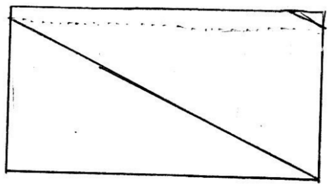


Figure 6: Mathematically correct models

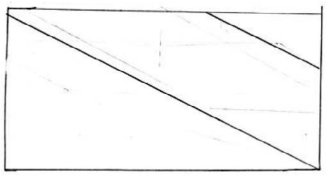


Figure 7: Mathematically incorrect models

Guiding to the Toilet paper tube task:
At the beginning of the second lesson, in order to switch from the first cycle of the *Oil tank task* to the second cycle of the *Toilet paper tube task*, the teacher asked the students to consider what objects were similar in shape to the oil tank, but smaller in size. The student responses included pencils, toilet paper tubes and so on. The teacher provided an actual toilet paper tube for each student and asked them to open the toilet paper tube. The students were asked to compare the length of each staircase and to produce a 2D drawing of the toilet paper tube (see the *Toilet paper tube task*, Figure 3). Almost all the students identified cutting the tube along the slit and were able to subsequently draw the parallelogram model.

Teacher's intentional switching (2):
Returning to the Oil tank task:
At the beginning of the final lesson, in order to switch from the second cycle of the *Toilet paper tube task* back to the first cycle of the *Oil tank task*, the teacher asked the students to compare the length of the staircase with reference to findings from the *Toilet paper tube task*. The students tried to solve the *Oil tank task* independently. Fifteen students made the parallelogram model of the oil tank and tried to calculate the length of the staircase.

During the final lesson, the teacher observed four different methods used to find solutions: (1) calculating the length of spiral stair (five students), (2) measuring the length of spiral stair in rectangle models and parallelogram models (four students), (3) translating the staircase in the rectangle models (three students) and (4) translating the staircase in the parallelogram models (six students). As a result of class presentations, students gained access to classmates' solutions and were able to solve the *Oil tank task* using parallelogram models or rectangle models of the oil tank.

Case of Australia
The Australian experimental class consisted of two 60 minute lessons (see Lamb et al., 2014). The class included 23 Year 6 students (aged 11 or 12) from an elementary school.

Showing the Oil tank task
At the beginning of the lessons, the teacher showed photographs of two oil tanks and framed the problem within the context of a fireman needing to climb to the top of one of the tanks as quickly as possible to extinguish a fire. The teacher then asked the students if the length of spiral stair was equal or not. The teacher showed 3D models of the oil tanks displayed in Figure 2 and asked the students to produce 2D drawings of the 3D models.

No student was able to draw the mathematically correct models of the oil tanks from the 3D models of the oil tanks. Eleven students (48%) drew the rectangular representation of the oil tank. However, in each case the students drew the staircase as a curved line on their 2D model. The remaining students (52%) were unable to draw a 2D model, tending instead to copy the 3D model provided for them.

Teacher's intentional switching (1):
Guiding to the Toilet paper tube task
In order to switch from the first cycle of the *Oil tank task* to the second cycle of the *Toilet paper tube task*, the teacher showed the students a toilet paper tube and asked them to predict what the toilet paper tube would look like when cut along the slit. No student was able to draw a mathematically correct 2D model. Six students (26%) drew a shape close to a parallelogram in which the spiral stair was curved. Other students pro-

duced shapes similar to a roll. To assist the students in finding the relationship between the *Oil tank task* and the *Toilet paper tube task*, the teacher asked them to cut the toilet tube vertically and confirm that the shapes created were parallelograms and rectangles.

As most students in the class were not able to first visualize and then draw 2D models from the 3D models of the oil tank or the toilet paper tube, at the beginning of the second lesson, the teacher used a concrete aid to demonstrate how the 2D models related to the 3D model. A rectangular piece of cardboard, rolled into a cylinder and marked with a red line, was cut at an angle and unrolled to illustrate the relationship between the 3D model and the 2D parallelogram model. Furthermore, the teacher cut a similar cylinder vertically and unrolled it to show the 2D rectangular model.

Teacher's intentional switching (2): Returning to the Oil tank task

In order to switch from the second cycle of the *Toilet paper tube task* back to the first cycle of the *Oil tank task*, the teacher again asked the students whether the spiral stair was the same or different for each oil tank. The students tried to solve the *Oil tank task* collaboratively. One group made another 3D model of the oil tank by using concrete 3D models. They opened the model and measured the length of spiral stair in the 2D rectangle model and the 2D parallelogram model. A student in the group explained, "I think they are all the same because the parallelogram and rectangle are almost the same size, so I expect they are the same".

During the last lesson the students were able to explain that both staircases were the same length for both the rectangle model and the parallelogram model through cutting and placing the pieces of their concrete 2D models on the whiteboard. The teacher demonstrated both models represented the same 3D model (Figures 8 & 9).

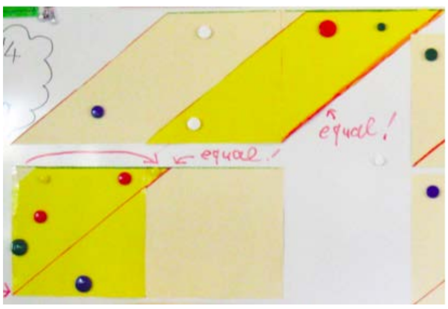


Figure 8: Solutions by both models

DISCUSSION

Using the *dual mathematical modelling cycle framework* (DMCF) to examine the same problem in both Japanese and Australian classes allows for comparisons to be made. The results from this study indicate that the teachers needed to change the method used to switch between modelling cycles intentionally to account for different levels of student ability. The Japanese elementary students in this study had more sophisticated visualisation skills and were able to move between 2D and 3D models of an ordinary helix on the side face of a cylinder as well as visualise the shape of the staircase in a 2D model. This allowed them to calculate the length of the spiral stairs and to compare the rectangle and parallelogram models, facilitating their understanding of the problem. As the Australian students had more difficulty with the problem, the teacher changed two of the switching methods. The teacher asked students to cut up the toilet tubes to confirm that parallelogram and rectangle models were equivalent, and demonstrated how concrete 3D models were related to 2D parallelogram and rectangle models. The changed method for the modelling lesson still depended on the students' understanding and promoted class discussion. It also remained grounded in the context, with a focus on the need to find the fastest route to the top of the oil tanks.

Use of the DMCF and its emphasis on switching between modelling cycles benefited both Japanese and Australian students by deepening their understanding as they moved between 2D and 3D models and the two cycles. The approach encouraged all students to participate at their ability level and to gain access to more sophisticated modelling approaches during whole class discussions.

Our future work will be to compare Japanese and Australian students' international switching by analysing the students' protocols, activities, and worksheets.

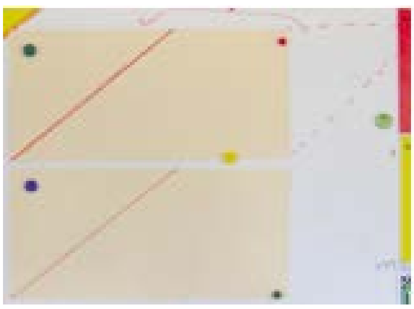


Figure 9: Relationship between both models

ACKNOWLEDGEMENT

This work was supported by JSPS KAKENHI Grant Number 25350204.

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Solving a problem by students with different mathematical abilities: A comparative study using eye-tracking

Miroslawa Sajka and Roman Rosiek

Pedagogical University of Cracow, Faculty of Mathematics, Physics and Technical Science, Cracow, Poland, msajka@up.krakow.pl, rosiek@up.krakow.pl

The main purpose of this study is to compare the problem solving processes of mathematically gifted and underperforming students by utilizing eye-tracking methodologies. We have found the following differentiators between the groups: (a) time of the analysis of the problem's wording, (b) the number and placement of fixations, (c) the number of fixations while analysing the text of the problem. We also prove that total amount of time of solving a problem is not an important differentiating parameter; speed is not a characteristic of mathematically gifted students.

Keywords: Mathematics education, problem solving, eye-tracking, gifted students, comparative study.

INTRODUCTION

The use of eye-tracking technology for the analysis of the learning process has become more and more widespread in recent years. Examining visual attention provides information not only on where the gaze is directed and how, but also constitutes a basis for further analysis and reflections on the ways of solving problems, reasoning, attention, and mental images (Just & Carpenter, 1976; Zelinsky & Sheinberg, 1995; Ball, Lucas, Miles, & Gale, 2003; Yoon & Narayanan, 2004).

Lai and colleagues (2013) reviewed 81 papers dedicated to the use of eye-tracking technology in research related to the analysis of the learning process, describing 113 studies carried out in the period of 2000–2012. The authors distinguished the following themes of studying eye movements and learning: patterns of information processing, effects of instructional design, reexamination of existing theories, individual differences, effects of learning strategies, patterns of decision making, and conceptual development. This research refers to the mainstream of examining patterns of information processing, and strategies, and individual differences during the process of solving mathematical problems.

In the field of didactics of mathematics, studies are still being undertaken (e.g., Andra et al., 2009; Chesney, McNeil, Brockmole, & Kelley, 2013; Merkley & Ansari, 2010; Schneider, Maruyama, Dehaene, & Sigman, 2012; Susac, Bubic, Kaponja, Planinic, & Palmovic, 2014).

Some research results indicate that the measurement of eye movements provides insights into otherwise unavailable cognitive processes and may be used for exploring problem difficulty, student expertise, and metacognitive processes (e.g., Susac et al., 2014). The authors have found that the number of fixations of the eyes represents a reliable and sensitive measure that can give valuable insights into the participants' flow of attention during equation solving. The authors claim that the more efficient participants developed adequate strategies, i.e., "knew where to look." They found a correlation between the number of fixations and the participants' efficiency in equation solving. What is more, they observed that the measures derived from eye movement data were more objective and reliable in comparison to the participants' reports.

Examining the differences between the performance of novices and experts during the process of mathematical problem solving is also the interest of other researchers who use eye-tracking as a research method. They have found quantitative and qualitative differences in the way of looking at a geometry problem (Epelboim & Suppes, 2001) and reading mathematical representations (Andra et al., 2009) by novices and experts in terms of eye movements.

An in-depth knowledge on effective strategies of reading mathematical problems has important didactic consequences. Students need to learn how to read mathematical problems, but this knowledge should be recognised by researchers and teachers beforehand. What is more, it can be useful for the authors of tasks, textbooks, and other didactical materials.

RESEARCH METHODOLOGY AND DESIGN

The aim of the research

The main purpose of this study is to find the differences and similarities in the process of solving the same problem by mathematically gifted students and non-gifted students, by utilizing eye-tracking methodologies.

The objectives of this study refer to the following comparisons in the two test groups of students:

> Aim 1 (A1). Comparison of the total time of solving the problem,
>
> Aim 2 (A2). Comparison of the time of analyzing the wording of the problem,
>
> Aim 3 (A3). Comparison of the number of *fixations* (the stopping of the eyeball at a certain point on the screen) while working on the problem,
>
> Aim 4 (A4). Comparison of the number of *fixations* while analyzing the wording of the problem.

Equipment used

The participants' eye movements of the left eyeball were recorded by the eye tracker SMI Hi-Speed 1250 as well as iViewX™ software. The sampling rate was set to 500 Hz, monocular. The data obtained in the experiment were processed by SMI BeGaze software.

All students attended the experiment in the same physical conditions, in the same air-conditioned room with the same intensity of lighting.

All of the study participants passed the calibrations with an accepted angular accuracy of less than 0.5° and were included in the eye-tracking experiment of solving the science problem. All respondents sat at a distance of 50 cm from a screen the size of 30 cm × 47,5 cm.

The participants' eye movement data, question responses and mouse clicking were recorded by *SMI Experiment Center 3.4* software. In addition to providing answers by using mouse clicks, all respondents were also asked to verbally confirm the selected choices. There was no time limit in regard to the duration of the experiment.

Study participants

The research was carried out in June 2014. The experiment included 52 fifteen-year-old students attending the last grade of junior high school (gymnasium) in Poland, all of which had already taken the final external exam after finishing junior high school.

The sampling of experiment participants was diversified in terms of abilities and mathematical skills, where 18 students were finalists in a regional science competition and therefore recognized as gifted in the field of science. The remaining 34 students attended various lower high schools in Cracow, having mixed abilities and mathematical skills.

Each participant of the experiment was interviewed twice with the use of a questionnaire, both before and after the experiment.

Problem

The problem was provided in the Polish language, as shown in the subsequent figures using the data generated by the *BeGaze* software. Figure 1shows an English translation of the screen.

This problem can be solved by children at the age of 12, but it is more appropriate for lower high school students (13–15 years old). The problem is nonstandard in comparison to typical school tasks. The main difference and difficulty in solving it lies in the application of a methodological approach based on analytical thinking, using reduction. If a student considers how many days pass until half of the pond is overgrown with duckweed, the answer to the problem appears evident.

What is more, the formulation of the problem activates "System 1" according to the psychological *dual*

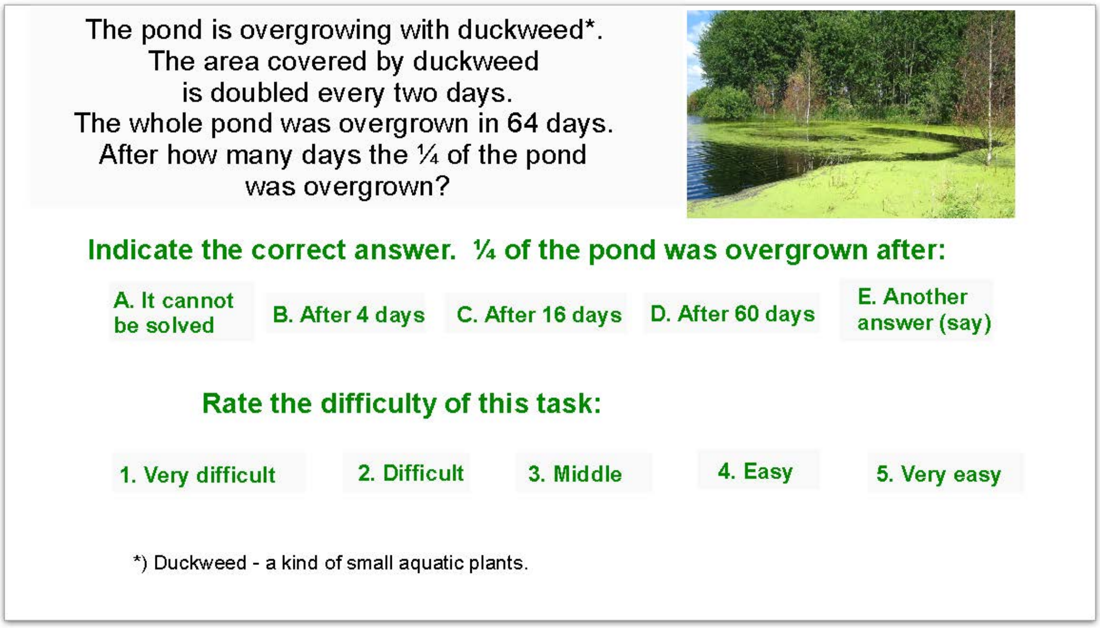

Figure 1: English translation of the Problem (slide 1)

process theory (Kahneman, 2011; Stanovich & West, 2000), and students have to overcome it. Answer C, "after 16 days," is a System 1 trap which forces quick, intuitive judgements with low mental effort which are frequently wrong. The Problem is analogous to the "lily problem" described by Kahneman.

The slide showed as Figure 1 was followed by two more slides with additional questions, the first of which suggested a method of reduction and provided graphical representations of the pond, as well as some hints and questions. The aim of slide 2 was to verify the answer provided to slide 1 and to suggest the proper method. After familiarizing themselves with this slide, all students were asked to check and correct their previous answer and rerate the difficulty of the problem.

The final slide was provided to the students in order to check their understanding of the method required to solve the problem. The students were asked to determine how many days it would take for the pond to be overgrown to 1/8 of the pond. They were also asked to assess six different methods of problem resolution, shown on the slide.

In this article we are focused only on the analysis of the answers to the initial Problem (slide 1).

Methodology

The analysis of all the answers to the questions presented in the three slides allowed us to select the students who correctly understood the whole problem and solved it perfectly. A ranking of the 52 participants was generated, taking into consideration the following criteria:

1. The correctness of answers to the whole problem (all questions on the three slides),
2. The mathematics score achieved on the final external exam after finishing junior high school.

The selection criteria for the comparative study was made by choosing a group of the best and worst performing participants from the ranking. However, we analysed the results of the groups in the context of general results.

Data for the analysis

The following sixteen "Areas of Interest" (AOIs) for obtaining the participants' data were defined within the slide area:

> Text of the problem, Picture – lake, Indicate the answer, Cannot be resolved, After 4 days, After 16 days, After 60 days, Another answer, Assign the difficulty, Very difficult, Difficult, Middle, Easy, Very easy, Explanation, White space.

Our analysis is based on numerical data, including graphical representations, such as: *focus maps*, *scan paths*, *AOI charts*, *key performance indicators*.

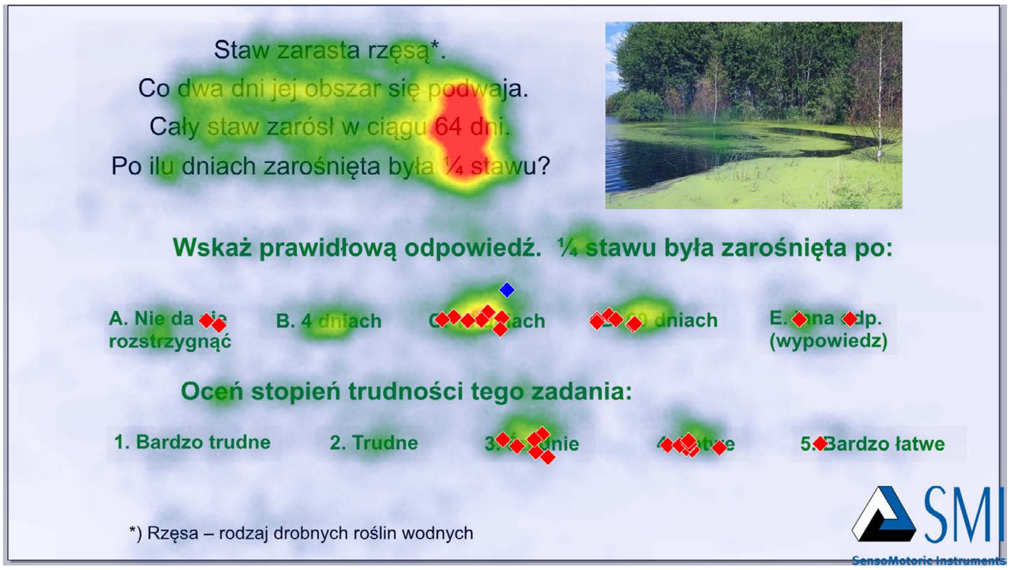

Figure 2: Heat map for all participants

RESEARCH RESULTS AND DISCUSSION

Overview of the general results

Only 5 students correctly solved the whole problem contained on the three slides. The correct answer, D, was provided only by 17 out of 52 students. The incorrect answer, C, "after 16 days," was selected by over half of the participants, i.e. 27 students. Despite the unsatisfactory general results, many participants rated the difficulty of the problem as "middle" (27) and "easy" (19).

The heat map for all participants is shown in Figure 2. Depending on the length of fixation time, the screen shows different colours – from blue (lack of fixations) through green, yellow, orange to red – representing the longest time of fixation.

The highest visual attention while reading the text of the problem was devoted to the most important phrases: "is doubled" („podwaja" at the end of the sentence); "64 days" and "¼ of the pond". The selected options are also visible as red symbols: ♦.

Two groups for comparison: „High Five" and „Low Six"

Only 5 students answered all of the questions from the three slides correctly. We call this group the "High Five". All of them were finalists of regional science competitions and they were recognized as gifted and interested in mathematics.

The second group in this comparative study was made by choosing a group of five the lowest performing participants from the ranking. This group consisted of 6 students and is called the „Low Six" group, as two of the students achieved the same mathematics score at the final external exam after finishing junior high school. The students from this group were the only respondents who did not pass the exam, achieving a result of below 30% of the available points.

A1. Total time of the analysis of the problem

The average total time for solving the problem by all participants of the study was 72 480 ms, the maximum time was 106 084 ms, and the minimum time was 32 890 ms.

The average total time for the "High Five" group was 72 150 ms, whereas the maximum time was 105 307 ms, and the minimum time was only 32 959 ms. Relevant individual differences can be observed (see Figure 3).

For the "Low Six" group, the corresponding values are the following: 57 275 ms; 69 981 ms, and 40 918 ms. The duration of solving the problem by the "Low Six" participants was more homogeneous (see Figure 3) and shorter than the average time of all participants.

A2. Time of the analysis of the problem's wording

In the two compared groups, we observe a crucial difference in the strategy of the analysis of the problem. The proportions between the visual attention devoted to analyzing the wording of the problem and the remaining text on the slide are significantly different.

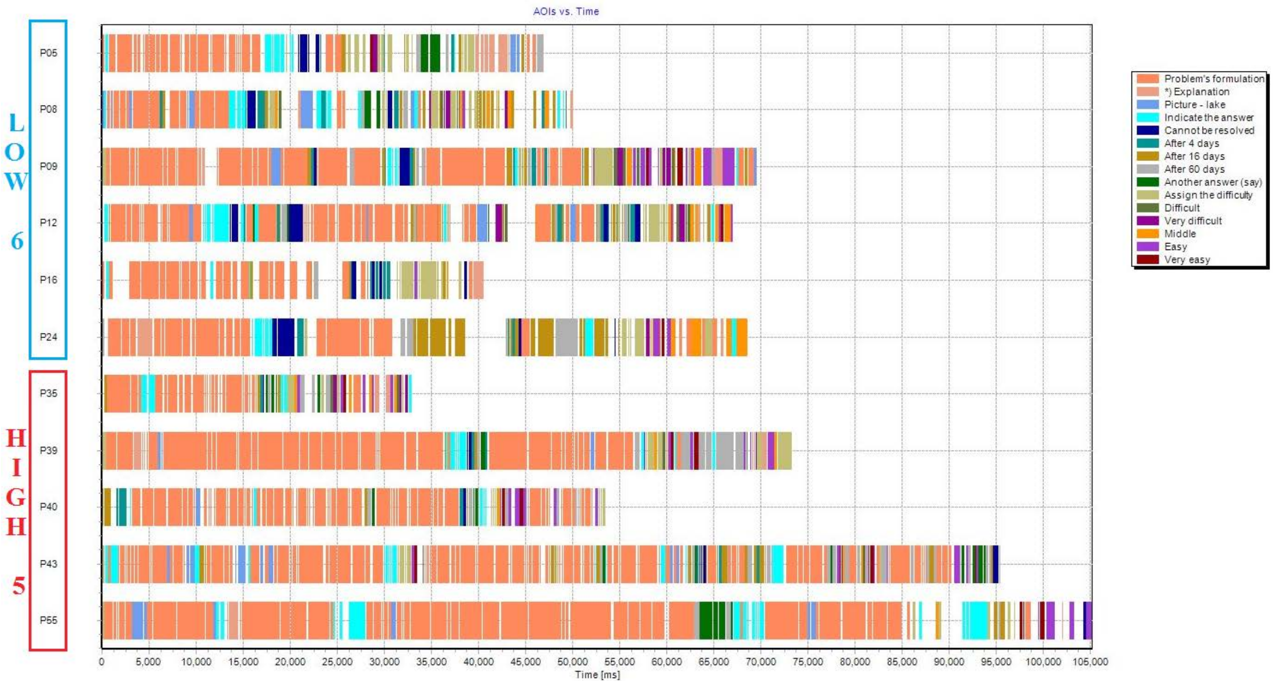

Figure 3: Sequence chart for "High Five" and "Low Six" groups

The average time percentage of the analysis of the wording of the problem in the "High Five" group is 65.9%. On the contrary, for the "Low Six" group it is only 36.5%. The proportions between visual attention devoted to the text of the problem and to the other parts of the screen are reversed for the two groups.

Figure 3 shows time (in milliseconds) spent by the participants' eyes at the defined AOIs respectively in the "High Five" and "Low Six" groups. The colors of the chart segments correspond to the sixteen respective AOIs described above. For example, the text of the problem is visible on the chart in orange. The sequence charts for both groups show an important difference in the way of looking at the screen.

A3. Number and placement of fixations while solving the whole problem

The respondents' visual attention is significantly different for the two compared groups. The heat maps (see Figure 4) show that students from the "High Five" group were concentrated on the wording of the problem and they achieved the maximum number of fixations on the area containing crucial information: "64 days", which had to be the starting point of discovering the correct answer.

On the other hand, the attention of the students from the "Low Six" group was more dispersed. They looked at the middle part of the screen as well – C and 3 an-

HIGH FIVE LOW SIX

Figure 4: Heat maps for "High Five" and "Low Six" groups

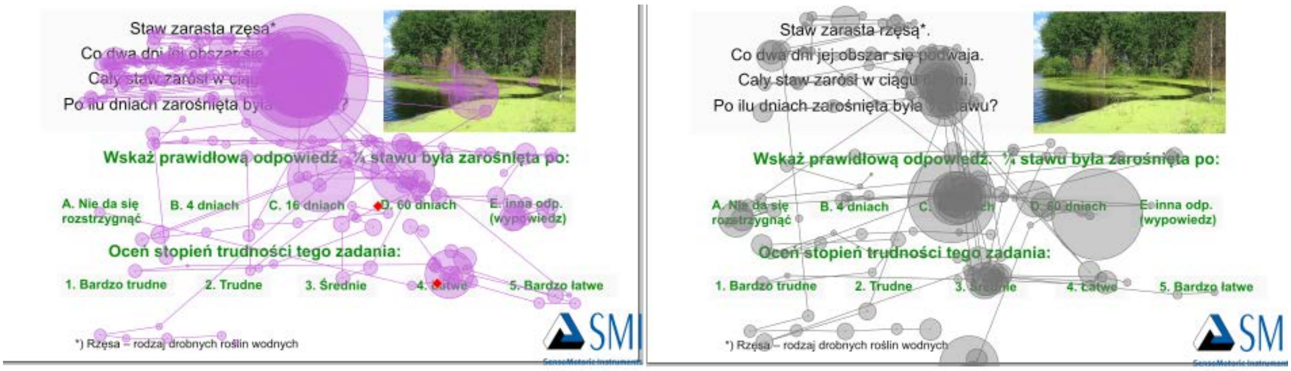

Figure 5: Scan paths of the representative participants from the "High Five" and "Low Six" groups

swers. This is a typical subconscious and intuitive way of looking.

The effect of dispersion can be observed individually, analyzing the students' looking paths, called *scan paths*. BeGaze™ software presents a clear graphic interpretation of data, showing the successive *fixations* (using circles) and *saccades*, i.e. paths of displacement between two consecutive *fixations* (using segments). In Figure 5, we present the *scan paths* of chosen two representative members from both groups.

A4. Number of fixations while analyzing the wording of the problem

The number of fixations while analyzing the text of the problem is the following for the "Low Six" and "High Five" groups respectively: Average number of fixations: 75,17 and 152,2; Maximum number of fixations: 110 and 225; Minimum number of fixations: 42 and 50.

SUMMARY

The following conclusions on the basis of the results of our research can be posed:

A1.

a) The average total time of solving the problem by the gifted students was the same as the average total time of all participants in our study. This parameter did not turn out to be a differentiator between gifted students and non-gifted students in the context of our research.

b) The total time of solving a problem by gifted students was very diversified. In this group we observed fast solvers (32 959 ms), average solvers, as well as slow ones (105 307 ms). Speed was not a parameter of mathematically gifted students.

A2.

The time of the analysis of the problem's wording was a differentiator between the group of gifted students and underperforming students in our research. Gifted students dedicated on average 65.9% of the total time of solving the problem to the analysis of the wording of the problem while the underperforming students devoted only 36.5% of their time for this purpose.

A3.

Analyzing the respondents' visual attention by observing the numbers and placement of fixations we observed significant differences between the two groups. The gifted students were concentrated on the text of the problem and they achieved maximum number of fixation at the area of the crucial information, which had to be a starting point to discover the correct answer. They did know where to look.

On the contrary underperforming students looked at various places of the screen, in a seemingly chaotic way. They also looked for longer periods in the middle of the screen, which is a natural way of looking. Their fixations were more often observed occurring at the areas on the slide without the wording of the problem.

A4.

The number of fixations while analyzing the text of the problem was also a differentiator between the groups of gifted students and non-gifted students. Both the average and the maximum number of fixations of the gifted students double those of the underperforming students.

On the basis of our research results, fixations were the visual symptoms of mental effort and motivation to solve the problem. It can be argued that the underperforming students were not sufficiently motivated to solve the problem or to make mental efforts.

The eye-tracking method allowed us to distinguish important differences in the strategy of reading a mathematical problem between gifted and underperforming students. It is be worth examining this topic further if the conclusions seem too broad, verifying them using different problems and a wider sample of students.

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A comparative analysis of mathematics textbooks from Kosovo and Albania based on the topic of fractions

Eda Vula, Jeta Kingji–Kastrati and Fitore Podvorica

University of Prishtina, Faculty of Education, Prishtina, Kosovo, eda.vula@uni-pr.edu

This study analyses the presentation of fractions in Kosovar and Albanian mathematics textbooks designed for students of one to fifth-grade. Physical characteristics of the selected textbooks, the presentation of fractions and the nature of the problems were analysed as well. Findings showed that Albanian mathematics textbooks covered more lessons on fractions than Kosovar textbooks. Textbooks from both countries focus mostly on part whole and operator construction. Also, the majority of problems focused on pure mathematics and single procedures. Few problems in Kosovar textbooks required problem explanation, while problem solving as a cognitive requirement was used more in Albanian textbooks. A common deficiency in all texts was the lacks of problems requiring special requirement.

Keywords: Textbooks, fraction representation, subconstruct, problems.

INTRODUCTION

Numerous studies have shown that there are enough similarities but also differences in curricula and textbooks of countries with different history, culture, language, economy and geographical stretch (Delaney, Charalambous, Hsu, & Mesa, 2007; Charalambous, Delaney, Hsu, & Mesa, 2010; Alajmi, 2012; Erbaş, Alacaci, & Bulut, 2012; Cheng & Wang, 2012; Özer & Sezer, 2014). It is shown also that in addition to the culture of a country, language and structure of the school, students' performance in certain countries is affected particularly by the educational background of teachers and textbooks (Cheng & Wang, 2012).

Kosovo and Albania have been carrying out innovative reforms to increase the quality of the system of education. Unfortunately, studies that show the degrees of student achievement as a result of these reforms are very limited. While, the data for improved student achievement on internationally comparable standardized tests in Albania (PISA, 2012) are welcome, for Kosovo, participation in international assessment programs still remains a challenge. Thus, both countries are making the efforts to reform school curricula and textbooks to achieve the educational objectives of European Union (EU) and Organization for Economic Co-operation and Development (OECD). Since majority of the population in Kosovo and Albania speak the same language, have the same culture and tradition, regarding the improvements on education there are efforts to unify Curriculum for Pre-University Education System, with the assumption that this unification will have impact on improving quality in schools and students' learning. By considering the textbooks as an important part of classroom life, as very important tools for quality assurance (Pehkonen, 2004; Nicol & Crespo, 2006) and as an important indicator that enable students to reflect on school curricula (Erbaş, Alacaci, & Bulut, 2012), it seems important for us to explore how fractions as one of the key concepts presented in elementary mathematics textbooks are used in Kosovo and Albania.

BACKGROUND AND RESEARCH QUESTIONS

Many researchers identified several factors which contribute to students' learning fractions. Teachers can improve students' fraction learning by placing more emphasis on examining and improving the design of their instruction (Brown & Quinn, 2007; Pitsolantis & Osana, 2013). Keijzer and Terwel analysed how low-achieving students learn fractions in the context of Realistic Mathematics Education (2004) and Reimer and Moyer found that the virtual manipulative helped students in the class to learn more about fractions (2005). However, the textbooks are the major written source which play an important role in guiding teachers in lesson planning and present them to their classes (Li, 2000; Alajmi, 2012). Textbooks, as an important resource in support of teaching and learning have received increased attention from the international education community over the last decades (Nicol & Crespo, 2006; Fan & Zhu, 2007; Charalambous, Delaney, Hsu, & Mesa, 2010; Alajmi, 2012; Erbaş, Alacaci, & Bulut, 2012). The TIMSS survey found that in almost all countries, the majority of teachers use textbooks as the main written source in deciding how to present a given topic in their classrooms (Beaton, et al., 1996). Textbooks influence what teachers teach, how they teach it, and how they measure the learning opportunities (Alajmi, 2012; Özer & Sezer, 2014). We could consider that textbooks help teachers with their workloads because the books provide ready and sensible structures for lessons and enough exercises for pupils (Pehkonen, 2004). They are an instrument for learning as well as an object of learning, and textbooks address both students and teachers (Erbaş, Alacaci, & Bulut, 2012). Because of their importance, many researchers focused on reviewing mathematical textbooks regarding the presentation of a particular mathematical concept or they focused on the cognitive demands of the material in the texts (Kieren, 1976; Behr & Harel, 1990; Alajmi, 2012). Fractions are among the most complex mathematical concepts that children learn in primary school. It was documented that teaching and learning of fractions has traditionally been one of the most problematic areas and one of the most serious obstacles in primary school mathematics (Behr, Post, Harel, & Lesh, 1993; Charalambous & Pitta-Pantazi, 2007). The misconceptions that students have about fractions, both in terms of fractions as numbers and how to operate with fractions relates particularly to the way in which fractions are represented and how they are taught (Barmby, Harries, Higgins, & Suggate, 2009). Various theoretical models have been proposed for understanding fractions in the elementary school (Behr et al, 1993; Charalambous & Pitta-Pantazi, 2007; Nicolaou & Pitta-Pantazi, 2011). Early subconstruct theories postulated that integrating the qualities of multiple perspectives were crucial to the understanding of fractions (Moseley, 2005). According to Behr and colleagues (1993), rational numbers can be interpreted as a part-whole comparison, an indicated division or quotient, a ratio, as an operator and as a measure. The part-whole construct refers to a continuous quantity or a set of discrete objects partitioned into parts of equal size. This perspective emphasizes situations which are related to the comparison of parts with the total amount. On the other hand the quotient construct considers a fraction as a result of a division of two whole numbers. The ratio perspective is based on comparing separate quantities and usually presented as *a:b* or *a/b*. Whereas the ratio is a relation between two sets, an operator is a relation between two states of a set. The operator construct reflects a function that transforms line segments, figures, or numbers. And lastly the measure construct identifies fractions as numbers or associating fractions with the measure assigned to some interval (Kieren, 1976; Behr & Harel, 1990).

Using multiple representations makes the fraction concept more concrete and understandable. But, it was shown that even in cases where students seem to understand the conceptual meaning (Hiebert & Carpenter, 1992) of the different perspectives of fraction representations they still struggle to connect fraction ideas to real world problems (Moseley, 2005; Alajmi, 2012). Textbooks should focus not only on problems that have low-level cognitive requirements, such as recall and reproduction (Li, 2000). Also, they should not present only the pure mathematics contexts but they should emphasize real-world problems dealing with fractions in all lessons (Alajmi, 2012).

This study reviews the similarities and differences of the presentation of fractions and problems of fractions in textbooks in Kosovo and Albania. Specifically, the study addressed the following questions:

1) What are the physical characteristics of first-to fifth-grade mathematics textbooks in the selected series in Kosovo and Albania?

2) What similarities and differences can be observed in the representation of fractions in the mathematics textbooks in Kosovo and Albania?

3) How are problems involving fractions introduced in these textbooks?

METHODS

Textbook sampling

Two elementary mathematics textbook series from Kosovo and Albania were chosen for this study.

Kosovo has a national curriculum. In this system only one textbook series is available, and it is published and distributed at no cost to students by the Ministry of Education, Science and Technology - MEST. Thus, all public primary schools use the same mathematics textbook series. Albania also has a national curriculum but any of fifteen commercial series of textbooks may be used in Albanian schools. Teachers together with the parents' council analyse each of these series of books and choose which textbooks will be used throughout the school year. For this study we selected the series of EduAlba publishing house, because these textbooks are used as secondary textbooks by some schools in Kosovo. Focusing on the fraction content, this study examined textbooks from both countries in detail, with respect to two aspects: presentation of fractions and the mathematics problems provided for students' practice. In this study we have taken into account only the basic textbooks and not the student workbooks or other materials.

Data Analyses and Procedures

In this study a framework and the coding system was developed in two stages. First we analysed the textbooks' physical characteristics, such as page size, number of pages and number of pages containing fractions while using a horizontal approach which gives readers an initial introduction of the textbooks. Secondly, for the vertical-analyses (Charalambous et al., 2010) we read each lesson relevant to fractions from each textbook carefully in order to analyse the meaning of tasks (Cheng & Wang, 2012). The meaning of tasks here indicated a meaning of a specific task, a definition, a picture, a question, an example, practical exercise or a problem that serves as a particular purpose in the lessons. Based on the meaning of tasks we developed the coding system which includes the following two dimensions: conceptual and contextual. Under first dimension we analysed the fraction meaning and presentation based on Kieren (1976) model of the five subconstructs: the part-whole, the ratio, the operator, the quotient and the measure construct. For the second dimension, Li's (2000) framework was used to analyse textbook problems provided to students. The problems selected from each textbook were those exercises or questions that did not have solutions or answers. Each problem was coded based on the three dimensions below:

1) Mathematical features: Single procedure (S) and Multiple Procedure (M);
2) Contextual features: Purely mathematical context in numerical or word form (PM) and illustrative context or story (IC);
3) Performance requirements:

    3.1 Response type: no explanation or solution process required (numerical answer or numerical expression only) (NES) and explanation or solution process required (ES)

    3.2 Cognitive requirement: conceptual understanding (CU); Procedurals practice (PP); Problem solving (PS) and Special requirements (SR).

Cognitive requirement mean 'the kind and level of thinking required from students in order to successfully engage with and solve the task (Stein, Smith, Henningsen, & Silver, 2000, p. 11), while special requirement (SR) refers to the problems that contain special or mixed cognitive requirements. For example, the problem asks students to write up a problem based on given information (Li, 1990).

After we selected the pages with fractions' content from each textbook and we coded them independently, the tasks that were coded differently were discussed and recoded again for compatibility. The researcher's agreement was 97 %.

RESULTS

Physical characteristics

Textbooks in Kosovo differ from Albanian textbooks. Textbooks in Kosovo are larger in size (28.5 x 20.5 x 0.8 cm) in each grade level. Albania textbooks have smaller sizes (27 x 20.5 x 1 cm). For the first time fractions are introduced in the second grade texts both in Kosovo and Albania. The number of pages devoted to fractions increased by grade to grade level in all two textbook series. In Kosovar textbooks, the percentage of pages grew from 2.34% on the second grade to 10.73% on the fifth grade, while in Albanian texts the percentage of pages grew from 3.15% to 24.89 %. Based on these facts the number of tasks per page related on fractions is bigger in Kosovar texts (5–6 tasks per page) than in Albanian textbooks (2–3 tasks per page). Also, the number of lessons on fractions increased each year from three lessons in the second grade to fourteen lessons in the fifth grade in Kosovar textbooks while in Albanian text books it increased from one lesson in the second grade to eleven lessons in the fifth grade. Addition and subtraction of fractions with same denominator for the first time are presented in third grade in the Albanian textbooks, though in Kosovar textbooks they are presented for the first time on the fifth grade. Also, in the Kosovar textbooks decimal numbers are not covered in the textbooks, whereas in the Albanian textbooks they are introduced in the fifth grade. The greatest difference in these textbooks is related to the quantity of real world problems included. It was found that the Albanian textbooks emphasize real world problems more than their Kosovo counterparts. Only two real world problems were found in the Kosovo fifth grade textbook, while in Albanian textbooks there are two real world problems in the second grade, twelve in the third grade, twenty-six in the fourth grade level and thirteen real world problems in the fifth grade level.

Fraction meaning and presentation

Based on the model of the multiple representations of fractions, the fraction constructs were identified in all textbooks (Table 1). Table 1 shows that the part-whole construct is used from the 2nd grade, when fractions appear for the first time, up to 5th grade in both series of textbooks. It was identified that the part-whole construct was used in a higher percentage in 2nd grades and 3rd grades, while in 4th and 5th grades the percentage of tasks presented by this construct is reduced. It seems that the ratio and quotient constructs were used only in a few tasks. In Kosovar textbooks there are only two tasks presented with the ratio construct, while in Albania textbooks, the ratio construct appears in 3rd grade and 4th grade, with a total of six tasks. It was found that only 5 tasks in the 5th grade Albanian textbook used the quotient construct. In Kosovar textbooks there were no tasks presented with the quotient construct. The dominant construct among tasks identified in 4th and 5th grade textbooks in both series of textbooks was the operator construct. Most of the tasks in 4th and 5th grade of both series of textbooks used the operator construct (about 70% in 4th and 87% in 5th grade Kosovar textbooks, whereas in Albanian textbooks, about 91% and 94% of all tasks). The measure construct was identified in three Kosovar textbooks (grade 3, 4 and 5) while in Albania only one task in the textbook of the 3rd grade was identified.

Classification of problems with fractions

Table 2 shows the percentages of problems classified in terms of problems' requirements (Li, 2000). It was found that there is a significant difference between the types of problems presented in the textbook series of the two countries.

From the third to fifth grade in Kosovar series of textbooks we have a higher percentage of problems or practice exercises which requires a single procedure (SP) in the solution process, while in Albania in third grade texts 76% of the tasks require multiple procedures (MP). Also we founded that 50% of problems require single procedures and 50% multiple procedures in the Albanian textbook of the 4th grade. In contrast, the Albanian textbooks of the 5th grade contained about 85% of the problems that required a single procedure for the solution. In Kosovar 3rd grade textbook, the majority of problems are presented through illustrative contexts (76.47%). In contrast, Albanian third grade textbooks have more pure mathematics problems (52.78%) than illustrative context exercises (47.22%). In the 4th and 5th grade textbooks we have approximately the same percentage of tasks with PM and IC. What is also important to mention (see Table 2) is the number of tasks that require only the numerical result (NES) of the assignment and no explanation on how the solution was reached (ES). Only in the Kosovo

Country	Kosova									Albania								
Grades	1		2		3		4		5	1		2	3			4		5
		N#	%	N#	%	N#	%	N#	%		N#	%	N#	%	N#	%	N#	%
Part whole	/	8	66.67	18	78.26	8	21.62	7	9.72	/	13	92.86	26	37.68	5	7.35	1	1.0
Ratio	/	0	0	0	0	2	5.41	0	0	/	0	0	5	7.25	1	1.47	0	0
Operator	/	4	33.33	4	17.39	26	70.27	63	87.5	/	1	7.14	37	53.62	62	91.18	94	94.0
Quotient	/	0	0	0	0	0	0	0	0	/	0	0	0	0	0	0	5	5.0
Measure	/	0	0	1	4.35	1	2.70	2	2.78	/	0	0	1	1.45	0	0	0	0
Total	/	12	100	23	100	37	100	72	100	/	14	100	69	100	68	100	100	100

Table 1: The fraction presentation based on the model of the five subconstructs

	Grade		Mathematical features		Contextual features		Performance requirements					
							Response type		Cognitive requirement			
			S	M	PM	IC	NES	ES	CU	PP	PS	SR
Kosova	2	N#	-	-	-	5	-	-	6	-	-	-
		%	-	-	-	100	-	-	100	-	-	-
	3	N#	4	3	4	13	17	-	15	-	1	-
		%	57.14	42.85	23.52	76.47	100	-	93.75	-	6.24	-
	4	N#	18	3	16	21	36	-	27	3	-	-
		%	85.71	14.28	43.24	56.75	100	-	90	10	-	-
	5	N#	31	3	37	18	43	5	35	15	1	-
		%	91.17	8.82	67.27	37.72	89.58	10.41	68.62	29.41	1.96	-
Albania	2	N#	-	-	-	5	-	-	5	-	-	-
		%	-	-	-	100	-	-	100	-	-	-
	3	N#	6	19	19	17	42	-	6	10	-	-
		%	24	76	52.78	47.22	100	-	37.5	62.2	-	-
	4	N#	18	18	22	25	46	-	29	3	20	-
		%	50	50	46.81	53.19	100	-	55.77	5.77	38.46	-
	5	N#	45	8	35	9	33	-	30	13	10	-
		%	84.91	15.09	79.55	20.45	100	-	56.60	24.53	18.87	-

Table 2: Problems Classified According Li's Framework (Li, 2000)

5th grade textbook about 10.4% have problems that required the explanation. Regarding cognitive requirements, it was identified that in textbooks from Kosovo as well as in Albanian textbooks, the largest number of problems requires conceptual understanding (CU) and procedural practice (PP). There are very limited requirements for problems which ask for problem solving in Kosovar textbooks. There was only one problem in each of the 4th and 5thgrade Kosovar textbooks, while in the Albanian textbooks it was a greater percentage of problem solving (about 38% and 19% of problems in the 4th and 5th grade respectively).

A common deficiency in all analysed texts is the lack of problems requiring special requirements (SR). Textbooks do not provide assignments in which students have the opportunity to practice more complex cognitive requirement or require contextualising fractions in everyday life.,,

DISCUSSION

This study has analysed the physical characteristics of the textbooks, fraction presentation and nature of problems used in Kosovar and Albanian textbooks. In both series of textbooks, fractions are presented for the first time from the 2nd grade. The number of pages covered by fractions is larger in the Albanian textbooks than in the Kosovar textbooks. But, it was observed that there is a difference in terms of the number of tasks, associated figures, tables or diagrams per pages. While in the books from Kosovo there are five or six tasks per page, in Albanian books every page has at most 3 tasks. Also, there are differences in terms of content. An introduction to fractions as decimal numbers is presented only in the Albanian textbooks. It seems that Kosovar textbooks covered less content and less representations for fractions. Thus, low fraction content coverage can have a negative effect on the student's abilities to use fractions in the future. According to Brown & Quinn (2007), students should spend the majority of their time learning fractions before the 6thgrade if we want to devote the time to the developing of their conceptual understanding of fraction relationships. Both series of textbooks used mostly two constructs for fraction presentation (part-whole and operator) and were focused more on rules and procedures, even though using multiple representations would assist students in developing more interconnected and viable representations knowledge for fractions (Behr et al., 1993; Charalambous & Pitta-Pantazi, 2007; Moseley, 2005).

It was found that the majority of problems in both series of textbooks are presented in purely mathematical contexts and that almost in all problems single-procedures are required. Also, it was found that the largest number of tasks requires contextual cognitive requirement with illustrative context. But, as Hiebert and Carpenter (1992) argue, there is a need to have a conceptual understanding, as well as the necessary skills for procedural practice, in order to be successful in mathematics .Only few problems were found in Kosovar textbooks that demanded problem explanation and problem solving. Based on Li's point of view, focusing only in the problems that have low-level cognitive requirements, such as recall and reproduction (Li, 2000) doesn't help students to be good problem solvers. On the other hand, Albanian textbooks pay more attention to the idea of including more practice examples that require problem solving compared to Kosovo. However, it is important to mention a common deficiency in all analysed textbooks, which is the lack of practice exercises that require special requirements. In this case connecting fractions to students' lives and with real word problems is an important issue. Students should be provided with opportunities to see fractions in their real lives and use these meanings in the learning process. Therefore, the texts should assist teaching and learning in order to enable the development of the meaning of fractions, understanding them (Moseley, 2005; Alajmi, 2012) and solving problems from everyday life contexts.

The study has highlighted some differences that textbooks from different countries have, and aims to encourage authors of textbooks to cooperate at the international level in order to improve the quality of mathematics textbooks. A comparative study of mathematic textbooks from Kosovo, Albania and other countries will help to share the best practices for fraction introduction in elementary textbooks. The recommendation of this study is to increase both the number and the variety of problems in all textbooks, especially those with a high level of cognitive demand. Finally, taking in consideration that the elementary mathematics textbooks are the main instructional resource and learning tool, the teachers' and students' views on textbooks are crucial for their improvement.

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The history of mathematics in the lower secondary textbook of Cyprus and Greece: Developing a common analytical framework

Constantinos Xenofontos and Christos E. Papadopoulos

University of Nicosia, Nicosia, Cyprus, xenofontos.c@unic.ac.cy

In this paper, we examine the ways the history of mathematics is integrated in the national textbooks of Cyprus and Greece. Our data-driven analyses suggest that the references identified can be clustered in four categories: (a) biographical references about mathematicians or historical references regarding the origins of a mathematical concept, (b) references to the history of a mathematical method or formula containing a solution or proof, (c) mathematical tasks of purely cognitive elements that require a solution, explanation or proof and (d) tasks that encourage discussion or the production of a project that would connect the history of mathematics with life outside mathematics.

Keywords: History of mathematics, textbooks, Cyprus, Greece.

INTRODUCTION

A number of recent studies highlight the importance of introducing elements from the History of Mathematics (HoM) in lessons as an alternative and supplementary means to establish more efficient teaching and learning of the subject (see, for example, Fauvel & van Maanen, 2000; Siu & Tzanakis, 2004). While recognizing the various pedagogical benefits of such an incorporation –as, for example, the development of more positive attitudes towards the subject (Farmaki et al., 2004), better conceptual understanding of mathematical ideas (Fasanelli et al., 2002), the elimination of racial discriminations (Michalowicz et al., 2002; Strutchens, 1995) and so on – this paper describes and discusses the developed framework that facilitates our analyses for the textbooks of the two countries under scrutiny, Cyprus and Greece, as well as some of our findings.

COMPARING MATHEMATICS TEXTBOOKS ACROSS COUNTRIES

Textbooks constitute the main teaching resource for most countries around the world (Mullis et al., 2012), especially for highly centralised educational systems, as are those of Cyprus and Greece. Textbooks play a significant role in the ways mathematics lessons are designed and delivered; their influence over the content of the lessons; the instructional approaches; the quality of activities assigned for pupils, in classrooms and for homework; and, the learning outcomes and achievement (Alajmi, 2012; Törnroos, 2005; Weiss et al., 2003; Robitaille & Travers, 1992). Drawn from the well-known trichotomic distinction of the curriculum in the *intended* (vision and intentions as specified in official curriculum documents and/or materials), the *implemented* (teachers' interpretation and enactment of the curriculum according to their perceptions and knowledge) and the *attained* (learning experience as perceived by learners and their resulting learning outcomes), mathematics textbooks could be seen as a mediator between the first two types (or what Schmidt, et al. call the *potentially implemented curriculum*). From this perspective, Mesa (2004, pp. 255–256) talks about "a hypothetical enterprise: What *would* students learn if their mathematics classes were to cover all the textbook sections in the order given? What *would* students learn if they had to solve all the exercises in the textbook?" Rezat and Straesser (2014) take this argument a step further and distinguish between three perspectives on textbooks analyses. The first perceives textbooks as *curriculum materials* that offer supplementary ideas for teaching. From another perspective, textbooks are approached as *artifacts* that are employed for the preservation and transition of acquired skills. Analyses from this perspective focus only on opportunities to teach and to learn. Lastly, textbooks can be seen as *instruments*. Such analyses take into consideration the actual use of textbooks in lessons.

Many studies make use of various frameworks for analysing mathematics textbooks, which include, among other things, their physical characteristics (i.e. size, number of pages, volumes, and so on), the structure of the lessons, the topics covered, the presentation of particular mathematical concepts, and the level of the cognitive demands of tasks (Bayazit, 2013; Alajmi, 2012). As far as cross-national comparisons of textbooks are concerned, a specific interest is shared between scholars, which shows how the cultural expectations and goals of different educational systems are manifested in the instructional materials produced (Haggarty & Pepin, 2002). Cross-national textbook analyses typically fall under three broad categories; namely, *horizontal*, *vertical*, and *contextual* (Charalambous et al., 2010). Horizontal analysis focuses on textbooks' general characteristics (i.e. content, structure, etc.), while the textbook is examined as a whole. Such examples include Stevenson's and Bartsch's (1992) analyses of Japanese and American textbooks, and the work of Campbell and Kyriakides (2000), who have investigated Cypriot and English elementary textbooks as part of the national curriculum of the two countries. In vertical analysis, the interest is on how the textbooks under examination treat certain mathematical concepts. From this perspective, Ding and Li (2010) discuss the ways the distributive property is presented in US and Chinese elementary textbooks, while Alajmi (2012) examines how elementary textbooks in the US, Japan, and Kuwait address fractions. Finally, contextual analysis is concerned with the role of the textbooks in classroom activities, either by the teacher or the pupils (see Remillard, 2005). Only few comparative studies employ more than one of the types of analysis above, as, for example, Haggarty's and Pepin's (2002) work with English, French and German textbooks, and their function in classrooms, and Charalambous and colleagues' (2010) framework developed for analysing learning opportunities provided by Cypriot, Irish, and Taiwanese textbooks.

THIS STUDY

In this paper, we analyse references included in the lower secondary textbooks of Cyprus and Greece in regards to the HoM. Both countries have highly centralised educational systems (Charalambous et al., 2010; Saiti & Eliophoto-Menon, 2009) in which schools are considered segments of the domain of government, and not of the community. The national curricula and continued textbooks are prepared by each country's respective Ministry of Education, more specifically, by the Pedagogical Institute, a department of the Ministry of Education. In both the Cypriot (MoEC, 2010) and the Greek intended curriculum (MoERA, 2002), explicit references are made to the importance of the HoM. Nevertheless, the purpose of this paper is to examine how the general curricular references to the HoM are actually transformed into learning opportunities in the Cypriot and Greek textbooks for both teachers and pupils.

In our study, textbooks are treated as artifacts (Rezat & Straesser, 2014) since we examine their content in relation to the HoM, and not in relation to other teaching material, or the ways by which they are utilized in classrooms. Furthermore, our approach could be regarded as a comparative vertical analysis (Charalambous et al., 2010), which focuses on how the two sets of textbooks, Cypriot and Greek, treat a particular concept, namely the HoM.

For the purposes of this project, we analysed the national textbooks of Cyprus and Greece for lower secondary education (grades 7, 8, and 9). The educational system of Cyprus is currently reforming its curriculum for all subjects and grades; consequently, only the textbooks for lower secondary education were available to us once our study began, therefore, we chose to focus our analyses on the three lower secondary grades. The Cypriot textbooks were launched between 2012 and 2013 by the Pedagogical Institute of Cyprus, and the Greek ones were introduced in 2007 by the Pedagogical Institute of Greece.

At the first stage of data collection, all references to the HoM in both textbook series were identified. We were interested in both encyclopedic pieces of information and mathematical tasks inviting pupils to interact with them and provide solutions or answers. We worked independently with the two data sets (Cypriot and Greek), trying to find ways of clustering the references identified. Due to the long distance, in person meetings were not possible; however, after individual progress we had online meetings to discuss our ideas, which were, interestingly, similarly handled. After combining our working ideas, we applied

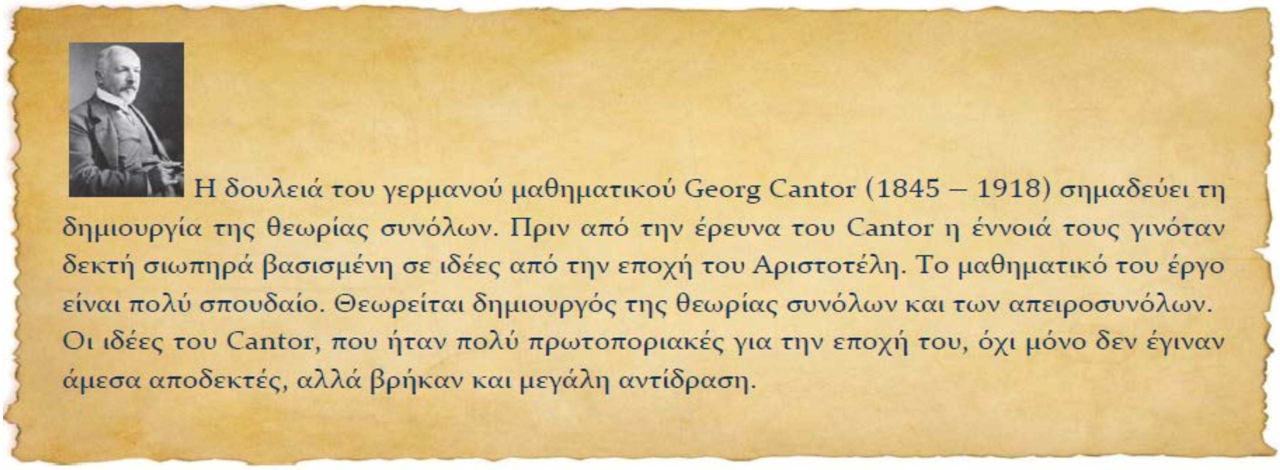
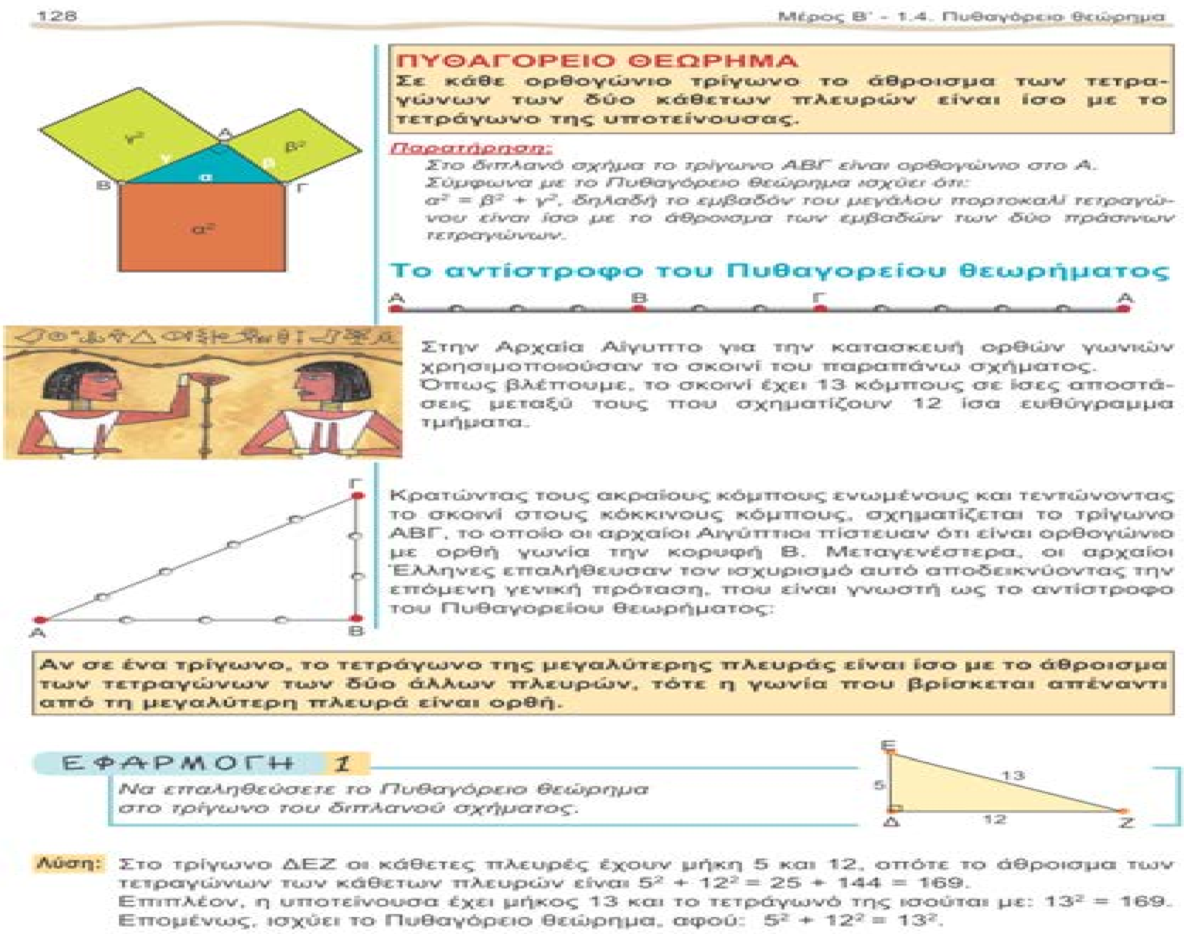

Η δουλειά του γερμανού μαθηματικού Georg Cantor (1845 – 1918) σημαδεύει τη δημιουργία της θεωρίας συνόλων. Πριν από την έρευνα του Cantor η έννοιά τους γινόταν δεκτή σιωπηρά βασισμένη σε ιδέες από την εποχή του Αριστοτέλη. Το μαθηματικό του έργο είναι πολύ σπουδαίο. Θεωρείται δημιουργός της θεωρίας συνόλων και των απειροσυνόλων. Οι ιδέες του Cantor, που ήταν πολύ πρωτοποριακές για την εποχή του, όχι μόνο δεν έγιναν άμεσα αποδεκτές, αλλά βρήκαν και μεγάλη αντίδραση.

Figure 1: An example of a biographical reference

a data driven analysis (Kvale & Brinkmann, 2009) as well as the constant comparison process outlined by Strauss and Corbin (1998), which led to clustering the references in two broad categories. Further analyses resulted in the division of the two categories into two subcategories each. The first category was concerned with references that did not pose questions to pupils; they'd rather provide historical pieces of information. In this category, we could find (a) simple biographical references about mathematicians or historical references concerning the origins of a mathematical concept and (b) references to the history of a mathematical method or formula including a solution path or proof. Figure 1 shows an example of the first subcategory and provides biographical information about


128 Μέρος Β' - 1.4. Πυθαγόρειο θεώρημα

ΠΥΘΑΓΟΡΕΙΟ ΘΕΩΡΗΜΑ

Σε κάθε ορθογώνιο τρίγωνο το άθροισμα των τετραγώνων των δύο κάθετων πλευρών είναι ίσο με το τετράγωνο της υποτείνουσας.

Παρατήρηση:

Στο διπλανό σχήμα το τρίγωνο ΑΒΓ είναι ορθογώνιο στο Α. Σύμφωνα με το Πυθαγόρειο θεώρημα ισχύει ότι: $\alpha^2 = \beta^2 + \gamma^2$, δηλαδή το εμβαδόν του μεγάλου πορτοκαλί τετραγώνου είναι ίσο με το άθροισμα των εμβαδών των δύο πράσινων τετραγώνων.

Το αντίστροφο του Πυθαγορείου θεωρήματος

Στην Αρχαία Αίγυπτο για την κατασκευή ορθών γωνιών χρησιμοποιούσαν το σκοινί του παραπάνω σχήματος. Όπως βλέπουμε, το σκοινί έχει 13 κόμπους σε ίσες αποστάσεις μεταξύ τους που σχηματίζουν 12 ίσα ευθύγραμμα τμήματα.

Κρατώντας τους ακραίους κόμπους ενωμένους και τεντώνοντας το σκοινί στους κόκκινους κόμπους, σχηματίζεται το τρίγωνο ΑΒΓ, το οποίο οι αρχαίοι Αιγύπτιοι πίστευαν ότι είναι ορθογώνιο με ορθή γωνία την κορυφή Β. Μεταγενέστερα, οι αρχαίοι Έλληνες επαλήθευσαν τον ισχυρισμό αυτό αποδεικνύοντας την επόμενη γενική πρόταση, που είναι γνωστή ως το αντίστροφο του Πυθαγορείου θεωρήματος:

Αν σε ένα τρίγωνο, το τετράγωνο της μεγαλύτερης πλευράς είναι ίσο με το άθροισμα των τετραγώνων των δύο άλλων πλευρών, τότε η γωνία που βρίσκεται απέναντι από τη μεγαλύτερη πλευρά είναι ορθή.

ΕΦΑΡΜΟΓΗ 1

Να επαληθεύσετε το Πυθαγόρειο θεώρημα στο τρίγωνο του διπλανού σχήματος.

Λύση: Στο τρίγωνο ΔΕΖ οι κάθετες πλευρές έχουν μήκη 5 και 12, οπότε το άθροισμα των τετραγώνων των κάθετων πλευρών είναι $5^2 + 12^2 = 25 + 144 = 169$.
Επιπλέον, η υποτείνουσα έχει μήκος 13 και το τετράγωνό της ισούται με: $13^2 = 169$.
Επομένως, ισχύει το Πυθαγόρειο θεώρημα, αφού: $5^2 + 12^2 = 13^2$.

Figure 2: An example of a historical reference to a method and its solution process

	No question(s) for pupils		Asks pupils to interact with it	
	Simple historical/biographical references	*Show solution/proof of a method/formula*	*Mathematical tasks*	*Encourage discussion/ project*
Cyprus	27	2	28	2
Greece	41	4	26	7

Table 1: Distribution of the historical references in the two textbook series

the life and work of the German mathematician Georg Cantor (Cypriot textbooks, grade 7, part A, p. 27).

Figure 2 shows an example of a historical reference to a method and its solution process. In particular, it explains how Ancient Egyptians used a piece of rope with 13 knots and 12 equal line segments to create right angles, a method called in the book, "the reverse of the Pythagorean theorem" (Greek textbooks, grade 8, p. 128).

The second category comprised tasks that invited pupils to interact with them and provide some sort of answer. This included (a) mathematical tasks of purely cognitive elements that require a numerical solution, explanation or proof and (b) tasks that encourage discussion or the production of a project that would connect the history of mathematics with life outside mathematics. Examples of these subcategories are illustrated in Figures 3 and 4 respectively. Figure 3 shows the method applied by the ancient Greek mathematician, Thales, for the calculation of the distance between a ship and the coastline. Furthermore, pupils are asked to prove Thales' method and explain how he could be sure that the distance was right (Greek textbooks, grade 9, p. 197).

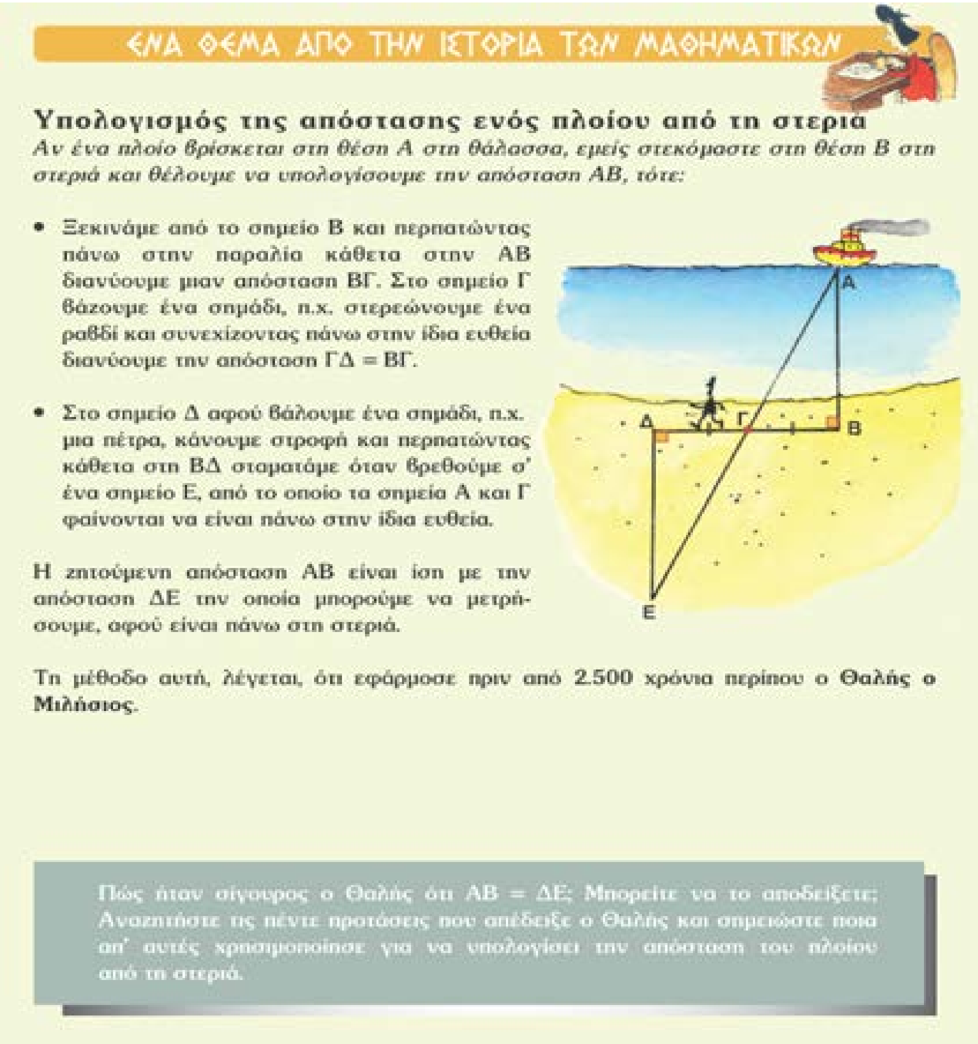
Figure 4 provides an example of a task that requires pupils to carry out a small project comparing and contrasting the number systems of Mayans and of Babylonians-Sumerians, as well as discussing the difficulties of these two systems, which eventually led to a need to establish the positional decimal system (Cypriot textbooks, grade 6, part A, p. 85). It is worth stating that there are no previous references to the two ancient number systems in the textbooks. Pupils are asked to conduct their own research and find information about them, since the textbook implies that there is no further instruction about these systems.

Table 1 shows the distribution of the HoM's references per country. No significant differences could be observed as regards the two broad categories and the references of the HoM from each country, χ^2 (1, N = 137) = 0.986, p = 0.32.

DISCUSSION AND CONCLUSIONS

Clearly, both countries value the HoM, a fact not only expressed in their curricula, but also reflected in their national textbooks. This could be attributed to a common cultural-historic heritage and the profound role of ancient Greek mathematics in the development of contemporary mathematical thinking. Nevertheless, the two countries' educational traditions share much in common, historically speaking (Koutselini-Ioannidou, 1997). In the case of their mathematics textbooks, Cypriot textbooks, launched a few years after the Greek ones, include several slightly modified examples. For instance, in the Greek textbook of grade 8 (page 10) we find a mathematical riddle from the tombstone of the ancient Greek mathematician Diophantus, inviting passers to solve it and calculate the age of his death. In the textbook, an equation showing the solution to the riddle follows. In the Cypriot textbooks (grade 7, part A, p. 173) although the same riddle is included, no solution is presented and pupils are asked to calculate Diophantus' age.

About half of the HoM's references in the Cypriot series and more than half in the Greek one constitute biographical and historical information that do not ask pupils to interact with them in any way. From our experiences as former pupils in the two educational systems, such information typically remains unexploited in classrooms, in a similar way to those arguing that the mathematics curriculum so overloaded with topics that teachers do not have much time to dedicate to the HoM. Also, despite comments of authors like Lawrence (2008) and Jahnke and colleagues (2002), who see advantages of the HoM in lessons as an


ΕΝΑ ΘΕΜΑ ΑΠΟ ΤΗΝ ΙΣΤΟΡΙΑ ΤΩΝ ΜΑΘΗΜΑΤΙΚΩΝ

Υπολογισμός της απόστασης ενός πλοίου από τη στεριά

Αν ένα πλοίο βρίσκεται στη θέση Α στη θάλασσα, εμείς στεκόμαστε στη θέση Β στη στεριά και θέλουμε να υπολογίσουμε την απόσταση ΑΒ, τότε:

- Ξεκινάμε από το σημείο Β και περπατώντας πάνω στην παραλία κάθετα στην ΑΒ διανύουμε μιαν απόσταση ΒΓ. Στο σημείο Γ βάζουμε ένα σημάδι, π.χ. στερεώνουμε ένα ραβδί και συνεχίζοντας πάνω στην ίδια ευθεία διανύουμε την απόσταση ΓΔ = ΒΓ.
- Στο σημείο Δ αφού βάλουμε ένα σημάδι, π.χ. μια πέτρα, κάνουμε στροφή και περπατώντας κάθετα στη ΒΔ σταματάμε όταν βρεθούμε σ' ένα σημείο Ε, από το οποίο τα σημεία Α και Γ φαίνονται να είναι πάνω στην ίδια ευθεία.

Η ζητούμενη απόσταση ΑΒ είναι ίση με την απόσταση ΔΕ την οποία μπορούμε να μετρήσουμε, αφού είναι πάνω στη στεριά.

Τη μέθοδο αυτή, λέγεται, ότι εφάρμοσε πριν από 2.500 χρόνια περίπου ο Θαλής ο Μιλήσιος.

Πώς ήταν σίγουρος ο Θαλής ότι ΑΒ = ΔΕ; Μπορείτε να το αποδείξετε; Αναζητήστε τις πέντε προτάσεις που απέδειξε ο Θαλής και σημειώστε ποια απ' αυτές χρησιμοποίησε για να υπολογίσει την απόσταση του πλοίου από τη στεριά.

Figure 3: An example of a mathematical task

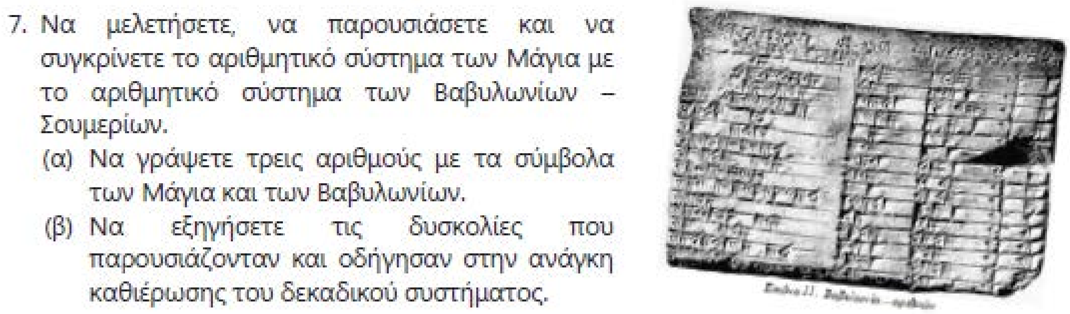

7. Να μελετήσετε, να παρουσιάσετε και να συγκρίνετε το αριθμητικό σύστημα των Μάγια με το αριθμητικό σύστημα των Βαβυλωνίων – Σουμερίων.
   (α) Να γράψετε τρεις αριθμούς με τα σύμβολα των Μάγια και των Βαβυλωνίων.
   (β) Να εξηγήσετε τις δυσκολίες που παρουσιάζονταν και οδήγησαν στην ανάγκη καθιέρωσης του δεκαδικού συστήματος.

Figure 4: An example of a project-based assignment

opportunity for pupils to realise how mathematical knowledge of the past has influenced our modern everyday life, both series include very few project-based tasks. Once again, from our experiences, Cypriot and Greek secondary mathematics teachers perceive such tasks as not "mathematical" and often ignore them, as a result. It is worth mentioning that both series are not restricted to ancient Greek mathematics; they include references from many civilizations, and this is an opportunity considered important for the elimination of racial discriminations (Michalowicz et al., 2002; Strutchens, 1995).

In closing, we cannot but emphasize the significance of collaboration between researchers from different countries in respect to comparative enquiries. Scholars must be aware that things with the same name might have a different meaning and function across nations, and they ought, therefore, to avoid assuming that colleagues elsewhere share the same understanding. Furthermore, we would like to invite colleagues from other countries to draw on our emerged framework of analysis about the types of historical references in mathematics textbooks, and to examine the ways in which HoM is included in other curricula, textbook series, and classrooms around the world.

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TWG11

Posters

Cooperation and innovation for good practices: Teachers and researchers understanding mathematics in PISA (TRUMP)

Sebastian Bauer[1], José M. Diego[2], Claudia Lazaro[3], Sue Pope[4], Tomas Recio[2], Ornella Robutti[5], Jaime Carvalho e Silva[6] and Ana M. Vieira[7]

1 University of Duisburg-Essen, Essen, Germany, sebastian.bauer.seuberlich@uni-due.de
2 University of Cantabria, Santander, Spain, diegojm@unican.es, reciot@unican.es
3 Spanish Federation of Mathematics Teachers' Societies, Spain, international@fespm.es
4 Association of Teachers of Mathematics, UK, suepope@atm.org.uk
5 University of Turin, Turin, Italy, ornella.robutti@unito.it
6 University of Coimbra, Coimbra, Portugal, jaimecs@mat.uc.pt
7 Portuguese Association of Mathematics Teachers, Portugal, anavieiralopes@gmail.com

We are briefly describing the proposal of an international project for enhancing mathematical literacy in different European countries (Italy, Germany, Portugal, Spain and UK), through the cooperation of teacher societies of mathematics and university researchers.

Keywords: Teacher education, comparative curriculum, international assessment.

INTRODUCTION

Our goal is to announce the proposal of the Teachers and Researchers Understanding Mathematics in PISA (TRUMP) project, submitted to Eramus + EU programme in 2014 (Key Action 2: Cooperation for innovation and the exchange of good practices).

The TRUMP project seeks to enhance adolescent's mathematical literacy at a European level. PISA 2012 (as others international projects) suggests that European mathematics education is often less successful in developing mathematical literacy than other OECD members (PISA, 2012).

Mathematical literacy is essential, in modern society, for effective citizenship. In this way, the TRUMP project aims to create a network of European teachers and researchers in partner countries that, informed by the PISA framework and its assessment, helps teachers' professional development.

Although the proposal presented in 2014 to Erasmus+ was declined, the potential of the project has been recognised and the current team is searching for new European members to collaborate with and to further develop the original proposal.

CURRENT TEAM MEMBERS

The current working team is led by the Spanish Federation of Mathematics Teachers' Societies (FESPM). This 25-year-old non-profit organisation comprises 21 Spanish mathematics teachers' societies, with about 6000 members who are mainly secondary education teachers. The group team includes also universities from England, Germany, Italy, Spain and Portugal, as well as professional associations of teachers of mathematics from Portugal and England.

The project is unique in the sense that it involves both universities and professional associations of mathematics teachers. We believe that universities will provide research and development expertise and teachers' associations will ensure wide participation and dissemination to practicing teachers, and long-term sustainability. Digital technologies will be used to support collaboration among different partners.

MAIN PROJECT ACTIVITIES TO BE UNDERTAKEN

The participant members will work within the PISA theoretical framework to devise and evaluate strategies for teacher professional development, drawing on existing materials and resources. Some examples are provided below:

1) Teachers and researchers from all the member groups will undertake bilateral visits to learn from teaching practices in the different countries.

2) Researchers from the member groups will develop an evaluative framework to assess materials, classrooms teaching practices and learner outcomes.

3) A network of teachers will be created and guided to trial and evaluate the programmes and educational resources emerging from the project.

4) Teachers in each participant country will adopt and adapt shared resources to their cultural and educational context.

5) Transnational meetings of group leaders will be held to share resources, review progress to date, and to agree on priorities for the subsequent stages of the project.

EXPECTED RESULTS

Within the results expected are the following:

1) Establishing a community of researchers to design the training programmes materials and evaluative framework for teachers' professional development and enhanced mathematical learning.

2) Establishing a community of teachers and researchers working together internationally and locally to enhance professional competences.

3) Enhancing and developing student mathematical literacy.

4) As a resulting outcome, the TRUMP project intends to contribute to reducing the number of low-skilled adults.

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A comparison between paper and pencil and computer based assessment

Alice Lemmo

University of Palermo, Palermo, Italy, alice.lemmo@unipa.it; alice.lemmo@gmail.com

In this study, we describe two tools designed for comparing paper and computer assessment modalities: a "comparison table" to control migratory process from paper to computer modality and vice versa and a "grid for observation of students" to codify students' behaviours in solving process.

Keywords: Paper and pencil based assessment, computer based assessment, comparability, students' behaviours comparison.

INTRODUCTION

A large number of Institutions for Assessment (OECD-PISA, PIAAC, NAEP, …) introduced a computer-based section. Ripley (2009) analyses the main approaches to the use of technology to support assessment. In particular, he defines the *migratory approach* to indicate paper based tests migrating to screen based environment. Research on migrated tests, involving comparative analysis on outcomes, finds no statistical evidence to suggest that the administration modality changes the coherence and consistency of the test (Kim & Huynh, 2007).

Quantitative studies give information on students' performances in scoring without dealing with cognitive aspects affected by this changing of environment. Threlfall and colleagues (2007) explore the effect on assessment of paper and pencil test items migrating into their computer corresponding ones. They show that, even if in most cases changing to the different environment seems to make little difference, for some items computer deeply affects how the question is attempted and what is being assessed. Furthermore, researches start from the assumption that it is possible to consider these assessment modalities as equivalent. The aims of this study is to develop a scheme to control, in terms of equivalence, migrating processes from paper to computer and vice versa and to analyse cognitive aspects influenced by these migrated tests.

FRAMEWORK AND METHODOLOGY

Comparing students involved in tests in different environments requires considering comparable questions. In migration process corresponding features of the questions probably change, such as syntax, vocabulary, editing, layout, and others. For this reason, it is necessary to define a monitoring tool in order to control any possible changes that might affect the comparability of questions. We define a *comparison table* to control the migration from paper to computer based question and vice versa.

We administer to couples of students two parallel tests of migrating items, one in computer and the other in paper environment. Every couple involves in only one of these tests; protocols are video taped with the purpose to focus on behaviours and heuristics processes.

Protocol analysis is performed through a *grid for observation of students' behavior*, inspired by the definition of problem solving episodes proposed by Schoenfeld (1985), which allows to identify differences focussing on strategies, planning and results, in qualitative terms.

CONCLUSION

Qualitative analysis on migrated questions shows that migration from one administration modality to another is necessarily influenced by the environment in which these questions migrate. It reveals the inconsistency in the definition of equivalence between questions in computer and paper environment.

In general, protocols analysis highlights that students achieve the same outcomes in both administration modalities. However, a deeper study on cognitive processes shows case studies in which administration environment greatly influences students' strategic choices in resolution process. In other cases, despite comparable heuristic choices, the organization of the problem-solving process is less linear in computer than paper modality.

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TWG12

History in mathematics education

Introduction to the papers of TWG12: History in mathematics education

Uffe Thomas Jankvist[1], Snezana Lawrence[2], Kathleen Clark[3] and Renaud Chorlay[4]

1 Aarhus University, Aarhus, Denmark, utj@edu.au.dk

2 Bath Spa University, Bath, UK

3 Florida State University, Tallahassee, USA

4 Université Paris-Diderot, Paris, France

CERME9 was the fourth time of the TWG on "History in Mathematics Education". For CERME9 the group had fourteen papers and two posters, and around twenty participants in the sessions. TWG12 covers a range of topics related to history in mathematics education, but on an overall scale, submissions to the group may be distinguished by either concerning "History in Mathematics Education" (HiMEd) or "History of Mathematics Education" (HoMEd). This time round, the group had five papers addressing topics of HoMEd, while the remaining nine papers and two posters were concerned with issues of HiMEd at various educational levels, which also included teacher education.

BULLETS IN THE TWG12 CALL FOR PAPERS

For CERME9, TWG12 in particular welcomed empirical and theoretical research papers and posters, but to some degree also methodological and developmental papers related to one or more of the following issues (bullets below) – although any paper/poster of relevance to the overall focus of the group was taken into consideration:

- Ways of integrating original sources in classrooms, and their educational effects, preferably with conclusions based on classroom experiments;
- Surveys on the existing uses of history or epistemology in curricula, textbooks, and/or classrooms in primary, secondary, and tertiary levels;
- Design and/or assessment of teaching/learning materials on the history of mathematics;
- The role of history or epistemology of mathematics at the primary, secondary, and tertiary level, and in pre- and in-service teacher education, from cognitive, pedagogical, and/or affective points of view;
- Investigations or descriptions of the historical instances of research cultures and cultures of teaching and learning in mathematics;
- Relationships between (frameworks for and empirical studies on) history in mathematics education and theories and frameworks in other parts of mathematics education;
- Possible parallelism between the historical development and the cognitive development of mathematical ideas;
- Theoretical, conceptual and/or methodological frameworks for including history in mathematics education;
- The potential role of history of mathematics/mathematical practices in relation to more general problems and issues in mathematics education and mathematics education research.

FOUR AREAS OF QUESTIONS FOR REFLECTIONS DURING SESSIONS

The work following the presentations of participants' papers and posters was orchestrated by four overarching themes cutting across the topics of papers:

Meta-level or methodological reflections on HiMEd and HoMEd

- What (if any) is (could be) the role assigned to epistemological/historical reflection in some major mathematics education theoretical frameworks: e.g. TDS; ATD; APOS; MKT; etc.?

- With regard to the local/global tension: Can large-scale surveys (e.g., history of algebra, historical development of geometry, notion of proof from Euclid to Hilbert, evolution of the concept of function, etc.) go beyond the "bird's eye view"? Can we elicit necessary conditions for such large-scale surveys to make any sense? Is the "epistemological narrative" the only way to organize historical material on a large scale?

HiMEd – the student perspective

- Which theoretical perspectives provide fruitful orientation for empirical studies designed to measure students' engagement/learning/etc. of mathematics (when history of mathematics is "used")? What measures are valued in such studies? What methods of analysis can be (should be) employed?

- What is the role of students' mathematical ability (or mathematical interest or prior mathematical experience) in successfully including (elements of) history of mathematics in the teaching of mathematics?

HiMEd – the teacher perspective

- What minimal/satisfactory level of command of history of mathematics can we reasonably attempt to achieve in teacher training?

- Sub-issue: *The able reader*: knowledge of available sources, distinction between primary and secondary sources (more generally, the ability to identify the nature of a source), ability to assess a document with a critical mind, deontological aspects (basically, citing one's sources, indicating alterations when altering a text).

- Sub-issue: *The epistemological toolbox*: what descriptive/analytical concepts do we wish to make available to teachers? Concepts such as: proof-generated concept, zero-definition, conceptual differentiation, analysis/synthesis, epistemic object/tool, etc. Beyond the toolbox, are there "facts" about the "nature of mathematics" that we find we ought to teach (cf. the wealth of literature in physics education on the "nature of science")?

- To what extent should we expose (future) teachers to elements of history of mathematics that have no direct connections with classroom contents (in particular to enrich their "image" of the parts of higher mathematics, which they studied but will not teach)?

HoMEd – the mathematical education landscape

- Lessons from history that can be learned from the construction of the curriculum: Who is the curriculum mainly for? What "big" problems or issues does it aim to resolve? Who benefits most in the short and long run? What are the preferences of areas and topics from mathematics that are being promoted – and why? Who become the developers (and carriers) of the curriculum and how?

- How are mathematical institutions built and are they linked with the new curricula or aims of the society?

- How are cultural values created, narrated, and developed within the new mathematics education landscapes?

- What are the elements by which the tradition in mathematics education, practice, and research is perpetuated?

- What are the outputs of the new curricula/institutions/new mathematical education landscape (the material, the ephemeral, i.e. new values)?

In the following, we give examples of some of the specific questions addressed for each of these four topics. We conclude the report with some selected reflections related to the areas of questions.

SELECTED CONCLUSIONS

TWG12 participants – within both the small group and whole group discussions – had much to offer regarding the several subquestions related to the role of large-scale and small-scale surveys of history of mathematics. Several participants shared the view that using a general survey of history (i.e., "global view") helps to create a cultural landscape, which includes and accommodates multiple tools, concepts, and ideas – and which establishes a meaningful lens to use from the outset. Group discussions during CERME9 consistently returned to the notion that accessing and reading general surveys of history of mathematics provides a good starting point from which to approach resources. However, it was also important to access different types of resources so that practitioners would be equipped to address different views that emerge from history of mathematics in mathematics education. Participants also offered several examples of general survey textbooks and sourcebooks that would serve practitioners.

Given the current educational landscape in several countries, particularly regarding curriculum reform, the participants of TWG12 spent a great deal of time discussing lessons that can be learned from the construction of curriculum over time. When considering the question, *For whom is the curriculum constructed?* participants believed that a country's mathematics curriculum is for the ministers of education. However, this also raised further questions in the group's discussion, such as, *What does 'curriculum' mean?* That is, there are several meanings and contexts that apply and what might be considered 'curriculum' by one may not hold for another. Our group also discussed history in a different way when considering the questions of the fourth topic area. For example, we thought of historical heritage, and questioned whether mathematics curriculum had caught up with what is needed. And, of particular interest to many of the TWG12 participants, we raised the question of: *How can history inform the decisions that are made with regard to mathematics curriculum?*

We look forward to exploring this issue in particular at the next CERME.

TWG12

Research papers

'Mebahis-i İlmiye' as the first periodical on mathematical sciences in the Ottoman Turkey

Mustafa Alpaslan[1], Gert Schubring[2] and Feza Günergun[3]

1 Ahi Evran University, Faculty of Education, Kırşehir, Turkey, amustafa@metu.edu.tr

2 Bielefeld University, Institute for Didactics of Mathematics, Bielefeld, Germany

3 Istanbul University, Department of the History of Science, Istanbul, Turkey

Mebahis-i İlmiye (1867–1869) was the first periodical on mathematical sciences published in 19th century Ottoman Turkey. The authors primarily attempted to introduce the mathematics required for the financial, societal, educational and military development of the country. This paper analyzes the periodical in order to understand (i) the fundamental reasons for its publication, and (ii) transmission of contemporary mathematics from Europe to Ottoman Turkey. Findings unveiled that the periodical had various reasons for mathematics education in the Ottoman society of the time. Moreover, it served the transmission issue to a certain extent.

Keywords: Mebahis-i İlmiye, Ottoman Turkey, history of mathematics education, reasons for mathematics education, transmission of mathematical knowledge.

INTRODUCTION

During the 18th and 19th centuries, the Ottomans transmitted mathematical knowledge from Europe on the grounds that it was the leading region of the time in terms of mathematical sciences. The mathematical knowledge was chiefly conveyed by translating European textbooks. Periodicals were salient in this process too, because editors could compile various kinds of sources, such as excerpts from journal articles, in accordance with the needs of the society. According to Günergun (2007), the earliest Turkish periodical introducing mathematical sciences was *Mebahis-i İlmiye* (Scientific Themes), issued between 1867 and 1869 in Istanbul, capital of Ottoman Turkey. It was published in Ottoman Turkish, which included Arabic and Persian words and was written in Arabic alphabet. The authors aimed to contribute to the teaching of pure and applied mathematics (other disciplines such as astronomy and physics). *Mebahis-i İlmiye* was a monthly publication of *Cemiyet-i Tedrisiye-i İslamiye* (Educational Society for Muslims) which was founded in 1865 by enlightened Ottoman state officials Yusuf Ziya Bey (1826–1882), 'Vidinli' Hüseyin Tevfik Bey (1832–1901) and Ahmet Muhtar Bey (1839–1919) with the goal of enlightening Muslim society by promoting mathematics and science education.

The idea of publishing the periodical belonged to Yusuf Ziya Bey, who was teaching basic arithmetic to apprentices in Grand Bazaar as an accountant in *Daire-i Askeriye* (Military Office) (Zeki, 1924). However, Hüseyin Tevfik Bey, a graduate of Mühendishane (School of Engineering) (Schubring, 2007), committed himself to the publication from the very beginning (Günergun, 2007). He was a remarkable mathematician and had a deep background in education: military attaché and vice-principal in *Mekteb-i Osmani* (Ottoman School) in Paris. During the time of *Mebahis-i İlmiye*, he gave courses on mathematics and mechanics in *Mekteb-i Harbiye* (Military Academy), and he taught mathematics to apprentices as a member of *Cemiyet-i Tedrisiye-i İslamiye*. He was well-known for his book originally titled "Linear Algebra" (first edition in 1882), in which he introduced the general notion of linear algebra independent of the terms 'associative' or 'non-associative' (Schubring, 2007). Given Tevfik Bey's education in the context of traditional Islamic mathematics and his later formation in European mathematics, he was the appropriate person for transmitting European mathematics into the Ottoman context of mathematics. Lastly, Ahmet Muhtar Bey, who was a graduate of *Mekteb-i Harbiye*, was teaching science in the time of *Mebahis-i İlmiye* in this school.

In her historical research, Günergun (2007) asserted that the periodical served the purpose of teaching mathematical sciences for different groups of individuals in the Ottoman society, which was considered as indispensable for the growth of national prosperity. Yet, there may be further reasons for publishing this periodical, which can be unfolded through mathematics education literature. In accordance with Niss' (1996) reasons for mathematics education in a society, this study intends to analyze *Mebahis-i İlmiye* with respect to the motives leading to its publication. Moreover, it aims to clarify the mathematical traditions that the authors relied on in the publication, which will also reveal the transmission of mathematical knowledge from European powers as the 'metropolis' to the Ottoman Turkey as the 'periphery' (Schubring, 2000). Hence, the following research questions are asked:

1) For what reasons did *Mebahis-i İlmiye* provide the 19th century Ottoman Turkey society with mathematics education? How were the reasons addressed in its content?

2) On which mathematical traditions did the authors of *Mebahis-i İlmiye* rely?

According to Niss (1996), the reason why a society is educated in mathematics is "a driving force, typically of a general nature, which in actual fact has motivated and given rise to the existence (i.e. the origination or the continuation) of mathematics teaching within that segment, as determined by the bodies which make the decisions (including non-decisions) in the system at issue" (p. 12). Niss makes a distinction between the reasons addressing *formative ends* serving the individuals' development, and those referring to *practical ends* resulting in practical outputs for the society.

Niss (1996) sets three categories of fundamental reasons for mathematics education: (i) *technological and socio-economic development of society*, (ii) *political, ideological and cultural maintenance and development of society*, and (iii) *providing individuals with prerequisites which may help them to cope with life*. The reasons may be driven by the desire for the societies' own welfare or their effort for competing against the other groups of societies of the time. Related to *technological and socio-economic development of society*, mathematics education aims to train individuals who would serve their country as labor force of high quality. Such an education attempts to develop individuals with high abilities, knowledge and dispositions for performing their role in the society. In other words, it requires individuals to have "knowledge, skills, flexibility, and attitudes so as to allow them to obtain, manage, and develop jobs in the present and in the future" (Niss, 1996, p. 25). This requirement connotes the indispensable relationship between science and commerce, manufacture and industry. Individuals should possess certain general qualifications (i.e., some applicable side of mathematics and geometry such as mensuration) as well as those specifically related to their vocation. The reasons of this first category are expected to produce practical ends. The global mathematics education in the 19th century substantially served this kind of reasons.

The initial needs for a mathematics education have virtually sprung from the reasons under *society's political, ideological and cultural maintenance and development*, for example, to meet the administrative requirements of the society. More precisely, this type of reasons indicate that mathematics education can assist individuals to become nationalist, collaborative, hardworking and dedicated to work for their society. Moreover, it enriches the individuals' mental capability and skills, especially those related to reasoning.

Providing individuals with prerequisites which may help them to cope with life refers to making individuals to acquire the required knowledge and skills for different aspects of their daily lives such as business life, education, personal development and so on. This kind of reasons are valid for addressing practical ends, to exemplify, an individual may confront with a task in his work which requires the application of basic mathematics.

Schubring (2000) defines the notion of 'transmission' of mathematical knowledge as the dissemination process of mathematical ideas from scientifically established 'metropolis' countries of the time to not yet scientifically productive countries in the 'periphery'. In this process, contrary to a received view where by 'transmission', it is understood to hand over concepts remaining identical, traditional and new mathematical knowledge becomes transformed according to the particular national, cultural and societal context of the peripheral country through an active role played by innovative individuals. Thus, in this conception, reception transforms the received. Herein, the reasons behind such transmission are important as well. In order to examine such a transmission process, selecting a country, which was not colonized in the imperialism period, is a good choice because this situation could enable the country to be independent in receiving the new mathematical knowledge.

THE 18TH AND 19TH CENTURY CONTEXT OF THE OTTOMAN TURKEY

Decay strongly felt in the economic, educational, social and military fields was the main cause of the decrease in the political authority that existed in the 18th and 19th century Ottoman Empire. There were wars with neighboring countries and nationalistic upheavals in various communities under the Ottoman rule (Günergun, 2008). The Ottomans fought mostly against European and Russian armies, and most of the wars were lost. The defeats drew the Ottoman administrators' attention to the superiority of the opposing military forces which developed new military organizations (Günergun, 2006). The losses were also linked to complete regression including the abuses in the administrational and financial areas (Somel, 2001).

As a result of the decay, Ottoman Muslim administrators led a variety of necessary reforms in the above-mentioned fields to preserve the borders and to increase the prosperity of the Empire. In this sense, teaching of mathematics was initially modernized through the foundation of modern engineering and military schools in the late-18th and 19th century (İhsanoğlu, Şeşen, & İzgi, 1999). Günergun (2008) states that learning and application of the European scientific, mathematical and technical knowledge behind the military reforms were studied in these military schools which were systematized and tutored by both Ottoman and European (e.g., French, English and German) professors, technicians and experts. The schools focused on reaching contemporary European mathematics and science (Günergun, 2008). Correspondingly, there was a need for new books in which western methods were utilized (İhsanoğlu et al., 1999). The Ottoman administrators and military officers translated and edited European mathematical texts (mostly French) including applied mathematics, geodesy, mechanics, ballistics, and so on (Günergun, 2008). Mathematical studies according to the traditional eastern methods had seriously decreased after 1850. It was clear that these attempts by Ottoman scholars contributed greatly to the formation of the Ottoman mathematical and scientific nomenclature which was formed of words in Turkish, Arabic, Persian and European languages in the 18th and 19th centuries (Günergun, 2006). Therefore, the Ottoman Turkey became a meeting point for the Eastern and the Western cultures of science (Günergun, 2008).

Mathematics education in the modern trend was initiated in 1775 with *Hendesehane* (School of Geometry) (Günergun, 2006). Technical training was first given here for military officers of artillery, fortification and navy. Various branches of mathematics were taught by French and Ottoman experts in the leadership of the French military expert François Baron de Tott. Some mathematical tools were imitated in parallel with the west. The sources for mathematics education were mostly European texts at the beginning. The school was named as *Mühendishane* after 1781. Günergun (2008) states that teachers of this school in the 19th century translated and composed some European mathematics textbooks for teaching, for example, Ibrahim Edhem Pasha's (1818–1893) translation of the geometry book by Adrien-Marie Legendre (1752–1833). Indeed, this was the time for replacing the medieval Islamic sources of mathematics by the modern European ones. *Mühendishane-i Bahri-i Hümayun* (Imperial School of Naval Engineering) was established in 1784. Çınari İsmail Efendi (died, probably 1790) and Gelenbevi İsmail (died 1790), who were teachers in this school, wrote translations and compilations regarding algebra and logarithms considering the European sources in addition to the traditional books on algebra. For instance, Çinari İsmail Efendi translated Cassini and Clairaut's tables (Günergun, 2006). In order to train cadets in military officership and engineering that was necessary for the modern army *Nizam-ı Cedid* (New Order), *Mühendishane-i Berri-i Humayun* (Imperial School of Military Engineering) was founded in 1795. Hüseyin Rıfkı Tamani (died 1818) was a prominent teacher in this school through his translated and edited geometry and engineering textbooks from French and English. For instance, he and Selim Efendi translated John Bonnycastle's (1750–1821) "Euclid's Elements" from English as *Usul-i Hendese* (1797). Another teacher of *Mühendishane-i Berri-i Humayun* was Hoca İshak Efendi (died 1836) who wrote books on arithmetic, algebra, geometry and mechanical drawings based on the recent European mathematics. To illustrate, he used French Etienne Bézout's (1730–1783) works in the first volume of his *Mecmua-i Ulum-i Riyaziye* (Compendium of Mathematical Sciences) (İhsanoğlu et al., 1999). In 1834, *Mekteb-i Harbiye* was founded to train the cadets for the new army. This time a number of Ottoman administrators and teachers were trained in Europe (Günergun, 2006). To illustrate, Hüseyin Rıfkı Tamani's son was Emin Pasha (died 1851), who graduated from University of Cambridge with his doctoral thesis "Calcul de Variation", was assigned as director of *Mekteb-i Harbiye* in 1841.

The modernization process addressed above was not easy to accomplish. There were uproars mainly from *Yeniçeriler* (Janissaries) and *Ulema* (the learned of Islam). The janissaries were opposed to any military reforms because they probably regarded new developments in the military as a threat for their existence. They had further become extremely influential in politics with their conservative character. They created telling crises of modernization, for instance, destroying the institutions of *Nizam-ı Cedid* (e.g., printing house and schools) after the military defeats in Balkans and Arabian geography, and deposing Sultan Selim III in 1807 (Abdeljouad, 2012). In 1826, the reformist Sultan Mahmud II (1808–1839) abolished the *Yeniçeri Ocağı* (Corps of Janissaries) on the grounds that the janissaries had become inefficient as warriors. He established a modern army titled *Asakir-i Mansure-i Muhammediye* (Muhammed's Victorious Army) which completely replaced the rebellious Janissaries in 1826.

FINDINGS

Mebahis-i İlmiye had two volumes formed of issues corresponding to 1867 and 1868. During the publication period, there were main topics serializing a mathematical subject or mathematics problems for the public. In Table 1, main topics in the periodical are illustrated with corresponding author(s) and brief contents.

Table 1 indicates that the topics in *Mebahis-i İlmiye* consisted of collections from both pure and applied mathematics. No primary research existed, hence *Mebahis-i İlmiye* can be characterized as a periodical rather than a journal. The authors apparently reflect-

Main Topics	Author(s)	Contents
Hesab-ı Müsenna (Dual Arithmetic)	Hüseyin Tevfik Bey	Oliver Byrne's dual arithmetic with explanations
Fenn-i Basite (Science of Sundial)	Ahmet Muhtar Bey	Construction and usage of Islamic sundials
Fenn-i Makine (Mechanical Sciences)	Hüseyin Tevfik Bey	The concepts of mechanical operation of machines
Mahsusat ve Gayrimahsusat (Perceptible and Imperceptible Matters)	Hüseyin Tevfik Bey	Physics: motion, movements of the earth, Newton's law of gravitation... Metaphysics: logic, philosophy of knowledge...
Arsa Taksimi (Partition of Lands)	Hüseyin Tevfik Bey	Division of lands with various geometrical shapes
Emsile (Examples)	Anonymous	17 problems asked to the reader and their solutions sent to the periodical
From Public to the Reader*	Anonymous	Problems sent by newspapers and their solutions by the periodical
Vocational Mathematics*	Hüseyin Tevfik Bey & Yusuf Ziya Bey	Mathematical knowledge and skills needed for industry or craft segments to increase effectiveness
Topics from European Science*	Hüseyin Tevfik Bey	Contemporary topics from European mathematics journals
Islamic Contributions to Science*	Hüseyin Tevfik Bey & Yusuf Ziya Bey	Topics from Islamic mathematicians' textbooks (e.g., al-Karaji's proof for $1^3+2^3+3^3+...+n^3 = [1+2+3+...+n]^2$)
Topics from Greek Mathematics*	Hüseyin Tevfik Bey	Diophantus's problem on five equations with five unknowns
Main topics with a '*' mark are categorized by the authors of this study considering the aim and content of shorter papers.		

Table 1: Main topics in *Mebahis-i İlmiye* by the authors and contents

ed their educational background and their lectures in *Mekteb-i Harbiye* when selecting and ordering the main topics such as *Fenn-i Basite* (Günergun, 2007). The periodical included recent European mathematics, ancient Greek mathematics and medieval Islamic mathematics.

Technological and socio-economic development of the society

Some of the articles under Vocational Mathematics* seem to stress the relationship between industry and science. It was stated in Hüseyin Tevfik Bey's *Sanayiinin Muhtac Olduğu Ulum* (Knowledge Needed by Industry) that blacksmiths should master mechanics, drawing, geometry and the relevant computation underlying metal working. Goldsmiths had to know basic chemistry, drawing and geometry. It was also noted that theoretical knowledge provided by science was a must for improving crafts and industrial professions. It is not solely dependent upon practical knowledge. In another article by Hüseyin Tevfik Bey, *Bakırcılık ve Demirciliğe Mütealik Bir Mesele* (An Issue in Copperworking and Ironworking), the optimum ratio between the radius (of the base) and height of a container was given to produce the container with the smallest possible surface area and the least raw material. *Kavaid-i İlm-i Hisab* (Rules of Arithmetic) was the serial of an arithmetic book written by Yusuf Ziya Bey for educating the students of the *Cemiyet-i Tedrisiye-i İslamiye*'s school. The content mainly consisted of numbers (integers, rational numbers, prime numbers, irrational numbers), the arithmetical operations, checking the results of the operations, extraction, ratio and proportion, and equations. This could be for training "'human calculators' for business and commerce" (Niss, 1996, p. 25).

Another relevant main topic was *Fenn-i Makine* (Mechanical Sciences). Based on his lecture notes in *Mekteb-i Harbiye*, Hüseyin Tevfik Bey explained the basic concepts such as velocity, time, rotation, power, resistance, motion, work and efficiency needed for geometrical and mechanical study of machines. He intended to teach the characteristics and working principles of machines and how to construct them.

Arsa Taksimi (Partition of Lands) included common problems for dividing lands of various geometrical shapes and their solutions according to the recent scientific methods of the time. Partition of triangular areas into two or three equal parts and curvilinear areas through integral calculus such as the trapezoidal rule, Thomas Simpson's (1710–1761) rule and Jean-Victor Poncelet's (1788–1867) rule. Exploiting such new European sources in the periodical illustrates the notion of transmitting mathematical knowledge. Moreover, some widely consulted measurement information about surfaces and solid matters was provided for workers.

In *Emsile* (Examples), there were questions which would familiarize the audience with the working principles of some technological tools of the time such as pendulum (#14), lenses (#15) and barometers (#17). These questions were selected for those asked to students of a French lycée, and they indicate an attempt to knowledge transformation.

Political, ideological and cultural maintenance and development of the society

From political and ideological perspectives, the authors emphasize Islamic Contributions to Science* through several articles. For example, *Hasan bin Ali bin Ömer el-Marakeşi'nin "Cami el-mebadi ve'l gayat fi amel el felekiyat" Nam Kitabından Tercüme Olunmuş Bir Meseledir* (A Problem Translated from Hasan ibn Ali ibn Omar al-Marrakechi's Book Titled "Cami el-mebadi ve'l gayat fi amel el felekiyat") by Yusuf Ziya Bey was the translation of a problem on latitude and declination from a book by al-Marrakechi, a Moroccan astronomer of the 13th century. The geometrical method in this problem based on length of shadow and geometrical path did not require observational tools or logarithmic scales. *Bir Zaman Ulema-yı Arabın Malumları Olan Havas-Adaddan Bir Mesele* (A Problem Known by Arab Scholars in Former Times) gave place to geometric justification of the sum of the cubes problem, $1^3+2^3+3^3+...+n^3=(1+2+3+...+n)^2$, which referred to al-Karajī's (953–1023) famous book titled al-Fakhri fi'l-jabr wa'l-muqabala (The Glorious Book of Algebra).

The periodical presented cultural perspectives by means of displaying the interdisciplinary characteristics of mathematics and multi-cultural face of mathematics. To illustrate, mathematics was linked to other disciplines such as astronomy and physics. Real life problems like al-Marrakechi's calculation of latitude and declination can be shown as an example of the link between mathematics and astronomy. As for mathematics and physics, *Fenn-i Muvazene-i Miyah Usulü ile Bir Dairenin Mesaha-yı Sathiyesini Tayin* (Finding the Area of a Circle with Fluid Mechanics Method) under Topics from Greek Mathematics* discussed a law by Archimedes and the related proof. It was about finding the area of a circle which served as a basis for a cylinder whose height was equal to the diameter of the circle. In order to find the area of this circle, the cylinder was filled with water. Then the water was transferred into a cube that had a length of the diameter of the circle. The area of the circle was equal to the product of the length of the cube and the height of the water in the cube. Mathematics in physics could also be distinguished in *Emsile* through some problems regarding the concepts of heat and temperature (#3), mass and density (#5), velocity (#4), and so on.

Mahsusat ve Gayrimahsusat (Perceptible and Imperceptible Matters) contained issues of physics and philosophy. Newton's law of gravitation, which could be considered as relatively new mathematics-related knowledge, was utilized to distinguish between perceptible and imperceptible matters which were explained as that could be observed (e.g., free fall of objects) and could not be directly observed (e.g., gravitational force). Herein, the necessity of asking the possible reasons behind the perceptible matters in daily life was stressed as well, for instance, why balloons fly rather than fall to the ground. Force, absolute and relative motion, the Earth's daily and yearly rotation were also explained in the same manner.

Mebahis-i İlmiye also took up interesting contemporary topics in order to develop curiosity for mathematics revealing the mystery of the universe. Hüseyin Tevfik Bey's *Arıların Peteklerinin Müseddes El-Şekl Olmasının Sebep ve Hikmetine Dairdir* explained why bees made hexagonal wax cells in their nests together with the related mathematical proof. This subject and all the others under Topics from European Science* seemed to contribute to the transmissions from the 'metropolis'.

In order to make the society more intellectual, *Mebahis-i İlmiye* allowed exchange of ideas (Günergun, 2007). The periodical published the answer of *Emsile #3* by Saadet Efendi, a teacher in *Mekteb-i Harbiye*; and the answer of *Emsile #7* by Zeki Efendi as a second grader in the same academy. In the latter, it was notable that Hüseyin Tevfik Bey stated that there might be alternative solutions. The authors of the periodical also published the answer to an interest payment problem by *İstanbul Gazetesi* (Istanbul Newspaper) in From Public to the Reader*. In *Mahsusat ve Gayrimahsusat*, Hüseyin Tevfik Bey criticized Resul Mesti Efendi's essay in a newspaper claiming that the earth does not move. This was in line with the periodical's modern view of science.

Providing individuals with prerequisites of life

Mebahis-i İlmiye included some basic mathematics required for individuals' day to day working life. Under Vocational Mathematics*, *Mesele: Acaba Ayakları Ne Vecihe Vaz Etmekte Ziyade Faide Vardır* (Issue: What is the Effective Position of Legs to Firmly Stand Up) by Hüseyin Tevfik Bey accounted for the ideal geometrical standing position for a soldier when on guard. In *Fenn-i Basite* (Science of Sundial), how to design, construct and use the Islamic sundial was displayed in order to help local timekeepers determine the five prayer times in a day and also the Mecca direction for prayers. Another topic here was *Fenn-i Makineden Dülgerliğe Dair Bazı Mebahis* (Some Issues about Woodworking, A Branch of Mechanical Sciences) in which matters of physics such as force (e.g., direction, magnitude) and resultant force for the construction of poles underlying the construction of wooden buildings were explained. The content of *Mebahis-i İlmiye* was also composed of mathematics serving the individuals' educational life and personal development. *Emsile #14*, *#15*, *#16* and *#17* were mathematical problems taken from French periodicals for lycée students published a decade earlier. Lastly, mathematical problems that would be encountered in everyday private lives were presented, for example, the interest problem in *İstanbul Gazetesine Cevap* (Answer to Istanbul Newspaper).

CONCLUDING REMARKS

Findings indicate that *Mebahis-i İlmiye* addressed all the three kinds of reasons for mathematics education (Niss, 1996) to a certain degree. The authors utilized transformation of the recent knowledge of both pure and applied mathematics from Europe, mainly from France, as the 'metropolis' of the time (Schubring, 2000). Reception occurred in the difficult social setting of conflicts between modernizers and traditionalists, and within the already existing culture of Islamic mathematics. An important aspect of this transmission was the development of a terminology for the modern mathematics in Ottoman Turkish language, since the traditional mathematics did not provide terms for the new developments in the field.

The development of an own terminology is essential for an eventual take-off.

Future research will, on the one hand, focus on identifying the public reading of this periodical and its reception in the 19th century Ottoman Turkey and, on the other hand, introducing an international comparative dimension on the transmission of mathematical knowledge to the other 19th century non-colonized countries (e.g., China) which have original mathematical cultures (e.g., the development of 19th century publications in mathematics). The further comparative studies may assess results of the project, begun in France in 2013, on *Circulations des mathématiques dans et par les journaux: histoire, territoires et publics*. The transmissions of mathematical knowledge from Europe to the Ottomans by *Mebahis-i İlmiye*, which is investigated in this paper, can shed light on a broad systematic investigation of the above-mentioned future research. Moreover, *Mebahis-i İlmiye*'s promotion of mathematics education can enable such a further study to reveal national, cultural and societal motives behind the transmission in a clear way.

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Was Euclid in Iceland when he was supposed to go?

Kristín Bjarnadóttir

University of Iceland, School of Education, Reykjavík, Iceland, krisbj@hi.is

In a seminar on new thinking in school mathematics, held in Royaumont, France, in 1959, one of the main speakers, Jean Dieudonné, summarized the new school-mathematics programme he had in mind in the sentence: Down with Euclid. The purpose of the article is to analyse the context in which this quote was expressed and connect it to geometry teaching in Iceland where Euclidean geometry instruction seldom had a firm ground. Euclidean geometry in an amended version gained new interest in Iceland by the introduction of the New Math in the 1960s.

Keywords: Royaumont seminar, Dieudonné, Euclidean geometry, New Math.

INTRODUCTION

One of the most renowned phrases connected with the Royaumont seminar in November 1959, where the reform movement, entitled New Math, was launched world-wide, was 'À bas Euclide' (Down with Euclid), attributed to Jean Dieudonné, who belonged to the Bourbaki-group. This seminar led to substantial alterations in mathematics teaching and geometry teaching in particular. In the following, some consequences of this reform movement will be considered with special respect to geometry teaching in Icelandic schools. The research questions are:

> In what context was the above quote expressed?
>
> What context did the New-Math geometry meet in Iceland?

The research method is historical: i.e., a careful analysis of a range of documents. The history is traced by referring to scholars' published work, legislation, regulations, reports, articles and mathematics textbooks, and the remembrance of the author of this article. Textbooks were analysed, their forewords as well as their mathematical content, and information about their lifetime was sought in official reports.

The importance of this study is contained in an analysis of some important seeds for development of school mathematics, sowed at Royaumont more than half century ago, but also in an analysis of an example of a dissemination process of mathematical ideas from scientifically established 'metropolis' of the time to a not yet scientifically productive one in the 'periphery' (Alpaslan, Schubring, & Günergun, 2015).

BACKGROUND

The New Math movement and the Royaumont seminar

In the aftermath of WWII, reforms of mathematics and science teaching were considered in many countries. A notable arena was the Commission Internationale pour l'Étude et l'Amélioration de l'Enseignement des Mathématiques, CIEAEM, the International Commission for the Study and Improvement of Mathematics Teaching, founded in 1950. Among its members were the Swiss psychologist Jean Piaget, mathematicians Jean Dieudonné and Gustave Choquet from France, and some outstanding secondary school teachers. The main concern of the CIEAEM was a growing attention to the student and the process of teaching, the relevance of psychology in mathematics education, the key role of concrete materials and active pedagogy, and Piaget's research of the relation between mental and mathematical structures as introduced by the French Bourbaki group of mathematicians, including Dieudonné, called *Mathématique Moderne*, Modern Mathematics (Furinghetti, Menghini, Arzarello, & Giacardi, 2008).

The actions of the CIEAEM, containing important germs of didactic research, were paralleled by the

New Math movement in the United States. World War II had focused national attention on the growing need for trained personnel to serve an emerging technological society (Osborne & Crosswhite, 1970), and several important school mathematics projects were launched. The actions by CIEAEM and the New Math movement had roots in common with the Bourbaki School: set theory, functions, relations and logic were to be placed in the new curricula, supported by the methodology of discovery. The reform movements gathered at a seminar on school mathematics reform in November 1959, held by OEEC, the Organisation for European Economic Co-operation, at Royaumont, France. The member countries were invited to send three delegates each, and the seminar was attended by representatives from all the invited countries except Portugal, Spain and Iceland. A questionnaire was sent out before the seminar and replies were reported from most countries, also from Iceland (OEEC, 1961, pp. 7, 135–140, 213–219).

Down with Euclid!

Among the guest speakers at the Royaumont seminar was Jean Dieudonné from the CIEAEM. In his speech, reproduced in full in the seminar's report (OEEC, 1961, pp. 31–46), he described the diverse curriculum that first year students at university should master: on one hand to be familiar with a certain number of elementary techniques in which it takes a long time to achieve proficiency, and on the other hand be already fairly well trained in the use of logical deduction and have some idea of the axiomatic method (p. 32). In the universities, new developments in analysis had been incorporated in the curriculum. Under the new overcrowded curriculum, most students emerged with the haziest notions about it.

Easing this squeeze could only be done in one way: The curriculum of the secondary schools had to be reorganized to eliminate any undue waste of time. Some elements of calculus, vector algebra and a little analytic geometry had recently been introduced for the last two or three years of secondary school, while such topics had been relegated to a subordinate position, the centre of interest remained as before pure geometry taught more or less according to Euclid with a little algebra and number theory (pp. 33–34). In Dieudonné's opinion, the day of such patchwork was over. Much deeper reform was required, and if he were to summarize the whole programme he had in mind in one slogan, it would be: *Down with Euclid*!

Recently, it had become possible to reorganize the Euclidean corpus placing it on simple and sound foundation – separating what is fundamental from a chaotic heap of results with no significance except as scattered relics of clumsy methods or an obsolete approach. The whole course could actually be taught in three hours: one of them occupied with the description of the axiomatic system, one by its useful consequences and possibly a third one by a few mildly interesting exercises (p. 35).

Actually, the whole system could easily be replaced by an axiomatic system producing two-dimensional linear algebra. The present process of teaching geometry was fantastically laborious, no complete system of axioms was ever stated and it was completely impossible to check the correctness of any proof. Dieudonné suggested the following list to take the place of Euclidean geometry (pp. 37–38):

a) Matrices and determinants of order 2 and 3.

b) Elementary calculus (functions of *one* variable).

c) Construction of the graph of a function and of a curve given in parametric form (using derivatives).

d) Elementary properties of complex numbers.

e) Polar coordinates.

Dieudonné's guiding principles were two: Firstly, that a mathematical theory could only be developed axiomatically in a fruitful way when a student has already acquired familiarity with the corresponding material, i.e. with constant appeal to intuition. The other principle was that when logical inference is introduced in some mathematical question, it should always be presented with absolute honesty without trying to hide gaps or flaws in the argument (p. 39).

In his outline of a modern curriculum, Dieudonné recommended to limit the teaching of mathematics up to the age of 14 to experimental work with algebra and plane geometry and to make no attempt at axiomatization. He referred to recent research and experimentation in educational circles, especially in Belgium, concerning the methods by which this teaching of geometry as a part of physics could be conducted. This development should be highly encouraged, provided it did *not* put the emphasis on such artificial playthings as triangles, but on basic notions such as symmetries, translations, compositions of transformations etc. (pp. 40–41).

The language and notations universally in use, such as $\in$ and $\Rightarrow$, should be introduced in these experimental mathematics as soon as possible, and objects should be called by their proper name like 'group' and 'equivalence relation' whenever such an object was naturally observed in some algebraic or geometric setting. This did not at all imply to develop in advance the abstract theory of those objects. The laws of arithmetic could also be developed, starting from the 'Peano axioms' (p. 41).

Dieudonné proposed detailed programme for age 14 with the idea of a graph of functions; age 15 when a statement of axioms of two-dimensional linear algebra should be given with both algebraic and geometric interpretation, by emphasis on the various linear transformations and the groups they form; age 16 with deeper study of the groups of plane geometry, and in particular the use of angles and of trigonometric functions; and age 17 with three dimensional geometry by use of matrices and determinants. The programme should lead up to and connect directly with the then present programme of the first years in the university (pp. 42–45).

GEOMETRY IN ICELAND

Earlier times

Iceland was a tributary of Denmark since the 14th century. Its population was 40.000–50.000 until the 19th century. There was no army and therefore no military academy and no need to teach geometry for that purpose. The sole Learned School adhered to Danish regulations of the Royal Directorate of the University and the Learned Schools. A certain number of graduates from the school had priority for grants at the University of Copenhagen, (Þorláksson, 2003, p. 382) until Iceland's sovereignty in 1918.

In 1822, Björn Gunnlaugsson (1788–1876), who had studied mathematics at the University of Copenhagen, came to work at the school, to stay there until 1862. In his inauguration speech, Gunnlaugsson emphasized the utility aspect of mathematics education. During summers of 1831–1843, he travelled around the country to make geodetic surveys as a basis for a scientifically drawn map which served as a basis for maps of Iceland into the 1900s. Gunnlaugsson taught the prescribed syllabus of arithmetic, algebra and geometry by textbooks that were stipulated for the learned schools by the Royal Directorate, but remarked in his reports of 1823 that he gave exercises in land-surveying in order to enhance the students' interest. Gunnlaugsson wrote his own textbook in geometry in Icelandic. The yearly cohort of learned-school graduates was 10–12 students which was not enough for such a publication. Danish legislation of 1871 prescribed streaming at the learned schools. The Icelandic school could only offer one stream, language and history stream, and mathematics teaching became severely reduced. Students who wanted to study mathematical subjects at tertiary level, such as engineering, had thereafter to spend an extra year in Denmark. That situation remained until 1919 (Bjarnadóttir, 2007, pp. 87–90, 108, 110–170).

The Jul. Petersen's secondary school geometry textbook

The Danish geometry textbook, *Lærebog i elementær Plangeometri* [*A Textbook in Elementary Plane Geometry*] (Petersen, 1870) was adopted in 1877 in the Icelandic Learned School for the lowest grade where the average age of students was 14 years but could be in the range 13–16 yrs. It remained on the reading list into the 1970s – in translation from 1943 – with breaks in the 1920s while a textbook by Daníelsson (1920) was in use in the late 1930 and in the 1960s during the influence of the New Math reform movement (*School reports for the Reykjavík School*, 1846–1976).

The content of Jul. Petersen's plane geometry textbook is probably typical of European textbooks in Euclidean geometry. In chapter one, several fundamental concepts are listed and the postulate that one and only one line may be drawn through two points. The structure: *Fundamental concepts and their postulates – Definitions – Theorems with proofs*, is explained. In next two chapters, enough definitions and the parallel postulate are presented in order to be able to present theorems and their proofs. Chapter four is devoted to triangles and chapters five to seven to constructions using circles and triangles. Chapters eight to twelve concern angles and arcs, trapezes and parallelograms, the loci of points, similar triangles, and measuring area, with appropriate definitions, theorems, proofs and constructions. One might interpret Dieudonné's speech so that the first three chapters sufficed as geometry teaching. Against that, one could say that objects for exercising proofs on were then lacking, such as the triangles, Dieudonné's 'artificial playthings'.

Even if this textbook managed to survive in the Reykjavík School for a century, it had notable criticism. In Denmark, the textbook was intended for the so-called *Mellemskole* [middle school] for age 11–14 (Hansen, 2002, p. 40). A reviewer said about Petersen's 1905 edition:

> … one reads between the lines the author's disgust against modern efforts, which … deals with making children's first acquaintance to the mathematics as little abstract as possible by letting figures and measurements of figures pave their way to understanding of the geometry's content … Working with figures … aids the beginner in understanding the content of the theorems, which too often has been completely lost during the effort on 'training the mind'. If the author knew from a daily teaching practice, how often pupils' proofs have not been a chain of reasoning but a sequence of words, he would not have formed his introduction this way … for the middle school it [the textbook] is not suitable.[1] (Trier, 1905)

In Petersen's obituary in 1910 it said:

> … People began to realize that the advantages of these textbooks were more obvious for the teachers than for the pupils … the great conciseness and left-out steps in thinking did not quite suit children. These books were excellent when the whole syllabus was to be recalled shortly before examination, but if the students were to acquire new material one had to demand a wider form for presentation. (Hansen, 2002, p. 51)

A student at the University of Copenhagen, Finnur Jónsson, later philology professor, wrote in 1883, criticizing Reykjavík Learned School and its regulations:

> … the new regulations have [prescribed] … that the [geometry] study is to commence in the first grade; in order to grasp it, more understanding, more independent thought is needed than first-graders master … [I] tutored two [first grade] boys in geometry … and for both of them it was very difficult to understand even the simplest items; but the reason was that they did neither have the education nor the maturity of thought needed to study such things, which is very natural. (Jónsson, 1883, p. 116; underlining KB)

The pupils of the Learned School were sons of farmers, priests and other officials who also made their living from farming, so the majority of the pupils came from farming communities where there were no primary schools. The novices were prepared for school at home and by priests in Latin, Danish and basic arithmetic, and had seldom met geometric concepts. Land was e.g. not measured in square units. Dieudonné's first guiding principle was that a mathematical theory could only be developed axiomatically in a fruitful way when a student had already acquired familiarity with the corresponding material, i.e. with constant appeal to intuition. This is in accordance to Jónsson's remark that the pupils did not possess "the maturity of thought" needed to study deductive geometry as presented in Jul. Petersen's textbook. The young pupils had not acquired familiarity with the corresponding material with the appeal to intuition that Dieudonné recommended.

Daníelsson's high school geometry textbook

The textbook *Um flatarmyndir* [*On plane geometry*] by Ó. Daníelsson (1920), intended for novices at the six-year Reykjavík High School, around age 14, may be interpreted as strictly adhering to the Euclidean tradition. It began by a section on limits to prepare proving the existence of irrational numbers. Next section was a list of definitions. The author admitted in his foreword that his experience was that students were relieved when that section was completed. The third section was on parallel lines, followed by exercises whereof there were five on computing angles, one of them in the hexadecimal system, and all exercises after that through chapter six out of fifteen, were on proving on the basis of the definitions and theorems introduced. Following exercises were alternatively on constructions and proving, and computations by recently proved formulas, such as Heron's formula on area. Eventually, *On plane geometry* was transferred up to the upper level. Geometry was again required for novices in 1937. From that time, Danish textbooks were translated, among them Peterson's *Geometry* in 1943 (*School reports*, 1846–1976).

1 Translations of quotes were made by the author of this paper.

The first two grades were dropped from the high school level in 1946 and after that there was no Euclidean geometry below the age of 16. There was shortage of trained mathematics teachers who had to seek their training abroad, traditionally in Denmark with which all connection was broken during World War II. Only five high school teachers in the whole country had graduate degree in mathematics in 1959. Training of engineers at University of Iceland began in 1940 due to the broken connection to Denmark. Mathematics had not been taught before at the university. Mathematics teachers might be trained in engineering or natural sciences. In the 1960s, the high schools had to cope with up to ten times as many students as at Daníelsson's time, beginning their mathematical training at the age of 16 by studying Petersen's (1943) *Plane Geometry* (Bjarnadóttir, 2007; *School reports*). The experience of the author of this article in 1959–1960 was that the main emphasis laid by a geologist teacher was on construction, the scattered relics of clumsy methods, according to Dieudonné, and the axiomatic structure of the content was scarcely visible.

Other school levels

The first primary school legislation in 1907 contained requirements on knowledge in computations of area and volume of common objects. These requirements were repeated in national curriculum documents issued in 1929 (*Námsskrá fyrir barnaskóla*, 1944) and 1960 (Menntamálaráðuneytið, 1960). By the introduction of the New Math, a draft national curriculum was made, but when it came to geometry, the authors claimed that experience was lacking to build geometry on (*Drög að námsskrá*, 1970). So, indeed, the only geometry taught at compulsory school level before the introduction of the New Math was mensuration, computing area and volume.

The studies at the University of Iceland were tailored after the Technical University in Copenhagen. The mathematical subjects were mathematical analysis and linear algebra and no Euclidean geometry, but they surely built on the high school training.

The New Math in Iceland

For the Nordic countries the Royaumont Seminar was a catalysing event. The Nordic participants agreed upon organising Nordic cooperation on reform of mathematics teaching. A committee, *Nordiska kommittén for modernisering af matematikundervisningen* (The Nordic Committee for Modernizing Mathematics Teaching), abbreviated as NKMM, was established. The committee produced model textbooks which were then translated into the various Nordic languages. Iceland did not have a member in that committee but learned about its activities through personal contacts of high-school and university mathematics teachers G. Arnlaugsson and B. Bjarnason with Svend Bundgaard who was guest speaker at Royaumont. Arnlaugsson and Bjarnason were the leaders of the introduction of New Math in Iceland (Bjarnadóttir, 2015). Their choices of textbooks for mathematics-teacher training witness that they were aware of Dieudonné's recommendations.

Bjarnason chose a Danish textbook: *Matematik 65* (Christiansen and Lichtenberg, 1965), for a special course to train high school mathematics teacher students, the first time a course of its kind was run in the academic year 1966–67 for only three students. Other courses were part of a programme for engineering students. Section V of *Matematik 65* concerned questions from geometry. The authors remarked that around the last turn of the century, David Hilbert had succeeded in composing such a system of axioms that could follow Euclid's thought and solve all Euclid's unsolved problems at the same time (pp. 309–311). They also mentioned Gustave Choquet's system of a so-called transformation geometry with few but strong axioms: 5 undefined concepts (plane, point, line, distance function and order relation) in a set-theoretical presentation; and 4 axioms (10 in total with sub-axioms): axioms of incidence, axioms of order, axioms for affine structure, and a folding (symmetry) axiom (Christiansen and Lichtenberg, 1965, pp. 312–320; Choquet, 1969, pp. 17–75). Choquet was also member of CIEAEM and guest speaker at Royaumont. Dieudonné may have referred to his work in that recently it had become possible to reorganize the Euclidean corpus, putting it on simple and sound foundation.

For the training of teachers at primary level, one of the three teacher students was entrusted to give a course in the New Math style in 1967. Bjarnason and Arnlaugsson chose a Danish textbook on geometry (Anderson Bo, Nielsen and Damgaard Sørensen, 1963) which was built on basic notions such as symmetries, translations, compositions of transformations, etc., as Dieudonné suggested. For the more advance students, an American textbook by Schaaf (1965) was chosen. These three books for training mathematics teachers at different stages were not in use for a long time, however. Not many educators were ready to interpret them, and the educational system was in flux. The training of compulsory school teachers was transferred to tertiary level and reorganized, as was the training of high school teachers. Arnlaugsson and Bjarnason became principals for new modern high schools and did not work further on promoting the New Math (Bjarnadóttir, 2015).

Some products of the NKMM for the primary and lower secondary school levels were translated into Icelandic. Primary level mathematics textbooks, written by Agnete Bundgaard, Svend Bundgaard's sister, and E. Kyttä, were translated year by year, beginning in experimental edition in 1966 (Bundgaard and Kyttä, 1967–1972). It had not reached the geometry in volume 5 when a draft national curriculum (*Drög að námsskrá*, 1970) was published, and people therefore did not know how to present New-Math geometry for primary level. For the lower secondary level the NKMM *Rúmfræði* [*Geometry*] by Bergendal, Hemer and Sander (1970) was translated with foreword by Arnlaugsson. It was based on set theory with e.g. lines defined as sets of points, but no axiomatization. Both products provided teachers with new ideas about geometry for compulsory level while both were in use for less than a decade.

During a six-year period, 1967–1973 various texts with new topics were tried in Iceland in order to replace Petersen's Geometry in high schools, such as NKMM-texts with emphasis on vectors, functions and their derivatives, and *Book T4* in the British SMP series (School Mathematics Project, 1966), which aimed at linking algebra and geometry by vectors and matrices to present transformations and their combinations, with a final chapter on algebraic structure of matrices and examples of groups of transformations. These texts included trigonometry. Influences from Royaumont were thus channelled to Iceland through various routes: from the Nordic countries, the United States and from Britain.

SUMMARY AND CONCLUSIONS

We may understand from Hansen (2002) and Dieudonné's presentation at Royaumont that it was customary in Europe to teach axiomatic Euclidean geometry to young children, even 11–13 year old. Dieudonné's reaction, such as to limit the teaching of mathematics up to the age of 14 to experimental work with algebra and plane geometry, and to make no attempt at axiomatization, must be considered in that context. This was less the case in Iceland. It may though be spotted in Jónsson's (1883) criticism on the teaching of geometry in the Reykjavík School. However, from *School reports for the Reykjavík School* (1846–1976) one may gather that Eucliean geometry was most of the time transferred from the beginners' stage at age 14 up to age 15 or 16. While the Reykjavík School was small, only enrolling 25 students a year, and Daníelsson was the head teacher in 1919–1941, Euclidean training may have been considerable, but less so later when the number of students increased out of proportions to trained mathematics teachers. The axiomatic structure of geometry was thus not much visible in the peripheral Iceland before the New Math reform as only few teachers were capable to interpret Petersen's century old textbook successfully with respect to an axiomatic system in the 1960s.

Within a six years period, 1967–1973, geometry, modernized in the spirit conveyed by Dieudonné and Choquet, had been implemented in the teacher training and at all school levels. Euclid might thus be interpreted to have arrived in Iceland in Choquet's modified versions, at least in the teacher training, at the time of the claim 'Down with Euclid'. The conclusion is therefore that Euclidean geometry was revived in Iceland by the New Math movement.

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Why is it difficult to learn from history?

Renaud Chorlay

Université Paris Diderot, ESPE de Paris – IREM & LDAR, Paris, France, Renaud.chorlay@espe-paris.fr

This paper is of a methodological nature. First, it aims at spelling out several structural differences which may stand in the way when researchers in mathematics education endeavor to derive didactically relevant information from the history of mathematics as written by today's historians. Second, the core of the paper aims at illustrating what historians actually do, with a methodological focus on the notions of "agency" and "puzzle-solving".

Keywords: History, epistemology, methodology, Euclid, Descartes.

In a paper of 1990 entitled *Epistémologie et didactique* (Artigue, 1990), Michèle Artigue reflected on 10 years of practice within the French mathematics education community, while stressing the *need* of epistemology for the working researcher. First, she underlined the need for epistemological awareness as an *experience* for the researcher, enabling him to distance himself from his personal mathematical culture; second, she pointed out that some *knowledge* of the history of mathematics was of a key component of didactical research, either to understand the historical development of some mathematical concept, or to understand the shaping of mathematics as a ruled cultural activity.

This paper will, to a large extent, directly echo this Artigue paper, and is hence somewhat dependent on the French context [1]. For these reasons, we will here use the adjective "epistemological" to denote the *endeavor* to derive from knowledge/awareness of the history of mathematics some insight that is relevant from a mathematics education research perspective. As Artigue did, we will focus on issues of method, although with a shift of emphasis. Instead of focusing directly on didactical concepts, we will mainly discuss research practices at the intersection of two autonomous fields of knowledge: math education research on the one hand and history of mathematics on the other hand; in this context, "history of mathematics" will denote the outcome of the work of historians of mathematics.

After spelling out some basic structural differences (and some similarities) between the two fields of inquiry, we wish to present several examples in order to illustrate what historians actually do. These examples will enable us to highlight two methodological aspects: the focus on agency (how actors engage with mathematics), and the riddle-solving (or puzzle-solving) aspect of historical research (which we would like to contrast against the "erudite description" view some may have of historical research). We wish to provide a basis for further methodological discussion, and for ever more fruitful interactions.

STRUCTURAL DIFFERENCES BETWEEN TWO FIELDS OF RESEARCH

RME (Research/researchers in mathematics education) and HM (history/historians of mathematics) are two different and autonomous disciplines: each one has its own empirical field of investigation, its own set of legitimate questions, its own way of validating claims, its own reference works etc. That fact may be self-evident; however, we feel this fact should be taken into account seriously in order to pave the way for fruitful collaborations. It is also a fact that RME and HM have often been speaking at cross purposes: when HM read what RME say about the history of mathematics, the typical reaction goes: "this is not history, but a sketchy reconstruction of history framed within a-historical categories; what really happened is really much more complicated than that, you know …"; which RME are usually fully willing to acknowledge while wondering why historians would deny them the right to make *heuristic* use of HM in a preliminary phase to their main investigation. For them, learning *about* history (which one of the things historians do) is a means to learn something *from* history (which is not what historians do). Reciprocally, RME are sometimes surprised by the lack of theoretical frameworks in the work of historians, since such frameworks provide the main tools to describe and analyze specific issues; and enable researchers to integrate their particular study to a growing and soundly-structured body of knowledge about the learning of mathematics in current educational contexts. Historians do not usually rely on explicit theoretical frameworks.

The purpose of this paper is not to claim that these usual misunderstandings are only the result of the relative isolation of the two communities, and that they would soon fade away if everyone decided to work together with an open mind. Quite the contrary, we think these misunderstandings point to differences which are *structural*, and our purpose is to sketch ways of living with this fact.

Since the intended audience of this paper is that of RME, we would like to point to some elements which show what historians actually do. Of course, our approach is descriptive and not normative.

REM and HM have at least this in common: contrary to what research mathematicians do, the object of their investigation is not *mathematics*, and this object is not studied primarily *mathematically*. Rather, they study how agents *engage* with mathematics, in a context which can be described; mathematics is necessary to make sense of this engagement and this context, but cannot possibly be the only background tool

Beyond this common agent-based approach, dissimilarities become striking: MER study learners, HM tend to focus on experts (of course, both parts of this statement call for qualification). MER has direct access to the living agents it studies, which means empirical data can be gathered, hypotheses can be put to the test in finely-tuned conditions, and cognitive processes can be investigated; HM have indirect access to the agents they study, and it is part of the trade to attempt to assess what biases it entails (critique of sources, careful methodological reflection on corpus delineation etc.). HM have to deal with events which happened once, but can be understood, compared and, to some extent, fit into narratives; MER has an experimental side to it, and can aim for invariants and reproducibility.

The fact that historians depend heavily on the availability of sources and do not explicitly rely on theoretical frameworks does not imply that their work is purely descriptive and erudite. To use Kuhn's phrase, a historians *solve puzzles*, just as any researcher does, whatever the field. We would like to illustrate this *agent-based*, *puzzle-solving* approach from three different angles.

A SHARED INTEREST IN MATHEMATICAL AGENCY

First, let us mention the kind of questions that historians aim at tackling. A very general and context-free list of questions can be found, for instance, in Catherine Goldstein's methodological paper (Goldstein, 1999, p. 187, trans. RC):

> At a given period in time, what were the networks, the social groups, the institutions, the organizations where people practiced mathematics or engaged with mathematics? Who were mathematicians? In what conditions did they live; in what conditions did they carry out mathematical work? How were they educated and trained? What did they learn?
>
> Why did they work in mathematics, in what preferred domain? What did this domain mean to them? (...) Where did mathematicians find problems to be solved? What were the form and origins of these problems? Why was some result considered as very important, or of lesser importance? According to which criteria? What was considered to be a solution to a problem? What had to be proven, and what did not require a proof (tacitly or explicitly)? Who decided so? When was a proof accepted or rejected? When was an explicit construction deemed indispensable, optional or altogether irrelevant?
>
> When, where and how mathematics were written? Who wrote, and for whom? For instance, were new results taught, were they printed, were they applied? What got transmitted? To whom was it transmitted, in which material and intellectual conditions?
>
> What changed and what remained fixed (and according to what scale, to which criteria)?

The variety of structural differences between history and didactics does not imply that no questions may be shared, in particular when one focuses on agency.

For instance, Goldstein's list strikingly echoes the list of questions which Guy Brousseau considered to be meaningful for RME when attempting to derive didactically-relevant insight from a study of mathematics from the past. Discussing Georges Glaeser's paper on the epistemological obstacles relative to negative numbers (Glaeser, 1981), he critically summed up Glaeser's "obstacle" approach, and then pointed to what he would consider to be the more relevant questions (as cited in Artigue, 1990, p. 252, trans. RC):

> This formulation shows what failed Diophantus or Stevin, seen from our time and our current system. We thus spot some knowledge or possibility which failed 16th century authors and prevented them from giving the "right" solution or the proper formulation. But this formulation hides the necessity to understand by what means peopled tackled the problems which would have required the handling of isolated negative quantities. Were such problems investigated? How were they solved? (...) What we now see as a difficulty, how was it considered at the time? Why did this "state of knowledge" seem adequate; relative to what set of questions was it reasonably efficient? What were the advantages of this "refusal" to handle isolated negative quantities, or what drawbacks did it help avoid? Was this state stable? Why were the attempts at changing it doomed to fail, at that time? Maybe until some new conditions emerge and, some "side" work be done, but which one? These questions are necessary for an in-depth understanding of the construction of knowledge [*pour entrer dans l'intimité de la construction de la connaissance*] (...).

In both list, we can see that a focus on agency does not mean that the object of study is a freely creative cognitive agent. Quite the contrary. Agents are born in a world which pre-exists, and constrains their actions. When it comes to mathematical activity, constraints come from a great variety of sources, ranging from the material environment (a Chinese abacus is not an electronic calculator) to epistemic values (rigor, generality, simplicity, accuracy, applicability, etc.) and epistemic categories (definition, justification, proof, example, algorithm, analysis/synthesis, principle, etc.). The historical contingency of these constraints does not imply that they have no a-historical components, be they mathematical properties (a rule such as "minus times minus equals minus" is not compatible with distributivity of × over +) or semiotic properties (an algebraic shorthand with no parentheses – such as Cardano's – has different properties from Bombelli's). Making historical sense of how actors engage with mathematics involves understanding how they act *within* a given set of constraint, what *meaning* they give to their actions, and in what respect these actions *alter* the system of constraints.

FOCUSING ON AGENCY: USES OF A DIAGRAM

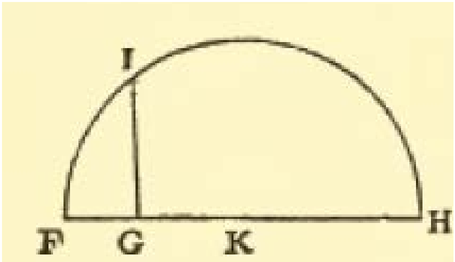

Figure 1

Let us now flesh out this notion of agent-based approach – this focus on mathematical agency – from another angle. We will use the example of this diagram to illustrate several methodological points.

The very same diagram (Figure 1) appears in two of the most influential works in the history of mathematics: Euclid's *Elements* and Descartes' *La Géométrie*. One could argue that not only the diagram is the same, but the mathematical content is the same; however, the parts these diagrams play in both works are strikingly different.

In Euclid's *Elements* (Heath, 1908), this diagram comes with proposition 14 of book II; a proposition which solves the following construction problem: to construct a square equal (in area) to a given rectangle. If the sides of the rectangle are equal (in length) to FG and GH, then the perpendicular IG is the side of the sought-for square, which Euclid proves using proposition 47 of book I (which we call Pythagoras' theorem). At the end of book I, a series of proposition established that, for any given polygon, a rectangle with the same area could be constructed (with ruler and compass only), hence prop. 14 provides the final positive solution to the problem of quadrature of polygons (i.e. to transform any polygonal area into a square). In turn, this fact implies that – at least for polygons – area is a well behaved magnitude: areas can be compared (since square areas can), and added (since the Pythagorean construction provides a means to add square areas). On this basis, a modern reader would conclude that a theory of measure in possible for polygonal areas; the modern reader also knows that this requires the set of real numbers. Euclid was very well aware of the fact that the theory of well-behaved geometrical magnitude (even line-segments, for which comparison and addition are straightforward) requires more than natural numbers and their ratios. The solution he presented in book V is a number-free solution, based on the notion of ratio of magnitudes and not of measure. The positive result of II.14 also points to open questions in the theory of magnitudes, in particular the extension of the theory beyond the case of polygons (the case of the circle being of prime importance).

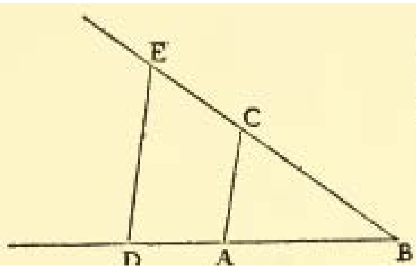

Figure 2

The same diagram (Figure 1) appears on the second page of Descartes' *La Geométrie* (1637). Along with another diagram (Figure 2), he aims at defining operations on segments; operations for which he would use the same names as for arithmetical operations. In Figure 2, if AB denotes a unit segment, then BE will be called the "product" of segments BC and BD. In Figure 1, if FG is the unit segment, then IG will be called the "square root" of GH. Descartes then adds that he would not only use the same names as those of arithmetical operations, but that he would also resort to the same signs as in algebra: letters for segments (known or unknown), symbols such as $\times$ and $\sqrt{}$ for the above mentioned constructions. The project is to use the means of algebra (rewriting rules, elimination in simultaneous equations, identification in polynomial equalities, method of indeterminate coefficients) to capture and analyze geometrical relations between segments; among such relations, those expressed by one equation in two unknowns capture plane curves.

This specific Cartesian project is quite different both from Euclid's, and from what we call either algebra or coordinate geometry. In the *Elements*, prop. II.14 solved an *area* problem; in terms of magnitudes, considering two line-segments could lead either to a new segment (by concatenation, which can be seen as a form of addition), or to an area (that of a rectangle, which can be seen as a form of multiplication), or to a ratio (which is not a geometrical entity, but not a number either). On the contrary, Descartes uses elementary construction (with an *ad hoc* unit segment) to define operations such as "time", "divide" or "root" as internal operations within the domain of segments; this enables him to make free use of algebraic symbolism while warranting geometrical interpretability. This system, however, involves no global coordinate system; it does not even involve coordinates, since no numbers play any explicit part in the system.

The fact that Descartes' system is an algebra of segments has other far-reaching consequences. Let us mention one of general epistemological importance. At first, when we read in *La Géometrie* that the solution of equation $z^2 = az + b^2$ (z being unknown, a and b known) can be expressed by formula:

$$z = \frac{1}{2}\mathrm{a} + \sqrt{\frac{1}{4}aa + bb},$$

we feel we are on pretty familiar ground. However, we need to recall that this formula is not a symbolic summary for a list of arithmetical operation on numbers, but is a symbolic summary for a geometrical construction program; a ruler-and-compass construction program which involves two concatenations, two multiplications (Figure 2), the construction of a "square-root" segment (Figure 1), and three midpoints. This, in turn, means that the algebraic manipulation of formulae and equations deals with the transformation and comparison of geometric construction programs. Here, the comparison with Al-Khwarizmi (ca 820 CE) is striking:

> Roots plus numbers equal squares; for instance, when you say: three roots and four in numbers equal one square.
>
> Procedure: halve the number of roots, you get one and one half; multiply it by itself, you get two and one quarter; add four, you get six and one quarter; take its root, which is two and one half, add it to half the number of roots, that is one and one half, you get four, which is the root of the square, and the square is sixteen. (Rashed, 2007, p. 106. Trans. RC)

With its purely rhetorical algebra and its use of generic examples, this excerpt from Al-Khwarizmi's *Algebra* may look less familiar than Descartes' formula. However, it presents a *bona fide* list of operations which enables one to solve an equation, in a numerical context. In the rhetorical context, algorithms are easy to express, but not so easy to compare, transform and calculate upon. One of the properties of Descartes' system is that its symbolic algebra allows for calculation to operate on algorithms; the fact that the basic steps of the algorithms involved are ruler and compass constructions and not numerical operations is irrelevant, and testifies to the meta-level function of symbolic algebra.

This interplay between the familiar and the not-so-familiar (yet understandable) may feel disorienting at first, but this disorienting effect is a positive effect, as Artigue stressed. It has a *critical* function, helping the researcher to distance himself from his own mathematical culture ; and a *heuristic* function, suggesting new viewpoints on seemingly familiar notions, for instance on the role of symbolism in algebra, or the role of real numbers in geometry (as measures and as coordinates). At least two other functions can be mentioned. First, it helps identify *problems* to which there are no straightforward answers. For instance, what should we consider to be the geometrical analogue of multiplication, at least for one-dimensional objects? In particular, should the analogue of the product be one-dimensional or two-dimensional? A long series of different – yet mathematically sound – constructions provide different answers to this question, including dimension changing solutions (going down with the dot product, or up with the exterior product). Secondly, it helps question the notion of *identity*. It could be argued that, from a purely mathematical point of view, Euclid and Descartes rely on the same content associated to Figure 1; this probably makes sense, but it is probably not very helpful, either to the historian of to the RME. Indeed, researchers in both fields aim at analyzing how content depends on – for instance – semiotic resources, or intended use.

To finish with this Euclid-Descartes example, we would like to make plain what it took to come up with such an example. On the one-hand it is relatively small scale: we did not need to include it in any large-scale narrative on the "stages" in the history of geometry for this sketchy comparison to serve the four above listed functions of epistemological inquiry; on an even smaller scale, the comparison with a short passage of Al-Khwarizmi could play a relevant part even with no background "big picture" on the history of algebra, or even on the *Kitāb al-Jabr*. On the other hand, to compare the uses of the same diagram required that its role in the whole structure of the works (the *Elements*, and *La Géométrie*) be analyzed. It requires some knowledge of history to make sense of highly sophisticated but largely forgotten theoretical constructs such as the classical theory of ratios, or the 17th century research program of construction of equations. This knowledge cannot derive from a quick look at short extracts from the original sources, and probably not even from a long look at the whole books; here we depend on professional historians such as (Netz, 1999) for Euclid, and (Bos, 2001) for Descartes.

SOLVING PUZZLES, DESIGNING RESEARCH-QUESTIONS

Finally, we would like to illustrate the fact that erudite analysis is not all there is to historical research. Finding answers and grounding answers through erudite analysis of documents come only in a second phase; in a first phase, historians strive to identify challenges, and craft non-trivial (and possibly innovative) questions. Let us give five pretty different examples.

In (Proust, 2012), Christine Proust studies the algorithm displayed in paleo-Babylonian tablets when working out the reciprocals of large numbers in the sexagesimal system. The clay-tablets display instances of calculations, but no general descriptions of the method (much less any justifications), which is why historians endeavor to come up with reconstructions of the algorithm. Pioneer in the history of Babylonian mathematics Otto Neugebauer (1899–1990) reconstructed an algorithm on the basis few tablets; an algorithm which required that additions be used along the way. However, in the *floating point* sexagesimal number system, and in the purely numerical context of these tablets, addition is not possible (whereas products and reciprocals make perfect sense)! On the basis of a much larger sample of tablets, Proust reconstructed a different algorithm; one which is fully compatible with a floating point arithmetic.

In his now classic *The Shaping of Deduction in Greek Mathematics* (Netz, 1999), Reviel Netz attempted to re-historicize the endeavor of the Greek mathematicians of the classical and Hellenistic periods, and help us question many things we implicitly take for granted. This difficult for two at least reasons: first, some of the basic elements of practice displayed in these texts – in particular, the practice of discussing lettered diagrams using only explicit axioms and formerly established results – is so familiar to us that we cannot imagine how a few men strove took to establish this specific cultural form on the background of other cultural activities. Second, because we feel we know that mathematics was a central intellectual activity in these periods, as many texts of Plato and Aristotle seem to indicate. Netz established that it was *not* the case, and discussed why Plato and Aristotle distorted our perception of historical realities. In a stimulating review of this erudite book, Bruno Latour emphasized the extent to which it echoed central methodological trends in the social history of science (Latour, 2008).

The question of the circulation of mathematics between different cultural areas – and not only different periods – is also a central field of investigation. In (Chemla, 1996), Karine Chemla discussed the introduction of "western" mathematics in 17th century China by Jesuit missionaries. It was usually thought that, in this period, the indigenous Chinese tradition of mathematics was to a large extent forgotten in China, and that western mathematics had been adopted passively. Actually, studying the Chinese sources leads to a more nuanced picture. In particular, when Jesuit Matteo Ricci and Chinese scholar Li Zhizao collaborated to write a treatise of arithmetic based on Clavius, they ended up with much more than a translation: Li added many elements from the indigenous tradition, in particular the *fangcheng* algorithm to solve simultaneous linear equations (similar to Gaussian elimination). This work of synthesis did not stir interest in the West; in China however, the introduction of western mathematics revived scholarly interest in classical Chinese mathematics, and triggered comparative studies of both traditions.

It is well-known that for the founders of the calculus the prime goal was the study of curves defined by ordinary differential equations, in a geometrical of physical context. Pen and paper, and formulaic solutions were not all there was to it, as is demonstrated by the deep and original work of Dominique Tournès. In (Tournès, 2003), he studied the intense work on graphical methods and graphing devices carried out from the very beginning (Leibniz, Newton, Jean Bernoulli, Euler), up until the advent of digital instruments in the second half of the 20th century. This work brings to light a great wealth of largely forgotten mathematical ideas and techniques; shows the continuity between the algebraic research program on the "construction of equations" (as in Descartes) and the late-17th and 18th century researches on ODEs; and documents the deep connections between the most theoretical considerations on the one hand, and the demand for effective approximation methods in the engineering communities on the other hand.

The distinction between "local" and "global" is now standard in the scholarly mathematical world, but it was not always the case. Studying the emergence of an explicit local-global articulation is tricky for a number of reasons: it concerns more or less all mathematics; the *meta* level terms "local" and "global" have definitions which differ in every specific mathematical context; actually they can be used with no definitions at all. Moreover, the question of the *explicit* is crucial. When, at the turn of the 20th century, some mathematicians began to explicitly express a such distinction, was the general context one in which it was actually clear to everyone that this mattered (though it went without saying), or one in which no clear distinction was made between local and global statements, resulting in a wealth of faulty proofs and ambiguously-worded theorems? These questions were addressed in (Chorlay, 2011), who provided answers based on a combination of quantitative and qualitative methods.

CONCLUSION

This short list of examples illustrates how historians endeavor to design non-trivial questions; what means they use to answer these questions; and what kind of answers they tend to consider relevant and innovative. Although we feel historians provide a great wealth of material of prime interest for mathematics education research, it is a fact that they do not usually provide this material in a form which directly meets the needs or wishes of the mathematics education research community.

To further this paper, we can identify at least three avenues for research: (1) to analyze papers in mathematics education research which depend heavily on historical analysis, such as (Sierpinska, 1985), (Katz, 2007) or (Dorier, 2000); (2) to discuss the relevance of standard conceptual tools such as "epistemological obstacle", "(mis)-conceptions", "historical genesis"; (3) to review the main theoretical frameworks in mathematics education research in order to identify which (if any) role they assign to epistemological or historical investigation.

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ENDNOTE

1. An altered version of this paper makes up the first, introductory part of a chapter in a collective volume in the honour of M. Artigue (scientific editors: B. Hodgson, J.-B. Lagrange, A. Kuzniak).

The contribution of history of mathematics on students' mathematical thinking competency

Kathleen Clark

Florida State University, School of Teacher Education, Tallahassee, USA, kclark@fsu.edu

This paper presents a small subset of results from a pilot study conducted at a small private university in 2012. The study sought to identify changes in students' mathematical thinking competency. In this paper, I present summaries of pre-instruction and post-instruction think-aloud interviews for one of 18 items on the think-aloud instrument, for two of the participants. I propose that history and philosophy of mathematics courses do have the potential to impact students' mathematical thinking competency and further studies such as the one presented here must be undertaken to expand what we know and how we can use the knowledge to enhance the teaching and learning of mathematics.

Keywords: History and philosophy of mathematics course, think-aloud protocol, mathematical thinking competency.

INTRODUCTION

In the past, much of the available research on history of mathematics courses has been primarily focused on how to teach such courses (e.g., Miller, 2002). Recently, however, scholars have attempted to "describe ways in which a history of mathematics course can help prospective teachers of mathematics develop knowledge they will need for teaching" (Huntley & Flores, 2010, p. 603). Although investigations into the development of mathematical knowledge for teaching constitute important contributions, there exists a need to examine the potential impact of a history of mathematics course on the more general undergraduate mathematics major population. I propose that what is missing from the research literature regarding the alleged influence of the history of mathematics is inquiry on the actual impact on learning, or as considered here, on students' mathematical thinking competency as a result of studying mathematical ideas from the perspective of the historical and philosophical development of those ideas.

The field of mathematics education boasts fewer empirical examples about how the study of the history and philosophy of mathematics contributes to student thinking about and understanding of essential mathematics concepts than it does theoretical studies on the same. The outcomes of the study discussed here hold promise for increasing awareness of what the history of mathematics contributes to learning and thinking about mathematics (if anything) and for contributing something further to the empirical base of previously well-known theoretical studies.

Based upon research that I have conducted with prospective mathematics teachers (Clark, 2012), I conjectured that the think-aloud task interview results would in fact reveal changes in the undergraduate students' mathematical thinking competency, using the definition provided in Danish KOM-report (Niss & Højgaard, 2011, pp. 52–53). In particular, I am interested in identifying evidence for which participants engaged in different aspects of this competency, such as:

> ...being able to *recognise, understand* and *deal with the scope of given mathematical concepts* (as well as their limitations) and their roots in different domains; *extend the scope* of a concept by *abstracting* some of its properties; *understand* the implications of *generalising* results; and be able to *generalise* such results to larger classes of objects. (emphasis in original; Niss & Højgaard, 2011, p. 53)

That is, the primary goal for the research was to link students' experiences from a history and philosophy of mathematics course to changes in their competency for thinking about mathematics with regard to important fundamental mathematical concepts. Thus, this qualitative, exploratory study set out to document and analyse changes in the students' mathematical thinking competency on three topics of interest (infinity, the complex number system, and the axiomatic structure in mathematics).

The results of the present research may enable instructors of history and philosophy of mathematics (HPhM) courses to consider the power of the course on improving mathematical understanding and could find it beneficial to revise courses to capitalize on this implication. Additionally, researchers interested in the "value-added" impact of a course on the history (and philosophy) of mathematics for prospective mathematics teachers may be able to make a stronger case for the inclusion of a history of mathematics course in teacher preparation programs that do not currently require it.

METHODOLOGY

The importance of such an empirical study—even an exploratory one—on changes in how students articulate their mathematical thinking as a result of an HPhM course is timely. The study is exploratory along two dimensions. The first dimension entails piloting the mathematical task think-aloud protocol, created to allow students to articulate their thinking about the mathematical ideas of infinity, the complex number system, and the axiomatic structure of mathematics. The second dimension, which is the focus of this paper, is the description of students' mathematical thinking pre- and post-instruction in the HaPM course and an attempt to qualify changes that occurred in their mathematical thinking competency with respect to these fundamental ideas in mathematics.

I heavily draw on Corbin's and Strauss' (2008) definition of qualitative analysis as "a process of examining and interpreting data in order to elicit meaning, gain understanding, and develop empirical knowledge" (p. 1). The nature of data analysis required me to search for changes between a student's thinking during the think-aloud interview post-instruction when compared to their pre-instruction thinking, and to qualify those changes using most of the aspects of the mathematical thinking competency (Niss & Højgaard, 2011): recognise, understand, and deal with the scope of the given concept; extend scope by abstracting; understand implications of generalising results; and generalising results to large classes of objects. For this reason, each student's pair of interviews was coded individually, according to their own phrasing, references (e.g., to the HPhM course material), and examples.

Participants

The particular context for the study was selected for two reasons, First, I was already familiar with faculty members at the university (one in mathematics education, one in mathematics), which would facilitate my multi-day visits to collect data, get to know the students in the course, and to conduct the think-aloud interviews. Second, students from a variety of majors took the history and philosophy of mathematics (HPhM) course at Private Christian University (PCU; a pseudonym). Thus, a course with a diverse student population as the one at PCU (e.g., first-year through fourth-year undergraduate students; mathematics and non-mathematics majors) was a valuable convenience sample to use.

Students were recruited from the HPhM course at the beginning of Fall 2012. The course instructor introduced the opportunity to participate in the research study during the first class session. Then, at the end of the second class session, I introduced myself and explained that the primary goal of the research was to investigate changes in students' mathematical think-

Participant	Year in school	Major	Highest level mathematics course taken in high school; college mathematics courses taken
Jenny	Senior	Elementary Education	Precalculus (high school); Intro to Mathematical Thinking (PCU)
Tabitha	Senior	History (Secondary History Education)	Intro to Mathematical Thinking (PCU)
Darren	Sophomore	Mathematics or Music (undecided in Fall 2012)	Advanced Placement Calclulus (high school); currently enrolled in two math courses (PCU)
Michael	Junior	Mathematics (recently changed major to Mathematics Education)	Mathematical Analysis II (high school); Intro to Mathematical Thinking (PCU)

Table 1: Student participants (Private Christian University, Fall 2012)

ing about mathematics concepts (e.g., the concept of infinity) that occur as a result of studying the history and philosophy of that concept. Finally, I explained the nature of the think-aloud interviews and answered students' questions about the research, interview process, and their potential participation. Of the 19 students enrolled in the course, four students volunteered to participate in the pre- and post-instruction "think aloud" interviews. Brief descriptions of the four participants are given in Table 1. The names that appear in bold indicate the participants on whom I focus for the results discussed in this paper.

Data collection

Mathematical task interviews were conducted using a think-aloud approach, in which student participants were asked to articulate their methods, interpretations, and thought processes while working on mathematical tasks. During the think-aloud process, the researcher intervenes as little as possible, with the exception of limited prompts such as "talk about what you're thinking" or simply "keep talking" (Young, 2005). The think-aloud interviews were conducted before and after classroom instruction and subsequent study of the three concepts of interest. Smartpen technology was used during each interview. The Smartpen was used by students during the interviews and enabled me to capture audio as well as written documentation of the students' responses to each of the tasks. Interview transcripts, once transcribed, were analysed for qualitative differences in students' mathematical thinking competency, as a result of studying historical details and philosophical perspectives of fundamental mathematics topics. In addition to the audio and written documentation, the course instructor's instructional plans and documentation related to the three selected topics comprised the third data source (e.g., lecture notes, assigned reading, homework assignments, class session activities).

Think-aloud interview instrument

A three-part interview instrument was constructed for this research, which was composed of 18 items. The instrument included items about the concept of infinity (five items), the complex number system (six items), and axiomatic structure in mathematics (seven items). However, due to space limitations, I discuss only one of the items in this paper: Item 1 (complex number system) asked students to *draw a Venn diagram to show the relationships among the different types of numbers that comprise the number system.*

In the following discussion, I present a summary of the pre- and post-instruction responses for both Tabitha and Michael. Additionally, I identify which aspects of the mathematical thinking competency were found in their responses. Finally, I provide a profile for Tabitha and Michael (based upon their pre- and post-responses) in an attempt to identify changes in their mathematical thinking competency resulting from their experience with a history and philosophy of mathematics course.

SELECTED RESULTS

The four students who participated in this study generated a surprising amount of rich data in their responses to the 18 items during the think-aloud interviews. In an effort to make sense of the various ways to describe the changes in students' mathematical thinking competency, this paper presents responses for only one item of the think-aloud instrument from two of the four participants.

The important results of this study are found in the qualitative analysis of the participants' pre- and post-instruction think-aloud responses. Tabitha's and Michael's responses to one item are presented to display the potential for HPhM course work to impact students' mathematical thinking competency. For this paper I selected an item in which the HPhM course might impact students' competency differently, given the range of student abilities and experiences represented.

To illustrate the scope of changes in the mathematical thinking competency for Michael and Tabitha, I present a summary of the key content of their responses, along with identification of aspects of the competency exhibited in their response. When necessary, I also provide a continuum rating to quantify the extent to which the aspect was evidence (e.g., "naïve or beginning understanding", "stronger example").

The case of Tabitha

To begin, I provide the summary of Tabitha's pre-responses to Item 1 (complex number system), with elements of the mathematical thinking competency (Niss & Højgaard, 2011) given in bold as well as an image from Tabitha's think-aloud interview (Figure 1):

(a) She expressed types of numbers as dyads: positive and negative; real and fake, or "unreal" numbers; ra-

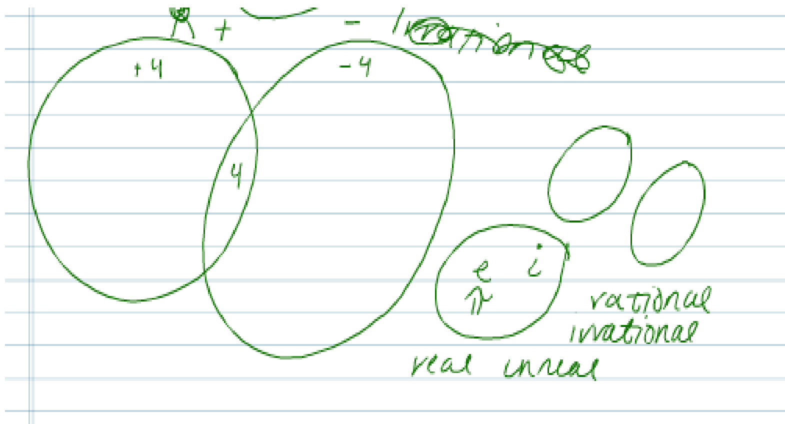

Figure 1: Tabitha's pre-response Venn diagram

tional and irrational, which belong to real numbers (though her Venn diagram did not reflect this) *[recognise a given math concept]*;

(b) Numbers that are not numbers: *e, i, π*; she later changed her mind: *π* is a number—but not *[recognise a given math concept (naïve or beginning understanding)]*;

(c) She first stated that *i* means "infinity" and then changed her mind and stated that *i* stands for "irrational" *[(attempt to) recognise a given math concept]*;

(d) +4 and – 4 mean the same thing, since they each just reduce to 4 *[(attempt to) recognise a given math concept]*; and

(e) She tried to seek connections throughout her discussion *[extend concept by abstracting properties]*.

Next, I summarised Tabitha's post-responses to Item 1:

(a) She identified whole numbers and fractions and those numbers that cannot be written as fractions: "so irrational" *[recognize and understand a given math concept]*; she also asked: "can prime numbers be irrational?" *[awareness of types of questions]*;

(b) In her Venn diagram she needed two non-intersecting circles one for rational, one for irrational *[recognise and understand a given math concept]*;

(c) She again listed dyads of numbers: rational/irrational; prime/composite; imaginary/complex *[recognise and understand a given math concept]*;

(d) Her post-response included imaginary versus real numbers (Figure 2) *[recognise and understand a given math concept; extend concept by abstracting properties]*;

(e) She also added infinity and zero to her diagram, noting that these particular numbers represented divine connections for her *[extend concept by abstracting properties; deal with scope of given mathematical concept in different domains]*.

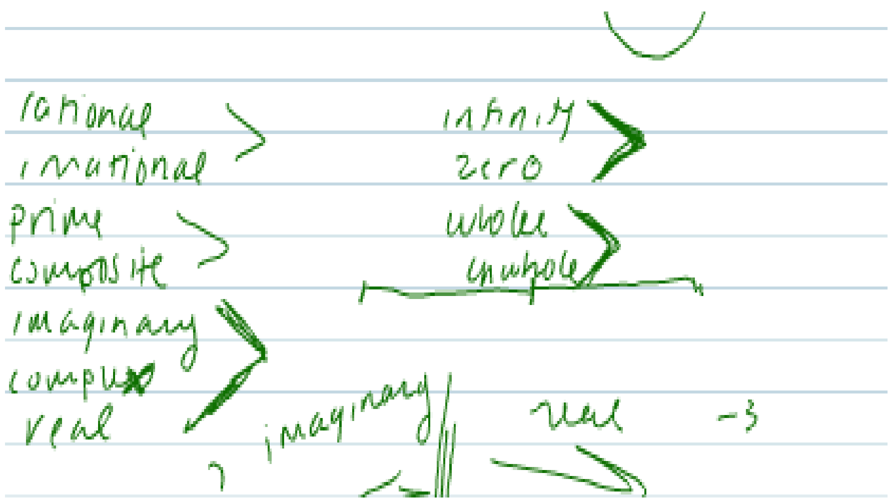

Figure 2: Tabitha's list if types of numbers needed for her Venn diagram (post-response)

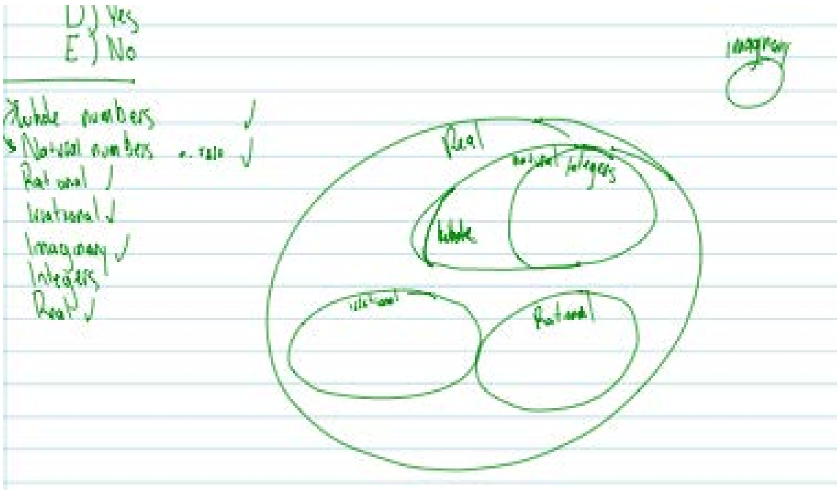

Figure 3: Michael's pre-response Venn diagram

Finally, I constructed a mathematical thinking competency profile for Tabitha (Item 1, complex number system), based upon the elements from the mathematical thinking competency (in bold in the summary). This profile shows growth from pre- to post-instruction, in that the variety of aspects from the mathematical thinking competency developed from several instances of only two aspects to multiple instances of four aspects of mathematical thinking.

Tabitha: Item 1 (complex number) pre-response

— Recognise a given math concept (four instances, including two at a beginning or naïve level of understanding)

— Extend concept by abstracting properties

Tabitha: Item 1 (complex number) post-response

— Recognise and understand a given math concept (four instances at varying levels)

— Extend concept by abstracting properties (two instances)

— Deal with scope of given mathematical concept in different domains

— Awareness of types of questions

The case of Michael

Michael, a mathematics major who recently changed his program of study (major) to mathematics education, possessed stronger mathematics content knowledge than Tabitha. However, the history and philosophy of mathematics course experience did appear to impact his mathematical thinking competency. Michael's pre-response summary and the identification of the relevant competency aspects for Item 1 (complex number system) included:

(a) He first listed kinds of numbers, including whole, natural, rational, irrational, imaginary, and integers (of the last category Michael asked: "where do these belong?") *[recognise and understand a given math concept (transitional understanding, not completely sophisticated)]*;

(b) He asked: Are you going to be paying attention to how big my circles are?...because there is this whole debate on if there are more rational or irrational numbers" *[extend concept by abstracting properties]*

(c) He was careful to make sure that the circles representing the sets of rational and irrational numbers did not overlap in the Venn diagram, but he did want the two circles to touch (see Figure 3) *[recognise and understand a given math concept (transitional understanding)]*.

In his post-response to Item 1:

(a) Michael began the discussion of his Venn diagram by beginning with the top of the hierarchy for the number system as he understood it: "...[we have] real numbers, then we have imaginary, then you'd have... complex numbers, which is a little of both" *[recognise, understand, and deal with the scope of a given math concept]*;

Figure 4: Michael's post-response Venn diagram

(b) He listed sets of numbers as he did before: whole, natural, integers; and real, irrational, algebraic, transcendental, and transfinite *[recognise, understand, and deal with the scope of a given math concept]*; and

(c) He felt there was more that he could do with classifying types of numbers: "I feel like I could do this forever; just drawing circles and breaking them up… because even with all of these you could define them into primes and which primes are seen more often, and…" *[extend concept by abstracting properties]*.

Using the summary above, I constructed Michael's mathematical thinking competency profile for this item:

Michael: Item 1 (complex number) pre-response

— Recognise and understand a given math concept (two instances; though not completely correct on each instance (e.g., "transitional understanding"))

— Extend concept by abstracting properties (one instance)

Michael: Item 1 (complex number) post-response

— Recognise, understand, and deal with scope of a given math concept (two instances)

— Extend concept by abstracting properties (stronger example than in pre-response)

Michael's understanding of the complex number system before participating in the HPhM course was fairly strong. Yet even with his understanding of the complex number system, his mathematical thinking competency was strengthened by the end of the course.

DISCUSSION

The pre- and post-response summaries, identification of mathematical thinking competency aspects, and construction of a competency profile for Tabitha's and Michael's work on Item 1 (complex number system) attempt to show that the HPhM course had some influence on students' mathematical thinking competency—at least from the perspective that the students themselves noted this while cognitively addressing the content before them. For example, Michael's ability to discuss and organise the different numbers of the complex number system (although "complex" was purposely not given to participants in the item or during the think-aloud interview) was already strong before the HPhM course. However, after the course, Michael revealed a more complete or nuanced understanding of the complex number system, which he stated was due to his reading of a required course text, *Zero: The Biography of a Dangerous Idea* (Seife, 2000). Classroom observations also revealed that Seife's book was used during classroom discussions and assignments.

Tabitha's post-instruction responses do reflect a better understanding of the types of numbers comprising the complex number system. However, except for a brief reference to learning about Venn in a class lecture, it is difficult to explicitly link the change in her thinking about the complex number system to the HPhM course. That said, I believe the course did influence her knowledge because as a Secondary History Education major, it is unclear from where she would have gained such knowledge during the semester of study. Instead, an attempt to identify changes in Tabitha's mathematical thinking competency shows promise for characterising the impact of HPhM courses on tertiary students' mathematical experience.

CONCLUSION

This paper reports only a very small subset of results from a pilot study conducted in 2012 and its purpose is to highlight the potential for history and philosophy of mathematics courses to enhance students' mathematical learning. In the current attempt, I tried to qualify the profile of two students' mathematical thinking competency, using student responses to identify aspects of the competency (Niss & Højgaard, 2011), for a single item from the interview instrument developed for this project. In the next step I will analyse responses to several additional items from the interview instrument. Then, in a future publication, all four student cases (Darren, Jenny, Michael, and Tabitha) will be presented, along with supporting data from classroom observations and instructor documentation, in order to reveal a more extensive landscape of changes in students' mathematical thinking competency resulting from their experience in a history and philosophy of mathematics course. In the interim, it is my hope that this initial report will promote discussion about similar empirical studies.

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Using calculus in economics: Learning from history in teacher education

Peter Collignon

University of Erfurt, Erfurt, Germany, peter.collignon@uni-erfurt.de

Historical awareness has an impact on teaching and learning mathematics. It includes knowing the historical development, the questions under investigation and the answers given to these problems. In this paper, the focus lies on calculus and its applications in economics. It shows how far the knowledge of a changing scientific understanding can be beneficial in teacher education. The paper covers this issue from an epistemological, historical and educational perspective and suggests a constructivist view for educational purposes.

Keywords: Calculus, constructivism, economics, epistemology, teacher education.

INTRODUCTION

Within teacher education, an expansion of scientific understanding can be achieved, which promotes an adequate epistemological view and modifies naive-realistic ideas. For this purpose, covering mathematical economics using methods of calculus is an unusual but promising approach. According to Fischer & Malle (1985, p. 107), the absence of a law-of-nature-character is necessary to allow learners the free use of mathematics describing "reality". In this way the modelling perspective demonstrates a human distance to reality. Jablonka (1996, p. 34f) states, that this view assumes an understanding of the underlying mathematical concepts separate from the context. In the following text, this consideration will be modified by discussing the usage of a certain mathematical concept (calculus) for modelling (economic) circumstances, which originally did not contribute to its genesis. However, this approach should go along with an adequate historical awareness of the evolution of the mathematics involved.

In most cases first-year students have not experienced the formal treatment of scientific issues in the classroom as a modelling process. Even more there is little or no experience with a systematic processing of economics applications. Dealing with economics in terms of mathematical modelling offers general education and provides insights into epistemological concepts and helps to foster an enlightened understanding of science. In view of the above and considering also the potential of a constructivist understanding of science due to Ernest (2007) and Lyotard (1979) it is promising to use a mathematical tool in connection with a subject that it is originally not meant for. This approach aims at prospective teachers primarily, but it applies to their educators as well.

APPLIED MATHEMATICS IN TEACHER EDUCATION

Mathematical modelling is listed as one of the six important competencies in Germany's "Bildungsstandards Mathematik". This is a consequence of the postulate of integrating more reality-based problems in lessons and lectures on mathematics. The focus is not the application of a given algorithm, but the mathematization of facts and problems whose relation to mathematics may be not initially obvious. Circumstances of the so-called "real world" are to represent formally by abstraction so that the representatives enable a quantitative analysis. The solution found for the model may be interpreted as a proposal for the solution of the real problem.

In many cases, such an approach is presented as a multiple passage of a modelling cycle that shows the interaction between "reality" and model as an idealized scheme. Critical validations should successively lead to a revised design of the model as an interpretation of reality.

Beginning their studies, prospective teachers have years of experiences in applying mathematical methods. For example, elementary calculus, stochastic and analytical geometry are known in principle. Introductory lectures at university address those issues again and there is the chance to add new aspects corresponding to the didactic spiral principle. This includes applications in general and should cover topics from mathematical modelling in particular.

MATHEMATICAL MODELLING IN ECONOMICS

In current mathematics education discussion the modelling cycle of Blum and Leiss (2005, p. 19) is widely accepted as a "model of modelling", but also different approaches are considered, e.g. an interpretation of the modelling process in the sense of taking the spiral principle or descriptions including a coexistence of cyclic and progressive courses into account. Meyer and Vogt (2010, pp. 142ff.) argue for an increased emphasis on the processual character and the inclusion of an appropriate terminology. Möller (2014) analyses the concept of the "rest of the world" and exposes the problems of the disjoint separation of mathematics and reality.

Facets of the reliance on mathematics

The field of economics shows a multitude of aspects that differ from those of the natural sciences. There are much less "canonical" formalizations than in physics, e.g.

Human action is involved, which refers to needs and wants. The scarcity of goods and services forces people to economize. Economic actions go along with conflicts and competition; they require decisions under uncertainty. Therefore, they are fraught with risks. Economic activities are goal- or benefit-oriented, often for-profit.

Economic issues, due to their reliance on mathematics, must take all aspects of the modelling process into account. Using mathematics within the "exact" and natural sciences often appears canonically, and in school they are rarely taught as a result of a modelling process. An in-depth understanding of their non-canonical nature, which manifests itself in the different models of the physics of classical mechanics, Maxwell's electrodynamics, the theory of relativity, quantum mechanics to recent developments in cosmology remains a domain of advanced students and experts.

Nevertheless, mathematical modelling is often demonstrated in the context of physical-technical issues. The decision for a certain model is closely related to the finding and definition of suitable parameters and their functional relationship. Frequently the number of parameters involved and their skillful assignments are looked upon as a quality criterion.

While early graders get involved in proportionalities relating to quantities, on the secondary levels the notion of function comes to the fore. It requires good education to impart the features of modelling in such a framework. This applies to teacher training, too.

At this point the comprehension of economic topics carries a valuable educational potential. The fundamentals of the conditions and circumstances – with respect to the subject of the application as well as the mathematical methods used – gives insight into the processes of gaining knowledge.

The values of goods and services are defined by their position on a common one-dimensional discrete scale; a price is assigned. So already the introduction of the quantity "money" is an early normative modelling item (cf. Möller, 2007, pp. 3ff.). It allows a comparison and measuring with monetary units, which implies the definition of an order relation for all the goods and services. Here a descriptive aspect becomes apparent: Aspects of economic objects can be characterized (via prices). Once the concept of functions is available, rates may be expressed by functional dependencies and forecasted under appropriate model assumptions. In accordance with and in extension to Henn (2000) the descriptive aspect can be interpreted as a summary of description, declaration and prognosis.

The role of calculus

Even if modelling using functional relationships is regarded as reasonable, there is initially no need to apply methods of calculus. Under what conditions and for what motives does this happen, at all? In order to answer this question one must distinguish between

- a use of a calculus not reflected upon (cf. Doorman & van Maanen, 2009, p. 4), which offers the possibility of using methods taught in school to discover certain properties of functions (local extrema, inflexion points, monotony etc.);

- a deliberately chosen approach, which allows an analysis of the model with methods of infinitesimal calculus in the first place with the aim of gaining knowledge.

The latter begins with the substitution of discrete quantities, such as money or lot sizes, with continuous ones. In this way, the real numbers (as well as real intervals) are involved. Functional dependencies described by tables or in terms of numerical sequences must now be expressed by functions defined on (connected) subsets of the real numbers. Often the mapping rules are chosen in a way, that allows an analysis according to infinitesimal calculus, i.e., they are continuous, differentiable etc. Providing this, it is possible

- to use methods from calculus on the mathematical model level;
- to interpret the available mathematical concepts relative to the real situation, i.e., an economic interpretation.

Corresponding to the second point, we often observe an interaction between the mathematical practice and the subject to be mathematized: Several economic concepts were generated or at least clarified by the use of a certain kind of mathematics. Marginal costs, defined by the first derivative of the cost function, are an example. The history of physics and its partial co-evolution with calculus offers a variety of relevant analogies.

STUDENTS' PREVIOUS KNOWLEDGE ABOUT THE CO-EVOLUTION OF CALCULUS AND PHYSICS

Modern calculus textbooks are characterized by an axiomatic representation and by offering physical applications. The representations are supplemented by graphics or pictures to support their understanding. The main concepts include the real numbers, the notion of a function and a limit. They are proposed by a formalised representation following a trias that is definition, theorem and proof, supplemented by a few examples illustrating the propositions. In this way the representation of calculus made its way during the first half of the last century (see also Jost, 1998). This outcome is still regarded as "modern" and results from a long historical non-linear development with discontinuations. Thus it may be an ambitious task for first-year students to look into history of calculus self-contained. In the case of master's students an independent research is more appropriate. The following sections outline the knowledge students should have to participate in a discussion about further applications in social sciences, especially in economics.

The discovery of the Archimedean palimpsest (cf. Netz & Noel, 2007) revealed his geometric view consisting of several singular perceptions of areas bordered by parabolas. His way can be characterized as singular efforts but he did not deduce a method which could have led to a general view. He took first steps onto the way to Riemannian integration, but however, he did not develop a general method like Riemann did.

New steps towards a theoretical approach of analysis were taken more than a thousand years later within occidental mathematics. Further efforts were made within the influence of natural sciences in particular the planetary movements as well as the ballistic investigations. These considerations corresponded to the notion of dynamics, e.g. the velocity, and both concepts involve the dependency between a geometric position and the time. Here Johannes Kepler (1571–1630) used the lists made available by Tycho Brahe (1546–1601) (lists that put the loci of the planets in dependency of the time) and deduced his laws of planetary orbits.

Newton (1643–1726) further developed and refined the theory of planetary movements by using an infinitesimal calculus that emphasized dependencies of time. However he did not yet apply yet our modern concept of a function. Around the same time Leibniz (1646–1716) generated an infinitesimal calculus by introducing infinitesimal quantities, not using consequently a functional approach either. However, both of them and some other contemporary scientists provided a basis for their followers in the 18th century, see also Sonar (2011).

Euler (1707–1783) picked up these conceptions and was the first to introduce the functional concept in a way in which we use it still nowadays. This means the infinitesimal calculus reached the next level and is since called modern. During the following decades the focus was on computational aspects but there was still a lack of theoretical foundations. The real numbers had no theoretical foundation and the concept of a limit was still missing.

Karl Weierstrass (1815–1897) bridged both gaps (real numbers and limits) using the epsilon-delta-statement for the definition of a limit and contributed a theoretical approach to the real numbers. The latter was refined later on by Dedekind and Cantor who defined the real numbers by Dedekind-cuts. Cantor contributed the concept of Cauchy sequences (equivalence classes of Cauchy sequences) and the continuum hypothesis (cf. Hairer & Wanner, 1996, pp. 172ff.). Their contributions established a completion of the rational numbers to the real numbers so they obtained a complete field of numbers.

Teacher students should be also aware of the following: In the classroom, but also in contemporary university lectures, especially in teacher education, one can observe some kind of (anti-) didactical inversion (cf. Freudenthal, 1983, pp. 305ff.), interpreted in terms of history. There is an introduction to the real numbers on secondary I level, which mainly consists of some examples of irrational numbers. As a matter of fact this is part of the recent history of calculus and it is, strictly speaking, mathematics from the 19th century. Regarding the concept of functions, the situation is similar: It is introduced before a discussion about infinitesimal concepts takes place on secondary II level. The figure below presents a very rough scheme that reflects teacher students' often observed reception of the history of calculus and the (experienced) order of teaching.

EDUCATIONAL POTENTIAL

There is no controversy about the necessity of gaining insight into economics and economics applications using mathematical concepts. However, this is defined differently or non-specifically. According to May (2001, p. 3),

> ... economic literacy can be paraphrased as the qualification (knowledge, competencies, skills, attitudes etc.) to manage living conditions determined by economics.

At the same time we find the statement:

> The intention of teaching economics is ... the (economically) responsible citizen.

Engartner (2010, p. 15) emphasizes the social meaning of economic literacy:

> Only an appropriately qualified citizen, educated in matters of responsibility, is in a position to follow the rapid process of social change at least rudimentarily.

Orientation towards the "basic experiences"

Sociological and economic issues affect the reality of young adults' lives to a larger degree than topics from physics do. From the mathematics educational point of view they are matters of the "basic experiences" of Winter (1996): Mathematics education is providing general education by

- realizing phenomena of nature, society and culture;
- knowing (and appreciating) mathematical issues, represented by language, symbols, images an formulas;
- acquiring heuristic competencies.

Mathematics education has a general and life preparatory function (Heymann, 2013, pp. 131ff.). Mathematical modelling offers the opportunity to realize this in two ways: firstly, on the basis of concrete facts to be modelled, then again by modelling itself on a meta-level. This provides access to epistemological issues.

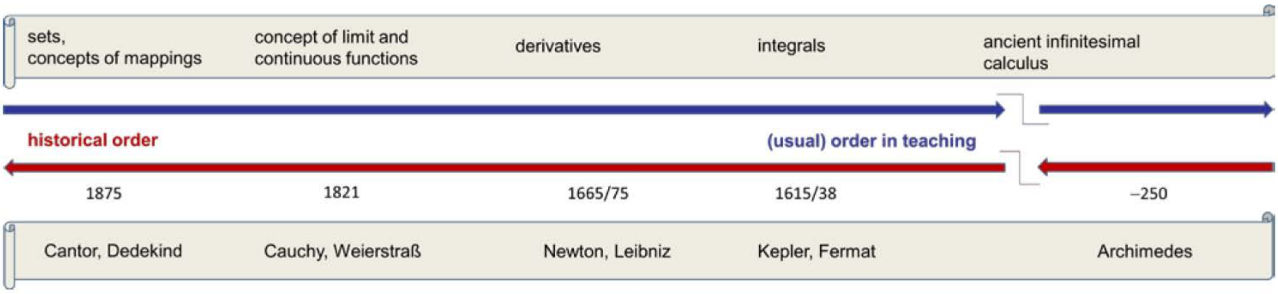

Figure 1: Didactical inversion – in terms of history

Epistemological aspects

Theory of science is not science philosophy, and the latter is not the same as epistemology. Especially the German language is differentiated here, although not uniformly (Poser, 2012, pp. 15ff.). It is not the role of an introductory mathematical lecture to explicate this in detail. But particularly with regard to teacher education one should not ignore the observation, that students, but also teachers, often show a naively realistic understanding of science. Zeyer (2005) confirms their socialization through their academic training in terms of an unconsidered positivistic attitude. In this context a proposal from Duit (1995, p. 905) deserves consideration.

> Knowledge acquisition is regarded as an active construction on the basis of existing ideas. The active, self-directed and self-reflective learner is in the center and the idiosyncratic processes of construction are always embedded in a particular social context.

In case of the natural sciences and pure mathematics there are justified objections to a strong emphasis of relativism, even from a didactic point of view. Regarding economics, the situation is quite different. It is obvious, that the economic denominations listed above are closely connected to human action. The implications and theories deduced do not have the significance of "laws of nature". This makes it easier for students who did not have any contact with epistemological questions until this point to identify the reference objects of many scientific discourses as models.

The history of economic sciences shows various paradigm shifts – even within the radical meaning of Kuhn (1992) –, and even the contemporary ones are more accessible to laymen than the development of physics in the twentieth century. This supports an open-minded modelling activity in economic contexts in which the distinction between "wrong" and "correct" solutions becomes less important and is put into perspective. The main idea of the paradigm and the paradigm shift relates to the facts to be modelled and the mathematics used at the same time. For example, the economic marginal analysis uses infinitesimal methods originally physically and geometrically motivated. In economics, however, quantities of discrete nature are often modelled as continuous ones with intent to apply methods of differential calculus, which emerged from completely different requests. Lectures and seminars on calculus offer the opportunity to point to the transfer of methods from natural to economic sciences and discuss how far this is an historic attempt to make economy "accurate".

Considering everyday experiences, and apart from the necessity of laboratory conditions, Newtonian mechanics may be a "canonical" modelling. Anyway, the use of infinitesimal methods in economics is based on model assumptions which are replaceable in a more obvious way, e.g. by the application of methods from discrete mathematics.

Brodbeck (1998, pp. 22ff.) refers to these observations as "social physics" and continues:

> The mechanical approach in economics shall primarily allow the application of mathematical and experimental methods in analogy to physics (...).

Prospects

Jablonka (1996, p. 187) concludes her meta-analysis of approaches to mathematical modelling and applications by declaring:

> The lack of examples from which we can learn where to ask more questions and to try different ways and to illuminate the problem from different perspectives and to weigh the outcome and evaluate critically clearly shows that a reflection beyond the scope of an analysis of purposes and efforts has little place in mathematics lessons, which target the comprehension of mathematical procedures and theories.
>
> As a resource to change the viewing angle self-consistent, and as an instrument to recognize and evaluate alternatives, to search for reasons and in the exchange of arguments, reflection is not knowledge, but an attitude in the assessment of mathematical methods to be aimed at an intentional creation of situational activity, which results in the insight, that recipients or persons concerned are authorized and applying operators are obligated, to adopt a critical attitude.

Teaching experience meets the high expectations linked to the potential of economic modelling against the background of the history of calculus outlined here. Qualitative investigations planned in this context will provide further indications.

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The role of history of mathematics in fostering argumentation: Two towers, two birds and a fountain

Paulo Gil[1] and Maria Helena Martinho[2]

1 Escola Básica e Secundária de Pinheiro, Lisbon, Portugal, pbastosgil@gmail.com

2 Universidade do Minho, Centro de Investigação em Educação, Instituto de Educação, Guimaraes, Portugal, mhm@ie.uminho.pt

This paper aims at discussing how a context of argumentation can be created in the classroom through the development of tasks based on the history of mathematics. These were proposed to a class of 8th grade. Through this experience, we found that students use different types of arguments that show different degrees of formality and kinds of reasoning; express and justify their ideas; and interpret and understand the opinions that are presented to them. The history of mathematics proved to be an enabling tool for mathematics learning, particularly for building a community of mathematical discourse, in which mathematical argumentation is reflected.

Keywords: History of mathematics, argumentation, Pythagorean theorem.

INTRODUCTION

The integration of the history of mathematics in mathematics classes has, over time, attracted the interest of researchers, mathematicians and mathematics teachers (Fasanelli, 2000). In fact, in recent years, such an integration has been prominently featured in literature and in educational curricula of different countries. In parallel, there are several studies that highlight the value of argumentation in mathematics (Pedemonte, 2002; Knipping, 2008) as well as a number of specific references in the curriculum guidelines that advocate the development of students' ability to communicate the way they reason. Taking as starting point these principles, we intend to show how the integration of tasks based on the history of mathematics in the context of the classroom can foster the development of mathematical reasoning, influencing students' ability to reason.

HISTORY AS A PROVIDER OF MATHEMATICAL ARGUMENTATION

A number of arguments justify the benefits of integration of the history of mathematics in teaching and learning mathematics. Tzanakis and Arcavi (2000) point out five fields in which such an integration may be particularly relevant to support, enrich and improve the teaching and learning process: the learning of mathematics; the nature of mathematics and mathematical activity; the didactical background of teachers; the affective predisposition towards mathematics; the appreciation of mathematics as a cultural endeavor. On the other hand, Jankvist (2009) proposes two sets of categories in which to place the arguments for using history (the "hows") – history as tool or history as a goal – and the different approaches to doing this (the "whys") – illumination, the modules and the history-based approaches.

Discussing and developing mathematical arguments implies creating conditions for students to learn how to reason mathematically, since a classroom culture that promotes the argument stimulates the participation of students in their own learning (Krummheuer, 1998). In this sense, the history of mathematics can play a useful role in creating a community of mathematical discourse, namely fostering contexts for argumentation. The different points of view that arose from different historical contexts provide an opportunity for students to argue and to develop the art of arguing, to justify their own opinions and to present their thoughts to their peers (Fasanelli, 2000). Thus, students develop not only deeper mathematical skills but also other sorts of skills such as the ability to discuss, analyze and "talk about" mathematics (Tzanakis & Arcavi, 2000). In fact, according to Siu (2007), using the history of mathematics in the classroom does not necessarily make students obtain higher scores in a particular subject, but it can make learning mathematics a more meaningful and lively experience, thus helping to learn more and more easily. It contributes to establish a context of teaching and learning that provides both teachers and students with different ways of using and acquiring knowledge about what is taught in its diversity. These different forms may arise, for example, by proposing and solving certain problems, since they allow not only the observation of the historical evolution of the concepts, but also promote reasoning and comparison of resolution strategies. Such a process promotes the exchange of ideas and reasoning schemes (Lakoma, 2000), which allows not only the development of argumentation situations, but also the establishment of connections with previously acquired knowledge or other areas knowledge, fostering, in particular, the students familiarity and personal involvement in mathematics (Tzanakis & Arcavi, 2000). Therefore, the history of mathematics appears as a way promoting communication development and, in particular, mathematical reasoning in the context of the classroom.

ARGUMENTATION IN MATHEMATICS CLASS

Argumentation refers to techniques or methods for establishing a speech statement, i.e. processes that produces a (not necessarily deductive) logical assertion on a given subject. The value of arguments in mathematics classes, arises not only associated to the idea of explanation and justification – to convince the other – but also to the discussion and evaluation of different ideas expressed in the classroom, for example while addressing a given task.

The model proposed by Toulmin makes possible a local analysis of the arguments. This model aims at capturing the "logical form" of a speech (Pedemonte, 2002). Toulmin (2008) describes the basic structure of rational arguments as a pair *data/conclusion*. The passage from data to conclusion can be put in question, and often is justified by a *warrant*. Although, sometimes the distinction between data and warrants is unclear, their functions are different: data transmit a set of information; warrants authorize the step of inference. This distinction allows Toulmin to provide the skeletal elements of a pattern that we call *simple argumentation form*. However, this elementary scheme may not be sufficient to analyze certain argumentative discourses. Toulmin, thus, adds three auxiliary elements of discourse analysis: *modal qualifiers*, *conditions of rebuttal* and *foundation*. In particular, in mathematical argumentation these auxiliary elements arise, respectively, as indicators of the strength of the argument, as exception conditions and warants to support the inference. We designate this by *complex argumentation*.

A classification of the different arguments produced is given by Reid and Knipping (2010), who establish four categories of arguments: empirical, generic, formal and symbolic. Moreover, they state that within these four categories several subcategories can be identified. Other cases may lie at the borderlines between them.

METHODOLOGY AND CONTEXT

This experience, as part of a broader research study, followed a qualitative methodology, involving students in a class of 8th grade and the respective teacher. It aimed at examining how the integration of history-related tasks in the context of the classroom may influence the students' ability for reasoning and argumentation.

The study had several phases: organization of the task; implementation (along two-class periods of ninety minutes each) and data collection; analysis of the arguments produced by students, local and global analysis of different discursive interactions, identification of difficulties experienced by students and their own assessment.

The task – *two towers, two birds and a fountain* – was structured in different parts. Thus, for Part I, students were asked, in small groups, to solve the following problem: Two towers, measuring 30 meters and 40 meters of height, respectively, are placed at a distance of 50 meters. Between the two towers there is a fountain to where two birds, flying down from the towers at the same speed, arrive at the same time. What is the distance between the fountain and each of the two towers? The problem is found on Fibonacci's *Liber Abaci*. In Part II, students were asked to read and review strategies of resolution of this problem by Fibonacci himself and Gaspar Nicolas (a Portuguese mathematician of the 16^{th} century).

Data collection was carried out through observation field notes, as well as audio and video records. A documental analysis was also carried on using documents produced by the students: resolutions of the tasks performed and their critical reflection upon them. The teacher put a strong emphasis on group work and discussion with the whole class, interacting with students whenever considered necessary.

The analysis of the arguments produced by students was done using local analysis as proposed by Toulmin and the classification of arguments proposed by Reid and Knipping (2010) model arguments. The evaluation made by the students was based on the five domains mentioned by Tzanakis and Arcavi (2000).

PRESENTATION OF RESULTS

We present in the sequel some examples of the work done by the students during the resolution of task's Part I as well as a brief reference to how students reacted when reading and discussing the different arguments documented in the historical primary sources (Part II).

Type of arguments (in solving the problem)

In Part I all groups tried to solve the problem and different types of argument were observed. Using unrepresentative examples, the arguments produced by students were not restricted to a simple enumeration. In some cases they can even be framed in the subcategory *crucial experience*. Taking into account that at a certain moment the two birds are flying down from the two towers at the same speed and arriving at the center of the fountain at the same time, and that both birds start from the top of two towers, 30 and 40m high, respectively, the students observed that "the [tower] 40 is higher than 30, so it must be closer to the fountain so that the birds arrive at the same time and with the same speed." (Nelson, group G2). Moreover, given the difference between the heights of the two towers, 10 meters, the students felt that this value would influence the distance from the fountain to each of the towers. The fact that students take into account the data of the problem, noting what should be the position of the fountain (closer to the highest tower), suggests that they are addressing the problem in general. In fact, the initial argument of this group lies between the empirical and the generic, as it is the result of a refutation.

Nuno: We depicted the towers [pointing to the towers drawn on paper] and the space between them is 50. If we put the fountain in middle, which is 25, gives this [pointing to the bird that is in the 30m tower] going down faster than this one [pointing to the bird that was in the 40m tower], because the building is taller.

Argumentation: local analysis (in solving the problem)

Yet in Part I, through the various types of arguments used, it was possible to carry out a functional reconstruction of various statements made in the different groups. Observe an excerpt of the dialogue between the teacher and the G4 group:

1 Teacher: What have you done?
2 Diana: We have depicted two towers with two birds each. A 40m high and another with 30m and the distance between them was 50m. Afterwards we put the fountain here. 20m from this [pointing to the tower with 40m] and 30m from that [pointing to the tower with 30m, Figure 1], because the difference between them [the two towers] was 10m. So the fountain has to be closer 10 meters to one of them than to the other.
3 Teacher: That is...
4 Diana: It is closer to this one [pointing to the tower 40m]. This one [pointing to the tower 40m] is 10m higher than that [pointing to the tower with 30m], therefore the fountain must be 10m farther to walk the same distance. And here [pointing to the tower 30m] has as least 10m, then [the bird] will walk less, then the fountain must be farther.
5 Filipa: That is, the fountain has to be closer to this [pointing to the 40m tower].

The excerpt shows that this group considers the following solution for the problem: the fountain must be located 20m from the highest tower and 30m from the lower (§2) one. The determination of these values was based on two preliminary conclusions, (§4) and (§5), which acted as new data in the argumentative discourse, to obtain the final conclusion. In this reasoning chain it is still possible to identify a warrant (§2) "because the difference between them [the two towers] was 10m. So the fountain has to be closer 10

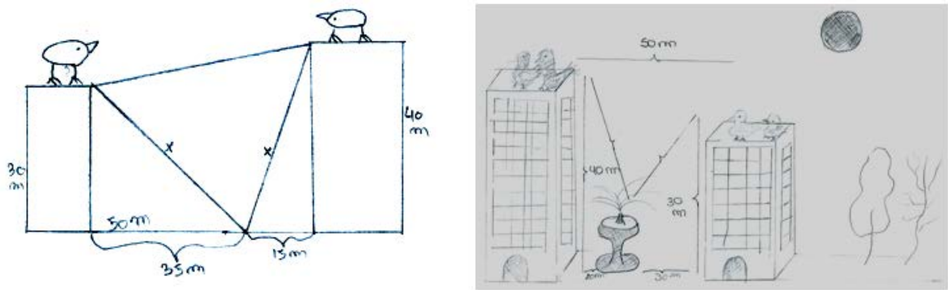

Figure 1: Illustration of the problem made, respectively, by groups G1 and G4

meters to one of them than to the other", whose role for this group of students was to legitimize the final conclusion. For the Toulmin model, the argumentative discourse corresponds to a simple form of argumentation.

Almost all groups obtained the wrong values mentioned above. Group G1, however, initially proposed a set of different values. Their initial solution took the form of a conclusion obtained after a refutation, although later they also refute the conclusion.

Nuno: We depicted the towers [pointing to the towers, Figure 1] and the space between them is 50. If we put the fountain in the middle, which is 25, results in this [pointing to the bird in the lowest tower] coming down faster than this one [pointing to the bird in the other tower], because the building is taller. As this is higher, we added 10m ahead [for 40m along the tower], so that they come down at the same speed [and arrive at the same time], we get 35 [25 + 10] with this [pointing to the tower 30m] and x [pointing to the way the bird would go] and use the Pythagorean theorem.

Through this passage, we can conclude that the first solution figured out by the students was 25m, i.e. the fountain would distance 25m of each of the towers. However, students refute this possibility, because the bird on top of lowest tower would reach quicker the center of the fountain, since the other tower was "a higher building". Although, at this moment, they do not mention the problem data given, they certainly consider it: the birds fly at the same speed and arrive at the same time to the center of the fountain. Therefore, considering the negation of the claim refuted as a new data, they claim that the fountain should "move forward"' 10m, i.e. come 10 meters closer to the highest tower. Accordingly, they add 10m to 25 (which was given as half of the distance between the two towers). Thus, for those students, the fountain should be located 35m away from the lowest tower and 15m from the highest one. It should be noted that, although not explicitly mentioned, the choice of 10m is related to such being the difference in height between the towers. According to the Toulmin model, this argumentative discourse is a complex form of argument.

In the final phase of the excerpt shown, students insist they still need to apply the Pythagorean theorem. Later this group will come to refute the consensual conclusion: that fountain is located 35m ahead from the lowest tower and 15m from the highest one. Note that this group using the same method, will refute the conclusions of the others. Thus, at the end of Part I no group had come up with the correct solution.

Reading and interpreting the reasoning argument present in primary sources

Although students have not found the correct solution, in Part II they were faced with three new resolutions coming from historical primary sources. They read, reviewed and discussed each of these resolutions, and arrived to the actual solution. They compared the resolutions, not only among themselves, but also with their own attempts. From what students wrote we are able to undertake a local analysis of the arguments, but also to identify the difficulties shown and the assessment they made of their work, namely, the confrontation of different resolutions.

From the claim accompanying the arithmetic resolution by Fibonacci, which reads "(...) in geometry it is clearly demonstrated that the height of either tower multiplied by itself added to the distance from the tower to the center of the fountain multiplied by itself is the same as the straight line form the center of the fountain to the top of the tower multiplied by itself; this therefore known (...)", the students were able to solve the problem algebraically. Observe the following:

1 Emanuel: 900, here 1600 (writing next to the respective towers, Figure 2).
2 Teacher: Then what does he say? He adds what?
3 Emanuel: The distance from the tower to the center of the fountain multiplied by itself.
4 Teacher: You have already the square of the distance from the lower tower to the center. How much it is?
5 Emanuel: 900.
6 Teacher: And what does he do? He add 900 to…
7 Emanuel: The square of the distance from the lowest tower to the center of the fountain.
8 Teacher: You know how much is this distance?
9 Emanuel: It is x!
10 Teacher: It is x?
11 Emanuel: Can be x, because I don't know the length (marks on the drawing x)
12 Emanuel: It'll be this (pointing to 900) plus x squared (writing next to the figure $900 + x^2$). Then there is also this 1600.
13 Teacher: And you agree with what is there?
14 Emanuel: (After a silence) Yes, it is the Pythagorean theorem.
15 Teacher: Let's continue. Could you do the same for the other triangle?
16 Daniel: Yes.
17 Emanuel: This is not x! (pointing the distance from the fountain to the highest tower).
18 Teacher: So here is…
19 Emanuel: $50 - x$. [The distance from the center of the fountain to the highest tower]

(Next, they write the expression $1600 + (50 - x)^2$ to denote the square of the distance from the highest tower to the center of the fountain).

20 Teacher: Yes, and what can you do with these two expressions?
21 Emanuel: We have to match! (they reply very promptly).
22 Teacher: And why did you match?
23 Emanuel: Must be the same distance, because the flights were of equal lengths. (Writing $900 + x^2 = 1600 + (50 - x)^2$). And now we solve.

Through this passage, we can conclude that the students translate, in current notation, the information given. Although the purpose of this task was to analyse the process of resolution presented by Fibonacci, this group, from reading the first part of the resolution of Fibonacci, chose to translate it algebraically, obtaining an equation. Thus, for this group, it was enough to solve the equation and thus get the solution.

After solving algebraically the equation, group G2 continued to read and analyse the Fibonacci resolution. They noted that the method used by Fibonacci was the false position, a method often used by the Egyptians to solve problems. Students refer this fact, because the procedure used by Fibonacci corresponds to choose a measure for the distance from the center of the fountain to the highest tower, which is not the problem solution. Then this is revised in order to obtain the correct solution. Students observe that Fibonacci appointed for 10 the distance from the fountain to the highest tower, introducing new data in the problem; naturally the distance from the lowest tower to the fountain is 40, since "the distance between the towers is 50 steps". Students justify Fibonacci's choice of this value (10) as being related to the difference between the heights of the towers. Next, students report that Fibonacci squared these values (heights of the towers and the distance from each tower to the fountain), as he "knew that what was shown in the figure was the Pythagorean theorem". Students continue their interpretation, noting that actually the result has to be the same. Therefore, they report that Fibonacci put the fountain 5 steps more ahead of the lowest tower. Implicitly, they account for the fact that the birds fly at the same speed and arrive at the same time to the center of the fountain, which shows that the birds travel the same distance. By the same procedure, which corresponds to the application of the Pythagorean theorem, students say that Fibonacci got a new value for the difference between the distance

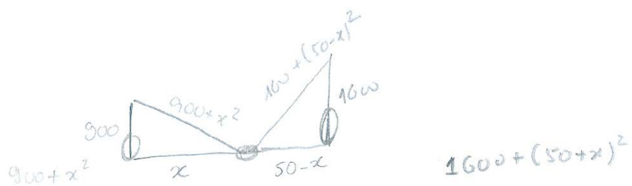

Figure 2: Illustration of the problem made by group G2

travelled by birds. Again the values obtained are not equal, and thus conclude "the fountain was not well positioned". They state that Fibonacci noted that when he "moves" the fountain 5 steps ahead the difference between the distance travelled by each bird was 300, i.e. the difference decreased in 500 steps. Since the birds travel the same distance, students report he used "the rule of three to see how many steps they had to walk so that the difference stays equal to 0". However, in Fibonacci's resolution there is no reference to this rule, but rather a numerical recipe – "multiply 5 by 300, and divide by 500".

In the local analysis of the arguments produced, it is observed that those are presented sequentially, which did not happen when students attempted to solve, initially, the problem. It was also noted a common concern with effective warrants and even with the corresponding foundations, which could legitimate the inference steps.

The second strategy to solve the problem is based on the similarity of triangles, Figure 3, *efz* and *gem* where *m* is the intersection point of *ef* and the parallel to *df* which contains *g*. Fibonacci started solving the problem by showing geometrically that the point *z* is the center of the fountain. Then he proceeded numerically. In this first phase, the students were challenged to read the beginning of Fibonacci's resolution and to find a justification for point *z* be the center of the fountain.

After some discussion students observed that *e* is, by construction, the midpoint of the segment *ga*, and *ez*, and also by construction, a segment perpendicular to the segment *ga*, and thus the triangles *aez* and *gez* are geometrically identical, since both sides are identical

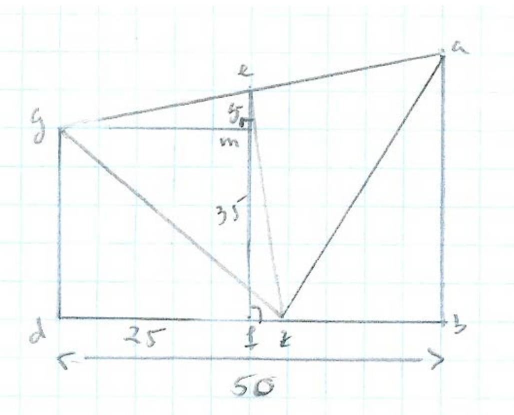

Figure 3: Illustration of the problem made by group G2 to the second strategy

and form an angle (90°). With respect to the arithmetic resolution, although the students were able to identify its main elements, they did not understand how Fibonacci arrived, in current notation, to the expression $[35x(35 - 30)] / 25$.

Daniel: Subtract this 35 by 30 will give 5; drawing this parallel line will get this 5 [pointing to the Figure 3]. Then multiplying 35 by 5 gives 175. Then dividing by 25 will [pointing to the segment]. But to divide by 25 will give the result of this [pointing to *fz*]. I'm not understanding why it gives the result of fz here [pointing to *fz*], if it is to change this [pointing to 5 and 35] for this [pointing to 25].

This excerpt reflects the difficulty expressed by students. Although the figure can identify the values reported by Fibonacci, it cannot justify the solution. Involving the whole class in the discussion the teacher points to the relationship between the triangles *efz* and *gem*.

1 Teacher: And these two triangles are equal?
2 Andrea: Which ones? This [pointing to the triangle *egm*] and this [pointing to *efz*]?
3 Teacher: Yes.
4 Daniel: No!
5 Emanuel: They do not have the same measures.
6 Teacher: Are they similar?
7 Teacher: [after a silence] What is required for two triangles to be similar?
8 Daniel: They must have two equal angles...
9 Nelson: Or the sides proportional....
10 Teacher: Or two proportional sides and the angle formed by them equal.
11 Daniel: They have a right angle [pointing to *egm*]... and also this [pointing to *efz*].

This extract, shows that students observed that triangles, *gem* and *efz*, are not geometrically identical (§4), since they have different sides (§5). Questioned whether they were similar (§6), students pointed out two cases of similar triangles, (§8) and (§9), and the teacher a third one (§10). Students observed the existence of the two right angle triangles (§11). For them to be similar triangles it would be necessary to find another pair of geometrically equal angles. Later, the students find that angles *gem* and *fze* are geometrically identical and therefore the triangles are similar.

The resolution proposed by Gaspar Nicolas, is written in the form of a recipe without any algebraic symbol. Although he considered the towers with 80 and 90 fathoms and the distance between both to be 100 fathoms, the students understood easily the calculations but question the procedure: "why to subtract 6400 to 8100", "why to add this difference to the square of the distance between the two towers" and "why to divide that result by 200". The teacher suggested a possible correspondence with the algebraic process used previously. With two unknowns, students were able to establish a correspondence between Nicolas's and their own algebraic process.

CONCLUSION

The experiment shows that different sorts of arguments were produced by different groups of students. Many of them were not developed sequentially, i.e. in a deductive way. The need for sharing ideas and exchange opinions was responsible for the emergence of not only parallel arguments, but also of new data which was suitably inserted into the argumentative chain. In simple forms of argumentation, we also observed the existence of inference steps for which the corresponding warrants or foundations were not explicitly presented. On the other hand, when analyzing the resolution strategies of Fibonacci and Gaspar Nicolas, students exhibit a strong concern for seeking warrants or some sort of formal explanations (eventually not explicit in the resolution shown) to legitimize certain inference steps. The analysis of the forms of argument used shows that students were not only able to express their ideas, but also to interpret and understand the ideas presented to them, as well as to participate constructively in the discussion. In assessing the experience, students highlighted the importance of solving the same problem through different processes and of becoming able to compare different resolution strategies. They also stressed the exercise influenced the predisposition to this discipline, while providing a broader view of its nature: "the geometrical resolution was the most difficult, but we learned that it was possible to solve the same problem geometrically"; "for me the most interesting was the resolution of Gaspar Nicolas, because I liked to try understand his thought and to solve an equation in words".

Experience has shown the potential of integrating the history of mathematics in the context of the classroom. This entails the need for a broader consideration of this problematic and its importance. Similarly, it seems necessary to outline a number of guidelines that promote this integration and scales up its potential namely, in what concerns the development of mathematical reasoning. This experience confirms that the development of reasoning in the mathematics class is a complex process: it requires a careful selection of the tasks, and entails the need for promoting a suitable environment for them, focused on the students' mathematical maturation.

ACKNOWLEDGEMENT

This work is supported by ERDF – European Regional Development Fund through the COMPETE Programme, by National Funds through FCT – the Portuguese Foundation for Science and Technology, within project FCOMP-01-0124-FEDER-041405.

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Teaching history in mathematics education to future mathematics teacher educators

Uffe Thomas Jankvist

Aarhus University, Department of Education, Aarhus, Denmark, utj@edu.au.dk

The paper describes a course the purpose of which is to introduce future mathematics teacher educators to the topic of using history in mathematics education. Excerpts from students' hand-in mini-projects are displayed and discussed in terms of the Danish mathematics competency framework.

Keywords: History in mathematics education; teacher education.

INTRODUCTION

The paper addresses the question: *how to introduce future mathematics teacher educators to the discussion of history in mathematics education, and how to prepare them theoretically for a potential use of history of mathematics in their own future practice*. The "answer" presented to this question is one by example, since the paper reports on a concrete design and implementation of a course. The theoretical framework adhered to in the paper is that of the Danish KOM-report (Niss & Højgaard, 2011), which lists eight mathematical competencies that all students of mathematics should come to possess, and an additional six didactic and pedagogical competencies that mathematics teachers should also develop (to be described later). Illustrative examples of students' reports from the course will be displayed and discussed. First, however, a bit of the background of the course participants and the educational setting is provided.

EDUCATIONAL BACKGROUND AND SETTING

To become a mathematics teacher educator of primary and lower secondary teachers in Denmark, it is often favoured by the teacher training colleges that the educators hold a master's degree in mathematics education (in Denmark a Ph.D. is not required), which the Department of Education at Aarhus University is the only provider of in the country. More precisely, to enter the master's program the student must already have a university bachelor's degree, e.g. in mathematics, or a vocational bachelor degree, e.g. as a primary and lower secondary mathematics teacher. The two-year master's program consists of courses in mathematics, courses in general didactics, a course in didactics of mathematics, several of these involving student projects, and finally a master's thesis. The course of our interest here is that in "didactics of mathematics". In this course, the mathematics educators of the department, of which we are presently four, are given the opportunity to teach in a mathematics education topic of their own choice. One of the ideas behind this is that students in this way also are confronted with recent research, in which the mathematics educators themselves are involved. The course counts 10 ECTS, and each topic takes up six lessons of two to three hours teaching and supervision plus student group work, etc. For each of the topics, groups of students hand in a mini-project. Based on a random draw, at the end of the course the student groups are examined in one of the four mini-projects. In the following, I present the design of the six sessions related to the topic of "history in mathematics education" – but first, a brief description of the theoretical framework of the paper.

KOM: THE DANISH COMPETENCY FRAMEWORK

In KOM – a Danish abbreviation for competencies and mathematics learning – a mathematical competency is defined as: "a well-informed readiness to act appropriately in situations involving a certain type of mathematical challenge" (Niss & Højgaard, 2011, p. 49). More precisely, KOM deals with eight mathematical competencies: *mathematical thinking competency*, i.e., to carry out and have a critical attitude towards mathematical thinking; *problem tackling competency*, i.e., to formulate and solve both pure and applied mathematical problems and have a critical attitude towards such activities; *modelling competency*, i.e., to carry out and have a critical attitude towards all parts of a mathematical modelling process; *reasoning competency*, i.e., to carry out and have a critical attitude towards mathematical reasoning, comprising mathematical proofs; *representation competency*, i.e., to use and have a critical attitude towards different representations of mathematical objects, phenomena, problems or situations; *symbols and formalism competency*, i.e., to use and have a critical attitude towards mathematical symbols and formal systems: *communication competency*, i.e., to communicate about mathematical matters and have a critical attitude towards such activities; *aids and tools competency*, i.e., to use relevant aids and tools as part of mathematical activities and have a critical attitude towards the possibilities and limitations of such.

In addition to the above mathematical competencies, KOM also provides a competency-based description of the profession of mathematics teachers, describing a set of six didactic and pedagogical competencies: *curriculum competency*, i.e., to evaluate and draw up curricula; *teaching competency*, i.e., to think out, plan and carry out teaching; *competency of revealing learning*, i.e., to reveal and interpret students' learning; *assessment competency*, i.e., to reveal, evaluate and characterise the students' mathematical yield and competencies; *cooperation competency*, i.e., to cooperate with colleagues and others regarding teaching and its boundary conditions; and finally *professional development competency*, i.e., to develop one's competency as a mathematics teacher.

DESIGN OF THE SIX LESSONS

In the following, I describe the content and purpose of the six lessons. For each lesson the students read a collection of texts, mainly research papers, which the students then worked with during the lesson. Also, supplementary texts were given for each lesson.

In *lesson 1*, students were to familiarize themselves with different arguments for and against the use of history (and epistemology) in mathematics education, potential dilemmas, and of course different approaches to involving history. The texts read included Fried (2001), Jankvist (2009) and Niss & Højgaard (2011). The purpose of this was to enable the students to more competently discuss concrete uses of history at different educational levels including teacher training. *Lesson 2* focussed on the role and use of theoretical frameworks in empirical studies related to a use of history in mathematics education. The students were presented with two studies (Jankvist, 2011; Kjeldsen & Blomhøj, 2012) which served as cases, and they then were to discuss the use of Sfard's (2008) framework of commognition in these (the students were already somewhat familiar with this framework). As supplemental literature for this lesson, students were also encouraged to look at Jankvist and Kjeldsen (2011) and the use of the Danish competency framework (Niss & Højgaard, 2011) in this. *Lesson 3* addressed the use of original sources in the teaching and learning of mathematics as well as different approaches to involving such sources (e.g. Barnett, Lodder & Pengelley, 2014; Jankvist, 2013). Also here, one purpose was to qualify the students to argue for and against a potential use of original sources in a particular educational setting. Supplementary texts for this lesson included Glaubitz (2011) and Jankvist (2014). The topic of *lesson 4* was that of history in mathematics teacher education and not least the teachers' potential benefits of being introduced to elements of the history of mathematics. Drawing on the topics of the previous lessons, the students were to compare an older empirical study (Arcavi, Bruckheimer, & Ben-Zvi, 1982) with a newer one (Clark, 2012), and discuss the outcomes of these in the light of interpreting results by means of the framework of *Mathematical Knowledge for Teaching*, MKT (e.g., Ball, Thames, & Phelps, 2008, students were already somewhat familiar with this), drawing also on a reading of Mosvold, Jakobsen and Jankvist (2014). *Lesson 5* was a workshop in which the students were to further relate the read texts to each other as well as to a concrete case of their own choice. This work eventually resulted in a hand-in mini-project report (approximately 12 pages plus appendices) for each student group. These reports were presented and discussed during *lesson 6*, where each student group would also have read the report of another group in order to provide constructive feedback and receive feedback themselves.

ILLUSTRATIVE EXAMPLES OF STUDENTS' MINI-PROJECTS

In this section, I provide three examples of students' hand-in mini-projects: one centred around an activity to compare different historical proofs of the Pythagorean Theorem; one dealing with the changing notions of the concept of function through the 18th and

19th century; and finally, one that applies the presented theoretical constructs of the course literature in an analysis of the inclusion of history of mathematics in a secondary school mathematics textbook.

Example 1: The Pythagorean Theorem

The idea of *Group 1* was to design an activity centred around four different proofs for The Pythagorean Theorem, more precisely that from Euclid, those of Liu Hui and Zhao Shuang, respectively, and finally, one from a modern day textbook. The focus of the group was mainly one of history as a goal, following a modules approach (Jankvist, 2009). The group explained:

> The case may illustrate to the pupils that mathematics develops differently depending on the culture in which the development is embedded. The difference may be seen from the many different proofs for The Pythagorean Theorem. The Chinese proofs are characterized by wanting to picture the approach and in this way convince the receiver. Euclid's proof is characterized by a more stringent way of proof and an axiomatic composition. The modern proof is also stringent, but makes use of modern notation which is easier for the pupils to follow. This amounts to an argument of history as a goal, since the different approaches to proving The Pythagorean Theorem may contribute to pupils' understanding of mathematics not being absolute, a point also made by Fried (2001). (Group 1, 2014)

Group 1 argued that their historical case in principle may be backed by all three of Fried's (2001) reasons for resorting to history:

> [i] By including the history of the proofs from different cultures and presenting the originators, the mathematical content may be 'humanized'. [ii] The history and the context may be more 'interesting' for some pupils that for others and the proofs may appeal to different pupils as well, so that more pupils may experience the mathematics as understandable, and hence may obtain an insight into the theorem and the associated concepts. [iii] Finally, examples and tasks from the history of mathematics may be used to provide pupils with insights into the use of the theorem in 'problem solving'. Such tasks might for example be the nine tasks, which Liu Hui described and solved, now named the *Sea Island Mathematical Manual*. These include tasks where one has to find the height of an island, or a tree, or the width of a river using The Pythagorean Theorem [...]. (Group 1, 2014)

Following Sfard (2008) and Kjeldsen and Blomhøj (2012), the group discussed how their historical case may introduce meta-discursive rules:

> Since the proofs are different, it becomes possible to ask the question: *What is a good mathematical proof?* That is to say, it becomes possible to make the pupils conscious about this meta-discursive rule by including history of mathematics in the teaching. By working with the historical sources, the pupils can become aware of the discourse they are themselves part of, and develop an understanding of that working with and developing mathematics is part of the discourse in force in the time of question. (Group 1, 2014)

Group 1 ended their project by discussing the potential anchoring (Jankvist, 2011) of meta-issues of their historical case in the in-issues of the four proofs.

Example 2: Concept of function

The case of *Group 2* was four different definitions of the concept of function; more precisely Euler's definitions from 1748 and 1755, respectively, Dirichlet's from 1837, and a modern definition relying on the notion of sets (e.g., see Kjeldsen & Petersen, 2014). The group aimed for a small module to be implemented in 9th grade of secondary school, since they found that the concept of function is one that is troublesome for pupils at this grade and the beginning of upper secondary school. Hence, an assumption of the group was that such a module might assist in easing the transition phase between the two educational levels (Jankvist, 2014). Unlike the previous group, this one used history as a tool:

> We intend a half-half relationship between mathematics and history, and we use the history of mathematics as a means to teach the pupils the concept of function, i.e. our 'why' is history as a tool. We use it [history] as a motivating and cognitive tool by offering different ways to introducing the concept. (Group 2, 2014)

In terms of approach, Group 2 intended a four-session module relying on the hermeneutic approach (Glaubitz, 2011):

> We find that the hermeneutic approach fits our case, because it is the contrasts between past and present that are to be examined consciously, and because it is the embracing of these tensions that provides the deeper understanding of both the mathematics itself and the history of mathematics (Barnett et al., 2014). Since we choose the hermeneutic approach we first present the pupils with the modern definition of the concept of function and afterwards the original sources. (Group 2, 2014)

The idea, the group explained, is that the pupils must relate the early definitions to the modern one. In relation to Dirichlet's definition, they said that the point of it and the modern one is actually the same, but the associated concepts have changed over time, e.g. set theory was not available at the time of Dirichlet. And by relating the modern definition and Dirichlet's to those of Euler, the pupils must obtain an idea of why Euler's concept of function is insufficient for us. Following this explanation, the group addressed the potential benefits of relying on original sources:

> One of the advantages of original sources is that they promote the reader's abilities to think like the author, and another is an understanding of the different context in which the sources are written (Barnett et al., 2014). If the pupils become aware of the historical context and try to understand what the author did, there is a chance that they also try to understand the mathematics. [...] Other advantages are, among others, to bring the pupils closer to experiencing the creation of mathematics and see the road of mathematical development, flow, errors, and success (Barnett et al., 2014). (Group 2, 2014)

Finally, Group 2 touched on the discussion of having a Whig approach to history, and even though they admitted that their purpose of using history as a tool may have such consequences, the important issue is that they did so in an informed and conscious manner:

> In our case we have chosen not to use all of the original sources, because even in the Danish translations they appear to be too difficult. Hence, we have chosen to use only the definitions, which are what the pupils get as 'original sources'. [...] We have found ourselves in the dilemma that either the original sources were too difficult, or we had to face that it was not possible to avoid being a 'little' Whiggist in our approach. Hence, we are aware of the fact that our approach is a little Whiggist, but we have found this difficult to avoid when the target group is secondary school. (Group 2, 2014)

Example 3: A textbook analysis

The case of *Group 3* was a Danish textbook system called *Sigma*, and in particular they looked at the books for 8th grade, which consist of one textbook for the pupils and one for the teacher. For their initial analysis of the books' use of history, the group relies on the constructs of Fried (2001) and Jankvist (2009). Firstly, the group discuss the textbook authors' purpose of using history and the degree to which they find this realized:

> In the teachers' textbook [...] we find the following statement in the chapter on Numbers and Algebra: *"We believe it to be important that the pupils learn about the development of mathematics and in particular that of numbers. Although such knowledge may not have a direct yield, it assists in providing the background for a part of the world, which we live in today. Without this knowledge, mathematics [...] appears as if it has always existed in the form we are introduced to today"* [*Sigma 8*, teachers' textbook, p. 6]. Here, knowledge of the history of mathematics is viewed as relevant in itself. Hence, generally speaking, we have to do with a goal argument [of using history]. The interesting thing then is how this is reflected in the pupils' textbook [...]. Through the entire chapter, we see a large focus on the history of mathematics. 10 out of 24 pages are dedicated entirely to history of mathematics, where the pupils are informed about the historical development of the numbers from hieroglyphs over Roman numerals to negative numbers and the number 0. Occasionally, the historical account is replaced by traditional mathematics exercises. However, there is almost no connection between the historical information and the exercises, since these can be solved without having read the historical account. Hence, the intention from the teachers'

book is not clearly implemented in the pupils' book. (Group 3, 2014)

Group 3 also provided another example, one on the Pythagorean Theorem, where the teachers' textbook provides an extensive account of Pythagoras, his school, and the presumed Babylonian origin of the theorem. Again, the implementation of this piece of history in the pupils' textbook is reduced to a cartoon drawing, a picture of a marble bust of Pythagoras, and several examples accompanied by modern-day notation. As pointed out by the group, the book misses an opportunity of applying excerpts from original sources here. Original sources, however, are part of the teachers' book, but the book authors' intention with this remains unclear:

> ... in the teachers' book [...] six excerpts from original sources on the proof of The Pythagorean Theorem are shown [...], but no suggestions as to how the pupils may be brought to work with these sources are provided – actually, there is no mentioning of the sources themselves, so it is unclear why they are included in the first place. (Group 3, 2014)

In further relation to the discussion of purpose of using history versus approaches to using it, the group pointed out that despite it being difficult to realize 'history as a goal' through mere 'illumination approaches' (Jankvist, 2009), this appears to be what is happening in the *Sigma* system. They continued:

> The teachers' textbook [...] contains quite a number of test exercises, but history of mathematics is not a part of any of these. In the teachers' book it is clear that history is used as a 'spice' and seen as a tool, not as a goal. In the notion of Fried (2001), what we are dealing with is a 'strategy of addition'. (Group 3, 2014)

With continued reference to Fried (2001), Group 3 went on to argue that the book system has a somewhat Whig approach to using history, in particular in the pupils' textbook. Following this, the group discussed the missed opportunities of the book system in relation to fostering Sfard's commognitive conflicts, with reference to Kjeldsen and Blomhøj (2012):

> In the teachers' textbook [...] it is stated that the pupils should become acquainted with the Roman numerals, although not to a very large extent: *"The positional number system should – once more – be examined carefully with the pupils, while the Roman numerals should not be examined as much – they merely serve to illustrate the advantages of the numeral system we apply today"* [*Sigma* 8, teachers' textbook, p. 7]. The authors' intention here is that of having one numeral system meet another in order to illustrate clearly the good idea of one of them. It is exactly in this meeting between two different discourses that the opportunity for learning arises, since the difference between the two discourses is made clear by the advantage of one of them. The intention here is for students to discover the ineffectiveness of addition in the Roman numerals as compared to our current positional system. Unfortunately, as seen before, this idea is not pursued in the pupils' textbook, which only contains little information about the Roman numerals, and not a single exercise where pupils are to work with these. (Group 3, 2014)

Finally, Group 3 attempted a small analysis of the (potential) role of history of mathematics in the Danish mathematics teacher educational program. Using their own knowledge of KOM, the group was able to deduce that from a curricular and rhetorical stance, history ought to play some role in Danish mathematics teacher education, but unsurprisingly it does not. As a potential explanation, the group pointed to the non-trivial relationship between mathematics, history of mathematics, and didactics of mathematics, and that teacher educators will need to possess knowledge and competencies not only in each of these, but also in their cross-sections.

CONCLUDING DISCUSSION

In the following, I discuss the students' yield in terms of 'history *in* mathematics education' as part of their future practice as teacher educators by relating this to KOM's didactic and pedagogical competencies (Niss & Højgaard, 2011). A basic assumption in KOM is that to be able to develop pupils' mathematical competencies, teachers must themselves possess the eight mathematical competencies. A similar assumption is that for future teacher educators to be able to develop teachers' didactic and pedagogical competencies, they must themselves also possess these. In the mini-projects, some of these competencies of course reveal themselves more clearly than others and some not at

	Curriculum Competency	Teaching Competency	Revealing Learning Competency	Assessment Competency	Cooperation Competency	Professional Development Competency
Group 1	✔	✔			✔	✔
Group 2	✔	✔			✔	✔
Group 3		✔			✔	✔

Table 1

all. The competencies of revealing students' actual learning and of assessment cannot be analyzed meaningfully based on the mini-projects. For that reason these are not discussed here, and they are marked out of the given summarizing table (see Table 1).

As for the curriculum competency, both Group 1 and Group 2 were well aware that the contents of their modules fit into the curriculum, and that their teaching sequences focus on a variety of the mathematical competencies as well. Group 1's module on proofs for the Pythagorean Theorem required pupils to develop both their reasoning and their thinking competency. Group 2's module also built on these two competencies, but draw in the representation competency to a large extent as well in the pupils' comparison of concrete functions as a means for evaluating the definitions of function. All three groups reveal aspects of teaching competency: group 1 and 2 in the active part of the competency, since they themselves plan and organize a teaching activity; and group 3 in the passive part of the competency, since they analyze already designed activities in textbooks. Group 2, in particular, also takes into account the pupils' "characteristics and needs" (ibid, p. 86) in designing their activity, e.g. by evaluating the difficulty of the original sources and by arguing for looking at the concept of function in relation to the transition problem between educational levels.

Having to do these mini-projects, the students, i.e. the future teacher educators, depend on, train, and develop their cooperation competency. In particular the KOM-framework mentions "the ability to bring the above-mentioned [...] competencies into play in mathematical, pedagogical and didactic cooperative projects" (ibid. p. 88), which is most certainly what is taking place when the students have to design these teaching activities in their groups and write up the mini-project reports, while drawing on the course literature and putting this into play in design considerations, etc. This matter of being able to apply the various theoretical constructs from the course literature is of course closely connected to the professional development competency, i.e. being able to develop one's competency as a mathematics teacher. The competency involves "being able to enter into and relate to activities which can serve the development of one's mathematical, didactic and pedagogical competency" (ibid, p. 88), which of course is a basis for the course design to begin with, but in relation to using history it is exactly the theoretical constructs from the literature which to a large extent assist the students in this. Also, this competency is "about keeping oneself up-to-date with the latest trends, new material and new literature in one's field, about benefiting from relevant research and development contributions" (ibid, p. 88), which certainly also took place since much of the literature had only recently been published.

The professional development competency is to some degree a 'meta-competency', as can be seen from its drawing on the other didactic and pedagogical competencies. Perhaps for this reason, in particular, it is essential for teacher educators. KOM's six teacher competencies are of course directed towards the teaching of mathematics, and teacher educators will have to teach mathematics as part of their future profession. So, in this sense, it is quite meaningful for the students to reflect upon the potential role of history and how it may best be used in various situations and contexts, i.e. not only in primary, lower and upper secondary school, but also in pre-service and in-service teacher education. Nevertheless, the teacher educators are also going to teach courses on the didactics of mathematics to pre-service teachers. And in this respect, the professional development competency appears relevant as well. The way the students are able to articulate their considerations and choices regarding a potential use of history using theoretical frameworks and constructs to make their arguments, is likely to be reflected in their own teaching of future teachers, and hence also in the pre-service teachers' professional development.

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Algebra in Dutch education, 1600–2000

Jenneke Krüger

University of Utrecht, Freudenthal Institute, Utrecht, Netherlands, jenneke.kruger@gmail.com

Algebra became part of mathematics education in the Netherlands in course of the seventeenth century. At first in the form of cossic algebra, but by the end of the century, the influence of the notation of Descartes was noticeable. In the eighteenth century, algebra was part of the basic curriculum of the Foundation of Renswoude. In the second half of the nineteenth century, algebra was seen as useful for a technical career. The number of topics in school algebra grew, but eventually algebra became mainly a subject in which complicated calculations were performed, which did not seem to serve a purpose outside the subject. At the end of the twentieth century, school algebra in lower secondary became a fairly informal way of solving 'practical' problems.

Keywords: Algebra education, Leiden Engineering School, Foundation Renswoude, HBS.

INTRODUCTION

When we consider examples of mathematics curricula in Dutch education it is obvious that the content of geometry didn't change very much for a long time. Some problems on geometry in the lecture notes from the Leiden Engineering School, dating from 1620 (Krüger & Van Maanen, 2014), were recognisable for a student in secondary education around 1960. With algebra it is a very different matter.

In 1600 Simon Stevin wrote the *Instruction* for the Duytsche Mathematique, as the Leiden Engineering School was called. In the *Instruction* Stevin specified the content of the course. He mentioned arithmetic, specifying calculations with decimal numbers, geometry, surveying techniques and fortification, but no algebra (Krüger & Van Maanen, 2014).

In the second half of the 18th century the Foundation of Renswoude provided talented orphans with an education for technical professions. A large part of the curriculum consisted of mathematical subjects: arithmetic, algebra, geometry (*Euclid*) and trigonometry (Krüger, 2014a). Algebra included operations with algebraic forms, formulating and solving linear and quadratic equations and also geometric, arithmetic and harmonic series.

About a hundred years later, in 1863, national legislation for secondary education was introduced by mr. J. Thorbecke, the liberal prime minister. The 'Hogere Burger School' (higher secondary school for citizens) or HBS as it was called, provided general education, with science and mathematics as important subjects and prepared students for admission to the Polytechnic School in Delft. The HBS was meant for sons of the middle class. Mr. Thorbecke advised on the content of mathematics for the HBS, the only subject on which he gave such an advice. He advised that algebra should contain quadratic equations, arithmetic and geometric series and the binomial theorem, not very different from the algebra in the Foundation of Renswoude. The HBS attracted many students. Towards the end of the 19th century the number of topics in school algebra had expanded considerably (Krüger, 2014b).

By the end of the 20th century the word 'algebra' had disappeared from textbooks, though algebra itself was very much part of the curriculum and caused a lot of debate in mathematics education (Drijvers, Goddijn, & Kindt, 2006).

Which role did algebra have in mathematics education from 1600 until 2000?

METHOD

Information on algebra in curricula from the seventeenth, eighteenth and nineteenth century was collected in the course of research on factors and actors which influenced the content of mathematics curricula in the past, from 1600 to the present (Krüger & Van Maanen, 2014). Archives, manuscripts and textbooks were an important source of data. For the twentieth century there are many sources: textbooks, articles in teachers' periodicals, the archives of the Ministry of Education, research journals, other publications on education research and curriculum development and the authors' own archives provide abundant data. For this paper only changes in the algebra curriculum are taken into account.

ALGEBRA IN THE CURRICULUM: 1600–2000

Seventeenth century: Leiden Engineering School, 1600–1681

Towards the end of the sixteenth century the Republic of the Seven United Netherlands had become rather successful in its war with Spain. The war was mainly fought around fortified towns, with help of new military techniques, which required mathematically well trained engineers. The expanding population in Holland and the strong economy also caused demand for military and civil engineers, surveyors and building masters. Prince Maurice of Nassau (1567–1625), one of the two commanders of the army and former student of Leiden university, arranged with the curators of Leiden university that an Engineering School would be attached to the university. Simon Stevin (1549–1620) wrote the formal curriculum, the *Instruction,* in which he stated the aim, the teaching language, the content, learning activities etc. (Krüger, 2010). The formal aim was to train military engineers as quickly as possible, the teaching language would be Dutch, hence the name 'Duytsche Mathematique' and the curriculum consisted of mathematical theory and practical exercises in surveying and in the basics of fortification. However, in a copy which the curators had made of the *Instruction* to have it printed, they slightly modified the aims into 'training in engineering or other mathematical professions'. The Duytsche Mathematique was going to be a course for engineers and also other mathematical practitioners.

Stevin did not mention algebra in the *Instruction*; evidently algebra was not yet useful for an ordinary engineer. But he mentioned the possibility of further study for those who had mastered the basic course in Duytsche Mathematique. Further study may have included cossic algebra, with different symbols for the unknown, its square, its cube, etc. and numerical values for known quantities (Figure 1), as used by Stifel in his *Arithmetica integra* (1544). The introduction by Viète, in 1591, of vowels for unknowns and consonants for known quantities, was an improvement, but the close link with geometry remained. Descartes used a more advanced notation in his *Géométrie* (1637) and allowed powers higher than third degree, which made algebra more useful (Figure 2).

The first teachers at the Engineering School were Ludolf van Ceulen, a respected mathematics teacher and advisor and Simon Fransz. van Merwen, a Leiden surveyor; not much is known about their interpretation of the curriculum. They both died in 1610 and were succeeded by Frans van Schooten Sr. (1581/2–1645), who was a student of Van Ceulen and Van Merwen and who had assisted Ludolf van Ceulen. His lecture notes in the library of Leiden university, *Mathematische Wercken* (BPL 1013) date from ca. 1620 (Van Maanen, 1987). They show the interpretation by Van Schooten of the *Instruction* and were also used by his successor, Frans van Schooten Jr. (1615–1660).

These lecture notes do not contain algebra, but the library of the university in Groningen possesses a manuscript by Van Schooten Sr., also a teaching text, on cossic algebra (Hs 443), which probably dates from ca. 1623 (Figure 1). The problems which are discussed in this manuscript were also used by Frans Jr. (Dopper, 2014). It is likely that Frans van Schooten Sr. taught cossic algebra in private lessons or to advanced students of the Engineering School. Frans van Schooten Jr. gave public lectures on algebra, using the cossic notation, though he knew the work of Viète and of Descartes very well. He produced a Latin translation

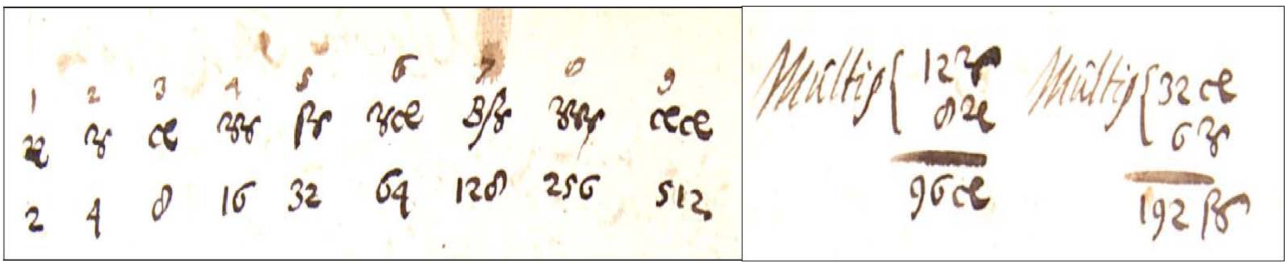

Figure 1: Hs 443, cossic symbols and examples of multiplication, by Frans van Schooten Sr.

of *Géométrie*, which made the work available to a much larger public.

Frans van Schooten Sr. was a mathematical practitioner, who worked as a teacher, surveyor and cartographer, also for the army. He had a good knowledge of mathematics and of the practice of surveying, mapping and fortification. Frans van Schooten Jr. was a very respected and well-known mathematician, but not so much a mathematical practitioner. He attracted a circle of talented private students, with whom he discussed Descartes' work. Dopper (2014) suggests that in his public lectures Van Schooten Jr. used cossic symbols instead of Descartes' notation for didactical reasons.

So there was some cossic algebra taught at the Engineering School, probably by Frans van Schooten Sr., certainly by Frans van Schooten Jr. At the start of the 17th century geometrical solutions for problems were common; gradually geometrical problems were more often translated into equations, algebra gained importance (Van Maanen, 1987, p.174–175). Van Schooten Jr. made occasional use of Cartesian notation in his *Mathematische Oeffeningen* (*Mathematical Exercises*, 1659), published shortly before he passed away. In 1672 Abraham de Graaf (1635–ca. 1717), a teacher of mathematics, astronomy and navigation in Amsterdam, published *De beginselen van algebra volgens de manier van Renatus des Cartes* (*The principles of algebra according to Renatus des Cartes*). In this book he treated the rules for operations, linear and quadratic equations and he gave many geometrical applications, but also other examples of use of algebra, using Cartesian notation (Figure 2).

So gradually algebra, with Cartesian notation, became more common.

However, surveyors, an important group of potential students for the Engineering School, were not particularly interested in new mathematical techniques, they did not feel a need to use algebra (Muller & Zandvliet, 1987; Morgenster, 1707; Van Nispen, 1662).

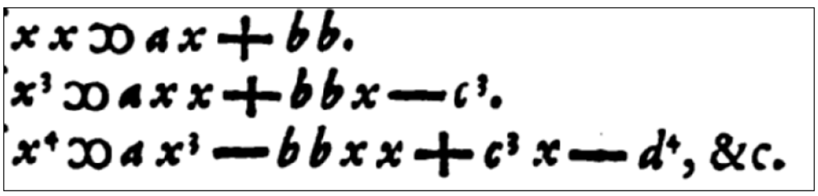

Figure 2: Cartesian notation in Algebra, by A. de Graaf (1672)

Eighteenth century

Several mathematicians contributed to the development of algebra during the seventeenth and eighteenth century (Grattan-Guinness, 2000; Struik, 1995). Examples are Wallis (1616–1703), who published his *Algebra* in 1685, Newton (1642–1727), Euler (1707–1783) and Lagrange (1736–1813). When in 1761 teaching started in the Foundation of Renswoude in Utrecht, algebra naturally was part of the curriculum. The Foundation was the outcome of the legacy of a Dutch millionaire, Maria Duyst van Voorhout, baroness of Renswoude. She bequeathed her capital to three orphanages, in Delft, The Hague and Utrecht, on condition that the money should be used to educate talented orphan boys, to prepare them for a technical profession (Krüger, 2014a). In her own words (translation by author):

> … select some of the most talented and suitable boys, at least 15 years old, to set them apart from the orphanage in order to teach them Mathematics, Drawing or Painting Art, Sculpture or Stone Cutting, practices in building dykes to protect our Country against floods or similar Liberal Arts…." (Utrecht Municipal Archive: *HUA* 771, inv. 1).

The Foundation thus aimed to educate selected boys for a wide range of technical professions. This meant that a large part of the teaching was on mathematical topics, with drawing as the second most important subject. In Utrecht the governors of the Foundation left the decisions on what mathematics to teach to the mathematics teacher. However, they took care to find an excellent teacher, with a good reputation for knowledge content and for teaching. Laurens Praalder (1711–1793) had no academic background; he had been teaching mathematics, surveying, navigation and other subjects, in the North of Holland and as mathematician and examiner at the Naval School in Rotterdam. In 1761 he accepted the position of mathematics teacher of the Foundation in Utrecht. Praalder indeed was an excellent teacher, who integrated mathematical theory with practical work and did physical demonstrations with his students. Through his extensive network he managed to find useful apprenticeships for many of his students. He also attracted private students.

The curriculum was divided into three phases, the first phase lasted ca. 2 years and was the same for every student; algebra was one of the topics in this phase. Praalder used his own teaching notes for lessons, but the students were also provided with books on all topics taught. During the first years of teaching in the Foundation, the governors bought mathematics books for the first and second phase, after consulting with Praalder. For geometry the choice was fairly simple; an edition of *Euclid* by Warius and a standard work on geometry and surveying by Morgenster, this book also contained chapters on trigonometry and logarithms. For algebra the choice was more difficult, as is clear from the different publications which were purchased during the first three years, for 11 students. Eight copies of a Dutch translation of *Elemens d'Algèbre* by Clairaux, eight copies of *Elements of algebra* by Hammond and six copies of *Algebra* by Venema, a Dutch author. After some years Venema became the standard book for most students, it offered a small amount of theory and many exercises. As Praalder taught the theory from his own notes, the students could use the book by Venema to make the exercises. Theory treated the principles of algebra, linear and quadratic equations and series.

A similar program for algebra was taught in 1776 by professor Hennert at the university of Utrecht, though ten years later he had diminished the amount of algebra taught (Bos, 1984). In 1789 the first three military academies were established in the Netherlands, their mathematics curriculum was similar to that of the Foundation. One of the students of the Foundation was appointed as a tutor to the military academy in Breda in 1789 (Krüger, 2014a).

Evidently algebra was considered useful as a tool for solving problems; geometrical and other problems. Formulating and solving linear and quadratic equations formed a major part of the content in teaching. Negative solutions of quadratic equations were sometimes ignored, so were complex roots. For most students this would be sufficient knowledge for their future career.

After Praalder retired, his colleague at the preschool of the Foundation, Dirk de West, was asked to take over the mathematics teaching in the Foundation. De West used copies of Praalders teaching notes, including the four bands on algebra, for his lessons. In later years four copies of Euler's *Elements of algebra* were bought, probably for the more advanced students, who could not be taught further by De West.

Nineteenth century

Characteristics of mathematics in the nineteenth century were: an increasing specialisation, more emphasis on the distinction between pure and applied mathematics and a striving for mathematical exactness (Grattan Guinness, 2000; Struik, 1995). The algebraic properties of numbers, the fundamental theorem of algebra, roots of equations, complex numbers and quaternions were some of the topics in which much development took place. The mathematics curricula of secondary schools were somewhat influenced by these developments.

In the Netherlands secondary education remained a matter of private initiative until 1863 (Krüger, 2014b). Industry and commerce required an educational system of good quality, which would offer subjects such as mathematics and science, modern languages and economics. Also the Royal Academy for Civil Engineers in Delft and the Royal Military Academy in Breda had problems with the relatively large number of students who entered with insufficient knowledge of mathematics and science. In 1863 the HBS, with a curriculum of five years, was Thorbecke's answer to these problems. The programme for algebra should enable the pupils to do well in the physics lessons. Also after the first three years of HBS the pupils should be able to pass the entrance exams of the Royal Military Academy and after passing the final exams a student should be ready for the Polytechnic School, the successor of the Royal Academy for Civil Engineers. The mathematics teachers at the HBS had to have a university degree or an equivalent teaching degree.

Mathematics was one of the 18 subjects taught in the HBS and one of the 16 subjects in the final exams. Mathematics was subdivided into several topics: arithmetic, algebra, plane and solid geometry, trigonometry and descriptive geometry. The final exams on mathematics consisted of four exams: arithmetic/algebra, geometry, trigonometry, descriptive geometry. Mathematical and science subjects took ca. 33% of teaching time, the same amount as the four languages and literature. Mathematics received ca. 18% of the teaching hours, more than any other subject (Steyn Parvé, 1868).

Until about 1890 most students at the HBS did not take the final exams. Many students left school after the first three years, to sit for entrance exams for a range of professional institutions or to enter the workforce. The content of mathematics in the first three years consisted of arithmetic, algebra and geometry, usually 2 hours each in every year. During the first three years of algebra the pupils were taught the main operations, also with fractions; roots, powers, negative and fractional exponents; linear and quadratic equations and sometimes also equations of degree higher than two, exponential equations and indeterminate equations. In year four and five there was more difference between schools in the choice of topics. Newton's theorem (an extension of the binomial theorem), factorizing, combinations and permutations, infinite series, convergent series, continued fractions, limits, theory of irrational numbers and complex numbers were topics which were taught, but not in all schools. The exam syllabus, introduced in 1870, mentioned arithmetic and geometric series, Newton's theorem, indeterminate and exponential equations. Teachers and textbook authors included topics outside the prescribed exam syllabus for various reasons, i.e. the topic occurred in entrance exams of some institutions or the topic was of educational or didactical value. There were many different textbooks available, most of them written by teachers of HBS or gymnasium (the former Latin school).

By 1899 the authors of a popular textbook on algebra explained that they had not included the theory of irrational numbers, as there was no time to treat the topic with the necessary exactness (Derksen & De Laive, 1899). Gradually, the topics which were not in the final exams would be treated in fewer schools.

In the Polytechnic School algebra would be used in analytic geometry and in analysis. There were occasional complaints about lack of skills of first year students, but overall the results of the first year students in these topics were satisfactory (*Onderwijsverslagen*, 1866–1876 [Education reports to Parliament, 1866–1876]).

Twentieth century

Groen (2000) presents a concise overview of changes in the mathematics curriculum of the HBS between 1900 and 2000. In 1937 a new mathematics curriculum for the HBS was introduced, after many fruitless attempts to modernize the curriculum. Differentiation and integration, number theory, functions and graphs were some of the new topics. However, the exam syllabus did not change and so the topics in the final examinations remained the same. As the final exams had gained considerable importance since the start of the HBS, this meant that the new curriculum did not have much influence on the lessons. In 1958 a new curriculum was introduced, this time including a new exam syllabus. Analytic geometry, differentiation and integration were part of the new curriculum; descriptive geometry was at long last abolished. There was less emphasis on extensive calculations in algebra.

In 1963 new legislation introduced a new structure for secondary education; the HBS was replaced by new types of school. The pupils should receive general education and preparation for entrance into higher education (Van Kemenade, 1981). The new curriculum started in 1968; it was influenced by the ideas of 'New Math', as happened in many countries. For a short time algebra was fairly abstract, with emphasis on formal language and sets. Algebra was mainly taught in the first three years of secondary schools; pupils in higher years were supposed to make use of algebraic techniques in differentiation, integration and analysis. In the 1970's, new teaching methods for lower secondary schools were introduced, exercises in algebra were based on realistic situations (contexts); the use of tables, graphs and simple formula's was emphasized (Kindt, 2000). In 1998 new exam syllabi for upper secondary schools were introduced, these contained topics such as graphs, functions, combinatorial analysis, matrices, analysis and differentiation and integration. It was a very full curriculum, but as at the same time the graphic calculator was introduced into the schools, the curriculum designers assumed that the use of a graphic calculator would save much time. This curriculum was supposed to improve transition to higher education and also to provide general education. One of the many reasons it didn't work as expected was the lack of algebraic skills with which students entered upper secondary school. Another reason was the lack of training of teachers in the use of graphic calculators in teaching (Krüger, 2014b).

CONCLUSION

From the mid-seventeenth century algebra was recognized as a valuable topic in mathematics education. The content expanded from mainly arithmetic with letters and modelling problems in linear and quadratic equations to a variety of topics at the end of the nineteenth century. Algebra in the curriculum of the HBS was considered useful for physics; it also was part of the preparation for technical studies. In the twentieth century the emphasis shifted, at first from elaborate calculations to a more formal approach of structures, following that with an emphasis on global understanding and modelling of 'realistic problems'. At the end of the twentieth century there were many complaints about (lack of) algebraic skills, made by teachers from universities and colleges, by high school teachers and gradually also by other people who felt the need to make their point. It is interesting to note that this type of complaint was also made in the nineteenth and the first part of the twentieth century. It is evident however that the introduction on a large scale of new learning materials, graphic calculators and software for algebra, is bound to have a large impact on the learning of algebra (Drijvers & Van Reeuwijk, 2006; Goos, Galbraith, Renshaw, & Geiger, 2003). These developments should have a large impact on teacher training and teaching methods as well. These aspects received insufficient attention from nearly all those who were involved.

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E-Dynamic.Space: A 21ST century tool to stage-manage and build experience in the field of the history of mathematics and its teaching

Caroline H. Kuhn

Bath Spa University, Bath, UK, c.kuhn@bathspa.ac.uk

This study aims to answer the question of how can the history of mathematics resort to a digital tool – E-Dynamic.Space – designed by teacher-students and intended to serve as a workbench not only to create supportive knowledge from historical material, which has proved to support the understanding of mathematics but, also to orchestrate both, their actual learning of the tangent line problem and their future mathematics teaching experience. It therefore explores aspects for the design of the teaching activities, and it analyses the 'whys' and 'hows' of including the historical dimension into the teaching experience. It is part of a bigger research project that looks at how can teacher-students favour from a historical informed pedagogy of mathematics that uses a personalised learning environment as a means to learn.

Keywords: Digital tool, PLE, orchestration, tangent line problem, teacher education.

INTRODUCTION

Digital technologies are the landmarks of the 21st century, ubiquitous and bearers of social identity for the majority, especially young people (Boyd, 2014). In what way can digital technologies support 'history in mathematics education', which is understood here as the learning of mathematics supported by the integration of elements from the history of mathematics. Researchers of mathematics education call for more research of digital technology (e.g., Hoyles et al., 2010; Trouche & Drijvers, 2014), and this paper extends their call to the integration of supportive knowledge created by teachers (Kuhn, submitted at the 7th ESU in History and Epistemology of Mathematics) using web-based tools, which have their own possibilities and difficulties. In this paper, digital technologies are understood in a broader sense, not only as mathematical software but also as web-based tools and social media, which brought together by the end user in a flexible digital environment will constitute what I will call from now on personalised learning environment (PLE) (Buchem, 2010; Kuhn, 2014a[1]).

Teachers have to keep up to date with young students' mind-set and expectations, and the advancement of digital technologies. On the other hand teacher-students need support in being prepared in a sensible manner for their job in the near future and to move confidently in this new ecology of digital resources (Luckin et al., 2012). There is evidence that shows how designing and developing a PLE will improve the digital skills of the end-user (Wild et al., 2008), teacher-students in this particular case.

How can teachers explore the affordances[2] of digital tools, take advantage and build experience in the digital world and in the field of history and mathematics, integrating them for the learning of mathematics and further teaching? Looking for possible answers I propose a PLE, E-Dynamic.Space (Kuhn, 2014a) as a 21st century self-management tool, designed and populated with new content created by teacher-students to support them in the design and organisation of the learning experience.

My proposal aims to address not only how to stage-manage the learning of mathematics but also to explore how teacher-students can create supportive material from historical sources, which has proved to support the understanding of mathematics (Kuhn, submitted op. cit.), using the E-Dynamic.Space as a tentative tool for constructing meaning or in words of Noss & Hoyles (1996), webbing[3] in the process of grasping and understanding the tangent line. The tangent line has been chosen as a starting point that will set the ground of a number of concepts to develop (as a mid term goal of the project) in order to craft a more unified and connected way of teaching the background concepts of calculus for GCSE and A-levels in the UK.

In a first stage of the project I will focus on the design of some of the teaching activities I propose for teacher-students and in a second stage, not addressed in this paper, I will look at how they can transfer these skills to their classroom practice and improve the learning experience of their pupils. Empirical evidence indicates that following the work of teacher-students during "a time long enough to be able to catch real changes (a) during a program, (b) immediately after the program, and (c) one or more years later, can assist in providing valid feedback mechanisms for professional development programs" (Trouche et al., 2013).

SOME THEORETICAL ASPECTS FOR THE DESIGN

Troublesome Knowledge

The introduction of analytic geometry revived the tangent line problem in early 17th century. Descartes in his 1637 work La Géométrie described the problem of finding a tangent line to a given curve at a specific point as:

> (...) the most useful and general problem in geometry that I know, but even that I have ever desired to know. (Smith & Latham, 1925, p. 95)

Reading this sentence in combination with my interest in the calculus as a rich topic, both historically and conceptually, made me wonder why would an intellectual of the calibre of Descartes find this problem so useful and worth knowing. Although I was motivated and thrilled to know more I encountered difficulties while finding my way into Descartes' ideas. I found myself confronted with some trouble, or maybe with troublesome knowledge? But what exactly is troublesome knowledge and what it has to do with the history of mathematics in mathematics education? The notion derives from a research project in the UK looking to identify key factors leading to high quality learning environments in higher education, very much in line with the aim of my own research. The idea is associated with threshold concepts, conceptual gateways that have the potential to open up new conceptual spaces transforming the way learners understand the subject matter (Meyer & Land, 2005). Threshold concepts, although usually attached to particular concepts, sometimes they are not necessary concepts in any rigorous sense but different ways of thinking and practicing with a threshold-like nature, all of them providing entrance in one sense or another to a new or different conceptual landscape (Meyer & Land, op. cit.). Transformative ideas, and it is in this sense that I am using the term.

These ways of thinking and practicing, often lead to what Perkins (2006) describes as *troublesome knowledge*, knowledge that is conceptually complex, alien or counter intuitive, thus challenging students' beliefs and intuitive knowledge but at the same time, developmental productive. This is in line with Barbin's idea within mathematics education, of *depaysement* or reorientation, challenging student's perceptions, making the familiar seems unfamiliar. History shows also how mathematics is a human understanding, a history of human beings disabling or extending established ideas, allowing the learner to see mathematics as much more than disconnected algorithms or discrete chapters, integrating the subject in a sociocultural context.

Why and how to use this historical knowledge in mathematics education?

The previous section introduced, in a general way, some of the reasons for using history of mathematics in teacher education. Adding to this Jankvist (2009) answers this question in a more focused and didactical oriented way connecting it with content knowledge, suggesting that history can be used as a goal or as a tool. In particular he refers to a cognitive tool for the learner (teacher-student in this particular case). In this latter sense he implies the idea of epistemological obstacles (Jankvist, 2013). Brousseau (1997) highlights in this regard that knowledge exists and it makes sense only because it represents an optimal solution in a system of constraints. For him, history can be illuminating in finding those systems of constraints. Sierpinska (1994) suggests: "epistemological obstacles are not obstacles to right or correct understanding: they are obstacles to some change in the frame of mind." (p. 121)

Dimensions of knowledge in teacher training that can profit from the history of mathematics

One of the aims of our community for the history and pedagogy of mathematics is to find ways in which teacher-students can profit from the history of mathematics for their learning/teaching experience. In each profession there are core skills and knowledge to be mastered. In mathematics education, Ball and colleagues (2008) have developed a theoretical framework, Mathematics Knowledge for Teaching (MKT), proposing the kind of knowledge demanded by the teaching profession. This framework has been explored recently by Clark (in press), cited by Jankvist and colleagues (2012). She contextualised it in the history of mathematics exploring how the history can add to teachers' MKT. In this work I will use three of the six dimensions of the model: knowledge of content and curricula (KCC); knowledge of content and students (KCS), and horizon content knowledge (HCK), in order to see how teacher-students' knowledge can potentially profit from and be enhanced by the history of mathematics. This choice responds in part to a call that Jankvist (op. cit.) has made to address the absence of clearer links to general mathematics education research frameworks. This theoretical construct –the MKT- seems to have productive implications for teacher education (Jankvist, op. cit.).

Epistemological obstacles and conceptual development, and its association with the Mathematics Knowledge for Teachers

Tracing the historical development of a particular concept, following Brousseau (op. cit.) is a way to understand the constraints of each time, hence to understand some of the epistemological obstacles involved in the development of an idea. Connecting epistemological obstacles with the didactical situation of teacher-students is possible through the idea of conceptual development, which has been researched for didactical purposes by different authors (e.g., Vosniadou, 1994). The general consent is that for conceptual change to happen there must be, in the student, a cognitive conflict or a 'stuck place' in words of Meyer & Land (2005), a difficult stage in the conceptual development as it confronts them with different epistemological obstacles (Brousseau, 1997) blocking any transformation in the cognitive realm. Teachers are responsible to identify the sources of those obstacles and free them up making the change possible. In this regards, teachers ought to develop knowledge of content and students (KCS) (Ball et al., 2008).

This 'stuck place' is similar to what happens to the collective culture of mathematicians throughout the development of an idea. Teachers can look closely at these epistemological obstacles in order to find inspiration and knowledge to identify possible sources of obstacles in their students. This kind of understanding can also improve teachers' knowledge of content and curriculum (KCC) allowing them to make a historical informed decision in relation to the breadth and depth they should teach in the different key stages. All of the above seems to add to a wider kind of knowledge, one that goes beyond the basic knowledge teachers need to deploy in class. Following Ball & Bass (2009), it is called horizon content knowledge (HCK), and they describe it as " (...) an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated" (p. 6). This kind of knowledge "confers a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment." (p. 6). There is evidence (Mota, 2008; van Maanen, 2009) that this knowledge will profit from the history giving teacher-students a wider breath of the mathematical cultural context of a particular idea to be taught.

Having explored the whys of using history in teacher education and looking at how the mathematics knowledge for teaching can be enhanced by the use of historical material, let us look at how can this material be integrated in the teaching experience. Taking into account the varied background of Bath Spa University PGCE students (PGCE responds to Post Graduate Certificate in Education and it is a one year program for students with different backgrounds that want to become teachers), I decided to follow Tzanakis and Arcavi's (2000) idea of *historical packages* in which a mathematical topic (in my case the tangent line problem) from the curriculum is taught by means of historical materials in a relatively short period of time; similar to Jankvist's approach with historical modules.

How can a teacher-student get involved with the history of mathematics in order to gain a deeper understanding of the epistemological development of the concept and also take advantage of the affordances of the PLE and its available tools? One way to do this is through the activities proposed below for which the didactical intention is underpinned by the idea of webbing described previously. There is also a wider mathematical aim and it is to explore in depth the development of the tangent line problem in order to gain a deeper understanding and a wider vision, in epistemological terms about the historical process of the definition of the derivative in terms of the limit; for that the tangent line is key. In words of Whiteside (1961): "It will be illuminating therefore, to discuss the particular methods invented to resolve the tangent-problem, and this will yield a truer perspective on the *elegant general treatments which were later abstracted from the particularised methods of the mid-century* [emphasis added, p. 348]." History shows that the starting point of that definition was neither limits nor the differentials or fluxions. It has been a process of successive abstraction (Lehmann, cited in Swetz et al., 1995), which is what we aim to trace with this module.

THE HISTORICAL TOUR: FROM EUCLID TO FERMAT

In this section I will describe briefly what teacher-students will explore during the sessions. The online sheets and the web-based tools are allocated in the PLE, which they will further populate with their own creations. The didactical intention is that the learner generates new supportive knowledge as a product of webbing while exploring the historical material, making sense of new chapters of the tangent line's history. Teacher-students will course from Euclid to Fermat and reflect around the systems of constraints of each period identifying the epistemological obstacles and the change in the collective frame of mind. In doing so they will become the appreciative tourist of the larger mathematical landscape as they advance in their epistemological tour.

We need to bear in mind that there is this unavoidable risk – clearly explained by Fried (2001) – of doing 'Whig history'. In order to address this issue (though not sure to completely avoid it) an initial reading of his paper (2001) is assigned to the group.

Time and allocation of session is to be determined

As an integrative and final activity for webbing the learning of the topic and also intended to develop the epistemological understanding of the concept, students will create an interactive timeline with at least two of the resources created by them through out the module. They need to add the group reflections where pertinent and illustrative, as well as the relevant comments posted in Padlet. Highlighting new frames of mind is important in this task. The interested reader is invited to follow the link[9] to explore the PLE with man examples of sheets and resources, as well as a time line crafted by the author to explore the affordances of the tool.

DATA COLLECTION AND THE PROCESS OF WEBBING THE UNDERSTANDING OF THE TANGENT LINE PROBLEM

As suggested by Barbin and colleagues cited in Fauvel & van Maanen (2000), we can evaluate the effectiveness of introducing a historical dimension into teacher education through an examination of each of the processes involved in the development of understanding, namely, the change in how teachers perceive and understand mathematics which generally is reflected in the way they subsequently will teach, and finally in the understanding and perception of their pupils about mathematics. None of those processes can be captured in a quantitative approach, instead a qualitative and holistic method is much more desirable for understanding more in depth how to best integrate historical material into the teacher experience. Therefore qualitative data will be gathered (with the proper software, e.g., Camtasia) in their online public and private spaces. There is also a reflective logbook with didactical prompts (still under development) for each student to document their learning; the process of webbing the tangent line problem making sense of the different frame of mind and the historical development of the concept studied. The prompts will trigger in the student the cognitive processes that will help them to describe their main struggle when trying to elaborate the resources. In particular the timeline is considered a rich intellectual artefact with the potential to uncover partial understandings of the student in relation to the epistemological advancement of the concept. What resources they choose, what they consider to be an illustrating example and how they justify it will reflect students' process of constructing meaning throughout the task. An important aspect of the learning experience will be the idea of extending the web of ideas and intellectual resources (at the beginning of the journey) and re-structuring it as a result of the connections made for the learner to be able to find and construct meaning through the sequence of activities he/she is doing including discussion and reflection.

Author	Resources	Task + Question + Reflection
Euclid	Book III, def. 2 and prop. XVI Online version of Oliver Byrne[4]	Go through the definition and work out the proposition in your group, post your work in padlet[5] for a common discussion. Have a look at other posts and comment on at least one+reflect
Apollonius	a. Module of the MAA: 'Tangent line then and now'	Read the extract about Apollonius method. Interact and explore the GeoGebra example and discuss with your peers your thoughts, difficulties and any 'aha' moment.
	b. Treatise of conic section. Heath translation 1896. Online[6]	Go through proposition I.33 and discuss, try to make sense of it with your peers. Find the analogue elements with the MAA method and document the process in your logbook. Pay special attention to any difficulty in understanding any of the parts, documenting it for further thinking in the group discussion.
	c. Working with online sheets in GeoGebra[7]	Work in pairs and interact with the sheets for finding the tangent line to a parabola. Produce your own example in GeoGebra, record the steps in the sheet and post it to GeoGebra Tube. Do one of the sheets posted by your peers, comment your experiences in Padlet (difficulties + ahas + findings + remarks). Reflect on the system of constraints you think could be present in that period and what implications do they have in the method you just did. Read and comment on one post in the wall
Descartes	a. Look at the video[8] by Jeremy Gray about the history of the calculus. b. The History of mathematics (Fauvel, J & Gray, J.) Section 11.A10	Watch with particular care the section where the method of Descartes to find the tangent line is explained. Read section 11.A9 to complement. Make notes for a discussion session with the group about the steps of the procedure. Try it your self with a simple curve ($y=x^2$) and document the process. Work in a small group for a richer and more reflective discussion.
	Online GeoGebra sheets in the PLE	Go to GeoGebra and do the sheet with Descartes' method. Take notes about things that were important for you to accomplish the task, key ideas. Think about your own difficulties along the exercise and write them down for a common discussion. Try to write about the new mathematical features introduced in his method and compare it to Apollonius' one. Reflect about the system of constraints in Descartes' time and think about the new frame of mind introduced then. Think about the implications and epistemological difficulties of a more general method comparing it to the Greeks (section 11.A10)
Fermat	The history of mathematics (Fauvel, J & Gray, J). A copy of the relevant text of module 9 of the Open University course: Topics in the history of mathematics	Read section 11.C – 11.C1b. Tinker with the method. Try to create your own example working with a curve that you feel comfortable with (use pencil and paper). Make notes about your process and document any difficulties. Think about the *adequate* method he used, trace the history of the term and give meaning to it in that context. What was the problem then? Can you see why Fermat could work with a wider range of curves? Go to GeoGebra, work through the sheet. Once you have understood the method create your own sheet with its animation and uploaded it to GeoGebra Tube. Try out one of the sheets of your peers and comment on his/her work. Reflect on the steps taken by your peer.

Table 1: Sources and questions in relation to the tangent line problem

A session will be dedicated to reflective writing and how to do it in a way it can enhance their own learning process. The three dimensions, specified above, of the MKT framework will be explained in detail and their reflections will be stated in terms that make reference to these dimensions so consistent evidence can be collected (still an idea under development). The prompts given in the activities are focused in the mathematical features that have been shifting from time to time and are intended to bring students to reflect on how those changes have transformed the tangent line problem into what we know today giving them a wider background knowledge or in words of Ball & Bass (op. cit.) enhancing their HCK. Important ideas to grasp along the learning experience are the optimal solutions in a system of constraints stated above by Brousseau and the change in the mind frame argued by Sierpinska.

The module has not been tried out yet therefore a real and fruitful discussion will be part of a next piece of research, where the data will be collected and analysed, and hopefully the analysis will shed some light to the rich discussion in relation to the benefits of the integration of history for the mathematics education community.

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ENDNOTES

1. Available at: http://portal.sinteza.singidunum.ac.rs/paper/114

2. Affordances are in this context related to the digital world and it refers to the potentialities and constraints of different modes that digital tools allow. What is possible to represent with the resources of a mode and what is not.

3. Defined by Noss and Hoyles (1996) as the fundamental motor for the construction of meaning.

4. The first six books of The Elements of Euclid, described as one of the oddest and most beautiful books of the 19th century. Available at: http://publicdomainreview.org/collections/the-first-six-books-of-the-elements-of-euclid-1847

5. Padlet is a web-based tool. It affords to have a collaborative discussion and upload files to it (https://padlet.com/)

6. https://archive.org/stream/treatiseonconics00apolrich#page/n9/mode/2up

7. http://hom.wikidot.com/calculus-1 (by Gabriela Sanchis, under Creative Commons Attribution ShareAlike 3.0 licence)

8. https://www.youtube.com/watch?v=OTMkCLtflHY

9. http://www.symbaloo.com/home/mix/13ePQJ81NS

The history of the fourth dimension: A way of engaging pupils in secondary classrooms

Snezana Lawrence

Bath Spa University, Faculty of Education, Bath, UK, s.lawrence2@bathspa.ac.uk

The National Curriculum in England has, over the past decade, been revised a multitude of times. Disengagement of pupils was one of the reasons for revisions. In September of 2014, a new curriculum in mathematics was introduced, aiming to give greater freedom to teachers and schools to construct a curriculum and teaching episodes that are engaging and appropriate for their students. This paper investigates how such episodes can be constructed through the investigations based on historical development of mathematical concepts and how they could easily link to the new curriculum, offering at the same time greater opportunities for pupils' engagement. The history of the fourth dimension is one such possible topic, and the paper suggests a way of using it in a secondary classroom.

Keywords: Fourth dimension, engagement, Schläfli, Stringham, Flatland.

INTRODUCTION

In the decade between 2004 and 2014, mathematics curriculum changed twice, about fifty reports on mathematics education in the country have been published, and seven different Secretaries of State for Education passed through the British Parliament. Each of the changes and reports suggested that the state of mathematics education in Britain is troublesome; the causes were identified, the evidence was given (in either a narrative or analytical format), and of course the suggestions to improve the situation were recommended.[1] The most troublesome of all troubles listed was the perceived irrelevance of mathematics and the lack of desire to engage with it.[2]

When exploring the issue of disengagement, teachers reported that it was the curriculum that narrows down the topics and the lack of choice to engage with different topics from the curriculum that was at the basis of the problem.[3] The organizations such as National Strategies (discontinued 2010) previously tried to help teachers devise teaching episodes and gave suggestions on pedagogy. The new curriculum instead offers an element of autonomy, meaning that schools and teachers are given freedom to choose and design the topics and teaching episodes appropriate to their environments. Likewise, the curriculum itself lists the skills and knowledge to develop in pupils, but gives no (or minimal) guidance as to the choice of topics.

The choice of topic described in this paper – the history of the fourth dimension – arose from two experiences: of working with gifted and talented pupils some years ago on the representations of the fourth dimension in mathematics (Lawrence, 2012) and the engagement reported in alternative curricula, such as, for example, Steiner system, which introduces projective geometry and the study of the fourth dimension in the final years of secondary school (Woods et al., 2013).

The paper thus first gives a historical overview of the topic, and then investigates the possible pedagogical approaches to develop material based on certain principles listed. It concludes by showing how an unorthodox topic such as this, can nevertheless be easily linked to the new curriculum and the skills and knowledge it aims to develop in pupils. The engagement is expected but not yet empirically proven; this is proposed as a possible future study, and some initial experiments from schools in which the teaching pedagogy is trialed are described.

HISTORY OF THE FOURTH DIMENSION

Whilst the concept of the fourth dimension was developed in the nineteenth century, the origins of it could be traced as far back as the antiquity in the most possibly wide sense of conceptual development. Aristotle for example, discussed it in *De Caelo*,[4] and Ptolemy denied and disproved it but nevertheless mentioned and contemplated upon it (Cajori, 1926, p. 397; Heiberg, 1893, p. 7a, 33). John Wallis, although writing this whilst considering geometric interpretations of quantities he was developing in the context of algebra, wrote (Wallis 1685, p. 126):

> A Line drawn into a Line shall make a Plane or Surface; this drawn into a Line, shall make a Solid: But if this Solid be drawn into a Line, or this Plane into a Plane, what shall it make? A Plano-Plane? That is a Monster in Nature, and less possible than a Chimaera or Centaure. For Length, Breadth and Thickness, take up the whole of Space. Nor can our Fansie imagine how there should be a Fourth Local Dimension beyond these Three.

With the French Revolution, some revolutionary mathematical thinking happened, and Lagrange in particular, spoke of three coordinates to describe the space of three dimensions, introducing time as the fourth, and denoting it *t* (Lagrange, 1797, p. 223).

The reader is reminded that this can by no means be an exhaustive study, but is a sketch of the history of the fourth dimension and the narrative given is but a thread that will later be examined in possible educational setting and application to teaching.

Let us then trace further historical development. The first such opportunity was the example of Möbius and Zöllner. Möbius (1827) first spoke about an object getting out of a dimension it belonged to in order to perform a spatial operation. If one had a crystal, structured like a left-handed staircase, how would one get its three-dimensional reflection? Zöllner (Johann

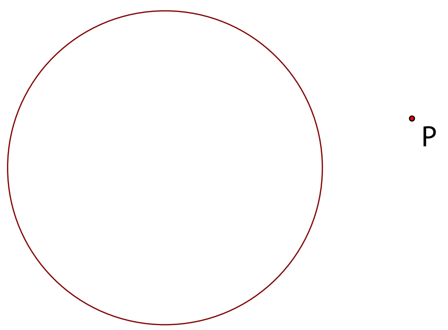
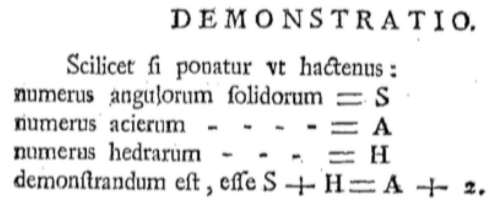


Figure 1: Zöllner illustrates that in order to perform certain operations in space, objects must exit their current dimension (Zöllner, 1878)

Friedrich, 1834–1882) further simplified this. If one has a circle and a point outside of it, how can one get the point into the circle without cutting or crossing over the circumference?

In 1852, Ludwig Schläfli (1814–1895), a Swiss mathematician published a book *Theorie der vielfachen Kontinuität, (Theory of Continuous Manifolds)*, in which he wrote about the four dimensions. Schläfli looked at *Elementa doctrinae solidorum* published in 1758, in which Euler described for the first time what was to become known as *Euler's characteristic*, the expression which conveys the information that in all convex solid bodies the sum of the solid angles and the number of faces is equal to the number of edges add 2.



DEMONSTRATIO.

Scilicet ſi ponatur vt hactenus:
numerus angulorum ſolidorum = S
numerus acierum - - - - = A
numerus hedrarum - - - = H
demonſtrandum eſt, eſſe S + H = A + 2.

Figure 2: Euler's characteristic first described in *Elementa doctrinae solidorum*, 119

We now usually denote Euler's characteristic by Greek letter chi and describe it for convex polyhedra $\chi = V - E + F = 2$, where V is the number of vertices, E is the number of edges, and F is the number of faces in a polyhedron. If we further analyze the formula we notice that we begin from the first variable which counts points (point we earlier took to represent 0^{th} dimension); the second variable which numbers the edges in a solid, (representing line, 1^{st} dimension) and the third variable, numbering the faces of a solid, (polygon is bound part of a plane, representing the 2^{nd} dimension).

Schläfli (1852) showed that this formula is also valid in four dimensions or indeed any higher dimension. We will get there – but let us first look at how he first defined a system which would describe any regular polytope in any dimension.

There is only one polytope in 1^{st} dimension, a line segment, and the Schläfli symbol denoting this is {}. Regular polygons in two dimensions are, for example, triangle {3}, square {4}, pentagon {5}, etc. Remembering that he only used these symbols to denote regular polytopes, we continue. In three dimensions the five regular polyhedra, Platonic solids, can be described as {3, 3} – tetrahedron has three-sided polygons that meet three at each vertex; {4, 3} – cube has four-sided polygons three of which meet at each vertex; {3, 4} – octahedron has three-sided polygons four of which meet at each vertex; {3, 5} – icosahedron has three-sided polygons five of which meet at each vertex; {5, 3} – dodecahedron has five-sided polygons, three of which meet at each vertex (Schläfli, 1852).

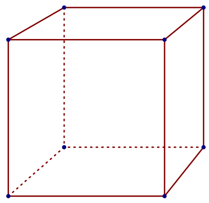


Figure 3

Schläfli realized and showed that Euler's characteristic can be represented in a slightly different form: he stated that $V - E + F - C = 1$, in effect stating that the number of vertices, minus the number of edges, plus the number of faces, minus the number of cells, equals 1. Cell in three dimensions is a solid (convex); so for a cube this formula would be:

8 (vertices) – 12 (edges) + 6 (faces) – 1 (cell) = 1

He then showed that the minus plus pattern continues with even dimensions (0th, 2nd, etc.) having positive value and odd (1st, 3rd, etc.) negative. This was a big breakthrough: by extending the validity of Euler's characteristic to the fourth and any other higher dimension, Schläfli showed that it was possible to calculate various characteristics of four-dimensional polytopes if we had certain other information. This also meant that the four-dimensional solids could be now identified, classified, and studied.

The mathematical description of generating the fourth (and higher dimensions) was first given in an elegant way by William Stringham (Stringham, 1880, p. 1):

> A pencil of lines, diverging from a common vertex in n-dimensional space, forms the edges of an n-fold (short for n-dimensional) angle. There must be at least n of them, for otherwise they would lie in a space of less than n dimensions. If there be just n of them, combined two and two they form 2-fold face boundaries; three and three, they form 3-fold trihedral boundaries, and so on. So that the simplest n-fold angle is bounded by n edges, $\frac{n(n-1)}{2}$ faces, $\frac{n(n-1)(n-2)}{2}$ 3-folds, in fact, by $\frac{n!}{k(n-k)!}$ k-folds. Let such an angle be called elementary.

Stringham tried to illustrate this in his paper in the following manner (Figure 4).

The study of the dimensions became something of a vogue in the 19th century and many a famous mathematician, from Graßman (1844), Riemann (1854), Clifford (1873) and Cayley (1885) to name but a few, wrote on it. But how to translate this into a classroom experience for teenagers? A novel from 1884 may give us some guidance on that.

NARRATIVE ABOUT LIFE IN DIFFERENT DIMENSIONS

The introduction to 'dimensionality' could be given and illustrated finely through the metaphor about *Flatland*, the 19th century English mathematical novel written by Edwin Abbott Abbott (1838–1926), a London schoolmaster and Shakespearean scholar. Flatland is a land that is flat. It is (Abbott, 1884, p. 2):

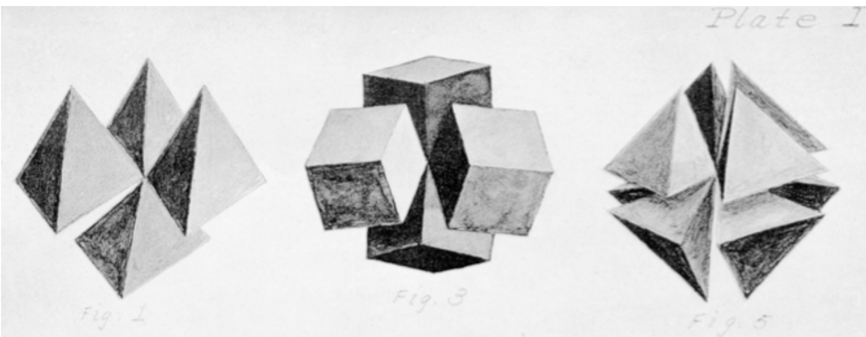


Figure 4: These represent 'respectively summits, one in each figures of the 4-fold pentahedroid, oktahedroid, and hexadekahedroid, with the 3-fold boundaries of the summit spread out symmetrically in three dimensional space' (Stringham, 1880, p. 6)

> like a vast sheet of paper on which straight Lines, Triangles, Squares, Pentagons, Hexagons, and other figures, instead of remaining fixed in their places, move freely about, on or in the surface, but without the power of rising above or sinking below it, very much like shadows – only hard and with luminous edges – and you will then have a pretty correct notion of my country and countrymen…

Flatland, whilst it gives opportunities for many discussions to be brought into the mathematics classroom, also offer a good introduction to the bigger questions that are not easily dealt with in mathematics education. Abbott for example raises the question of ethics and the place of women in the world as he knew it. The two dimensional beings who are stuck in the two dimensional reality are also stuck in the belief that women should be treated in a different way to men. One of illustrations from this strange world shows that very clearly.

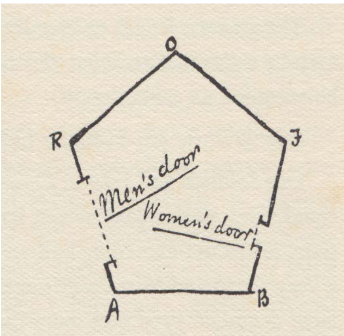


Figure 5: The most common construction for a house in Flatland, with separate doors for men and women

Apart from the social dimension, Flatland further offers ample opportunities to introduce some higher order thinking about 'big' questions in mathematics – what is the nature of space for example and how many (real) dimensions does it consist of? The question of dimensionality is in the book introduced with an almost mystical experience that the main protagonist, Square, has when he meets Sphere. First the Square saw Sphere through Sphere's intersection with the Flatland, but eventually Sphere spoke the Square. And the 'mystical' wasn't that after all as Sphere explains (Abbott, 1884, p. 77):

> Surely you must now see that my explanation, and no other, suits the phenomena. What you call Solid things are really superficial; what you call Space is really nothing but a great Plane. I am in Space, and look down upon the insides of the things of which you only see the outsides. You could leave this Plane yourself, if you could but summon up the necessary volition. A slight upward or downward motion would enable you to see all that I can see.

Of course Sphere is, similarly to Square, stuck in his own world of limited dimensions and when, towards the end of the novel, Square regurgitates the analogy between dimensions and speaks of projections of fourth dimensional bodies in three dimensions, the Sphere explains a simple "Nonsense!" – of course there are no higher dimension than that which he could experience.

What Flatland offers is an introduction to the discussion about dimensions in mathematics that can lead to asking pupils to imagine a life in dimensions different to the ones they are used to. What if they lived in two dimensions? How would they see the friend sitting next to them? Equally what if they lived in four dimensions? What would they and their friends look like?

SOME SERIOUS MATHEMATICS, BUT HOW TO DO IT IN THE CLASSROOM?

We will now turn to examining points we have so far mentioned on our way from Aristotle to Flatland and suggest a way of constructing a narrative and teach-

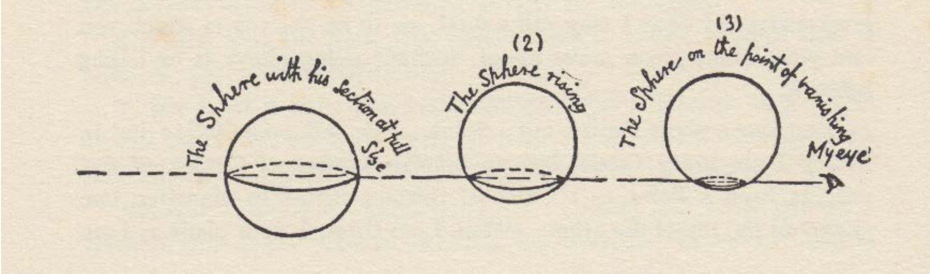


Figure 6: The diminishing sphere leaving projections in Flatland, its cross sections being circles, p. 72

ing episodes that could be used to engage teenagers in a classroom setting.

A Firstly, the trace we plotted of discussions about dimensionality stretch different times, cultures, and geographies. By analogy there could be some new mathematics born just right here and now, from their own thoughts and ideas. We mention philosophers and mathematicians that span twenty three centuries. The development of mathematical ideas can be employed in the classroom in various ways: from construction of timelines to open-ended discussions about the nature of mathematical inventions, and the contributions that are possible in this field.

B We begin with defining of the fourth dimension by musing about our ability to describe it mathematically. Both Wallis' and Lagrange's descriptions could be used in the classroom to give possible interpretations from different context and offer an insight into how new mathematics begins. In this way the way to model behaviors and attitudes may begin to emerge, and pupils may feel emboldened to pose new questions, mathematical exploration is thus brought closer to the classroom practice. A teacher can illustrate this by examples and geometric diagrams from original works some of which are mentioned above and listed in bibliography, and some interpretations illustrate also this paper.

C We meet with the study of Platonic solids. This is a rich field in the history of development of mathematical concepts that also spans centuries, and can be investigated in the classroom in a number of ways. For example, construction of Platonic solids, their representations in art (Emmer, 1982), and the study of Platonic solids by Euler are some instances that could be used as starting points for activities. Students can derive formula for Euler's characteristic and further investigate it in the light of Schläfli's extension of it.

D We come across various mathematical descriptions and symbols – the development of notation and formulae gives pupils opportunities to engage with the process and attempt to do the same/similar themselves in their own contexts. Teachers can work with pupils on Euler's, Schläfli's, and Stringham's algebraic formulations and discuss their different approaches: Schläfli symbols are also an interesting way of presenting geometrical entities.

E Finally the depiction of how life would exist in different dimensions and the consequences for our world are well narrated with the help of Flatland. We come to the practical construction of that narrative – suggestion is to mimic a dialogue by examining existing dialogues in the book and one such example may be that between Square and Sphere, appearing throughout the novel.

CURRICULUM LINKS

At this point it would be good to see whether we can, after all, establish some connections between the history of the fourth dimension and the new National Curriculum (DoE, 2014). We propose to state the NC

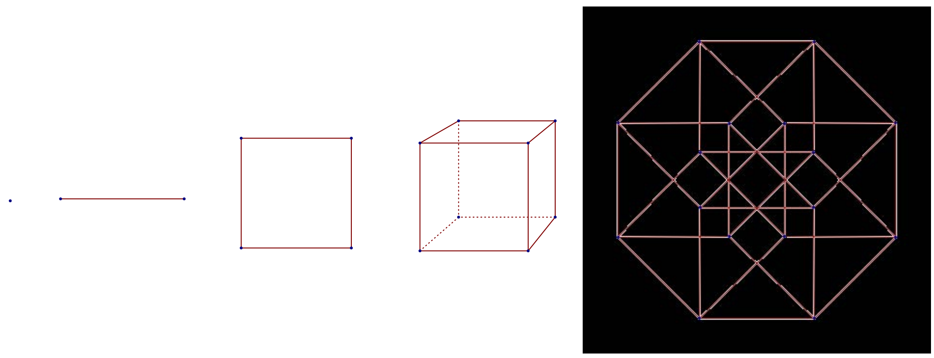


Figure 7: Here is an opportunity to use dynamic geometry software in illustrating different dimensions: starting from the 0th dimension, represented by point, we generate one-dimensional object – the line segment. Further by moving the line segment perpendicularly to itself we generate a square, a two-dimensional object. By moving the square perpendicularly to itself, we generate four-dimensional object, a tesseract (image by the author).

description and 'answers' from the 4th dimension as follows.

NC *(National Curriculum description): Mathematics is a creative and highly inter-connected discipline that has been developed over centuries, providing the solution to some of history's most intriguing problems.*

4dh *(4th dimension history lesson for the classroom):* Whilst mathematics is often described as a method and a way of exploring and solving the big questions of life and existence, the pupils in the classroom report a different experience (Smith, 2004). The history of the fourth dimension can give a concrete and a very tangible view of mathematical development that occurred in the pursuit of the question of existence of higher dimensions as we have shown on above examples.

NC: *Mathematics is an interconnected subject in which pupils need to be able to move fluently between representations of mathematical ideas.*

4dh: An example of dimensionality which can be described both visually and algebraically is the way Stringham described the n^{th} dimension. Stringham says that, the simplest n-fold angle is bounded by n number of edges, and so on. Stringham's formulations are related to number theory and figurate numbers. An investigation can be conducted on the similarities and certainly representations of the two corresponding strings of formulae. We can here return to Wallis' studies in context of algebra.

NC: *Move freely between different numerical, algebraic, graphical and diagrammatic representations, including of linear, quadratic, reciprocal functions.*

4dh: The descriptions of the fourth dimension include all of the above as we have seen in this short paper.

CONCLUSION AND FURTHER DEVELOPMENTS

From the listed history of the fourth dimension, the materials are being developed for the use in the classroom by the author and current cohort of trainee teachers studying with me, in the context of three very different educational settings.

Institution A is the first school where we have started working with the pupils. It is a mainstream boys' school, has a high achievement record in mathematics, and is comparable to specialized and grammar schools (i.e. the study of history, classics, and high achievement in mathematics are the norm). The school A poses the challenges in terms of producing the material that would engage and stretch pupils' abilities so the material developed aims to link geometry and algebra as seen from our mention of Wallis, Euler, and Stringham. It appears that pupils in this setting are very keen to engage with mathematics, and the study of the 4th dimension is seen as a way of stretching the pupils to study beyond the curriculum.

Institution B is a national institution promoting the study of mathematics for the gifted and talented pupils. Whilst many pupils in this setting have some idea about the dimensionality, exercises offering them possibility to represent them in different ways as described above given pupils not only the sense of achievement, engagement, and enjoyment, but their further interest and individual research is also noted.

Institution C is a special school for disabled pupils whose abilities in functional mathematics are low and hence the mathematical curriculum is narrowed down to teaching basic skills such as financial functioning. The challenge in this school was to provide a curriculum that engages and instigates an aesthetic appreciation and enjoyment of mathematical ideas lest mathematics is perceived by both pupils and teachers as a discipline which is only functional or arithmetic bound.

By working within these three very different institutions, we trialed the three aspects of mathematics that we have identified as possible principles of developing teaching programmes for the new curriculum. These are:

a) extending the most able pupils by modeling mathematical practices from the past

b) engaging pupils by learning mathematical skills and understanding via 'big' questions of mathematics

c) teaching mathematical appreciation via aesthetic experience rooted in mathematical concepts.

The preliminary conclusions are that researching a historical development of mathematical concept can give opportunities for multiple settings, differentiation in terms of possible levels of achievement in mathematics, and cover different aspects of modeling mathematical practice with different types of pupils.

Finally, after identifying the links with the curriculum so easily, an idea is forming that the new programme for secondary school mathematics can indeed be entirely based on historical development of mathematics, offering for the first time an educational programme that would be truly meaningful and engaging and give pupils a glimpse of the big questions of mathematics about the nature of space and time that many past mathematicians had been enthused by.

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ENDNOTES

1. See in particular http://mathsreports.wordpress.com/2013/01/05/homehome/ for summary and text of these. Accessed 1st October 2014.

2. For example Adrian's Smith's report on the state of mathematics education in 2004 was followed by the establishment of National Centre for Excellence in the Teaching of Mathematics in 2006. See Lawrence (2009).

3. See for example a project report on student disengagement in English secondary education by Sonia Sodha and Silvia Guglielmi (2009).

4. Aristotle says (2012, 268a, pp. 10–15): "A magnitude if divisible one way is a line, if two ways a surface, and if three a body. Beyond these there is no other magnitude, because the three dimensions are all that there are, and that which is divisible in three directions is divisible in all." He however rejected the possibility of an extension of this thinking: "All magnitudes, then, which are divisible are also continuous. Whether we can also say that whatever is continuous is divisible does not yet, on our present grounds, appear. One thing, however, is clear. We cannot pass beyond body to a further kind, as we passed from length to surface, and from surface to body." (Aristotle, 2012, 268a, pp. 25–30)

Teaching the concept of velocity in mathematics classes

Regina Dorothea Möller

University of Erfurt, Erfurt, Germany, regina.moeller@uni-erfurt.de

The genesis of the concept of velocity refers to a centuries-old search within the context of motion. This led to Newton's definition which is still taught in school. The historical development has shown that both mathematics and physics classes have their respective characteristic manner using this term. However, the mathematical potential for teaching this concept is by far not exhausted.

Keywords: Concept of velocity, historical development, teacher education.

INTRODUCTION

The causes to investigate the historical development of the concept of velocity are mathematical problems which lately can be found in German fourth grade classes.

Several pictures are given which show people or machines such as cars or trains each with information about the respective velocity: the train covers 150 km in an hour. The assigned task is to fill out tables in which the students write the distances for different time spans: 1h, 2h, and so forth. Above the assigned tasks one can read the title velocities. One can observe at once that like in other cases in which an introduction into the field is in the focus of interest – here a start with a series of different observable motions and a leading to the question how they can be quantified – there is an emphasis on the computational aspect under which the ideas that have led to the possible to computations, have disappeared (Doorman & van Maanen, 2009). Instead of an approach pinpointing the quantities in question there is a given table to be filled out by the students.

This kind of problem can be considered as an anticipation. There are at least three other kinds of anticipation tasks on the elementary level: one refers to the ratios such as a half, one quarter and three quarters and the second refers to the decimals within the context of magnitudes. The third is the appearance of tables in application problems. In these problems prices of products are given – like 1 kg of apples cost 75c – and the question is how much is to pay for 2 kg / 3kg / 5kg. These three kinds of anticipations occur in math classes because of the application principle. Students see these kinds of numbers and these kinds of questions in their daily life and math classes respond to this phenomenon by introducing these numbers and these tables without giving a rigid mathematical reasoning.

What kind of anticipation is done when students solve this kind of velocity problem? Since the students are to fill out tables, which are methodically a representation of functions one could argue that functions have arrived in elementary math classes since the inspection of Felix Klein (1905) to make functions a subject matter in math classes. It also could be understood as an example of an (anti-) didactical inversion (Freudenthal, 1983, pp. 305ff.). In any case it is obvious that this is not following the historical development and it is not showing an elementary approach which is possible at this stage of mathematics classes.

Since this kind velocity problem is not really an application task, since only tables need to be filled out, one could argue that they give a mathematical way to compute velocities which the students can observe in their daily lives because our modern world presents this phenomenon. Since the formula is given later at secondary I level one chooses tables to compute.

Having in mind the way magnitudes are introduced in math classes on the elementary level (finding representatives, studies of comparisons with chosen measurement objects and afterwards with agreed upon measurements objects, leaning the standardized measurement unity and at last solving of application problems), the velocity problems in focus do not show any such procedure although velocity is the first composed magnitude that the pupils encounter.

Since this is a mathematical concept one would expect a series of steps that lead to a definition. A possible approach would be the didactical triangle of Bruner (1960) in which he argues for an approach that encompasses the enactive, iconic and symbolic level. Another didactical theory of leaning concepts is given by Vollrath (1984) who outlines in general what kind of different steps lead to an understanding of mathematical concepts.

Another concern is the fact that the concept of velocity can be looked upon as a real mathematical modelling procedure (e.g., Blum & Leiß, 2005). Observable movements can be measured in two dimensions: length and time span. It could be arranged as a project for students in which the definition of velocity is the end product of their investigative endeavour.

This point also leads to the question why a fundamental phenomenon like the concept of velocity lacks any historical approach in textbooks. The students could measure for example the free fall of objects getting an idea of how scientists in the middle ages approached problems of velocity.

The paper focuses on the historical development of the concept of velocity with the idea that Newton might not have thought primarily of it as a function since he was still following Galilei's proportional theory. The development starts with some inherent philosophical aspects since in times of Newton the subject matter still belonged to the so-called natural philosophy.

PHILOSOPHICAL AND HISTORICAL ASPECTS

The concept of velocity is one with a long tradition, similarly like the history of calculus (e.g., Doorman & van Maanen, 2009). Embedded in the concept of motion already Greek mathematicians, especially Aristotle (384–322 BC), had ideas about velocity which he combined with his observations of the spheres and of the free fall of objects. Before the next step was done by Galilei (1564–1642), Nicole of Oresme (1360) used graphic representation of changing qualities. Later Galilei used experiments to argue for the statement that there is a quadratic dependency between the distance travelled and the falling time. Even later Newton (1642–1727) defined velocity using the concept of force that initiates motion. Leibniz (1646–1716) developed the differential and integral calculus also considering the idea of (planetary) movements.

In the following sections, two aspects are discussed: Firstly, the formal definition of the concept of velocity was preceded by a struggle for a clarification of the concept of motion. Secondly, the concept of velocity embodies a circular reasoning.

Although a lot of the work of Archimedes (287–212 BC) concerning mechanics is transmitted and gives an idea of his far-reaching mathematical understanding, we have no clear idea what he understood by velocity. Assured work is passed on us of Aristotle (384–322 BC), who investigated the phenomenon of motion qualitatively and verbally (Aristotle, Physics, 1829). He distinguished three types of motion: motion in undisturbed order, such as the celestial spheres, the "earthy" motion such as the concept of the rise and fall, and the violent motion of bodies that needs an impulse (cf. Hund, 1996, p. 29). Although his remarks touched the phenomenon of velocity, his conceptions proved wrong later on: "Aristotle came close to the concept of velocity when in the sixth book, the words 'faster' (longer distance in the same time, same route in shorter time) and the 'same speed' are explained." (Hund, 1996, p. 30)

Galilei (1564–1642) succeeded in a better understanding of the concept of velocity, as he did not rely on his direct perception: "The means of scientific evidence was invented by Galilei and used for the first time. It is one of the most significant achievements, which boasts our intellectual history [...]. Galilei showed that one cannot always refer to intuitive conclusions based on immediate observation because they sometimes lead to the wrong track" (Einstein, 1950, p. 17).

As Weisheipl (1985) points out, Galilei struggled with the Aristotelian concept of nature (pp. 8ff.). Aristotle considered nature as an active principle. "Nature is a source not only of activity but also of rest" (p. 22) which has an impact on the understanding of motion (p. 49ff.) He also still pondered over the idea of Parmenides "all change is illusion" and the one of Heraclitus "everything is flux". Galilei can be understood as being at the brink of Aristotelian sight of nature and the one later proposed in Newton's Principia and Descartes' Principia. Newton formulated the principle of inertia like this: "Everybody preserves in its state of rest or of uniform motion in a straight line, unless it is compelled to change that state by forces impressed upon it (Weisheipl, 1985, p. 69).

Another author, Palmerino (2004), argues for a new sight of the reception of Galilei's theory since like any other theory it was not at once in the way accepted in which textbooks nowadays present for example the free fall and projectile motion (Palmerino, 2004, p. 140). During the last decades it became apparent that the ideas that Galilei, Descartes and Newton presented at their times were different from each other. Also the European continent was not aware of Galilei's and Newton's theories and later Leibniz focused very much on his functional approach using variables. That is, the way the theory of motion and velocity is presented nowadays is much more formed by Leibniz than by Galilei and Newton although it was very much their ideas that came through eventually.

Only Newton (1643–1726) introduced the concepts of absolute time and absolute space and opened up the exact relationship between force and motion: The power does not get the motion upright (Aristotle), but it causes its change (acceleration). While Aristotle argued by inspection, Newton made an abstraction, as he looked upon length and time as not necessarily bounded materially.

In today's linguistic usage, we understand motion as a change in position in the (Euclidean) space over a certain period of time. Lengths and time periods are conditions for the quantification of such motions. On this basis, the (average) velocity is defined as the quotient of the distance travelled over the time required.

A circular argument is obvious on closer inspection: time depends on a movement and vice versa, because time is measured using motion (Mauthner, 1997, Vol. 3, p. 438). In the hourglass sand runs through, in an analogous clock the pointer moves, and for the period of a year, we follow the cycle of the earth around the sun.

Likewise, the idea of space is connected to motion, for only through movement we perceive the space. Mauthner said: "His [The people, note of the author] language makes it impossible for him to understand the metaphorical tautology of the preposition 'in'. Only rigorous reflection will enable him, to at least understanding the metaphorical of the preposition (in time). In space means something like 'in the space of the room', for the time as much as 'in the space of time' (ibid, p. 443)".

Even Piaget refers to this circular argument: "Speed is defined as a relationship between space and time – but time can be measured solely on the basis of a constant velocity" (Piaget, 1996, p. 69). For him the concepts of space, time and speed are mutually dependent.

MATHEMATICAL ASPECTS OF THE VELOCITY CONCEPT

Towards a functional terminology of velocity Oresme (ca. 1320–1382) sought the help of experiments to assign a rate of change to certain intensities and concluded: "All things are measurable with the exception of numbers .... (Pfeiffer, Dahan-Dalmedico, 1994, p. 228ff.)". The difficulties mathematicians had at that time are well summarized there.

The reasoning of Aristotle and Galilei were based on their observations of linear motions. However, both had also planetary motions in mind. For motions on a curved path you need two different aspects: the direction and the magnitude of a velocity vector. It is this distinction which led, in modern terms, to a vectorial description and thus to a further clarification of the concept of velocity, which is thus a generalization of the concept of velocity on a straight line. Bodies on a straight line have the same speed, in the same direction and the same quantity. Since then, the following statement is true: the change in force and velocity are vectors with the same direction (Einstein, 1950, p. 38). We observe an idea of permanence, because all statements that apply to velocities along curved paths must also apply to linear trajectories.

The cause of this observation is given by an idealized thought experiment, which confirmed the theory (cf. Einstein, pp. 25ff.); this is yet another idea that came to an effect only at the time of Galilei. Since then, the mathematical language has been used in physics to reason for not only qualitative but also quantitative conclusions.

As soon as you engage in quantitative calculations, one deals with quantities ("Größen"). With respect to the concept of velocity you have the dimension (the quotient of distance and time) and the measured value (an element of real numbers), which is a composite physical quantity.

Griesel analysed the subject matter of quantities on the primary level (length, weight, time periods) (1973, vol. 2, pp. 55ff.) as a technical background for the didactics of quantities. This presentation does not fit the quantity of velocity (and is not mentioned there) because it requires a description as an element of a vector space, which can be higher than one-dimensional.

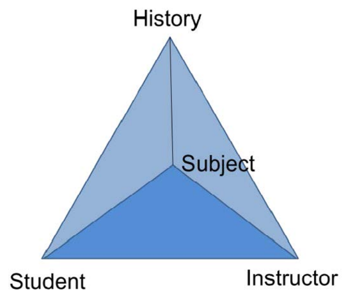

Figure 2: Extended didactical triangle

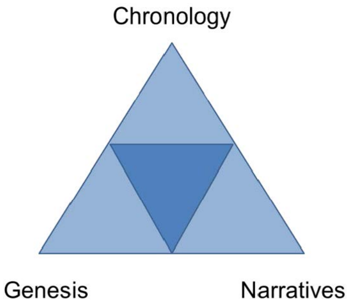

Figure 3: Aspects of history

However, Freudenthal (1977, vol. 1, p. 188) argues that one can interpret measure indications as function symbols; an idea that was not previously addressed in class. Another functional aspect occurs in two ways with the concept of velocity: The distance-time function leads with considering of the difference quotients to the average speed and the transition to differential quotient to instantaneous velocities that are themselves again functions, namely, the velocity-time functions.

It has taken over 2000 years for the concept of velocity to be defined consistently out of the concept of motion – in today's usage an act of mathematical modelling. It is therefore a prime example of a mathematical and interdisciplinary concept development – with both mathematical and physical – mechanical – representations throughout history. The intuitive conclusions drawn by Aristotle led to difficulties and proved much later untenable. Only an idealized thought experiment led eventually to a verifiable physical theory. The concept of velocity is an example of a mathematical modelling of a qualitative knowledge with more potentials as there are quantitative statements and other insights. The knowledge of such phenomena, the resulting misconceptions and the trodden paths of knowledge are essential components of mathematical education and exemplify scientific processes.

The genesis also shows a potential for didactical perspectives of mathematics education. Despite the scarce representation of this topic in mathematics class (in many curricula of the German provinces the concept of velocity is only mentioned only one time on the secondary level), there are a diversity of ideas which can be reflected.

DIDACTIC CONSEQUENCES

This example shows again, that the mathematical subject matter is not always the 'first philosophy' in mathematics education (e.g., Ernest, p. 213). Although of great importance for the development of mathematical and physical theory, it plays only the role of a physical application in mathematics classrooms. Hidden under algorithmic manipulations, the ideas that lead to the definition of concept of velocity do not appear to pupils in class. Since this topic also disappears during teacher training, there is an extra afford needed to pull the history into the light.

In the light of the use of language in math classes, Vygotsky (1964) elaborated on the relationship between everyday experiences and logical reasoning. Everyday experiences may interfere with scientific reasoning which in the case of the concept of velocity occurred historically for quite some time because of the lack to understand the invisible forces. It is therefore necessary for teacher students to learn to discriminate between the appearance or motions and its scientific reasoning.

This afford can be done in three ways: there is a chronology of historic development which can be shown on a line. There is the genesis which makes apparent who under which presumptions found a concept or an algorithm.

The genesis of the concept of velocity is influenced by interdisciplinarity for centuries. Therefore, it is wrong to neglect it in mathematics education by pointing to the physics education or to avoid it in the classroom at all. In mathematics teaching, it owns several important functions:

Basic everyday experiences relate to the phenomena of velocity and can be used on several grade levels by taking up Vollrath's idea and take seriously measurement processes as a basis of experience (Vollrath, 1980). This can be addressed in propaedeutic form at the primary level and in a quantitative manner at secondary levels.

Under the current relations of applications of mathematics education, the concept of velocity is a prime example of mathematical modelling of everyday phenomena of motions which can be quantified normatively, allowing measurements and calculations. According to the current understanding these are necessarily factual and methodological skills.

As a first composite quantity the concept of velocity can be addressed in lower secondary education at many occasions. Before getting to the usual algorithmization (Doorman & Van Maanen, 2009) teaching this concept allows for teaching the fundamental idea of measuring. This is a valuable and vital contribution to the implementation of the rules in the educational standards and curricula and guiding principles would be met. The implementation of the items listed would be an important part in promoting mathematics education.

To arrive at an overall conception of the concept of velocity in mathematics education, a strategy is needed which leads from an intuitive to a structured taxonomy (Vollrath, 1984, p. 14). Vollrath has given a general theory to teach mathematical concepts. The following steps outline the procedure scheme for the specific concept of velocity which teachers could have in mind before teaching the respective classes (Vollrath, 1984, pp. 202ff). It would mean to get a clear view on the subject matter and teach perhaps more than what is given by the textbooks:

1) The concept as a phenomenon (intuitive understanding of the concept)

They recognize that

— the feature of the motion is the continuous change of position of subjects and the objects.

— for describing motion two quantities, length and time intervals, are required.

— with a constant distance the one object is faster that needs less time. Conversely, at constant time, the object is faster that has travelled a longer distance.

2) The second term and his wealth aspect (constructive understanding of the term)

They recognize that

— rest and motion of a body are always give relative to a reference system.

— with a rectilinear and uniform motion in equal intervals of time, equal distances are covered.

— for a linear and non-uniform motion the same time intervals can be covered in different journeys.

— for a linear and uniform motion, velocity can be described as the ratio $v = s / t$.

3) The concept is a carrier of properties (substantive understanding of the term)

They recognize that

— for mathematical calculations of the body and the mass, they are idealized to a point.

— the ratio $\Delta s / \Delta t$ possess a constant (average) value relative to the total distance and total time.

— at constant velocity Δs to Δt proportional.

— the quantity velocity possesses the dimension "distance per time" with the units m/s or km/h.

4) The concept as a tool for problem solving (problem-oriented understanding of the term)

They can

- bring the definition v = s / t into relation to everyday life contexts.
- calculate uniform velocity.
- appreciate that the concept of velocity is a fundamental physical phenomenon that can be described quantitatively by mathematical means.

The discussion of the velocity concept in the context of the stage scheme shows the potential that this concept already has in mathematics education. From this more elaborated standpoint the potential for the interdisciplinary character can be developed further and more in detail.

The four steps in the concept development to enlighten teacher students in their knowledge of this topic also gives a good background to approach the topic in an ethical manner. It is essential for their teaching to have a grip on solid knowledge and cultural heritage in the subject matter they are to teach.

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Proportionality problems in some mathematical texts prior to fourteenth century

Antonio M. Oller-Marcén[1] and José María Gairín[2]

1 Centro Universitario de la Defensa de Zaragoza, Zaragoza, Spain, oller@unizar.es

2 Universidad de Zaragoza, Departamento de Matemáticas, Zaragoza, Spain

Proportional reasoning seems to be one of the oldest mathematical problem solving tools. Problems involving proportionality have consistently appeared in mathematics texts over time. Moreover, proportionality is still today one of the main topics in school arithmetic and several studies point out difficulties shown by the students in this context. In this paper, we perform an analysis of historical proportionality problems in order to describe the main situations where proportionality naturally appeared. Knowledge of these situations might be of interest in order to design and develop well-contextualized teaching and learning activities.

Keywords: Proportionality, arithmetic, problems, history.

INTRODUCTION

Rhind papyrus is the oldest known work that can be considered, in some sense, a Mathematics textbook. It is dated by the copyist Ahmose to the sixteenth century BC, although he acknowledges to be copying older material from around nineteenth century BC. In this papyrus, we already find problems like the following one (Chace, 1979, p. 102):

> "If 10 *hekat* of fat is given out for a year, what is the amount used in a day?"

This daily problem clearly points out the important applications that proportional reasoning had in several situations involving productive aspects, commerce or administration. This fact, together with the difficulties that many students show when they face concepts related to proportionality (Modestou, Elia, Gagatsis, & Spanoudis, 2008; Van Dooren, De Bock, Janssens, & Verschaffel, 2008) justify that proportionality and its applications are still important topics in Primary and Secondary school curricula in many countries.

THEORETICAL FRAMEWORK

Mathematics, just like any other human activity, takes place in a particular historic, social and cultural context that is, of course, changeable (Wussing, 1998). Consequently, the study of mathematical concepts and problems from a historical perspective must be an essential part of research in Mathematics Education. In this sense, the contributions of History of Mathematics to Mathematics Education are diverse and admit many different approaches (Fauvel & Van Maanen, 2000; Katz & Tzanakis, 2011).

Phenomenology can also be approached from a historical perspective. Freudenthal (1983) presents historical phenomenology as one of the four types of phenomenology which have interest from a didactical point of view. The idea is to pay attention to those phenomena lying in the origin of the concept that we want to study.

Since Mathematics can be conceived from a problem-solving point of view (Ernest, 1989), it seems interesting to focus a historical-phenomenological study on those problems arising in the origin of a particular concept. This interest is even clearer if we note that "problems have occupied a central place in the school mathematics curriculum since antiquity" (Stanic & Kilpatrik, 1989, p. 1). Moreover, Tzanakis and Arcavi (2000, p. 224) point out that historical problems that motivated and/or anticipated the development of a mathematical domain are interesting from a didactic perspective.

Many authors (De Lange, 1996; Van Reeuwijk, 1997) point out that contextualization is an important issue regarding mathematical work in the classroom. Obviously, the case of proportionality is not an exception. Thus, the knowledge of those situations in which proportionality has historically appeared can be useful in order to design and develop well-contextualized teaching and learning activities that might help the students to overcome their difficulties.

OBJECTIVE AND METHOD

In this paper, we perform an analysis of historical proportionality problems. In particular we are interested in studying the contexts in which these problems appeared. Our main goal will be to give a survey of the most important situations in which proportional reasoning arises in a natural way.

In order to do so, we will focus on the study of ancient texts following the steps of content analysis (Krippendorff, 1990).

It is difficult to make a selection of texts. First of all we must determine a temporal frame. We decide that the most recent work in our study would be the *Liber abaci*, edited twice during the thirteenth century, because it presents proportionality in a relatively closed form. Later medieval and renaissance texts on Arithmetic will not include any essentially new problems regarding proportionality [1].

On the other hand, since proportionality admits different conceptual approaches (Oller & Gairín, 2013), it is necessary to gather works coming from different traditions. Moreover, to ensure the representativeness of the sample, it seems interesting to search for texts that were *encyclopedic* in the sense that they tried to summarize the mathematical knowledge at the moment and place where they were written.

As a consequence of the previous considerations the following main sources were selected:

1) *Rhind papyrus*, ca. 1650 BC, from Egypt (Chace, 1979).

2) *Jiu zhang suan shu*, an ancient Chinese text commented in 263 by Liu Hui but containing older material (Kangshen, Crossley, & Lun, 1999).

3) *Lilavati*, written in 1150 by the Indian mathematician Bhāskara II (Patwardan, Naimpally, & Singh, 2001).

4) *Liber abaci*, written in 1202 by Leonardo of Pisa, *Fibonacci* (Sigler, 2002).

Quite often, ancient texts only present the statement of the problems or, even if they provide a solution, it is only of numerical nature. In these cases we do not have a clue about the actual solving method. Thus, if we want to avoid the mistakes [2] pointed out by Grattan-Guinness (2004) we cannot classify the problems according to their (modern) method of resolution. Instead, we decide to classify the problems according to the particular context and situation where they appear. This criterion is coherent both with the practical nature of proportional reasoning and with the didactical usefulness of the study.

CONTEXTS WHERE PROPORTIONALITY PROBLEMS ARISE

The practical and pedagogical importance of proportionality is made clear by the great amount of related problems that appear in the consulted texts. In the *Jiu zhang suan shu*, for instance, three out of the nine chapters are solely devoted to this topic. Also, in Fibonacci's text we find a chapter exclusively devoted to companies presenting thirteen problems with their detailed solutions.

This huge number of problems also represent a variety of problems ranging from distribution of food to purely mathematical and abstract situations. Hence, it is difficult to give an exhaustive and detailed classification of proportionality problems according to their contexts. Nevertheless, we have identified four categories that include most of the problems:

1) Exchange problems.
    a) Exchange of merchandise (barter).
    b) Buying and selling.
    c) Exchange of currency.
2) Distribution problems.
3) Loan problems.
4) Mixing problems.

EXCHANGE PROBLEMS

It is quite possible that the exchange and distribution of merchandise were the situations where proportional reasoning firstly appeared. In fact, texts like the *Jiu zhang suan shu* or the *Liber Abaci* use barter situations as an introduction to this topic. In cultures where currency existed buying and selling made sense but these situations can be seen as an Exchange of merchandise for Money. Finally, when different cultures, towns or countries interacted, the need to compare and interchange their respective currencies arose. We also consider these cases as exchange problems.

Exchange of merchandise

In order to perform an interchange, some criterion is needed to guarantee its fairness. Throughout our study we have found different criteria that we now exemplify.

1) Sometimes a table or some additional information giving exchange rates for different merchandise is given. That is the case, for instance, in the *Jiu zhang suan shu*, where a table indicating the exchange rates for different cereals is presented and used to solve problems such as the next one (Kangshen et al., 1999, p. 146):

   "Now 5 *dou* 2/3 *sheng* of millet is required as sesame seed. Tell: How much is obtained?"

2) In other cases, the monetary value of both merchandises is known and this information is used to deduce the fair way to interchange them. These kind of situations appear, for instance, in the *Liber abaci*, where we find problems like the following (Sigler, 2002, p. 181):

   "A hundredpound of pepper is worth 13 pounds, and a hundredweight of cinnamon is worth 3 pounds; it is sought how many rolls of cinnamon are had for 342 pounds of pepper."

3) Finally, we can find a unit that measures, in some sense, the quality of a certain good. This happens remarkably in the *Rhind papyrus*, where a unit called *pefsu* (Robbins & Shute, 1987, p. 51) measures the lack of quality. This gives rise to problems like this (Chace, 1986, p. 108):

   Example of exchanging loaves for other loaves. Suppose it is said to thee, 100 loaves of *pefsu* 10 are to be exchanged for a number of loaves of *pefsu* 45. How many of these will there be?

Buying and selling

Buying and selling processes can give rise to several different problematic situations, but the two main possibilities that we have found are:

1) The price of a certain quantity of merchandise is known and the price of some other quantity is sought. For instance (Sigler, 2002, p. 134):

   "Also a hundredweight of some merchandise is worth 14 pounds and 7 soldi; how much are 37 rolls of the same merchandise worth?"

2) The *inverse* situation; i.e., the price of a certain quantity of merchandise is known and the quantity of merchandise that can be bought with other quantity of money is sought. For instance (Patwardan et al., 2001, p. 78):

   "If 3/2 *pala* of saffron costs 3/7 *niskas* O you expert bussinessman, tell me quickly what quantity of saffron can be bought for 9 *niskas*."

Exchange of currency

The exchange of currency is a relatively recent commercial phenomenon. Among the revised texts, only the *Liber abaci* presents these problems. This is not surprising considering the historical and geographical context in which it was written. In fact, these problems will be popular in renaissance arithmetics due to the active commerce around the Mediterranean Sea.

Just like in the case of barter, there can be different pieces of information available at the moment of the exchange. The main possibilities are:

1) In the simplest case, the amount of one currency that is obtained in exchange of a certain quantity of the other is known (Sigler, 2002, p. 157):

   "Also a Genoese soldo is sold for 21½ Pisan denari, and it is sought how much 7 Genoese soldi and 5 denari are worth."

2) In some cases a third currency is involved that acts as a measure for the value of the two exchanged currencies (Sigler, 2002, p. 186):

"Also it is proposed that one Imperial soldo is worth 31 Pisan denari, and one Genoese soldo is worth 22 Pisan denari, and it is sought how many genoese denari 7 Imperial denari are worth."

3) The previous situation can be generalized so more than one currency acts as mediator (Sigler, 2002, p. 195):

"Twelve Imperial denari are worth 31 Pisan denari, and one Genoese soldo is worth 23 Pisan denari; and one Turin soldo is worth 13 Genoese denari, and one Barcelona soldo is worth 11 Turin denari; it is soought how many Barcelona denari are 15 Imperial denari worth."

4) Finally, the exchanged can be performed according to the actual value of the silver (for instance) contained in each coin (Sigler, 2002, p. 199):

"Again one indeed has 26/3 pounds of some common coin that has 2¼ ounces of silver, an one pound of silver is worth 149/20 Pisan pounds, and it is sought how many Pisan pounds will be had for the 26/3 pound of the coin."

DISTRIBUTION PROBLEMS

As we already pointed put, distribution situations are among the oldest situations where proportional reasoning arises. *Rhind papyrus* already presents this type of problems.

The "fair" way to perform the distribution is often implicitly assumed and the context determines if the distribution must be direct or inverse. We now give some examples.

1) Sometimes there is a benefit to distribute, coming from either work or investment. In these cases the distribution is made directly proportional to the individual contributions (Chace, 1979, p. 104):

"Suppose a scribe says to thee, four overseers have drawn 100 great quadruple *hekat* of grain, their gangs consisting, respectively, of 12, 8, 6 and 4 men. How much does each overseer receive?"

2) In other cases, there is a loss to be distributed (like a tax payment, for instance) and the distribution is inversely proportional to the "rank" of the payers (Kangshen et al., 1999, p. 166):

"Now given five officials of different ranks: *Dafu*, *Bugeng*, *Zanniao*, *Shangzao* and *Gongshi*. They should pay a total of 100 coins. If the payment is to be shared in accordance with ranks, the higher pays the less, and the lower pays the more. Tell: how much should each pay?"

3) Finally, there can be several factors involved in the distribution which will be made directly proportional to some of them and inversely proportional to the others (Kangshen et al., 1999, p. 315):

"Now given a problem of fair distribution of tax millet. The tax bureau is at County A, which has 20520 households and where millet costs 20 coins a *hu*. County B, 200 *li* away from the bureau, has 12313 households and millet costs 10 coins a *hu* there. [...] County E, 150 *li* away, has 5130 households, and millet costs 13 coins a *hu*. The total tax millet for the five counties is 10000 *hu*. A cart carries 25 *hu*; the transport cost is 1 coin a *li*. Assume the payment by each household is equal in cash and labour. Tell: how much millet shoud each county pay?"

These distribution problems will continue to appear consistently over time, leading to the so-called problems of companies that received much attention from thirteenth century on (Lamassé, 2001).

Other interesting and important situation where distribution problems can be found is that of inheritance [2]. This is particularly the case in Muslim tradition because of the strict and somewhat complicated rules determined by the Islam. In fact, the whole third chapter of al-Khwarizmi's *Algebra* (Rosen, 1986) is devoted to inheritance problems. A typical example is given by the following problem coming from the twelfth century Andalousian mathematician al-Hufi (Laabid, 2001, p. 321):

"Une femme a laissé après sa mort, son mari, sa mère, sa sœur germain, sa sœur consanguine, sa sœur utérine. Et une succession constituée d'un esclave et de 15 dinars. La sœur germaine a pris pour sa quote-part l'esclave et a remboursé pour

> les autres héritiers 5 dinars. (Il s'agit de calculer le montant global de la succession et la valeur de l'esclave, la quote-part de chaque héritier.)"

This type of problems do not appear in the sources used for this work and, strictly speaking, they could be considered as an example of benefit distribution problems. Nevertheless, their context is specific enough to study them separately.

LOAN PROBLEMS

In spite of the bad reputation that the loan of money at a certain interest had in several cultures, it seems that this was a quite ancient and common practice. Moreover, simple interest is still a topic in textbooks; although this is more due to tradition than to its practical use (compound interest is far more common).

The problems that we have found in this context are very similar, if not identical, to problems that appear nowadays. For example (Patwardan et al., 2001, p. 84):

> "If the interest on 100 for 4/3 months is 26/5, what will be the interest on 125/2 for 16/5 months?"

In most ancient problems the reader is asked to find the interest obtained by a certain amount of money after a certain time. In some cases the unknown quantity corresponds to other magnitude. The following example from the *Liber abaci* shows such a situation, although it is not strictly a proportionality problem since it involves an affine function (Sigler, 2002, p. 384):

> "A certain man placed 100 pounds at a certain house for IIII denari per pound per month of interest, and he took back in each year a payment of 30 pounds; one must compute in each year the 30 pounds reduction of capital and the profit on the said 100 pounds. It is sought how many years, months, days and hours he will hold money in the house."

MIXING PROBLEMS

The idea of mixing products of different qualities and prices naturally arises when working with raw materials. In this way one can obtain a product of a predetermined price or improve or worsen its quality. A very particular case is that of alloys. These problems were important in Europe during the thirteenth century (Williams, 1995). In fact, chapter 11 from the *Liber abaci* is entirely devoted to this topic (Sigler, 2002, pp. 227–257).

Again, a huge variety of different situations arise in this context. Nevertheless there are essentially two types of problems.

1) Knowing the weight and quality of the ingredients, the weight and quality of the resulting mixture is sought (Patwardan et al., 2001, p. 98):

> "O golden mathematician, four types of gold 10 *masas* of 13 carats, 4 *masas* of 12 carats, 2 *masas* of 11 carats and 4 *masas* of 10 carats are melted together to form a new one. Find its fineness. Is this is purified and 16 *masas* gold is obtained, what is its fineness? If the mixed gold when purified has 16 carats fineness, what is its weight?"

2) Inversely, knowing the weight and quality of the mixture some information about the ingredient is required (Sigler, 2002, p. 255):

> "A certain man wishes to make a bell with five metals, of which a hundredweight of one metal is worth 16 pounds, another truly 18 pounds, an another 20 pounds, another truly 27 pounds, and another indeed 31 pounds; he therefore makes a bell from them that weights 775 rolls and costs 651/4 pounds; it is sought how much he puts in of each metal."

SOME DIDACTIC IMPLICATIONS

As we already pointed out, the previous study might be useful in order to design teaching and learning activities involving proportionality and, in particular, to decide which situations can promote that students significantly construct the main concepts regarding proportionality.

From this point of view, we think that the situations that are more adequate to introduce the basic techniques of proportionality in the classroom are exchange problems. This choice is not only motivated by the historical importance of those situations but also by the fact that students are very familiar to them.

Exchange problems are particularly suitable to introduce outer ratios and what Cramer and Post (1993) call unit rate strategies to solve missing-value problems. In addition, these situations can be used to present the idea of co-variation that may lead to a functional view of proportionality (Bosch, 1994).

Distribution problems are as old as exchange problems and they can also lead to meaningful activities that help to present important aspects of proportionality.

For example, distribution situations are suitable to work at the same time with inner and outer ratios and discuss their relations and different meanings. Moreover, the idea of fair distribution can help to clarify the concept of proportional magnitudes and a distribution situation is a natural context in which inverse proportionality appears.

We find loan and mixing problems less useful, mainly because students are less familiar to them. Nevertheless, they are also interesting and can also be used under certain circumstances. For instance, mixture problems can be a nice context in which progressively start the difficult transition (Wagner & Kieran, 1989) between arithmetic and algebra.

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ENDNOTES

1. We must point out that we focus on arithmetic proportionality. Of course, we find proportionality in several geometric problems of practical nature and the theoretical frame can be traced back to Euclid and before. Nevertheless, we think that arithmetic situations are more useful in order to introduce proportionality in the classroom.

2. Namely, the confusion of history and heritage. History describes what happened (and did not happen) in the past and tries to explain it, while heritage focus on the question "how did we get here?" In our case, it would be a mistake to classify problems as 'direct proportion problems', 'inverse proportion problems', etc. since these are modern categories.

The history of the concept of a function and its teaching in Russia

Ildar Safuanov

Moscow City Pedagogical University, Moscow, Russia, safuanov@yahoo.com

The history of teaching the concept of function in Russian school and university mathematical education is described. For this reason, in the first section the brief survey of the history of the function concept is given. Finally, we conclude that the opinion about the introduction of a modern rigorous and precise definition of a mapping has not been reached not only in school-based education, but also at the undergraduate level of mathematics education.

Keywords: Textbooks, history of mathematical education, function.

BRIEF SURVEY OF THE HISTORY OF THE FUNCTION CONCEPT

The function concept is one of most complicated concepts not only of school mathematics but also at undergraduate level. The students' difficulties with this notion have been reported for several decades (see, e.g., Tall, 1990; Sierpinska, 1992; Bardini et al., 2014). Therefore, a need to elaborate upon effective approaches to the teaching of this notion still exists.

This survey is a first step to the research project aimed at elaborating new ways of teaching the concept of function using the genetic approach. The application of genetic approach to the undergraduate mathematics teaching is described, e.g., by Safuanov (2005). The genetic approach could be combined with methods based on the APOS theory (Dubinsky, 1991).

In order to understand the problems of teaching the notion of function, first of all we will trace the history of its teaching, at least in Russia. The changes in the teaching the concept of function have been largely determined by the history of the concept itself. Thus, in order to trace the history of teaching the concept of function, we should first make a short survey of the history of the concept itself. The other reason for surveying the history of the concept is our belief in the necessity of knowing the historical development of the concept for elaborating its teaching by genetic method.

We will briefly trace the history of the development of the concept of function following the fundamental monograph edited by Yushkevich (1970, pp. 139–148, and 1972, pp. 250–255), and also books by Klein (1977, pp. 286–292), Gleizer (1983, pp. 20–25) and the "Mathematical encyclopaedic dictionary" (1988, p. 617). We also trace the history of the development and teaching of the concept of function in Russia on the basis of the analysis of mathematical, didactical, educational and reference books and articles.

In the monograph edited by Yushkevich (1970) it is noted that "shoots of this concept though not yet realized or made explicit, were present in the Greek mathematics and natural sciences" (Yushkevich, 1970, p. 139).

The concept of function for the first time emerged in medieval Europe in connection with the renewal of attempts of mathematical studying of various natural phenomena...

For the subsequent development of the theory of functions crucial importance belongs to trigonometry and logarithms, on the one hand, and to the birth of symbolic algebra, on the other hand.

At the beginning of 17th century functions were still quite often defined verbally, graphically, kinetically or through tables, but already in the second half of the century their analytical expressions being to play a leading role.

The term 'function' for the first time appeared in Leibniz's manuscripts of 1673, in particular in the manuscript entitled 'The inverse method of tangents, or on functions' (Methodus tangentium inversa, seu de functionibus)... The terms 'variable' and 'constant' were introduced by Leibniz" (Yushkevich, 1970, p. 146).

The definition of function as analytical expression, for the first time was distinctly formulated by I. Bernoulli in the article published in "Memoires de l'Academie des Sciences de Paris" in 1718: "Definition. A function of a variable is a quantity composed in any way from this variable and constants". The notation $f(x)$ for a function was introduced by Leonhard Euler in 1734.

The further development of the concept of function is connected first of all with Euler's name. Yushkevich (1972) noted that "in the foreword to the 'Introduction to the calculus of infinitesimals' he (Euler. – I.S.) for the first time has distinctly expressed the idea that the analysis is the general science about functions, that the analysis of infinitesimals rotates around variable quantities and their functions" (Yushkevich, 1972, p. 250).

In the first chapter of the first volume L. Euler, correcting the definition of his teacher I. Bernoulli, "... has emphasized that functions are defined by formulas: 'A function of a variable quantity is the analytical expression composed somehow by its variable and numbers or constant quantities'. Thus the important step forward was done; the independent variable is considered as the set of all real and imaginary numbers so that functions of a complex variable were introduced on the equal rights with functions of a real variable" (Yushkevich, 1972, p. 250).

To compose analytical expressions one could use four elementary operations, computation of a root, exponential and logarithmic operations and, furthermore, "uncountable others, provided by integral calculus", meaning thus also the integration of differential equations.

Euler's definition of a function given in the first volume of the "Introduction to the analysis of infinitesimals", appeared to be too narrow for the calculus as a whole. Euler formulated the new definition of a function in his foreword to the "Differential calculus" (Euler, 1755): "When quantities depend on others in such a way that at the change of the last they are also changed the first are called functions of the second ones. This term has extremely wide character; it covers all the ways by which one quantity can be determined by means of others". In the quoted definition nothing is spoken about the way of calculation of values of a function.

Close to the modern definition of a function is Lobachevsky's definition: "...the general concept of function demands, that one should name a function of x a number that is given for every x and that gradually changes together with x. A value of a function can be given either by analytical expression or by a condition which gives means to test all numbers and to choose one of them, or, at last, dependence may be unknown" (Mathematical encyclopaedic dictionary, p. 617).

Thus, "... the classical definitions of a function given by Lobachevsky in 1834 and by Dirichlet in 1837 the second of which has passed to the latest textbooks ("If in some interval to each separate value x a unique value of the variable y corresponds then the variable y is called a function of x"), are hereditarily connected with a definition belonging to Euler" (Yushkevich, 1972, p. 254).

Felix Klein (1977, p. 291) on the other hand noted that, beginning with the development of Cantor's set theory "also functions defined for values of x from any set (not necessarily numerical) are considered..." In van der Waerden's classical textbook, we already see the quite modern definition of a mapping: "If to each element from some set M by any rule a unique (generally speaking, new) object ?(x) is put in a correspondence then this correspondence ? is called a function. If all objects ?(x) belong to some set N, the correspondence x??(x) is called also a mapping from M into N" (van der Waerden, 1930).

In the post-war decades (1945–1960), in connection with fast development of topology and abstract algebra, the most formal definition of a function (=mapping) was introduced into the world of mathematics: a function is a correspondence from one set into another (i.e., a subset of their direct product) where for any element x from the first set there is a unique element y from the second set such that the pair (x, y) belongs to this correspondence.

Klein (1977, p. 292) complained that "...the school mostly ignores all the development of a science which took place after Euler" and offered: "... we wish that the general concept of a function … has entered as the enzyme into all the teaching of mathematics at school; but it should be introduced not in the form of abstract definition but rather on concrete examples … in order to make this concept a living property of a pupil". He noted that "it would be desirable that among numerous teachers there was at least a small number of independently working people who would be familiar also with newest concepts of the set theory".

Despite Klein's appeal, formulated by him (and attributed by him to Euler) and more precisely by van der Waerden definitions of a function did not soon find the way to the educational practice and literature not only in secondary school but also in undergraduate mathematical education in our country. Up to the beginning of 21th century only the first part of Klein's appeal has been in essence executed: Euler's definition of a function has taken a strong place in school and university mathematical curricula. The second part – taking into account the development of mathematics after Euler and use of set-theoretic concepts has not been actually used until now at the either school or undergraduate level.

HISTORY OF TEACHING THE CONCEPT OF FUNCTION IN RUSSIA

In this preliminary survey of teaching the function concept we restricted the scope mainly by textbooks. Practices of teaching the notion of function have been largely determined by presentation of the subject in textbooks. These practices (of the last decades of 20-th century) have been partly described, e.g., by Dorofeev (1991).

Most probably, the first to mention the term "function" in Russian mathematical textbooks was Kotel'nikov who published in 1771 a book "On variable quantities" that was essentially the concise translation of Euler's "Introduction to the analysis of infinitesimals" (Prudnikov, 1956, p. 76). Note that the first German translation of Euler's "Introduction to the analysis of infinitesimals" appeared later, in 1788. Thus, Kotel'nikov's textbook was essentially the first translation of Euler's book into modern language. In this book, the definition of function was given where functions should be represented by analytical expressions. The similar definitions were given in the 3-rd volume of the textbook "Fundamentals of pure mathematics" published in 1812 by Fuss who was the disciple and grandson-in-law of Euler (Fuss, 1812, p. 278), and also in the 3-rd volume of the textbook "Course of Mathematics" by Osipovsky (Osipovsky, 1823, p. 2).

Generally, throughout the 19-th century the definition of function as analytical expression representing a dependence of variables prevailed (see. e.g., Scheglov, 1853, p. 310; Chebyshev, 1936, p. 8). The situation was slightly improved in the beginning of 20-th century when, following to the Klein's appeal, prominent mathematicians called to reform the mathematics teaching and, in particular, "to permeate the entire exposition of elementary algebra, beginning with elementary grades, by the concept of functional dependence" (Grave, 1915, p. 1). The textbooks with definitions not reducing functions to analytical expressions appeared (e.g., Shaposhnikov, 1908, p. 92)

In Soviet schools, up to the middle of 60-th the textbook of algebra written by Kiselev (first published before the October Revolution) was used with the following definition of a function: "That variable whose numerical values change depending on numerical values of another one is called a dependent variable or a function of that other variable" (Kiselev, 1964, p. 25). In the textbook there was a talk about tabulated and graphical presentations of functions; however the emphasis was made on the analytical expression of a functional dependence.

Similar definition was contained in the textbook of Barsukov for grades 6–8 used from 1956 to 1967: "If two variables are connected in such a way that to each value of one of them a unique value of another one corresponds, one speaks that there is a functional dependence between these variables. … If two variables are in a functional dependence, the variable that can accept any (admissible) values is called an independent variable… Other variable, whose values depend on values of the former one, is called a dependent variable or a function…" (Barsukov, 1967, p. 250).

Still in 1970, in the algebra textbook of Kochetkova and Kochetkov for grade 10 that replaced Kiselev's textbook, the similar definition is given (in the slightly more precise form), essentially ascending to Euler's one: "If to every value of a variable x somehow a certain value of another variable y is put in a correspondence one says that a function is defined" (Kochetkova & Kochetkov, p. 127).

The same situation is observed in undergraduate textbooks. In the textbook on higher algebra by Shapiro (1935, p. 5) also the functional dependence is stressed: "If two variables *x* and *y* are connected in such a way that to each value *x* a certain value of the variable *y* corresponds, then the variable *y* is a function of *x*: $y = f(x)$".

Similarly, in Sushkevich's textbook (1941) it is supposed by default that a function is an expression f(x), where x is a variable (p. 86). At the same time in his textbook there is (in a little archaic language) the quite modern definition of a group homomorphism with the requirement that to each element of the first group one has put in a correspondence a unique element of the second group, i.e. with the requirement to a homomorphism to be a mapping in the modern sense (p. 353).

Uspensky (2002, p. 163) noted that in Great Soviet encyclopaedia in 1956 a function was defined as a dependence of variables on other ones. He also mentions similar definitions of a function in authoritative undergraduate textbooks on calculus.

We explored a number of undergraduate textbooks on mathematics. Here are some results.

In the textbook of Stepanov (1953) on differential equations the definition of a function is absent. On the other hand, in the second edition of the textbook of Pontryagin (1965) the special appendix was added containing the modern definition of function as a mapping (pp. 292–293).

Note that the greatest mathematicians of the level of Luzin, Kolmogorov, Aleksandrov, Pontryagin, were apparently, first to realize the necessity of the introduction of the modern definition of a mapping into the scientific and educational literature. Such definition of a mapping is used in books of Luzin (1948), Aleksandrov and Kolmogorov (1948), Kolmogorov and Fomin (1954). Note that all these descriptions characterizing a function as a rule of correspondence were not strict definitions and left the concept of a function (mapping) undefined.

At the same time some great scientists still did not introduce the general concept of a function and its definition, limiting themselves to special cases. So did Mal'tsev (1956) and Gel'fand (1971) in their textbooks on linear algebra. Apparently, it was implicitly supposed, that mastering special cases of the concept of a mapping is enough for mastering the appropriate themes of mathematics, and it is not necessary "to multiply entities". Dorofeev (1978, p. 21) expressed similar educational ideas when he in a discussion article even protected a thesis about uselessness of the definition of a function: "Pupils have, basically, the correct substantial view of a function as a mathematical object, but experience significant difficulties when they encounter the definition of this object... This situation, namely the possession of a concept without knowledge of its exact definition is not strange at all... it is typical in the majority of kinds of human activity...".

Kolmogorov (1978, p. 29) in his reaction, however, indicated: "...Dorofeev ... at school in general ... allocates to any version of the set-theoretical definition of function a modest place (basically only for optional lessons). I think, however, that for school textbooks ... rules (composing the definition of the concepts of a function. – I.S)... should be given to pupils early enough and should be coordinated with some certain final definition".

Beginning with 1960s, as many researchers observe, owing to "bourbakization" of mathematics, set-theoretical concepts and, in particular, the general concept of a mapping (function) entered into curricula of secondary and tertiary school.

Kolmogorov supervised the reform of school mathematics teaching at the end of 60s. However, in the textbook for upper secondary school edited by Kolmogorov (first editions appeared in 1960s) a concrete definition of function is used, and authors consider only numerical functions: "A correspondence with a domain D where to each number x from the set D a unique number y corresponds by some law, is called a numerical function" (Kolmogorov et al., 1990, p. 20).

Thus, "Kolmogorov" reform did not aim at giving to teaching of mathematics abstract and formal character of what it was severely accused by opponents. The purposes were to eliminate archaic language and character of teaching, to correct the scientific level of mathematical education. The great attention was given to the didactical maintenance of the reform.

The general definition of a mapping has been introduced into the first textbooks corresponding to reformed curricula and published under the edition of Markushevich (1975) – the prominent mathematician and educator, the ally of Kolmogorov in the reforming of school mathematics. In 1960–70-s didacticians have developed also methods of teaching the concept of function at school (Kolyagin et al., 1977), and concluded that it is expedient to study the concept of function at school, consistently introducing concepts of sets, ordered pairs, direct products of sets, and correspondences.

The "Kolmogorov" reform met strong resistance in the school environment. Many of innovations were not accepted. It seems that "Kolmogorov" reforms were doomed to failure in conditions of inflexible, uniform and authoritarian education system. Some prominent mathematicians such as Pontryagin in 1980, attacked Kolmogorov reforms. They used as their mouth-piece the official magazine "Communist". The article in that magazine (as well as in the newspaper "Pravda") was equivalent to the denunciation. As a result, the general concept of a mapping, as well as other general set-theoretic concepts, has been expelled from school curricula, and Euler's definition of a function occupied a strong position in school mathematics.

In our opinion, this was a mistake, because the absence of the strict definition often complicates mastering the concept of a mapping by students in their further study in universities (Kolmogorov, 1991; Carlson, 1998; Bardini et al., 2014). We believe that, whilst not demanding from pupils the faultless possession of strict definitions, it is still necessary for them to attain an awareness of the modern definition of function.

Further development of methods of introduction and teaching of the concept of a function became possible in 1990-s when it was allowed to use alternative textbooks at school.

So, in the set of textbooks of Mordkovich for grades 7–11 the dialectic approach to the introduction of mathematical concepts is applied: "...the concept of a function ... should not, in a deep belief of the author, be introduced strictly from the very beginning, it should grow" (Mordkovich, 1998, p. 6). Strict definition of a function is introduced only in grade 9. Nevertheless, A. Mordkovich also considers only numerical functions, defining them in the language of variables.

We see a similar picture in the textbook of Bashmakov (1992, p. 11): "The variable y is a function of a variable x if such dependence between these variables is defined that for each value x uniquely determines the value of y".

Thus, the most widespread textbooks of mathematics for the senior grades contained definitions of functions which are similar to Euler's definition, as mentioned by Klein. We note also that Klein's advice was to introduce these definitions gradually, through examples. We see, that the majority of mathematical and educational community in Russia admitted that it is inexpedient to study the modern set-theoretical definitions of a mapping at school.

In undergraduate textbooks, since 60–70-s, the strict definition of a mapping basically has gained a strong position: one may mention textbooks of Skornyakov and Kostrikin on algebra, of Arnold on differential equations, of Zorich on calculus. In some textbooks for pedagogical institutes (for example, in the textbook of Kulikov on algebra) the strictness and formalism have got excessive character, complicating the learning of a subject. Nevertheless, in some undergraduate textbooks of the prominent scientists of the senior generation (for example, of Faddeev on algebra, of Gel'fand on linear algebra) authors still tried to avoid the introduction of the general definition of a mapping.

Thus, the final opinion about the introduction of modern strict definition of a mapping is not reached concerning not only school, but also the undergraduate textbooks.

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Using historical school book excerpts for the education of mature mathematics teachers

Ysette Weiss-Pidstrygach[1] and Rainer Kaenders[2]

1 Johannes Gutenberg – University Mainz, Institute of Mathematics, Mainz, Germany, weisspid@uni-mainz.de

2 University of Bonn, Institute of Mathematics, Bonn, Germany

The use of historical and cultural perspectives in university mathematics education can support the development of self-esteem and maturity. It can bring together students with similar interests. We present the concept of a seminar on the analysis of mathematical textbooks and of learning contexts based on the consideration of historical excerpts. Such a seminar can become a starting point for a community of practice with the potential to develop social recognition and personal appreciation of the individual interests and talents of its members and their joint activities.

Keywords: Historical school book excerpts, mathematics teacher education for secondary schools, community of practice, Bildung.

INTRODUCTION

The organizers (among whom were the authors) invited gifted student teachers in mathematics at mathematics departments nationwide to apply for a four-day spring workshop in beautiful surroundings in Bonn that was paid for by the university – including travel and entertainment expenses. Despite this exceptional opportunity, only a few students answered this call. A closer investigation of this fact informed the organizers that most students, even the best mathematics student teachers, do not feel particularly *gifted*. In individual cases, there might be individual reasons for this, but in general, we interpret this phenomenon by the fact that these students were brought up in a school system in which they had to produce a required output in situations that are created and determined by others. The students learnt to interpret the evaluation and assessment of the learning output as a degree of their own learning success – just as how they learn in social networks to take the number of 'likes' on their posted output as a degree of esteem.

Mathematics student teachers for the gymnasium usually study mathematics in courses together with mathematics bachelor students, who study only mathematics (and a minor) all day. In mathematical tests and evaluations, they often perform weaker. The student teachers in Germany study next to mathematics a second subject and pedagogy, where they learn completely different perspectives and methods. Instead of feeling like a jack-of-all-trades they experience their inferior results in mathematics assessments as inferiority in terms of their 'giftedness'. Other forms of assessment as well as feedback from fellow students and professors do exist, but it seems that they are not taken as much into account as written mathematics examinations, as far as the self-conception of the student teachers is concerned.

Notably the *von Humboldt Bildungsideal* is built on two notions: the *autonomous individual* and the *cosmopolitan or Universalist (Weltbürger)* – that is, a universally interested person that cares about the important questions of humankind. The university should be – both for students and professors – a place for autonomous individuals to become such a *Weltbürger*. Furthermore, how do we want the students to act later as teachers, when they educate their own students? What is their understanding of *Bildung* and education at school? Student teachers, who are about to become responsible experts for *Bildung* at school, not only need to get in contact with these ideas, but should also be given opportunities to work on their own *Bildung* and personal development.

These considerations constitute in mathematics education the need for appropriate learning opportunities. Working on *Bildung* has an impact on one's whole personality and is therefore likely to last longer in professional practice. Here the history of mathematics plays an important role in mathematics teachers' education.

Several projects use historical themes for an autonomous study of student teachers or teachers. There are classical seminars, where students give talks on historical themes; there are websites to supply students and teachers with historical sources that can be used for teaching at school[1]. In addition, there are book projects evolving from seminars with students on historical matters (e.g., Van Maanen, 1995).

In this article, we reflect upon the set-up of a seminar around historical excerpts from schoolbooks that allows a process of value creation in a community of practice. First, we explore this idea by conceptual considerations and secondly, we illustrate the concept using Euclid's proof of the irrationality of √2 and the development of analysis.

ENLIGHTENMENT IN MATHEMATICS TEACHER EDUCATION

How can we transfer the existing learning opportunities in the above-described direction of *Bildung*?

The normal procedure in academic studies for student mathematics teachers is that they follow basic courses in pure and applied mathematics, courses and lectures in mathematics education, courses in educational studies (pedagogy, psychology) and study a second school subject on equal footing to mathematics. Only in some universities (e.g. Mainz) are there general courses on the historical and cultural roots of mathematics, reading courses and seminars for chosen topics in the history of mathematics. Assessments in mathematics are mostly tests on problem solving and mathematical terminology. Assessment varies in the history of mathematics and in mathematics education – we find essays, coursework, oral examination, seminar papers, presentations and homework assignments.

The feedback and assessment provided is thus much broader than in written mathematics exams. The problem is not that these alternative forms of assessment wouldn't exist. In mathematics, students hand in weekly exercises (often duos or trios) that are not only discussed qualitatively with the collaborating fellow students but also in weekly tutorials. Moreover, the students give talks in seminars. In mathematics education, in the history of mathematics, and in pedagogy courses as well as in most second school subjects, students have to write homework essays and give presentations. All this seems not to essentially affect the student's sense of self.

To sum up, with respect to *Bildung*, we encounter problems partly related to the common practices of assessment in mathematics courses:

— Student teachers appraise their own abilities according to their results in mathematical tests.

— The categories *right* and *wrong* characterize the attitude to mathematics.

— Prerequisites for teaching new content in mathematics are thought of in the form of activity-free knowledge (the student knows...).

— The bureaucratic Bachelor-Master system insinuates that education is about measurable outcomes, credits, evaluations, quality management, etc.

Do we want our future teachers to have this attitude towards their students? What are they going to teach them? Here, enlightenment ideals can still make a difference to the by now common views on education or training.

Immanuel Kant gave "An Answer to the Question: What is Enlightenment?" (Königsberg in Prussia, September 30th, 1784):

> Enlightenment is man's emergence from his self-incurred immaturity. Immaturity is the inability to use one's own understanding without the guidance of another. [...] If I have a book to have understanding in place of me, a spiritual adviser to have a conscience for me, a doctor to judge my diet for me, and so on, I need not make any efforts at all. I need not think, so long as I can pay; others will soon enough take the tiresome job over for me. The guardians who have kindly taken upon themselves the work of supervision will soon see to it that by far the largest part of mankind (including the entire fair sex) should consider the step forward to maturity not only as difficult but also as highly dangerous.

In this quote, one can sense what the age of enlightenment intended by education. Since present ways of teaching leave the student's perception of self at the side of the educational system, the above-mentioned problems can be understood as a lack of enlightenment in mathematics teacher education.

What alternative ways and models are there to change the sense of self and the self-esteem of students in order to become individuals that are more autonomous?

COMMUNITIES OF PRACTICE AND A GROWING MATURITY

In order to become autonomous, mature individuals the students need to experience a sense of self in which their own personal values for their development arise, ones that are supported by a recognition of other like-minded people with similar developing interests – backed by universal values and rules. In that way, they can actualize themselves and determine their own actions.

Hence, we need a course that allows university teachers and students to develop a community of practice, which fosters their development as far as orientation, maturity, autonomy, emancipation, responsibility, self-actualization and self-determination are concerned. To this end, we withdraw our guidance in small steps and replace it by development through progress in the community. Such development of value systems has been described in the discourse on communities of practice. For instance in (Wenger, 2002; chapter 8), we find several procedures for measuring and managing the value creation of a community or network. Such value creation methods are nowadays widely used in management. When we compare the five cycles of value creation, we discover some similarity between this procedure and the five basic questions for the preparation of lessons as Klafki (1963) formulated them. We use the value assessment framework as tool to structure the seminar and its development. A constant reframing and reconsideration of how success, appreciation and development are defined characterize the framework.

A SEMINAR ON HISTORICAL EXCERPTS FROM SCHOOL BOOKS

Historical excerpts from schoolbooks form the basis for an activity in the framework of the seminar. The goal is to undertake small-step lesson planning, starting from the "historical" excerpt. How can such an existing historical excerpt be augmented to curriculum-relevant teaching that serves the *Bildung* of the students? In Germany, there are a handful of schoolbook series that are used extensively in school (Rezat, 2010; Otte, 1981). The historical insertions in these books are all of a similar kind. Since these historical references stem from books that teachers use in their daily teaching at schools, they constitute a link of this activity with the practice. Indeed, when the future teacher finds her or himself teaching with the help of such a schoolbook later, it might be an occasion to unfold the learned attitude again – together with students and colleagues. In addition, it would be possible to get teachers from school involved in the project. Therefore, the seminar at the university already has the goal of letting the participants find like-minded people, who are also interested in history of mathematics and the development of mathematical contents.

Designing the seminar and also tackling the aforementioned problems in self-esteem and predominance of normative results, means starting with sufficiently open but concrete tasks and leaving a lot of time for group discussions: "A key element of designing for value is to encourage community members to be explicit about the value of the community throughout its lifetime. Initially, the purpose of such discussion is more to raise awareness than to collect data, since the impact of the community typically takes some time to be felt." (Wenger et al., 2002, p. 60).

A motive of development and support for a community of practice uniting student mathematics teachers, mathematics teachers, mathematics educators, mathematics textbook authors, historians, educators, other social science people and workers in further education (teacher development) leads to various activities. This motive defines activities and possible actions during the running seminar as well as long term planning as the:

— organisation of student teaching to apply the developed material,

— establishment of connections to textbook authors,

— development of an internet page with additional materials to the historical excerpts,

- linking of related existing internet sources with the materials of the seminar,

- development of activities for teacher in-service training on the basis of the additional materials to the historical excerpts and some aspects of the use of history in mathematics classes.

For our project in Germany, the enormous influence of mathematics textbooks on teaching – especially at the beginning of the career – is essential. In other countries and periods, this might be different. For the development of a community of practice in the spirit we are aiming at, it is important that the joint activities generate joint creations, have a context that is relevant to the participants, are related to their personal experiences and manageable by any member (the excerpts are small – about one page). An important step in transforming the seminar into a community of practice is the emergence and display of immediate values and potential values of the community. Having this in mind, monitoring, structuring and back up of the first group's discussions are important.

The students work in groups on one excerpt. They choose the topic and the material from the books on different grounds, for example:

- individual historical or mathematical interests,

- relations to other school subjects,

- experiences from their practical lessons or teaching in school,

- experiences in tutoring students,

- the wish to be in a team with a best friend.

The activities and interactions between members of the seminar in the group discussions, the choice of the material and plans to unfold it are important for the immediate and potential value of the further community.

Intentions of the discussions are, for example for students to get to know their biases and preferences, to start a general discussion about the use of textbooks and the particular design of the textbooks they will deal with, the realising of interrelations between topics studied in history of mathematics courses, the introduction of existing materials related to the excerpts and the repetition of basics in lesson planning and textbook use.

According to Wenger and colleagues (2002), one of the tasks of the seminar leader is to understand which potentials and *knowledge capital* of the community can be put into use. It may be helpful to give introductory presentations to support structure and to display possible joint activities.

When dealing with a schoolbook there are essentially three perspectives that one can take: the high school student's perspective, the teacher's perspective, or the perspective of the schoolbook author. While being used to the first perspective, the students usually struggle with the two more mature ones. Here we see the major role of communities of practice.

The textbook analysis starts from the perspective of a student, e.g. reading of the chosen excerpts, solving related problems, clarifying prerequisites and presenting the results. The teacher's perspective comes with learning objectives, reflection on the assumed knowledge, time management, and representation of the solutions of the problems. The perspective of an author appears in questions about the way history is used, the accuracy of the presentation of the historical fact, the design of the historical excerpt and the learning objectives of the involvement of history.

The three perspectives are understood as levels of awareness (Mason, 1998). Cooperation of the students is organised by grouping around interests in mathematics, history or interdisciplinary questions related to concept development. For the analysis of the mathematical textbook students can take a historical perspective, asking questions related to the story of a mathematical problem, the history of an idea, the story of a mathematical area, the biography of a mathematician, the story of an institution, the history of a concept.

Questions arising from a mathematical perspective are:

- How do modern notations and representations differ from historical ones?

- Which problems led to the imposition of a term?

Examples	Proof that $\sqrt{2}$ is an irrational number. *Assertion*: $\sqrt{2}$ is irrational. *Proof by contradiction*: Assume the opposite is true.	
The proof on the right is the eldest historically handed down proof for the irrationality of $\sqrt{2}$. At the same time, it is an especially beautiful example of an indirect proof. 	Consequences:	Explanation:
	(1) $\sqrt{2}$ is irrational.	
	(2) $\sqrt{2} = \frac{p}{q}$	Can $\sqrt{2}$ be represented as simplified fraction $\frac{p}{q}$?
	(3) $2 = \frac{p^2}{q^2}$	squaring of the equation
	(4) $p^2 = 2q^2$	resolving to p^2
	(5) p^2 is even	property of even numbers
	(6) p is even	property of even numbers
	(7) $p = 2n$	p is even, hence divisible by 2
	(8) $p^2 = 4n^2$	squaring of (7)
	(9) $4n^2 = 2q^2$	inserting (8) in equation (4)
	(10) $2n^2 = q^2$	dividing the equation (9) by 2
	(11) q^2 is even	property of even numbers
	(12) q is even	property of even numbers
	(13) $q = 2m$	q is even, hence divisble by 2.
	(14) $\frac{p}{q}$ is not a completely simplified fraction	p and q are even, so both are divisible by 2 .
This is a proof by contradiction to the assumption that $\sqrt{2}$ is representable by a completely simplified fraction. Hence, $\sqrt{2}$ is irrational.		

Figure 1: Historical excerpt from a German schoolbook (Neue Wege, 9. Schuljahr)

— What is the mathematical statement of a historical mathematical text source?

The perspective involving reflection on historical, mathematical and cognitive development is the most complex and advanced. It includes theoretical frames for concept development with historical references; that is to say, a historiographic approach, a hermeneutic approach, mathematical awareness, use of history as a mathematical tool, 'Whiggish' approaches or cognitive genetic and historical genetic approaches.

Student work is organised in groups attached to an excerpt. There can be several groups working on one historical textbook excerpt. The joint product of the group activities consists of a joint essay and short lesson plan related to different perspectives, from a historical and from a mathematical point of view. Let us consider two examples.

Example from geometry – reproductions, excerpts from historical sources

Here, we consider a reproduction of a small part of a small historical source. We look (Figure 1) at the *historical proof* of the irrationality √2 from Euclid, which one can find in most textbooks in a similar way:

1) First students will study the proof from the perspective of a high school student. Corresponding mathematical questions from their perspective

could concern the logic of the indirect proof, the logic of the arguments, and proof of the used arguments, mathematical terms and notations.

2) From a historical perspective, it is natural to search for the primary source and to compare it with the reproduction. In this case, it leads to an interesting search for a translation, containing the mentioned proof. At this point, the discussion can be supported by translations with commentaries introducing the students to historical questions that are related to translations and the choice of documents we are referring to when citing Euclid.

3) From the perspective of a teacher, the students would discuss whether the presentation helps to support mathematical understanding, what additional materials with enactive and iconic presentations can be used, and how to evaluate the understanding of the proof. Historical questions could deal with the use of original sources, with additional materials about Greek mathematics, and interdisciplinary learning environments relating to history, mathematics and philosophy.

4) Taking the perspective of a textbook author one could start with the question of whether history in this excerpt is used as a tool or as an object (Jankvist, 2009). Looking at the presentation and the use of modern notations can lead to questions related to *Whiggish* (Fried, 2001) presentations and their justifications. Another aspect is the use of geometrical concepts in Euclid's books and their algebraization in modern school mathematics.

To support group discussions, the tutor could give introductions to Greek mathematics, an overview of Euclid's *Elements* or an introduction to Geometric Algebra. Both translations and secondary literature on Euclid are easily accessible to the students. In this case, one would initially not provide reading material but rather help students to organise and structure the found sources.

An example from analysis

We now have an example of another kind. The textbook authors give a historical overview over a longer period. This excerpt at hand starts with general remarks about the roots of mathematics and analysis. The first mentioned mathematicians reach from Archimedes to Riemann. The excerpt ends with general remarks on the contemporary role of analysis and its central place in the study of mathematics. The subsequent questions are challenging for high school students

Excursion

… After these preparations, in the 17th century, the Englishman Isaac Newton and the German Gottfried Wilhelm Leibniz independently laid the foundation for analysis with infinitesimal calculus.

Newton assumed variable magnitudes to be time-dependent and called them "fluents" (flowing ones). By the derivative with respect to time, he denoted their instantaneous velocities ("derivative"), which he called "fluxions" and marked them with a dot (e.g., $\dot{x}$).

Newton calculated the fluxions by limit considerations. Since such a practice did not fit to his own methodological ideals, at first he did not publish his results, but just mentioned them indirectly while arguing with time-independent geometrical magnitudes.

In that way, it happened that the Germen Gottfried Wilhelm Leibniz developed about ten years later his own theory for the notion of derivative. Leibniz regarded a curve as an "infinity-gon", such that a tangent would intersect the curve in an infinitely small line segment. Here he built, amongst other things, on the insights of Cavalieri. Leibniz introduced the notion of "differentials", which brought forth the notion of "differential calculus". The quarrel between Newton and Leibniz, about which one of them would first have discovered the notion of derivative, has found its way as priority dispute into history of mathematics.

In the course of time, analysis was first further developed without really substantiated foundations. Only in the 19th century could one work with it in a way that fits today's standards, for only since then have notions like function, limit or integral been clarified precisely. To this end, the mathematicians Joseph Louis Lagrange (1736–1813), Augustin Louis Cauchy (1789–1857), Karl Weierstraß (1815–1879), Carl Friedrich Gauß (1777–1855) and Richard Dedekind (1831–1916) contributed in crucial ways.

The integral, in the form it is taught nowadays at grammar schools, goes back to the German mathematician Georg Friedrich Bernhard Riemann. Riemann determined the area that is bounded by the x-axes and the graph of a function by the help of easy to calculate areas of rectangles. The idea of the so-called "Riemann integral" was later further developed by the French mathematician Henri Léon Lebesgue (1875–1941). …

Figure 2: Historical excerpt from a schoolbook (Lambacher Schweitzer, pp. 165–166)

... Prepare a presentation about the history of analysis. To this end, read up on more contributions of the mathematicians that are introduced in the excursion. Take the following points into account:
Which mathematician is credited with the discovery of the exhaustion method? How did Galilei determine the velocity resp. the acceleration of a ball rolling down an inclined plane? What is understood by the theorem of Cavalieri? Explicate Newton and Leibniz's approach to the notion of derivative, giving examples.

Figure 3: Exercise related to the historical excerpt in Figure 1

and for the student teachers in the seminar too. In the case of an overview, a restriction on a well-selected aspect is helpful. In the case of the development of calculus, Barrow-Green (2008) and Blåsjö (2015) give wonderful examples of how to illustrate development.

For the development of the perspective of a teacher or even a textbook author additional help by the tutor is necessary.

The excerpt offers at several places the opportunity to go deeper into history. This single excerpt can supply the seminar with questions and aspects to be explored for many semesters. For our example we restrict ourselves to just one of them.

RESUMÉ

We are aware that the development of a community of practice of mathematics teachers and mathematics educators interested in history is a long-term task.

At the present economising of university life and the strong dominance of normative value systems we consider this experiment nevertheless particularly important.

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ENDNOTE

1. For learning mathematics via historical sources see for instance: www.uni-due.de/didmath/ag jahnke/historische, www.fransvanschooten.nl, www.cs.nmsu.edu/historical-projects/ or www.pageaboutme.mathsisgoodforyou.com/.

TWG12

Posters

“It is necessary to understand where we have come from so that we can further the journey”: History of mathematics in the formation of teacher identity

Maurice OReilly

St Patrick’s College Drumcondra, CASTeL, Dublin, Ireland, maurice.oreilly@dcu.ie

The author draws on his work with pre-service mathematics, physics and chemistry teachers taking a capstone module including six hours of History of Mathematics (HoM). He reflects on this experience using two perspectives: Clark’s application of Ball’s mathematical knowledge for teaching (MKT) in the context of prospective secondary mathematics teachers (PSMTs) use of history, and ongoing work on both teacher identity and ‘Mathematical Identity’.

Keywords: History of mathematics, prospective secondary mathematics teachers, mathematical identity.

SUMMARY DESCRIPTION

Initial teacher education is an activity that necessarily occupies a hybrid space, drawing on several epistemologies. The tension between pedagogy/didactics, on the one hand, and the ‘disciplines’, on the other, is ever-present. In the HPM community, we believe that HoM should play a central role in teaching mathematics (e.g., Kjeldsen, 2012) and yet we struggle to find where and how to locate HoM in practice (e.g., Guillemette, 2012).

The context of this reflection involved 34 students in the second semester of their final (fourth) year of the BSc in Science Education at Dublin City University, meeting once a week for a 3-hour session background in over a period of twelve weeks in the spring semester of 2014. All students had significant background in mathematics, physics and chemistry (and majoring in two of these) and involving substantial school placement. The author accepted the invitation of the module coordinator to prepare and deliver two 3-hour sessions on HoM, and to design assessment associated with these sessions.

Students were required to choose a ‘fact’ from the MacTutor HoM Archive and respond to one of two questions online relating to their chosen fact, in fewer than 100 words. The first 3-hour presentation followed this exercise immediately, the intention being that students would have some short, yet intense, engagement in HoM in advance of a substantial presentation. The second more extensive exercise required students to make a connection between one reading (from a choice of seven) and one (or two) web links (from a choice of twenty) to HoM topics ranging in time from Plimpton 322 to the fundamental theorem of algebra. The second 3-hour session comprised a presentation on these twenty topics and a group discussion, and immediately followed the second exercise.

Details of the range of student responses emphasising their mathematical thinking and their expectations of using HoM in their teaching are presented. This discussion relates to the aforementioned work of Clark (2012) which uses Ball’s MKT framework.

It also draws on the author’s own work and that of his collaborators (Eaton, Horn, Liston, Oldham, & OReilly, 2014) on Mathematical Identity to give a different perspective on the data arising from the second exercise just outlined. Mathematical Identity developed out of the works of several researchers such as Wenger (1998), Sfard and Prusak (2005) and Kaasila (2007). Initial consideration of these data, through the lens of Mathematical Identity, indicates a promising approach to encouraging PSMTs to take HoM seriously in their future careers.

In keeping with the work of Eaton and colleagues, attention is drawn to the narrative elements of the data and in particular to the evolution of students' perception of mathematics and their experience of learning/teaching it over time. It seems that the inclusion of HoM can be a catalyst to enrich such narrative and consequently strengthen students' Mathematical Identity.

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Primary sources in the elementary school

Vasiliki Tsiapou

University of West Macedonia, Kozani, Greece, vana@semiphoto.com

The poster reports on the design and the implementation of a pilot teaching intervention – part of my ongoing research – with two historical texts. The sample was a small group of Greek students of the 6th grade. On the poster I present the texts, the design, and photos of student's activities.

Keywords: Primary historical sources, elementary education.

The historical texts belong to Liu Hui, a renowned Chinese mathematician of the 3rd century CE. The first one is his commentary (Lay-Yong & Tian-Se, 1986) on a problem of a circular field which is included in the ancient Mathematical Canon *'The Nine Chapters on the Mathematical Art'*. The area of the field is the unknown, but the data assumed π to be 3. The Canon offered four algorithms and the solution. Liu Hui proved the correctness of the first algorithm and gave a more precise value of π. The second text is the preface of Liu Hui's commentary on the canon. In this text Liu Hui reveals his reasoning and pedagogical considerations (Siu, 1993).

OBJECTIVES – DESIGN AND IMPLEMENTATION

The objectives are for students to: 1. make a transition to the more theoretically oriented geometry of middle school; 2. appreciate mathematics as a human creation and of different cultures; 3. engage in meta-level discussions about in issues and meta-issues of History (Jankvist & Kjeldsen, 2010). For addressing the objectives I designed a historical module (Jankvist, 2009) with the use of MKT (Ball, Thames, & Phelps, 2008) as the overall framework under which I tried to coordinate domain specific frameworks (i.e. proof in the elementary school's settings).

The implementation of the module had three phases: Introduction, Analysis and Synthesis (Jahnke, 2000). In the *introductory part*, I provided historical information about the socio-historical context and the mathematics at the time of Liu Hui. The use of History was under the conceptual dipole *'History as a goal'-'History'* (Tzanakis & Thomaides, 2011) and partly the second objective was addressed. In the *analysis part*, and for the first objective (*'History as a tool'-'History'*), students were reading small excerpts of the first text, and were decoding the commentator's guidelines. During the teaching students were encouraged to develop reasoning and communication skills (i.e. to explain and justify using deductive reasoning). In order to prove the geometrical face of the proof (the correctness of the first algorithm) students had tactile experiences and constructed geometric figures with conventional and digital tools. For the arithmetical face of Liu Hui's proof (the reason that π is 3.14), which is not highlighted in the Greek textbook, students made process pattern generalizations and filled in and interpreted tables and spreadsheets. In the *third part* I tried to address the last two objectives. I gave information about the commentator's philosophical context and with a more playful activity under the name 'the cards of philosophy', I engaged students in meta-issue discussions about the relationship between Liu Hui's work and philosophy *('History as a goal'-'History')*. In this activity I incorporated the second text and students discussed about Liu Hui's internal factors, by reflecting on the first text's activities. Also, following Liu Hui's concept of the discipline of mathematics as a 'tree', students created their own 'tree' connecting mathematical topics that they had used while they were dealing with the proof. In the end students presented to peers a synthesis of their work.

The integration of primary sources in the elementary school's mathematics may seem challenging for various reasons. Yet, if we want to stay true both to the commitments of mathematics education and of the History of mathematics (Fried, 2007) it is worthwhile to explore the possible ways to do so.

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TWG13

Early years mathematics

Introduction to the papers of TWG13: Early years mathematics

Maria G. Bartolini Bussi[1], Esther Levenson[2], Ingvald Erfjord[3], Eugenia Koleza[4] and Bożena Maj-Tatsis[5]

1 Università di Modena e Reggio Emilia, Italy, bartolini@unimore.it
2 Israel
3 Norway
4 Greece
5 Poland

INTRODUCTION

The aim of the Early Years Mathematics working group is to share scholarly research related to mathematics education concerning children aged 3–8. This age group spans preschool through the early grades of primary school, and takes into consideration that in different countries children begin primary school at different ages. During CERME9, there were approximately 30 participants (authors and co-authors), with 20 accepted papers and 7 accepted posters. Of the 20 papers, one paper was a literature review paper on early childhood education (Alpaslan & Erden); the rest were empirical studies (12 related to preschool and 7 to primary school). There were 6 posters reporting empirical studies conducted within the preschool environment and one poster which described a STEM project promoting popular science which involved both mathematicians and scientists (Sorokina)[1].

The structure of the timeslots was designed in order to stimulate interaction and collaboration among participants and was as follows. Related papers were paired and after a short presentation of the first presenting author, a prepared question was posed by the second presenting author which initiated a general discussion. The roles of the presenting authors were then exchanged. Each pair of papers was allotted nearly 40 minutes. Posters were also allocated a few minutes of presentation time within the working group. One timeslot was devoted to small group discussions about general issues that had been raised during the papers and posters presentation. This report draws on the outcomes of these small group discussions and of an extra meeting of co-leaders with a few participants who had volunteered to help in the preparation of the report. On the last day of the conference, our room was visited by more than twenty colleagues from other working groups, where a lively discussion ensued.

In addition to the presentations, we had the privilege of visiting a local preschool. Very early in the morning, a group of participants visited the *Materska Skola, Opletalova 14, Praha 1*, directed by J. Moravcova, accompanied by Michaela Kaslova, one of the participants of the TWG. This visit was really interesting as it explained, better than a report, the organization of preschools in the Czech Republic.

IS EARLY YEARS MATHEMATICS "MATHEMATICS"?

The above question was raised and discussed in the first session of the TWG and continued to be in the background of many discussions throughout the conference. Our shared answer was positive. Early years mathematics is mathematics when it fosters the development of mathematical processes. We can study the construction of mathematical meaning at each age. Some of the participants preferred to use the terms pre-mathematics or prepa-maths (see Fuchs et al., 2015; Kaslova, 2010) in order to emphasize that during the early years, the concern is for the processes which enable construction of mathematical meanings as opposed to the rote learning of terminology, techniques and calculation. It is important to raise public awareness about early years mathematics (e.g., through informal mathematics or new STEM projects). The papers and posters presented during

1 These short references hint at the authors of papers/posters in the TWG.

TWG13 sessions offer a wide variety of examples of mathematical meanings to be constructed during those early years.

Preschool mathematics

The construction of mathematical meanings in preschool was presented by several authors in a variety of cases: the give-N task (Rinvold & Erstad); comparison of whole numbers (Tubach); conceptual subitizing (Rodrigues, Cordeiro, & Serra); the meaning of double (Björklund); mathematical discourse (Lavi); the coordination of the audible, the visible and the tangible (Pimm & Sinclair); unsolvable problems (Tirosh, Tsamir, Levenson, Tabach, & Barka); measurement (Erfjord, Carlsen, & Hundeland; Skoumpourdi), shapes (Erfjord, et al.; Pettersen, Volden, & Ødegaard), and symmetry (Demetriou).

While discussing the various mathematical activities described in the studies, it became aparent that different countries have different curricula with specific mathematical goals for preschool. The group expressed the need to know more about the organization of preschool in each country. Two cases were explicitly discussed – Italy and Sweden – observing that in both cases the preschool curriculum had been developed (in Italy in 1991, in Sweden in 2010) following Bishop's (1991) analysis of the universal activities for mathematical enculturation: counting, locating, measuring, designing, playing, explaining (Svensson). In addition, it was noted that programmes for preschool teacher education and development must be encouraged, where clear mathematical goals and ideas for learning trajectories for children are pinpointed. A specific example of a test for diagnostic purposes was analysed (Kaslova). The group discussed the risk that this kind of testing may introduce (e.g., teaching to test).

Primary school mathematics

Several authors offered examples concerning mathematics in early primary school:

Grade 1: inclusive definition of squares and rectangles (Bartolini Bussi & Baccaglini-Frank); conceptualizing parallel and perpendicular lines (Vighi).

Grade 2: multiplication tables (Maffia & Mariotti); magic squares (Maj-Tatsis & Tatsis).

Grade 3: patterns (Ferrara); fractions on the number line (Robotti, Antonini, & Baccaglini-Frank); employing the bar model for solving arithmetic word prblems (Koleza).

In most of the above examples, argumentation was also emphasized.

The teacher's role in promoting mathematics was the explicit focus of some contributions (Delacour; Erfjord et al.) but was also discussed in other cases. The group agreed that it is important to reach a balance between structured activities controlled by the teacher and fostering conditions which will encourage children´s agency. The issue of agency in preschool was the explicit focus of one contribution (Erfjord et al.).

CONTEXTS

Part of the time during small group work was used to discuss the issue of different contexts. Without entering into a review of the related literature, there is a risk of misunderstanding the notion of context. However, defining what is meant by context that was not the aim of this working group. Instead several papers and posters related to context either when describing a specific task, a specific piece of mathematics knowledge or a specific intention of the teacher/researcher. More generally, there are also institutional contexts (see, for instance, the short reference above on curricula in different countries) and the sociocultural contexts (Jaworski et al., 2015).

We collected meaningful examples of different contexts (in all of the above meanings) concerning

- outdoor activities in northern countries in the very cold winter (Delacour in Sweden and Erfjord et al. in Norway), which was quite astonishing for researchers from southern countries but are, on the contrary, a part of the shared values of their culture;
- families as places for mathematical play (immigrant families in Germany, Solmaz);
- informal (out of school) activities (Math circles, STEM camps, talks with a mathematician, Sorokina).

When the focus was on school mathematics, we had many examples of using manipulatives, either con-

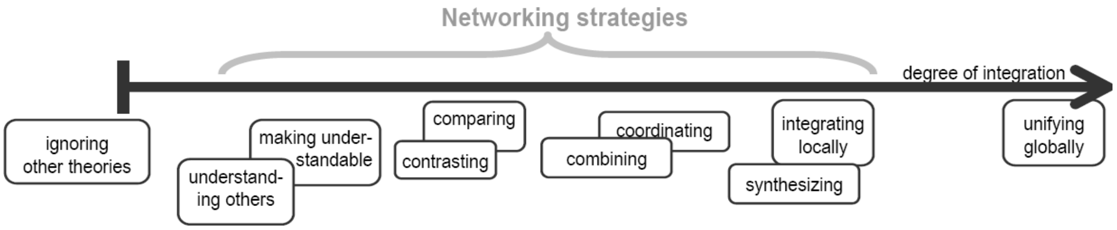

Figure 1: Networking strategies

crete manipulatives (cardboad collection, Demetriou; straws, Vighi; three dimensional models and folding solids, Pettersen et al.; blocks, Solmaz; virtual manipulatives (programmable robots, Bartolini Bussi & Baccaglini-Frank; softwares and apps, Pimm & Sinclair). Comparing concrete and virtual manipulatives was also investigated and discussed (Demetriou).

THEORETICAL FRAMEWORKS AND RESEARCH QUESTIONS

Several different theoretical frameworks were mentioned:

- variation theory after Marton (Björklund);
- levels of artefacts after Wartofsky (Svensson);
- semiotic mediation after Bartolini Bussi & Mariotti (Bartolini Bussi & Baccaglini-Frank; Maffia & Mariotti);
- discoursive approach after Sfard (Lavi; Vighi);
- semiotic bundle after Arzarello and colleagues (Ferrara);
- theory of didactic situation, didactical contract after Brousseau (Delacour);
- theory of knowledge objectification after Radford (Rinvold & Erstad);
- Interactional niche after Krummheuer (Solmaz);
- grounded theory (Arnell);
- sociocultural theories after Rogoff (Erfjord et al.).

In most cases, it seemed that theoretical frameworks are chosen with reference to either authors or groups of authors from the same country; this choice could be misunderstood as patriotism, it is not necessary the case. For instance, the semiotic mediation theory is useful in countries where the focus is on long term studies, which in turn depend on the institutional role of a teacher working for more than one year with the same group of pupils (Italy: Bartolini Bussi & Baccaglini-Frank, Ferrara). Hence, we have another interesting example of the influence of cultural contexts research on early years mathematics.

The relationship between context and theoretical framework is a big challenge for the diffusion of findings and the possibility of exploiting findings from different cultural contexts (Bartolini & Martignone, 2013). The presence of different theoretical frameworks shows that it is timely to discuss networking theories, according to the vision developed by Bikner-Ahsbahs and Prediger (2014, p. 170).

This issue might be focused in CERME10.

According to different theoretical frameworks, the authors of papers/posters have developed different kinds of empirical studies:

- intervention studies (short term and long term studies; attention to the teacher's role or focused on learners; examples of STEM in informal education)
- observation studies (observing learners; observing teachers; observing classroom processes)
- studies on teacher education .

It is difficult in some cases to make this clear distinctions, as in developmental projects the main idea of the activities is a result of a process of the engagement between researchers and teachers.

CONCLUDING REMARKS

The discussion sketched above left a problem open. During the last meeting with visitors from other TWGs we realized that there was some overlapping between the TWG13 and other groups, for instance the TWG02 on *Arithmetic and number systems*, where a major focus was on primary school arithmetic. One possibility to avoid overlapping is to interpret *Early year mathematics* as focused solely on preschool. This choice might be problematic because of the different institutional interpretation of preschool and primary school in different institutional context (a relevant example was offered by Vennberg, with the "transition" year between pre-school and primary school in Sweden). One of the arguments for continuing the TWG13 on *Early year mathematics* with focus on both is the importance to study also continuity between the two kinds of schools. Participants of TWG13 added that the exchange between researchers of both levels contributed to the general understanding of issues related to teaching and learning mathematics during these critical years.

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A few months after the conference, we received the very sad news of the death of Zişan Gûner Alpaslan and Mustafa Alpaslan in a traffic accident. Gisan was part of our group in TWG13 and as the first author of a paper she proved her talent in mathematics education and we learnt to know her as a very nice colleague. She will be sadly missed.

TWG13

Research papers

Using pivot signs to reach an inclusive definition of rectangles and squares

Maria G. Bartolini Bussi and Anna Baccaglini-Frank

Università di Modena e Reggio Emilia, Dipartimento di Educazione e Scienze Umane, Modena, Italia, bartolini@unimore.it

We present some fragments of a teaching experiment realized in a first grade classroom, to sow the seeds for a mathematical definition of rectangles that includes squares. Within the paradigm of semiotic mediation, we studied the emergence of pivot signs, which were exploited by the teacher to pave the way towards an inclusive definition of rectangles and squares. This was done to favor overcoming children's spontaneous distinction of these figures into distinct categories, reinforced by everyday language. The experiment is an example of an approach towards the theoretical dimension of mathematics in early childhood.

Keywords: Bee-bot, first grade, pivot signs, rectangles, squares.

INTRODUCTION

Rectangles and squares represent a paradigmatic example of the conflict between the perceptual experience and the theoretical needs of a mathematical definition (on this persisting conflict also see Hershkowitz, 1990; Clements, 2004; Fujita, 2012; Koleza & Giannisi, 2013), where squares are to be considered as particular rectangles (we will refer to a definition of rectangles that includes squares as being *inclusive*). Mariotti and Fischbein (1997) claim that "from the figural point of view squares and non-square rectangles look so different that they impose the need of being distinguished at least as much as triangles and quadrilaterals" (Mariotti & Fischbein, 1997, p. 224). Actually the difficulty of naming and classifying geometrical figures (and, in particular, squares and rectangles), according to inclusive criteria, seems to depend on different reasons:

— the implicit constraints of everyday language: for instance, both in Italian and in English (as well as in other European languages) the names "quadrato" [square] and "rettangolo" [rectangle] hint at a complete separation of the figures into two different classes (square and not-square rectangles);

— some widespread improper practices in school which reinforce the separation between squares and rectangles (for instance, activities with attribute blocks, where squares and non-square rectangles are classified in different sets).

Hence, teaching needs to orient learning towards an inclusive definition. The question is: *at what age?* We claim that, although this choice may create a discontinuity between everyday language and school language, it is possible from early childhood to sow the seeds of an inclusive definition, focusing on the experience of walking along or drawing a rectangular path, where the change of direction in the four angle vertexes has the potential to attract the students' attention. In the following, we report on some fragments of a long term teaching experiment, carried out within the theoretical framework of semiotic mediation (Bartolini Bussi & Mariotti, 2008). Additional details are discussed by Bartolini Bussi and Baccaglini-Frank (2015).

THEORETICAL FRAMEWORK

In order to design and to analyze the teacher's role in the classroom teaching process, we adopted the theoretical framework of semiotic mediation (Bartolini Bussi & Mariotti, 2008; Bartolini Bussi, 2013). The design process is represented by the reciprocal relationships between the tasks, the artifact, and the mathematical knowledge at stake. In this relationship the *semiotic potential* of the artifact is made explicit. The artifact is the *bee-bot*, a small programmable robot represented in Figure 1 (also see the next section). When children are assigned a task they engage in a rich and complex semiotic activity, producing traces (gestures, drawings, oral descriptions and so on), that we refer to as "situated texts". The teacher's job is to collect all these traces (by observing and listening to the children), to analyze them and to organize a path for their development into "mathematical texts" that can be put in relationship with the fragments of mathematics knowledge that are to come into play.

Figure 1: Bee-bot's back

The process of semiotic mediation also concerns the functioning of semiotic mediation within the classroom. The teacher acts as a cultural mediator, in order to exploit, for all students, the semiotic potential of the artifact (the bee-bot in our case). In this last process, Bartolini Bussi and Mariotti (2008) identify three main categories of signs: artifact signs, pivot signs, and mathematical signs. *Artifact signs* "refer to the context of the use of the artifact, very often referring to one of its parts and/or to the action accomplished with it. [...]"; *mathematics signs* "refer to the mathematics context" and *pivot signs,* which "refer to specific instrumented actions, but also to natural language, and to the mathematical domain" (ibid, p. 757). *Pivot signs* can be particularly useful for fostering a transition from situated "texts" to mathematical texts. Pivot signs develop and are enriched by their relationships with other pivot signs, hence building a *network* of pivot signs. Mathematical signs are not intended to suddenly substitute artifact signs; in fact the latter may survive for some time, especially for lower achievers or in cases in which the formal mathematical definition and the reasoning of the corresponding concepts require long term processes to be achieved.

Within this framework, our study addressed the following research questions:

(1): How might a long-term process of semiotic mediation that exploits the semiotic potential of the bee-bot with respect to the development of an inclusive definition of rectangles look for first graders?

(2): In particular, which kind of pivot signs (if any) can be identified and exploited during such long-term process?

THE CHOSEN ARTIFACT: THE BEE-BOT

The bee-bot (Figure 1) is a small programmable robot, especially designed for young students. Its ancestor is the classical LOGO turtle, originally a robotic creature that could be programmed through an external computer to move around on the floor (LOGO Foundation, 2000). It is not necessary to have any additional computer to program the bee-bot; this can be done simply pressing a sequence of command buttons on its back. When the programme is executed, the bee-bot moves on the floor: the execution of each command is followed by a blink of the eyes and by a short beep-sound. The bee-bot hints at many sets of meanings and mathematical processes, partly related to mathematics and partly related to computer science, for instance: counting (the commands); measuring (the length of the path, the distance); exploring space, constructing frames of reference, coordinating spatial perspectives, programming, planning and debugging. In a long term teaching experiment, all these sets of meanings are at stake, sometimes in the foreground and sometimes in the background. Focusing on any set of them depends on the adult's teaching intention. The bee-bot walks on the floor and traces paths that can be perceived, observed, described with words, gestures, drawings, sequences of command-icons and so on. Paths (either traced or imaginary, when no trace mark is actually left) constitute a large experiential base to "study" some plane figures, that can be traced using the available commands. These are polygons with sides measured by a whole number of steps and with right angles only. With the additional constraint of being convex, the bee-bot can be programmed only to turn "left" or "right" (with respect to itself), and therefore the convex polygons it can trace are always rectangles (including squares). Moreover, in experiences where "pretending to be the bee-bot" is essential, children embrace the robot's perspective: they move with the bee-bot and they see through its eyes. In particular, when walking along a closed convex path and ending up where they started, the children turn 360° in four equal "chunks" during which their orientation is perceived as essential (they find it important to end up facing the same direction as when they started).

THE TEACHING EXPERIMENT

Above we have discussed some features that define bee-bot's high semiotic potential with respect to the emergence of an inclusive definition of rectangles, characterized by the property of four right angles. Our teaching experiment was designed to capitalize on bee-bot's potential of fostering awareness of the "four right angles" property of generic rectangles (including squares).

Several sessions (15) were carried out in a first grade classroom at the beginning of the school year, for 4 months (more or less once a week) either in the classroom or in the gym, with a careful alternation of whole class or small group activity (with adult's guidance) and some individual activity. Each session was carefully observed by the teacher, by a student teacher or by a researcher (the second author of this paper), with the collection of students' protocols, photos, and videos. The tasks were designed by the whole research team, drawing on the initial intention and on some changes implemented "on the fly" based on episodes that occurred during the experiment. Due to space constraints it is not possible to report on all the details, so we have focused on particular sessions where the production of signs was very rich and fundamental for preparing the final summary texts and poster for the students (see Figure 6 in this paper, and Bartolini Bussi & Baccaglini-Frank, 2015).

Observing programmed bee-bots

In this session, students were given two bee-bots that had ahead of time been programmed with the same sequence. The task was: *Describe what they do*. The students watched the twin bee-bots move together, starting facing in the same or in different directions, and then moving separately. Then the memory of one of the bee-bots was erased (CLEAR command-icon) and the students were asked to reprogram it so that it would move just like the other bee-bot. The students' productions concerned both global and local aspects. Global aspects refer to the perception of a path as a whole (as if bee-bot had drawn it on the floor), whilst local aspects refer to special points of the path. An example of the former is the expression "it did an L"; an example of the latter is "they switched the turn". Both aspects also appeared in gesturing: the path is represented by a single pointer finger tracing a path in the air (tracing gesture), whilst turning is represented by moving the right hand (for a right turn) or left hand (for a left turn) up and to the right or left in a rotation (turning gesture). The *turning gesture* was mirrored by the student-teacher, as a pivot sign with respect to the notion "angle" in a path.

Pretending to be a bee-bot

During this session the students were asked to work in pairs: one pretended to be the bee-bot and the other gave the first commands to move according to some undisclosed (to the first student) path. The intention was to guide the children to focus their attention on the turn command. Typical words used were be "Straight Ahead" "Left" "Right" "Backwards" usually without quantifying the number of steps, and frequently combining a translation with a change of direction (rotation). For example, when a student said "left" the bee-bot student frequently would not only turn left, but s/he would also take a step in that direction, or even just take a step to his/her left without even turning in that direction. The student-teacher's intervention here was fundamental in focusing the children's attention on "turn" commands, which led to their beginning to explicitly consider rotations as important elements per se, without having to associate them to steps.

Constructing paths

Several activities were designed around tracing different kind of paths on the floor. When the aim was to produce particular letters of the alphabet, the students' attention was focused mostly on the "possible" and "impossible" letters: they empirically discovered that some capital letters (e.g., L, T. I) could be traced out, whilst others could not (e.g., B, A, D, O). In fact, neither acute angles ("sharp points") nor circular arcs ("fat curves") could be traced by the bee-bot. Children produced many examples of combinations of words, gestures and drawings, aiming at distinguishing the shapes (letters) which could or could not be drawn. There was a particularly rich production of words such as "angles", "(fat) curves", "diagonals", "(sharp) tips/points", "broken lines" and of related gestures and drawings. Suddenly, within this experience, an important event took place; this will be the seed of an inclusive definition of rectangles.

The main pivot sign: the "squarized" O

In a small group the following exchange occurred:

Student-teacher: ...Did you do an O?

Student: No. Then it could do like this this this and this [he gestures four consecutive right angles] a squarized O. Ah, then it can make a square!

We have translated a non-existing Italian word (*quadratizzato*) into a non-existing English word (*squarized*). Other students started talking about "squarized Os" and other possible "squarized letters", intending letters that include one or more squarized Os within them (e.g., P, B). These squarized 0s were acknowledged by the teacher and the research team as pivot signs, hinting at both the perceived path produced by the bee-bot (*artifact sign*) and at a *square* (a figure, interpreted as a *mathematical sign*). The importance of the four consecutive right angles suggested to orient children's attention towards this feature, that seemed to put in shade the length of each piece of the traced path (the sides) and to put in the foreground the four changes of direction, common to all squarized Os.

Focusing on the four right angles

In the students' complex experience, each right angle appeared with seemingly different meanings, that also affected the signs used. These, initially, were mainly dynamic and related either to the student pretending to be a bee-bot or to the bee-bot:

a) *Dynamic change of direction of the student pretending to be a bee-bot;*

b) *Dynamic change of direction of the bee-bot under the effect of the turn command.*

In both these cases, however, the angle was the external angle, i.e. the region swept by the gaze of either the student or the bee-bot while changing direction. When the researcher proposed to draw the paths in a "faster way: using a mark like the one the bee-bot

Figure 2: Sign for the right angle

would make if a marker were used", she chose to mirror a sign produced by a student "a turn like this" close to the turning point of the path (see Figure 2).

The sign had the potential to become a pivot sign with respect to the notion of "angle" (external angle): it recalls the command-icon on bee-bot's back, but it is somewhat decontextualized, since there appears to be no explicit mention to the bee-bot
In addition to these dynamic signs, as the teaching experiment went on, the children developed other signs, which lacked such dynamic components:

c) *Hands-meeting gesture referring to the point in the path traced by the bee-bot;*

d) *Gestures to interpret a static figure (referring to a dynamic experience);*

e) *Verbal utterance of the list of commands (uttered during or after the programming of the bee-bot);*

f) *List of commands written horizontally.*

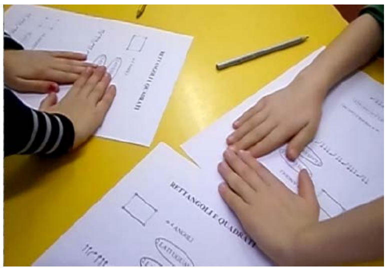

Figure 3: The students' gesture

First we describe the hands-meeting gesture (*type c*). While exploring figures that represented rectangles, including squares, a powerful gesture was realized by one of the groups of children and rapidly imitated by others: the two hands coming together at a right angle (Figure 3). The gesture emerged as the students tried to explain the property that all squarized O's (be they "allungati" [stretched] or "perfetti" [perfect]) had in common: all the four angles (internal angles) are equal and right. Moreover the gesture stresses the vertex as an important feature of the angle. Signs of *type d* were identified, for example, in the argument presented below (Figure 4), on how the angles of a square or rectangle have to be (as opposed to angles such as the ones of the parallelogram that was included in one of the worksheets).

Figure 4: Veronica's gestures

Veronica: "[in a square or rectangle] the angles go down straight...[in the parallelogram] they are a bit down to the right and a bit down to the left. It has to go straight, not like this and down, it shouldn't be a bit down like this one [she moves her pencil in the air along a slanted line with respect to a horizontal bottom line]. Instead it has to go straight like this and like this...it has to be straight like the line but a bit lying down [she marks the lower horizontal line]."

Signs of *type e* appeared when the students' attention was drawn to the "length of the path". Sometimes the turn command was in shade, as it did not lengthen the path perceived while the bee-bot spun around. However the number of commands for paths with angles, was not the same as the number of steps forward. So sometimes the turn command was still skipped (children 1, 2, 3, below). While in some of the children's utterances it was acknowledged (child 4, below) as a command like the others (it is represented by a similar button and it is executed with by a beep and a blink of bee-bot's eyes).

Child 1: Three steps then three then three then three we make a square, because it is the same ends, the same length.

Child 2: Instead, the other one has 1, 2-1, 2, 3-1, 2-1, 2, 3, it has two the same and two the same.

Child 3: The other was three, two, three, two. Not all equal.

In contrast

Child 4: Two forward, turn right, two forward, turn right, two forward, turn right, two forward, turn right. The segments have to all be equal.

Type f emerged in activities in which children had learned to represent traced paths as written sequences of commands, typically in a horizontal line, from left to right. Within these sequences they searched for regularities allowing them to distinguish different types of "squarized Os". Figure 5 shows signs left on the interactive white board after a discussion on "stretched squarized Os" (non-square rectangles) with respect to "perfect squarized Os" (squares). We note here how some students' language (in this and other occasions) seemed to be evolving into condensed pre-algebraic forms, such as a+b+a+b, that could eventually become expressions like 2a+2b for the rectangle and 4a for the square (a particular case in which a=b). In this teaching experiment, however, we did not pick up on these expressions, leaving them only as little germs to be nurtured by the teacher in future years (perhaps even during the second grade).

Focus on the shapes as wholes

Shapes as wholes were focused on from the very beginning of the teaching experiment, with either verbal descriptions alone or also with hand gestures. After the introduction of the idea of squarized Os, the adults involved in the experiment started mirroring students' utterances involving the words "rectangles"

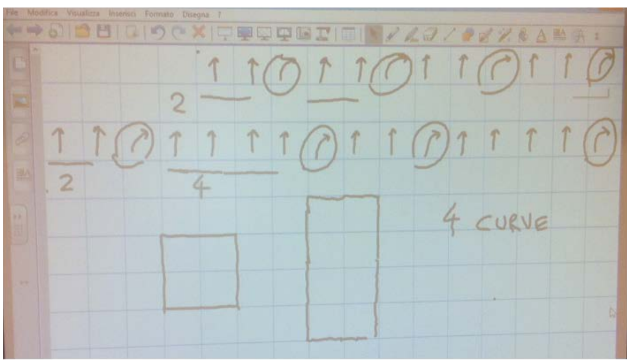

Figure 5: Agreed-upon signs for the programmed sequences and the paths

and "squares". As expected, when the attention was not brought to the word *squarized O* students spontaneously tended to partition the two situations, implying that "rectangles" had pairs of sides with *different* lengths ("equal in front of each other") while "squares" had sides that were "all equal". For some children this property seemed to persist when talking about "stretched squarized Os" with respect to "perfect squarized Os", while other children seemed to only differentiate "perfect squarized Os" from all other squarized Os, since they were special, being "all equal".

The shared meanings

We chose to build on what seemed to be the idea of this second group of students to reach a summary of the shared meanings. The most important step in this direction was a poster of "our" discoveries, a first step towards the development of "mathematical texts".

In this poster (Figure 6) several signs produced in the classroom are reconsidered, constructing a text where artifact signs (e.g. the figure of the bee-bot, the recollection of "giving it" a sequence of commands, the turns), pivot signs (e.g. the squarized Os, the small arrow to represent the external angles, and mathematical signs (e.g. squares; numbers; rectangles) are included.

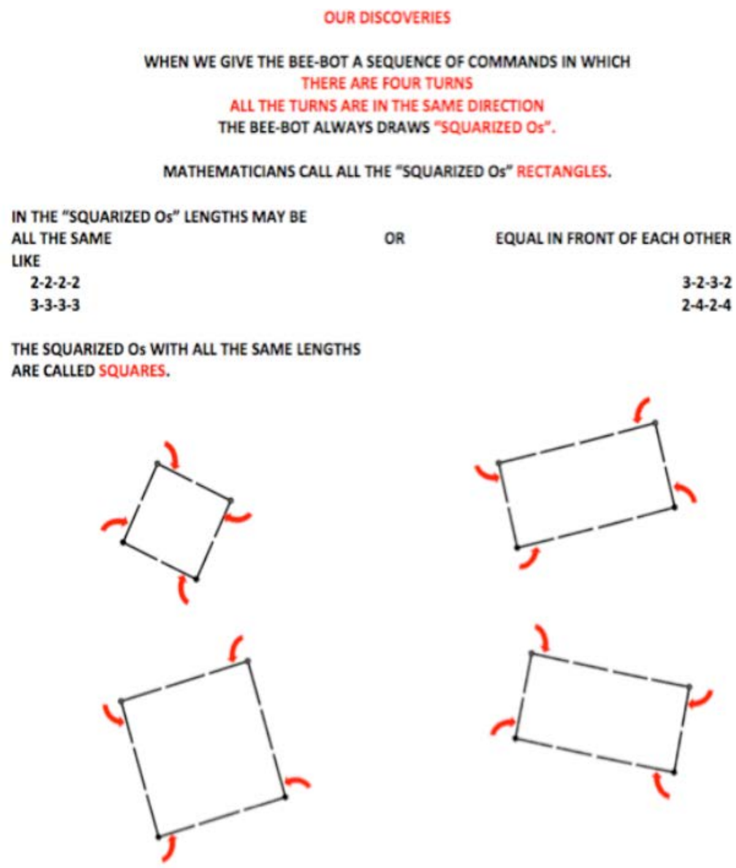

Figure 6: Poster of "our" discoveries

Is this text a mathematical text? Not yet: it is still a hybrid text, where the richness of the exploration remains present. What is important in this phase is that *all* of the students could identify this poster as having been produced by the whole class as a community. The choice of which signs to include was discussed by the research team, trying to collect signs that hinted at the individual and collective processes. The poster was discussed in the classroom; the students seemed very happy to find their ideas made public and to receive a reduced-size copy to glue on their notebooks. Some months later, a follow up questionnaire confirmed that (at least some) students had appropriated, and transferred it to a mathematical context, an inclusive definition of rectangles (other students were still on their way along this process). As mentioned before, the process is not to be considered finished. The teacher has planned to go on with the same group of students and deepen the inclusive definition for which she planted the seeds during this teaching experiment in the first grade.

DISCUSSION

The teaching experiment fruitfully exploited the semiotic potential of the bee-bot, joining different ways of representing the paths traced by the small robot, as sequences of commands, as wholes, as either physical or mental drawings, in both dynamic and static ways. During this long term process the students approached several pieces of mathematics knowledge, including counting (the commands), measuring (the length of the path, the distance), exploring space, constructing and changing frames of reference, coordinating spatial perspectives, programming, planning and debugging. The approach towards an inclusive definition of rectangles is only one aspect of this long and complex process.

A final comment on language. We do not claim that the inclusive (and decontextualized) definition of rectangles is already accepted by all the students (in fact we saw that this was not the case). Rather we find it important that students started becoming aware of the fact that theoretical mathematical needs may be different from everyday life needs. Moreover, we do not believe that the inclusive definition should be used also in everyday life. Rather it seems that, with this experiment, we have put the students in the situation of potentially seeing squares and rectangles within a same "family". What happened, indeed, was that the idea of "square" seemed to be overarching, in spite of the mathematical choices. The students seem to speak of the squarized O as the ancestor of rectangles (including squares) but, from the perceptual point of view they need to distinguish "perfect squares" from "stretched squares". This reminds us of the Chinese way of naming squares and rectangles: the sequences of ideograms for the words "square" and rectangle" contain two out of three of the same ideograms. Those that indicate "sides" and "shape" are the same, while the first indicates "exact" (for the square) and "long" (for the rectangle). This is represented in Figure 7.

正方形　長方形

Figure 7: Chinese characters for "square" and "rectangle", respectively

So, linguistically, a square is seen as a "shape with exact sides" and a rectangle as a "(same) shape with long sides". In this case language makes explicit that squares and rectangles are two kinds of a same thing, deeply related to each other and not partitioned into categories. The Chinese choice of the square as the most important shape may be related to the Chinese ancient culture, where it represents the Earth and the circle represents the Sky. This Iconic cosmology is shared by other ancient cultures.

ACKNOWLEDGEMENT

This teaching experiment was carried out within the PerContare project (ASPHI, 2011), with the support of the Fondazione per la Scuola of the Compagnia di San Paolo di Torino.

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The trouble with double: Preschoolers' perception and powerful teaching strategies

Camilla Björklund

University of Gothenburg, Gothenburg, Sweden, camilla.bjorklund@ped.gu.se

This is a study of three teachers working with the notion 'double' with 4- and 5-yearolds. The purpose is to analyse what aspects that are critical for children to develop conceptual understanding of the notion. Data for analysis consists of video documentations from the teachers' authentic work that is driven by the theoretical framework of Variation theory of learning. The results show that children perceive the notion as an operation, either absolute 'add one' or relative 'add equally many', but there are also children expressing awareness of the part-whole relationship. These qualitatively different ways of perceiving the same notion give valuable insights of emerging mathematical abilities and the analysis of the teaching acts reveal powerful strategies for learning the notion, in terms of simultaneous discernment of critical aspects.

Keywords: Mathematics, double, preschool, variation theory of learning.

BACKGROUND

This study is part of a developmental project conducted in Swedish preschool (National Research Council, grant nr 724-2011-751). The general aim of the project was to enhance preschool teachers' awareness of mathematics as a content and goal in early years education, and to find pedagogical strategies that are in line with both the traditional way of working with children in Swedish preschool, which is heavily appreciating play as children's natural way of learning and being, and with contemporary research and theories on children's mathematical development.

This study is a contribution to the field of knowledge concerning early mathematical learning and teaching. The notion chosen for inquiry is 'double', which is regarded complex in its own simplicity, as double means not only an operation on a quantity, but also a description of the relationship between quantities and is relative in its nature.

CHILDREN DEVELOPING CONCEPTS

Mathematics in the early years is often considered close to the child's experienced life, not least because of the common notions used on a daily basis to describe numerical and spatial relationships in the surrounding world.

Piaget, Inhelder and Szeminska (1981) studied young children's conception of parts and fractions, which showed that notions such as half and double are fairly difficult to operate with. The notion double seemed to be very hard to grasp. This was confirmed by Holmqvist Olander and Nyberg (2014) whose study of 6–7-yearolds revealed that children usually failed to separate the concept from the base amount. One interpretation was that the operation 'to double' demands reasoning on quantities that are not present.

Numerical abilities and understanding are according to Piaget (1952) built on earlier acquired competencies, where the most basic competence is to abstract the perceived properties of objects (such as color, shape or size) and perceive objects as parts of a set that can be compared to other sets of same or different number of items. Piaget (1952) claimed that children develop concepts from the concrete towards the abstract. In order to learn to calculate and manipulate quantities, the child is thereby often supported by concrete referents (Mix, Huttenlocher, & Levine, 2002). Simultaneously, these concrete referents have to be considered as neutral counters, meaning that features of the objects do not interfere with the numerical relationship. Vygotsky (1987; 1998), on the other hand, described concept development as beginning with abstraction that is enriched and further developed as the child encounters and makes meaning of the concept in more complex settings. This latter way of seeing concept development broadens the view of children's meaning making as they use notions according to their current understanding, which in itself is an abstraction based on the child's logical reasoning.

Concepts such as 'double' are consequently troublesome to deal with for young children and for teachers to organize learning for. Considering these conjectures outlined above, there is a need for a study that describes children's perception of this kind of concept. A theoretically driven analysis will then contribute to our understanding of how children learn and develop their concept knowledge and how to facilitate deeper concept development.

THEORETICAL FRAMEWORK

The theoretical framework chosen for this study was the Variation theory of learning. This pedagogical theory (Marton & Booth, 1997; Marton, 2014) is based on empirical research of learning activities within multiple knowledge areas and school forms. The theory contributes to our understanding of learning, in that it focuses on the object of learning, what is made possible to learn and how this is orchestrated in powerful ways. Variation theory unfolds aspects of a learning object that the learner needs to discern in order to understand a phenomenon in certain ways. In teaching acts, these aspects are put into play within carefully developed patterns that enables the learner to explore the learning object in ways he or she has not previously been able to. Such patterns are contrast, generalisation and fusion (Marton, 2014).

According to the Variation theory, the learner cannot explore for example the number three if there are no numerical contrasts present, such as groups of two or four items. Generalization further means to implement an idea to other phenomena, meaning for example that there may be three of different kind, but still remaining its numerical meaning. In order for learning to occur, aspects of a learning object have to be differentiated through contrast and then brought together as a coherent whole, in other words fused together. Variation and simultaneous discernment are thereby necessary conditions to organise for in any teaching act.

This theoretical framework has been successful in both theoretical analyses and empirical studies in different kind of knowledge areas. Most research has been conducted in classroom settings, but there are also studies from early childhood education, where traditional lesson plans are exchanged with play and routine care situations (Björklund, 2013, 2014; Reis, 2011). Variation theory of learning was considered a relevant approach to studying young children's concept development when the interest is a specific new notion, such as 'double', due to the analytical framework and the pedagogical focus.

AIM, DESIGN AND METHODS

In focus for inquiry was the notion 'double' and those dimensions and aspects that are critical for preschool children to discern the nature and complexity of. A design study driven by the theoretical conjectures outlined above was carried out where the learning object was explored in authentic preschool activities.

Three preschool teachers participated in the empirical study. They planned and conducted one pedagogical activity each for the 4- and 5-yearolds they were working with (4 children participating with each teacher, a total of 12 participating children). The activities were designed in accordance with the theoretical framework, in other words enabling aspects of the notion 'double' to be explored through patterns of variation (Marton, 2014). In all learning sessions, the children were encouraged to talk and explore the notion by themselves and together with peers under the guidance of their teacher, in order to facilitate so called meta-cognitive dialogues (Pramling Samuelsson & Asplund Carlsson, 2008).

The acts and interaction between teacher and children are documented with video. The video documentations constitute data for analysis and interpretation, consisting of the three activities (12, 17 and 23 minutes of video data) conducted by the three teachers in different child groups. The various expressions of the children, both in verbal and non-verbal manners, are analysed as expressions of their understanding. The data is further analysed for revealing which aspects that are enabled to discern in the teaching acts, hence considered critical to elaborate in order to develop the children's understanding. The study is thereby two-folded: first to find out how children experience and understand the notion and second to discuss implications for the teaching act.

RESULTS

The purpose of this inquiry is to discern critical aspects of the notion 'double' as they are expressed by preschool children in designed but authentic learning situations in preschool. Further, the analysis will reveal teaching strategies that appear to be powerful for concept development.

Two dimensions, *operational* and *part-whole relationship*, are found to be emphasized in the learning act. This is unfolded by the directed attention that the designed activities and the interactions enable. Within these dimensions, aspects of the learning object are brought fore, giving a broader picture of how the children perceive the notion and why difficulties may appear. This will be discussed in the following text.

Operational dimension

The notion 'double' may be interpreted as an action or operation. The study gives evidence of children perceiving the notion as an operation, either absolute 'double means you add one' or relative 'equally many'. When analyzing the effects these conceptions may have on children using and communicating this notion, we can see where some challenges may appear.

Excerpt 1: You add one

Maria (teacher): From the beginning we had one (pointing to a stick on a sheet of paper). We were about to double that one, then you went to get one more (pushes two paper sheet with one stick on each towards one another). And together they make two. Pretty easy, don't you think?

Niklas: If there is three, it will turn to four (showing three fingers on one hand, then four fingers).

The children are listening to the teacher's explanation of how to 'double' a quantity. They perceive that it has to do with an operation on a quantity. However, as shown in Excerpt 1 above, there seems to lack some aspects of the notion, as the child expresses this operation in terms of 'adding one', and offering a logical solution to what happens when you double the quantity of three, which according to this line of reasoning would end up four. One critical aspect seem to be the number of the added unit, which should be perceived as a whole unit, equally large as the original one.

Excerpt 2: You add equally many

Isabella (teacher): Now it is Elias' turn. You will get three camels, Elias (puts three camels in front of Elias). And then you can pick out any ones you want from this pile (pushes the pile closer to Elias), 'til you have twice as many [in Swedish, the same word is used for 'twice' and 'double'].

Elias picks out a blue figure, then a green one, he hesitates and looks up on the teacher.

Isabella: How many did you have from the beginning? Do you know, when you only had the camels, do you know how many you had then?

Elias: Three.

Isabella: And if you are about to double that, then it means that you need to add equally many.

Elias: I have only taken two! (Elias leans forward and grabs one more figure, leans back again and smiles)

Isabella: You had only taken two. Now...

Elias: I have equally many.

Isabella: Do you have equally many there? Good, then it turned out twice as many.

Elias: They are six together.

Elias seems to be occupied by the thought of adding items to solve the task, in other words, he perceives the notion as an operation on quantities. He thus becomes unsure of how many to add and gets support from the teacher who directs attention to the original quantity. Further it becomes necessary to recognize the aspect of equally many that is to be added, which we can see that Elias has acquired. He also perceives the original and added sets as one whole, as he sums them up, regardless of their original belonging or visual features.

Some children comprehend the idea of 'equally many' and are even able to reason about this aspect on a generalized level, thereby accounting for the relativity in the meaning of the notion. The number of objects that are to be added are depending on the original quantity. This aspect, the relativity of the phenomenon, is with high probability a necessary aspect to discern in order to develop conceptual meaning in line with the conventional way of defining 'double'.

Excerpt 3: Relative sets

Isabella (teacher): Let me ask Brenda, what did you have in the beginning, how many did you have?
Brenda: I had four in the beginning, those (pointing along the row of camels).
Isabella: How did you know how many to take to get the double?
Brenda: I knew because I looked at them (pointing at the camels), that I'd take four.
Isabella: Yes, you had four and thought you'd take another four. Exactly. And how about you Elias, which ones did you have or how many did you have from the beginning? Can you remember?
Elias: Three.
Isabella: Three. And how many more did you take?
Elias: Three.
Isabella: And Tindra, how many did you have from the beginning?
Tindra: Two.
Isabella: And how many more did you take then?
Tindra: Those two (pointing at the outermost two figures).
Isabella: That means that when you have three items, like Elias has, you add equally many and then it is double. And Tindra had two and took equally many and Elias had three and took equally many.

The teacher in excerpt 3 is summarizing the tasks and solutions that the children have just completed. Attention is directed towards the added set and the teacher tries to generalize the relative aspect of the notion by pointing out examples of 'double'. This does however not seem to be sufficient, as one child responds by pointing at specific items as an answer to the question 'how many', indicating that the child reasons about what they just did, not on the operation and its impact on the quantity.

Dimension of part-whole relationship

When encountering the notion 'double', there is one aspect that becomes very important to account for: What set is the question aiming at? This is apparent when children and teacher are giving different suggestions for how to solve the doubling problems. In other words, are teachers and children focusing on the original quantity (that is to be doubled) or the result of the operation? This addresses the part-whole relationship, which is essential in all phenomena of numerical character.

Excerpt 4: Two or one and one?

Maria (teacher): Now there are equally many [one] on each paper. How many are there together?
Nora: Two (showing two fingers on one hand).
Amanda: One.
Maria: How many are there together? (pushes the paper sheets next to one another)
All children: Two.
Maria: Two.
Amanda: And if they are not together it is one, one (showing both index fingers).

This excerpt shows that children may focus on either the new whole constructed by the two parts together or on the set that is added to the original quantity (see also excerpt 5). The teacher in excerpt 4 and 5 tries to visualize the idea of 'double' as an addition, constructing a new whole. Amanda's focus seem though to remain on the parts that were added, not necessarily on the sum of the parts. This visual way of combining units by pushing paper sheets closer together seems to facilitate the children's attention to the result of an act, but at least one child's responses in the excerpts reveal that the perception of separated parts is dominating. It is thereby important to recognize possible different perspectives, as they may have impact on how the children continue to follow a line of reasoning.

Excerpt 5: Subsets by features

Maria (teacher) empties the paper sheets and pulls them apart. She puts a boat and a pig on one sheet, then adds a stick and another stick on the same paper sheet.

Niklas: Now it is two. Now they are three. Now they are four.
Maria: How many items do we have on the paper?
Niklas: Four.

Agnes and Nora point at each item and count out loud.

Maria: And if we are doubling four, how many items do we have to put on that paper then? (pointing at the empty paper sheet)
Nora: Four!

Amanda: Two and two.

This dialogue is interesting in several ways. The answer to the teacher's question may be 4 or 8; she asks how many they should add to get double, but she does not point out if the intention is to double the original quantity or to compose a unit consisting of equally many as the original one. Anyway, the children seem to direct their attention to the 'equally many' aspect and answer, according to that, 'four more'. One child, on the other hand, expresses her understanding of the task as a part-whole relationship, though focusing on the features of the objects, rather than the numerical relation, since she groups two sticks into one whole unit and the other two objects as another whole unit 'two and two'. The features of the objects that are used interferes with the numerical reasoning (see Piaget, 1952; Mix, et al. 2002) and emphasize grouping of items according to their physical appearances. This draws attention from the numerical relationship and limits the child's opportunities to get involved in the line of reasoning about the meaning of the notion.

The challenge of the notion 'double' is probably the shift from perceiving a set of items as a whole unit, to being a subset of a new whole that is doubled in number. It is probably easier to explore notions such as 'half' that includes manipulation of a whole visible quantity, dividing it into subsets. Double indicates on the other hand manipulation towards an unknown quantity that is difficult to imagine without concrete manipulatives. This result confirms the statements of preschoolers' perception of the notion as shown by Holmqvist Olander and Nyberg (2014).

Fusion of operational and part–whole relationship dimensions

The use of 'double' as a description of a quantity is in fact quite complex. It includes a relativity, which means that the quantity has to be related to another quantity and it is this relationship that the notion describes, not any one particular quantity. One can see the complexity that lies within this in the children's attempt to describe in words what double means, such as 'that you want a lot', 'very, very much' or suggesting to use a measuring tape to know how much double is. However, these are expressions of some insights to the notion that it is not an absolute number of added items and it would consequently be powerful to introduce and pay attention to the idea of 'equally many'.

The following excerpt is an example where both dimensions are present and used deliberately in the teaching act to bring out those aspects that are necessary for the children to discern simultaneously.

Excert 6: Making play-dough

The teacher Julia has prepared two identical recipes but instead of measuring units such as decilitre or table spoons she offers blocks. The task is to double the recipe. They have started by putting blocks for each amount of measuring unit on the original recipe.

Julia: How many were there on the flour? They [the blocks] can remain there.
Hugo: Two.
Julia: Now you should put twice as many here (pointing at the recipe sheet).

Hugo puts two blocks on the recipe.

Julia: How many are there (pointing at the original recipe with two blocks)
Hugo: Two.
Julia: And how many there (pointing at the recipe Hugo put two new blocks on).
Hugo: Two.
Joseinfe: Are they equally many?
Hugo: Yes.
Julia: Are there twice as many then?
Hugo: No.
Julia: They are equally many. How many do you think we should put here to get twice as many?
Emelie: Three there and two there.

The teacher follows Emelie's suggestion, which gives two blocks on the original recipe and three on the doubling-recipe.

Linus: No, I think you should put like this (puts one block above each of Hugo' blocks, then starts to do the same on the original recipe)
Julia: But you cannot put any here (pointing at the original recipe).
Linus: (throws away the block he had put on the original recipe) This is how I think.

Julia and the children continue comparing the original with the recipe with added blocks, altering taking away and adding blocks on the double-recipe to see the relationship between equally many and twice as many. They work in similar way with groups of one and two units through the whole recipe.

The excerpt above shows a way of teaching where bringing in contrast is used to support children's directed attention towards aspects that the teacher has considered necessary to explore. The starting point is the children's suggestions, which are explored and then challenged by other children's suggestions. Critical in this teaching act seems to be the simultaneous visualization of the original quantity and the double, combining the idea of adding 'equally many' with the comparison of quantities. The simultaneous discernment of equal and double is a powerful strategy that seems to enable the children to make sense of the troublesome notion, troubles that are likely due to the ambiguous nature of the notion in that it includes both a dimension of operation (adding) and a dimension of part-whole (comparing sets).

SUMMARY

The notion 'double' is complex in its own simplicity. 'Double' includes not only an operation on a quantity, but also a description of the relation between quantities. The notion is also in this sense relative in meaning, as the number of objects are not fixed rather depending on the observed number of objects in a particular set. These aspects are critical to discern, but as this study has shown, they challenge the children's perception and understanding of said aspects. In the teaching act, the teachers become aware of the nature of the notion, in that it includes both a description of a quantitative relationship between sets and an operation on a set. This insight is valuable, since it gives a reasonable explanation for children's actions and interpretations of the notion explored in an activity.

Important for the development of conceptual understanding is presumably a simultaneous discernment of the operational dimension of the notion 'double', which can be seen when children are talking about effects of doubling a quantity, and the relativity of the part-whole relationship of the notion, meaning that the number of added units of a set is related to the original set, not an absolute number. The teachers tend to use the expression 'equally many more' to emphasize this, indicating (perhaps intuitively) that there has to be a differentiated set to depart from.

Pedagogical implications of the findings point attention to the question: Are the children focusing 'double' as a comparison of two quantities or an additive operation that changes a quantity. Or, is it how the teacher offers these dimensions to be discerned that becomes critical? This is put to the test by one of the teachers who design an activity where the children are enabled to compare 'equally many' with 'original whole unit' and 'doubled whole unit'. The children participating in this activity seem to be given better opportunities to follow the line of reasoning and explore the nature of the notion in powerful ways. This is conjectured by the theoretical framework (Marton, 2014), but more empirical research will be needed to confirm this.

It is worth noticing that children who are expressing their solutions in verbal terms show variations in abstraction. Even though they solve a problem ending up with the right number of objects, some children are very attached to the concrete objects, expressed in terms of '*these two* were added' aiming at two specific objects that were picked out to double the original quantity, even though the numerical abstraction of the notion would suggest that the specific objects or features of the objects are irrelevant for the quantity that is described (Mix, et al., 2002; Piaget, 1952). Other children creates their own line of logical reasoning and implement ideas that may also be considered abstracted, such as 'you add one'. In Vygotsky's terms, children are developing their already existing concepts and implement their logical reasoning in new settings which will develop their concepts even further and towards feature-independent reasoning (Vygotsky, 1987; 1998).

Results from this study help us to better understand the very basic conceptual knowledge and abilities necessary for developing understanding of the notion double. The study also indicates that pedagogical strategies start from which dimensions and aspects that the learning object contains and further which of these the children are enabled to discern.

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How the role of the preschool teacher affects the communication of mathematics

Laurence Delacour

Malmö University, Malmö, Sweden, laurence.delacour@mah.se

The purpose of this article is to study and analyse how two teachers implement an outdoor situation for children aged 4–5 in a Swedish preschool. The analysis and interpretation of the data in this paper has been influenced by situation theory and by the use of the didactic contract as a key concept. The term didactic contract is seen as a metaphor; consequently, I use a broader definition of the didactic contract to illustrate how the role of the teacher affects the communication of mathematics with children. Most of the text in this paper is adapted from my thesis (Delacour, 2013) and articles (Delacour, 2014a, b).

Keywords: Kindergarten, teacher, didactic contract, mathematics.

INTRODUCTION

During recent years, preschool mathematics has been on the national agenda. In 2011, a revised preschool curriculum was introduced in Sweden (National Agency for Education, 2011), in which the goals for children's mathematical development were made clearer both in scope and in content. How children are to create, explore, and use mathematics is not specified in the curriculum, however, as it is a goal-oriented document and is not concerned with teaching methods that can lead to different ways of communicating mathematics (Delacour, 2013).

In this study, two preschool teachers implement a situation in which the children investigate properties and what belongs together. I describe the situation, and compare the roles of the preschool teachers to analyse how they affect the communication of mathematics with the children. The following research question has been formulated: how do preschool teachers' implicit expectations affect children's motivation to participate in mathematical activities?

THEORETICAL FRAMEWORK

First, I take up the didactic perspective of the study, highlighting the theory of didactic situations. The key concept here is the didactic contract.

Theory of didactic situations

According to Brousseau (2000), it is common to see teaching as a transfer of a special knowledge to children, and to see the didactic relation as a communication of information. Teachers try to plan their lessons so that the children get as much out of them as possible. Brousseau states that in order to succeed, it is necessary to consider the factors that may contribute to the teaching of mathematics. Brousseau (1998) developed the *theory of didactical situations* (TDS) to highlight the didactic relation between the teacher, the children, and the mathematical content. This relation is termed the *didactic triangle*. The theory of didactical situations offers a model of knowledge, teaching situations, and the roles of the teachers and children in the classroom. Knowledge cannot be transmitted merely through explanation but, according to Brousseau, the teacher orchestrates learning and teaching in order to increase the child's knowledge. Knowledge needs to be learned through different contexts, and teachers can organize specific situations that lead to children's learning of a particular piece of knowledge (Perrin-Glorian, 2008). Teachers often try to find activities to introduce a new item of mathematical knowledge to children. When a teacher focuses on the "why" aspect of teaching, he/she will see a problem or a situation, not as a simple reformulation of knowledge, but as an environment that offers answers to the children, according to certain rules. What information do children need from the "milieu" to make choices, and to use one specific piece of knowledge rather than another? (Brousseau, 2000).

The didactic contract

Brousseau (1988) introduced the concept of the didactic contract as part of his work in the theory of didactical situations, and defined it as the teacher's behaviour (as expected by the child) and the child's behaviour (as expected by the teacher). Brousseau studied how the didactic contract affects mathematical learning. Teachers usually develop different practices to give children the exact assistance they need, while children try to meet their teachers' requirements by interpreting the teachers' signals. However, teachers should not provide detailed instructions on how to solve mathematical problems, because this practice results in children not learning what the teacher expect them to learn (Delacour, 2014b). Brousseaus famous example shows how teachers can arrange a situation to help children gain a greater understanding of numbers. The teacher takes up five tanks (more or less depending on the child's number perception) and ask the children to pick up as many brushes as there are containers. The child should collect all the brushes at one time. The teacher does not tell children what to do but it is the children who must find a strategy to solve the problem. Some children will use their fingers, others draw on paper, others use blocks. How the teacher is planning the "milieu" is important. How far brushes are from the containers, for instance, will affect how the children act. Eventually, the children will understand that counting is a useful strategy.

During a situation, the preschool teacher repeats, clarifies, or asks questions that allow the didactic contract to progress in a direction that the teacher has in mind. Sometimes it is the children engaged in the contract who influence the change, when they make a discovery or get an understanding of something and share it with the group (Garcion-Vautour, 2002). My interpretation of the preschoolers' expectations is that these are connected to the procedures related to the situation (Delacour, 2014a). The didactic contract is often invisible until one breaks it. A child may break the contract when he or she accepts learning or playing the game, as Chevallard (1998) called it, but does not fulfill the teacher's expectations. Learning is only obtained under the breach of the didactic contract according to Brousseau (1998). When children sit around a table and play memory game for the first time for instance, they turn upside down on the pictures without really understanding what will happen. They turn on the pictures they can access from their seats. Children believe that the teacher expects them to remain in their seats as they usually do when they eat, for instance. When a child breaks the didactic contract, getting up from his chair to reach the pictures that are further away and manage to get a couple, an individual learning occurs which then becomes collective.

DATA PRODUCTION AND ANALYSIS METHOD

This article is based on data collected for my thesis (Delacour, 2013) between the spring of 2011 and the autumn of 2012. The preschool teachers in this study were videotaped while they implemented a mathematical situation outdoors. They have been previously interviewed about their interpretation of the curriculum which have influenced my analyses of the video.

The teachers in this study work in two different preschools in a group consisting of four-year-old and five-year-old children. The preschool teachers were videotaped while they implemented a mathematical situation outdoors. In this article, two of these mathematical situations (filmed material) were analysed in terms of a didactic contract. The selection of the first video is based on the richness of the material, and the selection of the second is due to its variation from the first.

The preschools included in this study are located in two small communities of the same municipality. The schools show no major differences in staff composition, group size, or children's socio-cultural and economic background.

To analyse and interpret the data, I made a reflexive interpretation of the preschool teachers' implementation of a situation, and used an abductive analytical method, which meant that the reflection moved between data and theoretical analysis (Alvesson & Sköldberg, 2010). In the preschool teachers' implementation of the situation, my focus was on highlighting the relation between the teachers, the children, and the mathematical content. The analysis shows that the teachers had similar expectations about how the children should communicate, but obtained different reactions from the children.

ANALYSIS

In this part, I describe how two preschool teachers introduce a mathematical situation for the children, and highlight how the communication between teacher, children, and learning object progressed. I describe in which situation the groups was engaged in before the situation I analyses in this paper because I believe it affect the outcome. As the objectives in the Swedish curriculum are formulated as goals that the preschool should strive towards and there are no specific goals for the children to achieve I will focus mostly on the participation of the children rather than on the knowledge they acquired.

The mathematics situation

The preschool teacher Lotta collects four children in the schoolyard. She tells the children that she has found a lot of garbage in the forest, and asks them what they think about this. The children propose to remove the garbage from the forest, and then want to sort the garbage items in different piles according to their characteristics. Now the children are sitting around a white cloth that Lotta has laid on the ground. Lotta's situation is based on a play in which children present four items individually. The video shows how Lotta makes a window out of sticks, and puts four items in it. She uses the garbage that the children have already sorted. She asks the children to point out one item that is not associated with the other three. Lotta introduces a new concept: *having something in common.* She lets the children tell each other which one of the four items they think does not have something in common with the other items. The children argue with each other about the items they have chosen, and about how they are thinking. Lotta follows the discussion and asks an occasional question. Once the children understand the rules of the game, they take turns to choose four items.

The preschool teacher Malin asks eight children to sit on two logs in the forest facing each other. Malin has laid a white cloth on the ground. The children had previously been working with patterns. During the work with patterns, Malin had sent the children to collect a leaf and a pinecone each. Then she asked each child to tell what he or she had collected, and to lay one leaf, one pinecone, one leaf, one pinecone, and so on, on the white cloth. For the next situation that I will analyse for this paper, Malin has collected materials from nature in a box. She picks up one item at a time, and asks the children to tell what it is, what color it is, which tree it comes from, etc. Then she asks the children to come, one at a time, and choose two things and tell why they think the two things belong together.

The rules of the didactic contract

There is a strong tradition of morning meetings in Swedish preschool, when the children and teacher sit in a circle and do different activities (Rubinstein Reich, 1993). The children have been instructed in the preschool's rules, and know what is expected of them in a meeting. When Lotta and Malin laid a white cloth on the ground, the children expected that what was about to happen would be different from the meetings that take place in the morning and indoors. The children knew that free play was over, and that a teaching situation was beginning. The white cloth also provided an indication to the children about what was expected, and where they were expected to turn their attention. These rules were implied, as the children have been through similar situations in the past and can now recognize the teacher's actions, body language, and voice.

When Lotta introduces the situation, she does not need to tell them that they are allowed to talk, move, and exchange thoughts and ideas. There is an implicit understanding here of what mathematics is, as the teacher has previously communicated in this way to the children in this group. Mathematics in this preschool is something you do in groups, in which people help each other to solve various problems in a playful way related to the theme currently being worked on. This group has been working with environmental issues throughout the semester. When the preschool teacher asks questions, makes small comments, and calls for the children's attention on what is being said, the children are seen, and their actions are approved. They are encouraged to take the initiative and to listen to each other. The didactic contract is characterized partly by explicit rules but primarily by implicit rules. How the situation will proceed and what the children should do is not pronounced verbally. Lotta does not tell the children to come up with proposals of their own, or which garbage pieces they have to choose. In Lotta's group, once the children are involved in the situation, they introduce mathematical concepts, while noting properties of the items such as flat and thick, and different sizes, shapes, and materials. There may be concepts that some but not all of the children know. The children can then learn from each other. Lotta encourages the children to be active and investigative. She would rather offer realistic problem situations in which many solutions are possible according to Lotta. She encourages them to find their own way, to explore rather than to find an answer. The children's responses and behaviour are valued and praised with words like "good" and "talented." Both individual children and the entire group receive praise and encouragement.

David (5 years) puts three metal cans and a plastic bag in the window that Lotta (teacher) has made from sticks.

Olle (5 years): "This one, this one, this one." (pointing to the plastic bag)
Klara (5 years): "These two should be removed," (showing a can and a plastic bag) "for they are like that." (showing two jars, turned upside down)
Olle: "This one, this one, this one, this one." (plastic bag)
Lotta: "How were you thinking, Olle?"
Olle: "For it is metal, this one is not!"
Klara: "I think like this," (putting her hands on the plastic bag and showing a jar) "this one will be removed because they are the same thing," (the other jars) "for they are so." (showing that the two jars are upside down)

Olle takes a jar that was removed by Klara and turns it.

Olle: "But if you turn it right side up, what is it then? It will be the same, right?"
Klara: "But look, they are golden, so it should not be. It is silver."
Lotta: "Mmm right, and you think this one. What do you think, Eric?"
Eric (5 years): "Plastic and not plastic, and this one will be removed." (pointing to the plastic bag)

During the situation, the children reflect and draw their own conclusions. The children are given the opportunity to experiment with the items and discuss with each other.

This situation can be interpreted as the preschool teacher creating a situation in which the children can communicate. The children's diverse knowledge of the materials offered creates a dynamic. When they express their thoughts verbally, the children reflect with the help of their fellow students and the preschool teacher's questions.

Malin introduces her situation to the children by giving instructions on what they have to do. During the first part of the situation, in which the children work with patterns, they have to follow the teacher's instructions step by step, and no individual proposals are expected of them. In the part I analyse, Malin starts by asking the children to tell the names of the items she picks up from the box. She also asks different questions about the qualities of the items. Again, no initiative or individual proposals are expected. The answers children give can only be right or wrong. Next, when Malin asks the children to come one by one, pick up two items in the box, and explain their choices, she is expecting them to take initiative and come up with individual proposals.

Malin (teacher) calls the first child, Richard (5 years).

Malin: "You can pick up two things."
Malin: "You can stand there and tell what you chose."
Richard: "A berry and a leaf."
Malin: "A berry and a leaf. Why do you think the berry and the leaf belong together?"
Richard: "Because both are red."
Malin: "Because both are red. Thank you very much, Richard."

Six of the eight children use color to motivate their choice. Malin does not value or praise the children's motivation. She thanks them for their participation.

According to Blomhøj (1995), the teacher usually develops different forms of work in order to give children precisely the assistance required, and the children try to meet the teacher's expectations by interpreting the teacher's signals. The teacher cannot give detailed instructions on how the children should solve mathematical problems, because under this circumstance, the children do not learn. Instead, Lotta uses different signals to confirm that the children are acting as expected: She uses linguistic signals, such as telling the children they are doing well, repeating what the children say, asking them if they hear what a child says, and telling them that it is exciting to see how many solutions they can find; or she uses body-language signals, such as putting her hand on a child's arm, nodding, and looking satisfied. Lotta often uses the pronoun "we," as in, "We have to think now!" to signal that they will reflect all together. She sits with the children around the white cloth.

On the other hand, Malin gives complete instructions to the children, and tells them exactly what they have to do. She stands a bit back from the children, and calls one child forward at a time. It seems difficult for the children to suddenly start thinking by themselves. My interpretation is that the children are confused about what the teacher is expecting of them, and try to give a right answer, as they were previously required to do. Most of the children use the same motivation: "Because they are the same color." Malin uses signals and asks further questions to invite the children to talk, but the questions she asks require a right answer: Is it the same kind of leaf? What kind of leaf is it? What color is it? These questions may prevent the children from daring to come up with their own proposals. Malin does not involve the other children by asking them if they can see a different similarity than the one proposed by the child standing in front of them. Malin only involves all the children when she asks *right and wrong answer* questions. Her way of meeting the children does not seem to encourage discussion.

In Lotta's group, when the children do something other than what the preschool teacher has shown from the beginning, the situation evolves as the children want it to, and the teacher follows the children's behaviour. Lotta initially asks the children to point out one item that is not associated with the other three. After playing for a while, the children begin to point out two items that belong together. Lotta responds that it is true that the two items belong together, and the play continues with the new rules for the rest of the situation. Here, the children's actions govern the rules. Lotta's response can be interpreted as not wanting to inhibit the children's creativity by telling them to point out only one item, because she has designed the situation to get the children to explore, and not to judge their competency.

The didactic contract becomes visible

According to Brousseau (1998) and Blomhöj (1995), the didactic contract is not visible until it is broken. There is a difference between not following the didactic contract and breaking the didactic contract: The one who breaks the contract is involved and interested but does not act as expected, whereas the one who does not follow the contract loses concentration or is uninterested. When Lotta communicates mathematics with the children, she wants the children to discover that there may be many different solutions to the same problem. However, she does not communicate this to the children before one of them breaks the contract:

Olle (5 years): "Yeah but what is this? This one should be removed; they are not the ones."

Lotta (teacher): "It is great thought. And you others have also found a solution. Great God, how good, oh, great. What do you say? Should we collect?"

Olle: "What did you think? Did you think it was a bit difficult? You thought it was easy, and it wasn't."

Lotta: (puts her hand on Olle) "There are several different solutions. It was great."

Olle: "Did someone guess the right solution?"

Lotta: "There are many right solutions. You had a great explanation of how you thought, Olle, and David, you too, you also thought very well. And Eric, you were thinking the same way, and Klara too. Really good."

Olle thinks that the other children may have had difficulty guessing what he was thinking.

Olle does not accept the rules. He does not accept that there can be many solutions to the same problem. He thinks that the others should try to guess what he was thinking. He breaks the didactic contract, and when Lotta explains that there are many right solutions, the contract becomes visible.

Lotta makes the didactic contract visible when she repeats that there is more than one solution.

CONCLUSION

Preschool teachers prepare for a situation with an intention in mind. In one of the types of situation above, the children control the content of the situation and introduce various concepts. It is in the children's interests for them to control the shape of the situation when mathematics is being communicated. According to Lange, Meaney, Riesbeck, and Wernberg (2014), the teacher should be able to listen very carefully in order to build her suggestions on the children's interest. The preschool teacher reinforces what the children say by repeating and by asking questions that challenge them to explain, listen to each other, and move on. The teacher's scaffolding by using different feedback strategies, such as putting her hand on a child's arm, nodding, and looking satisfied; or by asking open-ended questions rather than giving instructions, will support children's learning (Bäckman, 2014). The preschool teacher's ability to follow the children's interest is critical in the establishment of a didactic contract by which both students and teacher are able to accept the rules of the game. The two videos examined here (and my previous interview) show a difference between the attitudes of the teachers and the responses they get from the children. Many factors can affect children's motivation to participate in a mathematical situation and it is difficult to know based on two observations how the "milieu" affects the children's motivation. In Malin's group, although the children did what they had to do, they were not motivated. My interpretation is that the kind of questions Malin ask confused the children about her expectation but I am aware that other factors can affects the outcome of the situation. The fact that eight children have to wait for their turn for instance, be quiet, sit still, and listen to each other can affect the communication in Malin's group. *Instruction*, which refers to experiences aligned primarily with the teacher's goals, and *construction*, which refers to processes young children actively engage in to acquire concepts and skills, should be integrated, according to Jie-Qi Chen (2012). However, this study shows that young children who are initially engaged with instruction may have difficulty then engaging with construction; and that the teacher needs to be clear about her expectations in order to help the children to break the didactic contract formed when instruction is used.

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The use of virtual and concrete manipulatives in kindergarten school

Louiza Demetriou

University of Cyprus, Department of Education, Cyprus, dimloui@hotmail.com

This study aimed to explore kindergarten students' ability in solving symmetry tasks. Specifically, I wanted to investigate how the use of virtual and concrete manipulatives can improve kindergarten students' (4,5–5,5 years old) ability and representations in symmetry geometry tasks. For this purpose, I used two intervention programmes. In group A (n=25) students used virtual manipulatives and in group B (n=25) concrete manipulatives. Three types of symmetry tasks were used. Analyses of the data showed that both virtual and concrete manipulatives can help students move to a higher level of structural development. Additionally, students improved their ability to a greater extent with the use of virtual manipulatives.

Keywords: Kindergarten, symmetry, virtual manipulatives, concrete manipulatives.

INTRODUCTION

The increased emphasis that has been given to the geometry during the last decades, has changed the content of traditional Euclidean Geometry by introducing new types of geometry (Jones, 2002). According to Sinclair (2008) one of the impacts of Euclid's Elements in school geometry is that symmetry plays a peripheral role in the curriculum. Furthermore, additional to the mathematical motivations for increased emphasis on symmetry, the psychological research suggests that young children show a strong capacity for attending to and identifying symmetry. According to Vurpillot (1976) the ability to detect symmetry develops early. This strong capacity for attending to and identifying symmetry should be developed through their school geometry experiences (Sinclair & Kaur, 2011). Additionally, the use of technology is growing within schools and gives teachers the opportunity to differentiate their lessons and children's experiences. Computers are important tools for exploration and discovery of mathematical concepts (Burns & Hamm, 2011). Nevertheless, Burns and Hamm (2011) found little research that supports the use of virtual manipulatives over concrete manipulatives and according to Steen, Brooks and Lyon (2006) research on the impact of virtual manipulatives is limited.

As a result, the purpose of this exploratory study was to investigate kindergarten students' ability in symmetry tasks as well as the role of virtual and concrete manipulatives in solving the specific tasks. Consequently, the research question addressed in this research is:

How can the use of concrete manipulatives and virtual manipulatives contribute to the understanding of symmetry by kindergarten students?

LITERATURE REVIEW

Students' understanding of symmetry

Children have intuitive notions of symmetry from early years. As Sarama and Clements (2009) argued, symmetry was the easiest transformation regarding visualization to young students. Additionally, according to Seo and Ginsburg (2004), pre-school children spontaneously constructing symmetrical figures in informal play. Vertical bilateral symmetry remains easier for students to handle than horizontal symmetry (Genkins, 1975) which in turn is easier than diagonal symmetries (Palmer, 1985). While Boulter and Kirby (1994) argued that analytical strategies may lead students into successful answers, Tzekaki and Christodoulou (2000) found that symmetry can be accessed by five and six year old students in a holistic manner. At the same time they found that five and six year old students were able to distinguish symmetrical and non-symmetrical shapes, but on the other hand they were unable to draw symmetrical shapes taking into account their relative position and size.

Virtual and concrete manipulatives

Virtual manipulatives are interactive, web-based virtual representations of dynamic objects that present opportunities for constructing mathematical knowledge. Learners could gain insight into mathematics using visual representations of concepts and relations. Results of available research on virtual manipulatives, offer potentially beneficial uses of technology in mathematics classroom. For example, in a study with two treatments, in order to teach symmetry and congruence, Johnson – Gentile, Clements and Battista (1994) found that Logo-based version enhanced the construction of higher level conceptualizations of motion geometry. Additionally, more recently, Sinclair and Kaur (2011) found that kindergarten children were able to develop an understanding of symmetry that showed awareness of the properties of reflectional symmetry through the behaviour of dynamic images.

Even though studies found many perceived benefits on the use of virtual manipulatives, Burns and Hamm (2011) found little research that supports the use of virtual manipulatives over concrete manipulatives. Concrete manipulatives are objects used as tools that allow students to experiment and explore mathematical concepts. Burns and Hamm (2011) found that fourth graders, who were just beginning a unit on symmetry, realized larger pretest–posttest gains when concrete manipulatives were employed.

Theoretical perspectives

In order to analyze the geometric learning of students interacting with virtual and concrete manipulatives I adopted the levels of structural development as proposed by Mulligan, Prescott and Mitchelmore (2003). Young children, who have learned to look for mathematical similarities and differences within and between patterns, will tend to look for similarities and differences in new patterns and broaden their structural understanding accordingly. In contrast, students who tend not to notice salient features of structure are likely to focus on idiosyncratic, non-mathematical features in all situations. Children's representations may classify into the following four broad stages of structural development:

- Stage 1 – Pre-structural stage: Most examples show idiosyncratic features and representations lack any evidence of spatial structure.
- Stage 2 – Emergent (inventive-semiotic) stage: Representations show some relevant elements of the given structure, but their spatial structure is not represented.
- Stage 3 – Partial structural stage: Representations show most relevant aspects of spatial structure, but the representation is incomplete.
- Stage 4 – Stage of structural development: Representations correctly integrate spatial structural features.

METHODOLOGY

The sample of this study consisted of 50 kindergarten students (4,5–5,5 years old) in an urban middle SES district in Cyprus. There were 25 children per class, with a wide range of academic abilities. The method of convenience sampling was used. In group A (n=25) students used virtual manipulatives, while in group B (n=25) students used concrete manipulatives. Firstly, students complete a pre-test in order to determine their initial notions on symmetry. After this, two intervention programmes followed. Students in group A were taught with the use of virtual manipulatives in groups of five students. Nine virtual math applets were used, with three kinds of tasks: recognition of symmetrical and non-symmetrical shapes and images b) positioning axes of symmetry and c) completing shapes and images in order to be symmetrical. The activities used in group B were the same as in group A but in this case students used concrete manipulatives. Equal opportunities were offered to all students to get involved and touch real materials such as geoboards, symmetry mirrors, cardboards, pattern blocks, folding papers.

Each teaching lasted 80 minutes and there were a total of 4 lessons for each group. At the end of the interventions the same test, as the initial, was given to students in order to see if they had an improvement. The test consisted of three parts. Figure 1 presents the three parts of the pre/post-test. Firstly, children were asked to identify symmetrical images and put them in a circle. Secondly children had to put symmetry axes in nine images. Finally, in the third part, children were asked to complete 8 shapes and images in order to be symmetrical. Data were collected in a quiet part of students' school environment in the form of personal interviews and the interviews were audio-recorded.

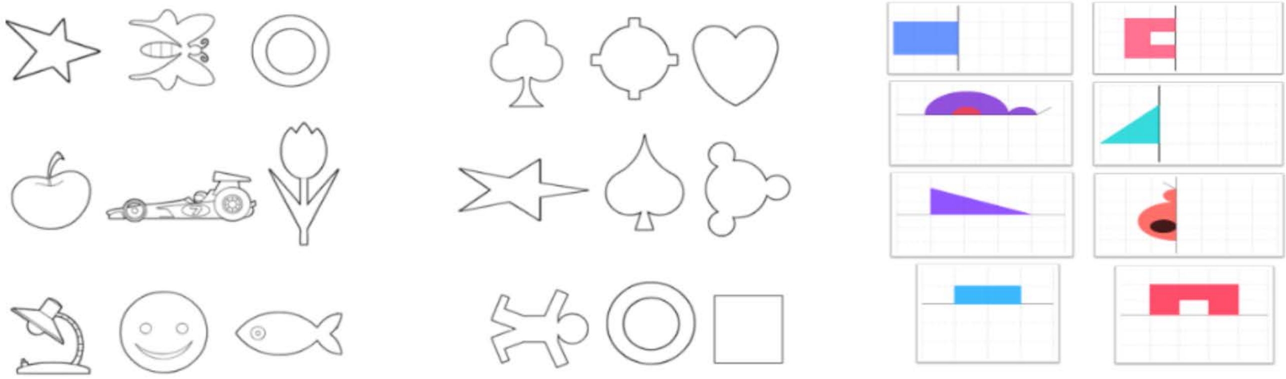

Figure 1: Pre/Post-test

Data analysis

Regarding the first part of the test, 0 was given to incorrect answers and 1 point to correct answers. At the second part of the test 0 was given to incorrect answers, 0,5 points to mainly correct answers and 1 point to correct answers. Coding for the third part of the test was based on the classification of representations according to the levels of structural development as proposed by Mulligan, Prescott and Mitchelmore (2003). The analysis of the data was qualitative and quantitative. Due to the limited size of the sample descriptive statistic (means and standard deviations) was used.

RESULTS

Means and standard deviations of pre-test are presented in Table 1. As we can see, the activity where children asked to complete symmetrical shapes have the lowest mean ($\bar{x}$=0.43, SD=0.23). In addition low performane appears in tasks where children asked to put axes of symmetry. Children seem to perform better in symmetry recognition tasks.

As mentioned above, the lowest performance appeared in tasks where children had to complete shapes and images in order to be symmetrical. The most difficult task in this category was task 5 where students asked to complete a symmetrical shape to create a triangle over a horizontal axis of symmetry.

Table 2 presents students' classification of representations in each task according to the level of the structural development. As we can see a large number of students seem to be at stage 1 (pre-structural stage) and stage 2 (emergent structure) since their representations do not present evidence of mathematical structure. Additionally none of students were at stage 4.

Post-test

As we can see in Table 3, after the teaching interventions mean of both groups was increased. Specifically in group A, the mean increased from 0.50 to 0.70 and in group B from 0.53 to 0.60. As it revealed students in group A improved their performance to a greater extent compared with students in group B.

	Number of students	Mean	Standard Deviation
Identify symmetrical images	50	0.64	0.17
Axes of symmetry	50	0.46	0.17
Completing symmetrical shapes	50	0.43	0.23

Table 1: Means and standard deviation of the three tasks in pre-test

	Task 1	Task 2	Task 3	Task 4	Task 5	Task 6	Task 7	Task 8
Stage 1	19	25	26	25	32	24	21	25
Stage 2	26	15	15	16	14	15	23	15
Stage 3	5	10	9	9	4	11	6	10
Stage 4	0	0	0	0	0	0	0	0

Table 2: Students' classification of representations in pre-test

		Number of Students	Mean	Standard Deviation
Group A	Pre-test	25	0.50	0.20
	Post-test	25	0.70	0.13
Group B	Pre-test	25	0.53	0.17
	Post-test	25	0.60	0.16

Table 3: Means and standard deviations for group A and group B

Noticeable seems to be the effect of teaching interventions in the third part of the test. Table 4 and 5 presents the number of students in each stage at pre and post -test.

Students in both groups moved from a lower stage of structural development in a higher level. In group A number of students which categorized at stage 1 and 2 at pre-test was between 19–23 while number of students which categorized at stage 3 and 4 was between 2–6. After the teaching intervention with the use of virtual manipulatives the number of students which categorized at stage 1 and 2 reduced (between 9–15) while the number of students which categorized at stage 3 and 4 increased to 10–16.

Additionally in group B, before the teaching intervention, number of students which categorized at stage 1 and 2 was between 20–23 while number of students categorized at stage 3 and 4 was between 2–5. After the teaching intervention the number of students categorized at stage 1 and 2 reduced to 13–20 while the number of students categorized at stage 3 and 4 increased to 5–12.

As we can see, students in group A increased their ability in completing symmetrical shapes to a greater extent than students in group B.

Qualitative data analysis

Through a qualitative analysis of the pre-test we can see some difficulties that students faced. Firstly in the first part of the test students ignored some aspects of the original shapes, which determined whether a shape was symmetrical or not, in at least one image. In the second part of the test, where students asked to put axes of symmetry, students were able to put vertical axes of symmetry but unable to put horizontal axes. Additionally only 2 of them were able to understand that shapes may have more than one axes of symmetry. In the third part, a large number of students at pre-test, transferred the initial shape in at least one task. Furthermore, they ignored some aspects of the original shapes. For example in task 2

	Tasks 1		Task 2		Task 3		Task 4		Task 5		Task 6		Task 7		Task 8	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Stage 1	11	3	12	3	14	9	13	5	17	5	14	2	11	4	12	5
Stage 2	12	9	8	9	6	5	8	10	6	4	5	9	12	10	7	6
Stage 3	2	9	5	11	5	8	4	8	2	10	6	9	2	8	6	9
Stage 4	0	4	0	2	0	3	0	2	0	6	0	5	0	3	0	5

Table 4: Students' classification of representations on each task: Group A

	Tasks 1		Task 2		Task 3		Task 4		Task 5		Task 6		Task 7		Task 8	
	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post	Pre	Post
Stage 1	8	4	13	7	12	8	12	7	15	10	10	5	10	4	13	7
Stage 2	14	13	7	6	9	8	8	10	8	10	10	8	11	14	8	10
Stage 3	3	7	5	10	4	7	5	7	2	3	5	9	4	6	4	6
Stage 4	0	1	0	2	0	2	0	1	0	2	0	3	0	1	0	2

Table 5: Students' classification of representations on each task: Group B

and 8 they ignored the empty square. At the same time in tasks 4 and 5 students ignored the diagonal line segment that they had to bring in order to complete the symmetrical triangle. At the same time, a large number of students seem to ignore the relative size and position of original shapes.

The qualitative analysis of the post-test showed that the biggest improvement was achieved by a girl in group A. During the intervention Maria faced many difficulties. Originally, she completely ignored the instructions and as she said, she was just trying to make images "to look good." In another attempt to complete the shapes, she ignored the initial position and the size of the shapes. However, at the end of those activities, she was able to complete symmetrical shapes with vertical and horizontal axes. In addition to this she was able to take into account the initial position and size of most shapes. In Figure 1 we can clearly see an example of Maria's improvement.

As we can see in Figure 2, at pre-test, Maria transferred all initial shapes. Most of her representations were categorized at the pre-structural stage. However, after the teaching intervention, Maria's representations improved significantly. In tasks 1, 2, 7 and 8 Maria's representations manage to reach the stage of partial structure. In tasks 3, 5 and 6, Maria's representations were classified at emergent structure stage.

The use of virtual manipulatives generated enthusiasm and motivated most of the students. For example John, a boy from group A, set more difficult tasks for himself. During an activity called "Creating Symmetrical Pizzas" John tried to complete a pizza (Figure 3) with a diagonal axis of symmetry even though he was not successful at it.

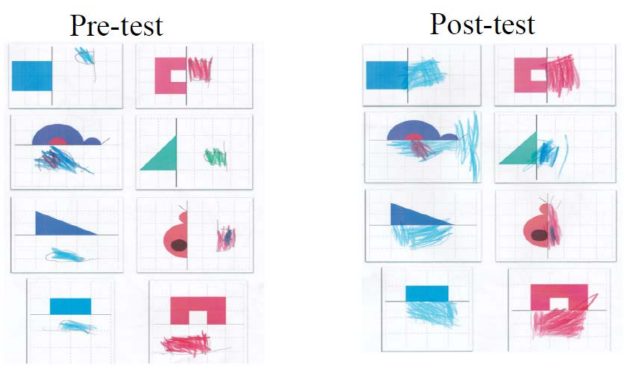

Figure 3: Pre/Post-test

John seemed to understand those activities from the very beginning. He was able to complete more complicated shapes with vertical and horizontal axes of symmetry without facing particular difficulties. Figure below presents two examples of John's activities.

John seemed to adopt an analytical strategy in order to complete those activities.

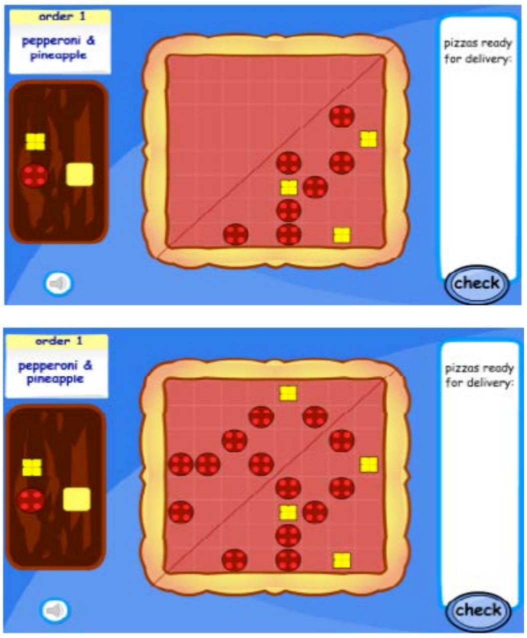

Figure 4: "Creating symmetrical pizzas"

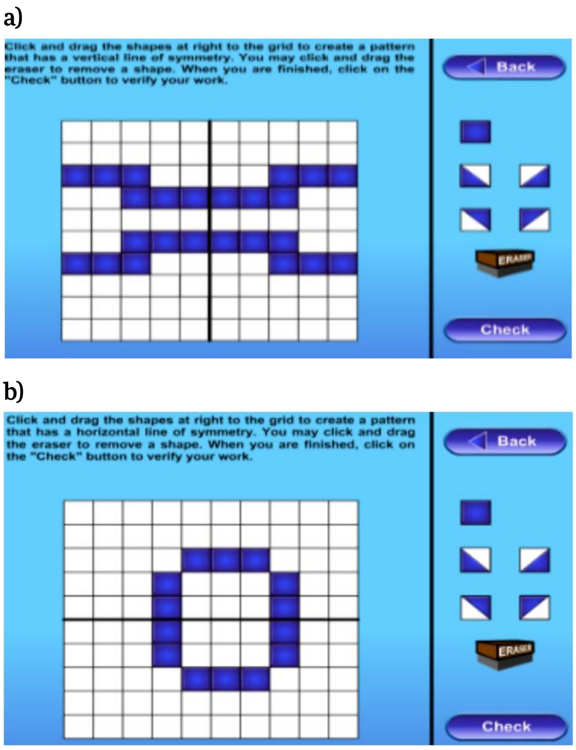

Figure 5: John's activities

Researcher: Can you tell me the way you think in order to complete this shape?

John: First, I count how many squares are painted on the one side.. at this line... 1, 2, 3 ... 4. So, on the other side I should put 4 squares. I have to put them next to the black line.

Researcher: Why you have to put those squares next to the black line (axis of symmetry)?

John: Well, since in the given example the first square is next to the black line, I strongly believe that I have to put another square to the other side right here (next to the black line). So when I have to fold this picture (he joined his hands and fold them) I am trying to imagine if the one side touch the other side. I remember the video we saw with the butterfly when the one side touches the other after we fold the picture.

CONCLUSIONS

As it revealed from the data analysis, kindergarten students had higher performance in identification of symmetrical images and lower performance in tasks where they had to put axes of symmetry and complete symmetrical shapes. These findings seem to be agreed with previous findings (Sarama & Clements, 2009; Tzekaki & Christodoulou, 2000). According to Tzekaki and Christodoulou (2000) kindergarten students, seem to access symmetry concept in a holistic manner. This is something that is confirmed in this study. During recognition activities, students ignored important details which determine if a shape is symmetrical or not. As a result, they made wrong recognitions. In the third part, most of the students transferred the original shapes and this is something that supports the holistic manner that students faced symmetry concepts. Additionally our findings seem to be in agreement with Genkins (1975) who argued that vertical bilateral symmetry remains easier for students to handle than horizontal symmetry since students at this study faced difficulties in putting horizontal axes of symmetry. As Tzekaki and Christodoulou (2000) argued children of 5 and 6 years old were unable to draw symmetrical shapes taking into account the relative position and size of shapes. This is something that is confirmed in our study since a large number of students from both groups were categorized at pre-structural stage and emergent structural stage according to Mulligan, Prescott and Mitchelmore's (2003) classification.

Even though both teachings improved students ability in symmetry, the results of our investigation, suggest that the use of virtual manipulatives can improve students' performance to a greater extent than the use of concrete manipulatives. As Yerushalmy (2005) argued computers may provide representations that are just as personally meaningful to students as physical objects. These results seems to be in agreement with Sinclair and Kaur's (2011) findings who found that kindergarten children were able to develop an understanding of symmetry that showed awareness of the properties of reflectional symmetry through the behaviour of dynamic images. At the end of the computer session, students in group A were able to recognize symmetrical and non-symmetrical shapes and images, to place axes of symmetry and to complete shapes and images. The number of students which classified at stage 1 and 2 reduced after the teaching intervention with the use of virtual manipulatives. Additionally, more students from group A moved to stage 3 and 4 according to Mulligan, Prescott and Mitchelmore's (2003) classification of representations. Char (1989) argued that a computer environment offered students greater control and flexibility comparing with the concrete materials. The flexibility of computer manipulatives allowed students to mirror mental "actions on objects" better than concrete manipulatives do and probably this is the reasons for the better students' performance in group A.

The results reported in this paper should, however, be interpreted with some caution. This study suffers from some limitations. First of all it is a study with small sample, so it is difficult to draw firm conclusions or to generalize the findings to other students or context. Also another limitation of our study is the limited time horizon. This study measured the impact of virtual manipulatives on a short term. As a result the real impact of virtual manipulatives may not become apparent during this short term. As a consequence we can see multiple directions for follow-up research. For example further research is needed to analyze the impact of virtual manipulatives with the use of high level statistical analysis, in large scale studies. These studies should also measure the impact of virtual manipulatives on a long term. Additionally another direction for future research consists of examining the applications of more recent technologies in kindergarten students such as touch screens.

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Distributed authority and opportunities for children's agency in mathematical activities in kindergarten

Ingvald Erfjord, Martin Carlsen and Per Sigurd Hundeland

University of Agder, Kristiansand, Norway, ingvald.erfjord@uia.no

The aim of this study is to investigate to what extent authority is distributed and opportunities for children's agency in mathematical activities. We are drawing on a sociocultural perspective on learning to analyse the distribution of authority among kindergarten teachers and children and exercise of agency in various mathematical activities in the kindergarten. Our analyses show that authority is distributed to some extent, through questioning and opening for children's contributions. Moreover, opportunities are given in which children exercise conceptual agency with respect to mathematical reasoning concerning geometrical shapes and measurement.

Keywords: Kindergarten, agency, authority, mathematics.

INTRODUCTION

Mathematics has not been explicitly mentioned in Norwegian curriculums for kindergartens until 2006 (Ministry of Education and Research, 2006). The 2006 curriculum emphasises that the children are supposed to develop their mathematical competence through play, experimentation and daily activities, and that the kindergarten teachers (KTs) are to facilitate and empower children's mathematical explorations. Today the KTs are supposed to be able to address mathematics as a subject in their daily enterprise. It is therefore of interest to study in what ways kindergarten teachers orchestrate mathematical activities in order for the children to engage in the mathematics.

This was the background for our investigation of how mathematical activities were organised, to what extent mathematics was the core of the activities, who was in charge of the mathematical content of the activities, to what extent the children had opportunities to appropriate mathematical concepts, and so on (see, e.g., Carlsen, 2013; Hundeland, Carlsen, & Erfjord, 2014). We have documented elsewhere (Erfjord, Carlsen, & Hundeland, 2012) evidence of a shift in the studied kindergartens regarding the didactic triangle characterising the kindergartens' enterprise; from a situation where a pedagogical activity (PA) was initiated with minimal mathematical input, to a situation where mathematics became the focal point in the activity – a so-called mathematical pedagogical activity (MPA). One important argument for this shift was found in the KTs' roles as mathematically competent adults and leaders of these activities.

In the study reported here, we thus had an initial hypothesis: In order for the mathematical activities to become effective with respect to mathematical outcome and experience on behalf of the children, the KTs have to distribute authority and facilitate children's exercising of agency in the adult-child interaction. Furthermore we had an initial hypothesis that if the KTs distributed authority amongst themselves and the children, the learning activities may gain flexibility and children initiatives, but may suffer from less mathematically goal-directed actions. This may lead to a situation where a planned MPA in effect becomes a PA. From these hypotheses we formulated the following research question:

> In what ways is authority distributed amongst kindergarten teachers and children in the observed mathematical activities, and in what ways are opportunities created for the children to exercise agency?

Agency is something that constricts or permits what one is free to do in a given situation. This question thus addresses an under-researched and important area in mathematics education, both the focus on distributed authority and exercising of agency as well as our focus on mathematical learning activities in the kindergarten setting.

Several studies during the last decade have documented the mathematical learning opportunities offered for children when they are participating in MPAs (see, e.g., Carlsen, 2013; Clements & Sarama, 2009). These studies document that learning opportunities are indeed offered for children in the kindergarten setting. Moreover, the children are nurtured in their ongoing process of appropriating mathematical tools when participating in MPAs. However, none of these have focused particularly on the interaction between KTs and children with respect to distribution of authority and agency.

AUTHORITY AND AGENCY

In order to address the research question, we adopt a sociocultural perspective on learning and development (Rogoff, 1990; Wertsch, 1998). Within this perspective interaction amongst adults and children as well as the use of tools are acknowledged as fundamental elements constituting the learning process. Tools such as mathematical language, paper and pencil, concrete materials, etc. are used as mediating artefacts in order for persons to communicate and interact in collaborative settings. In our study there are thus two main concepts that need a clarification, authority and agency. Cobb, Gresalfi, and Hodge (2009) define authority as "the degree to which students are given opportunities to be involved in decision making about the interpretation of tasks, the reasonableness of solution methods, and the legitimacy of solutions" (p. 44). Authority is thus a term used to address who is responsible when it comes to making mathematical contributions to an on-going problem-solving process. In our use of the term agency, we are following Lange (2009), that human agency is a term used about a child's "faculty to act deliberately according to one's own will and thus to make free choices" (p. 2588).

In order to align the view of authority, agency and a sociocultural perspective on learning, we are drawing on the work of Cobb and colleagues (2009) and their analysis of students' possibilities to exercise agency in particular classrooms. According to these authors, there are two aspects that are of importance when it comes to effectiveness in supporting mathematics learning; distribution of authority and opportunities for students to exercise agency. Furthermore, Cobb and colleagues (2009) argue that authority is closely connected with students' possibilities to exercise agency. According to Pickering (1995), agency concerns in what respect an individual's actions are emerging from free will or are influenced by others. Attributes of classic human agency are thus choice and discretion. Cobb and colleagues (2009) follow Pickering (1995) and differentiate between two forms of agency, i.e. conceptual agency and disciplinary agency. Conceptual agency is about "choosing methods and developing meanings and relations between concepts and principles" (Cobb et al., 2009, p. 45), that is attributes familiar within human agency. Pickering (1995) introduced the term disciplinary agency to emphasise that agency, when exercised within a conceptual practice like mathematics, is closely connected to the discipline in which it unfolds. In mathematics it is about employing established methods. According to Pickering, disciplinary agency is thus a specific disciplined pattern of human agency, e.g. routines with respect to symbol manipulations like $a(b + c) = ab + ac$. The notion of disciplinary agency then describes human passivity within a conceptual practice. Disciplinary agency thus "leads us through a series of manipulations within an established conceptual system" (p. 115).

Based on these notions of conceptual agency and disciplinary agency, Cobb and colleagues (2009) argue that in order for mathematical learning processes to be supported effectively, authority has to be distributed and students ought to be given opportunities to exercise conceptual agency. This argument is supported by Boaler and Greeno (2000) as well. Their study reveals that students need to be given possibilities to participate as creative agents in the mathematics classroom in order for mathematics teaching to be effective. Moreover, Boaler and Greeno argue that students need to be given possibilities to use their own language, to be given chances to think for themselves, to be given chances to make own interpretations and decisions.

Additionally, Cobb and colleagues (2009) argue that experience in exercising conceptual agency is needed in order for students to be able to reason about the usefulness of disciplinary tools in problem-solving processes. If authority is kept with the teacher, students are only offered possibilities to exercise disciplinary agency. Within a conceptual practice, like doing and learning mathematics in a kindergarten setting or in a mathematics classroom, there is therefore what Pickering (1995) refers to as a dance of agency. Conceptual agency and disciplinary agency may be intertwined and alternately take a lead. In our study the KT, the child, and the conceptual practice, i.e. a mathematical pedagogical activity (MPA), influence one another. We thus suggest talking about such activities as situations where authority may be distributed amongst the actors, giving each of them opportunities to act and exercise agency. When orchestrating a MPA, the KTs may have intentional learning goals of their mathematical activities. But since authority is distributed amongst adults and children, opportunities may be given for children to participate and possibly exercise conceptual agency.

The notions of authority, conceptual agency and disciplinary agency are used by Cobb and colleagues (2009) within a school context. As argued elsewhere (Erfjord et al., 2012), the Norwegian school context and kindergarten context differ to a large extent, both with respect to the nature of the two curriculums and with respect to organisational and structural nature. It is thus relevant to discuss how these notions may be employed in a Norwegian kindergarten context. Since kindergarten children are young and less autonomous than their counterparts in school, a KT may distribute authority in a different way than a teacher in school. The KT orchestrates mathematical pedagogical activities and she thus has most of the authority. However, she may ask the children several questions, suggest actions, asking for their opinions, their thoughts and ideas, asking for arguments and so on. In doing that some of the authority is distributed to the children, putting them in the position of being in charge for the mathematical interaction, how and in what direction this process may evolve. Nevertheless, as we will see from the analyses below, there is no doubt that the KTs have most authority in the cases we present.

To sum up, we use the term authority in line with Cobb and colleagues (2009), as something that may be given to others. In our case authority is given by the KTs to the children. When authority is given, opportunities are created in which children may exercise agency. However, as we will see, the children do not always take advantage of those opportunities to unfold their agency.

METHODS

Our study may be described as having a collective case study design (cf. Stake, 2000), because we study a number of cases in order to investigate the phenomenon of distributed authority and exercising of agency within MPAs in kindergartens. We were invited to kindergartens when the KTs argued to be orchestrating MPAs. These activities were videotaped. We analysed data from three kindergartens, and in this paper we present four illustrating excerpts to address the research question formulated for this study. We studied situations that from the outset were orchestrated by the KTs. Thus, the planned activities were led by the KTs and it was primarily the KTs that had the authority in the activities. Hence, the children's opportunities to exercise agency were limited. However, we seek to analyse to what extent authority was distributed, even though authority was mostly kept with the adults.

Our analytical process may be described in the following way: Firstly, we carefully analysed the conversations and actions between the KT and the children, paying attention to how the MPAs were orchestrated by the KTs. In doing that, we had the following questions in mind: Did the KTs invite the children to interpret the tasks? Did they accept and use the children's own wording of the situations, choices and actions to solve the tasks? The KTs actions indicate to what extent they distributed the authority in the MPAs to the children. Secondly, after having identified occasions where authority was given to the children, our next step was to analyse whether the children used their freedom to handle the situation based on their own free will. That is, whether they were able to exercise their agency in order to solve the tasks. Our attention was now particularly focused on the children's actions, whether they responded orally with own thoughts and actions or whether they only waited for the KT's suggestions. Thirdly, in order to study the distribution of authority, we paid particular attention to the teachers' conversations with the children. Did the KTs use suggestions or instructions? Did the KTs organise the MPA in a way that gave the children opportunities to act based on their own ideas? How did the KTs respond when the children presented their own ideas and acted in their own way? Fourthly, our analysis of the children's exercising of agency included paying attention to what the children did in the situations where authority was distributed. Did the children solve the problem using their own ideas? Did the children use their own language in their problem-solving process?

ANALYSIS AND RESULTS

The analyses in this study comprise excerpts from three kindergartens. The first excerpt is taken from Duckling kindergarten in which four children and one KT participate and where the mathematical theme is measuring. Excerpts 2, 3, and 4 are taken from Pinocchio kindergarten. Six children of age 4–5 years and three female adults were involved in planned activities with the children. The common theme for these activities was geometrical shapes: triangles, quadrilaterals and circles.

Excerpt 1

A group of 4–5 year-old children went on a forest trip together with their KT. When they arrived at the place where they used to stay, the KT gave four children the task to find themselves one tree each. Then she asked:

KT: Whose tree do you think is the thickest one?
Birger: Pedro's (5 sec.)
KT: Pedro's. Maybe you can try to hold your arms around the tree, like this (holds her arms as if she is folding a tree). Which one do you think it is? Do you think it is Pedro's tree? Can you imagine how we may find out whether it actually is Pedro's tree that is the thickest one?
Lisa: Because it looks so big
(The four children are holding their arms around their tree)
KT: Because you think it looks so big.
Birger: I fold the tree like this (he holds his hands around the tree)
KT: You fold it like that. Do you think there are other ways to find out whose tree is the thickest? How can we find out whether your tree is the thickest (Looks at Pedro)? Because that was what the others thought, that your tree is the thickest. How can we find out whether Pedro's tree is the thickest? Do you have any idea?

In this excerpt the KT seeks to distribute authority amongst her and the four children. She invites them into a discussion where they are supposed to reason about thickness of trees. The KT challenges the children to come up with their own ideas and opinions. The problem is how to compare thickness of trees. It is apparent from the excerpt that the children experience difficulties in responding to the KT's question. However, Lisa argues that it is Pedro's tree that is the thickest one because it looks so big.

The KT invites the children to contribute with their opinions. Authority is to some extent distributed between her and the children, when she asks the children to come up with their ideas with respect to deciding whether Pedro's tree is the thickest one. The children are supposed to compare thickness of the trees with their arms.

The children are given authority at various occasions. However, the children do not seem to exercise agency due to the difficulty level of the challenge given by the KT.

Excerpt 2

This dialogue took part in a sharing time with the children and the adults in a reserved small room. After a brief introduction where the KT gave the theme for the day, triangles, quadrilaterals and circles, she started off by giving the children a task:

KT: Can you see any shapes in this room? (The children move their heads and look around in the room)
Clara: I can see a "rounding" (The child points to one of the walls in the room)
KT: Where can you see a "rounding"?
Clara: There! (She points to a transparent plastic box, where a cross section of a rolled up poster has a circular shape)
KT: Yes. Great.

This kind of interaction pattern, with suggestions from the children and confirmation or clarification questions from the KT continued for a while. The authority is handled by the KT in the sense that she has organised the activity, and she takes decisions during the session concerning what are being discussed and how the topic is investigated. Also the agency is mainly with the KT. However, the KT's acceptance of the child's vocabulary, as for example the use of words as "rounding" for circle and "circle" for a cylinder, indicates that she transferred authority concerning way to express mathematics from her to the child. In opposite way she could have corrected the child's incorrect mathematics naming of the figures. The KT's invitation to the children looking for geometric shapes in the room, also gives some authority to the children in the meaning that they can select what to point at. As a consequence the object that is being discussed and interpreted as a triangle, quadrilateral or circle are chosen by the children.

Excerpt 3

This excerpt is from an activity that took part on the floor. The KT had prepared a closed cardboard box containing different shapes. On the floor three A4 sheets with big sketches of respectively one triangle, one quadrilateral and one circle were placed. The task for the children was to put their hand into the box and choose one of the shapes, describe the selected shape without looking at it, and finally place the shape on one of the sheets according to its shape. The KT asked one child at a time to do the task, and she interacted with questions to the child. One example of such interplay is the following:

KT:	It is your turn, Vicky
Vicky:	(Uses a couple of seconds to select one item from the box)
KT:	What have you found, Vicky? (She holds a ball in her hand within the box)
Vicky:	A bouncing ball (She shows the ball and bounces it at the floor)
KT:	What kind of shape has the bouncing ball?
Vicky:	A rounding
KT:	A rounding, yes. Can you place it on the "rounding" at the floor?

Compared with the activity analysed in Excerpt 2, this activity gives less possibility for alternatively solutions. In this sense the KT keeps her authority. However, the activity gives the children a possibility to describe the geometrical shape, firstly based on their tactile sense and secondly based on what the child see and is doing with the shape. Thus, the children have some freedom to describe this shape with their own wording. Similar to the activity above, the KT accepts the child's vocabulary when the child labels a spherical ball as being "a rounding". The KT also uses the child's label when she asks the child to place the "rounding" on the floor. The fact that the KT had selected a spherical shape, despite asking them to categorise the shapes as two dimensional (triangle, quadrilateral and circle), indicates that the KT probably intended the child to categorise the ball as a "rounding" or "circle".

Excerpt 4

This excerpt is taken from an outdoor activity which involved a walking trip up to a church close to the kindergarten. The children got one sheet of paper each with three columns headed with pictures of respectively a triangle, a quadrilateral and a circle. The children were asked to put a vertical mark in the correct column each time they discovered a thing with the particular shape. A road sign outside the kindergarten was one of the first thing noticed by the children. The road sign had a quadrilateral form with an inscribed triangle. The following talk took part between two of the children and one of the adults:

Clara:	See the road sign
KT:	Yes, What kind of shape is this?
Clara:	A quadrilateral.
KT:	Yes. Put a mark there (The KT points with her finger at the column "quadrilateral" on their sheets. The two children make a mark on their sheets in the correct column).
KT:	Can you see any more shapes on the road sign? (The children remain quiet for four seconds)
KT:	One – two – three (while the KT counts loudly, she points counting with fingers headed to the road sign). Can you see the shape within the road sign?
Ida:	A triangle.
KT:	Yes. Great. Put a mark under «triangle» on your sheets. (The two children put a mark on the correct column in their sheets)

In this activity we observed several similar examples to the one above where one or two children talked and got some help from one of the adults. In the activity, the children had been offered authority in deciding shapes and categorisation in the three types suggested by the KT in the sheet. The interaction pattern is similar to what we found in Excerpt 2, However, the outdoor area opens up for more options of things to categorise than the small room in Excerpt 2. The children's possibility to take the agency is bigger, and the KT has less possibility to prepare the activity in the open outside environment. However, several times during the outside activity, the KT asked the children as a group to look at particular things like a circular brick area outside the church. Such interventions gave more authority to the KT with an emphasis on particular things shaped as triangle, quadrilateral or circle.

DISCUSSION

We set out in this study to answer the following question: In what ways is authority distributed amongst kindergarten teachers and children in the observed mathematical activities, and in what ways are opportunities created for the children to exercise agency? The four excerpts show how the authority is distributed between the KTs and the participating children. The excerpts also exemplify how the children exercise their agency by participating in the mathematical learning opportunities that occur in MPAs.

In excerpt 1 the KT collaborates with the children in order to find out whose tree is the thickest one. The children come with various responses and the KT notices them and gives a response back, as paraphrasing and with a new question. The excerpt is characterised by the children's involvement in the activity, even though they are not so verbally activity. The KT lets the children imagine what they think, to estimate thickness of trees, as well as to come up with reasons for their ideas. This shows that the children are given mathematical authority in this case and that they exercise conceptual agency. They are given opportunities to participate and to give directions for the MPA. At the same time we observe that the KT is in charge of the activity. She controls the activity by actively asking questions to all children at the same time, but also individually.

However, our observations show a variety concerning the KTs' distribution of authority to children and to what degree children exercise agency during their participation in learning activities. In excerpt 2, 3, and 4, the distribution of authority is limited inside a planned frame for the activities. It concerns children's oral response to questions, selections of certain objects or participation in well-defined physical actions. The KT invites the children to take part in these actions, and it seems reasonable to argue that the children perceive this participation as voluntary. In that way we conclude that the children seem to exercise agency. However, it is also clear that the KT controls the activity, and does only allow responses that support her goals for the activities.

In the situations where authority is distributed to them, the children get opportunities to exercise conceptual agency. That is, the children contribute with ideas and arguments that may strengthen their opportunities to develop mathematical meanings and relations (cf. Boaler & Greeno, 2000; Cobb et al., 2009). As we saw, there is not so much disciplinary agency to be found in the analysed cases. That is, however, not surprising, given the Norwegian kindergarten's enterprise of being process oriented and situated within a social pedagogical tradition. Thus, disciplinary agency as the use of "established solution methods" (Cobb et al., 2009, p. 45) is not that prevalent in the kindergarten since these methods to some extent do not exist in any readymade matter.

We argue that authority, and hence, agency, ought to be distributed carefully within MPAs in order for the children to become supported in their mathematical learning process. Opportunities to exercise conceptual agency are needed, but the KTs need to orchestrate the MPA in such a way that the children are able to exercise agency. At the same time the KTs ought to control the activities in order to possibly reach mathematical learning goals. This is needed due to the limited mathematical experience of the children.

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Perceiving and creating in the mathematics classroom: A case-study in the early years

Francesca Ferrara

Università di Torino, Torino, Italy, francesca.ferrara@unito.it

This paper draws on recent research on the theorizing of embodiment in mathematics thinking and learning to adopt a non-dualist perspective that challenges the view that mathematical concepts cannot be perceived and created. This perspective brings out the intensive and immersive aspects of mathematical activity that feed the potential and the mobile in the classroom. Through the analysis of two 8-year old children, who reason on a figural pattern, I show how their ways of talking, moving and feeling allows them to mobilise and invent the mathematics they are learning. In so doing, I propose that perceiving is conceiving and creating is learning.

Keywords: *Creativity, embodiment, mathematics learning, materialism, perception, virtual.*

INTRODUCTION

In the last decade or so, lots of studies of embodied mathematics focused on the role and relevance of bodily activity in mathematics teaching and learning. Far from being an emergent generation of research, this corpus of work has started offering attempts to talk about and understand mathematical activity in *non-dualist* ways. Examples are studies of researchers like Nemirovsky, Radford, Roth and, more recently, de Freitas and Sinclair. No matter what their theoretical stances are, whether phenomenological, semiotic, philosophical, etc., they all embrace visions of a 'multimodal' or 'sensuous' mathematical cognition that recognise a special place to physical bodily aspects in the classroom, without assuming the existence of "two distinctive planes, one internal and one external" (Radford, 2013, p. 144). They all pursue a *participationist* view of teaching and learning that moves away from the constructivist tradition started with Piaget, which considers the mental schemas that students are expected to acquire. And beyond the mechanistic view still present in Lakoff and Nùñez's (2000) seminal embodied cognition theory, which fails to escape the mind/body split by inferring metaphorical mappings in the mind as sites of/for knowledge. Acquisitionist theories, based on structural concept formation, also entail levels of abstract thought confining mathematical concepts to abstractions, to de-personalised intangible and immaterial entities that *cannot* be perceived. Sfard's (2008) communicational theory shares the participationist commitment, by focussing on the ways in which students and teachers change their mathematical discourses, and conies the term 'commognition' to stress how thinking *is* communicating. But, sort of struck by all the 'fuss' about the body, and gesture in particular, it resists discussions about its participation in mathematics classroom discourse.

In this paper, I adopt a non-dualist participationist position that pursues a different vision of perception and creation in mathematics, according to which bodily activities are ways of thinking as well as of communicating (and feeling, I argue). In this way, I hope to contribute to the theorizing about the embodied nature of mathematics and of its learning.

THEORETICAL PERSPECTIVES

Sheets-Johnston challenges our ways of theorizing embodiment in thinking, shifting attention to "our being the animate organisms we are" (2009, p. 397) and "living moving bodies", which "*feel the dynamics*" of their everyday tactile-kinesthetic/affective experiences (2012, p. 393, emphasis in the original). There is no question then but that animate beings are not "*embedded* in the world" or "*embodied* in their actions, their emotions, their cognitions"—as Freedberg and Gallese (2007) would suggest on the basis of research on the mirroring system of the brain, which is in turn rooted in the realities of movement (Sheets-Johnston, 2009, p. 397, emphasis in the original). Animate beings, claims Sheets-Johnston, "are already living, and being already living, are already making sense of themselves and of the world in which they find themselves and of which they are a part." (p. 397). She is here suggesting a new image with respect to the 'I think therefore I am' *à la* Descartes. One that we might refer to as 'I move therefore I am', which entails that moving *is* thinking, so gesturing *is* thinking—as much as communicating *is* thinking (the other way around in the commognitive perspective).

Nemirovsky and colleagues (2013) stress a resonant point of view when they make a parallel between music and mathematics to investigate kinesthetic activity in museum exhibits where learners 'play mathematical instruments'. Fluent use of the instrument involves an interpenetration of perceptual and motor aspects of playing it. Kinesthetic activity is relevant here in two ways: "motor activity is involuntarily enacted as part of perceiving", and "partial motor and perceptual components have the power to elicit the activity as a whole over time" (Nemirovsky et al., 2013, p. 380). Working from a non-dualist approach to tool use, the authors again trouble the dichotomies between thinking and acting, *perceiving and conceiving*. In my own study on multimodality in mathematical activity (Ferrara, 2014), I examine kinesthetic activity in the context of motion detector use and I propose to see mathematical thinking in terms of floating intricate intensive entanglements of ways of perceiving, moving and imagining. Here, I follow Burbules (2006) in claiming that experiences engage our imagination "when we can interpolate or extrapolate new details and add to the experience through our own contributions", so that we may be "making guesses about things that are not immediately present to us" or "anticipating what will happen next in some sequence or development." (p. 41). Imagination, depending on students' active response and engagement in the activities, triggers feelings of *immersion*, senses of "as if", which make the experiences virtual experiences for the students. A key dimension of this quality of immersion that, for Burbules, "makes the virtual seem or feel "real" to us" at that moment, are "our posture, body tension, and startle responses" or "our relaxation, rhythmic movement, and kinesthetic sensations" (p. 42)—and he takes here any truly educational experience as being immersive, or virtual, as much as watching a film, hearing a story and listening to music. As I have stressed, this sense of immersion reconfigures mathematics learning as an alive and *genuinely creative* adventure.

A way of re-framing creativity in the mathematics classroom is offered by the new inclusive materialist approach of de Freitas and Sinclair (2013, 2014; see also Sinclair et al., 2013). Creativity is not studied here as

> a property or competence of a learner, as suggested by approaches that seek to measure the flexibility or fluency of a child's thinking. It does not exist independently of its exercise. In other words, it is not that individuals are creative or not creative, but rather that creativity flows across the learning assemblage in a somewhat impersonal way. (de Freitas & Sinclair, 2014, p. 86)

This conception of creativity is not bound to a "personal creativity as a characteristic that can be developed in schoolchildren" (Lev & Leikin, 2013, p. 1204). Indeed, it shifts attention away from the doer, and from the idea of giftedness and high ability in mathematics, to focus on the doing, without lapsing into reading actions as reflections of mental states. It "treats creativity as *an action taken* that emerges in context, without being exhausted by it" (Sinclair et al., 2013, p. 241, emphasis in the original) and *bringing forth the new*.

Thus the inclusive materialism centres on the process of creation of something new, looking at students' actions, with the other material actions in the classroom, as an expression of creativity. Interestingly, it relies upon a re-configuration of the contours of the learners' body, which enables to talk not only about the body *in* but also *of* mathematics. In fact, inspired by the French philosopher Gilles Châtelet and his notion of the virtual, de Freitas and Sinclair explore "how mathematics partakes of the material world" (2014, p. 1) and how this occurs "in operative, agential ways", troubling the tacit belief that "the mathematical concepts (multiplication, cube, zero) are taken for granted, while students collaboratively move towards them." (p. 40). Within a tradition that assumes that abstract thought and materiality are entwined, their philosophical position looks for "how bodies are assembled through activity" (p. 15). For de Freitas and Sinclair, "the body is an assemblage of human and non-human components, always in a process of becoming that belies any centralizing control." (p. 25). Their perspective "moves away from a theory of power as a totalizing, external force and follows power as it flows through sensation and affect, across the surfaces of bodies as they emerge in relation to these flows." (p. 41). In so doing, they open room for post-humanist discourses of subjectivity and agency, for which students are always in a process of *becoming* mathematical subjects through agential relations with the diverse dynamic materialities in the classroom, including the mathematical concepts.

Thus mathematical creativity (or inventiveness) is materially conceived of in terms of the process that "expresses and captures the temporal and dynamic moment when the new or the original comes into (*in-venire*) the world at hand", for example in terms of "the dance between the gesturing and drawing hand" (de Freitas & Sinclair, p. 88, emphasis in the original). Other than bringing forth what was not present before—a feature stressed by Châtelet (1993/2000) in his analysis of inventive moments in the history of mathematics, a creative act has also other characteristics (Sinclair et al., 2013). It is unusual: it does not align with current perceptual habits or practices that are taken as norms and the extent to which it is recognized as creative, depends on the context where it occurs. It is unexpected or unscripted: it is not directly of formally determined by the intentions of the individuals involved. It changes the way language and other signs are used and alters the meanings that circulate in a situation, so that its meaning cannot be exhausted by existing meanings. These qualities "point to the centrality of the body and its movement (actions)—rather than internal mental disposition—in creative acts." (p. 242). De Freitas and Sinclair (2014) discuss how "gesturing and diagramming can together bring about new ways of thinking, moving and imagining, and thereby give rise to inventive processes." (pp. 109–110).

In a different work, de Freitas (2014) claims that Châtelet shows us "how we might study a particular practice for how its lines of flight flourish and act generatively in unfolding new *intensive* dimensions." (p. 290, emphasis in the original). She draws on contemporary theories of perception to focus on the way the student's body, together with its potentiality, can be reconfigured, and the contours of the sensible and the intelligible recoded. She argues that we need to unpack the provisional nature of perception for emphasizing its *virtuality*, or virtual movements: "We never just register visual information from that which is in front of our eyes:", says de Freitas,

> we see potentiality, relationality, mobility, occurrence. [...] In other words, we live on "speculative investments," as though we were surfing "the front edge of a wave-crest" [...]. Perceiving an object entails a prehending of our body's potentiality to walk around the object, to reach out and touch the object, to see the object, to weigh it, to smell it. (p. 298)

In this paper, I want to use these arguments to focus on the *immersive* and *animated* ways in which children talk, move and feel in mathematical activities. These become means to look at how the children *perceive* and *create* in the mathematics classroom, giving rise to inventive moments that mobilise their doing mathematics as much as the mathematics they are doing. My non-representational monist view—aligning with those that challenge the body/concept and matter/thought dichotomy—questions the binary divide between perceiving and conceiving, creating and learning.

METHODOLOGY

In the analysis, I have chosen to focus on a particular pair of 8-year old children, Lara and Filippo, who deal with pattern activities to develop early algebraic thinking. The whole elementary class participates, during regular mathematics lessons, in a 5-year longitudinal study concerning the introduction of the concept of function and the use of variables. Thus the children had already worked on patterns in the previous grades. The selected data refers to the beginning of grade 3, when they are divided into pairs

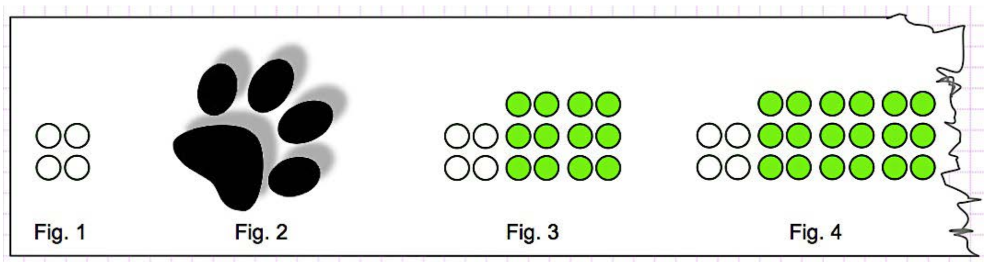

Figure 1: The figural sequence of the activity "Do you remember of Tobia?"

to face an activity called "Do you remember Tobia?", which re-presents a figural sequence to them. Tobia is the name of the imaginary dog that left the track on the ripped paper, having covered up term 2 of the sequence (Figure 1).

In grade 2, the children were asked to extend this sequence up to its sixth term and to find the second term. In grade 3, the purpose of the activity is to shift attention to the relationship between any term of the sequence and the term number—its position in the sequence. To this aim, the pairs are given the task of noticing any regularity in the pattern, and of explaining it.

Lara and Filippo sit on a desk, one in front of the other and turned around to face the other in the discussion. A master degree student, who participates in the lessons as an observer, films them by using a mobile camera. Data of their discourses come from the video-clips, while additional material is given by their written productions. A university researcher (the author) and the teacher are also present in the classroom. The researcher (labelled by R in the data) consistently takes on the role of the guest teacher, teaching the lessons in collaboration with the regular classroom teacher (T below), who has the role of an active observer. In the activities, phases of individual work alternate with pair work and collective discussions.

RESULTS AND ANALYSIS

The results are divided into three sections. The first two focus on the discovery of a remote term within the figural sequence. The last section centres on the way in which Filippo solves the new task about the position of a given total number of circles.

Perceiving numerosity in the particular: What about position twenty-five?

The activity began with Filippo and Lara examining the pattern of Figure 1, with the instruction of looking for regularities. Filippo started by focussing on the bottom row of term 3 and seeing that the number of circles there relates to the term number. In particular, he has found that if one counts the circles on the bottom row and divides this number by 2, one gets the position of the term in the sequence. This introduces in the discourse the new operation of division by 2, which Filippo and Lara share with the researcher as soon as she comes to the pair.

Filippo: For example, you look at this [*Points to term 3 of Figure 1 with the pen in his right hand*], you do, you count the circles below [*Runs the bottom row*], one, two, three, four, five, six [*Counts the circles*], then you do six divided by two [*Looks up at the researcher*] that gives, oh

Lara: Three [*Looks up at the researcher*]

Filippo: And this is, is [*Moves the pen twice around term 3*], oh, the position

Lara: Or you also take this [*Overlaps Filippo's voice. Points to term 4*], you can also take this [*Points to term 4 again*]

R: Oh, and does it also work here? [*Indicates term 4*]

Filippo: This one is equal. One, two, three, four, five, six, seven, eight [*Counts with the pen the circles on the bottom row of term 4. Lara joins him in counting*]. You do eight [*Looks up*] divided by two

Lara: It gives four [*Looks at the researcher*]

Filippo: And this one [*Moves the pen twice around term 4*] is in position four. This one [*Shifts the pen to term 1*], one, two, two [*Looks up at the researcher*] divided by two gives one [*Points with the pen to expression "Figure 1" below term 1. Looks up at the researcher. Smiles*]

R: Very Good! Oh, now I tell you: What about position twenty-five?

Lara: Twenty-five divided by two! [*Laughs. Looks at Filippo*]

Filippo: [*Looks at Lara surprised, looks up at the researcher, looks back at the sequence. Keeps thinking in silence for some seconds, suddenly mimes with his left hand a small rotation towards his torso. Looks up*] you do twenty-five, oh, times two

R: What do you use twenty-five times two for? Explain me.

This short passage shows that Filippo and Lara perceive the first structural relations in the figural sequence, between the numerosity of circles and the number of a given term like 3 or 4 – the children move the discourse beyond the recursive "adding six circles" (that emerged in grade 2), towards looking at the sequence in a functional way, by talking (for example using "position") and gesturing (around the term, to its bottom row). What they perceive is of a very different nature, since it introduces reasoning on the pattern in terms of whole numbers (numbers of circles), even if still per rows, but no longer strictly related just to the spatial structure. Dividing by 2 also comes to the fore as a means to manage the relations between numbers. A certain satisfaction can be grasped in Filippo's explanation about term 1.

Hoping to encourage the children to perceive more than one relation in the sequence, the researcher then introduces a new task for a new (remote) term: the "position twenty-five" task. Lara hurriedly says "Twenty-five divided by two", but Filippo keeps silent, marking his struggle (also expressed by his repeatedly changing gaze). The mathematics of the figural pattern is mobilised. Filippo responds thinking about multiplying by 2, but he gets confused about the kind of numbers in use when invited to explain. Thus he inquires "but do you say position twenty-five or the number twenty-five?". The situation breaks through with the answer "No, the position twenty-five", which prompts Filippo to insist on "you do twenty-five times two". When the researcher then asks "And what do I find?", discourse moves on.

Filippo: You find the number, the number of, of, oh, to put below [*Runs the bottom row of term 4 many times with the pen in his right hand*]. And you put them, at the beginning [*Moves left hand to term 4*] you put two of them and then two [*Indicates the bottom row with the pen, looks up*], then you put two of them and you go by two [*Jumps along the middle row*], and then you don't put any here [*Points to the empty space on the top row with left index finger*] and you always go by two [*Jumps along the top row with the pen*]

Filippo reasons in terms of numbers of circles on the 4th term of the sequence to think about the shape in a remote term like 25. In perceiving the row disposition and composition in term 4, gesturing on its rows with both hands ("always" referring to groups of "two" circles), he conceives of the structure in term 25 in terms of the spatial similarity that is granted by the algebraic structure of the pattern. Through gestures, the circles begin to be mobilised together with the numbers in the sequence.

Creating the new term: You skip the first two and you go

The teacher gets close and Filippo, excited, wants to tell her about term 25.

Filippo: In position twenty-five, you do, to discover that one [*Moves the pen many times around term 4 of the sequence*], this one [*Runs the bottom row of the term*], you do twenty-five times two, twenty-five times two, oh, then you put [*Points to term 4*], oh, wait, twenty-five times two, and, this number, oh, wait, I do no longer remember [*Smiles*]. You do twenty-five times two [*Pauses. Looks around, beats his head*], oh, what did I say? [*Looks at Lara, looks at the sequence*]

Lara: Twenty-five times two [*Pauses, looks at Filippo who points to term 4*], one, two, three, four, five, six, seven, eight [*Counts the circles on the bottom row of term 4*], you do eight divided

Filippo: Ah! You do twenty-five times two [*Looks at the teacher*], and you put the result here below [*Mimes with the pen the arranging of the first circles on the bottom row of term 4. Looks up at the teacher. Figure 2a*], you put the circles, all, of the result [*Continues the gesture outside of the paper. Figure 2b*]. When you arrive at the result with the circles, there [*Shifts the pen to a position towards the desk side. Figure 2c*], you stop and you go above [*Shifts the pen to indicating the middle row of term 4, keeps reference to it with left index finger*] and you always put twenty-five, no, always the result [*Mimes the arranging of circles on the middle row, moving to the desk side. Figure 2d*]. Then here [*Points with index finger and pen to the initial empty space on the top row of term 4*], you skip this, you skip the first two and you go [*Keeps the finger as a reference, mimes the arranging of the circles on the row with the pen. Figure 2e*], oh, and you put, you do [*Looks at the teacher*], oh, you take two away from the result and put those ones! [*Repeats the miming of the top row. Looks at the teacher, smiles*]

Figure 2: (a-c) miming the bottom row of term 25; (d, e) miming the other two rows

Filippo thinks of the pattern in terms of numbers as results of operations (no longer just numbers of circles). He uses his hands and fingers to enact the exact shape of the figure in the 25th position, with the reference of the 4th term. But he also gestures outside of the paper to talk about/imagine a term that would appear after the sequence––and, in any case, would be made of longer rows. Filippo mobilises the static diagram on paper through his gestures as a form of diagramming. He actualizes the virtual movement of the sequence, rather than only realising its logical possibility through numbers (that exists in the given only). Thus the circles are mobilised and the mathematics of the pattern is invented in the moving assemblage of the child, the pattern and the mathematics. This allows the creation of new mathematics as the new term is figuratively brought forth through Filippo's gestures.

Without distinguishing between perceiving and conceiving, as I am encouraged to do in materialist terms, I might say that, for Filippo, term 4 *is* term 25 here. In a similar way, without distinguishing between creating and learning, we might say that Filippo is starting to reason in algebraic terms and the children's discourse is moving to a more functional one compared to the previously discussed.

Perceiving and creating the unknown: What position is?

Filippo explains to the teacher the term in position Pippo that was introduced by the researcher as a challenge after term 25. The children faced the task with some tension with respect to using expressions like "the result of Pippo times two". During this interaction, the teacher poses the new task of having the total number of circles in a term: "I have a position, which I don't know, which has twenty-two circles, how can I discover what position is?", specifying that she means "as a whole".

Filippo: Twenty-two, oh, you take away four from twenty-two [*Mimes the operation moving his hands together in front of his torso from right to left. Figure 3a*] and you get eighteen, and it's the first group [*Mimes a grouping. Figure 3b*] of four [circles], eighteen. Then, you take away six from eighteen [*Mimes a block, with a vertical movement of his right hand. Looks at the teacher. Figure 3c*] and you get twelve [*Shifts right hand on the right, moves closer*] that, so, are four and a row of six [*Mimes the grouping again and a new block. Looks at the teacher. Figure 3d*]. Then [*Moves on the right again with both hands open to mime the remaining circles, bends his head*], take away six from twelve and it gives six, so they are four [*Marks a grouping with left hand*] and two rows of six [*Mimes a block with right hand*]. Then [*Pauses*], take away six from six, so it gives three rows of six [*Keeps still left hand, mimes the three blocks with right hand. Looks at the teacher. Figure 3e*], plus four [*Marks the grouping again with left hand. Look at the teacher. Smiles. Figure 3e*], and then, oh, there are no more [*Turns more towards the teacher*], you do, they are four plus three rows of six [*Mimes the grouping with left hand, the blocks with right hand*]

R: Ok, so what position is?

Figure 3: (a-c) miming the first operation; (d, e) miming the resulting blocks

Filippo: So, it's, one, two... three, four... five, six... four! [*Mimes the counting of the circles on the bottom row of the term, moving from left closer to the camera up to disappear almost entirely from its view. Smiles*]

This short episode shows how Filippo's ways of talking and moving become ways of diagramming the specific term in the sequence, and of imagining its position. It is as if Filippo had the term (big) in front of his eyes, could touch it and see it from various points of view, moving around it. His gestures, gazes, postures, smiles, all creatively tend to the term, with his body clearly marking the position and making space for the teacher. Like in the case of term 25, the invention of new mathematics is allowed as the given circles are mobilised, the imaginary blocks are animated and their position is created in the evolving body-material assemblage, through gestures of repeatedly subtracting six and the body shift closer and closer to the camera.

CONCLUSIONS

My goal in this paper has been to examine the children's ways of talking, moving and feeling as immersive and animated ways of perceiving and creating in mathematics, *the* mathematics. In a monist materialist perspective, the episodes have shown that the children's gestures play relevant roles as for the claim that their perceiving *is* conceiving and their creating *is* learning. Filippo and Lara begin to learn to think algebraically in the first and second episodes, when they create the shape of term 25 using the reference of term 4 of the sequence. Filippo sometimes gestures on the 4^{th} term as if the 4^{th} term was the 25^{th}, in other times he gestures beyond it for reaching the new imaginary term. Without these gestures, the diagram would stay static and the children would only use numbers to realise the possible given in the figure. In the third episode, Filippo is learning that a given total number of circles (not only the number of circles on the bottom row) can have a position in the figural pattern, when he creates the position for twenty-two circles without any reference to specific terms on paper. In the episodes, the child's hand gestures are never iconic representations of one term in the pattern. Rather, they are conceptions and creations that allow reducing distance between the physical and the mathematical. They transform what is static/possible towards the mobile/virtual––algebraic thinking here. At the heart of this virtuality, the figural pattern is a part of Filippo's body––and feelings––and *is* the mathematics in that mathematics is implicated by the pattern. As a consequence, the duality of the child-pattern and pattern-mathematics can be rethought of as *one*, the child-pattern-mathematics, which is a learning assemblage in the classroom.

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The status of early childhood mathematics education research in the last decade

Zişan Güner Alpaslan[1] and Feyza Tantekin Erden[2]

1 Bülent Ecevit University, Faculty of Education, Zonguldak, Turkey, zguner@metu.edu.tr
2 Middle East Technical University, Faculty of Education, Ankara, Turkey

This study explores early childhood mathematics education research from 2000 to 2013 in major journals of mathematics education and prior CERME-Early Years Mathematics papers. The comparison between the studies published in the journals and in CERME uncovers some interesting results: i) the majority of studies are from English-speaking countries in the journals whereas the majority of CERME papers are from non-English speaking countries, and ii) the focus of research topics in the journals seems to be significantly different than that in the CERME papers. Moreover, there are more empirical studies than any other types of research and studies mainly focusing on children appear to be more popular. Accordingly, there should be more diversified papers (e.g. regarding research topic) to deepen and enrich this field.

Keywords: Early childhood mathematics education, research trend, systematic review.

INTRODUCTION

Early childhood mathematics education (ECME) which refers to the education of the ages of 3- to 6-year-old children (NAEYC, 2010) is significant for young children (Clements, Baroody, & Sarama, 2013). The study of Clements and colleagues (2013) briefly positions its reasons in two main frames by stressing the benefits of early childhood mathematics (ECM) for minimizing the achievement gap among students from different backgrounds, and for being a precursor for children's further achievement. The studies also indicate that ECME enables children's acquisition of mathematical understanding and skills (Perry & Dockett, 2002) as well as their gaining positive attitudes towards learning mathematics (Stevenson & Newman, 1986). The contributions of early years to children's cognitive and affective domains play an important role for their further mathematics success (Campbell et al., 2001). Furthermore, its significance is stressed through perceiving early years as an opportunity to support and improve children's readiness for the learning of mathematics (Clements & Sarama, 2010). Hence, ECME has major interrelated goals concerning children. The first is related to content related goals, namely children's gaining a mathematical understanding and thinking. The second goal of ECME is about process goals, which include mathematical skills such as reasoning and predicting. The last one is about affective goals like enjoying learning mathematics (Clements et al., 2013). The above-mentioned outcomes and goals of ECME can undoubtedly be achieved through the high quality educational practices which rely on the research based policies, guidelines, and practices (Fox, 2007). In this sense, Sarocho and Spodek (2009) argue the significance of increasing number of research studies in early childhood education on the growing interest in ECME. Thus, the contribution of published articles in ECME research cannot be undermined because getting to know about the research trend becomes a priority for ECME researchers especially for the novice ones in terms of their guidance for the future of the field. However, the majority of these review papers reflect the status of this research field from a national perspective such as Australasian (Fox, 2007; Highfield & Goodwin, 2008), and American (Hachey, 2013). Besides, the remaining review papers seem controversial since they do not investigate the research papers on a certain database/journal and for a certain time period (Sarocho & Spodek, 2009). Rather, they examine the development of this field from a historical view. Therefore, this study aims to reveal the status of international ECME research from 2000 to 2013 in highly refereed mathematics education journals [i.e. Educational Studies in Mathematics (ESM), Journal for Research in Mathematics Education (JRME), For the Learning of Mathematics (FLM), Mathematical Thinking and Learning (MTL), The Journal of Mathematical Behavior (JMB), Journal of Mathematics Teacher Education (JMTE), and ZDM: The International Journal on Mathematics Education (ZDM)] and Early Years Mathematics WG papers in CERME6, CERME7 and CERME8. The review is guided by the following questions: (a) What were the nationalities of authors contributing to ECME research during these fourteen years? (b)What were the types of research design in ECME research during these fourteen years? (c)What were the research topics in ECME research during these fourteen years? (d)What were the research samples in ECME research during these fourteen years? In this way, the findings of this study will guide ECME researchers to better plan and improve the research field relying on the information about under-researched areas and also will call the need for the collaboration between policy makers and teacher educators to sustain the quality in this field.

METHOD

This review study was carried out in Turkey. The journals were selected considering the ranking of European Mathematical Society (EMS) in terms of some criteria such as recognition and citation of the journals (Törner & Arzarello, 2012). Therefore, this study only focused on the journals ranked as A plus and A category which are ESM and JRME in A plus journals; and FLM, MTL, JMB, JMTE, and ZDM in A journals. Among these journals, FLM was not included into analysis due to the limitation of online access. Besides, CERME papers were examined since the first meeting of Early Years Mathematics Working Group (WG) in CERME6. For the purpose of this study, the age interval for ECME (3 to-6-year-old) in NAEYC (2010) report was considered. Therefore, the papers that studied first and/or second grade children were not included. As a result, 49 published articles in the journals and 35 papers in CERME were analysed.

The nationality of contributing authors was quantitatively analysed in order to reveal the contributing countries to this research field. In this sense, the formula of Howard, Cole and Maxwell (1987) was utilised. According to this formula, each paper was given one point. In the case of multiple authors from different countries, the order of author (*i*) and the total number of countries (*n*) in the study were taken into consideration.

$$\text{Score} = \frac{(1{,}5^{n-i})}{\sum_{i=1}^{n} 1{,}5^{n-i}}$$

Research type of the published articles were determined based on the previous review study of Tsai and Wen (2005) which categorized the papers as empirical (e.g., quantitative and qualitative), position (i.e., advocating a certain issue related to the topic under investigation), theoretical (i.e., proposing or generating a theory the related field), review (i.e., summary of the previous research studies), and other papers (e.g., clarification of a specific curriculum or programme for the related research field). The interrater reliability was calculated by the authors of the study and it was found as .95.

Research topic was determined considering the working group names of the Eighth Congress of the European Society for Research in Mathematics Education (CERME, 2013). Whereas some of the WG names such as university mathematics education and history in mathematics education were not included, the determined sub-titles in Early Years Mathematics WG were additionally used. The adapted version of research topics were as follows: Teaching and Learning of Number Systems and Arithmetics (Topic1), Algebraic Thinking (Topic2), Geometrical Thinking (Topic3), Mathematical Potential, Creativity and Talent (Topic4), Affect and Mathematical Thinking (Topic5), Mathematics and Language (Topic6), Comparative Studies in Mathematics Education (Topic7), The Role of Natural Language in Early Years Mathematics (Topic8a), The Role of Play in Early Years Mathematics (Topic8b), The Role of Manipulatives and ICT Tools in Early Years Mathematics (Topic8c), The Transition between Manipulation and Different Kinds of Representation (Topic8d), Designing and Implementing Tasks for Early Years Mathematics (Topic8e), The Transition between Pre-primary and Primary School in Early Years Mathematics (Topic8f), Ways to Learn to Become a Learner at School (Topic8g), Early Years Mathematics and Learning Disorders; Early Identification and Intervention (Topic8j), Teacher Education and Development in Early Years Mathematics (Topic8k), Theoretical Approaches in Early Years Mathematics (Topic8l), Technology in Early Years Mathematics (Topic9), Social, Cultural and Political Challenges for Mathematics Education (Topic10), Stochastic Thinking (Topic11), and Argumentation and Proof (Topic12). The interrater agreement for research topic of the published articles was calculated as .90.

RESULTS

The results indicated that there were 49 published articles about ECME in the investigated journals during the last fourteen years. The distribution of published articles was respectively provided by year and journal. Besides, there were 35 papers in Early Years Mathematics working group in prior CERME conferences.

(For further details about results: https://www.academia.edu/7957956/CERME-9-_Appendix).

Contributing Countries

The formula of Howard and colleagues (1987) revealed the countries' contribution to ECME research in the selected journals from 2000 to 2013. The scores of the countries were analysed in terms of year and journal. There were totally 18 contributing countries to this research field during these fourteen years. The analysis by year indicated the high ranking contributing countries as the USA, Germany, and the UK. There was not any study about ECME in the years of 2000 and 2006 in the investigated journals. Although the total scores of the US, the UK and Germany were striking when compared to other countries, only the research articles from the USA were published during almost each year. As for Germany, the year of 2013 was efficient in terms of the increased number of published articles when compared to previous years. The analysis of contributing countries by journal indicated that the variation of the countries was higher in ESM (44.4%) and MTL (38.8%) than other journals. Authors from five countries contributed to ZDM, JRME and JMB, while JMTE had publications from four countries. Although the dominance of English-speaking countries appeared in each journal, the top ranking countries were from non-English speaking countries such as Israel, Germany, and Turkey in ESM and JMTE. Furthermore, the number of articles from Germany (40%) and Israel (26.6%) was striking in ESM. As for the contributing countries in previous CERME conferences, there have been eleven countries since the first emergence of Early Years Mathematics WG. As opposed to the majority of English-speaking countries in mathematics education journals, the top countries were Germany (37.1%), Israel (20%), and Norway (14.2%). The major contributing countries consistently continued to participate in CERME which is biennially organized. Although the number of countries in CERME6 and CERME7 was five, it increased to eight different countries in the recent CERME.

Research type

During the last fourteen years, almost all of the published articles were in the empirical category (87.75%). In particular, all of the published articles were empirical between 2001 and 2009. Among these empirical papers, qualitative research paradigm was the most popular (46.51%), and this was followed by quantitative (30.23%) and mixed type research designs (23.25%). Yet, other research types such as review (2.04%), theoretical (2.04%) and position (2.04%) were rarely presented in 2001, 2011 and 2013. As for the other research types, there were totally three papers published in 2010 and 2013 in order to provide an introduction into a mathematics education programme and/or a curriculum. The research type by mathematics education journals showed similar consistency in terms of the majority of empirical papers per journal. All of the papers were specifically empirical in JRME, MTL and ZDM. Each investigated journals published studies with different research paradigms including qualitative, quantitative and mixed research design. Among these papers, the dominance of qualitative research paradigm continued in ESM (54.5%), JRME (44.4%), JMB (66.6%), JMTE (66.6%), and ZDM (75%). However, the number of quantitative (50%) and mixed design research papers (40%) was interestingly higher than qualitative research papers in MTL. Other research types such as review, theoretical, position and other were published in ESM, JMB and JMTE. The dominance of empirical research papers (91.4%) was still seen in previous CERME Early Years Mathematics WG papers. Most of the papers were qualitative (80%) and it was followed by quantitative (17.14%), and mixed design (2.85%) studies. Besides, there were three research papers in the category of other to describe a certain curriculum or a programme. But, there were not any review, theoretical and position papers.

Research topic

From 2000 to 2013, the mostly investigated research topics were about number systems and arithmetics (22.44%), algebraic thinking (16.32%), teacher education in early years mathematics education (12.24%), designing and implementing tasks for early years mathematics (10.2%), and comparative studies in mathematics education (8.16%). The studies about number systems and arithmetics insistently became frequently explored topic almost each year except for 2000, 2002, 2005 and 2006. Even though the frequencies of popularly investigated topics varied, the studies about teacher education reached its highest proportion in 2011 (40%). Other research topics about geometrical thinking (6.12%), language and mathematics (6.12%), the role of play (4.08%), manipulatives and ICT tools (4.08%) in mathematics, theoretical approaches to early years mathematics (4.08%), affect in early years mathematics (2.04%), stochastic thinking, social and cultural challenges in mathematics education (2.04%) were not drawn attention by the researchers as much as popular topics. Furthermore, there were not any published articles about the role of natural language in mathematics, representations in early years mathematics, transition between preschool and primary school mathematics, technology in early years mathematics, ways to become a learner at school, and disadvantaged students in early years mathematics and intervention. As for the research topic by journal, the dominance of the research topics about numbers and arithmetics and algebraic thinking permanently continued as top research topics in JRME, JMB, and MTL. However, the top research topics per journal showed variance. For instance, designing and implementing tasks for early years mathematics got the highest proportion in ESM (20%) and JRME (22.2%), whereas all of the published articles in JMTE were about teacher education and half of the articles in ZDM were about geometrical thinking. There were CERME papers about various research topics. Although some of the popular topics in CERME papers were consistent with the publications in mathematics education journals, some of them were strikingly different. Among these papers, the most prominent ones were about the role of play in early years mathematics (14.2%), teacher education (14.2%) and designing and implementing tasks for early years mathematics (11.4%). It was followed by some of these research topics including geometrical thinking (8.57%), mathematics and language (8.58%), and the role of manipulatives and ICT tools in early years mathematics. On the other hand, the top research topics showed variance per each CERME conference. To be more specific, the top topic was designing and implementing tasks in early years mathematics (33.3%) in CERME6, teacher education (20%), the role of play (20%) and the role of natural language (20%) in early years mathematics became popular in CERME7. As well as the popularity of the role of play (18.7%) continued in CERME8, the use of manipulatives and ICT tools in early years mathematics (18.7%) were drawn attention.

Research sample

Research sample refers to the participants (e.g. children and teachers) of the studies in the investigated journals and proceedings. Considering the research sample in published articles, only empirical papers were further examined and analysed by year and journal respectively. The results illustrated that studying with children (74.4%) was the most popular method in early years mathematics education research during these fourteen years. The number of the studies with children has increased since 2010. Even though other samples were not preferred as much as children, there were studies conducted with pre-service teachers (6.9%) and the combination of children and teachers (6.9%), teachers (4.06%), the combination of children and parents (4.06%) and other (2.3%). The results for research sample by journal provided that studying with children continued its popularity in ESM (90.9%), JRME (77.7%), JMB (83.3%), and MTL (80%). Moreover, there were published articles in JMTE conducted with pre-service teachers (75%) and as well as the combination of children and parents (25%). As for ZDM, this journal published the studies conducted with children (50%) and teachers (50%). As for the combination of teachers and children, such studies were published in JMB, JMTE, and MTL. Similarly, the substantial part of the studies in prior Early Years Mathematics WG was conducted with children (60%) and the sample of children continued to be the mostly preferred one in all previous CERME conferences. Other research samples were not studied as much as children. But, there were studies conducted with teachers (14.2%), and the combination of children and parents (11.4%).

DISCUSSION AND CONCLUSION

This study presents the major contributing countries to this research field as English-speaking countries in the context of mathematics education journals. This may be related to the benefits of reporting the scientific papers in a more efficient way than the way non-English speaking researchers do. Yet, the contributions of non-English speaking countries particularly Germany and Israel were striking. Furthermore, these countries became the top countries in previous CERME conferences in terms of their contribution to the research field. Moreover, they provided a significant contribution to the field through their being a research community since the publications of such countries were constructed through a collaborative work.

In this study, majority of the published articles and CERME papers were empirical. Among these papers, qualitative research paradigm was the most utilized one. This may be related to the research sample of the published articles because another finding of this study indicates that studying with children was superior to studying with other kind of samples. Furthermore, using qualitative research methods is the most common way to study with children (Trundle & Saçkes, 2012). This trend was followed by quantitative and mixed research design. Although almost all of the quantitative studies were conducted with children, they mostly preferred to collect data in accordance with the target age group through using mathematical tasks and materials. As for the research type by journal, the majority of the empirical papers adopted qualitative research methods. Yet, the number of quantitative papers was higher than others in MTL. Moreover, as opposed to the dominance of empirical papers, other research types like review papers were rarely presented during these fourteen years and they were published in ESM, JMB, and JMTE. The difference among journals may be related to their aims and scopes. Although CERME accept various research types, there were not any review, theoretical and position study among the CERME papers.

The results also announce the top research topics in ECME research. These top research topics were different in the investigated journals and CERME papers. In the mathematics education journals, the top topics were about teaching and learning of number systems and arithmetics, algebraic thinking, teacher education, designing and implementing tasks for early years mathematics, and comparative studies in mathematics education. Its reason may be related to the goals of ECME such as promoting mathematical thinking and learning and improving qualification through various ways like teacher education and the implementation of tasks (e.g., Clements et al., 2013). Besides, these research topics consistently remained the top topics almost each year. This shows us the insistence of the researchers on these research topics to reach the stated goals of mathematics education. In spite of the significance of teaching various content areas such as geometry and measurement in ECME (NCTM, 2013), learning of number systems and arithmetics and algebraic thinking were the most frequently investigated topics. This may be related to the broad focus of 'numeracy' in early childhood curriculum rather than process skills and affect (Ginsburg & Golbeck, 2004) and as well as its misinterpretation as generally dealing with numbers (Sarama & Clements, 2004). However, other research topics such as geometrical thinking and affect in early years mathematics were not studied as much as the top topics. Besides, there was no study about remaining topics like technology in early years mathematics, and learning disorders; early identification and intervention in the selected journals and CERME papers though these topics are highlighted to achieve ECME goals, and to provide high-quality education to meet the needs of all learners (Highfield & Goodwin, 2008). Moreover, the top research topics varied per journal. For example, the topic of designing and implementing tasks for early years mathematics was the mostly studied one in ESM. This may not be surprising because the journal focuses on the practitioners in mathematics education through dealing with didactical, theoretical and pedagogical issues. The top research topics (e.g. the role of play and the role of natural language in ECME) in prior CERME were in general different from the investigated journal articles. This may be related to the dynamic in Early Years Mathematics WG and the research interests of group members. Furthermore, CERME papers filled the gap in terms of investigated research topics since there was not any study about the role of natural language in the journals examined. Last but not least, the top ranking research sample was children in both mathematics education journals and CERME papers. Moreover, it is good to reach the studies conducted with other samples. The combination of both children and parents and children and teachers is particularly a good indicator of understanding the relevant research topic through considering the context because the need for such research which explores parents' educational interactions with their children and the role of parents' on children's learning mathematics is pointed out (Ginsburg & Golbeck, 2004). Moreover, the results per journal indicated that some journals like JMTE and MTL publish the studies conducted with the combination of various sample. The findings of this study for each journal particularly introduce the aims and scope of each major journal in the field for ECME researchers who would like to contribute to this field. Thereby, this may play a facilitator role for future studies to fill a gap in the field as well as to determine the right journal in accordance with the kind of topic, design and sample of the studies. In spite of the evolution of this research field in the past decade, the results may also suggest some further research directions for the improvement of this research field. First of all, the research community in ECME should be internationalised through increasing the number of studies from various countries. Besides, cross-national and cross-cultural studies may contribute to the advancement of this research field. Second, other research types should be drawn attention by the researchers as well as empirical papers because the position, theoretical and review papers play an important role in contributing to the research field. Third, the goals of ECME should be accomplished through studying various research topics rather than the research topics focusing on only some goals of ECME. Particularly, there should be more research topics such as "technology education" for the improvement of children's mathematical learning, and "transition between preschool and primary school mathematics" for the development of mathematical proficiency (NAEYC, 2010). Last, there should be more studies conducted with various participants like children and parents based on the contributions of family environment to children's ongoing mathematical learning (Ginsburg & Golbeck, 2004). Furthermore, there is a need for studies conducted with pre-service and in-service early childhood teachers since the findings of such studies may guide teacher educators for the professional development of early childhood teachers in ECME (Lee & Ginsburg, 2007). The findings of the study were limited to the investigated journals in mathematics education and CERME Early Years Mathematics Working Group (WG) papers. Therefore, further review studies could be replicated through including highly refereed early childhood education journals and other international mathematics education conferences as well.

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The bar model as a visual aid for developing complementary/variation problems

Eugenia Koleza

University of Patras, Department of Primary Education, Patras, Greece, ekoleza@upatras.gr

In this paper, we report the preliminary findings of a study that considered how third grade students represent multiplication and division problems using the bar model as a tool. Initial results indicate that students are able to create visuals, representing problem's structure, and based on these visual representations they can formulate division problems given a multiplication one.

Keywords: Multiplication, division, complementary problems, bar model.

INRODUCTION

Use of diagrams is considered an efficient strategy in teaching and learning mathematics and especially in mathematical problem solving. "From the most elementary class to the most advanced seminar, in both introductory textbooks and professional journals, diagrams are present, to introduce concepts, increase understanding, and prove results. They thus fulfill a variety of important roles in mathematical practice" (Mumma & Panza, 2012, p. 1). Diagrams make plain the quantities in the story context and the relationships that exist amongst them, limit abstraction and thereby aid in the problem-solving process (Bishop, 1989). *Draw a diagram* is a well known strategy for mathematical problem solving (e.g., Polya, 1957; Schoenfeld, 1985), grounding in the belief that generating a diagram enables deeper understanding of the situation described and facilitates the conceptualization of the problem structure (van Essen & Hamaker, 1990). Not all diagrams are beneficial or can easily be used by the students. Vosniadou (2010) distinguishes between external representations that are *perceptually based* (grounded on everyday observations) *depictions* and those that represent *conceptual models* (theory –based). Pictures used in mathematics and science textbooks, number lines and bar models that are usually used on mathematical problem solving, are conceptual models.

In our study, we used the bar model as a visual support for the resolution of simple (one operation) multiplication problems and the formulation of the two corresponding division problems.

Children's difficulty to understand the close relation between multiplication and division was a problem noticed very early in mathematics teaching literature. Nevertheless, in most western curricula, multiplication and division, as well as addition and subtraction are taught separately. Paraphrasing Herscovics (1989), it is as if students are taught the syntax of operations, without the semantics. In other words, students know the algorithms (the rules of the 'grammar') but they do not understand the meaning and their relations. This situation explains the difficulty that pupils often face when solving equations in algebra with an operational view of equality (Wagner & Parker, 1988). "The 'one-thing-at-the-time' design, provide fewer opportunities for 'making connections' compared to those adopted in eastern cultures" (Sun, 2013, p. 13). "In Chinese elementary schools, addition and subtraction are introduced simultaneously, and subtraction is introduced as the reverse operation of addition. Division is also introduced as a reverse operation of multiplication" (Cai, 2004, p. 110). As Cai refers (p. 112), "representing quantitative relationships in different ways will not only help students develop deeper understanding mathematics, but also will help them develop their flexibility of using equations to solve application problems". Giving an example of a variation in a multiplication problem, Sun (2011, p. 104) explains that "within the problem set, there are two concepts of multiplication and division behind three similar problems made with 4, 6, and 24. Example problem: How many trees do 6 lines need so that each line can have 4 trees? Variation problem 1:

How many trees will each line get if we plant 24 trees in 6 lines? Variation problem 2: How many lines do we plant if we plant 24 trees in order so that each line has 6 trees? Clearly, the intent of One Problem Multiple Changes is to enable students recapitulate the general relationship of multiplication and division, and the meaning of equal from the problem set 4 × 6 = 24, 24 ÷ 4 = 6, 24 ÷ 6 = 4 [...]. The task draws students into a space of relations as opposed to directing attention to the object itself."

THEORETICAL FRAMEWORK

The 'model method', also known as graphical heuristic, consists of the use of rectangular bars to represent numbers rather than abstract letters to represent unknowns in word problems. This method is often used in education systems of many countries under various names: tape diagrams – in Japan (Murata, 2008), strip diagrams – in US (Beckmann, 2004), or bar models – in Singapore (Hoven & Garelick, 2007). This special kind of diagrams "are clearly designed *to help children decide which operations to use*. Instead of relying on superficial and unreliable clues like key words, the simple visual diagram can help children understand why the appropriate operations make sense" (Beckmann, 2004, p. 43). Cai and colleagues (2005) consider the 'model method' as one of the big ideas related to algebraic thinking in the Singaporean elementary curriculum. "Children solve word problems using the 'model method' to construct pictorial equations that represent all the information in word problems *as a cohesive whole, rather than as distinct parts*. To solve for the unknown, children undo each operation. This approach helps further enhance their knowledge of the properties of the four operations"(p. 8). In other words a basic property of the bar model is that it can support an exploration and visualization of the 'doing' and 'undoing' processes in mathematics. According to Hall and colleagues (1989) when the structure of a problem is recognized, a formal representation of this relationship may be constructed. Departing from meaningful tasks, students may construct personal meanings by the use of the bar model. In an initial phase the bar model may serve as a *model of the mathematical structure* of a word problem. Later, through a process of vertical mathematization, reflecting on the relationship between their actions upon a diagram and the effects of those actions, students may generalize and abstract those actions to successfully solve problems of the same semantic structure. In this way, bar model becomes, to those students, a *model for the mathematical structure.* Bar model is particularly useful for problems that involve comparisons, part-whole calculations, ratio and proportion.

In this paper, we will restrict to the use of the bar model as a visual support in order to formulate multiplication and division problems.

The Singapore education system's approach concerning the use of diagrams, has close relation with the one several soviet researchers (Bodanskii, Mikulina, in Davydov, 1991/1969) has used in their studies concerning algebra word problems. For example in Figure 1 (on the left) is the diagram of the problem: "*In the kindergarten, there were 17 more hard chairs than soft ones (labeled M). When 43 more hard chairs were added, there were 5 times more hard chairs than soft. How many hard and soft chairs were there?"* as presented in Bodanskii (1991/1969, p. 302), and on the right the bar model representation.

In both cases, the common element is the concept of "unit". Units are not simply single discrete entities, but instead may be composed of one or more 'shapes' (in the diagram) of various types (Davydov et al., 2000). Taking as a "unit" the number of soft chairs, the sum 17+43=60 is translated as 6 units, giving, thus, the answer of the problem. The problem solver is able to reason from the very diagram that was created as a *model of* the situation given. The algebraic equation that correspond to this visual solution- the *model for* this kind of relations- is just a step further.

The research question behind our study was:

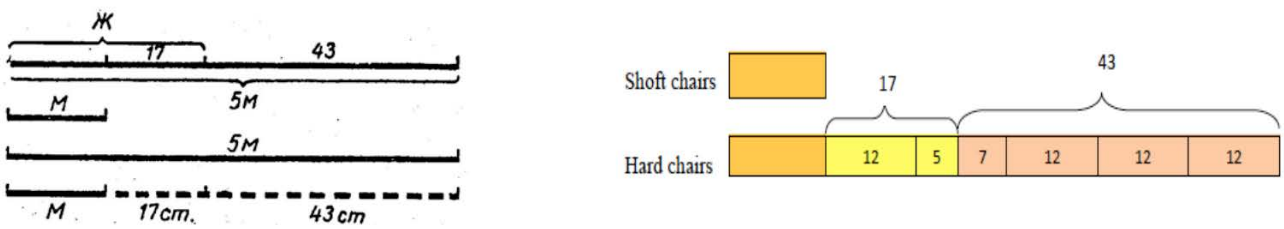

Figure 1: Similar representations of the same problem

Can the bar model support students in the resolution and formulation of multiplication and (complementary) division problems?

METHOD

In this study, 19 third graders (aged 8) were involved, belonging in a class of a primary school in Patras. During a period of eight 45-minutes sessions the students followed the regular lessons (teaching of the multiplication table) that were enriched by the bar model as a way to solve and represent problems, and mainly as a way to connect multiplication and division, in a unified scheme. This connection was designed only for the needs of our experiment, given that these two operations are taught separately in the Greek mathematics curriculum. Tasks given during instruction were arithmetic and algebraic tasks (given a problem, the task asked for his complementary ones-focus on relations) with three (group, restate and vary) of the five semantic relations (the other two were change and compare) identified by Marshall (1995). Data was collected and analyzed from field notes, photos, pupils' written work, and final tests. Learning gains were assessed by means of a word-problem test just after the end of instruction and a second one a month later.

Hereafter, the problems given in the tests.

Vary-problem in the short-term test: *Grandma has in her sac 45 sweets in 5 bags. How many sweet are there in each bag?*

Vary-Problem in the final test: *Grandpa gave to each one of his 5 grandkids 12 euros. How many euros he gave in total?*

For each problem was asked to the students: "*Departing from the problem, with the same story and the same numbers make two other problems and 'design' them*".

Teaching students for short periods of time could not serve as a basis for a solid understanding of their thinking and how it might be influenced by the use of a certain visual representation. Our 'teaching experiment' has not the characteristics described by Steffe and Thompson (2000), but it was an exploratory one which aimed only to the evaluation of the bar model as a visual aid for the resolution and formulation of complementary multiplicative-structure problems. "Any researcher who hasn't conducted a teaching experiment independently, but who wishes to do so, should engage in exploratory teaching first. It is important that one become thoroughly acquainted, at an experiential level, with students' ways and means of operating in whatever domain of mathematical concepts and operations are of interest" (Steffe & Thompson, 2000, p. 274). In other words, we wanted first to explore the role this specific visual representation may have in helping students to understand how multiplication and division are connected, and their eventual difficulties, in order to design our 'teaching experiment'. Because "incomplete understanding [...] can result in inappropriately designed artifacts or artifacts that result in undesirable side effects."(March & Smith, 1995, p. 254)

DISCUSSION

The first of the 8 lessons (the corresponding textbook objective was the multiplication table of, 2, 5 and 10) began with a teachers' question.

T: *What is a problem? Who can make a problem?*

S: *I have 80 candies and I give 10 candies to each one of 8 children*

T: *Is this a problem? What are you asking? In order to have a problem you must ask for something. For example, 8 children bought 10 candies each one. How many bought in total? Lets make a picture of the problem.* (The teacher designs the bar model). *In order to have a problem, we must 'hide' something.*

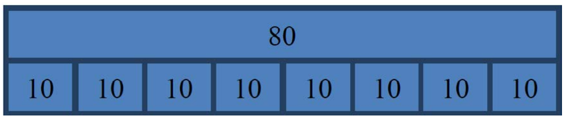

Figure 2

T: Can someone else make another problem, with the same story?

S: We can 'hide' the number of candies that each child will take.

(The teacher changes the previous model)

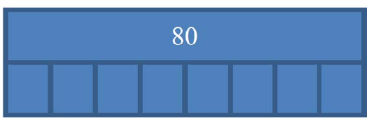

Figure 3

T: *Another problem?*
S: *We can 'hide' the children.*
T: *So, I will delete the squares.*
T: *If I had 40 candies, and 10 children, how many candies could have each child?*
S: *4*
T: *Can you 'design' the problem?*

During the next 2 sessions students worked with the multiplication table (according to the official schedule), and solved multiplication problems following a 'didactic contract' of 4 precise steps that had emerged from the previous lesson: (1) we read first and we 'design' (make a representation of) the problem, (2) in order to make the 2 'inverse' (complementary) problems, we "hide" a number from the design, (3) we write (in words) the 'inverse' problems, and (4) we solve the problems.

The complementary/inverse problems were introduced by the teacher as the two variations of an initial problem, and the bar model was presented as a tool of organizing problems' data. Each complementary/inverse problem was creating by 'hiding' a number on the bar.

During this process, students faced two major difficulties: (1) Expressing in natural language the problem that they had already represented by the bar model, and (2) representing by the bar model the quotative/measurement division problem-variation. For example, while the multiplication problem «*If I save 8 euro in a week, how many euros I will have after five weeks?*» and –'hiding' the 8- the corresponding partitive division problem "*If I saved 40 euros in 5 weeks, how many euros I saved in a week*" were easily represented by the bar model, it was not the same for the 'quotative division' problem. The rule of 'hiding' could not be applied: while in the "partitive division' problem they had to 'hide' the numbers, in the 'quotative division' one, they had to 'deconstruct' the bar model. In the following pictures (Figure 4) we see (to the left) the incomplete diagram made by a student (reproduced in the middle), and the one suggested by the teacher (to the right).

Teacher's proposition in order to face the difficulty was to reformulate the problem as "*How many 8s we want in order to make 40?*"

We present three different treatments of the "quotative division" problems representation during instruction

a) An arithmetic treatment: no use of the bar model

"*30 ducks fly in groups of 5. How many groups of ducks are there?*"

Figure 5

b) The 'quotative division' as a subtraction: A "filling-in" use of the bar model. "*I have 45 stamps in pockets of 9 stamps. How many pockets I have?*" The thought behind the model was:

"*I take off each time 9 stamps from the 45 stamps*"

Figure 6

The 'quotative division' as the inverse of the multiplication: An 'algebraic' use of the bar model.

Figure 7

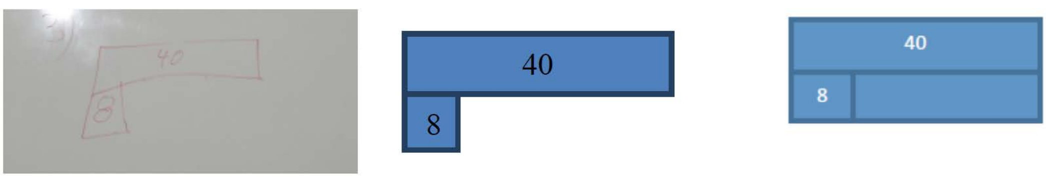

Figure 4

After instruction a final test was given. Hereafter, we comment students' competency in using the bar model for the resolution of the 'vary-problem' given in this final test (see page 4).

From a total of 19 students,

a) 13 students formulated correctly the two division-problems. The two main strategies observed during instruction, appeared also in the final test:

— a mixed ('filling-in' and algebraic) strategy -MS(10 students), and

— an 'algebraic' strategy -AS (3 students).

We give an example of each case.

(MS) *Fey*

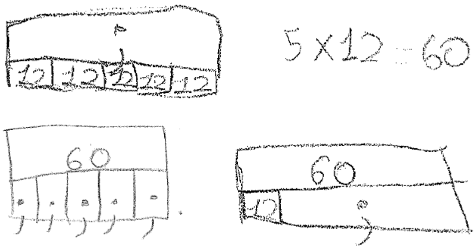

Figure 8

(AS) *Joseph*

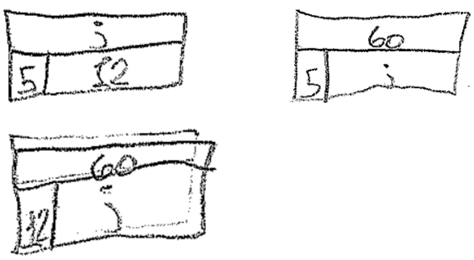

Figure 9

The 3 students who used the 'algebraic' strategy were the higher achieving students of their class. This outcome confirms similar results by Booth and Koedinger (2007) according to which, higher-achieving middle school students do benefit from the diagrams while low-achieving students perform better on story problems that do not have accompanying diagrams.

b) 3 students formulated only the partitive-division problem.

"*Grand papa has 60 euros and he wants to distribute them to his 5 grandkids. How many euros will have each one?*"

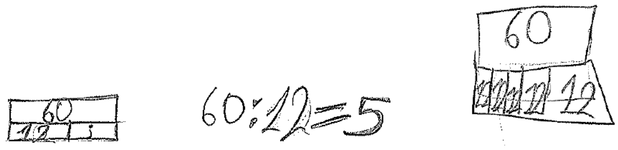

Figure 10

c) The rest 3 students were not able to use the bar model at all. For example for the multiplication problem the diagram proposed was

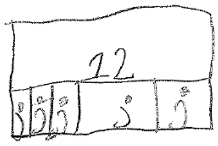

Figure 11

or

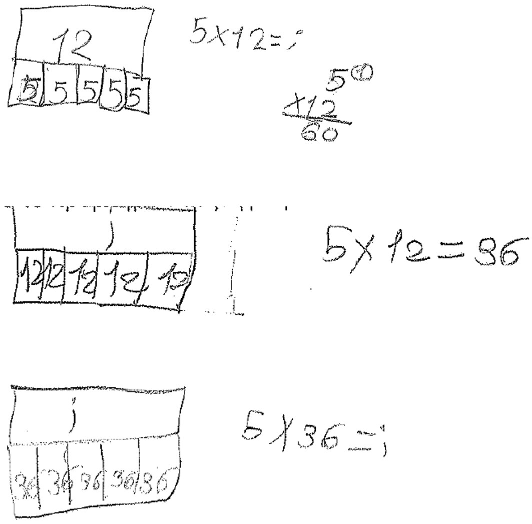

Figure 12

CONCLUSIONS

The results has shown, that the bar model is an effective model, but only for multiplication and partitive division problems. Though most students (13/19) after a relatively short term instruction (eight 45-minutes sessions) could formulate and represent by the bar model a multiplication and the corresponding division problems, further research is needed about the kind of representation that is more appropriate for the quantitative division problems.

The fact that a representation may be "clear" for a multiplication problem does not mean that it would be a useful tool for the formulation of the corresponding division problem. For example, lets take the array model.

This model in a concrete (on the left) or in a more abstract (on the right) version, under certain conditions may be useful for students to understand the relation between a multiplication and the two complementary-division problems. It is useful, if all information is on the representation, but not if the students must construct the representation by themselves, especially in case of big numbers.

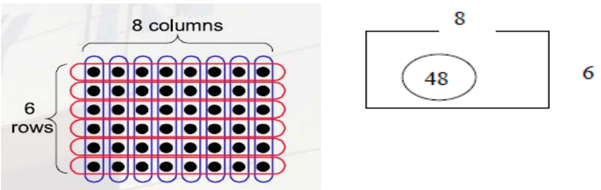

Figure 13

On the other hand, the use of the bar model, as our research has shown, is a high demanding task, because the level of abstraction needed. For example, in the problem: *Sarah made 210 cupcakes. She put them into boxes of 10 each. How many boxes of cupcakes were there?*, students must put on the representation information that does not exist.

The mathematics symbolization "…..?….", is not evident for the young students.

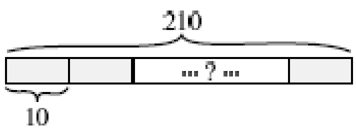

Figure 14

A combination of an array and a bar model representation could be eventually easier to use for students. For example, in the problem: *Sarah had 12 apples to hand out to her class. Each group of students in the class got 3 apples. How many groups were there in the class?* This kind of representation would be:

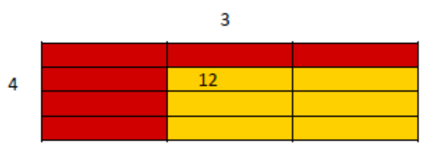

Figure 15

The data collected do not permit us to know if the students who had used the algebraic strategy (AS) were acting in a pure algorithmic way, or 'with understanding'. Eventually a confrontation with the strategies used by the same students in additive-structure problems may offer a more complete explanatory framework. A more accurate analysis of the relationship between the instrument and students' meanings is required, and that is what is going to be done with the data analysis of the whole teaching experiment.

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Introduction to arithmetical expressions: A semiotic perspective

Andrea Maffia[1] and Maria Alessandra Mariotti[2]

1 University of Modena and Reggio Emilia, Modena, Italy, andrea.maffia@unimore.it

2 University of Siena, Siena, Italy, mariotti21@unisi.it

An early introduction to arithmetical expressions is realized in a teaching experiment involving an artefact based on the rectangle model for multiplication. Children elaborate signs, strictly related to the activity with the artefact, which evolve to mathematical ones: the enchaining of different representations in many semiotic systems is described according to Theory of Semiotic Mediation. In particular the teacher role in selecting and elaborating specific personal signs, in order to make them pivot signs, results as crucial. Elaborated tasks reveal to be good triggers for a relational approach to arithmetical expressions.

Keywords: Arithmetical expression, multiplication, rectangle model, distributivity, semiotic mediation.

DISTRIBUTIVITY IN GRADE 2

In Italy there is a long tradition for which students are expected to know all the times-tables from 1 to 10 at the end of grade 2, often justified with sentences such as "if they do not learn times-tables in the second year, they will never learn them". It is well known that recall of results of one-digit numbers multiplications is more difficult when the operands are closer to ten. Psychologists refer to this phenomenon as *operand size effect* (e.g., McCloskey, 1991). Operations' properties can be used to reconstruct more difficult results relying on easier ones ($8 \times 2 = 2 \times 8$; $3 \times 8 = 3 \times 5 + 3 \times 3$), this kind of strategy could be particularly useful when times-tables have not yet been memorized completely. In other words an early introduction to operations' properties may promote flexible calculation strategies instead of rote memorization.

The use of operations' properties requires a relational approach to calculation in order to establish the equivalence between different calculation procedures; establishing such an equivalence requires to grasp the relationship between two arithmetical expressions organizing the relationship between two operations (i.e. multiplication and addition) in a highly structured way. The classic symbolic representation seems hardly accessible to very young children, thus distributive property is often introduced by graphical representations (Ding & Li, 2014; Izsák, 2004): rectangles have been largely used as model for multiplication (for a large review see Izsák, 2005) from Euclid's *Elements* since more recent western textbooks (Ding & Li, 2010).

The aim of this paper is to describe the emergence of symbolic arithmetical expressions as numerical representation of the distributive property: working with a specific artefact (a rectangular model for multiplication), pupils begin using the artefact, pass through an iconic representation and arrive to make sense of a structural relationship between different numerical expressions.

THEORETICAL FRAMEWORK AND METHODOLOGY

The work here presented is a part of a wider teaching experiment on multiplication, implemented in a grade 2 class in Italy. The general theoretical framework is the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008) and the teaching sequence is centered on the use of an artefact called "geometrical times-table". In this table, rectangles are organized, increasingly, in columns and rows as showed in Figure 1.

According to the TSM approach, the didactic intervention was implemented in *didactical cycles*, comprehending individual activity with the artefact, small group work and whole class *mathematical discussion*.

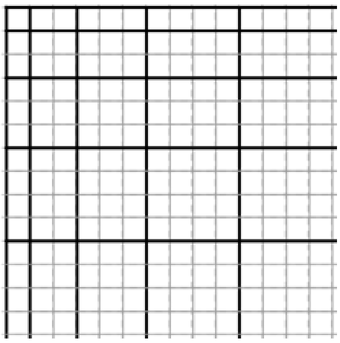


Figure 1: Geometrical Times-Table with five columns and rows

An artefact is defined as a *tool of semiotic mediation* when the teacher uses it intentionally to mediate a mathematical content to students (Bartolini Bussi & Mariotti, 2008). The artefact is both related to the personal meanings of its user while solving a particular task and to the mathematical knowledge underpinning the task and/or the construction of the artefact itself. This double relation is called *semiotic potential* of the artefact (ibid, p. 754). When students accomplish a task using the artefact, they produce signs strictly related to this activity; those signs, to which we refer as *artefact signs*, can be very different from the ones usually used by mathematicians while working with the mathematical knowledge related to the task, because pupils can be unaware of this knowledge. Anyhow students are expected to become able in using culturally determined *mathematical signs*. According to TSM we expect an evolution from artefact signs to mathematical ones. In order to describe such evolution we use the notion of *semiotic chain*, the set of dynamic relations among artefact signs and mathematical ones. In the description of the evolution through semiotic chain, a specific role is played by another type of signs named *pivot signs*. These signs are defined with respect to their function in promoting the relationship between the other two categories of signs (ibid, p. 756).

In the case of geometrical times-table, each one of the rectangles can represent a multiplication: the sides are the factors and the area is the result. Rectangles can be cut, moved and pasted and, in particular, some of results of these actions can be related to specific operations' properties. For instance, commutative property can be related to observation of different positions of rectangles with sizes of the same length. In the first part of the teaching sequence, children explored the artefact and were asked to explain how numbers and operations could relate to it. Finally, after a couple of months, a didactic cycle was released aimed at introducing pupils to the distributive property.

In the first task, students are asked to cut two pieces of paper with the same dimensions of two rectangles of the same row (both sides smaller than five). They also have to paste the two pieces of papers along one of the sides in order to form a new rectangle, then they are asked to look for a rectangle with the same dimensions into the table. Children are expected to use the obtained rectangle dragging and rotating it on the table, trying to fit it inside the borders of one of the rectangle in the table. We imagine a relation between the pasting of two rectangles in a bigger one as the sum of two multiplications to obtain another one, i.e. it is a transformation of an arithmetical expression according to the distributive property: $a \times b + a \times c \rightarrow a \times (b + c)$. In our a priori analysis we expected that pupils notice that all the rectangles have the same height and that the final rectangle has a width that is the sum of the two of the others, this constitute the germ of the mathematical meaning of the distributive property. In order to work on the other direction of the transformation ($a \times b + a \times c \leftarrow a \times (b + c)$), a second task has been implemented: each student received a copy of a letter by Giovanni, an imaginary child who lives in another city. In his letter, Giovanni explains that he has to calculate 3×7 but he only remembers multiplications with factors smaller than five. In our a priori analysis of the task we expected children to use the signs produced during the past activities to decompose 3×7 in two smaller multiplications, eventually using the table to find the right ones.

Using TSM lens, we analyse the activity of children in the progression of the didactic cycle, while facing the two tasks involving the use of the geometrical times-table and in the following collective discussions. According to TSM, we will look for semiotic chains, identifying the three different types of signs and their mutual relations, in order to describe the development of the semiotic mediation process. Thus our research questions can be articulated as follows:

(1) which semiotic chains can describe the semiotic mediation process from the use of geometrical times-table to symbolic representations of the distributive property?

(2) What is the role of the teacher in triggering this process?

The two tasks have been implemented in a grade 2 class of 20 students, in Tuscany in a period of two weeks. All lessons have been videotaped by the first author who, even if he was not the teacher, interacted with the pupils during the different work phases. Following the analytical model for studying videotape data by Powell , Francisco and Maher (2003), all videos have been viewed several times and a description of the events has been written. According to a priori analysis of the semiotic potential of the artefact, we selected critical events, then transcribed and coded. The different signs produced by teacher and students have been classified according to TSM (artefact signs, pivot signs or mathematical signs) and semiotic chains identified. Videos and written productions made us able to reconstruct the storyline interpreted in the next section.

DATA ANALYSIS

Combining tiles of the same row: the emergence of a semiotic bundle

After a previous activity, pupils became familiar with the artefact, with cutting and moving rectangles and recognizing them on the table. A shared system of artefact signs was established around the key word 'tile' that, at this point, has a complex meaning: it refers to one of rectangles (either on the table or cut on the paper), but also to the multiplication between the two numbers that represents the dimensions of the rectangle, specifically it refers to both the multiplication and the result. The first task of a new cycle asked to select two tiles in the same row of the table, to cut them and to combine them into a new tile and to identify on the table a rectangle that corresponds to this new tile. Working on this task children engaged in finding where their rectangles (to which they refer as "tiles") were inside the table. When all the students completed this task, the teacher asked for comments. A child made an intervention:

Lor: That…when two tiles are far (he points at the table) you can calculate the result and then you know it.
Researcher: And how do you calculate the result?
Lor: Between these two (he points two rectangles in the fourth row) you do nine times four, it is thirty-six (he points the 4×9 rectangle) plus twenty (he points the 4×5) it is fifty-six.
Researcher: […] Well, Lor gave an example and he said that when the results of two tiles are known, it is possible to discover the result of another tile. I have understood this way, you have to say to me if I understood correctly. You said that if I know the result of two tiles (he does the gesture in Figure 2a), I can do the addition (gesture in Figure 2b). Isn't it?
Lor: No. It is that, if you do these far two, you calculate them!

(a) (b)

Figure 2: Researcher gestures during discussion

The researcher elaborates the statement of Lor generalizing it, and also passing to the interpretation of the calculation – addition of multiplications – as the combination of tiles. The semiotic process of interpretation is accomplished by enchainment of words, graphical representations, mathematical symbols and gestures. The word "addition" is combined with the gesture of joining the fingers, with the intention of relating the combination of tiles (gesture) and the operation of adding numbers. The pupil seems not to follow the researcher comment and repeats his statement stressing the fact that he is referring to tiles which are far one from the other. Maybe this child looks at the pragmatic scope of the activity as to find the results of the union of two tiles which cannot be compared directly (because they are far). He is able to produce a new personal example without cutting new pieces of paper and using numbers which lead to a multiplication out of the table. The researcher continues the discussion trying to change signs and to work on another, simpler, example.

Researcher: What do you obtain?
Lor: Fifty-six.
Researcher: And what do you need this number for? What does it represent?
Lor: A tile.
Researcher: It is another tile, it is what I was saying: if you put together two tiles, then you find another one. Let's do an example […] if I, for instance, take the tile two

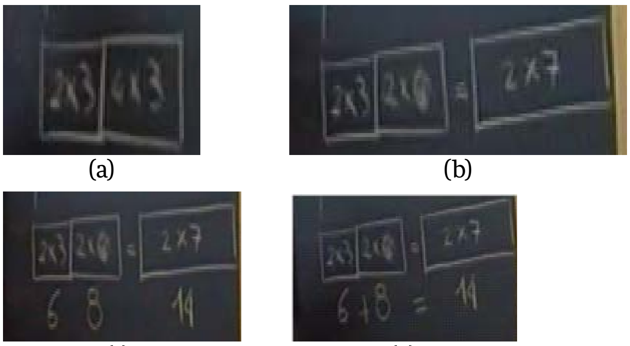
times three (he draws a 2×3 rectangle on the squared blackboard, he writes "2×3" inside it). Do you all agree that this is the two times three tile?

Chorus: Yes!

Researcher: And I put together the four times three tile (he draws a second rectangle juxtaposed to the previous one, Figure 3a) this is big as which tile? Putting all together?

The sign "addition" has been replaced with the artefact sign "put together" which refers directly to the activity done (i.e., the pasting of the pieces of paper) with the aim of bridging the gap between the idea of combining the tiles and that of adding multiplications. It is also introduced a graphical icon (Figure 3a) together with written and oral mathematical signs. The researcher refers to the graphical representation using the word "tile", an artefact sign. Arzarello (2006) defines a *semiotic bundle* as a collection of semiotic sets (set of signs, modes of producing, relationships with meanings) and relationships among them. In the last excerpt there is a system of different related signs involving the ones used by students and new ones introduced by the researcher, it can be described as an example of semiotic bundle. In particular we observe a *genetic conversion* (ibid, p. 281), namely the oral and gestural signs, produced in the previous part of the discussion, are converted in new graphical ones enlarging the bundle.

After these transcribed episodes, the researcher changes the example and asks children to develop the interpretation process, focusing on the dimensions of the rectangles, then on the height and width of the resulting rectangle. When they agree on the answer, he synthesizes their intervention saying that the two tiles, together, equal [1] the two times seven



(a) (b) (c) (d)

Figure 3: Graphical signs on the blackboard

tile and he draws it (Figure 3b). Finally, the researcher asks the children to say how many squares there are inside each one of the rectangles and he writes these numbers under the drawing, obtaining the signs represented in Figure 3c.

Some students begin to notice that six plus eight is fourteen. The researcher decides to rephrase one of his previous sentences.

Researcher: So, when I put together the squares inside this tile (he points the 2×3 tile) with the squares of this tile (he points the 2×4) I obtain the squares of this entire tile (pointing 2×7). Isn't it?

Non: It is true!

Researcher: Which operation does "put together" correspond to?

Chorus: Six plus eight!

Mab: Equals fourteen.

Researcher: (writes the symbol + and = between the numbers, Figure 3d) So, what does this mean? If I know the results of two little tiles (he points 2×3 and 2×4), I can put them together (he points the numbers 6 and 8) and what do I find? (he points 14) The result...

Chorus: Of a tile!

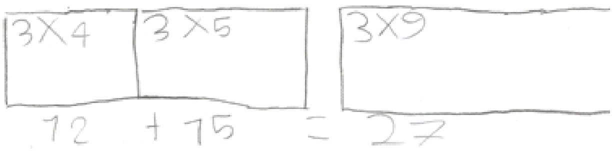
Researcher: And how do we find this tile? It has the same height, and this? (he points the base of the rectangle)

Sim: It is large as the two together.

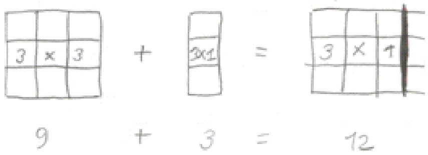
Researcher: It is large as the two together. Do you all agree?

Chorus: Yes!

"Put together" becomes a pivot sign that is explicitly related to the operation of addition, but in the same time it is related to the combination of tiles and more specifically to the addition of the width-dimension of these tiles. As suggested by the children, the researcher also introduces the symbol + in the discourse, it is not between the operations of multiplications but between the respective results. At the end of this discussion, the children are asked to produce some personal examples. Two children cannot find a way to accomplish this final task, another child just copies the example on the blackboard. The others give one or even more original examples imitating the pivot sign of Figure 3c to represent the distributive property (Figure 4a). Four children transform the drawing adding the sign + between the rectangles (Figure 4b). All these texts are made of two lines: the first line is composed of pivot signs and the second one of mathematical signs, as each text were a kind of Rosetta Stone: establishing an explicit relationship between the two lines they solicit the translation of artefact signs into mathematical ones, and so they may function as a resource for the unfolding of the semiotic potential of the artefact. The different texts produced by the pupils show interesting differences; for instance, the kind of text in Figure 4b (where the first line text includes both artefact and mathematical signs) presents an hybridisation between the different semiotic systems that provides evidence of the movement from personal meanings, strictly related to



(a)



(b)

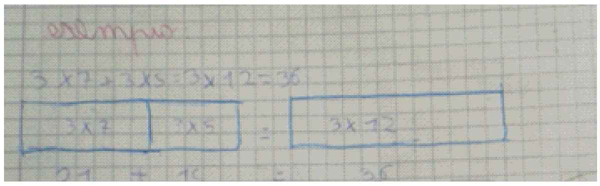
Figure 4: Fra's (a) and Mal's (b) productions after the class discussion

the use of tiles, to mathematical meanings expressing the relationship between operations. In the following days the teacher asked the students to give examples using the version with the + between the rectangles (Figure 4b). The teacher recalls this bundle, used by some students, and decides to foster the sharing of this type of signs that has the potential of linking the text composed of artefact signs with the text made of mathematical signs.

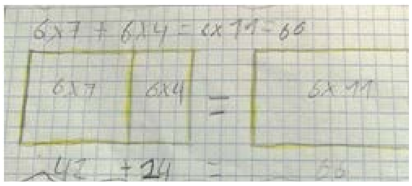
Translating rectangles into arithmetical expressions

After a couple of days, the teacher proposed a variation to the task: she gave them an example writing it on the blackboard, saying them to copy it on the notebook (Figure 5a) and then to invent some personal examples. In this way the teacher enriched the text introducing a new line of mathematical text, in parallel with the previous ones. Such a text is provided as an alternative 'translation' of both the artefact text and the mathematical text. Though at this moment just few students create new text including this new expressions (Figure 5b), the availability of a translation key from a semiotic system to another will play a crucial role in the further activity.

A week after, the second task was given. The students were asked to read the letter, discuss some possible answers in small groups and then to report their solutions to the whole class. A proposed solution consisted in decomposing the rectangle 3×7 in 3×4 and 3×3 or in 3×5 and 3×2 using the graphical pivot signs developed in the previous activities to represent this process. It is interesting to notice that, while reporting this second solution, Fra comes to the blackboard and begins from the expression 3×5+3×2 (written just in symbolic mathematical signs) and only after he completes it drawing rectangles (with the same height) around the two multiplications, without caring about their lengths.



(a)



(b)

Figure 5: Mir's copy of the teacher example (a) and Lor's personal production (b)

A while after, Mir asks to go to the blackboard saying that he would like to try a new different solution (Figure 6a). As matter of fact, from the mathematical point of view, what Mir writes does not differ from Fra's proposal; the difference is only in its representation: Mir's representation eliminates any reference to the artefact using genuine symbolic register, as it is confirmed by the following exchange.

Researcher: Ok, Mir can you explain me a thing. What do all that equal signs mean?
Mir: Three times five equals fifteen. Three times five plus three times two equals fifteen plus six.

The new signs are strictly related to the signs used till this moment, as shown by the activity of Fra, but

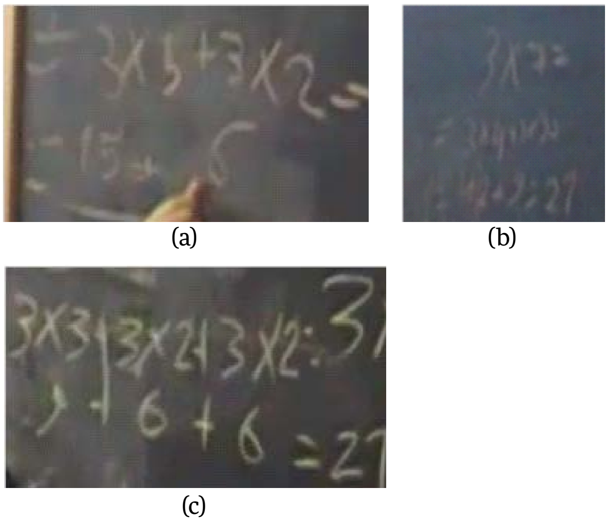


Figure 6: Arithmetical expressions are used to represent the proposed calculations

have lost all the graphical qualities shared with the artefact. The text now relies completely on mathematical culturally determined signs, its meaning is not explained but the direct linkage with previous signs, condensed in our Rosetta Stone, allows students to interpret it. When Mir receives a good feedback for his representation, other students ask to come to the blackboard to use the same representation (Figure 6b), among them, at the very end of the discussion, Lor comes to the blackboard and suggests another solution (Figure 6c).

After this discussion, students are asked to write individually a letter to answer to Giovanni. The majority of the given examples about the order in which Giovanni has to perform his calculation are expressed by arithmetical expressions, only three students just use a representation with rectangles. In many productions the two kinds of representation appear together (Figure 7), showing evidence of the process of appropriation of a relational meaning of arithmetical expressions and its relation with the use of the artefact.

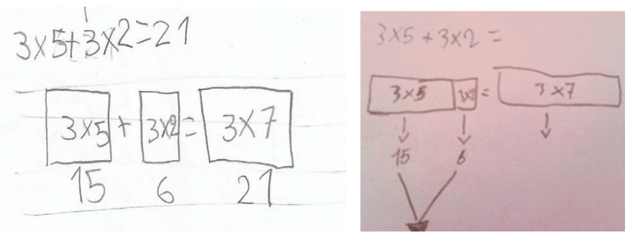


Figure 7: Sob's and Sim's combined usage of expressions and rectangles

DISCUSSION

The usage of artefacts to introduce the distributive property is a diffused approach: in some studies it is possible to find pre-constructed paths implemented in textbooks (Ding & Li, 2014) or in instructional materials (Izsák, 2004). However "how students might be supported to make transitions from concrete to abstract representations remains largely unknown" (Ding & Li, 2014). Our study shades light on such a complex process, showing the crucial role played by the semiotic dimension as it is modeled by the TSM. In our experiment we introduced an artefact with strong representative features based on the rectangular model for multiplication, with different tasks specifically designed for this experiment. The analysis of the data shows the expected process of semiotic mediation. The transition from artefact signs to arithmetical expressions is guided by the teacher and the researcher choosing tasks and orchestrating the mathematical discussions.

From a theoretical point of view, TSM gave suitable aids to design the activity with the artefact and provided useful analytical tools. In particular the distinction of different kind of signs gives many insights on the evolution of students' productions and on teacher's interventions. As clearly shown in the previous analysis, a sensible handling of the pivot signs in the collective discussion allows the teacher to foster the development from personal meanings to mathematical ones. It has to be noticed that many different kinds of signs go under the label "pivot signs". These signs are more or less related to the artefact or to mathematical symbols and they belong to different semiotic systems, sometimes used in parallel or generated one from the other, briefly they constitute a semiotic bundle (Arzarello, 2006). This construct grasps this semiotic richness that may explain the potential move from representing the combination of rectangles (tiles) towards representing the relationship between arithmetical expressions. Moreover, the use of semiotic bundles as pivot signs aimed at relating the activity with the rectangles to mathematical signs, was intentionally exploited by the teacher through the production of hybrid texts explicitly relating – as in a Rosetta Stone – the two different systems of signs.

It is also meaningful to remark how asking for individual productions after the first discussion allowed the teacher to observe emerging personal representations produced and shared by the children; these signs (specifically the introduction of the sign + between the rectangles), appropriated by the teacher, became fundamental to foster the evolution towards mathematical signs.

The proposed kind of tasks seems to be very promising for introducing the usage of expressions in very first grades. The selection of appropriate signs leaded to a quite natural introduction of this mathematical representation in a context in which it is useful and meaningful. Moreover, the need of conventions about the order of operations computing appeared as natural in this context and children showed a good structural control of the expressions. We have also to stress that, even if it was a minority, there were children who had difficulties in following this approach. It is needed further research and more cases to give more relevance to these findings.

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ENDNOTE

1. Actually, the word used by the researcher is "è uguale" which, in Italian, means both "equal" and "look the same as".

Investigations in magic squares: A case study with two eight-year-old girls

Bożena Maj-Tatsis[1] and Konstantinos Tatsis[2]

1 University of Rzeszów, Rzeszów, Poland, bmaj@ur.edu.pl

2 University of Ioannina, Ioannina, Greece

The paper presents the results of a case study of two eight-year-old girls working together on an activity involving – among others – two magic squares. During the activity we have observed the girls' participation in the task, which led them to the discovery of some properties of operations and, moreover, to mathematical reasoning. Additionally, there were differences in the way the two girls perceived the given activity at particular moments, a fact that may be related to their general views of mathematical activity.

Keywords: Mathematical reasoning, investigations, magic square.

INTRODUCTION

Mathematics teaching and learning is a process that begins from the early years of childhood and takes place in formal and informal settings. Children, even at a small age have access to powerful mathematical ideas, such as mathematization, connections, argumentation, number sense and mental computation, algebraic reasoning, spatial and geometric thinking, data and probability sense (Perry & Dockett, 2002). During most of their time, and especially during play, children are engaged in informal mathematical thinking, which may include reasoning and argumentation (English, 2004). Although there is a consensus on the importance of that informal mathematical knowledge and its contribution to the child's further development, the research on reasoning processes in informal settings is rather limited (e.g., Ginsburg, Inoue, & Seo, 1999). Having in mind these considerations, we designed a case study aiming to study the reasoning processes that will occur, together with the mathematical concepts that may evolve by engaging two girls in a series of mathematical tasks. Particularly, our research questions were the following:

- What aspects of mathematical reasoning can be observed during the particular activity?
- In what ways has the particular activity contributed in the girls' understanding of properties of mathematical operations?
- Which were the characteristics of the girls' participation in the activity?

THEORETICAL FRAMEWORK

The process of learning mathematics can be viewed by many different perspectives. Among them, there are those that focus on the child's activity while doing mathematics and comparing that activity with that of a mathematician. Ponte (2001) talks about "a parallel between the activity of the research mathematician and the activity of the pupil in the classroom" (p. 53).

One of the important activities of the students who are doing mathematics is the mathematical investigations, in which the students rather than solving a problem with clearly-framed questions, are faced with a situation in which the conditions might not be completely clear, thus they might have to search for regularities and relations or even formulate some questions by themselves (Ponte, 2001). During these processes it is highly probable that the students will use some mathematical reasoning in their work. Lannin, Ellis and Elliot (2011) connect mathematical reasoning with nine essential understandings. Among them we find developing conjectures, generalizing to identify commonalities, generalizing by application, investigating why, justifying based on already-understood ideas, and validating justifications. This framework has proved much helpful for the purpose of our research.

Mathematical activities like those described before can be also observed in students at the early stages of their education. In NCTM's (2000) *Principles and Standards of School Mathematics* in the "Reasoning and Proof Standard for Pre-K through Grade 2" we read that the ability for mathematical reasoning "develops when students are encouraged to make conjectures, are given time to search for evidence to prove or disprove them, and are expected to explain and justify their ideas" (p. 122). It is also a known fact that children do use mathematical notions in their informal everyday activities before they enter the formal school system (Ginsburg et al., 1999). English (2004) claims that children during their play are engaged in mathematical reasoning; moreover, from a researcher's perspective, there is an interest towards "the thinking behind children's mathematical responses" (p. 14).

The importance of children's reasoning processes lies in the fact that they are strong facilitators of their learning, even more than specific contents of mathematical knowledge (Perry & Dockett, 2002). But we have to stress here that none of the previous can be achieved without the help of the teacher who – among other actions – has to ask the right questions and choose the proper tasks. Ponte (2001) offers a detailed description of the expectations for the teacher in an investigation class. These vary from the careful selection and design of tasks to decisions concerning time management and class organisation. What is important, however, is that the tasks should be designed in such a way that conjecturing, justifying, generalising, etc. will come up naturally during the students' participation in the activity.

DESIGN OF THE STUDY AND METHODOLOGY

The design of our study was based on our theoretical framework; most of the tasks were taken by a textbook which is aimed to promote interest in mathematics (Lankiewicz, Sawicka, & Swoboda, 2012). Our choices were driven by the following assumptions: the problems should be accessible to a wide range of students on the basis of their prior knowledge; they must be solvable, or at least approachable, in more than one way and without the use of tricks; they should illustrate important mathematical ideas; they should serve as first steps towards mathematical explorations and be extensible and generalisable (Schoenfeld, 1994).

The students were given seven tasks in total and the session, which took place in one of the students' house, lasted for two hours. Both the authors of the paper were present in the session and both were known to the two girls. The first author, who will be referred to as Researcher in the transcripts, was the one who provided help and guidance to the students. Her roles were consistent with those described by Ponte (2001) and NCTM (2000) and can be summarised in the following:

a) propose challenging questions for the students;

b) support and evaluate students' progress by promoting a balanced participation in the activity;

c) think mathematically by asking new questions and by becoming involved in mathematical reasoning;

d) supply and recall information;

e) promote students' reflection.

The two students, Ania and Magda, were both eight years old at the time of the study, and they were fellow-students in the second grade of a public primary school in Rzeszow, Poland. They both had good marks

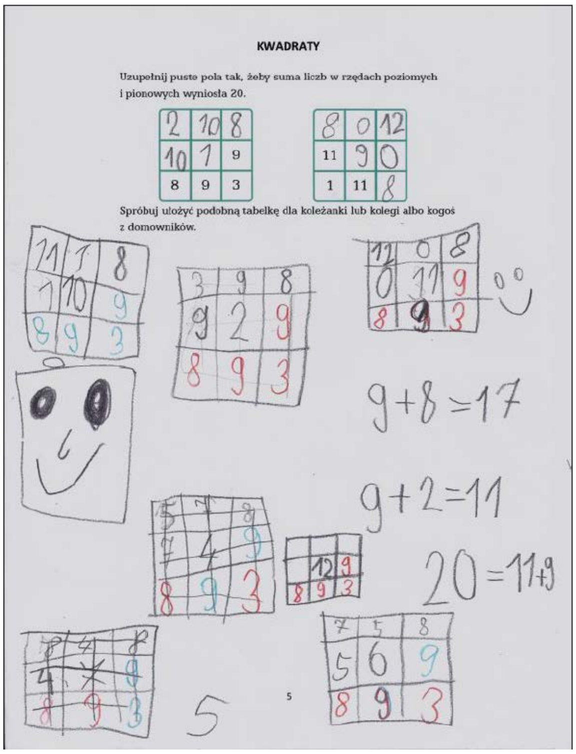

Figure 1: The worksheet containing the magic squares and part of solutions

in mathematics and they volunteered to participate in the research.

The analysis in the present paper focuses on a task related to a particular type of magic squares, which are partially filled, contain equal sums horizontally and vertically (but not diagonically) and the same number can appear more than once. This task was chosen because it fulfils the assumptions mentioned before and, particularly it is aimed to promote conjecturing and justification. Moreover, it has led to a rich discussion and engagement of our students. This was the fifth task in the row and Figure 1 shows the worksheet that was given to the students, together with some of the girls' solutions.

In the top of Figure 1 we read: "Complete the empty fields so that the sum of the numbers in rows and columns is 20". And then: "Try to create a similar table for your friend or somebody from your family".

The analysis of the girls' activity was done according to our research questions and was based on our theoretical underpinnings. Particularly, we firstly tried to locate any manifestations of Lannin and colleagues' (2011) essential understandings that are related to mathematical reasoning:

— developing conjectures,

— generalizing to identify commonalities,

— generalizing by application,

— conjecturing and generalizing using terms, symbols, and representations,

— investigating why,

— justifying based on already-understood ideas,

— refuting a statement as false,

— justifying and refuting the validity of arguments,

— validating justifications.

The analysis of the episode has shown that not all of the above were manifested in our students' interactions, which was somehow expected, since some of these understandings (e.g. generalising by the use of representations) were rather advanced for our eight-year-old participants.

Another useful analytic framework was Brandl's (2011) mathematical giftedness model, which consists of abilities specific to mathematics and general personality traits. The former include mathematical sensibility, memory, structuring, generalising and the reversion of mathematical processes. The latter include intellectual curiosity, willingness of exertion, joy in problem solving, perseverance and frustration tolerance. Although the model refers to mathematical giftedness, we have found it useful for characterising the participation of our students. Finally, throughout the discussion we have located the conjectures that are related to properties of the specific magic squares, as well as numbers and addition.

RESULTS

The task which is the focus of the study was the fifth in the row. For the purpose of the present paper we focus only on the first part of the task, which was to fill in the missing fields of the two magic squares. Mathematical activities of particular interest for our research are written in italics. Our notes are written in brackets. The discussion that follows took place few minutes after the worksheet was given to the students, since they needed some time to comprehend the task:

Ania: It will be here 12, for example 12 [the sum of 9 and 3 which are in the last column], 2 plus.... in order to be equal to 10, then 2+8. We have to put 8.

Magda: And here we can [put] 1 and add 10 [in the middle column].

Ania: [checking] Uhm. Yes. And here 8 [writing what she calculated before]. And here it would have to be 2 [in the top left].

Magda: and 10 [writing 10 in the last field, i.e. the middle in the first column].

Researcher: Is it ok?

Ania: Yes. 8+9+3 equals 20; 10+1+9 also equals 20; 2+10+8 also equals 20.

Researcher: Magda, how did you know that there [in the middle field] has to be 1?

Magda: Because I knew here and here I knew that it will be [showing the middle row and middle column]

Ania started with the last column in which only one number was missing; thus, she chose to begin with the easiest part of the task. Magda focused on the middle field; in this case the situation was open because both middle row and column had two empty spaces. Magda probably chose 1 because that was the number that added up to 10 (because of 9 in the bottom field). She *made a conjecture*, while Ania *validated* it. Both girls were engaged in solving the task and were *monitoring each other*. The researcher asked the question "why" in order to make Magda justify her choice. But she had difficulties in expressing her way of thinking. Up to that moment the girls were not aware that 1 in the middle field is not the only solution.

Thus later on, the question of the researcher "Can another number be in the middle?" surprised the students, since they thought that they had completed the first square. It created a cognitive conflict, since they were probably never faced a problem with more than one correct solution. It made them thinking for a while and the first answer of both of them was "no". After that, Ania had a second thought. She *made the conjecture* that in the middle you can put the number 2. But she quickly *refuted the hypothesis*: "If here would be 2, then not. Here has to be like that". She showed the other numbers in the middle column: 9 and 10 and she *concluded* that 2 does not fit to them. Because the completed square was misleading the students, the researcher asked them to draw another one – the same with the one given in the task. Then she repeated the question: "Can something else than 1 be in the middle?"

> Ania: No, because for example if here was 2...
> Researcher: yes...? [showing interest]
> Ania: then here it would equal 11 [with 9]... [thinking for a while] ... and here (the last column) you have to add 8 for sure. Here you have to add 8 for sure! [repeating and writing 8]
> Researcher: Yes..?
> Ania: And if here was 11, then 11, then you would have to add nine...? [unsure]

Ania was engaged in solving the problem. It was a real *challenge* for her and she demonstrated *intellectual curiosity*. She was not convinced that number 2 is not adequate although she had *rejected* it before. Therefore, she wanted once again to check if number 2 can be put in the middle of the square. This resulted in *discovering* that in the last column has to be 8 "for sure", which is later expressed in the sentence: "Here always has to be 8, because it can't be a different number". This is an expression of *generalization* accompanied by an *explanation* which is not justified. While Ania was trying to *investigate* the situation with number 2, Magda seemed to not be interested in the problem anymore. She proposed to move to the second magic square given in the task (see Figure 1). Ania firmly answered: "No, Magda. Wait, now we do that" which showed her *perseverance*. This happened few times during that task, which demonstrates Ania's *willingness of exertion*.

After filling the square the researcher wanted to engage Magda, so she asked "When are we sure that the square is correctly filled?". This provoked a *justification* by Magda: she drew lines on all columns and rows and *explained*: "When all squares [she means sums] will be correct. You have to make operations". After that the girls were convinced that number 2 can be put in the middle of the square. Ania also added: "But 1 also can be" by which she wanted to stress that there are two correct solutions. After that Magda proposed to check number 4. The solution made them enthusiastically state: "Here can be 4 as well!" Another discovery encouraged their further investigations: Magda noticed that she can use the previously filled squares to fill the next ones:

> Magda: [filling the square with 7 in the middle filed] It's so easy. Look [writing 4 and 8 at the top and showing the previous square with 4 in the middle; laughing]: From that. Because here everything is opposite!

By *identifying commonalities* she made a *conjecture* which later resulted in *generalizing by applying* it into another pair of squares (2 and 9 in the middle). The outcome inspired her to continue. Ania *developed* Magda's *observation* and "Everything is opposite" was elaborated to:

> Ania: ... every number is changing with something. (...) for example 7 with 4, 4 with 7 [showing squares with 4 and 7]; 9 with 2, 2 with 9 [showing the squares with 2 and 9] 1 with 10 [the first square with 1 in the middle], 6 with 5 [the square with 6 in the middle]".

The researcher moved their focus to the sum of these pairs of numbers and this led them to a common *discovery*:

Magda: 7+4 is 11 [square with 7 in the middle]. Here is also 11 [square with 4 – showing 7 and 4]
Ania: Every square 11, 11 here also 11. All it has 11.

The second cognitive conflict appeared when the students were discussing what numbers could be put in the middle of the square. The first ideas that "all" numbers can be put (Ania) and "even 100" (Magda) as a synonymous of a big number were quickly *rejected* and *reformulated* into "all up to 10!", which was clarified by Ania: "it means all one's [she means one-digit] numbers together with 10". To the researcher's question about number 11, both girls answered "no". The justification of Magda was that "If in the middle would be 11 and here 9, then it would already be equal to 20 [in the middle column] ... and here [empty field] we have one more. It seems that zero is not treated by the students as a number: you have to add something in the field, but there is nothing to add. But as soon as they realized that zero can be put in the empty field, they *applied* that understanding: "And 11 will be changing with 0!":

Researcher: So, can 12 be in the middle?
Ania: yes
Magda: yes
Ania: And 11 will be changing with 0!
Researcher: Uhm. And what will 12 be changing with?
Magda and Ania: hmm
Ania: 12? So I will do it a small one [drawing a new square]
Researcher: Ok, so quickly and then we will move to the next task
Ania: it's a pity. It's so nice that task... 12. But 12 cannot be because...
Researcher: Why 12 can't be?
Ania: Because ... because 12+9 is already 21!
Researcher: ok...
Ania: So 12 can't be
Researcher: Magda, can't it be?
Magda: No
Researcher: Hm. So what numbers can we put there in the middle?
Ania: up to 11.
Magda: from 1 to 11 [simultaneously]

Number 11 in the middle was immediately rejected, while with 12 they were more cautious. Only the thought about exchanging with another number made them remember the initial condition about sum of 20 which was used by Ania in her *argumentation*. The work at the first square was completed by the range of numbers that can be put in the middle in order to fill the whole square. The range was partially complete, since it did not include zero, although it was mentioned and used.

In the second square the students mainly *applied* their own *discoveries* (with some modifications according to the new conditions) and used the *argumentation* developed in the first one. It is interesting that Magda started filling the square by number 9 (which means that you have to add 0) which was the extreme case in the first square and it took them some time to accept it. Generally, they were working with a big enthusiasm and *complemented each other*. After filling the given square they did not draw any other squares because they expressed everything verbally:

Magda: Here for example 8 [in the middle] and add here 1 [above 8]
Researcher: Aha (...)
Researcher: And if here we put 7 [in the middle], then what will be here? [above]
Magda: 2
Researcher: And if 6?
Ania and Magda: [loudly] 3!
Researcher: How do you know that 3?
Ania: Because when it was 7 then it was 2. And if we decrease it more then it will be 3
Researcher: And if we put 5?
Magda: Then 4 (...)
Researcher: What's the biggest number we can put in the middle?
Ania: 20?
Researcher: Can we put 20 in the middle?
Magda: For me 9, because 10+11 it would be already 21.

We can notice that both students made a significant progress; they were more confident in *making conjectures* and *giving justifications*. Moreover, their discoveries gave them satisfaction which can be described as *joy in problem solving*.

Summing up, the particular activity had invoked many aspects of mathematical reasoning, which may be categorised according to our methodological framework into understandings related to mathematical reasoning (Lannin et al., 2011) and general personality traits which support mathematical activity (Brandl, 2011). The former may be further categorised into *interactive understandings*, which are directly oriented to the partner and *personal understandings* which have a more personal character (although they might be also directed to the partner or the researcher). Characteristic cases of interactive understandings in our study were the monitoring of each other and the development/elaboration of the other's observation; personal understandings included the development of conjectures and the generalization by application. During the episode that we analysed a number of conjectures were articulated, validated and eventually elaborated:

1) In the middle field you can put 1.

2) In the top right field always has to be 8.

3) In the middle field you can put 2 (in the sense of another solution for the square)

4) "Everything is opposite" – the addition of two numbers is commutative.

5) The sum of the two numbers in the upper fields of the middle column is constant and equal to 11.

6) 11+0 equals 11. Any number plus zero equals that number.

7) In the middle field of the first square you can put all numbers from 0 to 11.

8) The sum of the two numbers in the upper fields of the middle column is constant and equal to 9. Decreasing one number makes the second increase.

9) The biggest number we can put in the middle of the second square is 9.

It is obvious that some of them are related to properties of the specific character of our magic squares and some properties of addition: identity property, commutative property and associative property.

The personality traits that we located in our mathematical activity were: perseverance, willingness of exertion, intellectual curiosity and joy in problem solving. Moreover, we have noticed significant differences in our students' participation regarding not only their understandings of mathematical reasoning, but also the personality traits. Particularly, Ania was more willing to investigate the situation, developing new conjectures and perseverant. Magda, on the other hand, usually wanted to move on when she was faced with a novel situation; however, whenever she developed her own conjectures she experienced satisfaction. Summing up, we may say that Ania was better in proposing new ideas, while Magda was better in justifying their conjectures and monitoring Ania.

CONCLUSIONS

Our case study has provided us with interesting and valuable data that go in line with the relevant literature. We have observed that our students have been able to articulate sound mathematical conjectures, which supported their mathematical reasoning. Moreover, both students were engaged in the task and have demonstrated signs of mathematical sensibility and intellectual curiosity, but in a different degree. As we mentioned, Ania was better in proposing conjectures, while Magda was better in monitoring; thus, it seems that the students' roles although not identical, were somehow complementary to each other. This fact was very helpful for the "flow" of the activity and the outcome of their investigations. The researcher's interventions, mostly in the form of questions, were also vital for the girls' investigations, promoting their mathematical reasoning and fostering their reflection.

The magic squares have thus proved a useful tool for promoting our students' investigations and we believe that it can also be used for the discovery of properties of numbers and addition.

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“How do you make numbers?”: Rhythm and turn-taking when coordinating ear, eye and hand

David Pimm and Nathalie Sinclair

Simon Fraser University, Burnaby, Canada, djp3@sfu.ca

In this paper, we examine an environment involving a young girl, an adult and a touchscreen application (TouchCounts), in which engagement with number draws on all of the audible, the visible and the tangible. We broadly frame our analysis in terms of the conversation (both verbal and non-verbal) that occurs, seeking to gain insight into the nature of number – and in particular of its ordinal aspect – in this complex assemblage. We propose that much can be learned from our analysis about the nature of counting in more traditional environments, as well as about the particular forms to which TouchCounts may give rise.

Keywords: Kindergarten, touchscreen application, audible, visible, tangible.

INTRODUCTION

Over thirty years ago, Stephen Brown observed, “One incident with one child, seen in all its richness, frequently has more to convey to us than a thousand replications of an experiment conducted with hundreds of children” (1981, p. 11). Nowhere, perhaps, is this as true as in the arena of early number. Our starting point here is with such an incident, involving a five-year old girl – for argument’s sake, we’ll call her Katy. But the full encounter, which lasted just over an hour, also involved an adult, Nathalie (second author), some geometry tasks with pencil and paper, and, from about half-way through, an iPad with the numerical App *TouchCounts* (Sinclair & Jackiw, 2011), the pertinent aspects of which we describe briefly in this paper (see also Sinclair & Pimm 2014). We wish to explore the potential of framing this interaction as a *conversation* about number, albeit one involving an entanglement of the audible, the visible and the tangible.

We are also interested in examining how children’s number activities within this particular entanglement relate to findings from prior research on children’s development of number. Elsewhere, we have described how the design of this App supports the development of subitising (Sinclair & Pimm, in press) and finger gnosis (Sinclair & Pimm, 2014), in addition to offering more expected opportunities for children to work with cardinal aspects of number (Sinclair & Heyd-Metzuyanim, 2014). In this paper, we follow the suggestion of Coles (2014), who argues for the importance of ordinality in the early development of number, based on recent neuro-science findings, as well as on the work of Caleb Gattegno (1974). [1] Ordinals convey a sense of time and sequence, of ‘the next one to be named’ and ‘the one to be said after that’. Success with intransitive counting primarily involves being able every time to generate stably the same set of words in the same order. Transitive counting, which may be over-emphasised in early schooling (see Tahta, 1991), relies centrally upon intransitive counting and can actually be seen as a ‘mere’ application of it, a subordinate practice. Among other things, ‘ordinality’ refers to the capacity to place numbers in sequence: for example, that 4 comes before 5 and after 3 in the symbolised sequence of natural numbers, as well as in the parallel ordering of the number words. We here interpret attending to any kind of sequencing of numbers (not just counting by ones) to be attention to the ordinal.

In particular, Coles points to the need for research to explore the potential of an increased emphasis on ordinal aspects of number with children of primary school age, or younger. In current classrooms in Canada (and elsewhere), children are generally offered concrete resources and materials (such as Dienes Blocks, a move which Tahta, 1991, terms *metaphoric*). The neuro-science suggests such schoolwork on linking symbols to objects may reinforce the very way of thinking that underachieving students need to overcome in order to become successful at counting and arithmetic. Coles hypothesises that what these students need is support to work with number words and symbols in their relationship to other number words and symbols (which Tahta terms *metonymic*). Our research focus and central question involves understanding better the particular entanglement of the audible, the visible and the tangible plays out in young students' developing ordinal sense of number.

THEORETICAL FRAMING

In Jackiw and Sinclair (2010), user interaction with *The Geometer's Sketchpad* is discussed in terms of a pedagogic conversation, one in which it makes far greater sense to frame the student as the *teacher* rather than the software acting as one. The authors rhetorically ask, "If *Sketchpad* cannot speak, in what sense can it participate in discourse?", yet go on to claim, "there is a coherent and well-defined linguistic trajectory to users' interactions with *Sketchpad*, an explicit interplay and evolution of language [...]" (p. 159). The 'language' they discuss is all written and the user issues commands by means of verbal menu selection, mouse play and/or keyboard entries.

In the excerpt we discuss here, we wish to go further and examine the 'coherent and well-defined linguistic trajectory' we see in Katy's interactions with *TouchCounts* broadly inspired by the field of Conversation Analysis (see, e.g., Sacks, 1992), specifically focusing on sequential aspects of turn-taking, the core characteristic of naturally occurring conversation. The focused conversation (in the conventional sense) between Katy and Nathalie in the first part of the encounter proves Katy to be well-accustomed to taking turns and a respecter of that: indeed, there are almost no instances of overlapping speech. But when she 'converses' with *TouchCounts* on the iPad, she takes most of her turns with her index finger, changing things significantly.

This latter 'conversation' is about the spoken sequence of number names in English, as well as the visual numerals on the discs that are generated by her finger in the varying rhythms she adopts. Unlike with the *Sketchpad* setting mentioned above, there is no written language here: the only visible symbols (which are not part of the English language) are the numerals appearing on the successively generated discs. But the associative *rhythm* – parallel structure in the number words said, repetition in her finger touches, pulse in her attention – is so evident that it becomes for us one of the main phenomena of interest in this episode. Temporality and sequentiality lie at the heart of ordinal awareness. As Tahta (1989) claims, "Time becomes manifest [...] in the experience of rhythm and repetition. The medium for language, and so eventually for counting, is at first *sound*" (p. 20). Staats (2008) convincingly argues that, "repetition creates ideas that transcend the sentence", a phenomenon she links to Roman Jakobson's poetic function of language, one where "the form of the message calls attention to itself" (pp. 26–27). [2] While 'the number poem' [3] may be used as informally to refer to the sequence of number words, we doubt users of this expression are attending to Staats' assertion "Any time a repetition causes listeners to attend to the form of the statement, or to use the form of the statement to construct meanings, the poetic function of language is in play" (p. 28).

METHODS

The encounter at the heart of this paper occurred in an elementary school in a rural part of British Columbia, where the second author worked for about an hour with each of five kindergarten children, in a one-to-one setting separate from their customary classroom and teacher, which was videoed as part of the research project. The first part of each clinical interview focused on symmetry tasks, while the second part involved the use of *TouchCounts*, which was new for each student. Because this was the beginning of Katy's work with *TouchCounts*, it gave us an opportunity to see her first encounters with counting tasks on the iPad, seeing how she made sense of the App and how that interacts with her feel for number. Katy was the first student to be interviewed, as well as the youngest (having recently turned five). Although portions of this session have been analysed in a paper for the previous CERME (Sinclair & SedaghatJou, 2013; there is also more discussion of methods there), that paper focused on cardinal aspects of number. Here we offer a more ordinal focus and closer attention to the entanglement of the audible, visible and tangible.

Brief description of TouchCounts

We briefly describe only the Enumerating World of *TouchCounts*, which starts almost blank, except for a horizontal bar representing a shelf (Figure 1a). In this world, a user taps her fingers on the screen to summon numbered objects (yellow discs). The first tap (when initially turned on or after pressing Reset) produces a disc containing the numeral "1". Subsequent discrete taps produce sequentially numbered discs. As each tap summons a new numbered disc, *TouchCounts* audibly speaks the English word for its number. As long as the learner's finger remains in contact with the screen, it 'holds' the numbered object, but as soon as she "lets go" (by lifting her finger), the numbered object falls to and then "off" the bottom of the screen, as if captured by some virtual gravity. If the user taps and releases a numbered disc above the shelf, it falls only to the shelf, and comes to rest there, visibly and permanently on screen, rather than vanishing out of sight "below". (Thus, Figure 1b shows a situation in which there have been four taps below the shelf – these numbered objects are in the process falling – and then the "5" disc was placed above the shelf.)

Discs dropping away (under 'gravity') mirror the way spoken language fades rapidly over time, with no trace left – the impermanence of speech. Also, with discs disappearing, any sense of cardinality goes too: the disc labelled '2' is simply the second one to have been summoned (in the absence of the presence of the disc labelled '1'. So the Enumerating World with 'gravity' enabled (it is an option) is almost entirely an ordinal one. However, the shelf feature allows the

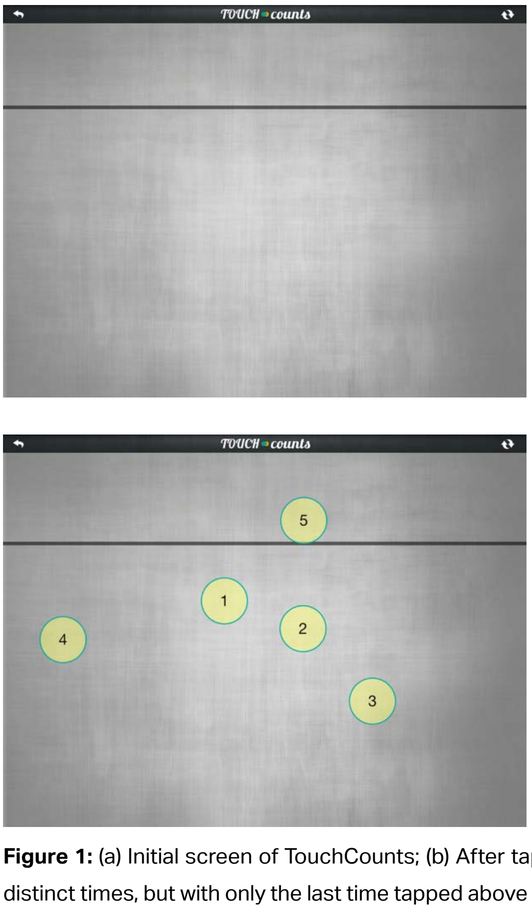

Figure 1: (a) Initial screen of TouchCounts; (b) After tapping five distinct times, but with only the last time tapped above the shelf

user to 'store' some objects for longer. Since a new numbered object is created each time a finger is placed on the screen and then lifted, one cannot 'catch' or reposition an existing numbered object by retapping or dragging it. Depositing discs on the shelf are as if they have been written down, inscribed on the screen as one might write on paper.

Fingers can be placed on the screen either one at a time or simultaneously. Thus, with five successive taps, a user sees five sequentially numbered objects appear one after another on the screen and hears these numbers announced by their English number name one by one. [4] However, if she places two fingers on the screen simultaneously, she sees two numbered objects appear simultaneously, but only hears the higher-numbered one explicitly named ("two," if these are the first two taps). Thus, repeatedly tapping two fingers on the screen produces the sequence of number names "two, four, six, eight, ...". This feature is barely drawn on in our excerpts.

Notably, *TouchCounts* 'takes care of the counting', both in terms of making sure that the sequence of numerals is given correctly on the screen and in terms of ensuring that the number names are said in the requisite order, in response to the fingered requests of the user.

DATA, ANALYSIS AND RESULTS

In engaging with the data, we looked for interaction in conversation (including the iPad as converser) as well as elements of rhythm and repetition. The ten-minute sequence from which the excerpts discussed in this paper have been taken begins:

Nathalie: Let's do some, just some numbers first, OK?
Katy: Numbers.
Nathalie: Yeah.
Katy: OK.

From strokes to pops

Without prompting, Katy's hand then approaches the screen and the index finger of her right hand touches the top of it and then slides down to the bottom (we call this a 'stroke'). A yellow disc appears under her finger with the numeral '1' on it and at the same time "one" is spoken by the iPad. Her finger moves back to the top of the screen, before touching again and slowly stroking downwards. The iPad says "two", followed by

Katy's echo. This happens repeatedly for three, four and five, while for six, seven and eight she says the number simultaneously with the device (Figure 2a).

She says nothing for nine (and it subsequently becomes clear she is not always sure nine follows eight). The appearance of the disc bearing '10' attracts her attention, perhaps because of its double digits. Katy bends over to look closely at the screen and she then says "ten" *after* the iPad and with a rising intonation (Figure 2b). She keeps her head down while continuing to make subsequent numbers at a good speed, but now only the iPad recites them: she is so quick that the iPad only gets to say the 'four-' of "fourteen" before "fifteen" comes. After "seventeen", several fingers fall on the screen at once, and then 'twenty-one' is heard. This produces a pause in the action and Katy's lips spread into a smile. All her other fingers are tucked away, as the rhythmic stroking continues along with the chorus of named numbers, which Katy begins to repeat with the iPad at "twenty-three" (only she consistently says "tenny" for "twenty"). At "twenty-seven", Katy looks up, no longer watching the screen (see Figure 2c), while she continues stroking and saying numbers.

Even in this first small episode, some transcription issues are evident, mostly concerning the coordination of ear, eye and hand. Each of the elements is a material event occurring in time in the world. Consequently, each of them occurs in time and takes a certain length of time and can occur in conjunction with the others. To transcribe them there would need to be a 'channel' (or 'register', though not in the technical, linguistic sense) for each, arranged horizontally (like a musical score for several instruments), where time is calibrated across them, but even then it would be difficult to show shifting subordination of one to the other (as occurs when Katy looks nonchalantly up while continuing to swipe). What we have had to attempt here is a narrative retelling, making observations about our linked phenomena of interest.

The *TouchCounts* timings are predictable, in that the numbered disc appears virtually instantly when the finger is lifted from the screen, while each number word takes a certain length of time to say. The words above twelve are generated syllabically, e.g. fif-teen and fif-ty, while "twenty-three" uses the same "three" as "three". There is an appreciable gap between, for instance, the "twenty" and the "seven" in the saying of "twenty-seven", a material fact that Katy makes good use of later by inserting her finger stabs to prevent the second part of each number word being said aloud.

Nathalie asks her to tap above the shelf, which she does, swiping her finger down to make 'thirty-one'. After 'thirty-two', Nathalie asks Katy to let go of the disc above the shelf (not *on* the shelf), which Katy does, noting that "it stops the number". When she taps again (above the shelf) she says "pop", and then taps several times again. At 'thirty-nine', she starts placing the discs on the shelf side by side, going from the left to the right, until she gets to 'forty-six', at which point she returns to the left edge of the line, makes 'forty-seven', looks up at Nathalie and puts her hands on her lap. Nathalie asks Katy to reset. Instead of hitting the reset button, she creates numbers in quick succession, so that the iPad says "forty-eight, forty, fifty, fifty, fifty".

Nathalie asks Katy to put "just five" on the shelf. Katy tries several times, without success, then puts her head down and taps intently on the screen below the shelf, saying the number names aloud with the iPad (one, two, three, four), then moving her finger up to place five on the shelf. Nathalie asks Katy to put "five and ten up here". On her first try, she places five below

Figure 2: (a) Katy stroking; (b) Attending to the disc; (c) Stroking while looking up

the shelf, so has to begin again. On her next try, she very quickly taps four times below the shelf and once above. She then puts 'eight' above the shelf and realises her mistake, but continues tapping quickly on the screen, so that the iPad says "twenty, twenty, twenty, twenty, thirty, thirty, thirty, thirty, thirty, thirty, thirty, thirty, thirty, thirty-nine, forty, forty, forty, forty, forty, forty, forty, forty, forty, forty, fifty, fifty, fifty, fifty, fifty, fifty-five". Katy smiles. We conjecture reassurance can derive from the regularity of repetition and rhythm.

The language of 'friends'

In the moment, Nathalie spontaneously tries to 'humanise' the mathematical task she wants Katy to attempt, namely to produce all the multiples of five alone on the shelf. She does this by saying, "Imagine five and ten are your best friends and they're the only ones you want to come over to your house (*points to the region above the line*). So you just want five and ten and not the other people in your class." The language of friends and home and the translation of what is happening on the screen into these terms, however, is precisely the sort of metaphoric shift that Tahta suggested can take attention away from the ordinal, away from links between the number words and symbols themselves. Additionally, this move transfers the 'I want' framing of the task into what 'you [Katy] want'. After a few tries, Katy succeeds in placing just five on the shelf (Nathalie: "Five is your friend."), then taps below the screen to make six.

When Katy spontaneously asks, "What kind of number is going to come after?" (there had been no talk about 'kinds of number' prior to this), we assume she is asking 'friend or not friend', but it could also signal that, at the moment at least, she is unsure that "seven" follows "six". Also, following the placing of "eight", Katy asks "Is nine going to come after?". She eventually places 5 and 10 above the shelf successfully. She stops. Then Nathalie asks her which other numbers are her friends. She mentions several (seven, one, two, "all of the numbers") and then starts tapping, making eleven, twelve, thirteen, four, fif, six, twenty, twenty, twenty, twenty-five.

From "pop" to "drop"

She starts at one again above the shelf, puts two below the shelf saying "drop", then places three above the shelf and four below (repeating "drop") and continues going over and under the shelf until she gets to seventeen above, pauses then continues until she reaches twenty-two. She then returns to the left side of the shelf, having filled the shelf with (mostly) odd numbers and continues to tap-drop, seemingly more focused on the rhythmic motion of her hand than on the specific value of the number names. She starts going very quickly, saying "drop" again so that only "forty, forty, forty, forty..." can be heard from the iPad, then "fifty, fifty, fifty...", then "sixty, sixty, sixty...". Katy says, "I'm doing a pattern", but does not go into any more detail, alas, as to which pattern she is seeing. She continues in this way into the seventies, the eighties and the nineties, until the iPad crashes. After this, Nathalie shows Katy how to use many fingers at a time to make "friends". She gets to two hundred and five, says "I don't want no friends" and presses reset while smiling. She then says she knows how many friends she wants, and makes one, two, three, four, five, six successively with her finger, then stops, saying, "That's how many friends I want."

DISCUSSION

Rhythm is the essence of counting, its heartbeat. It is there from the beginning, but changes over time. Katy's first rhythms are the slow strokings of the screen with her index finger, which she watches swim down the screen, as if each numbered disc were worthy of her attention. Here, she seems to be in conversation with the motion of the discs, more so than with the iPad's oral naming. Indeed, she notices the shift from single to double-digit numerals at ten (commenting "a one and an o"). Then tactile rhythm turns into taps, a new gesture that seems less interested in the individual number than in the succession – and this is where Katy starts chiming in with the voice of the iPad, no longer even needing to see what she is making happen. With the task of placing five on the shelf, a new rhythm develops over time, which is the four quick taps below the shelf and then a fifth above, this perhaps giving rise to the rhythmic 'above then below' alternation with which she will later play.

Rhythm is also there in the structure of the number names. It is there in Katy's large, alternating rhythmic gestures, gestures that are binary. The odd and the even, we might think: it is there in her gestures, but we hear none of it in what is said, merely in the pattern of the 'pops and drops', to use Katy's words. Were she attending to the numerals on the discs that were alternately produced above and below the shelf, she might have seen something recurring in the digits' place, attending to what changed, but changed regularly (but there would have been ‘noise’ here, as she occasionally places her finger on top of an already existing disc, which then does not make a new one). Had she allowed the iPad to say every number fully, she might have heard something similar: the same stem, first by itself, then followed by the number words one to nine, before the next stem changed, a decimal rhythm underlying a repeating refrain. But no, instead she edited [5] the conversation by means of her own interruptions, only allowing the iPad room to say the same thing each time: sixty, sixty, sixty, ... (ten times), “seventy” (ten times), “eighty” (ten times), then “ninety”.

Gattegno (1970) claimed, “To stress and ignore *is* the power of abstraction that we as children use all the time, spontaneously” (p. 12; *italics in original*). This episode with Katy illustrates his claim consummately: she consistently *stresses* the common decade stem and consistently *ignores* the variation that follows (by editing it out of the conversation, by making in vanish, aurally) and does so across more than sixty numbers one after another. She manages to do this by complex finger movements deployed in a highly rhythmic manner. Hers is no small achievement. In relation to Staats’ comment about language drawing attention to itself, we assert that Katy’s dextrous manipulation of her side of the ‘conversation’ does precisely that (namely drawing attention to the repetitive pattern of the number words in the decades).

CONCLUSION: THE TRIPTYCH OF THE SENSES

Deleuze (1981/2003), at the end of his book on the art of Francis Bacon, offers a brief discussion of the interaction between the hand and the eye, as well as degrees of subordination of one to the other. Deleuze distinguishes four ‘values of the hand’, which he terms the *digital*, the *tactile*, the *manual* and the *haptic*, teased out first along the degree of subordination of the hand to the eye (*digital* more subordinate than *tactile*, where the hand is reduced to a finger). With the *manual*, the direction of subordination is reversed (Deleuze writes of “the insubordination of the hand”, p. 155) and with the *haptic*, the link between eye and hand are relatively severed.

But it is not so much in the detail of these distinctions that we wish to dwell. Rather, it is that here, with *TouchCounts* present, we are looking at dynamic interactions *among* hand, eye and ear. And we are concerned with numbering not painting. But Deleuze’s idea of mutual entanglement and relative subordination among senses and their organs remains. We are interested in the particular way that this specific entanglement is shaped, which is distinct from other counting environments, and we can extend Deleuze’s concern with the hand and the eye to include the ear also. The lifting of Katy’s eyes while still making numbers provides a clear instance of a subordination of the eye to the hand and ear, one that is easy to observe (whereas instances when the ear is or is not attending are less apparent).

Nowhere in this episode does Katy seem to be counting objects, but rather is simply counting [6]: the ‘number poem’ is being made by her index finger – *TouchCounts* answer to Katy’s question as cited in our title. This one-to-one correspondence, in which *TouchCounts* takes care of saying the number names in the correct order, releases Katy from having to worry about *what*, if anything, is being counted, allowing her to dwell in the rhythmic succession that *is* counting. Later, the eye and the hand are both subordinated to the ear (in the same way that Hewitt, 1996, speaks of subordinating and coordinating actions and attention to achieve a central goal), as Katy achieves the (presumably desired) repetitive aural effect of the exaggerated looping back and forth tapping of her hand.

To return to Coles’ work and renewed attention to ordinality, not only does number involve coordination of the audible, the tangible and visible, as illustrated in this episode, but *TouchCounts* affords the stressing and ignoring of each of these aspects in various ways. In particular, it allows the subordination of the aural to the tactile – with the IPad ceding conversational ground to her with every tactile interruption – with the possible exception of when it ends the conversation abruptly, by crashing just prior to the unspoken “ninety-three”.

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ENDNOTES

1. Such ideas are also present in Walkerdine (1988) and, particularly, Tahta (1991).

2. Jakobson's poetic function relates to his metaphoric and metonymic 'axes', as drawn on in Tahta (1991) by the latter's use of these terms, as mentioned above.

3. Sometimes, 'the number poem' refers to rhymes like 'One two, buckle my shoe' or 'One, two, three, four five, once I caught a fish alive', that incorporate the number words in order as part of the text, while sometimes this expression simply refers to the number words being recited in order by themselves (see also Tahta, 1991).

4. Other language options are available with *TouchCounts*, including French and Italian.

5. For more on editing, see Hewitt (1997), although he sees it as a teacher function to affect student attention.

6. In *Symbols and meanings in school mathematics*, Pimm (1995, pp. 64–66) distinguishes between *transitive* and *intransitive* counting – the former connecting the counting of things to the fact that the verb 'to count' can take a direct object (an answer to the question 'What are you counting?'), while the latter label refers to counting (reciting the number words in order) where there is no such direct object (where the answer to the same question is "Nothing").

Learning interventions supporting numerosity in three year old children

Reinert A. Rinvold and Hans Erstad

Hedmark University College, Elverum, Norway, reinert.rinvold@hihm.no

This paper reports from a preliminary qualitative case study and accompanying theoretical considerations in preparation for a study of interventions designed to support the learning of numerosity in three year old children in a Norwegian kindergarten. Teaching interventions are analysed and discussed both from the perspective of children's learning and the teacher's mediation. Luis Radford's theory of objectification is used as theoretical framework. This is a new approach, as most research papers in the field use quantitative methods and a cognitive approach. Our study indicates that preschool teachers face challenges in taking the perspective of the children they teach, and that knowledge of children's learning phases and assessment of children is needed to improve teaching.

Keywords: Numerosity, objectification, give-N task, parallel individuation system.

INTRODUCTION

Already at the age of about two children begin to learn the sequence of counting words, but further development takes time. More than a year is often needed from when children can count small sets of objects properly until children respond correctly to questions involving the use of numbers (see Sarnecka & Carey, 2008, p. 664). We hypothesize that this time can be reduced by planned learning interventions conducted by preschool teachers or other competent persons. In a preliminary qualitative case study over a period of three months we followed learning interventions in a Norwegian kindergarten conducted by a preschool teacher who was supervised by two researchers. Twelve children took part more or less throughout the study. We emphasised activities which use and give meaning to counting and number, but also the learning of counting skills. The interventions in the preliminary study started up with low quality. During the study we observed improved instruction and many signs of learning. Interventions of mixed quality proved much better than no interventions at all. A follow up study is planned and will build on the findings from the preliminary study.

Norwegian children can attend kindergarten from the age of one until they start school the year they turn six. As much as 90 % of Norwegian children between one and five years old, attended kindergarten in 2013. As remarked by Hundeland, Erfjord and Carlsen (2013), "the Norwegian kindergarten is regarded by OECD (2006) as situated in a social pedagogy tradition, i.e. an educational institution where core enterprises are upbringing, care, play and learning". Norwegian preschool teachers usually do not support children's learning of counting and number through planned and systematic interventions. Now and then they initiate simple counting activities, and sometimes they initiate conversations about numbers with children based on free play situations. The preschool teacher in our study does not have mathematics as a subject in his education. However, since 1995 mathematics has been compulsory in preschool teacher education in Norway.

From the adult expert perspective, numerosity is the same as the cardinality of sets, but we use the former term to underline the very different perspective of the child and the educated adult. The learning of numerosity means to catch the meaning and cultural use of number. This is complex and diverse, starting early in children's lives. In the research literature, the understanding of the numerosity N has been operationalized by the so called give-N test (Wynn, 1990; 1992; Sarnecka & Carey, 2008). In give-4 the child is asked to retrieve four objects, for instance plates, from a heap or a larger set. Research has shown the give-N tasks to be far from obvious for the child. Wynn (1992) found a progressive development in children from first managing give-1, then give-2 and so on up to give-5 or give-6. This has been confirmed by several others (see Sarnecka & Carey, 2008). When give-5 or give-6 is mastered, children seem to respond correctly to give-N, for N as large as the child is able to count. We will analyse excerpts of interaction of such challenges both for the children and for the preschool teacher trying to support them. Finally, possible improvements in the design of interventions will be discussed.

RADFORD'S THEORY OF KNOWLEDGE OBJECTIFICATION

The research literature on young children's acquisition of number has been dominated by psychologists, cognitive scientists and neuroscientists. Their perspective is cognitive with emphasis on understanding, mental models and what happens in the brain, and they use quantitative methods. The give-N activity has been seen as a test of "understanding the cardinal principle" or being a cardinal principle knower, Sarnecka and Negen (2012). The cardinal principle was introduced by Gelman and Gallistel (1978), and states that the last counting word in a correct applied counting procedure, is the cardinality of the set. The latter refers to a concept as possessed by mature adults. We claim that the child does not experience concepts, but activities which gradually become mastered and turned into procedures. To possess numerosity as a concept comes much later in the child's development. In place of speaking about understanding, the noticing of regularities in and connections among the activities is emphasized. This approach is based on Luis Radford's variant of cultural historical activity theory. In his theoretical framework, mathematical objects exist in the culture among us. According to Radford (2008, p. 222), "[...] mathematical objects are fixed patterns of reflexive human activity incrusted in the ever-changing world of social practice mediated by artifacts". Both counting and give-N activities are examples of cultural activities. Objectification is the crucial moments of the child's learning when the child notices important aspects hidden in cultural activities.

> The term objectification has its ancestor in the word *object*, whose origin derives from the Latin verb *obiectare*, meaning" to throw something in the way, to throw before". The suffix – *tification* comes from the verb *facere* meaning "to do" or "to make", so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something apparent – e.g. a certain aspect of a concrete object, like its colour, its size or a general mathematical property. (Radford, 2006, p. 6)

The child does not notice an abstract property, but a pattern or regularity in a cultural activity involving language, physical actions and artefacts. An example is the give-3 activity where a child is asked to fetch three plates from the shelf. A child can take part in such activities before being able to do it alone. During the learning process what children have objectified can vary significantly. Following Radford (2005), we use the concept 'layers of objectification' to classify alternative objectifications of the same activity. The original use by Radford was to study how students in school objectify the generality of geometrical number patterns. Later on Lorange and Rinvold (2014) have applied the theory to the study of students' strategies in expanding fractions to a common denominator. One layer of objectification of give-3 is when a child is using 1–1 matching to take one plate for each member of his family. A more advanced layer of objectification for the same activity is to remember the number word three and take one plate for each of the three first counting words. Different layers correspond to different strategies for doing the task depending on the regularities, skills and semiotic tools the child has grasped. Counting aloud, silently or using their fingers, are variants which give rise to more finely grained layers of counting. Layers are not necessarily linearly ordered. We have seen the same child demonstrate different layers of objectification of counting within one learning session. An example is perfect counting of six toy cars and recital of the counting words one-two-three when asked how many noses the boy has. Fragile and uncertain mastery and understanding in a learning phase is one possible explanation, but also social uncertainty in responding to demanding tasks together with other children.

Children who master the give-N task for all N, or in practice 5 or 6, are called cardinal principle knowers or CP-knowers. According to Sarnecka and Negen (2012), recent empirical studies have found that CP-knowers show an implicit understanding of succession and equinumerosity. The competency of succession is described as knowing that adding one item to a set, means moving one word forward in the counting sequence. Sarnecka and Carey (2008) found that CP-knowers performed well on two tasks intended to operationalize this competency, and that the non-CP-knowers performed only slightly better than chance. Sarnecka and Wright (2013) came to a similar conclusion for a task intended to measure the competency of equinumerosity. The latter means that two sets with a 1–1 correspondence must be labeled with the same number word. These results indicate that objectification of give-N is an important milestone in the learning of numerosity.

THE PARALLEL INDIVIDUATION SYSTEM

Children younger than one year have been shown to be sensitive to small numerosities. This ability does not, however, mean that the children posit the concepts of the numbers one, two and three. Visual discrimination does not by itself gives understanding of language. *Verbal subitizing* (Benoit, Lehalle, & Jouen, 2004) in which a child immediately gives a number word when shown a set of one, two or three concrete objects has to be learnt. This learning process does not have to be difficult, but is complicated by the fact that many children meet the number words only in counting and not in number talk. Some children may never have been shown for instance three dolls and told that this is three dolls. Subitizing is believed to be based on what is called the parallel individuation system in the brain. According to Sarnecka and Negen (2012, pp. 247–248), this system represents and tracks individuals (objects, sounds, or events). Concrete objects and their properties are represented, so the parallel individuation system is not an abstract number system.

> This system privileges spatiotemporal information to initiate a mental index, or object file, for each item. Although inherently non-numerical in nature, these representations afford numerical content by retaining information about numerical identity – mentally stored items can be compared on a one-to-one basis with visible objects in the scene to detect numerical matches or mismatches. (Hyde, 2011)

Learning the colour red is mediated by pointing to red things and saying for instance “this apple is red”. We expect that verbal subitizing can be learnt similarly, but that generalization is somewhat more demanding than for colours. The parallel individuation system does not let the child see twoness, but the mental image of for instance two dolls. The possibility of comparing a given set of concrete objects to some standard visual representation makes generalization possible. Mental images also let the child solve some matching tasks without numerical thinking.

METHODOLOGY

In the preliminary case study we followed twelve children in a Norwegian kindergarten for a period of three months. The kindergarten was chosen because it is located in reasonable distance from the university college where the researchers work, and has a preschool teacher interested in cooperating in this project. The participants were all the children between three and five years with no known learning difficulties who wanted to take part and whose parent granted permission. At the start of the study the ages varies from 3 years 2 months to 5 years 7 months. The children took part in weekly intervention sessions of 10 to 20 minutes duration. In the first part of the study as many as seven children took part in each session. Later on some sessions had only three or even two children. All sessions were videotaped and observed by at least one of the researchers. An experienced male preschool teacher conducted all the sessions. Sometimes an assistant was present too, whose contribution was limited. Before each session the researchers gave the preschool teacher an outline of what to do, but he had quite a bit freedom and was not given precise instructions. The content of the sessions were different kinds of counting tasks and activities. Most often one child at a time completed a task or answered a question. In the paper we study samples with the four youngest children, two boys and two girls. They were Siri: 3 years 2 months, Nina: 3 years 6 months, Ole: 3 years 7 months, and John: 3 years 10 months. No testing or systematic assessment of the children was done, but all of them had some previous experiences with counting. Already in the first session we noted that Ole and John were relatively poor in counting.

ANALYSIS OF LEARNING INTERVENTIONS

We present selected excerpts from three sessions, focussing on one task from each of them. Each excerpt is a dialog between the teacher and one child. Information about movements, gestures and other relevant facts is put in square brackets. The tasks are analysed from both teacher’s and child’s perspective.

The pure give-N task

In this task a piece of paper with a numeral printed on it is laid on the floor. One of the children is asked to say which number it is, and then the teacher repeats the number. One child is asked to collect that number of plastic bears from a bucket and then place the bears beside the numeral. We follow the first give-N task given to Siri.

Teacher: It's the number four. Can you fetch that many bears?
Siri: [Collects two bears without any signs of counting and puts them on the floor]
Teacher: How many bears have you put on the floor? [He moves one of them a bit and makes counting gestures towards each bear] How many is this?
Siri: [after some silence] Two [without visible or audible counting]
Teacher: one – two [while pointing to each of the bears]
Siri: two – three

The teacher thinks that counting will help the child, but Siri resorts to a more primitive objectification layer of counting. When the teacher says one – two, she turns back to the practice of reciting the number words. The teacher emphasises counting, and he probably expected her to count aloud before answering "two". But, her answer was satisfactory. Both verbal subitizing and counting internally are legitimate strategies. He could have challenged her better by taking away the bears and giving her a give-2 task.

The collecting plates to the bears task

One week later Nina is part of the group in place of Siri. Now the children are given a situation which is modelling an everyday activity. This enables them to rely on information and strategies they already know.

Teacher: In one of the houses live a mammy bear, a daddy bear and a tiny baby bear. [moving the mammy bear] Now you must come and eat. Can you Nina come and fetch a plate to each of the green bears. The green bears must have one plate each. [while holding a transparent plastic bucket with small circular plastic pieces]
Nina: [Takes four plastic pieces from the bucket, one at a time, without looking at the bears. She loses one piece on the floor, begins to retrieve it, but stops.]
Teacher: They must have one plate each.
Nina: [She lays down one plate for each bear] porridge [takes the last plate and gives it to a bear belonging to another family of bears]

Nina is not looking at the bears while collecting plates, so she either takes them randomly or relies on some other information. The former is less probable, since she is taking time collecting the plates. The individuality of the bears can be remembered by the names mammy, daddy and baby or their medium, large and small sizes. This is within the limit of the parallel individuation system, so 1–1 matching based on memory is possible. That she takes one plate too many, may indicate that she relies on numerical information, but has not developed a firm grasp of numerosity.

As number words are not mentioned, this is not really a give-N task. It could have been so if the teacher had introduced number words, for instance by asking the child how many bears there are in the family and how many plates are needed. Later on in the same session Ole gets a give-2 task in which the number word 'two' is used both before and after the task.

Teacher (T): The bear family got visitors. [He places two bears beside the green family] The mammy bear asked if the two visitors also wanted porridge. Can you put on plates to them Ole?
Ole and T: [Ole takes some plastic pieces from the bucket. Simultaneously, the teacher repeats "place plates to the two, two".]
Ole: [Ole looks at the bears and returns some plates to the bucket until he has two. Then he walks to the bears, takes one plate in each hand and places one plate in front of each bear.]
Teacher: Excellent. There they got two plates.

That Ole first collects several plates in the bucket, may be because the visual presence of five bears overrides the number word 'two'. The teacher's repetition of the question initiates a change to collection by visual 1–1 matching. One week previously Ole was successful with give-2 tasks without direct visual support. The available visual resources lead Ole into a more primitive layer of objectification of give-2.

In the same session, John, the mathematically weakest child, gets a give-4 task.

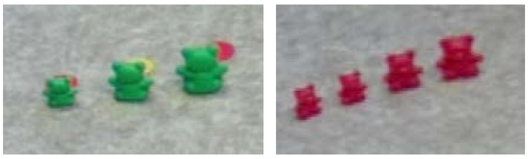

Figure 1: The green and red bear families

Teacher: John, the red ones must also have plates. [points to each of the plates in front of the green bears]

John: No. [takes three plates one by one from the bucket, staring into the bucket, places one plate carefully in front of three of the bears]

Teacher: How many have you there? Can you count them?

John: [points twice to the plate in front of the first bear]

Teacher: You must count them. [Counts to three aloud while both he and John point to each of the three plates] How many more plates do you need then? One [while John is taking one plate from the bucket]

John: [Places the plate in front of the fourth bear, then looks at the teacher]

Teacher: Good!

Since John is weak in counting and numerosity, it is unlikely that he relies on numerical information when taking out three plates. The sizes of the red bears (Figure 1) indicate that the family has two adults and two children, but this information does not seem to be used. A likely explanation is that the parallel individuation system is utilized, but that the capacity limit of three makes John unable to take four plates at first. The second time he correctly takes one plate, but this does not necessarily mean that John is a 1-knower. The teacher's utterance 'one' comes while John already is about to take one plate, so it is likely that John's action is based on spatiotemporal memory.

The garage task

In a session more than one month later the teacher has placed a heap of cars on the floor beside a garage. Ole, John and Siri are sitting on chairs placed so that they see both the garage and the heap of cars. The garage task is a challenging give-N task giving less support and meaning. Parking places and cars are part of daily life, and children enjoy playing with toy cars. However, it is not natural at all to park a given number of cars, randomly chosen from a larger set. The children's interest in toys cars can also be a distracter. The attention of some children was lead toward the cars themselves rather on the numerical information they were given.

Teacher: Then I first want that John takes two cars [taps three times with his hand on top of the garage] and puts them on top here. Can you take two cars and put them on top of the car house?

John: [gets up, walks to the heap and almost without hesitation takes three cars one by one into his left hand. After a very short break he takes another car and holds the cars to his chest. Then he continues and takes a last car. Then he takes a long break while looking at the last car.]

Teacher: Two cars do we want up there [taps three times with his hand on top of the garage] two cars.

John: [rises and walks to the garage, then places all the five cars one by one carefully on top of the garage, then turns in the direction of his chair.]

Teacher: [gestures John to return] Can you count how many cars there are here? Can you count them together with me? [Takes John's finger] one – two – three – four – five [while he moves John's finger to each of the cars]

John: [starts moving towards his place]

Teacher: Wait a moment. How many should you have? Two cars

John: [Begins to take up one car from the heap.]

Teacher: [Stops him] We must take away some cars. [Takes away three cars] Can you count now? How many cars are placed on top of the roof now?

John: one – two [while pointing to the cars one by one. Then looks at the teacher]

Teacher: Two, yes. It was this you should have. Exactly, two cars

Possible explanations for John's inappropriate response to this give-2 task are that he ignores the word 'two', or that he interprets it as 'many'. He may also take the cars he liked, or he takes into account how much space is available on top of the garage. The teacher's question of counting the cars is intended to draw attention to the numerical aspects of the task. John, however, again tries to return to his place, so this does not seem to be of any help. The next action by the teacher is to remind John that it is two cars he should have. This initiates John to start a new give-2 attempt by picking up a car. Unfortunately that attempt is refused by the teacher before we know what John would have done. John seems to have objectified give-N as a procedure for selecting some objects and putting them in a given place. If John had not been stopped, he may have taken up two cars and been rejected again because five cars already are on top of the garage. The teacher's strategy of reducing a larger set into a set of two members is far above the level of the child, and shows that he is not able to take the perspective of the child.

DISCUSSION

The most striking observation from the selected excerpts is the challenges the preschool teacher meets in mediating the children's learning. The same is true also for the rest of the videotaped interventions. When he tries to meet their needs, his response is often irrelevant and leads the children into more primitive layers of objectification. The level of tasks is not adapted to what each child has managed earlier. In order to help children struggling with the most basic tasks, the teacher should systematically remember or make written assessments of what each child has achieved so far regarding counting and give-N. More basically, the teacher needs to be aware that the child experiences cultural procedures and does not possess conceptual knowledge of numbers.

One conclusion is that give-N tasks are of different kinds and of varying difficulties. In the most demanding give-N tasks numerosity is given by a number word, and the child has to rely on that information in order to carry out the task. For the most demanding give-2 and give-3 tasks, verbal subitizing is a possibility if mastered by the child. In the easier tasks, the objects are visible to the child or have names supporting 1–1 matching while the child collects the objects. In the follow up study we plan first to give a child the easy type, but will also challenge the child with pure give-2 and give-3 tasks as preparation for the mastery of give-4. While working with the easy tasks, the teacher has to support the connections to audible and visible counting, but also give praise and appreciation to children who can tell the number of objects in sets by internal processes. Both internal counting and verbal subitizing are competences which are valuable in the learning of numerosity. Beyond the limits of the parallel individuation system, counting is indispensable. This is in accordance with Sarnecka and Negen (2012, p. 252), who claim that it is not possible to learn the meaning of large numbers in the same way as the small numbers have been learned. Similarly, Sarnecka and Carey (2008, p. 664) report that children who only master give-1, give-2 or give-3 (subset-knowers), do not use counting when solving these tasks, even when asked to do so.

In a follow up study we will take into account both the findings from this paper and other results which were not included here due to considerations of space. We will try both collaboration between children and more physical activities. Give-N has been described as collecting a set of physical objects. A variant of give-N is to repeat some physical movement a given number of times. This is used in collecting N objects, as the child fetches an object N times. Non-collecting examples are jumping N times or clapping N times. Board games in which the player moves a piece as many steps as a dice shows, is another possibility. Such a game may be tried with children who do not master the collection form of give-2 and give-3, by using a dice showing only one, two or three. To show the actual number of spots on a dice is an easier alternative than communicating N by a number word. Three different approaches can be combined by first asking a child to say the number of spots on a dice, and then apply counting to move a piece that many steps on a playing board.

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Coming to see fractions on the number line

Elisabetta Robotti[1], Samuele Antonini[2] and Anna Baccaglini-Frank[3]

1 Università della Valle d'Aosta, Department of human and Social Sciences, Aosta, Italy, e.robotti@univda.it

2 Università di Pavia, Dipartimento di matematica, Pavia, Italy

3 Università di Modena e Reggio Emilia, Dipartimento di educazione e scienze umane, Modena, Italy

The aim of this paper is to present a didactical sequence that fosters the development of meanings related to fractions, conceived as numbers that can be placed on the number line. The sequence was carried out in various elementary school classes, containing students with certifications of mathematical learning disabilities (MLD). Thus, our didactical aim was to make accessible to all the students of the class, including MLD students, meanings related to fractions using a common didactical sequence for the entire class. Our research is based on a range of different perspectives, from mathematics education to neuroscience and cognitive psychology. We discuss how such perspectives can be combined and provide the theoretical bases to design the didactical sequence, which will be outlined, and which allowed us to implement and strengthen inclusive education.

Keywords: Fraction, number line, artifact, mathematical learning disabilities.

INTRODUCTION AND LITERATURE

The concept of *fraction* is a very difficult one to master: frequently students are unable to reach an appropriate understanding of it, as described for example by Fandiño Pinilla (2007), and they can even come to fear fractions (Pantziara & Philippou, 2011). When children encounter fractions – typically in Italian school this happens in third grade (at 8–9 years of age) – it is the first time they have to treat sets of digits differently than how they treat those given in decimal positional notation representing positive integers. The numerator and denominator of a fraction are two numbers, each of which is bound by the rules that apply to positive integers, but that together represent a new, *single*, number. Learning to see the numerator and the denominator of a fraction together, as a single number is one of the most difficult – if not *the* most difficult – cognitive aspect of fractions (Bobis, Mulligan, & Lowrie, 2013).

Others disciplines, besides mathematics education, such as cognitive psychology and neuroscience, have also been very active in investigating the phenomena of (difficulties in) understanding mathematics (incuded fractions), even if the different interested fields of research have not yet reached sufficiently common grounds for conducting scientific and interdisciplinary studies. In this paper, we consider some results from research in neuroscience and cognitive psychology to ground important design decisions taken during the elaboration of a teaching experiment constructed around the learning of fractions in primary school. In the following paragraphs we will illustrate reasons why it is important to learn (and therefore teach) fractions, both from a didactical (math education) point of view and from the perspective of cognitive science.

IMPORTANCE OF FRACTIONS AND DIFFICULTIES IN LEARNING THEM

From the logical and epistemological points of view, the notion of *fraction* can be seen in different ways: as a linguistic representation of the decimal number obtained from the division indicated (but not calculated) by the number corresponding to the numerator and the one corresponding to the denominator; as an operator where the denominator indicates in how many equal parts a given unit is divided (*each part is called a unit fraction*) and the numerator indicates the number of these to consider.

$$3/4 = 3 \textit{ times } 1/4 \rightarrow 3 \times 1/4$$

Frequently, at least in Italian education, the conception of fraction as an operator is not explicitly identified as a rational number. Only when it is transformed into a decimal number is it placed on the number line.

From the point of view of learning mathematics, fractions constitute an important leap within domain of arithmetic because they represent a first approach to the idea of extension of the set of Natural Numbers. In this sense, fractions need to assume a specific position on the number line (Bobis et al., 2013; Bartolini Bussi, Baccaglini-Frank, & Ramploud, 2013). Teaching the notion of fraction is, therefore, a quite delicate issue and it is ever so important to explore insightful ways of structuring didactical activities around it. In this respect, particularly insightful approaches have been provided, for example, by Bobis, Mulligan and Lowrie (2013). Even if certain basic aspects of the concept of fraction, particularly when seen as the perception of the variation of a ratio, seem to be innate (McCrink & Wynn, 2007), the learning of fractions presents obstacles, not only of a didactical nature. In fact, research in mathematics education (e.g., Bartolini Bussi et al., 2013), has shown how learning about not only the semantic aspects but also the lexical and syntactical ones of fractions involves the overcoming of different epistemological and cognitive obstacles such as:

— Assuming that the properties of ordering natural numbers can be extended to ordering fractions (e.g. assuming that the product/quotient of two fractions makes a greater/smaller fraction).

— Positioning fractions on the number line using the pattern of whole numbers (Iuculano & Butterworth, 2011).

From a cognitive point of view, fractions seem to demand more working memory resources than representing whole numbers (Halford, Nelson, & Andrews, 2007). Moreover, fraction knowledge also requires inhibitory control and attention (Siegler et al., 2013), so that the numerator and denominator are not treated as independent whole numbers (Ni & Zhou 2005). With this in mind, it is clear that for a student with MLD (even when “D” stands for “difficulties” instead of “disabilities”) the learning of fraction will be a particularly arduous task. In fact, recent studies suggest that dyscalculia, a particular kind of MLD, is rooted specifically in weak visual-spatial working memory and inhibitory control (Szucs et al., 2013).

Our present work on fractions is part of a broader body of research (Robotti, 2013; Baccaglini-Frank & Robotti, 2013; Baccaglini-Frank, Antonini, Robotti, & Santi, 2014) that has the objective of building inclusive curricular material, grounded theoretically in research in mathematics education and in cognitive psychology, appropriate for *all* students, including those with MLD.

CONCEPTUAL FRAMEWORK

A large number of studies associated short-term memory (STM) and working memory (WM) with mathematical achievement for students and expert (see reviews in Raghubar, Barnes, & Hecht, 2010). Moreover, non-verbal intelligence, addressed to general cognition without reference to the language ability (DeThorne & Schaefer, 2004) also seems to be strongly related to mathematical achievement (Szűcs et al., 2013). These (and similar) findings suggest that non-verbal intelligence may partially depends on spatial skills (Rourke & Conway, 1997). Thus, spatial processes, performed on the base of spatial skills, can be potentially important in mathematical performances, where explicit or implicit visualization is required. Moreover, research in cognitive science (Stella & Grandi, 2011) has identified specific and preferential channels of access and elaboration of information. For students with MLD these are the visual non-verbal, the kinesthetic-tactile and/or the auditory channels.

Studies in mathematics education as well, although with different conceptual frameworks, have highlighted how sensory-motor, perceptive, and kinaesthetic-tactile experiences are fundamental for the formation of mathematical concepts – even highly abstract ones (Arzarello, 2006; Gallese & Lakoff, 2005; Nemirovky, 2003; Radford, 2003). In this regard, within a semiotic perspective, Bartolini Bussi and Mariotti (2008) state that the student’s use of specific artifacts in solving mathematical problems contributes to his/her development of mathematical meanings, in a potentially “coherent” way with respect to the mathematical meanings aimed at in the teaching activity.

Thus, in this paper we aim to describe examples (activities) of inclusive math education (Ianes & Demo, 2013), constructed referring to the math education domain as well cognitive psychology and neuroscience domains.

The goal of the activities we will describe, was to realize a sequence that would favor, for *all* children (including those with MLD) the development of mathematical meanings of fractions as numbers that can be placed on the number line. The sequence of activities was designed, realized and analyzed taking into account the following principles:

— the importance of an epistemological analysis of the mathematical content

— the role of perceptive and kinesthetic-tactile experience in mathematical concept formation as well the visual non-verbal, and auditory channels of access and elaboration of information, in particular in children with MLD

— the role of social interaction, verbalization, mathematical discussion;

— the teacher as a cultural mediator.

Following these principles, as we will describe later, particular artifacts (like paper strips, rulers and scissors) were identified with the intention of using them to help mediate the meanings at stake in the activities.

METHODOLOGY AND SEQUENCE OF ACTIVITIES

The sequence of activities was designed by 22 primary school teachers and 1 supervisor (the first author) composing a study group. The activities were carried out during a pilot experimentation, which involved 22 classes (nine 5th grade classes, six 4th grade classes, and seven 3rd grade classes), before being revised for an upcoming full-blown study. In this paper, we will report on the pilot experimentation carried out in the 3rd grade classes. Students worked in small groups.

The sequence of activities asked to work with differents artifacts such as A4 sheets of paper, squared-paper strips or represented squared-paper strips in notebooks. At first, students were asked to represent fractions on squared-paper strips, then to represent squared-paper strips in their notebooks and to represent, upon these strips, fractions. At last, students were asked to represent fractions on the number line (see below). As described below, the teachers also included moments of institutionalization and discussion (Bartolini Busssi & Mariotti, 2008) based on some critical episodes.

The activities concerning the sequence are:

1) Partitioning of the A4 sheet of paper: this activity involves dividing the A4 sheet of paper, chosen as a unit of measure, in equal parts, by folding and using the ruler; the procedure allows for the introduction of "equivalent fractions" as equivalent surfaces, and of "sum of fractions" for obtaining the whole (the chosen unit, that is, the A4 sheet).

2) Partitioning of a strip of squared paper. This activity involves three sessions:

    A) Given a certain unit of measure, position it on the strip; then, position some fractions on the strip (1/2, 1/3, 1/4, ...) according to the given unit of measure (see Figure 1). The objective is to represent, on the same strip, different fractions, introducing reciprocal comparison.

    B) Given different units of measure on different strips, on each strip a same fraction is represented (1/2). The objective is to make explicit the dependence of the unit fraction upon the chosen unit of measure (1/2u).

    C) Given a squared strip, choose appropriate units of measure to represent different fractions on that strip (e.g., 1/3 and 1/5). The objective of this activity is to find the *lcm* (least common multiple) between denominators as the appropriate unit of measure.

3) Placing fractions on the number line. The fractions, considered to be lengths of segments with origin in 0, are placed on the (positive) number line using the idea at the basis of the operator conception of fractions (developed in point 2). Since the right endpoint of the segment on the number line is labeled with a fraction, it will also assume the meaning of "number", as do all the other whole numbers on the line. Different fractions will be associated to a same point on the line, and will be used to revisit the meaning of "equivalent fractions".

ANALYSIS AND DISCUSSION

In this section we present an analysis of points 2 and 3 of the sequence, and in particular the transition from point 2 to point 3, which we consider the most significant in order to place fractions on the number line. Our objective is to highlight how the meaning of fraction evolved, thanks to the use of the tools (squared strip of paper, and number line) and to the designed tasks.

ACTIVITY 2, SESSION A. A certain unit of measure is given (for instance, a unit measure corresponding to 15 squares). The students are asked to position it on the *strip of squared paper* and to place and color on the strip unit fractions like 1/5,1/3, etc.

Figure 1: Four strips of squared paper where students had postioned a certain unit of measure and had defined unit fractions and colored fractions (4/5, 2/3, 5/3, 7/5)

With respect to the kinesthetic-tactile aspects that characterize activity 1 (partitioning of the A4 sheet of paper in equal parts), the manipulation of the artifact "squared strip" becomes a prevalently perceptive experience, in which the main channel for accessing (and possibly producing) information is the visual non-verbal one. Therefore the task (implicitly) requires the use of a procedure in which the fraction is conceived as an operator: the students partition the strip and produce linguistic signs associated to the name of the fraction expressed in verbal language ("Un mezzo" – tr. "One half"), in verbal visual language (the writing "un mezzo" – tr. "One half") and arithmetical language ("1/2"). The teacher institutionalizes the relationship between the different signs (partitions of the strips, visual verbal, visual non verbal, and arithmetical signs) in terms of rational numbers. Thus, the construction of meaning related to the notion of rational number, is based on the interplay between different types of semiotic sets (Arzarello, 2006). Note that the task was completed by all groups of students.

ACTIVITY 2, SESSION B. Each group of students is asked to choose a unit of measure, reproducing it on a strip and placing the fraction 1/2 on the strip. Then, their strips are compared. The dependence of the fraction on the unit of measure, observed comparing the results of the different groups of students, becomes explicit during a classroom discussion, from which we include an interesting excerpt:

Student 1: Maybe we made a mistake.
Student 2: No, we did not make a mistake, I am sure I folded the unit in half, so it's ½.
Student 3: We shouldn't look at the length, because each group chose a different unit of measure. [...]
Students 4: Because doing ½ is cutting in half, so if the units are different the halves are different [...] we have to be careful because to understand which counts more we can't put them one on top of the other like we did for the placemats.

Here a shared meaning is being developed for the fraction as an operator on a chosen unit of measure (1/2u). Note that the kinaesthetic-tactile approach in which the strips were put one beside the other is no longer effective for comparing fractions.

ACTIVITY 2, SESSION C. The task asks to choose a unit of measure to represent on the same strip different unit fractions like 1/3, 1/6, 1/8, 1/2, 1/4.

The parameters defining the situations are such that the situation makes it necessary to choose 24 squares (corresponding to the *lcm* of 2, 3, 6, 8 and 4) as the unit of measure. In fact the children do not simpy looked for the unit of measure spontaneously, generally using trial and error methods, but they also check the efficiency of their choice. Moreover, positioning on a single strip different fractions, makes the ordering of fractions quite similar to that of the other numbers that are perceptively evident (Figure 2).

We note here that for the different fractions on the strips (Figure 2), the teacher asks to also associate color to the verbal, figural and arithmetical representations. The reason is that, as suggested by Stella and Grandi (2011), the verbal channel is not the preferred one for most students with MLD. Color becomes a tool supporting working memory and possibly also long term memory, through which the meanings developed

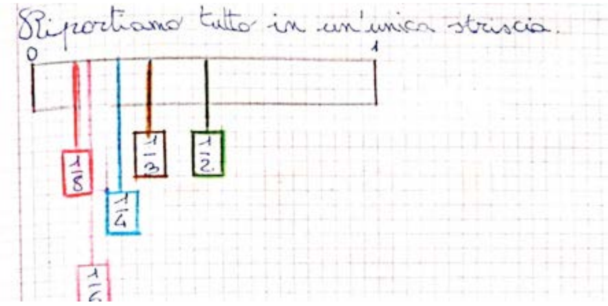

Figure 2: different fractions on the same strip

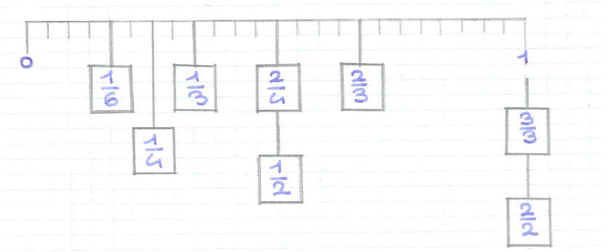

Figure 3: Fractions on the number line

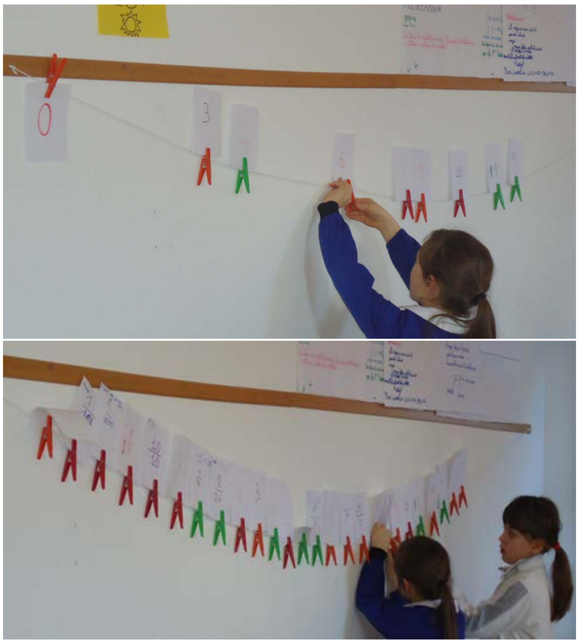

Figure 4: String on the wall where the position of the unit is made to vary dynamically sliding the corresponding label attached with a clothes' peg

can be recalled and used (Baccaglini-Frank & Robotti, 2013).

ACTIVITY 3. The objective of this activity is to place and order fractions on the number line (Figure 3) that has been partially constructed in Activity 2 and that is now used by the teacher as a tool of semiotic mediation (according to Bartolini Bussi & Mariotti, 2008). This transition is fundamental: the representation of the artifact "strip of paper" becomes a mathematical sign that represents the mathematical object "the number line".

Actually, from now on color is no longer used and the labels are referred to points on the number line. We can therefore claim that fractions here have assumed the role of rational numbers. The teacher could take advantage of this transition to construct a new tool of semiotic mediation, developed from the preceding artifacts. The "narrow strip" now becomes a concrete artifact (Figure 4): it turns into a piece of string on the wall, upon which 0 is placed at the left end and the position of the unit is made to vary dynamically sliding the corresponding label attached with a clothes' peg. The dynamic component of this artifact recalls certain software (such as AlNuSet, GeoGebra, Cabri2...) of course with evident differences, including the fact that as the unit (the position of the paper card with written "1") is made to vary, the positions of the other whole numbers and fractions do not vary dynamically at the same time or automatically, as a consequence of the new placement of the unit: their motion requires a specific action in order for the numbers on the line to maintain the desired mathematical relationships. This feature can actually be exploited to foster the students' appropriation and *active control* of important mathematical meanings at stake, such as the density of rational numbers or the ordering of fractions on the number line during activities like this one.

CONCLUSIONS

We have outlined particularly significant (and delicate) passages of the sequence of activities, showing how the transition was guided. Initially the students were exposed to a somewhat traditional conception of fractions as operators in the context of partitioned areas (the "part-whole" meaning described in Bobis et al., 2013). This idea was soon re-invested in a slightly different context: the areas became strips that gradually lost their "fatness" and were narrowed down until they become (oriented) segments indicating distances from the origin of the number line. The power of an approach like the one described resides in how such a transition can be gradual and continuous, if the teacher manages to keep alive the situated meanings that emerge throughout its unraveling. This is in fact what happened, and the children (including those certified with MLD) came to deal with fractions as numbers on the number line, without hesitating to compare them, place equivalent fractions on the same point, and add them, according to meanings they had developed using the strips of paper that still had an area.

In summary, the analysis of the teaching intervention has shown that students have elaborated personal meanings consistent with the mathematical meanings related to fractions. In particular, the strip was used as instrument of semiotic mediation to develop the meanings related to fractions as operators and, then, to the ordering of fractions, to equivalent fractions and finally to equivalence classes. The use of the strip, the string and color (for a certain period of time), has had a key role in favoring the construction of the number line as a mathematical object. On the number line fractions, associated with points, could assume the role of rational numbers being representatives of equivalence classes. Finally, it is possible that this kind of construction of meanings related to fractions might also support the management of procedural aspects involved in operations with fractions, as various researches both in mathematical education and in cognitive science have already suggested (Siegler, 2013; Robotti, 2013; Robotti & Ferrando, 2013). Further studies are needed to explore and to confirm this hypothesis that we consider significant both for research and for teaching.

We would like to greatly thank all the teachers of the "Questione di numeri" project who have realized, together with the first author, this research study.

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Asking productive mathematical questions in kindergarten

Per-Einar Sæbbe[1] *and Reidar Mosvold*[2]

1 University of Stavanger, Faculty of Education, Department of Early Childhood Education, Stavanger, Norway, per-einar.saebbe@uis.no

2 University of Stavanger, Faculty of Education, Department of Education and Sports Sciences, Stavanger, Norway

Although the trends seem to be shifting, researchers have given far less attention to the work of teaching mathematics in kindergarten than to children's learning. This paper aims at contributing to this under-developed area by focusing on one particular task of teaching mathematics: asking productive mathematical questions. From analysis of a situation that involves Lego play, we attempt to decompose the different kinds of mathematical questions asked and thus contribute to the further conceptualization and understanding of this particular task of teaching mathematics in kindergarten.

Keywords: Kindergarten, mathematics, teaching, productive mathematical questions.

INTRODUCTION

Children's learning of mathematics has been studied for decades; the teaching of mathematics to children below school age has been studied much less. Whereas numerous theories have been developed in order to describe different aspects of mathematics learning, few theories describe mathematics teaching. Almost three decades ago, Lortie (1975) called for a language to describe the work of teaching, and a language and theory of teaching is still called for – especially in kindergarten.

When developing a practice-based theory of mathematical knowledge for teaching (MKT), Ball, Thames and Phelps (2008) focused on "recurrent tasks of teaching". In the work of teaching mathematics, teachers are faced with different challenges, and these are referred to as tasks of teaching. An example is "asking productive mathematical questions" (Ball et al., 2008, p. 400). Teachers are continually challenged to ask questions that stimulate further mathematical thinking among children. Is this challenge similar in kindergarten? Identifying and investigating such tasks of teaching might provide a common foundation for further conceptualizations of knowledge needed for teaching mathematics (Hoover, Mosvold, & Fauskanger, 2014), and this paper represents an attempt to further investigate the task of asking productive mathematical questions in a Norwegian kindergarten context. We approach the following research question: How can the task of asking productive mathematical questions be manifested in a Norwegian kindergarten context?

Based on studies in the US, Ginsburg and Amit (2008) argued that teaching mathematics in kindergarten is similar to teaching mathematics in school. Other studies suggest that the work of teaching mathematics differs across kindergarten contexts (Mosvold, Bjuland, Fauskanger, & Jakobsen, 2011). Further investigations of the work of teaching mathematics are thus needed, both in order to develop more comprehensive theories of teaching mathematics in kindergarten and to learn more about similarities and differences in the work of teaching mathematics in different kindergarten contexts. In our attempt to approach this challenge, we focus on the challenge of posing productive mathematical questions. This is arguably a central task of teaching mathematics – also in the kindergarten context – and we aim at contributing to the further unpacking of this task. We draw upon exemplary data from a Norwegian kindergarten context, where a kindergarten teacher interacts with six children in an activity involving Lego play. The activity is analyzed with a focus on tasks of teaching as conceptualized by Ball and colleagues (2008). In the following section, we present some trends from research on teaching mathematics in kindergarten as well as previous research and theories related to asking questions.

THEORETICAL BACKGROUND

Traditionally, research on early years mathematics has had a strong emphasis on children and their learning and understanding of mathematics. In the last decades, however, research on the early childhood mathematics teacher has flourished. Some studies focus on the knowledge and beliefs of the teachers (e.g., Schuler et al., 2013), whereas other studies investigate the actual work of teaching mathematics in a kindergarten context (e.g., Carlsen, 2013). In mathematics education, a large amount of research has focused on the knowledge needed for – or used when – teaching mathematics (see e.g., Rowland & Ruthven, 2011). Relatively speaking, much research has focused on mathematics teachers in school; far less research has focused on teaching mathematics in kindergarten. The study by Ginsburg and Amit (2008) represents one of few examples of the latter, and these researchers argue that teaching mathematics in kindergarten is mostly similar to teaching mathematics in school.

When investigating mathematics teaching, there are different possible approaches (for an overview, see Thames, 2009). One possibility is to identify and describe issues regarding the mathematical content of what is being taught; another possibility is to identify the work of teaching that is distinctively mathematical. Our study, although related to both of these two, represents a somewhat different approach in that we analyze the work of teaching in an attempt to identify the mathematical demands. In doing this, however, we also investigate the nature of the mathematical tasks of teaching that are involved in the work of teaching – in particular related to asking productive mathematical questions.

A goal with mathematics teaching at all levels is that children (or adults) learn to think mathematically; some describe this as a process of mathematizing. In order to reach this a goal, an environment needs to be created where conjectures can be put forward and discussed without fear of being ridiculed, and children need to be engaged in such mathematical discussions (Lampert, 1990). When creating this kind of environment, the kinds of questions mathematics teachers ask are of importance. By asking the right mathematical questions, the teacher can create a supportive atmosphere in which the children further develop their mathematical thinking and start thinking like mathematicians (Mason, 2000). In a mathematics classroom, teachers ask different kinds of questions. Some questions are open and aimed at stimulating further inquiry, whereas other questions are more closed – oftentimes serving as control questions (cf., Carlsen, Erfjord, & Hundeland, 2010). There also seem to be cultural differences in the questions mathematics teachers ask in classrooms. In their study of questions asked in 1st grade mathematics classrooms in Japan, Taiwan and the US, Perry, VanderStoep and Yu (1993) found that teachers in the Asian countries asked more questions about problem solving strategies and conceptual knowledge than their colleagues in the US. Subsequent studies seem to confirm these findings (e.g., Hiebert et al., 2003), and international assessments like TIMSS and PISA show that children from these Asian countries outperform children from most countries in the Western world.

In a kindergarten context, Carlsen and colleagues (2010) found that the kindergarten teachers' frequent use of questions enabled children's participation in the learning activities. They did, however, also find that kindergarten teachers often asked questions that were not true questions. This coincides with a more recent study where Carlsen (2013) found that a kindergarten teacher mainly used structuring questions in her orchestration of a mathematical activity involving the telling of a fairy tale. From these studies, it can be argued that the questions asked by kindergarten teachers oftentimes serve as a means of reaching joint attention, and numerous studies in psychology contend that joint attention is of vital importance in children's learning. When Bruner and his colleagues started investigating this issue in the late 1950's, they mainly analyzed newborn babies or young children with a focus on their eye gaze (Bruner, 1995). A narrow understanding of the concept of "joint attention" would thus simply be whether or not an individual is looking where someone else is looking (Sigman & Kasari, 1995). A broader definition includes responsive and initiating behaviors as well as facial expressions and gestures. Baldwin (1995) defines joint attention with the mutual awareness in mental focus that two or more individuals have when looking at the same thing. A key issue then is that the mutual awareness must be in the mental focus – not only that two people stare at the same thing. Sigman and Kasari (1995) argue: "joint attention must involve an integration of information processing and emotional responsiveness" (p. 190). Studies like that of Tomasello and Farrar (1986) show that joint attention has a central role in children's early language learning, but it is arguably important in early years mathematics learning as well.

Given that joint attention has a central role in young children's learning, a natural follow-up is to ask about the role of the adult in this. The process of reaching joint attention has been referred to as a tutorial process (Wood, Bruner, & Ross, 1976), where an adult or "expert" helps somebody who is less skillful. A crucial feature of such interactions, it can be argued, is the adult's ability to make joint attention (Bruner, 1983). In earlier works, Bruner focused on joint attention in relation to language development and learning (Bruner, 1983); later he described the role of the adult in terms of scaffolding (Bruner, 1995). In this study, we follow Baldwin's (1995) understanding of joint attention in that it includes a mutual awareness of mental focus. We suggest that the questions posed by kindergarten teachers – and thus also the task of asking productive mathematical questions – could be understood in terms of reaching joint attention. When we investigate the task of asking productive mathematical questions, we therefore suggest that the process of joint attention needs to be an integrated aspect.

METHODS

In order to investigate kindergarten teachers' mathematical questions, we asked a kindergarten teacher for permission to video record an everyday activity that involved something he associated with mathematics. The kindergarten teacher decided to organize a situation of Lego play with six children from his kindergarten class (his class included a total of 18 children aged 3–6 years). The kindergarten teacher and the children sat around a table that was filled with classical Lego bricks of different shapes and colors; the children could play freely with the bricks. The first author recorded the session as illustrated in Figure 1.

The kindergarten teacher had 17 years of experience, and he finished his education before mathematics was introduced as a required course in Norwegian kindergarten teacher education. The six participating children were between 3,11 and 5,4 years of age (the decimals represent months). We refer to the children by fictitious names; the kindergarten teacher is referred to as "Teacher". The Lego play activity lasted 22 minutes, and the researcher had the role of a passive observer. Neither the kindergarten teacher nor the children seemed to take much notice of the researcher or his camera. Afterwards, the video was transcribed verbatim, and the transcripts were coded by the use of conventional content analysis (Berg & Lune, 2012), where the unit of analysis was the kindergarten teacher's questions.

RESULTS AND DISCUSSION

The activity begins with the kindergarten teacher gathering the children around the table to introduce the Lego activity. He explains that they are going to play with the Lego bricks, and someone (the first author) is going to record the activity in order to learn more about what they are doing. The children eagerly start playing with the Lego bricks. Shortly after, a first question is posed to the teacher:

8. Kaja: (holds up one red and one blue brick and turns to the teacher) Have you found one of those or one of those?

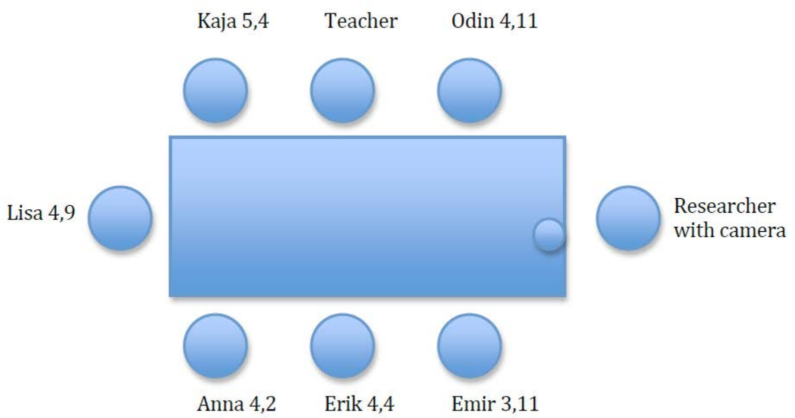

Figure 1: The placement of the participants in the Lego-activity

9. Teacher: One of those? (scratches his chin) What do you mean with "one of those"?
10. Kaja: Two of those (holds up the bricks again)
11. Teacher: Yes... (Odin hands the teacher a red brick, and the teacher holds it up) What does it look like?
12. Kaja: Table?
13. Teacher: Do you see what they look like?
14. Kaja: Triangle!
15. Teacher: Triangle, yes. But look, I'll show you something funny. If you ... put them together (puts two blue and one red brick together on the board). If we had one more of those, what would it have become then?
16. Kaja: Taaaaaable ... round [shape] (looks at the teacher)
17. Teacher: A round shape, or simply a circle.

Among the mathematical tasks of teaching, Ball and colleagues (2008) listed "responding to students' why-questions". The question posed by Kaja in the beginning (8) is not such a why-question, but we notice how the teacher uses the question as a starting point for posing another question (9) in order to direct the children's attention towards the mathematical concept that can be used to describe the bricks. His question can be described as an invitation to use more precise concepts than "one of those", and this could serve as an example of a productive mathematical question in a kindergarten context. When Kaja first responds by holding up the bricks instead of providing a more precise concept (10), the teacher asks her to describe what the brick looks like (11). As a response to this question, Kaja eventually says: "triangle" (14). The teacher confirms her answer and repeats what she said (15) – although the brick is not a triangle but a triangle-like shape with one round edge.

Even though there are six children in the activity, Kaja (5,4) is the only active participant in this part of the discussion; Odin (4,11) also contributes by finding the desired bricks, but his contribution is non-verbal. When Kaja responds to the teacher's follow-up question by saying that it would become a table (16), he introduces the concepts of "round shape" and "circle" as alternatives (17). This can be seen as an example of the task of using appropriate mathematical language, and we can also say that the dialogue in this excerpt indirectly contains the challenge of "choosing and developing useable definitions" (Ball et al., 2008, p. 400). Instead of using a more correct mathematical definition for a triangle – where the sides have to be straight line segments – the teacher decides to accept Kaja's description of the shape as a "triangle" (15).

Later in the discussion, when Kaja finished building her circle shape with two red and two blue bricks, Lisa (4,9) asks for the same kind of bricks:

41. Lisa: I need one more red and two more blue.
42. Teacher: Triangles like that? (points towards Lisa's board) But isn't it a little bit strange that ... they. How many are there here? (picks up Kaja's board to show)
43. Kaja: One, two, three, four (counts out loud while pointing)
44. Teacher: Four triangles. But isn't it a little bit strange that those triangles make a ...
45. Lisa: Circle
46. Kaja: Circle
47. Teacher: How is it that a circle can become a triangle? Or, triangles become a circle?
48. Kaja: We just ... (points to the board)
49. Lisa: The triangles have such round there.
50. Teacher: That edge ... side is a little bit round, yes.

We notice that the teacher follows up on Lisa's request by directing her attention using the word "triangle" and pointing (42) to reach joint attention. The teacher continues to ask questions while pointing at Kaja's board – which is finished – focusing on the quantity. We assume that the teacher is aware that the children know how to count, and his question about how many bricks can then be seen as an invitation to count. Ball and colleagues (2008) identified a similar task as: "connecting a topic being taught to topics from prior or future years". Kaja counts and finds out that four "triangles" are needed in order to build a circle (43).

Within a sociocultural kindergarten tradition (like the Norwegian kindergarten), the kindergarten teacher's knowledge and ability to ask questions are significant. Carlsen, Erfjord and Hundeland (2010) argued that the teacher's questioning is of vital importance for children's learning, and they found that almost half of the questions posed were open questions where the children are encouraged to present a solution themselves. The teacher in our study also poses a lot of questions, and through this questioning we gain knowledge of the children's development of number concept as well as their knowledge of shapes.

The teacher continues to pose questions, and his next question can be seen as an effort to re-focus the children's attention towards the seeming paradox that four "triangles" can make a circle (44). Both Kaja and Lisa are now referring to the shape as a circle (45 and 46) rather than table or round shape – indicating that they have adopted the kindergarten teacher's use of a more precise concept. When asked about how the "triangles" can become a circle (47), they do not respond verbally but points to the board instead (48). This situation could have been used to reach a more precise definition of a triangle, but the kindergarten teacher does not go in that direction. Instead, this part of the dialogue ends by the teacher confirming that the "triangles" have a round edge (50). Instead of stating that these particular "triangles" are in fact circular sectors – or quadrants – and four of them put together thus make a circle, the teacher leads the children into a mutual reflection about the shape of the bricks and how they can be used to create a new compound shape (circle). This also illustrates another task of teaching related to selecting representations for particular purposes – in this case using the Lego bricks to discuss shapes. Presentation and discussion of non-examples like this can be important in children's formation of solid concept images that go beyond the prototypical triangle (Levenson, Tirosh, & Tsamir, 2011). In this case, the children identified these particular Lego bricks as triangles although they are in fact non-examples. The children rely on visual reasoning, and the kindergarten teacher could have used the situation as a starting point for discussing more examples and non-examples and helped the children towards where they notice differences between shapes (ibid). From these examples, we have seen that the task of asking productive mathematical questions includes a number of intertwined challenges – or tasks – for the teacher.

CONCLUDING DISCUSSION

Ball, Thames and Phelps (2008) identified "asking productive mathematical questions" as one of the recurrent mathematical tasks of teaching. We argue that asking productive mathematical questions is indeed a relevant task of teaching mathematics also in a kindergarten context. From our analysis of this play situation in a Norwegian kindergarten, however, we suggest that the task of asking productive mathematical questions is highly complex. In this concluding discussion, we point at three issues that add to the complexity. First, there are different types of questions that might be asked to facilitate children's further reflection and exploration of mathematics. Second, there are different possible purposes that underlie the asking of questions. Third, the task of asking productive mathematical questions is often intertwined with other tasks of teaching, and the kindergarten teacher needs to address these tasks instantly as they appear.

In their analysis of kindergarten teachers' questions, Carlsen and colleagues (2010) found that the teachers posed open questions, asked for arguments, invited to problem solving, re-phrased children's utterances, and made conclusions. These types of questions could also be identified in the play situation analyzed here. When we have focused specifically on the mathematical demands, however, we can observe the following aspects in the questions:

- encourage use of more precise mathematical language
- confirm use of more precise mathematical language
- describe what children observe with their own words
- mathematical reflection about a more compound problem
- mentally complete an observed pattern or unfinished shape
- encourage reflection about observed patterns and connections
- invite to count

This list represents an attempt to decompose the task of asking productive mathematical questions, but it can also be seen as an attempt to identify the teacher's underlying purpose in asking these types of questions. It is difficult, however, to make conclusions about purpose from observations of activities and conversations only. In this study, our focus was on tasks of teaching that could be observed from discussions between the kindergarten teacher and the children. We have thus analyzed the observed work of teaching without bringing in the voice of the teacher concerning his intentions. Introducing the teacher's voice from a follow-up interview could have been interesting, however, but that would be beyond the scope of this paper – where our focus was strictly on unpacking the observed tasks of teaching. Interviews with the kindergarten teacher could, however, provide further information about teachers' beliefs and knowledge that would also be relevant to investigate.

Although we have investigated the data with a focus on the task of asking productive mathematical questions in particular, we have also seen that several other tasks of teaching are oftentimes intertwined in this task. When asking children to use more precise mathematical language, for instance, the kindergarten teacher could also face the task of choosing and developing usable definitions. Being faced with the apparent paradox of how four "triangles" could make a circle, the kindergarten teacher would have to make decisions about whether or not he should go into a discussion about the proper definition of a triangle. If he were to go into such a discussion with these children, however, he might also have had to deal with the concept of polygons and straight line segments, and this would probably be beyond the topics he intended to teach. On the other hand, avoiding the more precise definition at this stage could lead to misconceptions that would have to be dealt with later on, and this illustrates the complexities involved in the work of teaching mathematics in kindergarten.

An additional challenge that can be seen in several parts of this dialogue is related to joint attention. At all stages, but perhaps in particular with smaller children, the teacher is faced with the challenge of reaching joint attention. Several questions, comments and even gestures made by the kindergarten teacher can be seen as acts of reaching joint attention, and we argue that the issue of joint attention is embedded in all tasks of teaching mathematics in kindergarten.

In this paper, we have tried to contribute to the unpacking of one particular task of teaching mathematics: asking productive mathematical questions. We have seen that this task is complex, as we have already discussed, and we believe that it is also context specific. Asking productive mathematical questions to kindergarten children probably involves other kinds of challenges than asking such questions to children in lower secondary school, but the task of teaching is still relevant at all levels. We thus support the argument made by Hoover and colleagues (in press) that mathematical tasks of teaching "can serve as a common foundation for conceptualizing and measuring mathematical knowledge for teaching" (p. 101) – also in kindergarten – but we suggest that further studies are needed in order to investigate and unpack the tasks of teaching mathematics in different contexts and at different levels.

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Kindergartners measuring length

Chrysanthi Skoumpourdi

University of the Aegean, Mytilene, Greece, kara@aegean.gr

The present paper will clarify kindergartners' informal ability and spontaneous strategies when using different materials in order to measure and quantify length. The results validated the fact that there are some sub components of young children length measure informal ability which are of some interest. These are kindergartners' strategies for measuring length, their confusion on the concept of length with the perimeter and the area, the nature of the measurement result, as well as, the relation between the measuring material and the measured object.

Keywords: Length measurement, measurement unit, measurement materials, kindergartners.

INTRODUCTION

Geometric measurement is a crucial topic in mathematics (Smith, van den Heuvel-Panhuizen, & Teppo, 2011) due to the fact that it connects mathematical domains with each other (number and geometry) as well as, real life. Length measurement is a fundamental concept for students' geometric measurement learning. On one hand it is a comprehensible attribute for young children and on the other hand numerous other concepts rely on it. Students' main ideas, concepts and skills on area and volume measurement are constructed through length measurement (Tan-Sisman & Aksu, 2012).

Learning length measurement is a slow and evolving process and young children do have the capacity to understand and learn this attribute (Sarama, Clements, Barrett, van Dine, & McDonel, 2011). There are eight main concepts that are fundamental for children's understanding of length measurement (Clements & Sarama, 2007): 1. Understanding of the attribute of length. 2. Conservation of length. 3. Transitivity. 4. Equal partitioning of the object to be measured. 5. Iteration of the unit; the placing of the unit end to end alongside the object and the counting of these iterations. 6. Accumulation of distance; the number words of the counted iterations signify the space covered by the units up to that point. 7. Origin; any point on a ratio scale can be used as the origin, 8. Relation between number and measurement. These concepts are developed according to age and in a sequence that is not commonly accepted yet, since they are influenced by instruction and experience.

Length measurement is included in most curriculums (ΠΣΝ, 2011; Smith, Tan-Sisman, Figueras, Lee, Dietiker, & Lehrer, 2008) from pre-kindergarten and its study usually promotes a sequence of instruction that is based on Piaget's theory of measurement. These series include: use of words that represent quantity or magnitude, (direct and indirect) comparison of length, measuring of length with nonstandard units and with manipulative standard units, as well as, measuring with a ruler and other standardized tools. However, recommendations to this progression, propose the beginning of the instruction with standard units and rulers, for an initial understanding of measurement, and a later introduction of nonstandard units (Clements, 1999), as a result of the fact that children are able to use standard measuring tools before they fully understand them (van den Heuvel-Panhuizen & Elia, 2011). Therefore, Sarama and Clements (2009) proposed an early childhood learning trajectory of length measurement, which follows a developmental progression: 1. Pre-length quantity recognizer. 2. Length quantity recognizer. 3. Length direct comparer. 4. Indirect length comparer. 5. End-to-end length measurer. 6. Length unit relater and repeater. 7. Length measurer. The revised learning trajectory proved that (Sarama et al., 2011) the 7th level does not exist and the 4th and 5th levels are developed simultaneously.

Most of young children pass into and through the end-to-end length measurer level in which they use several units or begin to iterate a unit while leaving gaps between units. The students' errors that are most reported (Tan-Sisman & Aksu, 2012), during the learning procedure of length measurement, are: overlapping units, mixing units of length with other units of measurement, confusing the concept of perimeter with area, incorrect alignment with a ruler, starting from 1 rather than 0, counting hash marks or numbers on a ruler/scale instead of intervals and focusing on end point while measuring with a ruler.

Research results, on what young children are able or incapable to accomplish in length measurement, are not always in agreement. Several researchers (Barrett, Jones, Thornton, & Dickson, 2003; Clarke, Cheeseman, McDonoug, & Clarke, 2003; Clements & Sarama, 2007) proved that children as young as preschoolers know about the existence of length, distance, area, weight etc and they have an informal understanding of them. They confirmed that kindergartners are capable to perform direct comparison of objects and order objects according to their length. They explained that preschoolers are not capable of reasoning about the attributes of length, distance, area, weight etc and measure them accurately and that kindergartners are not able to use units to determine the length of an object.

Sarama and colleagues (2011), demonstrates that although no pre-kindergartner was able to consistently measure objects end to end, or iterate a unit along a length, kindergartners were able to do it. They were capable of arranging units end to end (without spaces or overlaps between units), count the number of units, as well as to state correctly the total length measured in terms of that unit. On the contrary, none of the children of kindergarten used a ruler consistently to measure length, despite repeated introduction in various teaching experiments. A different study (Castle & Needham, 2007) indicates that, at the end of the first grade, less than half of the class demonstrated the ability to iterate units. The above results signify that research outcomes are not entirely consistent with young children's capabilities to measuring length.

Kindergarten children face a great difficulty in length measurement tasks when tasks are of high complexity and require the mental use of the length measurement unit and unit iteration, as well as, complex ordering abilities (van den Heuvel-Panhuizen & Elia, 2011).

In this paper, the informal ability and the spontaneous strategies of kindergartners to realize measurement tasks using a variety of materials are investigated. We wondered if kindergartners were able to use different materials in order to quantify length. The research questions were the following: 1. Have kindergartners the ability to use the provided materials to measure the length of an object? 2. Which strategies they use to measure length and how are they related to the different materials? 3. Which of the materials were used more effectively?

METHOD

In order to investigate kindergartners' informal ability and spontaneous strategies to quantify length with the use of various materials for measurement, data were collected from 20 kindergarten children (10 girls and 10 boys) (4 years and 9 months to 6 years and 6 months) from a public kindergarten in Rhodes, Greece[1]. This kindergarten was chosen for two reasons. The first was that there is a frequent cooperation with the teacher who likes integrating innovations in her teaching and her students are used in teaching experiments. The second was that the children in that classroom were able to compare two objects directly and recognise their equality or inequality and they were also able to order objects according to their length.

For the purpose of the study, kindergartners were asked to measure the length of several objects' iconic presentations by using eight usual measurement tools and to note on a piece of paper, their size. The images were related to a story familiar to students and had rectangle-like-shape (stair, paper clips, a part of a road, frame, mirror) or oblong irregular shape (tower, lighthouse, shark). The materials used for measurement, were nonstandard manipulative units, standard manipulative units and standardized measurement tools. Specifically, the materials were of four types: 1) discrete items, such as 1cm cubes, 2cm cubes, same sized matchsticks, paper clips and anglegs which may be considered as measurement units, 2) continuous material, such as a piece of string, in which the appropriate size must be generated by the child, 3) combined materials, such as Cuisenaire rods, which may be considered as measurement units or continuous material and 4) composite material, such as a ruler, in which its (conceptual) organization into units or 'chunks' may be used for quantifying the size

1 Kindergartens in Greece include students from 3 years and 9 months to 6 years and 6 months old.

of an object in a standardized way. The length of the different iconic representations was measured with different types of materials.

The tasks were the following:

Task 1: Can you measure the length of the shark with these (2cm cubes)?

Task 2: Can you measure the length of the stair with these (Cuisenaire rods[2])?

Task 3: Can you measure the length of the frame with these (1cm cubes)?

Task 4: Can you measure the length of the tower with these (anglegs[3])?

Task 5: Can you measure the length of the frame with this (string)?

Task 6: Can you measure the length of the lighthouse with these (matchsticks)?

Task 7: Can you measure the length of the road with these (paper clips)?

Task 8: Can you measure the length of the mirror with this (ruler)?

The research was a case study and the unit of coding was each child's strategy based on the type of material used. Each child was interviewed alone in a separate room and the process was videotaped. Principles of ethics were taken into consideration.

RESULTS

The students' strategy was recorded in relation with the type of the material used for the measurement of each object (Table 1).

In the tasks 1, 3, 4, 6 and 7, where discrete items were used, the students acted in three different ways, which were common in all five tasks: *linear placement* strategy, *perimetrical placement* strategy and *spatial placement* strategy. The *linear placement* strategy was of two kinds: *linear placement without gaps* and *linear placement with gaps*. In the *linear placement without gaps* strategy, the children arranged the discrete items, in a linear way, either end-to-end with the object (Figure 1 & 2) or not (Figure 3). In particular, they either connected them in a line before arranging them or just placed them linearly in a systematic way.

Figure 1

Figure 2

Figure 3

Although, there were children who measured length in a right way, not all of them gave the right quantitative answer. They noted other numbers or series of numbers. In these cases they tried to count the items they used, but not always in a systematic way. In one particular case, a child, while counting, unitized the object, along the anglegs, by rhythmic movements of his index, whereas not always in a constant way. Except the quantitative answers, children notated either the name of the object to be measured or letters. There were also children that did not answer anything or mentioned the colours of the items used. These types of answers were also provided in all of the following tasks.

The *linear placement* with gaps strategy was the same with the previous one, but students left gaps between the linear arranged items or they began placing the blocks below the top of the object and stopped just before the end of it. The gaps between the 1cm and 2cm cubes, in most of the cases, were due to the lack of cubes connection.

The position of the discrete items in a not linear, but perimetrical way, drove to the *perimetrical placement*

2 The Cuisenaire rods are math learning aids for students that provide a hands-on way to explore math concepts as length measurement.

3 The AngLegs come in different lengths allowing students to explore math concepts such as length measurement.

strategy in which object's perimeter was covered by the items (Figures 4–6). In some occasions, only a part of the object's perimeter was covered because of the items' limited number (Figure 7). A kind of perimetrical strategy was the *perimetrical dragging* strategy, used by only one child. She was always dragging one of the items perimetrically to the object.

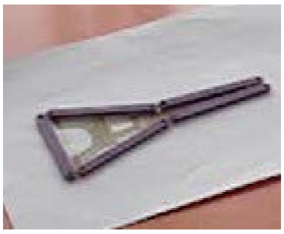

Figure 4

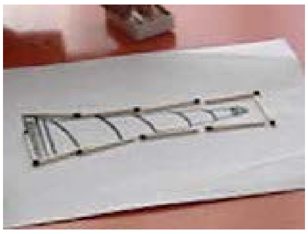

Figure 5

Figure 6

Figure 7

Children in many cases placed the items on the objects' surface instead of placing them linearly. Like perimetrical strategy, in *spatial placement* strategy, items were not adequate to fill the surface and for this reason kindergartners covered just a part of it (Figure 8). The spatial placement strategy differed when children measured with matchsticks and paper clips, due to the fact that their placements were of two kinds: horizontal (Figure 9) and vertical (Figure 10). These placements may have been favoured by the type

Figure 8

Figure 9

Figure 10

of material, which was not symmetrical, like cubes, but elongated.

In task 2, combined materials were used, and students acted in the same three ways as mentioned above: *linear placement* strategy, *perimetrical placement* strategy and *spatial placement* strategy. All strategies were as formerly described whereas with a significant differentiation. Since there was lack of materials in the same size, children had to decide what to select: a big one to manipulate it like a continuous material or small ones (with the same size or not) to manipulate them as counting units. The children chose one, two or three equal or unequal sized items that were: long as the object's length, smaller than the object's length, bigger than the object's length (Figure 11). Similarly,

Figure 11

Figure 12

Figure 13

there were occasions that the children chose many unequal sized items placed perimetrically to the object (Figure 12) or on its surface (Figure 13).

The children, in order to reply to the query, counted the pieces of the materials used. In one particular case, a child placed his finger at the one end of the object, while having underneath the already placed materials, counting 'one', then skipped his finger counting 'two', 'three', ... However, the distance skipped was not always stable. This student was the one who unitized the object, along the anglegs and the string. *Perimetrical dragging* strategy was used by a child, also in this task.

In task 5, in which continuous material was used, the students acted in two different ways: *linear placement* strategy and *perimetrical placement* strategy. In the *linear placement strategy*, the children, in most cases, left the string along the object without stretching it. The string was not placed end-to-end to the object but exceeded the length of it from both sides – *middle linear placement* strategy (Figure 14). In a case, a child put the string along the object's length from its end – *edge linear placement* strategy – and a pair of scissors was requested in order to cut the string. The lack of scissors made him unitize the object along the string by rhythmic movements of his index, which he simultaneously counted. This student was the one who unitized the object, along the anglegs. The *perimetrical placement* strategy was the positioning of the string on the object's perimeter. As with the other materials, here too, only a part of the perimeter was covered by the string, because of its limited length (Figure 15).

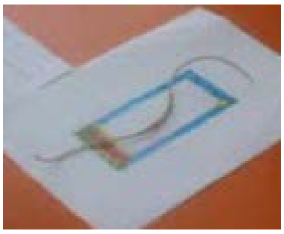

Figure 14

Figure 15

Figure 16

Figure 17

In task 8, a ruler was used as a composite material and students acted in the same way as before: *linear placement* strategy as well as *perimetrical placement* strategy. The *linear placement* strategy was of two kinds: *edge linear placement* strategy and *middle linear placement* strategy. In the former, the ruler was placed on the object's edge (Figure 16), and in the latter, in the middle of the object's length (Figure 17), just like the string's placement. In the perimetrical placement, two children placed the ruler in each side of the object. Students' notes in this task were in general arithmetic. Some children counted the ruler's units in a one to one correspondence with number words, usually inconsistently. In a case, the edge linear placement directed to the right answer – it was the student who unitized objects, along with the anglegs, the string and the cuisenaire rods. The rest of the students' noted a number by chance without counting, or a number that they noticed on the ruler.

Generally, students were consistent in both of the strategies they used and the kind of response they provided, sometimes regardless of the material used for the measurement and occasionally in accordance with the material used. For example, the child who dragged a single material and rearranged it perimetrically to the object performed it in all tasks. In addition, the child who unitized with his finger the

Tasks with discrete items	**Strategy**
1)'2cm cubes' to measure a shark 3)'1cm cubes' to measure a frame 4)'Anglegs' to measure a tower 6)'Matchsticks' to measure a lighthouse 7)'Paper clips' to measure a part of a road	*Linear placement (without gaps or with gaps): For 'Anglegs' Linear placement (with 2 pieces equal to tower's length or 2 pieces sorter than tower's length) *Perimetrical placement (or dragging) *Spatial placement: For 'Matchsticks' & 'Paper clips' *Spatial placement (horizontal or vertical)
Task with combined material	**Strategy**
2)'Cuisenaire rods' to measure a stair	*Linear placement (with 2 or 3 different size pieces equal to stair's length or with 3 different size pieces longer than stair's length) *Perimetrical placement (or dragging) *Spatial placement
Task with continuous material	**Strategy**
5)String to measure a frame	*Linear placement (edge placement or middle placement) *Perimetrical placement (or dragging)
Task with composite material	**Strategy**
8)Ruler to measure a mirror	*Linear placement (edge placement or middle placement) *Perimetrical placement (or dragging)

Table 1: Tasks, types of material and strategies used for measuring length

linear placed materials, like counting discrete quantities of objects, was constant in his strategy no matter what the material was. Children who arranged items linearly performed it in all tasks. However, a child, who measured most of the objects in a correct manner and rightly quantified the lengths, was puzzled with the use of the string and the cuisenaire rods. He held the edge of the string at the edge of the object and could not decide what to do with the other end. Finally, he placed the string perimetrically to the object. Regarding the rods, he used three different sized items which in total had the same length with the object, while he answered "three". The same child placed the ruler correctly on the object, counted with his finger but his answer was wrong.

It was also observed that perimetrical placement strategy and spatial placement strategy were influenced by the object's image. In one case, covering the perimeter was the same as covering the surface of the object, due both to the objects' shape (shark) and to the material used for measuring (2cm cubes). Furthermore, when the image was rectangle-like and empty inside (stair, frame, mirror), children covered its perimeter whereas when the image was oblong irregular and coloured inside (tower, lighthouse, shark, a part of a road), children tended to cover its surface.

Children quantified length in two different ways: a) counted the (equal or different sized) materials, b) unitized materials by their finger. Their responses varied, except the quantitative ones, which represented the amount of materials used in measurement. The rest of their responses were related with the colours of the materials, letters of the alphabet, number series and the name of the object to be measured.

DISCUSSION-CONCLUSIONS

This study indicated that kindergartners have an informal ability and use several strategies when measuring with a variety of materials. There were children who used the provided materials and successfully measured the length of objects' image, as it is also reported by another research (Sarama et al., 2011). They aligned units that were of equal size without gaps and they counted them. Other times, they placed the materials linearly and unitized them in order to quantify the length. In the latter, the child coordinated discrete and continuous quantity, which is a high cognitive level.

Although research mentions that children are able to use standard measuring tools before they fully understand them (van den Heuvel-Panhuizen & Elia, 2011), in our research we clarified that the majority of students used the measurement materials and tools in a not systematic fashion and were not able to determine the length of an object. This finding is in accordance with other research results, mentioning that kindergartners are not able to use units to determine the length of an object (Barrett, Jones, Thornton, & Dickson, 2003; Clarke, Cheeseman, McDonoug, & Clarke, 2003). They measured length by placing multiple materials that exceeded the length of the object or by attempting to iterate a unit, leaving gaps between units. Some children arranged enough materials to lay from one end to the other across the object without being concerned about their size. When children laid units leaving space between them, they spread them out to make them reach from one end to the other. In many occasions, although all materials were enough to reach from one end of the object to the other, children semi-covered (or cross along) the perimeter or the surface of the object. Children's notes were numbers, letter expressions, colours etc. In a few instances, they combined the numerical data with the qualitative in their verbal communication, mentioning the unit of measurement.

Three were the main strategies that kindergartners used, in order, to measure the length: linear placement strategy, perimetrical placement strategy and spatial placement strategy. These strategies acquire special features depending on the material. With the use of discrete materials, the linear placement strategy was of two kinds: linear placement without gaps and linear placement with gaps. On the contrary when the materials were continuous, like string and ruler, the linear placement strategy was defined as: edge linear placement strategy and middle linear placement strategy. The perimetrical placement strategy was transformed in a case, to the perimetrical dragging strategy. Finally, the spatial placement strategy differed when children measured with non-symmetric items, creating two kinds of placements: horizontal and vertical. The measurement of length by outlining or filling the object by measurement tools has also been mentioned in a study (Castle & Needham, 2007) as a first graders approach. The researchers state that this strategy may indicate their confusion of the different dimensions or that it may be a result of their linear thinking and of a difficulty to conceptualize the object in more than one dimension.

The effectiveness or not of these strategies, seems to be an indicator that students possess an informal knowledge for measuring and quantifying length, in which the teaching process can be based in order to be developed in formal knowledge. The understanding of kindergartners' informal ability to cope with length measurement and the sub components of that capability, can inform teacher's instructional design, in order to support children's learning. This is a crucial point both for helping students to develop their length measurement ability and also to realize the importance of measurement unit, connecting a number to length and specifying what this number represents. For instance, they must understand that a length of 3 is different from the length of 3 cubes, different from the length of 3 anglegs and different from the length of 3 unequal rods: 1 red, 1 green and 1 yellow. Likewise, that the length of five 1cm cubes is the same as the length of 5cm on the ruler. Further research is required to investigate how children interpret their notes regarding the objects' length, should they understand what they have written and what is written by others.

The results of this study, although it was investigative in nature and involved just one kindergarten class, are interesting, because they explore some sub-components of young children length measure informal ability. These can improve our knowledge for kindergartners' ability to cope with length measurement, as well as, to understand why the research results often differ. These are, except of the most reported students' errors, the personal effective strategies that even a kindergartner can use for measuring length, the confusing of the concept of length with perimeter and with area, the kind of the measurement result and most important, the relation of the measuring material with the measured object.

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Familial studies in early childhood that involve mathematical situations

Gizem Solmaz

Middle East Technical University in Ankara, Faculty of Education, Ankara, Turkey, gizem.solmaz@metu.edu.tr

This paper aims at giving information about "interactional niche in the development of mathematical thinking (NMT)" (Krummheuer, 2011b). Considering the benefits of playing games, some games were developed within the scope of erStMaL-FaSt that enable children to play with their families. The aim of this study is to analyze mathematical development of children with the immigration background. In a play setting, Tunç played with his father and mother. This paper provides a description of this play setting as well as its results. Based on empirical evidence of a familial play setting, the study provides further details of NMT. This study was conducted with a Turkish family that lives in Germany.

Keywords: Measurement, familial studies, interactional niche, immigration background, mathematics in early years..

THEORETICAL BACKGROUND

The place of geometry and measurement in our lives is unquestionable. Our environment is full of geometrical shapes and we measure objects around us in our daily life. Indeed besides matures even children who are two years old can use mathematical concepts. For instance, a child at the age of two can decide effortlessly whether his friends or his cookies are bigger (Heuvel-Panhuizen & Buys, 2008). No matter how complex these mathematical concepts can be seen, in fact they are used automatically. Children start to measure objects at very early ages (Clements & Sarama, 2007). While measuring, they use very different methods. For example, they make measurements with a ruler or compare objects by using their bodies (Clements & Sarama, 2007). In addition to these, by using their hands they can measure and comment about the length of a table (ebenda). When children engage in practical activities with support from adults, the mathematics embedded in the activities can be addressed and learnt by the children. Thus, it brings a question to mind "What is the role of the parents who spend most of their times with children?" In regards to children's experience with mathematics and geometry, Vygotsky states that, long before going to school children can improve their arithmetic skills and interact with people around them (Baroody & Wilkins, 1999). Children can make division, multiplication, addition, subtraction and can decide size of shapes, images and so on (ebenda).

METHODOLOGY AND THEORETICAL PERSPECTIVE

Considering the benefits of playing games, some games were developed within the scope of erStMaL-FaSt that enable children to play with their families. The aim of this study is to analyze mathematical development when children with immigration background and their parents play games. In order to achieve this goal, games were developed by using mathematical contents like shapes, patterns etc. The most important feature of these developed games was to explore how children learn mathematical subjects. Children were recorded with cameras while playing determined games within the scope of the project. Researchers later watched recorded videos and analyzed the play situations. Project erStMaL-FaSt is a project of the IDeA (Center for Research on Individual Development and Adaptive Education of Children at Risk). This research center is constituted by the German Institute for International Educational Research (DIPF) and Goethe University Frankfurt. The financial support is provided by the Ministry of Higher Education, Research and the Arts from the state of Hessen (for more information: http://www.idea-frankfurt.eu/de).

For the longitudinal analyses and comparisons among different mathematical learning situations, the concept of "interactional niche in the development of mathematical thinking" (NMT) (Krummheuer, 2011b), is used. Krummheuer interprets NMT as, "interactional niche in the development of mathematical thinking" (NMT) consisting of the provided "learning offerings" of a group or society, which are specific to their culture and will be categorized as aspects of "allocation", and of the situational emerging performance occurring in the process of meaning negotiation, which will be subsumed under the aspect of the "situation" (Krummheuer, 2012). NMT – Family, is an elaborated version of NMT offering analytical concepts to capture familial mathematical learning opportunities in early childhood and elementary school ages. Three components of NMT-Family are shown below, which will be elaborated later on, from the aspect of the design of FaSt. NMT-Fast, a developing subtopic of NMT, gives the opportunity to research the importance of families' attendance for pre-school children's learning of mathematical concepts. NMT-Fast consists of 3 components as outlined below in Table 1.

Content: The play situations by erStMal-FaSt are set up to elicit the families' chances for interactive negotiations. In these play situations, from the situational view, the rules of play and/or mathematical topics might be chosen as themes in the period of emerging negotiation.

Cooperation: The collaborating process of child and adult provides more opportunity to cleanse their thinking and to make their performance more beneficial and effective. A different leeway of participation comes up due to this cooperation. Krummheuer uses the term "Leeway" as a colloquial meaning of "room for freedom of action" (Krummheuer, 2012). "Leeway of participation" ("Partizipationsspielraum", Brandt, 2004) is one of the interactionist approaches by which child discovers and explores his/her cultural environment while co-constructing it. Therefore, this is a concept which belongs to the situational aspect. Brandt (2004) defines that the participants interactively fulfill varied margins of leeway of participation that are contributing or limiting the mathematical development of a child (see also Krummheuer, 2011a; 2012).

Pedagogy and Education: Learning paths are described and delineated with developmental theories and theories of mathematics education to study cognitive development. According to folk pedagogy, in the concrete interaction, the participating adults and children become situationally active and operant. Each individual's cognitive development is constitutively bound to the attendance in various social interactions. In the course of these interactions and participations in mathematical discourses, a "support system" which is called as "Mathematics Learning Support System" (MLSS) is proposed as a concept for the learning of mathematics evolving from Bruner's concept of a Language Acquisition Support System (LASS) (Bruner, 1986). MLSS occurs in different ways in the phases and routines of interactions between child and families. During these play situations they convey their knowledge by exemplifying explanation to the statements during negotiation of meaning. Either "the right given instructions" or "the wrong given instructions" by families bring about some different types of support. While the given instructions are being negotiated by children and their parents, they set new interpretations that support the development of the child or influence him/her unfavorably (see Acar Bayraktar & Krummheuer, 2011; Acar Bayraktar, 2014). In respect of all these three components, one

NMT-Family	**component: content**	**component: cooperation**	**component: pedagogy and education**
aspect of allocation	mathematical domains: Geometry and Measurement	Play as familial arrangements for cooperation	developmental theories of mathematics education and proposals of activeness for parents on this theoretical basis
aspect of situation	interactive negotiation of the rules of play and the content	leeway of participation	folk theories of mathematics education, everyday routines in mathematics education; MLSS

Table 1

Tunç	5;3 years old	Single Child	His first language is German also he can speak Turkish
Father	worker	Higher Education	both German and Turkish
Mother	unemployed	Higher Education	both German and Turkish

Table 2

chosen scene will be introduced as an example to show how measurement abilities in the interactional niche in a familial context emerge. In this study, data was collected through video recording. This study is a part of a bigger project which is erStMal-FaSt.

THE PLAY: "WHO WILL BUILD THE HIGHEST BUILDING?"

The chosen play is from the mathematical domain of 'measurement', which is developed according to specific design pattern (Vogel, 2014). Aim of the game is to build the highest building. At the beginning, all blocks are placed on the table. All players take one dice themselves, on which pictures of blocks are present. The

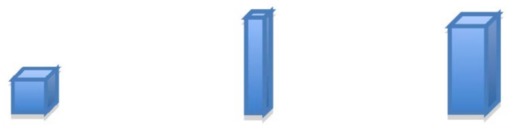

Figure 1

youngest player is to start first. He rolls his dice, then looks at the picture on the top of dice. The figure on the dice should be identified with the matching block from the pile on the table. The player is supposed to set each matching blocks on top of each other so that the highest building can be built without toppling down. In this way, all players try to set the blocks on top of each other by rolling dices in turns. In total, 6 rounds are played the blocks that are already placed cannot be moved or their places cannot be changed. During the game, if any of the player's building topples down, he/she fails and leaves the game. In the end, the player with the highest building wins the game. In the play setting at this study, Tunç plays with both his mother and father. Their play takes about 15 minutes.

The required information about Tunç's family is in the Table 2.

In the play situation, firstly, his father gives Tunç a little information about the game where he demonstrates for Tunç how to pick up the block from the box after he rolls the dice. Then he rolls the dice and asks Tunç about the figure that has occurred. Afterwards he asks Tunç to find this figure in the box, to show him the figure and to put the block on the table. Tunç puts the cube shaped block in front of him. His father wants him to re-roll the dice. He requests Tunç to put the new block onto the first cube shaped block. Then the father confesses to Tunç that they will build a big house by putting blocks on top of each other and that they will continue like this until the blocks drop onto the ground. Then his father explains to Tunç that he has to set the blocks by himself without any help from neither his mother nor his father. His mother adds that everyone will have their own blocks and whoever gets the highest one will win. Afterwards she asks who will start first. Tunç says "me" and starts the game by rolling the dice. Block A comes on the top of the dice. Tunç takes it from the pile and puts it on the table. Then, his mother rolls the dice, gets Block A and puts it on the table. Then father rolls the dice, gets Block B and puts it vertically on the table. Hereafter comes up the transcript of the chosen part of the scene: Table 3 on the next page.

Line 13, 14, and 15: Tunç, by showing a block that he picked up to his father asks him whether it is big. Tunç, by asking this question, may have tried to get a response to whether the block he found is appropriate or not. Or maybe he asked just because he waited to be approved by his father. A possible indication of the latter interpretation is that immediately after posing the question, he responded "Yes" without waiting for his father's feedback. Tunç has put his block horizontally just like his father did it with the previous block. The reason behind this may be that he thinks his father did right or that he may wait for his father's approval by putting it similar to the way his father put. If we look from another perspective, he may have behaved this way because his father did not listen to what his mother told him. In other words, he may have copied his father.

01	V9111600000000000000931005062395012.avi		
02	01:31	father	**yes**
03	01:32	mother	*looks at Tunç's father* it cannot be such high *takes a block b*
04			than *from the box, regulates* this is high now this is much
05			higher
06	01:37	father	but we must lay a strong foundation, don't we? If I put it so,
07			it can be easily dropped, can't it Tunç? *holds block b*
08			*vertically, makes it horizontally* because of this, father puts
09			so *puts block b horizontally on the table in front of him*
10	01:38	Mother	(not understandable)
11	01:50	Tunç	rolls the dice, handles the box and looks into takes one
12	01:52	Father	**yes!**
13	01:53	Tunç	is this big? No
14	01:54		*takes a block b sets horizontally on his first block a on the*
15			*table in front of him.*
16	01:56	father	ok... *looks at Tunç* it is your mother's turn
17	01:58	mother	*rolls the dice, picks a block b from the box, sets it on her first*
18			block on the table in front of the her
19	02:00	Tunç	*looks at his mother* you have two blocks
20	02:05	father	*rolls the dice* **yes!** *takes a block c, looks at it from side to*
21			side, smiles.
22	02:13	mother	you must put on it you cannot put next to it like that....
23	02:18	father	*put block c vertically next to his first block b*
24		mother	you cannot put two next to each other. like that
25	02:19	father	why cannot be next to each other?
26	02:19	mother	you cannot put two next to each other *hand up vertically* you
27			should put like that
28	02:24	father	why so that I win? *Smiles*
29	02:25	mother	no *smiles*
30	02:27	Tunç	rolls the dice, takes a block c , puts vertically on his corpus
31	02:28	father	**yes!**
32	02:29	mother	*points Tunç* look he does it right
33	02:30	father	**vaov Tunç!**
34	02:31	mother	*rolls dice* I get always small blocks *picks block a from the*
35			box, puts it
36	02:33	Tunç	*(german not understanble)*
37	02:36	father	it's my turn now *rolls the dice* **yes** *picks a block c from the*
38			box, put it vertically on his corpus
39	02:43	Tunç	*looks at his mother* you roll always small *rolls the dice* **yes**
40	02:47	father	what's it, Tunç?
41	02:48	Tunç	*shows figure on the dice to his father*
42	02:49	father	**huge! vaoov!**
43	02:50	Tunç	*looks for a blocks into the box*
44	02:51	father	*pushes the box to Tunç, asks to him* which block matches
45			with the figure
46	02:53	Tunç	*takes out a block c from the box, shows to his father* this
47	02:54	father	yes
48	02:55	Tunç	*puts a block c vertically on his corpus, regulates it*
49	02:56	father	**yes!**
50	02:57	mother	*looks at Tunç* make it robust then not to fall down *rolls dice*
51	03:03	father	but you cannot regulate it so much that Tunç after you put it
52			once, you cannot touch it again
53	03:08	mother	why not? *handles a block b puts it horizontally, then*
54			vertically, then horizontally again
55	03:17	father	*smiles* mother cheats Tunç

Table 3

Line 22: The mother says "you must put it on you cannot put next to it like that. The mother, by saying that, may have tried to remind the father of the rules of the game. The word "must" may cause one to think that there is a rule that has been mentioned. In other words, by saying "you must put on it you cannot put next to it like that" she may has reminded him that he is required to put them over each other. The mother, by doing so, may have tried to remind Tunç about the rules of the game again because as she was talking to his father, Tunç was also listening to them. So, with this reminder, it can be assumed that the mother may have opened up a leeway of participation for Tunç.

Line 25: The father asks the mother why he cannot put the blocks side by side. With this question, the father may have opened up a leeway of participation for Tunç because, through this question he could think about what possibly could happen if he puts blocks side by side. It can be supposed that there could be two reasons why the father asks this question. First, he may not have realized that he disobeyed the rules and tried Tunç to realize that also by waiting an answer from mother. Second, he may have thought that the way he placed the block is correct. This way, by asking the question, he may have wanted the mother to reconsider about the subject.

Line 28: The father asks the mother "Why? So that I win?" He smiles. With this question, he may have tried to tell Tunç that he could win if he places blocks vertically like himself. Another possibility for this situation is that he may have tried to encourage Tunç by acting like they are racing to win the game. On the other hand, the father may have tried to finish the discussion between him and the mother.

Line 34, 35: The mother rolls the dice and says "I always get small blocks". By saying this, the mother may have tried to get the attention of Tunç to compare the blocks with each other because the mother has got block A. In other words, Tunç could make the sense; block A was the lowest block, out of this. Namely, it could be assumed that the mother may have constructed a leeway of participation with this reaction. In contrast to this, the mother may have said this just to indicate which block she got.

Line 39: Tunç turns to his mother and says: "You always roll small". Then, he rolls the dice and says: "Yes". Tunç, by saying this, may have tried to show his mother that he knew the blocks coming to her were low. On the other hand, he may have tried to remind her that she always got low blocks and that she cannot win the game. In addition to these possibilities, it can be examined that, in line 34, by saying 'I always get small blocks', the mother could have opened up a leeway of participation for Tunç. Through this, Tunç thought about that what his mother tried to say and finally he decided that her mother got small blocks.

RESULTS

To sum up, I argue for indication of a developmental niche that has emerged for Tunç. His mother and father are reminding him about the rules of the game and even explaining why they put the blocks vertically or horizontally. Thereby it seems that Tunç has understood which way to place the blocks to get a longer structure. Also his father supported Tunç during the game by saying "yes". From this point of view, it can be said that the father's supportive actions increased Tunç's motivation.

Children at the age of 5 can easily decide which structure, created by placing objects over each other, and is longer than the others. Furthermore they can discover the connection between length and numbers. They can also perceive length differences between objects even though there is no discernible difference (Cross, Woods & Schweingruber, 2009). When this case is taken into consideration, it is possible to say that Tunç's perception about measuring has developed properly compared to his age. Also his mother and father have been helpful to him to apprehend the subject with their reminders and explanations.

Starting off from this analysis we can configure 3 components that we mentioned before:

Component "Content": There were 3 goals for the game, which are selected for Tunç: Measurement, matching shapes and figures and balancing the blocks statically. Pre-school aged children may have some difficulties with length measurement. They may not be able to compare the length of any object or building. But most of the children aged between 4 and 5 are able to measure with manipulatives, e.g. ruler, given to them. They can also make the connection between length and number of the components (Cross, Woods & Schweingruber, 2009) Based on this information it can be assumed that Tunç has succeeded in comparing the lengths of objects without using any tools. At this point, his mother and father remind him with examples when they place the blocks vertically. Therewith they will get a longer structure.

Component "Co-operation": During the game his mother and father accompanied Tunç. But the game has not been directed by a specific person. The father was generally quite dominant, and he gives some responses like yes, huge and vaov. In addition to the father, the mother was also considerably involved in the game. From time to time, they reminded each other about the rule of the game. The father gave suggestions to Tunç from time to time which seem to have encouraged Tunç. Also, when Tunç asked a question to his father, he gave time for Tunç to think and answer. For example, when Tunç asked his father if the block is big by poking the block at his father, the father waited for an answer from Tunç. Afterwards, Tunç found his own answer by saying "No". It can be said that the father has opened the leeway of participation of Tunç. On the other hand, the mother has succeeded getting Tunç's attention to the game by interfering with the father from time to time. Her action has caused Tunç to question the playing style of the father. Thus, he placed the blocks just like his father did. From this point of view, both the mother and the father have enabled Tunç to an opened leeway of participation.

Component "Pedagogy and Education": In the case of this selected game, Tunç has succeeded in building a long structure by observing his mother and father.. Vygotsky asserted that there is an important role of "social environment" in which child's cognitive development included (Kozulin, 2003). Children start to learn from people around them and their social world (ebenda). Starting off from this information, it can be said that Tunç has been able to explore how to build a longer structure with contributions from his mother and father. Studies have shown that preparation of learning and capacity of thoughts affect learning. This capacity (potential of learning) is shown with zones of proximal development, ZPD (Cook & Cook, 2007). French (2007) stated that with adult's help, children reach an extensive learning capacity which is different from the one earlier. At this point, we may assume tell that a child playing with an adult who directs him becomes effective in his learning period. If we analyze the example with Tunç and his parents in relation to ZPD, we see indication of such extensive learning possibility. Tunç has apprehended how to get a longer structure with support from his father and mother and in fact he even won the game by getting the longest structure.

All in all, in the chosen situation, it can be assumed that through his father's and mother's reminders, the developmental niche has emerged for Tunç.

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NMT-Family	component: content	component: cooperation	component: pedagogy and education
aspect of allocation	Measurement, matching shapes and figures and static balance between blocks.	Playing with father, and Mother	Vygotsky's Perspectives about children's developmental skills
aspect of situation	negotiation between father, mother and Tunç	Opened leeway of participation	Zone of Proximal Development and everyday routines in mathematics education; MLSS

Table 4

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Preschool teachers' understanding of playing as a mathematical activity

Christina Svensson

Malmö University, Malmö, Sweden, christina.svensson@mah.se

In this paper, preschool teachers' documentation of their learning about playing as a mathematical activity are examined using Wartofsky's three levels of artefacts, primary, secondary and tertiary. Playing is one of Bishop's six mathematical activities and was a new consideration for most teachers and in contrast to how play is generally conceived as a tool for learning. Wartofsky's three levels of analysis provided insights into how the teachers were able to visualize and understand playing as a mathematical activity.

Keywords: Artefact, mathematical activity, preschool, preschool class.

BACKGROUND

In this paper, I analyse preschool teachers' reflections on their learning about Bishop's (1988a) mathematical activity Playing using Wartofsky's (1979) three levels of artefacts. This investigation is a part of a wider study investigating preschool teachers' mathematical learning in a professional development project designed around the introduction of a revised curriculum (Skolverket, 2010). The project involved teachers completing written documentation about their reflections and this paper explores how this documentation was analysed. According to the Swedish curriculum (Skolverket, 2010), the preschool teachers are responsible for that activities in groups of children are performed in a way to stimulate and challenge children in their mathematics development. Skolverket (The Swedish National Agency for Education) (2008) highlighted how, ten years after instigating a curriculum for preschools, mathematics still had an inconspicuous position in preschools.

Previous research has suggested that preschool teachers perceive mathematics only to be about counting and measuring (Clements & Sarama, 2007; Ginsburg, Lee, & Boyd, 2008). Consequently, there have been suggestions that mathematics in preschool should include patterns (Björklund, 2008), and geometry (Clement & Sarama, 2007; Ginsburg et al., 2008). More broadly there is an argument made in favour of encouraging preschool children to think and make many mental relationships rather than to teach them specific subject content (Kamii, Miyakawa, & Kato, 2009). Counting, measuring, patterns and geometry are mathematical content and do not necessary include the expectation of mathematical thinking, such as Playing. This is in contrast to curricula such as the Swedish preschool curriculum (Skolverket, 2010) which emphasis mathematical thinking. In a background document to the curriculum (Utbildningsdepartementet, 2010), mathematics in preschool is discussed on the basis of Bishop's (1988a) six mathematical activities Counting, Measuring, Locating, Designing, Playing and Explaining, which perceive mathematics as a cultural activity, developed in all cultures. Bishop (1988a) considered the activities to be processes that lead to the development of mathematics. These six fundamental activities, he claimed are universal for two reasons. Firstly, because they seem to have been performed by each culture group that has ever been studied, and secondly because they are both necessary and sufficient for the development of mathematical knowledge.

Bergen (2009) asserted that Play can be defined as the medium for learning process for all ages because many qualities of play enhance learning process. She continues with saying that play is valuable for children primarily because it is a medium for development and learning, and it is important to understand that playing and playfulness is a quality which is valued by mathematicians, engineers and scientists. Helenius, Johansson, Lange, Meaney, Riesbeck, and Wernberg (2014b) also discuss that Play is an important means for learning and continues by saying that; with for mathematicians Play describes as a necessary component of their creativity in problem solving. Thus, enhanced playful learning at every level of education that prepares professionals for scientific, mathematical and engineering fields is warranted (Bergen, 2009). Playing is the mathematical activity, which deals with aspects of mathematical thinking. Bishop (1988b) considered that Playing involves designing and participating in games and pastimes, which have more or less formalised rules which all players must follow. Bishop (1988b) included games for adults and claimed that Playing is an important activity for developing mathematical thinking for all ages. He considered Playing as characterized by thinking hypothetically (imagining a potential action to take in the game and is the beginning to think abstractly), modelling (abstracting something from reality) and abstracting (identifying the relevant features to focus on within a situation), guessing, estimating, assuming or adopting.

The role of play in education is a major concern of early childhood educators (King, 1979) and so even in Sweden. Play has a long history in preschool curriculum in Sweden, which could be the means of that preschool teacher are unlikely to naturally connect it with mathematical thinking. Wernberg, Larsson and Riesbeck (2010) claimed that the early learning of mathematics needs to be problematized so that it not only consists of numbers and calculations. They argued that the learning of mathematics in preschool should be based on playing. They considered Playing to be the most important means for learning in preschool, because it promoted interactions between teachers and children or between children and children. But the concept of learning by Playing needs to be more nuances to be categorized as a mathematical activity Playing (Bishop, 1988b). Helenius and colleagues (2014b) mean that, for play to be considered mathematical, it must include all or most of the following three parts. The first is that the participants must abide the implicit or explicit rules of the play. The second is that if the rules change there needs to be negotiation by participants. The third is that negotiating the rules contributes to forming the boundaries of the play situation and thus what aspects of reality can be suspended and what aspects are modelled in what ways. "However, young children are unlikely to know the rules of mathematics so for play to count as mathematical, there must be abiding by group negotiated rules, but these may not necessarily be about mathematical content knowledge per se" (Helenius et al., 2014, p. 7).

As a result, it is possible that Swedish preschool teachers have difficulties understanding Playing as a mathematical activity. In research about the impact of a professional development project based on Bishop's (1988a) six mathematical activities, Helenius, Johansson, Lange, Meaney, Riesbeck & Wernberg (2014a) noticed that preschool teachers focused more on counting and measuring. Very few teachers labelled the activities that they described as Playing. They argued that changes in teachers' understanding take time and "a more explicit discussion of Bishop's six activities could be beneficial for future professional development programs" (p. 10).

Previous research on professional development in Sweden shows that, teachers claimed that they wanted to acquire extended knowledge and understanding of representations, and ideas in many different areas in mathematics (Doverborg, 2006). Furthermore, she found that continuous reflection with colleagues about videos, photographs, observations and interviews contributed to teachers' extending their knowledge and views on mathematics and pedagogical discussions. Alnervik (2013) focused on sharing and discussing pedagogical documentation between colleagues as a tool for visualising and developing their teaching practices. Pedagogical documentation is mentioned in the foreword of the curriculum as a way of making preschool practices visible and subject to discussions and evaluations of the quality and quantity of preschools (Skolverket, 1998). However, it is only when documentation is used for reflection that it becomes pedagogical, which has also been emphasized in published literature to provide guidelines for the use of documentation (Taguchi, 1997).

This research investigates teachers' documentation of their reflections on their learning about Playing as a mathematical activity. In the next section, Wartofsky's (1979) three levels of artefacts are discussed, as Alnervik (2013) used this in analysing the pedagogical documentation of preschool teachers in her research. The professional development was a project for developing preschool teachers' understanding of mathematics, including Playing as a mathematical activity. The project provided data for my research when it was necessary to determine whether teachers faced a conflict between Playing as a mathematical activity and other conceptions of play in order to improve the possibilities for learning opportunities within the professional development. Thus, the research question is: *How can Wartofsky (1979) three levels of artefacts in the analyses inform how preschool teachers developing understanding of Playing as a mathematical activity?*

WARTOFSKY'S THREE LEVELS OF ARTEFACTS

Wartofsky's (1979) three levels of artefacts have been used as tools for understanding the learning that occurs in mathematics classrooms (Hemmi, 2010; McDonald, Le, Higgins, & Podmore, 2005; Radford, 2008). There are several ways to classify artefacts (Säljö, 2005; Wartofsky, 1979). Säljö (2005) defined artefacts as tools that mediate between the individual and the social practice. He divided artefacts into two groups: *intellectual tools* like discourses and systems of ideas and *physicals tools* like texts, maps and computers. He also classified them into *primary tools* like a hammer and *symbolic tools* like those used for communicating ideas. Radford (2008) argued that we think with and through cultural artefacts, like the wooden ruler, the number line, and the mathematical signs on a piece of paper. A similar argument can be made for pedagogical documentation. Alnervik (2013) used Wartofsky's (1979) three levels of artefacts, primary, secondary and tertiary, to analyse teachers' pedagogical documentation to identify different perspectives. Artefacts, such as a die, a photograph or a film about a mathematical activity are unlikely to support changes in teaching practices without a task, a reflection and a conversation between colleagues and a re-reflection for reconstructing or developing the educational work focusing on the activities (Alnervik, 2013).

Following the Soviet tradition, Wartofsky (1979) focused on historic-cultural objects as artefacts. His interest was to define the ways humans worked to create cultural artefacts. He claimed that perceptions are culturally conditioned and represent a person's internal mental model which, through different representations, indicates that possible changes have occurred. Wartofsky (1979) connected the function of tools for use in internal mental models of human understanding by going from the practice and use of the tool to the theoretical and imaginative understandings of the tool. He provided a strategy to do this by separating artefacts into three levels, primary, secondary and tertiary, although he also acknowledged the importance of linking the three levels. The artefact or tool itself does not determine whether it is primary, secondary and tertiary, but rather how it is used.

Primary artefacts: These are tools, which are being used, in a special context. To utilise the tools means possessing the knowledge of how the tools operate, including the skills needed to operate them.

Secondary artefacts: These tools act as a model or pattern in which the artefact is used to describe how people are supposed to do something i.e. seeing something that was not apprehended earlier.

Tertiary artefacts: These are tools or aids for thinking, where the artefacts help people to see the environment in a special "new" way. These artefacts can be considered as a way to "create, understand and analyse the world" (Säljö, 2005, p. 98). Hence, new ways of viewing the world can be discovered and these can contribute to altering and understanding a practice in a new way (Alnervik, 2013).

METHOD

This paper reports on a part of a larger study aiming at investigating how preschool teachers use their understandings of the six mathematical activities (Bishop, 1988a) in their teaching practices during a professional development programme. This programme was part of Matematiklyftet (Skolverket, 2013) and used materials developed specifically for teachers in preschool and preschool class. Preschool class is a "bridging" year between preschool and school and school starts in the year when children turn seven, preschool is for one to five year olds. The material is based on collaborative learning (Timperley, Wilson, Barrar, & Fung, 2007) and Bishop's (1988a) six mathematical activities. Ninety preschool teachers and preschool class teachers, divided into four groups, actively participated in this project during a period of eighteen months, 2013–2014. The mathematical activity Playing (Bishop, 1988a) was the focus of three meetings with the teachers. The teachers were given mathematical tasks and questions to discuss, in order to develop their understanding of Playing. Before the first meeting the participants were supposed to have read the prescribed texts and looked at video(s), from the professional development material. During the first two meetings, the texts and videos were discussed and between the meetings the teachers implemented tasks in their preschools that could be linked to *modelling, abstraction* and *hypothetical thinking*.

Following Helenius and colleagues' (2014) advice about the need for more explicit discussions of Bishop's six activities, teachers got the opportunity to discuss mathematics in preschool and preschool class with their colleagues at the professional development. To base the discussions on teachers' own practices, the teachers brought with them documentation in the form of photographs or videos. Every third meeting was a reflection meeting where the teachers discussed their documentation. During these discussions the teachers used a reflection protocol as support for their discussions so that they could further develop the mathematical activities. As one of the facilitators of the professional development, I kept my own notes of the meetings and these were also analysed.

In this paper, the purpose is to look at how Wartofsky's (1979) three levels of artefacts were used to identify what understanding of Playing as a mathematical activity teachers displayed in their documentations during the professional development. Bishop's (1988b) description of modelling, abstraction and hypothetical thinking as key features of Playing was used in identifying how teachers visualises this mathematical activity in their work in preschools. Wartofsky's (1979) three levels of artefacts were chosen as a way of seeing whether a conflict between understandings of Playing and other conceptions of play in the Swedish preschool was evident in the teachers' pedagogical documentation. In order to operationalise Wartofsky's (1979) three levels of artefacts so that they could be used in analysing the teachers' documentation, descriptions of how they could be used in the data analysis were developed:

Primary artefact: Tools as a photograph, video recording or a note from the teacher to visualise Playing as a mathematical activity.

Secondary artefact: Tools in the documentation from the teacher, where the notes described what occurred in photographs or video recordings in form of the three concepts; modelling, abstraction and hypothetical thinking.

Tertiary artefact: Tools that aid teachers' thinking to see the environment in a special "new" way of Playing as a mathematical activity.

ANALYSIS

In the following sections, I provide a description of a teacher's documentation and how it was analysed using Wartofsky's (1979) three levels of artefacts.

Pedagogical documentation as a tertiary artefact

A teacher in a preschool class chose to video record a play situation where three children, six years old, played "doctor". This is a common play situation in preschools and preschool classes and models children's perceptions of a real hospital. Consequently it was considered to be an example of the *modelling* component of Playing as a mathematical activity. In the video, one child is a patient, one child was a doctor, and one child was a nurse and secretary. The teacher talked to the children and asked them questions during the episode captured on the video.

The teacher discussed with the children different aspects of Playing by asking questions. From this, she then added comments to the documentation.

The teacher asked the children question about objects they were using as part of their hospital game but which did not necessarily look like they did in reality. Such a situation occurred when the child, who acted as a secretary, sat down at the table and pressed the buttons of a calculator. The child answered that the calculator was a typewriter on which he wrote prescriptions. This was analysed by the teacher as illustrating the *abstraction* component of Playing because the child had only focused on specific aspects of a typewriter, the buttons, as being important for what he was doing. After that, the teacher asked the doctor about the patient's condition. The doctor informed the teacher the patient had a very high heart rate, so he had to stay at the hospital for a few days. When the patient heard that his condition was very serious, he lay down on the sofa again. The child who was acting as the patient imagined what it was like to have such a bad condition so he followed the rules of the game situation in order to ensure that he received further treatment from the doctor. This was considered to be an example of *hypothetical thinking* because he had to work out what would be expected of him, if he really was a patient with a bad heart condition the rules changes and there was negotiation by participants.

In her documentation, the teacher showed how her choice of implementing this situation and her conversation with the children, illustrated how she had perceived and worked with the three components of Playing as a mathematical activity. At the reflection meeting, my notes suggested that she had observed this play before, but had not seen it as a mathematical activity. During the reflection meeting, she also, of raised how she felt about her own presence during the children's "free play", and how this affected her further development and planning of this situation.

Her documentation of Playing was analysed as a tertiary artefact, as the teacher showed that she had gained new understanding, that might contribute to her changing how she saw this situation in the future Playing by providing her with new ways of understanding what she did.

Pedagogical documentation as a secondary artefact

The example of a secondary artefact comes from the documentation of two teachers who decided to work together. They took turns in participating in the children's free play and simultaneously photographing it. The teachers provided texts to accompany their photographs about what the children said and did.

However, they did not mention modelling, abstracting and hypothetical thinking (Bishop, 1988b) in their documentation. The children did discuss how chairs should be placed in order to resemble a ferry and in so doing organized the chairs from a model they had been aware of earlier and the rules contributes to forming the boundaries of the play situation. This situation could have been identified by the teachers as *modelling*. The children also put toys in a swimming pool to represent a pool on the ferry that could give massages. This second situation could have been identified by the teachers as *abstraction* like aspects from the reality. Later, the children informed the teachers that a chair was missing and it meant that not all the children could get a massage at the same time. This last situation could have been identified by the teachers as *hypothetical thinking* because of how the children pretended that the game could not continue if not all of the children participated.

This documentation was categorised as a secondary artefact because an individual reflection from the teachers on their roles, which was present in the first pedagogical documentation, was left out. The notes that they made allowed them to explain what the children had done but not reflect on it, thus making it a secondary artefact. Without the reflection component, it seems that teachers can remain unaware of how the components of Playing were present in the situation. It is interesting to note that when the teachers presented their documentation to their colleagues at the reflection meeting, they then could discuss the different aspects of Playing in the situation but this was not part of their documentation.

Pedagogical documentation as a primary artefact

In this documentation, the teacher video recorded a situation where a four-year-old child was supposed to sort material, containing teddy bears in different colours and sizes. The teacher asked questions during the documentation, which were related to the numbers and the sizes of the teddy bears, such as "how many teddy bears do you have?" and "do you have more orange teddy bears than red?" However, the child played with the material and responded with statements such as "this is a ring [of teddy bears], dancing around a Christmas tree". However, this situation could be an example of *modelling*, as the child presents a possible model of a real-world situation. The child continued with "I get to do this instead when I throw out the Christmas tree" which could visualized be a *hypothetical thinking* as she presents a possibility of something occurring. The teacher used the video camera, as a direct tool to produce a description of a mathematical situation or a play situation with math material. In the documentation, the situation was neither explained nor reflected upon. Hence, it was categorised as a primary artefact, a tool as a video used in a specific context to visualize mathematic in an organised situation.

COMPILATION OF THE RESULTS

A total of thirty-seven sets of documentation were collected from the teachers. Several teachers chose to do the documentation together, others chose to resist. About a quarter of the documentation were categorised as primary artefact (Wartofsky, 1979). In many of these, it emerged that the teachers perceived the situations as playful environments which they connected to mathematical learning situations. In other documentation categorised as primary artefacts, the teachers described the situation as a representation of Playing (Bishop, 1988), similar to sorting the teddy bears, but which contained no explanation or reflection. The rest of the collected data was analysed as secondary and tertiary artefacts.

During the two meetings before the reflection meeting, when Playing (Bishop, 1988a) was discussed in different ways, the teachers expressed that they had difficulties to seeing the children engaged in modelling and the abstraction. The facilitators therefore chose to focus the meetings on these two parts, by showing many different representations of modelling and abstraction. However, in the results it is hypothetical thinking that many educators had difficulties with seeing. Consequently many of their sets of documentation could not be classified as indicating a tertiary artefact, because this lack of awareness hindered the teachers from changing their understanding about their practices.

Many of the teachers found that it was in the collegial learning, when they discussed their documentation of their own activity that contributed to opening up their view of Playing as a mathematical activity. In addition, the teachers identified the importance of the facilitators being involved in the discussion of their documentations in the reflection meeting in order for them to see Playing as a mathematical activity.

CONCLUSION

Analysing how teachers in preschool and preschool class were able to visualize Playing as a mathematical activity is very complex. By using Wartofsky's (1979) three levels of artefacts primary, secondary and tertiary it became possible to detect how preschool teachers developed their understanding of Playing as a mathematical activity. Helenius and colleagues (2014a) showed that teachers struggled to catch the sight of the mathematical activity Playing in their practice. The same phenomenon appeared in the documentation from this professional development. This suggests that teachers' conceptions of play were connected to the curriculum understanding that learning occurs through play or that mathematics maybe is only about counting and measuring. Wartofsky's (1979) three levels of artefacts could inform analysis of preschool teachers developing understanding of play as a mathematical activity. More research in this area is needed on how the mathematical activity Playing (Bishop, 1988b) is made visible and teachers understanding of the activity to enhancing playful learning in early childhood education for prepare professionals in the scientific, mathematical and engineering fields.

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Unsolvable mathematical problems in kindergarten: Are they appropriate?

Dina Tirosh, Pessia Tsamir, Esther Levenson, Michal Tabach and Ruthi Barkai

Tel Aviv University, Tel Aviv, Israel, levenso@post.tau.ac.il

Not all mathematical problems have a solution. This paper describes a decomposition problem, set in a real-life context, for which no mathematical solution exists. It describes the strategies children use and how the rea-life context impacted on children's solutions. Results indicated that young children accept the possibility that a problem may not have a solution and that some turn to the context in order to find a practical solution instead.

Keywords: Unsolvable problems, kindergarten, real-life context.

Problem solving is an inherent part of learning mathematics at every age. In the U.S., the Curriculum Focal Points suggests that the content and concepts learned in kindergarten should be addressed in a context that promotes problem solving, reasoning and critical thinking (NCTM, 2006). In England, the Statuary Framework for the Early Years Foundation Stage (DCSF, 2008) states that "Children must be supported in developing their understanding of problem solving, reasoning, and numeracy in a broad range of contexts…" (p. 14). In Israel, the preschool mathematics curriculum stresses that problem solving should be interweaved throughout all of the main content strands (INMPC, 2008). Some major, related questions are: What types of problems should we present to young children? Should we only present problems that have a solution? Are young children capable of handling mathematical problems that have no solution? The INMPC (2008) suggests that problems, especially for young children, be set in a real-life context. How might such a context impact on the way young children approach a mathematical problem that has no solution?

One of the major content strands of preschool mathematics education is to promote number conceptualization and skills, including number composition and decomposition. The INMPC (2008) states that between the ages of 4 and 6 years, children should learn that a given amount of objects may be divided into two or more groups and that those groups may or may not have equal amounts of objects. Being able to flexibly compose and decompose numbers allows children to apply various strategies when solving number problems in various contexts and assists in the development of number sense (Baroody, 2004). This study describes kindergarten children's engagement with a number decomposition problem set in a real-life context that has no mathematical solution. Specifically, it investigates: What strategies do children employ when faced with a composition and decomposition problem? Do young children recognize when a problem has no mathematical solution? How might the context of a problem impact on the way children approach the absence of a solution?

NUMBER DECOMPOSITION AND PROBLEM SOLVING

Children's ability to compose and decompose numbers is at the heart of this study. Two prerequisite skills for composing and decomposing numbers are verbal counting and object counting. Both of these skills are almost always mastered by age five (Baroody, 2004). Related to quantification, but distinct from counting, is the process of subitizing (Baroody, Lai, & Mix, 2006). Subitizing refers to the immediate recognition of the number of items in a collection without counting the objects. It involves recognizing that three dots in a row has the same number of dots as three dots set up in a triangular array and that both of these sets have exactly three dots. As the child develops, these conceptions may be mentally operated on, such as mentally decomposing a pattern of five into two and three and combining them back together.

In addition to subitizing, children develop and use various counting strategies when solving arithmetic problems. Initially, when children are requested to add two numbers or to combine sets of objects, they use a "counting-all" strategy. For example, when combining three items with four items, children will count 1, 2, 3 and then 1, 2, 3, 4 and then finally, 1, 2, 3, 4, 5, 6, 7. This strategy develops into several more sophisticated strategies such as "counting-on-from-first" and "counting-on-from-larger". Eventually, procedural counting knowledge develops into declarative knowledge in the form of number-facts, which are remembered and retrieved (Baroody, 2004). Children take advantage of number facts, along with their knowledge of decomposing numbers, to count in a more efficient manner. In the above example, children may decompose the four into three and one, use a known fact such as "three and three makes six", and then add the leftover one, to make seven (Verschaffel, Greer, & deCorte, 2007).

Not all arithmetic problems present the same cognitive challenge. The easiest problems to solve are those that the child can use and manipulate physical items in a step-by-step procedure set out by the story problem (Sarama & Clements, 2009). For example, if the child is requested to find out how many candies Danny has if he received three candies from his grandma and four candies from his grandpa, the child can physically act out this story using substitute items, in order to figure out how many candies Danny now has.

Setting mathematical problems in real-life contexts is another issue discussed by mathematics educators. The realistic mathematics education (RME) approach, emphasizes that context problems must be imaginable to the students (e.g., Streefland, 1990). Such problems can serve as an introduction to a mathematical topic as opposed to merely demonstrating the applications of a mathematical topic at the end of a learning sequence. Students, however, do not always take into account the real-life context of the problem. When posed with the following problem:

> 450 soldiers must be bussed to their training site. Each army bus can hold 36 soldiers. How many buses are needed?

a considerable amount of students wrote that 12.5 buses would be needed (Verschaffel, De Corte, & Lasure, 1994).

While the bus problem is solvable (e.g., hire 13 buses), not all problems have a solution. For example, De Corte and Verschaffel (1985) posed the following problem to first graders: Pete had some apples; he gave 4 apples to Ann; how many apples does Pete have now?" (p. 11). More than half of the children did not recognize that the problem was unsolvable. In other studies with similar unsolvable situations, students gave answers which were irrelevant (Reusser & Stebler, 1997). It was as if students felt that if a problem is posed, a solution must exist. When does this belief develop? Does it exist already in kindergarten? In this study, we pose a problem which has no solution and investigate children's strategies when attempting to solve this problem.

METHODOLOGY

The participants in this study were 19 kindergarten children, ages 5–6 years old, attending three different kindergarten classes. All three kindergartens were located in the same middle-low socio-economic neighbourhood. Their teachers had participated for two years in one of our professional development programmes (Tirosh, Tsamir, & Levenson, 2011). The children were scheduled to enter first grade the following term. The study was conducted in March, the second half of the school year.

Birthday parties are part of children's reality. Every child experiences birthday parties in the kindergarten class. For this task, four empty party plates were placed on the table. In addition, eight cards were placed on the table, with different amounts of candies on each card. One of the eight cards was left blank (See Figure 1). The child was told, "Four children are coming to your party and you want to give each child seven candies on their plate. Can you arrange it so that there are seven candies on each plate? You can arrange the candies however you wish, but there have to be seven candies on each plate." At this point, the interviewer let the child work out the problem, adding comments as needed, such as letting the child know that it was permissible to place more than one card on a plate. When it seemed that the child had finished, the interviewer asked, "Are you finished?" When the child acknowledged that he or she was finished, the interviewer asked, "So, does every child get seven candies?" The problem has no solution; twenty-seven is not divisible by four; seven candies cannot be placed on each of the four plates.

Figure 1: The *Birthday Party* task cards

Analysis of the results began by considering correctness. First, we noted if a child who claimed to have placed seven candies on a plate, did indeed place seven candies on that plate. Second, when the child completed the task, did he or she recognize that not all plates had seven candies or did the child claim that each plate had seven candies? Third, we noted what strategies were used when composing seven. Did the child use a counting strategy or were number facts recalled? Finally, we were interested in investigating how the context of the birthday party might influence the children's solutions, and how the children would deal with a problem that had no acceptable mathematical solution.

FINDINGS

This section begins by summarizing the final arrangements of the cards on each plate, highlighting the issue of the blank card. It then describes one child's strategies when engaging with the problem. Finally, we describe how children dealt with the fact that this problem had no mathematical solution.

Table 1 summarizes the frequency of the final arrangements of candy cards on individual plates. The order in which the plates were filled was not recorded as children went back and forth moving cards from one plate to another. From the results we see that most children (11 out of 19) attempted to place seven candies on as many plates as possible, leaving only one plate with six candies.

An interesting aspect of this problem is the blank card. The blank card was meant to represent zero candies. In an abstract context, seven may be decomposed into seven and zero. However, in the birthday party context, there is no added value to placing the blank card on any of the plates. After all, if we place the card with seven candies on one of the plates, then that plate now has seven candies. Why add the blank card? And yet, eight children added the blank card to one of the plates. It could be that for these children, there was

Plate	Plate	Plate	Plate	Final Status	Frequency (claims not every plate has 7 candies)
7+0	6+1	5+1	4+3	7, 7, 7, 6	5(-)
7	6+1	5+1	4+3	7, 7, 7, 6	4(4)
7	6	5+1+1	4+3	7, 7, 7, 6	2(1)
7+1	6+1	5	4+3	8, 7, 7, 5	2(1)
7	6+1	5+3+0	4+1	8, 7, 7, 5	1(1)
7	5+1+1	-	-	7, 7, -, -	1*(1)
7	6	5	4	7, 6, 5, 4	1(-)
7	6	5	3	7, 6, 5, 3	1(-)
7+1	6+0	5+4	3+1	9, 8, 6, 4	1(-)
7+1	6+5	4+1	3+0	11, 8, 5, 3	1(-)

* This child stopped in the middle realizing that the problem was unsolvable.

Table 1: Frequency of arrangements of candy cards per plate (acknowledged that there were not 7 candies on every plate)

an implicit understanding that all of the cards had to be used, even if it did not make sense according to the story. If that was the case, then the blank card could have essentially been added to any of the plates. Yet, of the eight children, five placed it on the plate with the card with seven candies, representing the decomposition of seven into seven and zero. Perhaps they were trying to place two cards on each plate.

Regarding strategies for solving the problem, it was sometimes difficult to tell which strategy a child was employing. Counting was the most obvious, as children usually counted aloud or at least pointed and touched each candy, probably counting silently in their heads. Estimation and subitizing were more difficult to discern, although we gathered that many children used this strategy, especially those who placed seven candies on the first plate and six on the second, without counting the candies on either plate. Recalling number facts was most apparent for the decomposition of seven into four and three. For example, one girl looked at all the cards, chose the card with four candies on it and placed it on the first plate. Then she carefully looked over the rest of the cards and deliberately chose the card with three candies on it, placing it on the same plate as the four candies. She did not say anything. Because of the difficulty in accurately identifying which strategy was used, we do not quantify and summarize the number of times each strategy was used but instead illustrate in the following example how one child employed several different strategies when solving this problem and how the birthday party context impacted on her final answer.

Shula begins by counting the candies on the cards (lines 1, 3, and 4) and choosing which to place on plates. However, after she counts four candies (line 4) she adds a single candy without counting the total.

1 Shula: (Shula uses her finger to count the candies on the card:  .) 1, 2, 3, 4, 5.

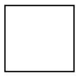
2 Interviewer: You can place more than one card on the plate.

3 Shula: (Shula counts the seven candies on the card ) 1, 2, 3, 4, 5, 6, 7 (and places the card on the first plate).

4 Shula: (Shula counts the four candies on the card ) 1, 2, 3, 4 (and places it on the second plate. She then places the card with  on the same plate.)

She then seems to randomly place cards on the other plates (lines 5 and 6), without counting, perhaps just to fill up the plates with something. Later (line 10), when she does count them, we learn that perhaps she thought she was placing seven candies on each plate, possibly by estimating the amounts.

5 Shula: (Shula places  and  on the third plate.)

6 Shula: (Shula takes the three cards that are left:  , ,  and puts them on the fourth plate.

Shula realizes that not every plate has seven candies (lines 8 and 12), but claims that it does not matter.

7 Interviewer: So, does each friend have seven candies?

8 Shula: Here (pointing to the second plate with four and one) there are 1, 2, 3, 4, 5. It's not important.

9 Interviewer: It's not important?

10 Shula: (Shula counts the candies on the other plates, miscounting the 5 and 3 and concluding that there are seven candies on that plate.)

11 Interviewer: So, does everyone have seven?

12 Shula: No.

13 Interviewer: Who doesn't have seven?

14 Shula: (Shula points to the second plate with 5 candies.)

15 Interviewer: So, what should we do?

16 Shula: Leave it.

Shula uses a combination of strategies to solve the problem. She counts, sometimes correctly and sometimes incorrectly. She may also have used estimation, placing the card with a lot of candies (six candies) on the same plate as the card with only one candy. Only when asked if each plate has seven candies, does she go back and count the candies on that plate. It is interesting to note that after Shula acknowledges that not all plates have the required seven candies, she does not try to rearrange the cards and work it out. Nor does she complain that there are missing candies. Instead, she says that it is not important and we should leave it.

In real life, 27 hard candies cannot be distributed equally among four plates. Of the 19 children who participated in this study, eight children answered correctly at the end that not all plates had seven candies. Four of those children, like Shula in the vignette above, acknowledged this fact without further comment. They did not seem bothered by not having seven candies on each plate; they made no comments or gestures that might be interpreted as frustration or exasperation; nor did they intimate that the problem was unsolvable. They seemed satisfied that the task was completed. One of those children placed the card with seven candies on the first plate, the card with five candies and the cards with one candy each on the second plate (i.e., 5+1+1), and then sat back and said, "There aren't enough." When the interviewer asked her how come, she pointed and replied, "Because here this is zero and six and here four and three." This child did not count and instead seemed to use a combination of subitizing (she recognizes without counting that there are three candies on the one card and four on the other) and number facts (e.g., 3+4=7). When the interviewer asks, "so only two friends will get candies?", she does not respond further and at her own initiative, she stops the activity, leaving the other two plates empty.

Of the four other children, three made it clear that one candy was missing. One child, trying over and over to solve the problem said, "Something is not right," before explicitly stating that one candy was missing. Another child turned over the blank card to make sure that there wasn't a picture on the other side. When asked what he was looking for, he pointed to the card with one candy on it as if to say that he was missing one candy. The third child said that a card was missing and stated "like that one" pointing to the card with one candy on it. The interviewer responded, "You are right, so, what do you say we go to the store and buy candies?" The boy nodded his head in agreement and laughed. The fourth child suggested removing the plate that had less than seven candies, leaving the remaining three plates with exactly seven candies.

Of the 11 children who incorrectly claimed that each plate had seven candies, many simply did not count the candies as they distributed the candy cards. Some seem to have relied on the strategy of estimation. That is, the card with seven candies and the card with six candies both looked like a lot of candies and thus, they did not feel the necessity for counting. One boy began by placing the card with seven candies on one plate, the card with six candies on the next, and the card with five candies on the third. Only at that point did he stop and realize that the five candies were not enough. After he thought about it and added two extra cards, with one candy on each to that plate, he placed the card with four candies and the card with three candies on the fourth plate, counting the candies on that plate to make sure. He did not go back and check the cards on the first two plates (the second plate had six candies) and so claimed that each plate had seven candies. Other children counted incorrectly. Incorrect counting most often occurred for the cards with seven and five candies, possibly because the candies were not pictured in a row, making it more difficult to keep track of the counted candies. In general, children engaged enthusiastically with this problem, used various strategies to solve it, and displayed various levels of competency when it came to counting.

DISCUSSION

The mathematical context of the problem posed to children in this study was the decomposition of seven. As described above, many children did not recognize that the problem was unsolvable, possibly because they estimated and did not count or because they miscounted. One factor which contributed to the difficulty of counting was the physical setup of the task. If children had been given actual candies and could physically move the items, they might have found it easier to count (Sarama & Celements, 2009). However, if they had been given candies, the problem would have lost the focus on decomposing seven. Furthermore, with physical candies, children might have distributed the candies by placing one candy at a time on each plate until all the candies were used up. It would then be quite obvious that one candy was missing and thus the impact of having an unsolvable problem might have been lost.

When children did recognize that every plate could not be filled with exactly seven candies, they did not seem very disturbed by the insolvability of the problem. None claimed that every problem must have a solution. Only one child tried over and over to move the cards around and solve the problem. In addition, unlike the students in the studies reviewed in the background (e.g., De Corte & Verschaffel, 1985), none of the children in this study gave a non-relevant solution.

In analyzing the reasons behind the results, we offer a few possibilities. First, it could simply be that this problem was not abstract and children were able to actually see the pictures of the candies and see that there were not enough. However, in a previous study of students' handling of unsolvable problems, Reusser and Stebler (1997) found that when students were told to explicitly make a sketch of a real life problem, the results hardly improved. What did improve the results were students' past experiences with other unsolvable problems. Reusser and Stebler (1997) surmised that when students are made to solve "so-called" real- life problems that are essentially routine mathematical exercises disguised as word problems, they become immune to the actual contextual limitations of the problem. The students in that study were fourth and fifth graders, old enough to have absorbed implicit norms which indicate that every mathematical problem given in school, must have a solution. The children in our study were quite young, not yet enculturated to school mathematics problems. They were most likely young enough to rely on their natural problem solving sensibilities, sensibilities we wish to preserve and strengthen. We suggest that engaging young children with unsolvable mathematics problems is one way to provide experiences which children may reference in the future.

However, not every unsolvable problem may encourage children's insight into or acceptance of the insolvability of that problem. The context, and how close it is to the child's real world, may also impact on the child's ability to see that a specific problem is unsolvable. The solution to is not a whole number. The problem presented in this study cannot be solved. However, there are also cultural norms related to the practice of birthday parties in kindergarten. Four children accepted the final status of the plates without further comment. For them, it could be that as long as everyone received some candies, the problem had a satisfactory ending. Or perhaps they felt that the child with fewer candies might get an additional piece of cake. One child stated that one plate should be removed. Perhaps for that child, equal sharing is more important than filling up all the plates. Perhaps, from that child's perspective, it would be better to invite only three friends than have four friends who cannot get an equal amount of candies. Three children stated that one candy was missing; one of those three playfully agreed to go with the interviewer to buy more candies. That is a solution which is quite realistic. In answer to the the title of this paper, kindergarten children can engage with unsolvable mathematics problems.

ACKNOWLEDGEMENT

This research was supported by THE ISRAEL SCIENCE FOUNDATION (grant No. 654/10).

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"If she had rolled five, she'd have two more": Children focusing on differences between numbers in the context of a playing environment

Dorothea Tubach

TU Dortmund University, Institute for Development and Research in Mathematics Education (IEEM), Dortmund, Germany, dorothea.tubach@tu-dortmund.de

Mathematically rich games can offer children important informal learning contexts for playful mathematical activities in kindergarten. At the same time, these activities can be used as starting points for formal learning in primary school. In the reported qualitative study, 20 children were observed dealing with three developed playing and learning environments in the last year of kindergarten and the first year of school. This paper concentrates on children's mathematical activities and insights while dealing with a playing environment in kindergarten. On the one hand, the focus is on how children interpret relationships between numbers. On the other hand, two different types of play in the game are outlined. The results will be discussed regarding connecting points for learning processes in primary school.

Keywords: Kindergarten, playing, games, dice, relational understanding.

PLAYING GAMES AND LEARNING MATHEMATICS BEFORE FORMAL SCHOOLING

In recent years, empirical studies have provided evidence that playing mathematically rich games can have a positive impact on children's mathematical learning, including in kindergarten (e.g., Stebler, Vogt, Wolf, Hauser, & Rechsteiner, 2013). Characteristics of mathematical learning situations in games have been worked out and conditions for their development have been investigated, whereby mathematically rich games can provide a meaningful context for mathematical activities and are open for different individual strategies (Stebler et al., 2013). An adaptive guidance through adults can help to unfold the mathematical potential of the game as a learning situation for the child (Schuler, 2011). At the same time, mathematical activities can themselves – independent from a game – be considered as play. This is the case if the mathematical activity has characteristics of play, e.g. it moves between the poles of rules and freedom (van Oers, 2014).

While current studies emphasize the importance of playing games for mathematics learning in kindergarten, playing as a mathematical activity, as described by van Oers, can be important for both learning locations: kindergarten and primary school [1]. However, there are fewer findings on how to use the idea of 'mathematical play' as a bridge between kindergarten and primary school at present, as well as *how* mathematical learning in kindergarten is effected in the context of a game (van Oers, 2014). The purpose of this paper is to ascertain the children's mathematical interpretations and characterize the playing activity in the context of a playing environment to outline opportunities for connected learning processes in primary school.

Playing and mathematics learning

As mentioned above, playing games seems to provide an important mathematical learning opportunity. But how can play or playing be characterized? Following van Oers (2014), play is understood in this paper as an activity that is characterized by *high involvement* of the actors, oriented on *rules* and allowing *some degree of freedom*. Accordingly, a characteristic feature of playing activities is that players voluntary adhere to rules and adopt a specific role, e.g. in a game, they engage themselves in a fictitious competition. The rules of the game can be explicit or implicit, predefined or negotiated in the process of playing. Van Oers (2014) differentiates between four different functions of rules: (1) social rules, namely how to interact and deal in play; (2) technical rules, concerning how to use play objects properly; (3) conceptual rules, regarding how to act based upon specific concepts; and (4) strategic rules, in terms of how to improve the playing process. Van Oers (2014, p. 62) defines the degree of freedom in a positive way, “as the freedom *to* change, to resist, to produce extravagant ideas and so on” [original emphasis].

Ginsburg (2006) distinguishes the observable mathematical play of children according to different types: “Mathematics Embedded in Play” and “Play Centering on Mathematics”. The former type can arise by playing mathematically rich games. In particular, a game can imply mathematical aspects, although the core is the competition among the players. In contrast, “Play Centering on Mathematics” occurs in operative discovering, exploring and inventing patterns and structures. Children play with mathematical objects by varying them and, thus gaining insights into mathematical relations: They discover the impact of a change of objects and how to react accordingly with another specific change (Steinweg, 2001). Both types of play can offer children a context to verbalize their strategies and interpretations, as well as negotiating mathematical meanings. The playing environment “Who has more?”, which is addressed in this paper, links both types of play according to Ginsburg (2006): it enables “Mathematics Embedded in Play” as a game with mathematical aspects. While the choice of structured game materials it is designed to allow play with mathematical objects. Therefore, “Play Centering on Mathematics” can be realized. Here the question is how can the playing activities of the children in the context of the playing environment “Who has more?” be characterized?

Mathematical play and relational understanding

The pivotal activity of mathematically-centered play involves establishing relations between mathematical objects of play. Wittmann and Müller (2009) highlight the ability to identify and use relations between single numbers by counting and calculating as a central objective for learning processes in kindergarten. The epistemological perspective broadens the view for the principally relational nature of numbers. Numbers and relations between numbers are not concrete objects, but rather can only be represented by symbolic or concrete objects. Only by dealing with the concrete and abstract objects and construing differences between them can children acquire the concept of numbers and their relations (Steinbring, 2005).

Due to the relevance of relational understanding, the playing environment “Who has more?” is focused on relations of differences. It is intended to give children the opportunity to identify and use relations of differences in the interactive context of a game and based upon the game material. The main questions addressed in this paper are as follows:

1) How do children in kindergarten interpret relations of differences in the interactive context of the playing environment?

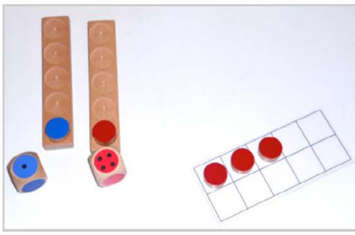
2) How can the children’s ‘mathematical play’ in the context of the playing environment be characterized?

METHODS

The research method is oriented by Cobb and colleagues’ method of “design experiments” (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). Accordingly, three complementary playing and learning environments for kindergarten and primary school were designed, focusing on the exploration of relations between numbers. In spring and summer, children in the last year of kindergarten were involved in playing environments. At the beginning of the first school year (in autumn and winter), they were involved again, albeit now in the context of learning environments. Overall, about 20 children were observed by video over two cycles dealing with the playing and learning environments in kindergarten and primary school. The sequences presented in this paper are from observation in kindergarten. To reconstruct interactive processes of understanding, a qualitative approach is chosen, oriented in the interpretative classroom research (Krummheuer, 2000). Essential for the mathematical analyses is the epistemological triangle, as described by Steinbring (2005), which enables identifying specific reference contexts that children use by interpreting differences. The epistemological analysis particularly focuses on the reconstructions of the interactive process of constructing knowledge based upon actions and interactions.

According to the rolled number, counters are put in the block of 5

Comparison: There are three red counters more.



The red player is allowed to take the three counters and put them onhis or her ten-frame.

Figure 1: Rolling and comparing

Construction of the playing environment "Who has more?"

In the following, one of three designed playing environments is exemplified: "Who has more?" The learning environment building on this playing environment is not considered here (for further details, see Tubach & Nührenbörger, 2014). "Who has more?" (WHM) is a game for two children, whereby each child receives a wooden block of five, a ten-frame and a dice (with the numbers 0 to 5) in the color blue or red. The game material is completed by small round gaming pieces, called counters (with a red and an alternate blue side).

Rule of the game: Both players roll their dice and put the appropriate number of counters in their respective blocks of five, according to the number rolled. The player with the higher number of counters is allowed to take the difference of counters (the ones that he or she has more) and puts them on the ten-frame (see Figure 1). Afterwards, the blocks are cleared and the dice are rolled again. The first player to fully fill the ten-frame wins the game.

At the heart of the playing environment is the comparison of two numbers and the determination of their difference. Children have the opportunity to gain insights into relations of differences. Meanwhile, children collect, structure and determine the number of counters on their ten-frame and gain experiences in composing and decomposing of numbers.

UNDERSTANDING OF MATHEMATICAL RELATIONS BY PLAYING "WHO HAS MORE?"

Maya (5,7) and Leon (6,4) play the game WHM with their guiding adult in kindergarten, a few months before entering primary school. Maya is already acquainted with WHM from previous game experience, whereas Leon is playing for the first time. From this game, two sequences are selected in which the children are engaged with the comparison of two numbers. One focus of the following analysis is on the children's interpretation of differences. The leading questions are: How do children interpret differences based upon the game material? Which reference contexts do the children use for the interpretation of differences in the gaming process? The second focus is on the characterization of the playing activities in these sequences. This will help to gain deeper insights into the mathematics activity in the gaming process, as well as understanding and clarifying the role of this game for mathematical and playing activities.

Focus 1: Interpretation of differences – Comparison of 3 and 2

Maya (M) and Leon (L) start the game WHM. Maya has chosen the red color and rolls number 3, while Leon rolls number 2 with the blue dice. Both children put the corresponding number of counters in their blocks of five (see Figure 2). They agree that Maya has one counter more. The following scene begins as the guiding adult (GA) asks once more.

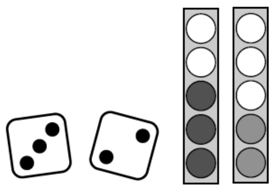


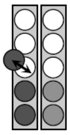
Figure 2: Initial situation

1 GA Ok. But how do you see that it's <u>one</u> more?

2 Maya 'Cause, 'cause <u>three</u> (*takes her dice with the rolled number 3 and covers one dot with her finger*)

3 GA Why aren't there <u>two</u> more?

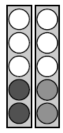
4 Maya If, if you (.) because I didn't roll two. Leon rolled two and I have three <u>one</u> more (*lifting her third red counter and putting it back again*)



5 Leon # If she had rolled five, she'd have two more.

6 Maya # then I can take it (*takes the counter and puts it on her ten-frame*) [...]

7 Leon Ah I know, I know (*points at the blocks*) Look here, one, two (*tipping the blue counters*) and if it was there (*tipping the third now empty field of M's block*), it's <u>always</u> the one <u>above</u>, which is one more. And if she had five and I had two, then would be one, two, three, four, <u>five</u> (*tipping the fields from bottom up in M's block*) So one, two, <u>three</u> (*tipping the empty fields of M's block*) more. Three (.) more. Three pieces more (.) If she had rolled five.



In reply to the GA's question (l. 1), Maya turns to her dice and covers one dot. Therefore, instead of three, there are only two dots visible. By showing with the dice pattern how to change the number 3 to achieve 2, Maya relates the two rolled numbers 3 and 2: if you cover one dot of three, both dice show the same number of dots (l. 2). As the GA asks once more why there were not two more, Maya emphasizes that she rolled another number than Leon: she demonstrates the difference between the rolled numbers with the help of the blocks of five and lifts her third counter. Hence, she creates an *equality* of the red and blue numbers of counters. She emphasizes the difference 'one' by putting her counter back on the block (l. 3). Therefore, she interprets the difference as the change needed to establish an equality of two numbers; in this case, by removing or covering the difference. Leon, however, constructs an example of a number pair (5 and 2, l. 7), which he regards will have the difference 'two' (l. 4): if Maya had rolled number 5, she would have had two more. Accordingly, Leon increases the minuend to achieve a greater difference. In the following, he gains a new insight into how to compare two numbers ("I know"): he divides the counters in the blocks into 'lower' and 'upper', stating that the counters above are the additional ones (l. 7). Consequently, he matches the equal number of red and blue counters and highlights that only those counters that cannot be matched have to be considered. These counters can subsequently be counted one-by-one. By separating the numbers of counters into 'lower' and 'upper' and focusing on the 'upper', Leon creates a new initial situation for comparing: instead of 3 and 2, he compares 1 and 0. Here, he unconsciously uses the mathematical relation that the difference remains the same if minuend and subtrahend are increased or decreased by the same amount. He illustrates this insight with an own number pair of 5 and 2 and, thus, probably picks up on his assumption in line 5 to check it. He determines the difference by first identifying the fields where red counters would be located. The counting procedure of the fields seems to represent or replace the concrete action of putting counters in the block. He then counts the upper three empty fields in Maya's block where counters would be located, thus concluding that the difference is three. He succeeds in matching the equal number of blue and red counters, which are no longer important for determining the difference, whereby he can compare 3 and 0 rather than 5 and 2.

Conclusion: The children's interpretation can be characterized by their different reference contexts. Maya's actions indicate that she uses a reference context for interpreting the difference, which can be called "equalizing". She answers the question of how many counters have to be removed to establish an equality between the two numbers of counters. Expressed algebraic: $a\text{-}d\text{=}b;\ b\text{+}d\text{=}a$. She represents this change dynamically in the material. In contrast, Leon interprets the difference in a spatial-static way, which can be described as "matching", whereby he matches the same number of red and blue counters and determines as difference the number of counters that cannot be matched. Accordingly, he changes minuend and subtrahend by the same number to create a 'to zero' comparison: $a\text{-}b\text{=}(a\text{-}c)\text{-}(b\text{-}c)\text{=}d\text{-}0$, with $c\text{=}b$. By interpreting the arrangement of counters, however,

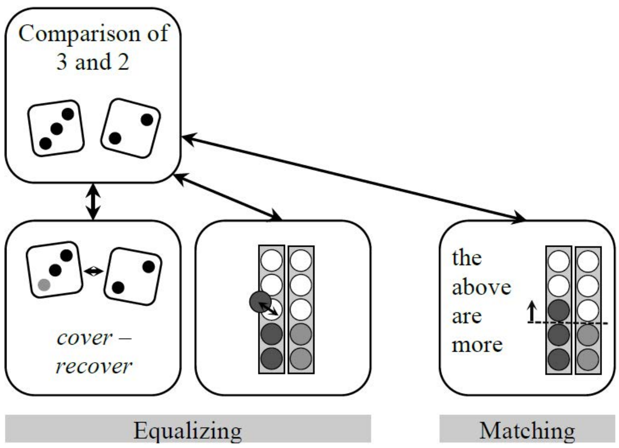


Figure 3: Epistemological triangle: 3 and 2

it is not necessary to identify the specific number of the equal change as subtrahend ($c=b$); rather, it is sufficient to remove or separate two unspecific yet equal amounts. This view on the arrangement of counters in the blocks of five helps Leon to determine the difference of his own example of a number pair.

The interpretation of "matching" builds upon the equality of two numbers to focus on the 'inequal'. "Equalizing", however, concentrates on inequality to determine the necessary change to establish equality between the two sets of counters. For both interpretations, it is not necessary to quantify the number of the equal counters; rather, a qualitative estimation 'equal' is sufficient.

Comparison of 5 and 2: What does Maya have to roll?

In the following, Maya and Leon win further counters. Now Maya has nine and Leon has seven counters collected on the ten-frame (see Figure 4). Leon rolls number 5 and is happy. Maya protests that number 5 would not be possible, since Leon would win too many counters. Hence, the question arises of what Maya has to roll for Leon to win.

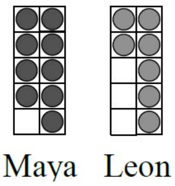


Figure 4

8 GA What does Maya have to roll so that you've won <u>immediately</u>?
9 Leon Mhh, one or zero or two.
10 GA Then you could win?
11 Leon 'Cause, 'cause wait (*takes his dice with the rolled number five and covers two dots with two fingers*), she's got to roll two.

12 GA # Why?
13 Maya # Heh?
14 Leon Because then I have three more (*shows his dice with the two covered dots*)

Leon answers the GA's question by giving three number pairs (5,1; 5,0; 5,2) with a difference greater than or equal to three (l. 9). Apparently, he assumes that he can win with every difference that is not less than three. In response to the question of whether he would win then (l. 10), Leon takes his dice with the rolled number 5 and covers two dots with two fingers so that only three dots are visible (l. 11). He concludes that Maya has to roll number 2 and then he would have three more. This approach leads to the following interpretation: Leon covers as many dots from the dice pattern of 5 so that three dots are left. The number of covered dots represent the number that Maya has to roll. In opposite to Maya, who covered the difference, Leon covers the subtrahend and the three dots left represent the difference.

It is remarkable that, on the one hand, Leon uses the dice pattern of 5 to *construe the difference* of 5 and 2. On the other hand, he uses it to *construct differences* due to the context of the game required to find a number pair with 5 for the given difference 3.

Conclusion: Leon's interpretation of the difference shows an expansion of his previous idea, whereby he decomposes the minuend 5 in two parts: 2 and 3. This enables both the statement that Maya has to roll number 2 to win three counters and the determination of the difference 3, if Maya had rolled number 2. Therefore, he compares 5 and 2 by "subtracting" 2 from 5. Expressed algebraic: a-b=d. Here, the activity of covering dots of the dice pattern has a double function: the construction of differences and the construing of differences.

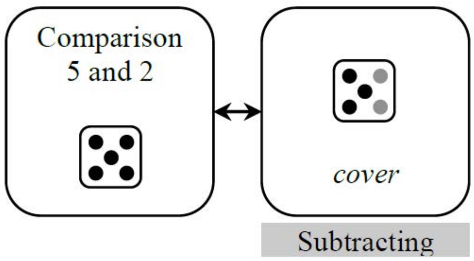


Figure 5: Comparing 5 and 2

Focus 2: The playing activity – Mathematically-centered play while playing a game

The playing activity of Maya and Leon in the context of "Who has more?" can primarily be described as "Mathematics Embedded in Play" (Ginsburg, 2006). Mathematical aspects are relevant, although the main issue is to win counters to fill the ten-frame the first. In addition and more surprisingly, "Play Centering on Mathematics" (ibid.) can be identified as a mathematical activity focusing on differences (see Figure 6). In this section, both mathematical playing activities are analyzed by the following characteristics: *involvement*, *rules* and *degree of freedom* (van Oers, 2014).

Children seem to be highly *involved* by playing WHM. They engage themselves in the fictitious competition and are pleased with a 'good' rolled number, for ex-

"Mathematics Embedded in Play" in the game	⇄	"Play Centering on Mathematics" esp. Differences
Put counters in blocks of five, according to the rolled number, determine the difference and fill the ten-frame.		a) Vary the use of materials b) Vary numbers c) Vary the interpretation of differences

Figure 6: Mathematically-centered play in the context of playing a game

ample. The game is framed by different explicit technical and conceptual *rules*, while further rules are negotiated in the gaming process (e.g. that the ten-frame has to be filled exactly). The children are *free* to choose whereupon they put their attention: social or mathematical aspects. Another analysis of WHM (Tubach & Nührenbörger, 2014) showed exemplarily that children keep the rule that 'the ten-frame has to be filled', aside from which they try to interpret the rules in their favor, e.g. manipulate the dice to get a full ten-frame quicker. Therefore, playing the game WHM embeds mathematics.

Nevertheless, this deviation from the intended playing course allows another type of play, namely "Play Centering on Mathematics". This playing activity can be recognized in the variation of using the game material, the variation of numbers and the interpretation of differences in the game material. In the following, these characteristics of mathematically-centered play are illustrated based upon the two sequences:

Variation of using materials: The reinterpretation of the game material is the key aspect for mathematically-centered play: Maya uses the dice pattern, which originally randomly determines the number of counters to put in the block of five, to represent the difference. In the second sequence, Leon also uses the rolled number as a representation to construct a number pair for a given difference. Furthermore, Leon regards the blocks of five 'as if' counters would lie there. Therefore, it is possible for him to cope with fictitious number pairs, not only to determine differences but also to test assumptions (5 and 2 have the difference 2).

Variation of numbers: Maya decreases the rolled number 3 by one and establishes a relationship to Leon's rolled number 2. Leon tries to increase the difference and increases the minuend of a number pair (5 and 2 instead of 3 and 2).

Variation of interpretations of numbers and differences: The analyzed different reference contexts show that children vary their interpretation of differences and also shorten the process of comparing two numbers in their blocks of five.

Children are obviously *involved* in these varying activities, as they are based upon their own ideas. They play together rather than against each other and inspire themselves towards new ideas, pick them up and develop them further. The *rules* are not explicit and predefined, but rather implicit and arise in the interaction in the playing process. In the context of the mathematically-centered play, transparency in own ideas seems important (social rule). At the same time, the children's ideas and interpretations are oriented on the properties and functions of the game material, e.g. they choose realistic numbers (technical rules). Their mathematical considerations are oriented on the game, which means that the game builds the conceptual framework for the mathematical activity (conceptual rule). The *degree of freedom* is apparent in the conceivable variations of how children (re-)interpret materials and their view on differences, as well as how they change numbers.

By means of the selected scenes, two types of play can be reconstructed, namely playing the game WHM and therein realizing mathematically-centered play with numbers, materials and interpretations.

CONCLUSION

The selected sequences of Maya and Leon indicate that the playing environment "Who has more?" offers rich possibilities to discuss interpretations of relations of differences. The epistemological analysis shows that children use individual reference contexts to compare two structured numbers of counters. However, these reference contexts are not used constantly, but rather are varied and become more sophisticated. This is reflected by the representations used and the ways of interpreting differences. As representation for differences, children use:

a) dice patterns, covering dots to represent a second or third number

b) blocks of five, to compare two linear structured (real or imagined) numbers

In addition, three different interpretations of differences could be worked out:

a) Equalizing: The difference (d) is the change needed to achieve equal numbers of counters: $b=a-d$ or $a=b+d$.

b) Matching: Equal amounts are matched to determine the number of counters that cannot be matched.

c) Subtracting: The two amounts are related so that the smaller number is interpreted as a part of the minuend, which can be removed: $a-b=d$.

It becomes clear that the recurring new situations for comparing and discussing in the game provide occasions to use not only individual different strategies but also make situational different and new interpretations, which become more differentiated (Nührenbörger & Steinbring, 2009; Stebler et al., 2013).

The analysis of the playing activities in the game also provided evidence that "Who has more?" not only enables "Mathematics Embedded in Play" but also allows "Play Centering on Mathematics", even while playing the game (Ginsburg, 2006). This mathematically-centered play can be distinguished from the social play where mathematics is embedded, in that it involves another intention (instead of winning, the focus is on varying numbers, materials and interpretation), it follows other rules and children are involved in other roles (explorer instead of competitor) (van Oers, 2014). Essential for mathematical play is the reinterpretation of the game materials as representations of mathematical relationships. Mathematically-centered play gains space in the process of playing the game. Thus, the space for mathematically-centered play is always limited by the game. To maintain the game process, there are only short periods for mathematically-centered play; for example, a new rolling of the dice interrupts the play with mathematical objects. If children's play stays focused on mathematical relations, e.g. they try to find further number pairs with a given difference, the game fades into the background or disappears.

Hence, connecting points for the arrangement of learning processes in primary school can be deduced: here, the mathematically-centered play can achieve more space in the context of a learning environment in primary school. Accordingly, children's experiences to construe and construct differences can be picked up, to play with the same material but independent from game, to construct differences, namely finding number pairs with a given difference.

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ENDNOTE

1. German kindergartens (depending on the federal state) normally have recommenations but not an obligatory curriculum for mathematical education. Compulsory education begins with school entry for children aged 5 to 6.

From speaking to learning of parallelism and perpendicularity relations

Paola Vighi

University of Parma, Mathematics and Computer Science Department, Parma, Italy, paola.vighi@unipr.it

This paper studies the interaction between language and mathematical learning, with the aim of investigating whether the spontaneous language of younger pupils and class communication may improve understanding and learning. Simple artefacts, based on 'parallelism' and 'perpendicularity' of segments, were prepared and presented to two first grade classes in a primary school, through a narrative. Two different linguistic approaches were adopted: traditional mathematical nomenclature presented by the researcher in one class, spontaneous language suggested by the pupils in another class. Starting from this idea, similar didactical itineraries were realised in both classes. The results of the experiments are analysed and compared.

Keywords: 1st grade, spontaneous language, traditional language, parallelism, perpendicularity.

INTRODUCTION

Much research deals with the problem of the relationships between language and thinking. As is known, Piaget (1983) and Vygotsky (1987) have very different and divergent positions on this. From an epistemological point of view, the problem is in term of a priority among experience, language and mathematical knowledge. For Wittgenstein (1978), the language is necessary to communicate, but the word used to describe an object do not convey immediately its meaning, which must be constructed. Lakoff and Nunez (2000) see the linguistic activity as an effect of the cognitive activity and strictly dependent on it. Moreover, Sfard (2001) confers a big importance to the communication in mathematics; specifically, she deals with this topic in term of "metaphors".

In particular, in matter of the role of the language in learning processes, research underlines two fundamental and apparently opposing aspects: the specificity of mathematical language and the role of natural language in mathematical communication. In the present paper, I want investigate whether (and how much) the spontaneous language used to describe some geometrical configurations can improve the understanding and learning of some geometrical concepts. The main hypothesis is that the transition from the observation of geometrical configurations to the use of a spontaneous language to describe them can improve the learning of mathematical concepts. The choice of the mathematical topic "parallelism and perpendicularity" is motivated from the following reasons: these relations are uncommon in the 1st grade of Italian primary school, so it is possible to observe the role of spontaneous language without the influence of scholastic language. The geometrical context is suitable at this age, since it relies on visual mediators, objects and drawings. The history of humankind (Keller, 2004) shows drawings, made in Paleolithic, in which parallel and perpendicular segments are present (Lascaux caves). The training with parallel and perpendicular segments in 1st grade is encountered in writing some capital letters (E, F, H, T, ...). I found only a research on this geometrical topic (Meyer, 2010), but it was conducted in different manner and with students aged 9–10 years.

THEORETICAL FRAMEWORK

The present research compares the role of children's spontaneous language with that of traditional geometrical language in the acquisition of knowledge. The starting point is the following quotation:

> [...] *it is necessary to postpone the systematic use* [...] *of specific words*, i.e. the typical concepts of geometry (circle, square, sphere ...) [...]. Children must also *learn to speak about spatial events* with their own words (i.e. to describe a path, a fig-

> ure, a movement) (Speranza, Vighi, & Mazzoni Delfrate, 1988, p. 14)

In other words, at the beginning, pupils use their spontaneous language to describe geometrical situations and then the teacher promotes the passage to the specific language.

Starting from the idea that the origins of all languages are “rooted in the child’s first experiences”, Gawned (1990, p. 31) elaborated a theory based on four consecutives stages: the origins of the language of mathematics, as well of all languages, are rooted in the child’s first experiences (*real-word language*); the *classroom language* has an important formative effect on children’s mathematical understanding and learning, the teacher manages the *specific domain of the language of mathematics* as long as the *construction of meaning in mathematics* occurs. These four stages are not strictly sequential, but in practice, this model appears quite rigid.

Geometry allows one to work with concrete objects and their description by natural language. Therefore, I prepared some artefacts in the meaning of “theory of semiotic mediation” (Bartolini Bussi & Mariotti, 2008): an artefact is a “tool of semiotic mediation” when the teacher intentionally uses it to mediate a mathematical content:

> In particular, the teacher may guide the evolution towards what is recognizable as mathematics. In our view, that corresponds to the process of relating personal senses (Leont’ev, 1964/1976, p. 244 ff.) and mathematical meanings, or of relating spontaneous concepts and scientific concepts (Vygotsky, 1934/1990, pp. 286 ff.) (Bartolini Bussi & Mariotti, 2008, p. 754).

Duval (1993) introduced another important aspect, the ‘semiotic representation registers’ and their role in knowledge: he provides a rich theory on it, based on the assumption that ‘there is no knowledge without representation’. Following this theory, the introduction of artefacts promotes the transition from “visual semiotic register” to “verbal semiotic register”:

> We may distinguish three main groups of semiotic representations: material representation (in paper, card, wood, plaster, etc.), a drawing (made either with pencils on a sheet of paper, or on a computer screen, with use of a geometrical software, etc.), and a discursive representation (a description with words using a mixture of natural and formal languages). Each register bears its own internal functioning, with rules more or less explicit. Moreover, students have to move from one register to another, sometimes implicitly, sometimes back and forth (Dorier, Gutiérrez, & Strässer, 2004).

Following Nonaka and Tageuchi (1995, quoted in Lester & William, 2002, pp. 494–495) the dialogue is fundamental in moving from a tacit knowledge to an explicit one. Speech helps to build concepts; subsequently the word used is a symbol of the concept itself.

Sfard (2001, p. 26) integrates the previous considerations:

> The conceptualization of thinking as communication is an almost inescapable implication of the thesis on the inherently priority of social origins of all human activities. Anyone who believes, as Vygotsky did, in the developmental priority of communicational public speech (e.g. Vygotsky, 1987) must also admit that whether phylogeny or ontogenesis is considered, thinking arises as a modified private version of interpersonal communication.

Following these hints, the research chose to take advantage of the “need to find similarities” to lead pupils speak in their “real world language” about the relative position of segments.

The research questions

1) Is it possible to extend ideas of Speranza and colleagues (1988) to the descriptions of binary relations of parallelism and of perpendicularity? Could the use of personal words (or locutions) enhance a child’s understanding of the concepts of parallelism and perpendicularity? The transition from spontaneous language to geometric language could improve the learning?

2) Do very young pupils perceive the relations of ‘parallelism’ and ‘perpendicularity’ from a qualitative point of view?

THE EXPERIMENT AND ITS METHODOLOGY

The paper presents the results of a research performed in school year 2013/14 in different Primary Schools, which will continue into the following year. In particular, it reports only on an experiment that involved two 1st grade classes[1], class 1A (25 pupils) and class 1B (23 pupils), having the same mathematics teacher. All sessions of work were recorded using a video camera.

The researcher (the author of the present paper) prepared some artefacts, made of simple materials (cards, straws, buttons, glue, adhesive tape): eight square cards, with 23 cm. sides, each containing two straws in different positions. I used pleasant colours, red and green, for caterpillars, and a 'neutral' colour, light grey, for the background. Obviously, straws would be unsuitable for representing segments, since they are 3D objects, but the age of pupils (6–7 years old) allows this choice. Furthermore, the superimposition of the straws it is not possible, while this operation is permitted with segments in the plane. A small button (as the head of caterpillar) is glued at one end of each straw; in this way, another didactical variable, the orientation of the segment, is introduced. Children suggested this feature during a previous activity involving segments representing caterpillars (Vighi, 2008). The researcher produced some cards about parallelism (cards 2, 3, 6), other cards about perpendicularity (cards 5, 7, 8), and others (cards 1 and 4) with straws neither parallel nor perpendicular (Table 1).

The experimentation took place with the presence of both, the teacher and the researcher, in classroom. The activity starts by presenting a narrative regarding a green caterpillar, named Pelù [*P.*], and a red caterpillar, named Mangiamela [*M.*], who stroll together. It consisted of four (class1A) or five (class1B) *activities*:

1) <u>Presentation of the cards</u>. I adopted two completely different ways of working in class. In class 1A, the researcher presented the cards and she immediately introduced (and 'imposed') the corresponding traditional locutions: "*P.* and *M.* are parallel", "*P.* and *M.* are perpendicular" and "*P.* and *M.* are neither parallel nor perpendicular". In class1B, the researcher invited pupils to observe straws representing caterpillars placed on the cards and to describe them. Possible objections to the chosen methodology are: in class 1A the imposition of 'locutions' without explanation can create problems of understanding, in class 1B the choice of a 'name' for each card can obstruct the work by analogy and the recognition of common properties in same card's configurations. I was aware of this, but the aim was also to observe if and in which way pupils would overcome these problems.

2) <u>Choice of common linguistic expressions</u>. This activity, developed only in class1B, characterised the research. The children proposed many and different 'names' for each card and the teacher wrote all the linguistic expressions proposed on the blackboard; at the end, she asked for a vote to

1 Specifically, the experimentation took place in the Primary School of Vicofertile, a small town near to Parma (Italy). I acknowledge the teacher G. Barantani for kindly giving us the permission to perform in her classes and for her collaboration during the experiment.

Card 1	Card 2	Card 3	Card 4
			
Card 5	**Card 6**	**Card 7**	**Card 8**
			

Table 1: Cards

choose common locutions. Thus, the "language of classroom" (Gawned, 1990) was constructed gradually.

3) Drawing and writing in the notebook: the teacher placed each card in a very visible position and wrote the corresponding 'linguistic expression' on the blackboard. The task was to copy the card into the notebook and write its name, using a page for each caterpillar's position. In class 1A children wrote the 'locutions' imposed by the researcher (very long to write and difficult to read and to pronounce), in class 1B they wrote the 'names' obtained by vote.

4) Individual interviews: outside of the classroom, the researcher proposed two different tasks to each child. In the first the researcher presented the cards one by one to the pupil, asking him to remember to relative locutions and to say it out loud (transition from the representation to the language); when it was necessary, the child was allowed to bring her/his notebook to recall a forgotten linguistic expression. In the second part of the interview, the researcher gave two straws to the child, with the task of placing them following each locution previously introduced (transition from the language to the representation).

5) Activities in gymnasium: children worked two by two, reproducing the different mutual positions of caterpillars with their bodies, following the tasks suggested by the researcher.

The first four activities occurred of two-three weeks apart, the last at the end of school year.

RESULTS AND THEIR DISCUSSION

As expected, the *activity 1* had completely different effects in the two classes. In class 1A, the experiment confirms that the absence of a relationship between a word and its meaning created obstacles (sometimes insuperable): often, it compromised children's performances. I observed difficulties in remembering the word 'parallel' or 'perpendicular' and in pronouncing and writing them. Nevertheless, the methodology adopted allowed us to find possible relations among the cards described with the same locutions, without a specific request of recognition of common geometrical properties. In fact, it happened sometimes in the case of parallelism, rarely in case of perpendicularity: a possible explanation is that the visual perception of the first relation is 'stronger' than that of the second. In class 1B, the activity had a completely different development: each card suggested an image that evoked a common name or a linguistic expression and, in turn, each locution sent again to a mental image.

As regards the *activity 2*, children of the class 1B suggested many locutions, as shown in the following list of 'descriptions of each card' (DC):

- DC1: triangle, mountain, crocodile's mouth, their faces are far, they meet, they collide with each other, if you rotate it looks like an 'A'.
- DC2: pencil with rubber, they are horizontal, in a line, letter 'I', they are going to the same side, falling down the tree, looking like as one caterpillar, *M.* is going towards *P.*, *P.* works harder more since it ahead, letter 'I' rotated.
- DC 3: tree trunk, crocodile's closed mouth, horizontal stairs, pole, they are as a strip, they are horizontal, two lollipops, railway track, they do not meet since they are going straight, train, ladder with rungs.
- DC 4: beak, elephant's trunk, mouse's snout, mouse's whiskers, one straight and the other slanting, railway track, crocodile's closed mouth, railway, they do not meet since they are going straight, a boat.
- DC 5: as a 'T', *M.* helps *P.* to bring an apple from the tree, *P.* climbs up *M.*, turning it in another way it looks like a 'T', each one goes home, one goes up another one goes right, competing in a race.
- DC 6: slanting giraffe's neck, slanting vertical line; curve; escalator in a supermarket, slope, steps, competing in a race, slanting railway track, slanting vertical line, rows.
- DC 7: slanting; letter 'Y', *M.* is falling down, *P.* is standing up, a kind of 'L', turning it looks like a 'L', *P.* is below and *M.* is on it.
- DC 8: letter 'L', two attached lines, car seat, half rhombus, two colliding cars, *M.* sniffs the feet of *P.*, looks like a hut slightly tilted.

The locutions underlined represent the results obtained by votes. Afterwards, each locution became the only linguistic expression used in reference to the corresponding card. This list of possible names for the cards is a very rich material to study the role of the language as a vehicle of pupil's perceptions. In a previous research paper (Vighi & Marchini, 2014), they were analysed in terms of "intrafigural, interfigural or transfigural space", using the same adjectives introduced by Piaget and Garcia (1983) but with reference to the "stages of learning". I also observed the presence of "independent space" as reported by Speranza (1994); in particular, I remarked that the majority of the poll choices reveal an idea of intrafigural space (Vighi & Marchini, 2014, p. 115). The previous list contains metaphors (Sfard, 1997), linguistic constructions describing a subject in terms of another unrelated object. There are also expressions referring to movements of caterpillars (they collide, they are going to the same side, etc.), maybe influenced from the narrative.

Drawings made during the *activity 3* gave other information: it was possible to observe if pupil's drawings respect parallelism, perpendicularity, distances between segments, their orientation and their length. At this age, the ability to draw is limited, so the final drawings very often were quite different from the cards configurations. In particular, some names influenced the drawing: for instance, the name "letter Y" induced to draw two segments with different lengths and not perpendicular, the name "letter L" suggested drawing segments parallel to the sides of the card by a rotation of the segments, etc.

The qualitative analysis of films produced during *activity 4* was very interesting. The individual interviews allowed us to verify not only the children's learning and understanding, but also to study their behaviour, abilities, difficulties and progress. Regarding of the first part of the interviews, 'cards names' recollection, as expected, in class 1A were observed more difficulties than in class 1B. Indicatively the percentage of positive answers was respectively 30% in class 1A and 80% in class 1B. In the first class, very often children said: "I don't remember", "I don't know". Moreover, after the revision of their notebooks, some of them were unable to deal with the task. Often pupils remembered only the name of two or three cards and they used the locution 'neither parallel nor perpendicular', systematically, in the other cases. Therefore, the lack of connection between a word and its meaning clearly emerged (Wittgenstein, 1978). Some pupils overcame the problem by proposing personal *locutions* very similar to those that emerged in the other class: for instance, "the beak of a bird" for card 4, "the equal symbol" for card 3 etc. In contrast, in class 1B children gave correct answers very often. Sometimes they firstly remembered their own locution and only in a second time chose the common name. The case of Benedetta, a little girl of class 1B, is significant: in front of card 5, firstly she said her own description "*M.* helps *P.* to bring an apple on the tree" and only in a second time she remembered the common locution "as letter T" associated to the card. The behaviour of the pupils in class 1B can be explained also in term of 'evoked concept image' (Tall & Vinner, 1981): for instance, when card 1 was shown, the child spoke the locution "crocodile's mouth" almost immediately, since the 'mental image' and 'name' were strictly related.

About the second part of the *activity 4*, the work with straws, pupils particularly appreciated the possibility of 'manipulating the caterpillars', putting them in different positions. In both classes, I observed the attempt to remember the straws arrangements in the cards and to place them in the same way, but with very different results: it was very difficult in 1A, not in class 1B. Often in the first class, children put straws randomly, without relationship to the words suggested by the researcher, some pupils remembered only one relation, more often parallelism, only children with a 'very good memory' could deal with this straws activity without problems. An important observation emerged about the task "put the straw neither parallel nor perpendicular": sometimes its meaning became 'a change of straws orientation with respect to the sides of the card'. In fact, another didactic variable is present in our artefacts, the 'form of the background'[2]. In class 1B, both the activities of 'writing with straws' and of 'reading straws arrangements' appeared easy to execute. A possible interpretation, as previous written, is that the 'locutions' chosen after the vote (those underlined in the list) became the 'proper names' of the cards and that these 'locutions' derive from the natural or scholastic language, in any case from lived experiences.

2 I decided to study its role preparing also eight round cards with straws glued 'in the same positions' of the straws placed on the squared cards. I presented the same activity based on round cards in another class and I compared the results.

Lastly, pupils appreciated the *activity 5*: they considered it as a play activity and the suggestion to act out the caterpillars with their own bodies stimulated their motivation to remember the different arrangements of the straws.

CONCLUSIONS

Next year, the activity will continue and the researcher will show some films produced in other classes[3] on the same subject to class 1B. The comparison of very different locutions, adopted to describe the cards of Table 1, will pose the problem of the use of different linguistic expressions for the same thing. For instance, the card 3 was named "railway track" in class 1B, but in other classes the names proposed were "equal" and "road": the question will be on the use of only one name well accepted and understandable for all children. In a second occasion, the teacher will organize a meeting between her classes 1A and 1B, with a comparison of the locutions. In this way, gradually, it is possible to stimulate the need of a common language and, later, to touch the topic of the 'role of the scientific language'.

The activity proposed offers the possibility of rich visual and linguistic experiences about parallelism and perpendicularity. Pupils remembered more their own locutions easily, while the 'imposed locutions' created a real situation of difficulty. In class 1A some pupils tried to overcome by finding common properties when the 'name of the card' was the same. Other children memorized all the 'names' or many of them. In particular, one child refused to memorise and he repeated this question: "What is the meaning of this words?"

The answer to the first research question is affirmative: the experimentation confirms that the use of personal locutions favours the concepts learning. For instance, in class 1B, in the case of cards 3 and 6 the votes gave, respectively, "railway track" and "slanting giraffe's neck": a possible interpretation of the use of these two different metaphors is that pupils are unaware of the likeness (i.e. parallelism) of the two cards. But, when a child of class 1B describes the card 6 as "slanting railway track", he shows to transfer the 'name of card 3', "railway track", to the card 6, making an adjustment by the use of an adjective: it documents the recognition of a common property of the straws arrangements in both cards. Similarly, the confusion between "railway track" and "slanting giraffe's neck" shows a grasp of the property of parallelism present in both the cards. Hence, an apparent mistake becomes a symptom of generalisation and of the recognition of an analogy between the caterpillar's positions in cards 3 and 6.

Referring to the second research question, in both classes I observed that only a part of children perceived the presence of invariants and, consequently, they distinguished parallelism, perpendicularity or the incidence of two segments. As regard parallelism, the constant distance between straws plays an important role: sometimes a child indicates this property with her/his hand, comparing it with the cases of cards 1 and 4 and highlighting the difference of the distances between the straws. As regard perpendicularity, I could observe the clear role of writing capital letters: the idea that cards 5 and 8 present arrangements of the straws similar to 'letter T' (even if in 'not canonical position') or to 'letter L' supports the pre-conception of perpendicularity. The case of letter 'Y' (card 7) is different: if a child makes it by attaching two straws with as much contact as possible, he obtains a right angle, but in fact, the shape of the letter Y, without a right angles, prevails. However, the observation "turning in that way it looks like an 'L'", referred to card 7, suggests the recognition of right angles. Sometimes the role of capital letters was negative: since different letters have different names, some pupils did not recognise the analogy among cards 5, 7 and 8.

In conclusion, our research gives concrete and meaningful situations that highlight the invariants in the concepts of parallelism and perpendicularity. The proposed activity appears suitable for young pupils. Usually the teaching of geometry in Italian schools starts in the third class (pupils 8–9 years old), when the parallelism is necessary to work with parallelograms and the perpendicularity is associated with right angles. However, as I reported above, these concepts are already present in writing capital letters, so they can brought forward without problems.

3 Classes 1A of Fognano (Parma) school and 2A of Vigatto (Parma) school. I thank the teachers Barbara Riccardi and Lucia Ferrarini for their collaboration.

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ENDNOTE

1. Work done in the sphere of Research and Experimentation in Mathematics Education Unit – University of Parma – Italy.

TWG13

Posters

Preschool class children's mathematical meetings

Sofie Arnell

Linköping University, Department of Behavioural Sciences and Learning, Linköping, Sweden, sofie.arnell@liu.se

This paper presents a minor part of an ongoing study about how young children meet mathematics. The purpose of this study is to investigate how and when children meet mathematics and engage in mathematical activities in preschool class and also to see how their meeting with school's more formal mathematics is arranged.

Keywords: Mathematical meetings, mathematical activities, preschool class children.

INTRODUCTION

Mathematical knowledge develops through mathematical activities, when shifting from act to thought and backwards. Bishop (1991) listed six fundamental activities that constitute the foundation of a culture's mathematical development [1]. At an early age, children are involved and engaged in informal mathematics connected to these, but in different degrees (Seo & Ginsburg, 2004).

In Sweden, almost every six year old goes to preschool class, a voluntary type of school between preschool and elementary school. Preschool classes are supposed to follow most parts of the public school curriculum, but not the section about subjects and syllabuses, since preschool classes do not arrange regular lessons and subject teaching (Skolverket, 2011). When children enter school and meet more formal mathematics, we can see a large variation in mathematical skills between the children (Fredriksson, 2009). But we know less about how this meeting is arranged. Therefore, it is important to investigate in what ways children actually meet and engage in (informal) mathematical activities in preschool and to see how they meet more formal mathematics in school.

METHOD

The data collection has an ethnographic design (Hammersley & Atkinson, 2007). It includes observations of and informal dialogs with 20 preschool class children in their everyday classroom. The study began in September 2014 and will last to October 2015, when the children have entered school. After the summer vacation the observations will especially concern lessons in mathematics, but also "non lessons" such as lunch breaks.

ANALYTICAL FRAMEWORK

Throughout the data collection and the analysis process a grounded theory (GT) approach (Glaser & Strauss, 1967) is used. The research question is relatively wide and the constant comparative method in GT helps to avoid making premature conclusions by staying close to the data. It also helps to systematically sort the data to find categories that later will build up the theoretical model, grounded in data.

RESULTS

Preliminary results show that preschool class children meet mathematics in several ways, in this study constituted by different categories. An inside and an outside perspective represent each of these, where the inside perspective includes mathematics children meet through own mathematical engagement and the outside perspective includes mathematics children meet trough encouragement and involvement from others, such as teachers and peers.

One category, of how preschool class children meet mathematics, is called *notices quantities*. In this category the inside perspective, entitled *own quantity engagement*, contains the subcategory *quantity indicating*. This includes situations when the child, on its own initiative, meets or engages in mathematics by showing interest for quantities, for example by telling or informing about how many, being curious about how many and then exploring it, and also by asking questions with a specific number in it. The outside perspective of this category, entitled *quantity invitations from others,* includes two subcategories, *quantity challenging* and *quantity clarifying*. Quantity challenging involves situations when someone else, a teacher or a peer, challenges the child to explore quantities, for example by letting the child answering questions about quantities, count something, add two quantities up or take away something from a quantity. These tasks can be given both on a concrete and on an abstract level. Quantity clarifying involves situations where someone else, usually the teacher, confirms, develops and clarifies the child's quantifying answers. This category also involves making quantities visible for the child, for example by keeping statistics on the whiteboard or explaining the mathematical content in an activity, both individually and for the entire group.

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ENDNOTE

1. The six fundamental activities are: counting, locating, measuring, designing, playing and explaining (Bishop, 1991).

Pre-school child and natural number

Michaela Kaslová

Charles University in Prague, Faculty of Education, Prague, Czech Republic, michaela.kaslova@pedf.cuni.cz

This work represents a part of a longitudinal research project realised in Czech kindergartens during last 15 years. We analysed data obtained from 580 children 5 to 6 years of age, 6–7 months before they began primary school. The research was focused on the natural number in different roles and situations. The variability of children´s reactions is presented and a particular deficiency in the pre-school curriculum is described. The results can help elementary school and kindergarten teachers.

Keywords: Natural number, kindergarten, configuration, quantity.

INTRODUCTION

Pre-school children differ from primary school pupils in that they are in the stage of concrete reasoning and at a pre-operational level, their vocabulary is relatively rich but not yet stabilised because the development of notions is at its initial stage; their process of generalisation is limited by the characteristics of this evolutional stage including egocentrism (projection, identification, impossibility to differentiate their needs, desires from reality) and syncretic perception (the object as a whole); children's experience could be deformed by accompanying emotions. The philosophy of education in families has changed during last twenty years, as has the teachers' work at kindergartens. What is the situation in reality?

Natural number can play different roles; it depends on the context in which it is used. For example, it can be in the role of quantity (number of objects/of physical units) or without any mention of quantity. Research shows that the development of the concept of natural number is not yet independent of the character of counted objects (colour, position, distance, configuration, material, shape, weight, size, visibility, ...); see Piaget (2000) and Anderson (1987) cited in (Atkinsonová et al., 2003), Hejný, Kuřina (2001), Kaslová (2010) and others . This process of generalisation depends on different aspects. It is a relatively long process, not unified for six-year-old children. We assumed a certain starting level of the development of this notion. The pre-mathematical test (or PM-Test) is composed of 4 parts in which a natural number is in the role of: 1) a member of spoken series, the orientation in its order (a, b, c, d), 2) quantity – its independence of colour (a), shape (b), position (c), visibility (d) of counted objects or by mixture of these characteristics (m, n), 3) quantity in changing conditions (a, b); 4) quantity objects in the structure/ rhythm (a, b, c).

METHODOLOGY

The PM-test was realized in the form of a dialogue between a questioner and 1 or 2 children. The PM-T had 15 subtests and it was accompanied by common checking activities. Subtest (2c) focused on natural number in the role of quantity of spots. I observed the independence of their quantity on their mutual position. The PM-T supposes that the pre-school child is able to draw more than 6 different representations of natural number "six". The child had to observe a picture (6 spots on paper 10 cm x 10 cm); the configuration of spots corresponded to the configuration on the dice. The child received white paper of the same measures and had to draw the same number of spots but in a new position (using only one colour). We assumed that 90 % of children would be able to create a minimum of 6 "new draws" with a success and 50 % of children will use the strategy of modification (changing only one position of a spot).

SELECTED RESULTS

5 % of children were unable to discover a new configuration; 10 % worked well but at most they created only 5 different configurations; 4 % of children created more than 30 different configurations (max. 68); 11 % of children in 20 % of their pictures, at maximum, created a new configuration, dividing an old configuration in two parts and transferring one part to a new posi-

tion; 7 % of children created a configuration different from the model but in the majority of cases by making a quantitative mistake; 88 % worked with 0 – 3 mistakes at maximum. 22 % of children were able to use a good spontaneous self-check, corrections included at one correction, only 6 % of children self-corrected all their mistakes (complementation, cancellation) – half of this group created more than 30 configurations; 71 % children worked in a special rhythm: after 3 to 5 new configurations, they repeated one of the "old" configurations ("smaller or bigger" drawing). 94 % of children did not use the strategy of modification, in the majority of cases. This means that the configuration was made as a new picture – a new whole. This shows that the "whole perception" (syncretism) still predominates.

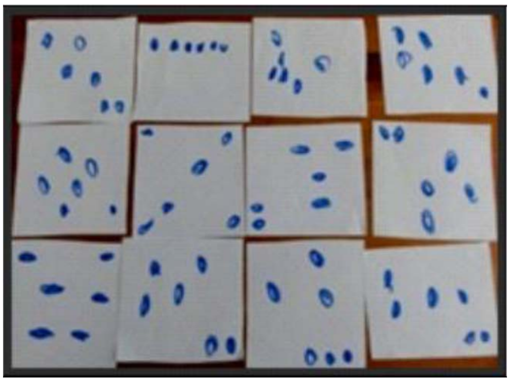

Figure 1: Example of children's work

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The development of numerical thinking in children aged two to five

Irit Lavi

The University of Haifa and Oranim college, Haifa, Israel, iritla1@gmail.com

In this study, I will trace the development of young children's numerical discourse. I present a theoretical model constructed to reflect the hierarchical nature of primary numerical discourse. This model will be tested and refined in a longitudinal study among 2 to 5 year old children. Every child will participate in a series of number-related activities, to be performed either with the help of concrete objects like cubes, candies est. or as specially designed iPad tasks.

Keywords: Young children, numerical discourse, developmental model.

This research rest on the *communicational approach,* rooted in Vygotsky's socio-cultural theory, which emphasizes the social aspects of learning (Vygotsky, 1962), and puts communication at the heart of any learning (Sfard, 2008). In this framework, the term "numerical discourse" embraces both interpersonal communication on numbers, and the intrapersonal communication (child's thinking about numbers). This non-dualist vision implies that the object called number, rather than being a self-sustained, mind-independent entity, is a discursive construct, that is, emerges from discourse and is built through communication (Sfard, 2008, 2012, in press). Numerical discourse is using number words while telling stories about the world. As opposed to Piagetian approaches, in the communicational approach there is no ontological gap between the world of abstract mathematical objects and the mathematical discourse as such. Numbers are part and parcel of the talk about them. Hence, numbers are discursive constructs that emerge for a person through using number words while participating in a numerical discourse. Learning mathematics is the process of extending and modifying one's mathematical discourse. This process, called *individualization* happens while the student imitates the expert in the discourse, and results in the learner's autonomous participation in the discourse. It is through the process of individualization of numerical discourse as presented by others that the child will build new objects called numbers (Sfard, 2008).

This longitudinal study will follow 16 children with middle and upper class socio-economic backgrounds. Data collection will continue for two years, and the participants will be split into groups based on their age. The data will be collected from conversations of the researchers, the participants and their parents as they perform predetermined activities. Each activity will be completed twice, once with concrete visual mediators and once with a specially designed programme on a tablet. During each meeting, the children will be shown a task involving one-to-one correspondence, rote counting and comparison. Conversations with the participants will be recorded on video and will be precisely transcribed.

In presenting the poster I will provide a brief overview of the method of study, the individualization process of numerical discourse and the developmental model. In assembling the model we based on (1) our understanding of numerical discourse (2) the hierarchical nature of this discourse (3) historical data and (4) past findings about rudimentary numbers learning (Sfard & Lavie, 2005).The underlying assumption in this model is that the initial numerical discourse as I know it from observation and experience is built from two different discourses: (1) Quantity discourse, which contains comparison words such as, "bigger than", "smaller than", "a lot", "few", "equals" etc.; (2) numerical discourse, which is comprised of number words. I will argue that at some point, these two discourses join together. As a result of this amalgamation, two new discourses are produced: Numerical quantity discourse and Arithmetical discourse which deals with numbers and operations performed on them.

The challenge of this study is to develop and refine this current version of the model that describes the formation of the numerical discourse using the characterisation of mathematical discourse as consisting of vocabulary and syntax, visual mediators, routines and endorsed narratives (Sfard, 2008, 2012) and analysed the data by these characterizations. I hope that this unique analytic approach would shed new light on the vastly investigated subject – the numerical discourse. Another challenge of this research is to identify the individualization process as accrued through the use of iPad and material objects. These two alternating learning environments were chosen because these are the environments children play in nowadays, and in which they feel comfortable.

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Visualisation of shapes and use of technology in kindergarten

Geir Olaf Pettersen[1], Monica Volden[1] and Elin Eriksen Ødegaard[2]

1 UiT The Arctic University of Norway, Faculty of Education, Tromsø, Norway, geir.olaf.pettersen@uit.no

2 Bergen University College, Faculty of Education, Bergen, Norway

In 2006, the Norwegian government added mathematics and technology as a subject in the national guidelines for kindergartens in Norway. The activity in this case is a visualisation task about the connection between 2D and 3D shapes. The activity starts with a short introduction from one of the researchers, where she together with children explored different 3D shapes with collapsible folding shapes. After a while, iPads were taken into the activity, using an application where they could see 3D shapes fold and unfold. The activity lasted for around 50 minutes where the children freely combined the work on the concrete folding shapes and the ones at the iPads. Preliminary findings indicate that the children were focused on the appearance more than the characteristics of the shapes. Further, technology gave some new opportunities, but also some limitations due to tactile feelings.

Keywords: Kindergarten, technology, geometrical shapes.

RESEARCH QUESTION

How do children utilise technology when exploring the transition between 2D and 3D shapes? Does technology add something to this activity?

The aim of the study is to develop knowledge about children's use of and explorations with interactive devices, like touch screens seen in relation to traditional play material for construction.

THEORY

The case study will be discussed in light of explorative and a playful approaches to mathematics (van Oers, 2010), where children experience different geometrical shapes. Van Hiele's model will work as a framework for the analyses of children's understanding (van Hiele, 1986). Clements and Sarama (2007) argue that use of different concrete models will help children develop spatial skills. The study reported here will support this by using an iPad application.

METHODS

The method is an explorative case study (Cohen, Morrison, & Manion, 2007) where the researchers together with kindergarten teachers designed and carried out the activity. The case study is explorative due to lack of research on kindergarten activities that combine virtual and concrete objects.

Figure 1

Two boys and two girls aged five were studied and the activity was recorded with two video cameras and transcribed. It was analyzed by the researchers and discussed with kindergarten teachers for validations.

EARLY FINDINGS

The children used the names of 2D shapes like circles, squares, rectangles and quadrilaterals, but not always in a very accurate way. For instance, a boy referred to a square as a long quadrilateral, although still pointing at a rectangle.

Figure 1 illustrates the activity. To the left, they compare the shapes included in the cylinder. Initially, they interpreted the group of shapes as a smiling face and pointed out the eyes and mouth. To the right, they compare concrete shapes with virtual shapes on an iPad. They used the color function to make the virtual shape look equal to the concrete shape. The children were focused on the appearance (e.g., color, proportions) more than the characteristics (e.g., angles, parallel sides) of shapes. Our findings indicate that the combination of technology and traditional collapsible folding shapes can give children richer experiences when they use the iPads to understand how to fold and unfold 3D shapes. We observed that this iPad application can help children investigate several shapes more effectively than with traditional solid shapes. Despite this, the tactile aspect must not be underestimated. Holding and touching the physical shapes is important for children´s investigation of shapes.

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The role of the structure in early mathematics learning: Research with children aged four

Margarida Rodrigues[1], Maria Cordeiro[2] and Paula Serra[3]

1 Instituto Politécnico de Lisboa, Escola Superior de Educação, UIDEF, Lisboa, Portugal, margaridar@eselx.ipl.pt

2 Associação Infantil e Juvenil "O Nosso Recreio", Lisboa, Portugal

3 Externato "O Poeta", Lisboa, Portugal

This poster reports a broader project that aims to research the preschoolers' cognitive capacities in mathematics learning, and integrates two studies that were carried with four-year-old children aiming to understand: (i) the children´s ability to subitize, and (ii) how the process of emergence of algebraic thinking develops. The results show the importance of structure in early mathematics education.

Keywords: Structure, early mathematics learning, pattern.

SUMMARY

Mathematics has been an important subject area within the Portuguese curricula for preschool education. Moreover current research has shown that young children can generalize mathematical ideas much earlier than previously supposed (Mulligan, 2013). Structure has a key role in the process of generalization. With appropriate designed and implemented learning experiences, young children are able to develop forms of reasoning involving the process of generalizing (Papic, Mulligan, & Mitchelmore, 2011).

Conceptual subitizing plays an advanced-organizing role (Sarama & Clements, 2009). In spatial patterns, some arrangements lend themselves to grouping facilitating the sudden recognition of the number using consciously strategies of decomposition linked to numerical structure.

This poster will present the results of two studies integrated into a broader project which aims to research the preschoolers' cognitive capacities in mathematics learning. Both studies were carried with four-year-old children. One of them intended to understand the children´s ability to subitize and the other aimed to understand how the process of emergence of algebraic thinking develops.

Both studies were developed in private schools in Lisbon and adopted a qualitative research methodology under the interpretive paradigm, emphasizing meanings and processes. The researchers took the dual role of teacher-researcher: each one conducted the study with her own children's group and in her own natural environment. Participant observation and document analysis (audio and video recordings, images and documents produced by the children) were used as data collection methods.

The research data of Cordeiro's study showed that children can subitizing up until four and begin subitizing sets of five and six items with different spatial arrangements. Children do perceptive subitizing, but few children begin to show signs of doing conceptual subitizing which contributes to numerical structuring. Children are able to identify the number of dots in patterns, eventually become familiar with them and even start making mental relationships between numbers, composing and decomposing them, and developing their number sense. The most common spatial arrangement of sets in the cards (corresponding to the domino game) is the easiest to identify, followed by rectangular, and after by linear and circular arrangements. The structure marked by the use of two colors or by the use of two groups in the spatial arrangement of sets, in the cards, contributed to the emergence of conceptual subitizing, decomposing collections into smaller recognizable collections, and either using addition to determine the total.

The results of Serra's study indicate that children master the concept of repeating and growing patterns, and they are able to identify the unit of repeat, create and analyze patterns of various simple repeating and growing patterns, evolving from simpler forms to more complex forms, in the case of pictorial repeating patterns. Children are aware of the pattern's structure when they identify either the unit of repeat or the regularity of a pictorial growing pattern. In this last case, children used a correspondence analysis of change indexing the figure number with the changing aspect of the dependent variable.

These results offer possible implications for the existing research in the area stressing the role of structure in early mathematics education, namely in the field of mathematical relationships.

THE WAY OF PRESENTING

In the poster, we will illustrate the main results related with structure by exhibiting some of children's productions, communicating them in a pictorial format. Next to visual data there will be short sections: (1) Introduction; (2) Theoretical framework; (3) Methodology; (4) Results; (5) Final considerations; and (5) References.

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Collaboration between scientists and teachers in the context of mathematics education of young children

Maria Sorokina

Warsaw University of Technology, Baltic Institute of Mathematics, Warsaw, Poland, mv40ina@gmail.com

Educators are always seeking alternative and effective approaches to education. The approaches, in the context of young (from 3 to 8 years old) children education, discussed in this poster are based on joint work between scientists and teachers involved in the process.

Keywords: Young children education, scientists, teachers, STEM.

AIM

Nearly from birth, young children develop everyday mathematics including informal ideas of more and less, taking away, shape, size, location, pattern and position. Mathematics helps children make sense of their world. All children can be successful with mathematics, provided that they have opportunities to explore mathematical ideas in ways that make personal sense to them and opportunities to develop mathematical concepts and understanding. Children need to know that teachers are interested in their thinking, respect their ideas, are sensitive to their feelings and value their contributions.

Young children have an endless supply of energy hence they can study till they have fun. It is very hard not to kill creativity and interest at that age. Apathy as a result of misunderstanding is a most common problem for the young children. No answer after another question "Why? What? etc." makes it happen. How we can solve that problem will be shown on the poster.

BACKGROUND

Mathematical education for young children is not new (Fuson, 2004; Denton & West, 2002; Vygotsky, 1986). In the 1850s, Froebel introduced a system of guided instruction focused on various "gifts." It included blocks that have been widely used ever since to help young children learn basic mathematics, especially geometry (Brosterman, 1997). In the early 1900s, Montessori, working in the slums of Rome, developed a structured series of mathematics activities to promote young children's mathematics learning. If children are capable of learning mathematics, and if we choose to help them learn it, what kind of mathematics should we teach and how should we teach it? The decisions stem from our educational values and goals, but should be informed by psychological research.

METHOD

The author of the poster proposes to seek for a solution as collaboration between science and education. For four years, the author has been involved in project Futurum2020. The project provides education for children. We use several method in the case of young children (3–8):

1) Scientists come to the classroom once a week. Topics, which children have mastered during that week, are discussed. A scientist shows some visualisations and gives some additional topics (graph theory, logic, etc) (For example, one can use Martin Gardner's ideas, 1970–1980)

2) A Math circle was organised for those children who enjoy math. Exciting topics that are normally outside the school curriculum were discussed. Some brilliant children get their first research problems. Simple ones, but real, with no known-in-advance solution.

3) STEM camps were organised 4 times a year. Time and space in the camp was filled with the atmo-

sphere of creativity. Solving problems took place both in classrooms and on the mountain paths thus allowing diving deeply into the amazing world of math.

Different generations of children and adults gathered at the same place but had their own educational programme. Scientists organised several lectures for teachers.

RESULTS

Organising camps in such a way helps to make transition between pre-primary and primary school "smooth" and unstressful. The results are amazing! During the first months of the programme, teams (a scientist and a teacher) found children with learning disabilities and helped them by co-working on finding different ways to approach math facts.

This way we showed that collaboration between a scientist and a teacher is very important and useful because it looks like a symbiosis in nature: teachers study mathematics, scientists study psychology. QED

THE WAY OF PRESENTING

The poster included the purpose, method and results, also examples of collaboration between scientists and teachers were discussed.

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Preschool class – one year to count!

Helena Vennberg

Umeå University, Department of Science and Mathematics Education, Umeå, Sweden, helena.vennberg@umu.se

I will present results from a case study of teachers' experiences and opportunities to assess pupils' mathematical development and thereby identify pupils' mathematical difficulties during the preschoolclass year. The study is part of a development and cooperation project between NCM (National Center for Mathematics education) in Gothenburg and Umeå University.

Keywords: Preschool class, intervention, assessment, teachers.

BACKGROUND

The Swedish preschoolclass (age 6 years old) is often described as the link between preschool and school. Curriculum for preschool (age 1–5 years old) in Sweden has an ambitious mathematical content (Skolverket, 2010) that is in many ways linked to the compulsory school curriculum in mathematics (Skolverket, 2011). But for preschool-class there is no specific curriculum in mathematics. In preschool and school in Sweden there are different views on what develops pupils' knowledge and/or competencies. Knowing mathematics can be understood in two different types of knowledge (Wedege, 2010). Knowledge *developed* in everyday life (Type 1), with representatives as D'Ambrosio and Bishop, and knowledge *wanted* in everyday life (Type 2) were Kilpatrick and Niss are predominant. Preschool in Sweden are influenced by Type 1 whereas school are strongly influenced by Type 2. When it comes to developing mathematical knowledge, preschools assess their activities which is different from the school context were the pupils' knowledge are assessed. This circumstances lead to a lack of clear guidance for the mathematical work in preschool class. In view of the lack of clarity that surrounds goals in mathematics for preschool class, questions arise regarding on what grounds teachers meet and asses students' knowledge and when do teachers in preschool-class discover misunderstandings that may exist in mathematics?

AIM

The twofold aim of my study include to describe preschool-class teachers' perception of pupils' mathematical development and analyze if raised awareness of pupils' mathematical development changes the possibility of early identification of difficulties in pupils mathematical development and prevent creation of pupils with special educational needs in mathematics. For this purpose two questions are posed:

When and how will pupils' mathematical development become visible to preschool-class teachers? When and how will pupils' difficulties in their mathematical development become detectable to preschoolclass teachers?

METHODS

Data was generated by following 14 teachers in nine preschoolclasses with a total of 200 students from four different schools from November to June. Three of the classes and three teachers were part of a control group where I followed their regular activities. The remaining 11 teachers and six preschoolclasses were part of an intervention, designed to build structured activity in mathematics where pupils, meet, use, develop and reason with different representations of numbers (Sterner, 2014).

The study includes several data collecting elements. At four occasions the teachers were asked to assess their pupils. The assessment was based on a matrix, influenced by Kilpatrick's framework (2001) on mathematical proficiency and the similarities in curricula goals for preschool and primary school. All the pupils were interviewed and tested with the Van Luit's "Early Numeracy Test" (2005) – in November and in June. The scores are used mainly as control points in relation to the teachers' assessments in regards to number sense, as being one of the key factors in knowing mathematics.

Interviews were held with the teachers in November and at the end of the preschool class year, in June. Observations were conducted in the preschool-classes on several occasions.

RESULTS

The intervention group had more activities and communication between teachers and pupils and in-between teachers. Increased communication between teachers led to raised awareness regarding their own knowledge of mathematics and of pupils' mathematical development. Raised awareness of pupils' mathematical development enhances the teachers' possibility to assess pupils' and to detect and correct gaps in pupils' math skills. The teachers describe a feeling of security in their knowledge and assessment but at the same time they express a sense of greater difficulty in assessing, even when the assessment were more accurate.

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TWG14

University mathematics education

Introduction to the papers of TWG14: University mathematics education

Elena Nardi[1], Irene Biza[1], Alejandro S. González-Martín[2], Ghislaine Gueudet[3], Paola Iannone[1], Olov Viirman[4] and Carl Winsløw[5]

1 University of East Anglia, Norwich, UK, e.nardi@uea.ac.uk
2 University of Montreal, Montréal, Canada
3 University of Brest, Brest, France
4 University of Agder, Kristiansand, Norway
5 University of Copenhagen, Copenhagen, Denmark

Research on university level mathematics education is a fast developing field as evident in the growth of the CERME *University Mathematics Education* (hereafter UME) Thematic Working Group. TWG14 was launched in CERME7 (Nardi, González-Martín, Gueudet, Iannone & Winsløw, 2011). After CERME8 (Nardi, Biza, González-Martín, Gueudet, & Winsløw, 2013), its leader team – in collaboration with TWG14 participants and others – worked towards a *Research in Mathematics Education* Special Issue on *Institutional, sociocultural and discursive approaches to research in university mathematics education* (Nardi, Biza, González-Martín, Gueudet & Winsløw, 2014) which focused on research that is conducted in the spirit of the following theoretical frameworks: *Anthropological Theory of the Didactic*, *Theory of Didactic Situations*, *Instrumental and Documentational Approaches*, *Communities of Practice and Inquiry* and *Theory of Commognition*.

The work of the group at CERME9 cemented and furthered this work but welcomed contributions from across the board of research approaches: the teaching and learning of advanced topics; mathematical reasoning and proof; transition issues "at the entrance" to university mathematics, or beyond; challenges for, and novel approaches to, teaching (including the teaching of students in non-mathematics degrees); the role of ICT tools (e.g. CAS) and other resources (e.g. textbooks, books and other materials); assessment; the preparation and education of university mathematics teachers; collaborative research between university mathematics teachers and researchers in mathematics education; and, theoretical approaches to UME research.

The critical – and growing – mass and quality of the work presented at TWG14 has led to the launch of an ERME Topic Conference, INDRUM2016, a conference of the newly launched *International Network for Didactic Research in University Mathematics* (France, Montpellier, March 31 – April 2). Its two broad themes are *teaching and learning of specific topics in university mathematics* and *teachers' and students' practices at university level*. In anticipation of INDRUM2016, in this report, we outline briefly the main focal points of the 45 papers (31 long and 14 short contributions) that comprise the set of CERME9 TWG14 papers published in these proceedings in accordance with these two broad themes. We note that several papers fit both themes and that we have opted to classify the papers according to what we see as their main research focus and contribution.

TEACHING AND LEARNING OF SPECIFIC TOPICS IN UNIVERSITY MATHEMATICS

The 16 papers classified under this theme (8 long and 8 short papers) address a range of mathematical *topics*, elaborate discussions of mathematical *reasoning, logic and proof* and introduce research into the teaching of mathematics to students in *other fields* (here: engineering and economics).

With regard to *mathematical topics*, contributions regarded topics in calculus and Complex Analysis. *Breen, Larson, O'Shea and Pettersson* analyse student data from Ireland and Sweden to discuss concept images of inverse functions, particularly in relation to the predominance of the models of "swapping x and y", reflection and reversal. *Ghedamsi* offers a *Theory of*

Didactic Situations (TDS)-based analysis of a teaching session on sequence convergence in order to examine the ways in which a university calculus teacher attends to students' prior knowledge in calculus and facilitates the transition from school to university mathematics. *Grønbæk and Winsløw* deploy an *Anthropological Theory of the Didactic* (ATD) lens to discuss the teaching of complex numbers using Maple sheets and demonstrate the institutional constraints – Maple sheets cannot create an appropriate media/milieu dialectic – which lead to the development of disconnected practices. The short papers also covered a range of topic-specific research: the transition from informal to formal understanding of the concept of order in abstract mathematics (*Akdemir, Narlı and Kaşıkçı*); improper integrals (*Cortés and Velasco*); differential geometry (*Dana-Picard and Zehavi*); differential equations (*Fardinpour*); linear independence of functions (*Wawro and Plaxco*); and, abstract algebra (*Mili and Ascah-Coallier*).

With regard to *mathematical reasoning, logic and proof, Hausberger* introduces the innovation of the *banquet*, a pocket-size algebraic structure aimed at helping students reflect on mathematical structures and the axiomatic method. *Bridoux and Durand-Guerrier*, through an a-priori and a-posteriori analysis of two tasks in an exam paper taken by students of a Computing Sciences module that aimed at improving students' proof production, find that the course did improve students' proof fluency, although they also observe that many difficulties remain. In their short paper concerning students' conceptions of logic, *Kazima, Eneya and Sawerengera* also highlight some of these difficulties, mainly focusing on issues of language.

With regard to research into the teaching of mathematics to students in *other fields*, a relatively novel strand, *Biehler and Kortemeier* analyse students' work with a typical electrical engineering task in relation to an expert solution and conclude that it is counterproductive to try to separate the mathematical and "real world" (engineering) parts of the problem. *Kürten and Greefrath* report aspects of a "bridging" course aiming to reduce engineering students' difficulties with mobilizing school mathematical skills. *Mkhatshwa and Doerr* investigate economics students' reasoning about marginal change (instantaneous rate of change) and in her short paper *Selinski* explores student noticing of exponential and power functions in university financial mathematics.

TEACHERS' AND STUDENTS' PRACTICES AT UNIVERSITY LEVEL

The 29 papers classified under this theme (23 long and 6 short papers) also address a range of teaching and learning issues: *curriculum and assessment; innovative course design in UME; student approaches to study; relating research mathematicians' practices to student practices; views and practices of mathematics lecturers;* and, *methodological and theoretical contributions to UME research.*

In the cluster of papers on *curriculum and assessment, González-Martín* deploys a combination of theoretical frameworks (ATD and the *documentational* approach) to investigate the use of textbooks by pre-university teachers (particular focus: the topic concept of series of real numbers) and to observe that the textbook is a central tool for the teachers, who align with its presentation and organisation. *Dibbs* describes the outcomes of the use of formative assessment in a calculus class and concludes that regular participation in formative assessment is the best predictor of achievement. *Raen* compares the assessment of student competencies through closed book examination and talk aloud interviews. She concludes that different methods reveal different competencies and that therefore a mixture of assessment methods is desirable. *Thoma and Iannone* use two different frameworks, the MATH framework based on Bloom's taxonomy, and a framework based on functional linguistics and Sfard's *commognitive* approach, to analyse tasks from an examination in abstract algebra. They find both frameworks useful in highlighting different, and often complementary, aspects of the tasks. In their short paper *Derouet, Henríquez, Menares and Panero* also deploy a priori analyses of examination tasks in order to compare final secondary assessments in different countries.

With regard to *innovative course design in UME, Biza and Vande Hey* deploy the *Communities of Practice* approach to study the process of – and the pedagogical benefits deriving from – involvement of two undergraduate students in a project of resource development for statistics. *Mesa and Cawley* report the 3-year implementation of Inquiry-Based Learning (IBL) in a range of courses. Drawing on data from teacher logs and a *Mathematical Knowledge for Teaching* (MKT) framework, they discuss challenges of the IBL approach. *Nardi and Barton* present a *commognitive* analysis of a "low lecture" episode (student-led inquiry oriented discussion on open-ended problems) to illustrate crucial steps of student enculturation into mathematical ways of acting and communicating, including a shift away from the lecturer's 'ultimate substantiator' role. *Rämö, Oinonen and Vikberg* take a similar approach to report the shifting of an introductory course on linear algebra from a "lecture based" format to a new "extreme apprenticeship" format.

In the growing area of *student approaches to study, Farah* investigates the role of students' personal work in mathematics and highlights the influence of institutional differences on student approach. *Gómez-Chacón, Griese, Roesken-Winter and Gónzalez-Guillén* report similarities in the learning strategies employed across two cohorts of engineering students, in Spain and Germany. *Liebendörfer and Hochmuth* identify different factors which support or hinder the autonomy of first year students and observe that student teachers are not convinced about the need of university mathematics for teaching at school. *Lehmann, Roesken-Winter and Schueler* reveal that mathematical competencies and beliefs about physics are substantial for engineering students' success in technical mechanics. In their short papers in this area, *Griese, Lehmann and Roesken-Winter* focus on what obstructs or facilitates examination success in first year engineering and *Švecová, Kohanová and Drábeková* explore issues concerning the mathematical literacy of first year students.

Three papers documented the *interplay between research mathematicians' pedagogical and mathematical practices and the influence of these on learner practices. Cooper* proposes a commognitive configuration of MKT (MDT, *Mathematical Discourse for Teaching*) as a tool to identify – and make optimal pedagogical use of – differences in the student teachers' and a mathematician's discourses. *Ouvrier-Buffet* presents a model of how research mathematicians practise the construction of formal mathematical definitions and highlights the pedagogical potency of epistemological analyses of mathematicians' practices. *Kondratieva* also favours epistemological analyses and discusses the pedagogical potential of exposing students to mathematical problems with different, more or less advanced, solutions to problems as opportunities for building mathematical connections.

In the populated area of studies on the *views and practices of mathematics lecturers* (6 long and 3 short papers), *Bergsten and Jablonka* investigate the views of mathematics lecturers on the transition problem for engineering students and observe that, despite the engineering context, lecturers see this transition as apprenticeship into becoming a mathematician, namely able to produce mathematics. *Hernandes Gomes and González-Martín* highlight differences in how teachers in engineering and in mathematics address rigor, approximation and modelling differently and how these views influence their teaching. *Gueudet* deploys the *documentational* approach to study teacher preparation and communication practices. She traces the interaction of teachers with resources in a goal-oriented activity that produces *documentation systems* (structured set of all the documents they develop) and identifies features of these systems. *Mali* studies how teachers with different disciplinary backgrounds use examples and representations in their teaching. *Petropoulou, Jaworski, Potari and Zachariades* deploy the *Teaching Triad* construct to investigate lecturer practices and rationales. They illustrate a case of a lecturer who shows sensitivity to students' needs and draws students into mathematical culture through mathematical challenge. *Viirman* offers *commognitive* analyses of how lecturers' epistemological and ontological positions on mathematics are articulated in their teaching discourse. The three short papers in this area touch on ways to enable student meaning making (*Didis and Jaworski*), UME conceptualisations of pedagogical content knowledge (*Khakbaz*) and tackling the difficulties of the transition from school to university mathematics (*Kouvela, Biza and Zachariades*).

Finally, *Kaspersen, Pepin and Sikko* propose a *methodological advance* in the study of the transition from higher education to the world of work through proposing an approach to purposeful sample selection for measuring student teachers' beliefs and practices. An advance of a *methodological as well as theoretical* character is put forward by *Tabach, Rasmussen, Hershkowitz and Dreyfus* who use a transcript of an excerpt of four undergraduate students' interaction while working on a specific initial value problem, to demonstrate a local integration of two theoretical and methodological perspectives on knowledge construction, namely *Abstraction in Context* (focusing on individuals) and *Documenting Collective Activity*.

IN CLOSING

While our presentation of CERME7 and CERME8 papers was in accordance with slightly different themes – for example in CERME8: *transitions*, *affect*, *teacher practices* and *mathematical topics* – some comparative observations across the three sets of papers are apt. As we noted in the Editorial of the RME Special Issue (Nardi et al., 2014), there is a clear surge of sociocultural and discursive approaches – and the number of papers using ATD and TDS is also remarkable. An emerging focus seems to be also on systematic investigations of innovative course design and implementation and there is certainly a rise in the number of studies that examine the teaching and learning of mathematics in the context of disciplines other than mathematics, such as engineering and economics. Furthermore, this time we welcomed more colleagues from outside Europe and also noted the rise in the number of papers on assessment and examination. We also observed the further strengthing, maturity and increasingly more robust theorizing of studies into teaching practices. Finally, we noticed in several papers the establishing of promising liaisons across different theoretical perspectives. We now look forward to cementing these developments further in future CERME conferences, in the rich presence of UME at the upcoming ICME13 and EMF2015 conferences – and of course INDRUM2016!

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TWG14

Research papers

The construction of the 'transition problem' by a group of mathematics lecturers

Christer Bergsten[1] and Eva Jablonka[2]

1 Linköping University, Linköping, Sweden, christer.bergsten@liu.se
2 King's College London, London, UK

This paper presents views of university staff about what has become called the 'transition problem' when students start studying mathematics at university. The data are from a focus group interview with eight experienced university lecturers at a Swedish university department that offers mathematics courses for engineering students. We use the portrayal of the problem in the literature as an axis for the discussion.

Keywords: Undergraduate mathematics, lecturers, transition problem, focus group.

DIMENSIONS OF THE 'TRANSITION PROBLEM'

In some places mathematicians seem to have for a very long time bemoaned a lack of university entrants' knowledge (Thwaites, 1972); it clearly became an international concern in the 1990's (ICMI, 1997) as part of a *transition problem*. Based on our reading of the literature, we have grouped the issues mentioned in a range of Swedish and international studies into eight dimensions [1].

(1) Pass rates and participation: In Sweden pass rates in undergraduate mathematics courses were at 70% for engineering students at the beginning of the 21st century (HSV, 2005, p. 45). Relatively low pass rates were also reported from other countries (Dieter & Törner, 2010; EC, 2000; De Guzmán, Hodgson, Robert, & Villani, 1998). Further, there is a perceived need to increase participation in higher education of under-represented groups in terms of gender, ethnicity and social class (Pampaka, Williams, & Hutcheson, 2012).

(2) Alignment of curriculum: A couple of studies reveal discrepancies between the mathematics faculties' expectations and the actual school curriculum (e.g., Brandell, Hemmi, & Thunberg, 2008). These relate to factual knowledge or use of formularies and tables, routine skills or problem solving, and computational fluency or use of technology. Swedish students' perceptions of their pre-knowledge tend to be more positive than that of their teachers (HSV, 2005, p. 34). Mismatches are also described in other countries (EC, 2000; Hourigan & O'Donoghue, 2007; Hoyles, Newman, & Noss, 2001; Kajander & Lovric, 2005).

(3) Changes in level of formalisation and abstraction: Mathematics at university entails specialised technical language, which students perceive as more cumbersome than at school. As depicted in the Swedish report issued by HSV (2005), 75% of the students find mathematics courses difficult, and for 85% the university creates new challenges. In a study in France, Spain and Canada, the tasks were perceived as more "abstract" (De Guzmán et al., 1998). The authors also point out that tertiary mathematics includes "unifying and generalising concepts" (pp. 752–753), often described as a switch from intuitive to formal mathematical thinking or from informal argumentation to mathematical proof (e.g., Brandell et al., 2008).

(4) Unclear role of mathematics for the students' career paths: HSV (2005) reports that only 40% of students in the second semester thought they have made much use of their mathematical knowledge in other subjects. Rather than 'directly' useful, students in another Swedish study described mathematics as generic problem solving technology (Bergsten & Jablonka, 2013). De Guzmán and colleagues (1998) see an underestimation of the role of mathematics in a range of subjects in the French, Spanish and Canadian context. A 'utilitarian trend', noticed in the UK (Hoyles et al., 2001), can cause conflicting messages as to the purpose of studying 'pure' mathematics.

(5) Differences in teaching and classroom organisation: In Swedish classrooms one finds a dominance of lessons devoted to individual work scaffolded by the teacher (Skolinspektionen, 2010), providing little experience with the lecture format common at universities where many students point at the increased pace as a characteristic difference to school (HSV, 2005). Similar views are reported by De Guzmán and colleagues (1998, p. 750). Also the character and function of assessment differ (Gueudet, 2008).

(6) Change in expected learning habits and study organisation: To study at university requires a higher degree of autonomy than in secondary school (Wingate, 2007; cf. De Guzman et al., 1998). This is also acknowledged by many of the engineering students in Sweden (HSV, 2005). When students' expectations do not match the reality they meet, including a change of the didactic contract (Gueudet, 2008), students can experience stress (Jackson, Pancer, Pratt, & Hunsberger, 2000).

(7) Differences in atmosphere and sense of belonging: A university setting with large group lectures increases the social distance between teachers and students, an issue also expressed by Swedish students (HSV, 2005). The anonymity at a large university can be 'quite a frightful experience' (De Guzmán et al., 1998, p. 755). As students often change groups, a sense of belonging cannot be developed as easily as at school.

(8) Differences in pedagogical awareness and education of teachers: The image of the university mathematics teachers held by Swedish engineering students is rather positive; while some complaints were raised, a large majority appreciated the engagement as well as the knowledge of their teachers (HSV, 2005). Similar perceptions were revealed in the study by De Guzmán and colleagues (1998). Nardi, Jaworski and Hegedus (2005) describe a spectrum of *pedagogical awareness* among undergraduate mathematics teachers, including four levels labelled as naive and dismissive, intuitive and questioning, reflective and analytic, and confident and articulate (p. 293).

From these studies it is evident that the shifts between the two institutional cultures concern the curricular content, forms of pedagogy as well as the identity of the students as learners of mathematics and as beginning university students. The outcomes of the studies suggest that the shifts of criteria for what counts as mathematics as well as for the appropriate study habits that help the students to acquire this 'new' type of mathematics are often neither coherent nor explicit.

This paper draws on data from a focus group interview to investigate how lecturers who teach first year undergraduate mathematics courses talk about the transition problem, and how their views match the dimensions portrayed in the literature as outlined above. While most previous studies have been framed by curriculum discussions, exam results, or student responses, we hoped that an exploration of the views of experienced lecturers who teach first year mathematics courses at university, may open up dimensions of the transition problem hitherto hidden.

METHODOLOGY

As part of a larger project [2], where around 70 engineering students at two universities in Sweden were followed and interviewed during their first year of study, their lecturers of the mathematics courses at one of these universities were invited to a focus group interview, moderated by one of the authors, to discuss the transition problem. The eight university lecturers/professors of mathematics who volunteered to participate all work at a mathematics department. The audio-recorded session lasted for about 80 minutes and was organised by prompts concerning the beginning mathematics studies at university [3]. As participants knew each other as colleagues and were a homogenous group in terms of their extensive teaching experience and involvement with undergraduate students, we hoped the interaction between them can develop freely into a shared opinion of the group and would also expose issues of disagreements (Morgan, 1997). The participants (L1 to L8 below) were between 40 and 65 years old, one female and seven males. The purpose of the ongoing project and the focus group interview was known to them, and shortly reviewed at the outset.

We used the dimensions of the transition problem as outlined in the literature review as a thematic framework, and indexed parts of the conversation that related to these themes and re-narrated the lecturers' statements. This also helped to identify new dimensions and views that differed from how the transition problem is portrayed in literature. Thus, after discussing some general issues about the focus group interview, some subheadings in the presentation below of our analysis of the interview transcript relate directly to some of the dimensions from the literature review (indicated by dimension number), while some categorise other topics emphasised by the lecturers. We also looked for expressions of emotions, disagreement and take-up of topics by group members. We believe that there was some interactive synergy in the discussion, which justifies our choice of conducting a focus group.

Our analysis draws on some analytical frameworks that have been used for analysing knowledge in education. Bernstein (1971) sees identity as the subjective, interiorised consequence of a discursive specialisation. This specialisation can for example be that of a pure mathematician, an engineer or an applied mathematician. In a more structuralist interpretation, pedagogic practices are an attempt to shape and distribute forms of consciousness, identity and desire (Bernstein, 2000, p. 203). For the purpose of the study, the concepts of classification and framing that describe the relations between discourses (and groups of actors) and how these are established by distinct pedagogic practices, are of relevance.

THE 'TRANSITION PROBLEM' IN THE EYES OF THE FOCUS GROUP

General framing of the problem

To the opening question about whether there exists a 'transition problem', the answer was unanimously, 'Yes'. Even though the opening prompt of the moderator was not phrased in a way that would essentialise the problem by talking about the 'so-called problem' and 'if there is such a thing', it might have been suggestive; but it did not suggest any specific way to talk about the problem, as, for example, in terms of their own experiences as teachers or in more general terms concerning the structuring of the university courses in relation to conceptions of the school curriculum. The group agreed that it was nothing new, 'This has existed all the time; one talked about this already when I started here as a doctoral student' (L8). Some shared their memories from the student point of view and one lecturer suspected that the problem might have increased in magnitude.

During the discussion the group consistently referred to experiences with their students. The curriculum was taken as a given, although some changes introduced earlier were mentioned. None of them referred to 'us' (as teachers or as an institution) having a problem. The participants did not phrase this as the students causing a problem for them. Instead they referred to their interpretations of students' knowledge and experiences and gave very specific examples. Only in one episode about marking criteria, the lecturers talked about themselves (in terms of an inclusive 'we'). When referring to students, the participants in most cases talked about 'students' and 'them' or 'one' (indefinite pronoun) as a homogeneous group and occasionally used passive voice (such as 'calculation rules have been forgotten'). In many of their statements, however, three of the lecturers did not generalise to all students, but said 'many students' and occasionally 'some students'. There was agreement in the group that there are many 'good students' who do not have problems with the transition.

A reading of the transcript with attention to individuals showed that none of them seemed to have changed their perception during the session. Also, there was not much evidence of argumentation amongst the group members. This does not entail that they held uniform views about of what dimensions the problem consisted, but that they mentioned different aspects and others agreed (often immediately with 'yes, yes') or provided additional examples. Some aspects brought to the discussion by L1 and L2 were picked up by the group and discussed in length. We looked for expressions of strong emotions, but felt there were no indications; most statements were to the point. The lecturers were engaged and eager to contribute and share their views.

Topics and themes

Most of the issues were discussed in contrasting them with what the lecturers knew or suspected about curriculum and pedagogy at school. In one prompt, they were explicitly asked about the differences, where some of the issues mentioned as being problematic were repeated. The problems raised were said to be very common, also among students who later 'show to be very capable' (L1).

Computational facility and problem solving strategies (#2 in the literature review)

Computation appeared as a key word in many statements by the lecturers. What they found lacking in students included general computational facility:

L6: minus signs brackets and such basic stuff can go wrong

L2: the probability that it goes wrong at least once is pretty high if you have to make several computational steps

When talking about computation, this was analysed as including both to calculate correctly and to have a strategy. In this context, they mentioned that many students appear not to have learned to think systematically, have no meta-strategies such as a habit to control results, do not know how to work through tasks that include more than one critical step and how to structure a solution when methods are not given:

L8: you get a problem and then you need to adapt .. restate and do things with it before you can get to a point where you can apply that old standard method ... at school this is more direct ... it works to shove that into it directly

The questions students ask in lectures were said to be mostly of the type, 'How did you do that?' that is, more about computational details than about conceptual issues.

Dependency on guidance and instruction (#5 and #6 in the literature review)

The lecturers said that at university students are asked to a much greater extent to approach new problem situations and find out how to use known methods rather than solving tasks by applying given methods, 'you must find out yourself what method to use' (L3). In relation to this requirement, they noticed that

L1: many students don't seem prepared that you may have a good idea and then we try it out and test it to see where it leads

L8: they just sit there if they don't see the whole way ahead they don't start but instead raise the hand and ask

The phenomenon illustrated by these statements had increased, according to the lecturers, and was given much attention in the discussion. Differences in the view of understanding were also raised: at school it means being able to follow the reasoning of others while at university one must do it oneself. One lecturer framed the inactivity of students in the face of more complex problems as a matter of 'maturity':

L5: it is also about being mature ... one can't require at upper secondary that people have the same maturity maybe one needs one semester to level it out

Organisation and academic study skills (#6 in the literature review)

Other aspects of an observed decrease in expected maturity of new students compared to earlier concerned generally more messy classes, students who do not bring things or return assignments in time, who take teaching for granted as a kind of service, and that as a teacher one must explicitly emphasise how many hours students are expected to work.

Formularies and electronic calculators (#2 in the literature review)

The group also mentioned that the common use of formularies at school creates problems at university where one must have some foundational knowledge available.

L2: we require that they should know some terms ... while at upper secondary it is required that they should know how to find [it in] a dictionary [...] you experience more and more difficulties to learn any ... rule and remember it

Another issue concerned the fact that at this department electronic calculators were not allowed at exams, while at upper secondary they were commonly used, 'that itself is a big step from upper secondary' (L2). This problem was linked to a lack of seeing meaning in mathematical objects that earlier had been available as buttons on a graphic calculator. Students, however, generally did not complain about the change.

L2: the elementary functions sooner or later must acquire some meaning they don't for many students when they come here [...] it's much more common that they complain that they were allowed to use them at upper secondary than that they're not allowed to use them here

Mathematical rigour (#3 in the literature review)

When asked about the differences between school and university mathematics, "rigour most of all" (L8) was mentioned, but when discussing the level of rigour the participants used in their lectures, it was agreed that such emphasis had decreased:

L1: much less than before
L3: you argue for your theorems by examples that make things likely

Nevertheless, how the examples were presented still supported a rigorous approach:

L2: the reasoning ... the examples they see in lectures there the solutions are as rigorous that they no doubt would pass as solutions [on exams]

The moderator also presented students' solutions to exam tasks, which the participants were asked to mark with the intention to initiate a discussion about the level of rigour they expected from the students. They could not reach full agreement on the accuracy of the presentation in a task they classified as a 'one-point task' (see below) and hence were not sure whether to give it a zero. While one lecturer compared the solution with one to another task and found it 'better', another qualified the discussion as 'nitpicking'. About one solution that contained calculations with approximate values for π and e, they found it unlikely that one of their students had produced it.

Incoherence in students' mathematical knowledge

An effect of the observations that 'a student can be very good at some things but [at the same time] maybe knows nothing at all about other things' (L3), implied that, in contrast to how it was earlier, 'it takes longer time to discover who are really strong' (L8). In the written exam results problems showed themselves through 'lots of simple mistakes' (L6), and that despite the adjustments of the level that had been made, results had generally decreased. However, in topics that were completely new to the students, this effect had not occurred:

L2: in linear algebra one is not so much disturbed by things one does not remember from upper secondary

Assessment and knowledge criteria (#3, #5 in the literature review)

There was a long discussion about the assessment practice for the written exams. Eventually the discussion revealed well-established practice. One aspect concerned the organisation of the tasks in written exam papers for summative assessment of the course. In most first and second year mathematics courses these included seven tasks to solve during a given time (commonly four or five hours) with full solutions to be handed in. Each task was marked with 0, 1, 2 or 3 points, where a solution obtaining 2 or 3 points was considered a pass on a task. A common criterion for a pass on the course was to obtain at least three 'pass tasks' and at least 8 points. However, the order of the tasks on the exam paper gave different 'weights' to the points given on each task. In this context, the intention of a task on behalf of the examiner was also critical, as was said after a long discussion about how to mark one specific task:

L1: if I had been the examiner on this task I would have considered beforehand what I want to test with this task, if I want to test the understanding of graphs yes then maybe this is a task for the upper part of the exam paper and then you can let a way of reasoning pass that we know or ... want to test a rigorous mathematical reasoning then it ends up further down and then there will be no points for the B task

The overall result for the specific student being assessed thus influenced the marking:

L2: actually we assess the exams the solutions differently if it is about a pass or a pass with distinction ... this we all do a little ... that we set up higher formal requirements for solutions if it is about a pass with distinction

This marking practice was termed 'holistic assessment', as explained by L7:

L7: if you make a holistic assessment of the whole exam paper and you look at this task in its context and compare to other tasks there are good things in it and pass or not is maybe not decided from this particular one but from a holistic evaluation of the whole paper

When asked whether students would be aware of this practice, one lecturer replied, 'I don't think so' (L2).

However, the holistic approach had been the practice at this department for a long time and several lectures acknowledged that it was somewhat hidden to the students: 'I think not many students know this practice' (L7).

How to overcome the problems

Typical for students who do well is that they work a lot with the course. The formulation that they 'get it' (i.e. the method, the theorem) was used here. It was emphasised, though, that also students with the highest school mark often have a very uneven knowledge base. However, when asked about what positive things they observe today, the group of lecturers agreed about a good 'spirit' in students and that most of them in the end overcome most of the problems pointed out.

L1: enthusiasm is actually something I think has become better the last years

L8: when they eventually get going and go through our courses then in the end they do pretty well … and I don't think we in any way produce worse engineers than we did some years ago …the end product I think is at least comparable … even if they maybe had to struggle more on the way

DISCUSSION

Much of what this group of lecturers discussed is implicated in the dimensions of the transition problem as portrayed in the literature. Not mentioned by the group as problematic was the lack of experience of students with the lecture format. As to the differences in teaching, the issue was only discussed in relation to the students' behaviour and not in terms of differences in teachers' pedagogical strategies. Change in expected learning habits and study organisation were touched upon, while differences in atmosphere and a sense of belonging were not discussed. The group focussed on differences in mathematical activities but did not talk about the role of mathematics for the engineering students' careers. Interviewing lecturers allowed for a differentiation of the transition problem and opened up some new dimensions.

The institution aims at introducing their students into a strongly classified (Bernstein, 1971) canon of traditional undergraduate mathematics. The lecturers expected, for example, avoiding inappropriate levels of approximation and not relying on arguments derived from graphs of functions. They also saw school mathematics as strongly classified (applications and modelling were not mentioned) but different in knowledge structure and pedagogic relation. In addition, they did not differentiate between different groups of students, such as from different engineering programmes.

The lecturers shortly talked about decreasing pass rates, but were not sure about any trend before a member with access to the data reported a decrease. In relation to the performance patterns in the assessments, they mentioned an increased 'incoherence' in the levels of individual students' knowledge. The discussion about the 'holistic assessment' is related to this observation about the increased unpredictability of the students' knowledge. The assessment practice is based on the assumption that there is only one dimension of mathematical competence that amounts to the students' performance. 'It takes longer time to discover who are really strong', reveals an assumption about an essential generic mathematical competence hidden behind a range of more or less virtuoso performances, a form of 'mathematicality'.

The 'pedagogic relation' (Bernstein, 2000) was by the lecturers depicted as one with students who depend on the expertise of their lecturers but even more so on their teachers at school level. In the eyes of the lecturers, the positions made available to the students change substantially: While at school they are constructed as dependant learners who learn how to use a range of techniques with the aid of calculators and formularies but with no authorship in producing some original piece of mathematics, at university they grant the students authorship to create some mathematics through combinations of techniques and mathematical argument as acceptable by academic mathematicians without calculation aids and formularies. This is an apprenticeship into becoming an academic mathematician. Despite the vast majority of the students being from engineering programmes, the lecturers do not conceptualise their teaching as apprenticeship into users of techniques for mathematical modelling in some of their students' future engineering fields.

The focus group came to the unexpected (with respect to the literature) conclusion that overcoming the transition problem is a matter of the students' own work and natural development as they become more mature and used to studying mathematics at university. They also maintained that the level of competency of the engineers who graduate from the institution has not in any significant sense dropped. As compared to earlier, however, today the students have to 'struggle more on the way'.

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ENDNOTES

1. These only partly overlap with the 'groupings' in De Guzman and colleagues (1998) and Gueudet (2008).

2. The project is funded by the Swedish Research Council; see www.vr.se.

3. These prompts were: Is there a transition problem? How does it show? How common are these 'problems'? How does it show in exam results? How much emphasis is made, in lectures and exams, on the formal aspects of mathematics? How are students informed about the assessment criteria? What type of questions do students ask in lectures? What is typical for students who do well? How does (upper secondary) school mathematics differ from university mathematics? Differences in knowledge criteria? Other issues? What is positive today?

Conceptualizing and studying students' processes of solving typical problems in introductory engineering courses requiring mathematical competences

Rolf Biehler, Jörg Kortemeyer and Niclas Schaper

University of Paderborn, Paderborn, Germany

The project KoM@ING aims to investigate the mathematical skills which are required in technical subjects of engineering bachelor courses. Our subproject is especially interested in the first-year-course "foundations of electrical engineering". In order to do research on this subject we developed the concept of a "student-expert-solution" (SES) which was generated by analysing expert interviews. The SES is supplemented by the required resources and a didactical reconstruction, for example, typical mistakes, alternative solution approaches and learning goals. The SES of a basic engineering exercise will be presented. We find some discrepancies with the standard modelling cycle, as well as some surprising problem solving strategies.

INTRODUCTION

We are interested in the competence expectations that are implicit in these tasks, i.e., expectations that are set up by the instructors of engineering courses, often based on years of experience, but not on any explicit theory of engineering competence. Originally, we intended to focus on mathematical aspects only, but it turned out that a more holistic approach is more appropriate. Based on this, we are analysing how and how well students in first year standard university courses on electrical engineering solve tasks given to them in homework assignments and written examinations Our project is part of the KoM@ING-project, where the modelling and assessment of mathematical and engineering competences is the focus. Our subproject chose a qualitative approach. The tasks we are analysing require knowledge and cognitive resources from mathematics on the one hand and from electrical engineering on the other. The mathematical knowledge is partly based on school knowledge and on the mathematics that students learn in the separate courses on higher math in the first year parallel to the engineering courses. It is well known that there is a mismatch between the mathematics learned in the separate courses, the mathematics at school level and "the contextual mathematics" required in solving engineering tasks (see, e.g., Redish, 2005). The mathematical practices in engineering contexts look far different from those in purely mathematical contexts. The tasks given to the students cannot directly be regarded as mathematical modelling tasks in the sense as this is discussed in mathematics education.

We focus on five tasks from the final exam of the second part of the "foundations of electrical engineering"-course (called the "GET-B" exam), which electrical engineering students are to take after their first year. All of the students' written work was scanned. Moreover the same tasks were given to eighteen pairs of students and their work and communication was video-recorded. With nine of the student pairs we used stimulated recall for extending the base of our analysis. In order to analyse the problem solving processes and the written work of the students we need a didactically oriented task analysis and theoretical frameworks on which we can base this analysis, in other words a "normative solution" of our task.

THEORETICAL BACKGROUND

We consider the following three theoretical frameworks as relevant: The first approach is the modelling cycle by Blum and Leiß (2007), which divides the solving of a mathematical modelling exercise into seven steps: (1) understanding of the task and the underlying situation, construction of the so-called "situation model" (2) simplifying and structuring of the situation: construction of the so-called "real model", (3) translating into a mathematical problem (entering the "world of mathematics"), (4) mathematical work, (5) interpretation of the result in the real world, (6) validating and (7) presenting of the results. The cycle consists of two parts, the "rest of the world" with steps (1), (2), (6), (7) and the mathematics with step (4). The changes between the two worlds happen in step (3) and (5). This modelling cycle description is considered as an idealisation, probably only applicable in school contexts. Nevertheless, this approach is useful for us as a tool to show important features of our "modelling example", which differ even on an idealised level.

The second approach is problem solving by Polya (1949), who divides problem solving processes into four phases: understanding the problem, devising a plan, carrying out the plan and looking back. The third approach is the description of ways of mathematical argumentation and mathematical resources in physics by Bing (2008) and by Redish and Tuminaro (2007). They distinguish between four framings that describe how students justify their results to exercises: calculation (algorithms give exact results), physical mapping (math should represent physics correctly), invoking authority (using of results of the physics-course) and math consistency (similarities to other physics problems solved with math).

We consider the first two approaches as "draft" process models, which will have to be extended and adapted to the specific tasks we are analysing. The third approach discusses the role of mathematical resources and knowledge in solving problems from physics and we use this approach to identify and characterize resources needed by the students. In other words, we consider that the development of theoretical descriptions has to be based on empirical results as well. We ask the task designer and electrical engineering experts to solve the tasks from the perspective of students who well understood the contents of the electrical engineering course. Based on further consultation of subject matter and didactical experts, we (re-)construct what we call the "student-expert-solution" (SES). The SES is used as a basis for sharpening the theoretical description and analysis of the solving processes, resulting in what we call a "theoretically enhanced SES" (TESES). We use this as a tool for understanding first year engineering students' solving problems.

METHODOLOGY

In order to get a detailed solution for the exercises we conducted interviews with the task designer and electrical engineering experts from the institute at the University of Paderborn, which is responsible for the GET-B, using the Precursor-Action-Result-Interpretation (PARI) method, a task-based interview technique conducted with experts of the task (Means et al., 1995). The aim of these interviews is to identify the explicit and implicit expectations of competences. The half-structured PARI-interview consists of three phases: In the first phase, experts have to do the exercise without any interruptions, but they are told to think aloud while writing down solutions. In the second phase, the interviewer goes through the written solutions with the expert in order to reconstruct the reasons for the way the exercise was solved and identify the used resources. In this phase experts need to justify each step of their solutions and make explicit the knowledge they used. The last phase is a didactic reconstruction of the exercise, which consists of two parts. In the first part the experts' view on students' expected solutions is solicited. This part contains questions on alternative solutions to the exercise, typical mistakes of students after their first year and possibilities to validate the results. In the second part the interviewer asks for the reasons for assigning the exercise and possible variations for exercises on the topic, aimed at making explicit the implicit competence expectations.

The interview is the foundation of the student-expert-solution (SES), which is the best solution an electrical engineering student could achieve with the knowledge presented in electrical engineering and mathematics lectures prior to the exam (i.e., the knowledge after their first year of studies). In the next step the student-expert-solution is subdivided into categories, i.e., phases and cognitive resources in a deductive approach based on the three mentioned theoretic approaches. This document consists of a two-column table: the SES in first column and related theory-based comments in the second column; it is called the "theoretically enhanced" student expert solution (TESES). The TESES is used as theoretical instrument to analyse the transcribed solution processes of our pairs of students. The participating students attend degree-relevant courses in electrical engineering or industrial engineering, and they were at the end of their first year when the study was conducted. Nine pairs were filmed while they were working on the exercises and were talking about the way they solved them.

EXEMPLARY RESULT: A SES FOR ONE TYPICAL TASK

A sample exercise

For illustration, we present one of the exercises of the mentioned exam, which deals with magnetic circuits, and its theoretically enhanced student-expert-solution. This section gives the problem setting and a short overview of the solution. The exercise consists of six subtasks and starts with the following sketch of a magnetic circuit:

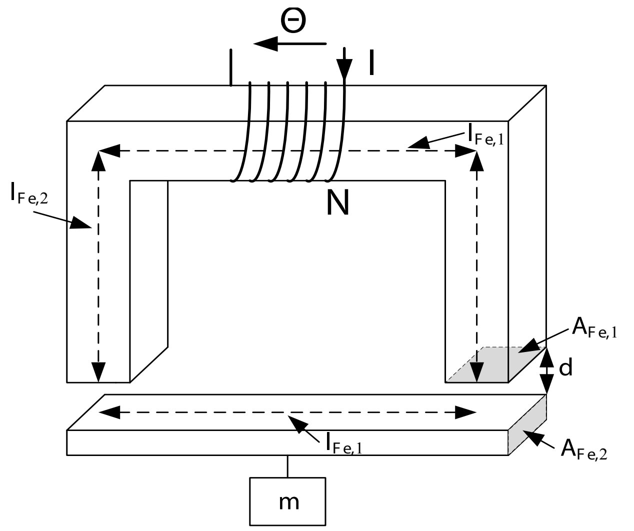

Figure 1: Sketch of a magnetic circuit consisting of two iron cores

The magnetic circuit consists of two iron cores with different cross section areas. The winding on the U-core has N=100 windings und is flown through by an electric current of I=10A. At the places of minimal distance between the two cores there should be a joint that behaves like an air gap. The exercise gives the following data for the iron cores: $l_{Fe,1}$=50 cm, $l_{Fe,2}$=30 cm, $A_{Fe,1}$=150 cm², $A_{Fe,2}$=60 cm², μ_r=1000.

Subtask 1: Sketch the equivalent electric circuit diagram of the magnetic circuit und simplify it as much as possible. Solution: The three parts of the U-core (the upper part and the left and right parts) and the lower iron core each give constant reluctances and can thus be summed up to one reluctance R_{Fe}. R_L, the reluctance in the air gap, is dependant on the width of the air gap d and has to be doubled, because there are two joints between the two parts.

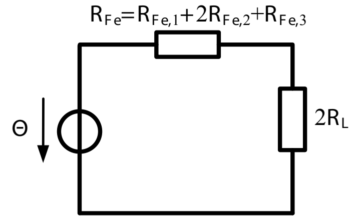

Figure 2: Equivalent circuit diagram

Subtask 2: Describe the total reluctance $R_M(d)$ as a function of the variable d. Solution: Because of subtask 1 the total reluctance can be written as the linear function of d: $R_M(d)=R_{Fe}+2R_L \cdot d$ using calculated values for R_{Fe} and R_L.

Subtask 3: Describe the inductance L of the setting as a function of d, the width of the air gap; assess the width of the air gap for which the inductance is maximal, and calculate the value of this maximum. Solution: The formula for L is $L(d)=N^2/R_M(d)$, i.e., the width of the air gap d is part of the denominator and the fraction becomes maximal, if $d=0$. We get the maximal value by insertion of $d=0$.

Subtask 4: Calculate the magnetic flux density b_L in the air gap.

Solution: We get the formula for the magnetic flux density by combining two formulas for magnetic fluxes.

Construction of the associated theoretically enhanced student-expert-solution

The first tool to analyse the exercise is the modelling cycle by Blum & Leiß (2007). We have developed several modifications of this cycle in order to better describe students' processes including identifying the resources required during the solving process. We summarize first and go into details later:

— Instead of constructing a real model as suggested in the modelling cycle students rather need strategies to understand conventionalized sketches and use them to mathematize the electrical engineering problem.

— Instead of entering the "world of mathematics," they enter into a "mathematics of physical quantities" with special resources: those resources are not solely based on pure mathematics learned in school or university mathematics courses.

— The authenticity of the problems offers strategies to validate the results.

We subdivided the solution process of subtask 1 and 2 into four phases. The two subtasks can be preliminary assigned to what is called (1) understanding of the task and the underlying situation, construction of the so-called "situation model" and (2) simplifying and structuring of the situation: construction of the so-called "real model". However, the competence requirements are quite different. The students do not do idealisations and simplification themselves, but they have to understand the given sketches as a "real model" whose idealisations will remain largely unconscious to them. The following idealisations, which were expressed in the interviews with the experts, are implicit: We only look at the magnetic behaviour of the test arrangement in an idealized static situation. The inductance of the windings, leakage fields and non-linear magnetic behaviour of the test arrangement are disregarded. Dynamic effects caused by the motions in the system are disregarded: Because of the changing of the distance between the two iron cores the energy changes and hence a non-linear ratio between power and force arises. The students are socialised into a world of certain real models. Idealisations often stay implicit and the students are often not aware, that there are idealisations. That is similar to physics students who use mass points without being aware of the idealisation, or geometry students that use dimensionless points. Students are to learn to "read" the sketch of Figure 1 and later find the fitting equations for this figure.

In contrast to saying that students should draw a picture or diagram of their own choice for understanding a situation (Polya, 1949; Blum & Leiß, 2007), the diagrams of Figure 1 and Figure 2 are very conventionalized in electrical engineering and constitute a specific "notational system", which is part of the tools of the discipline. Students who have not understood this might have difficulties if they approach the tasks and try to develop their own idealizations based on general physical knowledge – they could try to understand all the physical mechanisms and then be overwhelmed by the real situation. A further requirement is that students are familiar with the technical terminology (concepts) of electrical engineering in order to understand terms like the "magnetic flux" or "reluctance". Subtask 1 of the exercise requires using the "method of the equivalent electric circuit diagram", which helps to eventually mathematize the situation. The sketch of the test arrangement (Figure 1) has to be translated into an equivalent circuit diagram (e.g., Figure 2) using special rules for translation, which were expressed in the previous section. Equivalent electric circuit diagrams form a second notational system in electrical engineering. They are part of the acquisition of a domain-specific "graphical language" (similar to free-body force diagrams in mechanics or Feynman diagrams in quantum mechanics).

The third phase consists of setting up the equation for calculating the total reluctance with the help of the equivalent circuit diagram, which was generated in the first subtask. Once again, this is a translation task into which students have been socialized – idealized electrical or magnetic circuit diagrams are translated into sets of equations using the so-called Kirchhoff rules, just like free-body force diagrams are translated into vector equations using Newton's Laws, and Feynman diagrams are translated into path integrals using Feynman rules. The modelling of physical situations as idealized graphical diagrams and subsequent "mathematization" using sets of algorithmic translation rules is a common theme among the most powerful theories in physics and engineering. As mentioned above the reluctances in the iron and the air gap have to be added to execute this step, which requires forming of a set of equations between known and unknown physical quantities. There are also some differences to the modelling cycle in this step: A set of equations between physical quantities (numbers with units instead of just numbers) has to be set up. The student does not enter the "world of mathematics", but instead the "mathematics of physical quantities" with electrical engineering meaning. We are also convinced that many modelling problems at school level equally do not enter the "world of pure mathematics," but remain contextual mathematics with quantities.

In the next phase the total reluctance is calculated using the previous derived formula. As an example, in Figure 3 we reproduce the calculation of the constant part of the total reluctance (i.e., the reluctance of the iron core). The shorthand $\mu=\mu_r\mu_0=1000\mu_0=1000\cdot 4\pi\cdot 10^{-7}$ Vs/(Am) is used for the

$$\begin{aligned}
R_{Fe} &= R_{top} + 2 \cdot R_{left} + R_{bottom} \\
&= \frac{1}{\mu}\frac{l_{Fe,1}}{A_{Fe,1}} + \frac{2}{\mu}\frac{l_{Fe,2}}{A_{Fe,1}} + \frac{1}{\mu}\frac{l_{Fe,1}}{A_{Fe,2}} \\
&= \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V\cdot s}}{\mathrm{A\cdot m}}}\frac{0.5\mathrm{m}}{0.015\mathrm{m}^2} + \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V\cdot s}}{\mathrm{A\cdot m}}}\frac{0.3\mathrm{m}}{0.015\mathrm{m}^2} + \frac{1}{4\,\pi \cdot 10^{-4}\frac{\mathrm{V\cdot s}}{\mathrm{A\cdot m}}}\frac{0.5\mathrm{m}}{0.006\mathrm{m}^2} \\
&= 26.526\frac{10^3\mathrm{A}}{\mathrm{Vs}} + 2 \cdot 15.916\frac{10^3\mathrm{A}}{\mathrm{Vs}} + 66.315\frac{10^3\mathrm{A}}{\mathrm{Vs}} \\
&= 124.673\frac{\mathrm{kA}}{\mathrm{Vs}}
\end{aligned}$$

Figure 3: Calculation of the total reluctance

permeability of the metal, where μ_0 is the vacuum permeability and μ_r is the given relative permeability.

In the fourth phase, the resources of "mathematics of physical quantities" are needed, i.e., the management of units as well as techniques and strategies for the transformation of fractions containing physical quantities. Often these have neither been part of school nor university mathematics, and students tend to make many mistakes when attempting to apply these resources. One central resource is the "management of units", which includes the "handling of powers of ten". Those resources are typically required in physics and physics-related subjects if tasks require to calculate numerical values of physical quantities.

We define the "management of units" as the manipulation of base and derived physical units. All physical units in this exercise can be expressed in the base-units meter (m, for lengths), kilograms (kg, for masses), seconds (s, for time), and Ampere (A, for electrical current). E.g., the unit Volt (for electric potential) can be expressed as $(m^2kg)/(s^2A)$ using base units. The handling of powers of ten, which are expressed by letters in front of units, is also part of the management of units. The students have to translate the letters like k or m (for milli) to powers of ten, in this case 10^3 and 10^{-3}, respectively. The powers of ten then have to be multiplied, divided and potentiated using the rules for potentiation. For example, they need to realize that $6cm^2$ is equal to $6{\cdot}(10^{-2}m)^2=6{\cdot}10^{-4}m^2$, instead ignoring that the power of ten is squared alongside the unit and arriving at $6{\cdot}10^{-2}\,m^2$. In the last step the base units have to be translated in a summarising unit and the powers of ten into the right letter in front of the units.

The techniques and strategies for the transformation of fractions containing physical quantities include handling of algebraic and arithmetic terms with fractions. As seen in Figure 3, there are also compound fractions (especially due to units), and fractions have to be transformed in order to be able to add them.

Also resources of electrical engineering in a narrower sense are needed to solve the second subtask: students must know the formula for the reluctance, which is $R=1/\mu{\cdot}l/A$, where μ is the permeability, l is the length, and A is the cross-sectional area of the conductor. They have to insert the right values for each part of the iron core as well as for the two air gaps in order to apply the formula to the situation.

For the third subtask, students need to recall the formula for the inductance. Initially, students have to find a formula, which only contains known parameters from the problem setting, in this case the formula $L(d)=N^2/R_M(d)$. The maximum value can be determined using knowledge presented in the GET-B-lecture, namely, that the value for the inductance decreases the farther the two iron cores are moved away from each other. So the inductance is maximized if there is no air gap, i.e., the width of the air gap is zero. Alternatively $L(d)$ can be interpreted as a function of d, where the students can use techniques from mathematics to find the maximum (minimising the denominator). These two different types of reasoning for finding a solution are often applicable, i.e., reasoning by either calculation or physical mapping, see Bing (2008).

In the fourth part, students need to recall the formula for the magnetic flux density b_L, which is $\Phi=b_L{\cdot}A_{Fe,1}$, i.e., the product of the magnetic flux density and the relevant area. In this formula only $A_{Fe,1}$ is known, but there is another formula to calculate Φ, namely $\Phi=(N{\cdot}I)/R_M$, i.e., the product of the number of windings and the electric current divided by the total reluctance. In the second formula all physical quantities are known, and by combining the two formulas students can calculate the value for b_L. They get $b_L \cdot A_{Fe,I}=(N \cdot I)/R_M$ and with the help of this, $b_L=(N \cdot I)/(R_M \cdot A_{Fe,I})$.

This example shows the typical characteristics of "equation management". Initially students recall the relevant formulas containing known and unknown physical quantities. Then they start to transform these equations in order to get unknown quantities with the help of known quantities. This can either be done systematically by writing down all possibly relevant formulas at the beginning of the solution process, or step-by-step by starting with one formula and in each step trying to replace unknowns with the help of known physical quantities. This task is not rule based like solving systems of linear equations. It is not necessary to derive all the equations from general formulas in electric field theory presented in the lecture. This is a didactic reduction compared to the lecture, which was communicated in the exercise classes accompanying the lectures.

After finishing the calculations experts and students used various metacognitive strategies to validate their results:

— Validating of the result with the help of its unit in dimensional analysis: As the units of all physical quantities are known, one can check, if the units on the one side of each equation are the same as the units on the other side.

— Validating of the result with the help of its magnitude: In many cases the lectures, the problems in the exercise classes or previously done experiments show the order of magnitude of the resulting value.

— Check whether all information was used: it is a tacit agreement in the GET-B-course that all information that is mentioned in a problem is needed to solve it.

In contrast to the modelling cycle the implicit assumptions of the model are not questioned after finishing the exercise. On a positive note, in contrast to many school students' behaviour, validation takes place in a limited efficient manner, because the students expect tasks and results to be realistic, whereas in school mathematics often unrealistic, unauthentic assumptions and questions are prevalent.

PRELIMINARY RESULTS OF ANALYSING STUDENTS' SOLUTIONS

As we have not yet completed the analysis of students' work, we like just to point out two surprising results, where students employed special strategies that we did not anticipate. In the third subtask of the problem, the maximization of the inductance, all student pairs at first took the detour that when they read the word "maximum," they thought they needed to differentiate the term and find the roots of the first derivative. This does not lead to a solution, because the maximum is at the boundary of the interval; and since the differentiation itself is not easy, this approach could lead to further mistakes. This may be considered as a didactic obstacle in the sense of Brousseau having origin in school mathematics. In the fourth subtask, some pairs described the physical processes with the help of differentials like dA or dV, i. e., by application of university mathematics. Such argumentations are often used in exercises from field theory, which contain applications of Stokes' and Gauss' theorems for integral vector calculus. The magnetic flux through an infinitesimal area was expressed as $d\Phi=b_L \cdot dA$, where b_L is a function of the position. Students then used $b_L \cdot dA$ to mathematize the problem by seeing that the total flux can be computed using a surface integral. We observe a typical use of "differentials" in modelling physical problems that is not legitimated by mathematical theory. Since b_L is constant here, the integral method leads to the same simple formula as above, but in more general situations, the integral is mandatory. The phase of equation management was successfully preceded by a phase of deriving equations from more general principles. While unnecessary in this case, the approach was welcome as it shows further competencies. It depends on the course whether these are part of the expected range of competencies or whether students are just trained in "equation management."

FIRST CONCLUSIONS AND OUTLOOK

In summary we can state that our analysis shows that it is helpful to modify the modelling cycle as a theoretical tool for describing solution processes of students in engineering contexts:

— The component "cognitive resources" has to be added to the modelling cycle.

- One cannot distinguish between mathematics and the "rest of the world".
- Electrical engineering "does not exist" without mathematics.
- The setting up of equations is inseparably interwoven with the process of getting mathematical results. A division into two separate phases (setting up the model, mathematical solution) as in the modelling cycle is not adequate.

Furthermore, we have just started to use the TESES we described for analysing students' work. We will validate and extend our research results by developing additional student-expert-solutions for the remaining exercises of the GET-B exam, which require the higher mathematics taught at university level to a much greater extent than our above example. The content of these exercises includes for example ordinary differential equations of first and second order (in a task on oscillating circuits) or complex numbers (in a task on alternating current), which are the result of applying Kirchhoff rules to electrical circuits with time-varying currents through resistors, capacitors and inductances.

We will moreover analyse task solutions from written examinations to the GET-B-course in order to confirm, refine and enhance the results from the analysis using the SES and the video studies with the students. We also plan to interview teaching assistants from other universities. Although the content of the lectures is nearly identical between different German universities, there seem to be many differences in the expectations of competences, which can be made explicit by these interviews.

ACKNOWLEDGEMENT

Project funded by the German Federal Ministry of Education and Research (BMBF) under contract FKZ 01PK11021B. We would like to thank the reviewers for their thoughtful feedback.

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“No longer a divide between students and staff”: Learning through participation in statistics resource development

Irene Biza[1] and Eugenie Vande Hey[2]

1 University of East Anglia, Norwich, UK, i.biza@uea.ac.uk;

2 Loughborough University, Loughborough, UK

In this paper, we analyse the diaries of two intern mathematics undergraduate students who were involved in a project on statistics resource development. Our purpose is to use the theoretical perspective of communities of practice (Wenger, 1998) to investigate how internship functions as a learning environment for these students. The results indicate how the two students shaped their own practice within the internship by establishing a mutual engagement and a shared repertoire and by integrating the joint enterprise of the task to their own work. The overall social environment of the internship, consisting of other students and staff, shaped the practice of the two students and became a learning environment for them.

Keywords: Communities of practice, participation, statistics education, undergraduate mathematics education, learning resource development.

INTRODUCTION

In the last fifteen years higher education has started addressing the demand for statistics professionals in business, economics and research with curricular changes that improve the quality of undergraduate statistics education (Bryce, 2002). These changes demand further development of learning and teaching resources that will cover not only the statistical content but also the contextual needs of a range of disciplines in which statistics is applied (e.g., in engineering, psychology, finance, etc.). In this spirit a team of staff members and students in a UK university has been working since 2012 on the development of resources that can be used in statistical teaching and research with the second author of this paper as a co-ordinator. The students are mathematics undergraduates involved in the resource development either in summer internships or in the context of their final year dissertation project. Although there is little research on student involvement in curricular development in undergraduate mathematics education (Croft, Duah, & Loch, 2013), it appears that there is potential benefit of this involvement to students' learning and to resource development practice (Croft et al., 2013; Biza & Vande Hey, 2014). The study presented in this paper aims to contribute to this area of research by using the theoretical perspective of *communities of practice* (Wenger, 1998) to identify how the internship functions as a learning environment for the involved students. Specifically, we take a case study approach by concentrating on the *Excel Data Generators* (EDGE) project, which is a section of the overall resource development activity we described above, and on two intern students (Beth and Pauline). To this aim we analyse the diaries Beth and Pauline kept across the internship to investigate the following research questions: (a) How do students shape their own practice across the internship? (b) How is this practice related to the overall social environment in which it takes place? (c) How does this practice contribute to students' learning?

THEORETICAL PERSPECTIVE – ELEMENTS FROM THE LITERATURE

In this paper, we see students' learning occurring in the social context of their interaction with staff and other students and we draw on the theoretical perspective of *communities of practice* suggested by Wenger (1998). Communities of practice are formed by people who engage in a process of collective learning in a shared domain of human endeavour. In order to form a *community*, people need to be involved in activities with the same objectives, share a concern or a passion for these and learn how to achieve them better as they interact regularly. The *practice* is the source of coherence for the community, and defines the community through three dimensions: *mutual engagement, joint enterprise* and *shared repertoire*. Mutual engagement gives substance to the practice that "exists because people are engaged in actions whose meanings they negotiate with one another" (p. 73).The joint enterprise "is the result of a collective process" (p. 77) towards a common understanding of what the aim of the mutual engagement is. Shared repertoire includes resources created and used in a community of practice including "routines, words, tools of doing things, stories, gestures, symbols, genres, actions" (p. 83). Research discusses several communities of practice related to the teaching and learning of mathematics, especially at university level: undergraduate students, mathematicians and mathematics education researchers (e.g., Biza, Jaworski, & Hemmi, 2014). Usually, in this research lecturers reflect on their teaching or/and students participate and potentially shape a learning environment without being involved in its design. In our study students join the resource developers' community of practice: they work closely with members of staff and other students in resource development (e.g., data generator spreadsheets, problem/lab sheets guidance, video demonstrations, etc.) and engage in activities that are not included in their usual university practices. In this sense, students who usually are users of learning resources find themselves on the other side as developers of these resources.

Recently there is a growing interest in students' engagement as partners in course design and its contribution to students' learning. Croft and colleagues (2013), for example, identified several students' benefits from their involvement in producing screencasts for other students: "increased and deeper understanding of mathematical topics, improved technological skills, improved study habits, improved personal and organizational skills, and enhanced communication skills" (p. 1053). On the other hand, although lecturers saw opportunities in this collaboration for their professional development, they expressed reservations regarding students' lack of mathematical maturity and concerns over the mathematical integrity of the produced content. However, the study concluded that collaboration between students and lecturers during the resource design and production "may help lecturers overcome reservations, whilst preserving the benefits for students" (p. 1045). Solomon, Croft, Duah and Lawson (2014) see undergraduate students' internships as a pathway for improving dialogue between students and staff that challenges traditional hierarchical roles and relationships. They claim that projects that support these internships establish a *boundary-crossing setting* in which there is a "potential for expansive learning to take place" for the students and challenges for "staff perceptions about pedagogy" (p. 332). Similarly, in our evaluation of students' learning through their involvement in resource development projects we have considered not only the learning per-se, but also how and in what extent this learning is a result of students' participation (Biza & Vande Hey, 2014). This evaluation revealed that students explicitly linked their participation to the solidification and organization of their knowledge in terms of statistical thinking and reasoning. Also, this participation contributed to their learning about how statistics is taught and learnt. Students contributed to the whole process by introducing new practices and bringing in 'student' perspective. We also found out how students merged, or sometimes experienced the conflict of, multiple perspectives such as student vs developer and mathematician vs non-mathematician (ibid). In this paper, we pursue this investigation further: we consider the internship as an apprenticeship to the resource developers' community of practice, consisted usually by tutors, and we use the communities of practice theoretical perspective to investigate how this internship functions as a learning environment for the involved students.

THE CASE OF THE STUDY: THE EDGE PROJECT-BETH AND PAULINE

The Excel Data Generators (EDGE) project is the development of an Excel based tool (called the EDGE tool) and materials that support its use in research or teaching and learning (e.g. guidance, online resources, video demonstrations, lab sheets, etc.). The EDGE tool can generate data according to statistical models whose parameters can be set by the user, and perform statistical tests. It can be used for teaching/learning resource development, such as generation of datasets with specific characteristics for problem sheets or assessment; creation of individualised tasks with similar datasets, demonstration of statistical methods with a range of datasets; experimentation with variation and randomness; etc. The instructor can design the context of the problem and use the tool to generate relevant data. The full EDGE tool includes 19 Excel spread sheets for data generation and statistical tests including: one sample t-test; two sample t-test; chi-square test; tables of t-distribution and chi-square distribution critical values; simple linear regression; one-way ANOVA; etc..

Beth and Pauline are the two intern students involved in the EDGE project. Beth had completed her mathematics degree (BSc with first class honours) just before the internship. Additionally, she was one of the final year students who worked on the development of learning resources for a module on Statistical Modelling in the context of her final year dissertation the preceding academic year. This module was aimed at second-year students in mathematics and Beth produced a lab sheet and three videos on multiple linear regression in R (free software environment for statistical computing and graphics, http://www.r-project.org/index.html), together with an assessment tool to evaluate the effectiveness of these resources (Biza & Vande Hey, 2014). With this background Beth entered her internship with prior experience on working with staff on resource development and with skills on R and video creation. Pauline had just completed the second year of undergraduate studies in mathematics and she had attended a module on Introductory Probability and Statistics in her first year and a module on Statistical Modelling in her second year. She did not have any experience on resource development at the time she started the internship, but in her personal statement attached to her internship application she expressed strong interest in Statistics as a subject she wanted to take further.

The internship lasted five weeks during the summer break. Beth and Pauline were mainly working in a study area in which computers, learning resources (books, leaflets, etc.) and specialised software (such as LaTex, R, Excel, and Camtasia) were available. Elizabeth, the lecturer who coordinated the resource development, visited the study area almost every day and spent some time with Beth and Pauline on their project. Elizabeth, both students and Alicia (researcher in mathematics education with statistical teaching experience) had weekly working meetings (lasting approximately an hour on average) in which the produced materials were discussed. In the same area other intern students (up to 6 in total) were working on statistical and mathematical resource development. Each afternoon the whole group of interns, Elizabeth, Alicia and other members of staff and PhD students had informal coffee/tea breaks of approximately 45 minutes in which they discussed the progress of their work as well as other topics. This provided an informal context of discussion between students and staff on topics that were very often outside the strict boundaries of the projects. The resources produced by Beth and Pauline included: a fully developed set of Excel spread sheets; four video demonstrations of the use of the EDGE tool made with Camtasia software; and, two lab activities on one-sample and two-sample t-test with their accompanying handouts and EDGE (excel) files.

METHODS

In parallel to the resource development, data were collected for research purposes and towards the evaluation of students' experiences. These data included students' weekly diaries and audiotape of the regular working meetings. Due to space limitations, in this paper, we report outcomes only from Beth's and Pauline's weekly diaries. In these diaries, produced by the end of each week, interns were asked by the first author to include: the activities of the week with some outcomes, if they existed; items that they have learnt or they want to improve (e.g., a statistical concept or method, creation of spreadsheets or videos, use of statistical language, time-management etc.); what went well and what didn't go so well and why; examples in which working together or individually helped them to understand statistics better; contradictions in the collaboration (e.g., cases in which the collaboration didn't work very well) and what they did to overcome them, if they did anything, or why they didn't take any action; and, any other item they want to write down and reflect on it. These diaries, which were each around one A4 page long, were analysed according to how the students collaborated with each other (including the organisation of their work), with other students and with staff; and, how this participation affected their learning.

RESULTS

Beth's and Pauline's collaboration

Beth and Pauline mentioned in their diaries how they organised the work between the two of them as a team and how also they interacted with the other students in the same study area and with the staff members. In terms of the teamwork, early on they decided to work individually in some parts and together in others but try to be consistent all the time, especially in terms of formatting and symbolisation. As Beth described in her first week's diary:

> This week myself and [Pauline] started working on the EDGE generator spread sheet. We focused on the first four sheets (one sample t-test, two sample t-test, simple linear regression and two-way chi-square test) and divided up responsibility for them. [Pauline] worked on the one-sample t-test and I worked on the two-sample t-test sheet. We talked a lot together about how they worked in order to fully understand what we were doing and also to keep each other involved with what the other one was doing. [Beth, W1]

In the above excerpt Beth, who is more experienced, described how they "divided responsibilities" (who was accountable for what) by building a shared understanding of what they "were doing" and what the "other one was doing" (mutual engagement). In other occasions they worked together and helped each other:

> After this, we worked together on developing the simple linear regression sheet and the Chi-square sheet. I think we worked very well together on these and both welcomed suggestions from one another. [Beth, W1]

So, from the first week, they both engaged with the resource development and they established the rules of this mutual engagement. They were working individually and together – and at the same time they maintained a consistency in the outputs of their work by using the same formatting and terminology (shared repertoire):

> Now, we are creating two lab sheets, one for the one-sample t-test and one for the two-sample t-test. [Pauline] focused on the first lab and I focused on the second lab. [...] We did make sure that the formatting was exactly the same and conferred with each other on specific terminology so that the lab sheets were consistent. [Beth, W1]

This was evident in Pauline's diary as well:

> Although we wrote the lab sheets separately, checking with each other periodically has been important to ensure the layout and terminology within the labs is consistent. [Pauline, W1]

This pattern of work was followed across the internship, as Beth mentions:

> We worked separately on these spread sheets, each picking a different one, so that we would not overlap with editing. We still discussed what we were doing with each other and would ask each other questions if we were stuck in order for both of us to understand how each spread sheet works. [Beth, W4]

It seems that this work distribution worked for them as Pauline acknowledged:

> I feel we both worked really well together in deciding who would work on which part of the project and ensuring that we both knew exactly where we were at and what needed to be completed in the timeframe. [Pauline, W5]

Beth's and Pauline's collaboration with staff

The role of both Elizabeth and Alicia was to offer feedback and assess students' work. But Elizabeth's role also included the co-ordination and the management of the overall project, its objectives and enterprises. Both Beth and Pauline knew from the very beginning what was the overall aim of this project. However the aim of their task became more evident when Elizabeth informed them that she would use these resources in her module the following year:

> [Elizabeth] informed us that she plans to use this lab sheet for her second year statistics students. Even though the lab sheet will be available to other staff throughout the University to use, having a clear audience for the resource made it a lot easier when returning to the lab sheet after the meeting. [Pauline, W2]

Elizabeth was accountable for the establishment of this enterprise and both Beth and Pauline had to align to her plan. It was difficult to them to proceed without a confirmation from her that everything was on the right track. When Elizabeth was busy with other tasks and not available to confirm that everything was alright they "both felt that [they] were lacking a bit of direction" [Beth, W2] and that "[i]t was difficult to know if [they] were covering what she wanted"

[Pauline, W2]. We were interested to see how both students saw their relationship with staff at the end of the internship. Beth, who already had the experience of previous projects, wrote:

> The interaction between myself and members of staff has been very positive. I have always felt like I could approach both [Alicia] and [Elizabeth], and am not afraid to ask questions. Also, having lunch and tea breaks with both [Alicia] and [Elizabeth], as well as many other members of staff and PhD students, has made all of us interns feel very comfortable, and I no longer see a divide between students and staff. [Beth, W5].

Whereas, Pauline described how her relationship with staff had developed through the internship and how this can be beneficial for the following year of her studies:

> I also feel I have built a good relationship with [Elizabeth] and [Alicia] as this internship has progressed. To begin with it still felt as they were lecturers, and there was a boundary, but as we have been working so closely together I have become more confident in voicing my opinion and feel happy to ask questions whenever I have an issue within my work. This is a skill I aim to take with me into my third year at university and beyond and I believe it will have a positive effect on my module grades and hopefully my dissertation. [Pauline, W5]

Beth's and Pauline's collaboration with others

Throughout the internship Pauline and Beth had many opportunities to receive feedback from other interns and other staff members. Sometimes this was happening informally in the coffee/tea breaks or on purpose. In Week 4, for example, the whole group of interns and supervisors ran a presentation of their projects to three Human Biology lecturers who were interested in using these resources in their teaching. Beth and Pauline were very satisfied by their presentation on the EDGE project and also by the fact that it attracted "some interest from other departments" [Pauline, W4]. This was the first time that they had some formal feedback from outside their team and their department and this was a strong motivation for them.

Additionally, the interns working in the same study area organised between themselves a practice of resource exchange and getting feedback that increased gradually towards the end of the internship. In Week 3, for example, Beth and Pauline shared the resources created by that time with the others. This helped them to make the content user-friendlier. Also, the appreciation of other interns on their work, especially Andy, a very experienced Masters student, encouraged Pauline:

> After [Andy] letting us know how impressed he was with the EDGE tool (which uplifted my spirits a lot as it seems to have taken a long time to complete this!) he gave us comments on the lab sheet and ideas for further improvement. [Pauline, W3]

Both students claimed that the overall social interaction in the study area was very positive. In Week 3 the whole team of interns was already getting along and they had started socialising, as Beth mentioned: "We're all getting on very well and were successful in winning a pub quiz last night!" [Beth, W3]. At the end of the internship, Beth acknowledged: "All of the interns have got along very well, and it's much nicer to work in an environment where everyone is happy and having a laugh." [Beth, W5]. Whereas, Pauline concluded:

> Overall I have really enjoyed this internship [...] It is definitely interesting how close you can become with a group of people when you are in a fairly small space with them for 40 hours a week! [...]. [Pauline, W5]

Contribution to learning

Both Beth and Pauline found their involvement in the project beneficial to their learning of statistics and other general skills. At the beginning Beth with her experience and her familiarity with this type of projects, worked as a bridge between Pauline and the lecturer (Elizabeth). As Pauline had not attended the module offered by Elizabeth the previous year, she had difficulty putting together the terminology she knew already with this used by Elizabeth. But Beth who had the experience from both modules was able to help Pauline to make the link:

> I found [Elizabeth] uses some different terminology to what I am used to in statistical modelling,

> however [Beth] has explained the differences to me which has made this clearer [Pauline, W1]

Beth helped Pauline to establish statistical knowledge that was familiar to her:

> Although simple linear regression is something I am familiar with, I found working together was the best way to tackle this as I found myself struggling with getting my head around some parts of this process and being able to collaborate with [Beth] helped my understanding. [Pauline, W1]

Or, to expand her knowledge in new items:

> I was unfamiliar with the code before doing this, so I am glad we both had a part in the completion of this sheet. It has expanded my knowledge of the different commands available in Excel and I have a good understanding of the calculations carried out throughout this sheet [Pauline, W1]

Pauline was keen on grasping the opportunity to learn new things although there were more experts than her to undertake the tasks. Whereas Beth found that explaining to Pauline (and other students) improved to her own knowledge.

> We [Beth and Pauline] have decided to work together on the videos as [Pauline] is very keen to learn how to create them and I have made them before. I think this will be good as I can help [Pauline] if she gets stuck and she may notice new things for us to do when making the videos. Next week I am going to help teach [Pauline] and [other intern] how to make Camtasia videos, which will improve my knowledge and usability with this software even further. [Beth W2]

Both Beth and Pauline acknowledged that their knowledge of statistics improved throughout the internship:

> Beth: Throughout this internship I would say my knowledge of statistics as a whole has improved, and through the creation of the EDGE tool, I have become a lot more comfortable with various tests. My understanding of significance and power has increased as we spent a long time creating and editing the lab sheets which involved a lot of interpreting p-values and manipulating various values to see the effect on significance and power. I have also acquired some knowledge of survival analysis [...] [Beth, W5]

> Pauline: [...] I feel as if I have taken a lot of knowledge away from this project. I have basically covered the bulk of [Elizabeth's] Statistical Methods module, learning about multiple statistical tests that I hadn't come across before. [Pauline, W5]

Also, they mentioned how their presentational skills improved alongside other skills:

> Beth: This has reassured me that I am capable to present to groups of people, and that I don't need to be nervous about presenting or having an interview. [...] My ability with Excel has naturally improved a lot and I feel these skills will help me in the future. Also, creating the videos has reinforced my understanding of this software [Beth, W5].

> Pauline: I really enjoyed learning how to use Camtasia to produce the tutorial videos. [...] It is a skill that I hope will come to use in my final year here [...] I also increased my confidence in using Latex and came across new programs such as GeoGebra. I am able to come away from this internship with a lot more confidence not only in myself, but in my ability to research new areas. [Pauline, W5]

CONCLUSION

In this paper, we analysed the diaries of two intern students who were involved in a project on statistics resource development to our aim to investigate how the internship, in this case, and students' involvement in curricular development, more generally, functions as learning environments. In this study we considered these students entering a resource developers' community of practice and we obtained some insight into how students form their own practice across the internship (research question (a)). Specifically, from early on, both students established a *mutual* engagement: they established a shared understanding and distribution of responsibilities; they shared their practice with other students and staff; they negotiated the meaning of their work with others and they valued this negotiation. At the same time they established a *shared repertoire* that they maintained later by bringing together different terminologies and being consistent in formatting and symbolisation. Finally, the *joint enterprise*, although among staff's responsibilities, became gradually part of their work. Also, we identified how this practice is related to the overall social environment in which takes place (research question (b)) through the students' regular interaction with each other and with other students and staff. Finally, we have evidence that, according to those students, the working environment of the internship became an environment of learning (research question (c)) through participation and communication. If we draw on the more recent work of Wenger, McDermott and Snyder (2002) on *cultivating communities of practice* – in our case the resource development community of practice Elizabeth aimed to establish – we identified different levels of participation: *core* members were Elizabeth and Alicia who knew the aims of the project and co-ordinated the actions; Beth was an *active* member, who was familiar with this type of work and understood the rules; Pauline started as a *peripheral* member but very keen on drifting to the centre and becoming *active* with Beth's and others' support; other students or staff acted *outside* the community but involved occasionally when the community shifted to their area of interest and expertise and asked for feedback. According to Wenger and colleagues (2002) the quality of a community is established when all its members regardless their level of participation feel *full members*. From Beth's and Pauline's self-reports in their diaries we have evidence of them experiencing this sense of membership. However, we cannot claim that the participation to the project did not have some drawbacks. Students' frustration, for example when Elizabeth was not available to assist them was an indication that the teamwork was not always efficient. From the lecturer's perspective on the other hand several downsides challenge the success of this experiment (see also Croft et al., 2013, about lecturers' resistance), such as disproportionate time investment; integrity of resources; limited contribution to lecturer's professional development; and lack of institutional value on projects like this. The evaluation of the overall experience will be more comprehensive when we combine the students' perspectives with staff views as well and this is the next step of our analysis.

ACKNOWLEDGEMENT

We would like to thank the Teaching Innovation Award-2013 scheme for its support to the EDGE project and all the intern students and staff members who participated in the EDGE project since its launch in 2012.

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Contributions and limits of a specific course on manipulation of formal statements for fresh university students

Stéphanie Bridoux[1] and Viviane Durand-Guerrier[2]

1 Université de Mons, Mons, Belgium, stephanie.bridoux@umons.ac.be

2 Université de Montpellier, Montpellier, France

In this paper, we present preliminary results from an on-going project about an innovative course for first year students in Computer Sciences aiming to improve students' abilities in manipulation of formal statements and writing proofs. The data analysed show that although the course seems to improve syntactic abilities, the semantic control of formal statements remains problematic for most students.

Keywords: Didactics of mathematics at university level, logical analysis, proof, syntax, semantics.

INTRODUCTION

It is well known that the manipulation of formal statements is a main difficulty for many students starting university. In particular the widespread use of symbolic register to characterize the notions minimizes the use of natural language and faces students with the obstacle of formalism (Dorier, 1995). As a consequence, an important question is to find mathematical organisations [1] that allow students to both grasp the meaning of these notions and become familiar enough with formalism in order to be able to engage in proof and proving (Durand-Guerrier et al., 2012). In this paper, we describe such an attempt carried out at the University of Mons (Belgium), and we present elements of an explorative research aiming to investigate the effectiveness of such an innovation.

BACKGROUND

In the teaching of mathematics at university, definitions play a crucial role, in particular in the dialectic between proof and definition. From an axiomatic perspective, the role of definitions is to introduce new concepts relying on concepts already known, and avoiding contradiction. Ouvrier-Buffet (2006) showed the relevance of the activity of construction of definition for the process of mathematical conceptualization. However, in university courses, in France as in Belgium and in many other countries, the most common method of teaching is the axiomatic one (definitions, examples, theorems, proofs) in particular regarding formalising, unifying and generalizing (FUG) concepts such as the concept of vector space (Dorier, 1995). FUG concepts unify and generalize objects, tools or methods that exists previously in the students' background in various forms by using a new formalism. These concepts are difficult to introduce because the distance between previous and new knowledge is very big. Bridoux (2011) highlighted the nature of formalising, unifying and generalizing of the main notions of Topology, whose definitions require an intensive use of mathematical and logical formalism.

Durand-Guerrier (2013) stresses the importance of taking into account the logical complexity of the statement at stake in the tasks given to students in order to allow students to recognize the productiveness of operational formalism (the syntax) and to articulate it with the meaning of the mathematical objects involved in the tasks (the semantics). At the University of Mons (Belgium), in order to take into account this perspective, an innovative course was implemented in 2012 to replace the traditional mathematical courses for students in Computer Sciences.

Although these students are *a priori* well educated in mathematics, we hypothesize that they will face the classical difficulties identified in the literature when using mathematical language and notations, producing examples or engaging in proofs (e.g., Thomas et al., 2012). In this paper, we present the first stage of a project aiming to explore the following research question: How students involved in the innovative course deal with formal statements, in particular how do they take in consideration the relationship between syntactic and semantic aspects?

METHODOLOGY

The data was collected during a written examination. Due to the limited place, we choose two items for which the manipulation of formal statements was the key. The methodology that we use is a qualitative one relying on the consideration of the distance between *a priori* analysis, including a mathematical and a logical analysis of the tasks, and *a posteriori* analysis. The *a priori* analysis opens the possible answers that could be expected from students and the foreseeable difficulties. This *a priori* analysis has been done afterwards, for the purpose of the research, and reveals an underestimated logical complexity of the statements and the tasks considered. We first describe the main characteristics of the innovative course, and then present the data and the participants. In the next section, we provide the *a priori* analysis of the two items chosen for this paper, on which *our a posteriori analysis* relies. The last section of this paper is devoted to our conclusion.

DESCRIPTION OF THE DIDACTICAL INNOVATION INTRODUCED IN MONS UNIVERSITY

The teachers of the department of mathematics at UMONS took the opportunity of the introduction of the LMD [2] reform in Belgium to reconsider the content to be taught to first year students, and also the mathematical abilities that students should have developed at the end of their studies. The general objective for the first year at university is the deepening of concepts studied in upper secondary school in classical domains such as: Calculus, Algebra and Linear Algebra (e.g., Convergence of numerical sequences – Groups – Linear mapping – etc.) and at the same time helping students to master the following points: giving meaning to concepts, without neglecting technical and operational aspects; being able to use relevant knowledge even when they are not explicitly required; developing flexibility between the various representations, frames and registers of mathematical objects as suggested by Douady (1987); making explicit reasoning and providing justifications. All these aspects involve the manipulation of formal definitions and the properties studied during the courses. According to these aims, there are many tasks devoted to the manipulation of definitions during the tutorial classes and the assessment of students.

The course named « Mathématiques pour l'informatique 1 » takes place in the first semester. It comprises 20 hours of lectures and 40 hours of tutorials and deals with logic, set theory, methods for proof and proving (mathematical induction, proof by contradiction, proof by contraposition, etc.), and some topics in arithmetic. A characteristic of this course is to attempt working with students on the syntax and the semantics of quantified statements, involving more or less familiar or intuitive concepts. The quantified statements we work with involve simple inequalities, classical functions or classical sets and subsets such as intervals. At this stage, convergence of numerical sequences or infinite sequences of sets had not been introduced [3]. Here is an example of tasks proposed to students in this course.

Let X and Y be subsets of $\mathbb{R}$ and $f: X \to Y$ be a function.

a) *Define « f is injective ».*

b) *Define « f is surjective ».*

c) *Is the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^4 + x^2$ injective ? Surjective? Justify your answer using the definitions provided in the previous points.*

This kind of task is generally not given in secondary school. As a consequence, the teacher has to use didactical means such as explanations on the methods, orally as well as in writing. During the tutorial classes, the students are asked to present their work to their peers in order to facilitate exchanges between students and to allow collective discussions on reasoning and other difficulties.

PRESENTATION OF THE DATA AND OF THE PARTICIPANTS

In accordance with the aim of the course, exercises involving reasoning and manipulation of definitions were included in the assessment. We present below two items, the first one requires to prove by contraposition the statement that characterises non positive real numbers; the second one requires the definitions of injective and surjective function and their use for studying two given functions. In the first exercise, we will mainly observe the mastery of the syntax, while in the second one, we will focus more on the articulation between syntax and semantics (Weber & Alcock, 2004, Durand-Guerrier, 2011).

52 students were assessed in January 2014 and the assessment lasted 4 hours. Around 80% of the participants attended the whole course. Our data consists in the assessment papers submitted by the students. The seven questions given to students challenged them to manipulate quantified statements. It is not possible to say how much time the students spent on these two particular questions. The assessment was a closed book exam.

A PRIORI ANALYSIS OF THE TWO EXERCISES

In this paragraph, we present the *a priori* analysis of the two chosen exercises taking into account their logical structure.

First exercise: A proof by contraposition [4]

Let us consider the proposition: $\forall x \in \mathbb{R}, (\forall \varepsilon > 0\ x \leq \frac{\varepsilon}{2}) \Rightarrow$
$\Rightarrow x \leq 0$ *(1)*

The goal is to prove this statement by contraposition

- *Provide the proposition to prove in order to achieve this proof*
- *Write down the proof*

In the first item, it is required to provide the statement that will be proved, that is providing the contrapositive of statement (1). During the course, students were encouraged to perform this step prior to engage in the proof in order to avoid confusion between negation, proof by contraposition and proof by contradiction. The statement (1) is a universally quantified conditional statement whose antecedent is a universally quantified statement.

Global structure of the contrapositive

Let us denote $A(x)$: $(\forall \varepsilon > 0\ x \leq \frac{\varepsilon}{2})$ and $B(x)$: $x \leq 0$ respectively the antecedent and the consequent. Statement (1) can be formalized as: $\forall x[A(x) \Rightarrow B(x)]$. We hypothesize that students will recognize this structure. The global structure of the contrapositive is hence: $\forall x[\neg B(x) \Rightarrow \neg A(x)]$. It is possible that some students could apply incorrectly the following rule: *while negating a statement, change all the quantifiers* (this rule is valid only for quantifiers in prenex position) providing: $\exists x[\neg B(x) \Rightarrow \neg A(x)]$. Also some students may confuse negation and contraposition (or proof by contraposition and proof by contradiction) and propose: $\exists x[\neg B(x) \wedge \neg A(x)]$.

The next step is applying the negation to $A(x)$ and $B(x)$.

Negation of A(x) and B(x)

$B(x)$ is an atomic formula; taking its negation is elementary: $\neg B(x)$: $x > 0$.

$A(x)$ is a complex formula comprising a variable in the scope of a bounded universal quantifier: $(\forall \varepsilon > 0\ x \leq \frac{\varepsilon}{2})$. Taking the negation provides a statement $(\neg A(x))$ with a bounded existential quantifier: $(\exists \varepsilon > 0\ x > \frac{\varepsilon}{2})$.

Two rules are involved:

a) While negating a universally quantified statement, change the quantifier to the existential one.

b) Negating "$\leq$" provides "$>$" and vice-versa, that applies correctly to "$x \leq \frac{\varepsilon}{2}$", but not to "$\forall \varepsilon > 0$".

Although the treatment of bounded quantifiers had been discussed during the course, some students may apply this rule incorrectly, providing "$\exists \varepsilon \leq 0$,".

The method to prove the contrapositive consists of providing a number strictly superior to a given number. It had been widely discussed during the course, and the mathematical knowledge required is elementary. So in this exercise we aimed to test the reasoning structure and the consideration of the logical structure (mainly the manipulation of syntactic rules). Although this exercise might appear rather easy, this brief *a priori* analysis shows the complexity of the first item, whose understanding should help students to successfully write down the proof.

Second exercise: Manipulation of definitions

First, students are asked to provide the definition of an injective function and the definition of a surjective function. The formal definitions are expected, this should be clear for students due to the focus of the course. Indeed, during the course, the two classical definitions of an injective function were given, both in natural language and in formalized language:

$$\forall x \in D_f \forall y \in D_f (x \neq y \Rightarrow f(x) \neq f(y)) \text{ (1)}$$

$$\forall x \in D_f \forall y \in D_f (f(x) = f(y) \Rightarrow x = y) \text{ (2)}$$

Various examples of functions had been studied and the necessity of identifying the logical structure and the mathematical needed to prove that a given function is injective (surjective) or not had been considered. This is required in the two following items, for which the students should indicate if the given function is or not injective, resp. is or not surjective, and then prove their claims.

Type of proofs expected from students

It should be obvious that to prove that a given function is not injective it is enough to simply provide a counterexample. To prove that a given function is injective students need to provide a proof by generic element using definition (1) or (2). Likewise, to prove that a given function is not surjective, students need either to determine the set of all the images, or identify an element b with no antecedent. To prove that a given function is surjective, students have to consider a generic element b of the outputs set and find an antecedent. This leads to prove that for any b in codomain, the equation $f(x) = b$ has at least one solution.

In the course expectations, critical review of the reasoning produced by students is emphasised. A strong requirement of the course is also making explicit the logical structure of formal statement that should appear in the students' work. Another requirement concerns justifications: students have to explicitly indicate the results of the course.

Mathematical analysis

a) $g: \mathbb{R} \to \mathbb{R}: x \to g(x) = x^4 - 1$

Injection: Here $D_g = \mathbb{R}$. is an even function; hence it is not injective. It is also possible to prove this by providing a counterexample, i.e. an element satisfying the negation: any couple of element $(-a, a)$ with $a \neq 0$ is suitable. Note that attempting to prove that function g is injective leads to the class of counterexamples $(a, -a)$ with a in $\mathbb{R}^*$ (nevertheless, some students could provide an erroneous proof).

Surjection: As $Im\, g = [-1, +\infty[$, while the set of outputs is $\mathbb{R}$, function g is not surjective. It is also possible to prove directly that any given element of $]-\infty, -1[$ is not reached, i.e. that equation $x^4 - 1 = b$ with b an element of $]-\infty, -1[$ has no solution.

So in this question only elementary knowledge of functions, equations and real numbers previously studied at secondary school is involved. As indicated in the course description, similar functions had been studied during the course.

b) $h: 2^{\mathbb{N}} \times 2^{\mathbb{N}} \to 2^{\mathbb{N}}$
The function h associates to each couple of subsets of the set of natural integers the intersection of the two subsets. No similar function has been studied during the course, but the set theory knowledge that is necessary to prove the function is surjective had been studied, and numerous examples had been treated in the corresponding chapter, so that *a priori* the repertoire of students is sufficient to provide a counterexample for injection, in order to prove that function is not injective.

Injection: It is possible to find different couples of elements with the same image; example: $h(\{1\}, \{1\}) = h(\{1, 2\}, \{1\}) = \{1\}$. Hence, h is not injective.

Surjection: $\forall X \in 2^{\mathbb{N}}\ X = h(X, X)$; as a consequence h is a surjection.

Nevertheless, as considering couples of sets as inputs for a function is a rather abstract process, it is likely that many students will not be able to provide a correct proof. In particular, considering a pair of pairs of subsets in order to engage in a general proof of injection (in case non injection is not recognized) is difficult in terms of formal manipulation as well as conceptually. We hypothesize that such a proof will not appear in the students' work.

This brief *a priori* analysis of this second exercise emphasizes the fact that here both syntactic and semantic aspects of proof are closely and dialectically intertwined (Durand-Guerrier & Arsac, 2005). So although syntactic forms are clue aspects of mathematical reasoning, a semantic control on objects is needed in order to elaborate the expected proofs.

$\forall x \in \mathbb{R}, x > 0 \Rightarrow \exists \varepsilon > 0\, x > \frac{\varepsilon}{2}$	21	$\forall x \in \mathbb{R}, x > 0 \Rightarrow \exists \varepsilon \leq 0\, x > \frac{\varepsilon}{2}$	16
$\exists x \in \mathbb{R}, x \leq 0 \wedge \exists \varepsilon > 0\, x > \frac{\varepsilon}{2}$	2	$\exists x \in \mathbb{R}, x \leq 0 \vee \exists \varepsilon \leq 0\, x > \frac{\varepsilon}{2}$	2
$\exists x \in \mathbb{R}, x > 0 \Rightarrow \exists \varepsilon > 0\, x > \frac{\varepsilon}{2}$	2	$x > 0 \Rightarrow \exists \varepsilon > 0\, x > \frac{\varepsilon}{2}$	2
$\forall x \in \mathbb{R}, x > 0 \Rightarrow \forall \varepsilon \leq 0\, x > \frac{\varepsilon}{2}$	1	No answer	6

Table 1: Students' answers for the contrapositive

A POSTERIORI ANALYSIS OF THE TWO SELECTED QUESTIONS

First exercise: a proof by contraposition

Statements provided by the students are given in Table 1.

For this item, 42 students provide a conditional statement, among them 21 provide a correct answer; 6 change the universal quantifier in prenex position in an existential one, and 19 negate "$\forall \varepsilon > 0$" in "$\forall \varepsilon \leq 0$". Two students provide a conjunction, which is not the negation, and two students provide a disjunction not equivalent to the contrapositive.

We observed that the statement provided by students for the contrapositive had an impact on the way they shaped their proof, as in the following example, where the student fulfils the expectation of the teachers by writing down the proof in detail:

Given $x \in \mathbb{R}$. Let us consider $x > 0$. Let us show then: $\exists \varepsilon \leq 0, x > \frac{\varepsilon}{2}$. *Let us take $\varepsilon = -1$. Let us show $x > \frac{\varepsilon}{2}$, i.e. $x > -\frac{1}{2}$; true as x>0 by hypothesis.*

Table 2: An example of an incorrect proof in the first exercise

We interpret this as an indicator that this student remains in a syntactic point of view, neglecting the possible semantic control that could have lead him to come back to (and may be modify) the statement provided for the contrapositive. This example illustrates the importance of the relations between syntax and semantics in proof production, in line with consideration by Weber & Alcock (2004).

Second Exercise: manipulation of definitions

Providing the definition of an injective (resp. surjective) function

Most students (45 out of 52) were able to provide the definition of an injective function. Among them 38 choose the definition considering two distinct elements in the antecedent of the implicative statement. For the definition of a surjective function, we have 38 correct answers; all students, except one who used the image set, provided the definition with quantifiers. Wrong answers are either incorrect statements in natural language like "f is surjective if every real number has at least one image" or incorrect quantified statements like "f is injective if $\forall x_1 \in C\, \forall x_2 \in D\, x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$".

Manipulation of definitions

a) The first function considered in the *a priori* analysis is $g(x) = x^4 - 1$.

As already said this function is not injective. 36 students gave the correct answer, first writing down the negation of the statement (which is a teacher's expectation) and then giving a counterexample taking two opposite numbers. 5 students did not write down the negation, but provided a counterexample. This confirms that the rule "to prove that a universal statement is false, provide a counterexample" is available at that level for most students. What is often problematic is the relationship to the negation of a given statement (Njomgang & Durand-Guerrier, 2011) – here it is difficult to know if those who provide the negation make the links, or just fulfil the teacher's expectation.

Function g is not surjective. We obtained 27 out of 52 correct answers. Half of these 27 students began to write down the negation of the definition with a correct logical structure. Then, they provided a counterexample: mainly saying that −2 has no antecedent, sometimes without mathematical justification; 3 students provided *Im g* without mathematical justification. These students seem to master the mathematical knowledge involved. It is not the case with the 8 students who assumed the function is surjective, providing an antecedent for each given real number, without controlling its effective existence. In the following example, the student does not take in account the fact that $\sqrt[4]{y+1}$ is defined only on the interval $[-1, +\infty[$.

Function g is surjective, i.e. $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, g(x)=y$. *Given* $y \in \mathbb{R}$. *Let us take* $x = \sqrt[4]{y+1}$. *Let us show* $g(x) = y$ *i.e.* $x^4 - 1 = y$. $(\sqrt[4]{y+1})^4 - 1 = y$, *i.e.* $(\sqrt[4]{y+1})^{\frac{4}{4}} - 1 = y$, $y + 1 - 1 = y$, $y = y$.

Table 3: An example of an incorrect proof of the false claim that *g* is surjective

The second function considered in the *a priori* analysis is h: $2^{\mathbb{N}} \times 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$ that associates to each couple of subset of the set of natural integers, the intersection of the two subsets. Function h is not injective. We obtained 20 correct answers; 18 students did not answer this question. As in the previous exercise, most of the students began to write the negation of the definition. It appears that those who were able to identify that the elements of the inputs set were couple of subsets succeeded in providing the proof. Recognizing the nature of these elements was a difficulty for many students: some identified them as a subset of $\mathbb{N}$, or as a pair of numbers. In the following example, the logical structure of the proof is more or less adequate [5], but the proof is invalid due to the involvement of inappropriate objects.

h is not injective, thus $\exists x_1, x_2 \in 2^{\mathbb{N}}, f(x_1) = f(x_2) \Rightarrow x_1 = x_2$. *Let us take* $x_1 = (0,1)(1,2)$ *and* $x_2 = (1,2)(0,1)$. $f(x_1) = f(x_2) = 1$.

Table 4: An example of incorrect proof

Function h is surjective. We obtained 13 correct answers; 20 students did not provide an answer. Here the nature of the mathematical objects at stake is problematic for many students.

Concerning the results for both questions, we are able to say that students successful in the first question are, except one, successful in the second one. One perspective is to study whether the difficulties are the same in each question, for example for the sense given to the objects or for the manipulation of formal statements.

CONCLUSION

In this paper, we have focused on the elaboration of an innovative course for students attending the first year university in Computer Sciences in Belgium (Mons University). The aim of this course is to work with students on formal statements and proof and proving in mathematics. A main issue of this course is to emphasise some logical aspects that are often not elaborated upon by teachers, such as explicitly teaching and insist on *how to negate* or *how to take the contrapositive of* a conditional statement. In addition, specific attention is paid to the production of examples and counterexamples.

As a first exploration of the impact of this course on the students' logical and proving abilities, we have analysed two of the exercises focusing on manipulation of formal statements including definitions that were given in the assessment at the end of the course. A first result is that a rather large number of students attending the course were able to provide the required definitions, to recognize the logical structure of statements and to correctly negate a statement involving quantifiers. Referring to the existing literature that attest of strong difficulties with negation for university students (e.g., Njomgang Ngansop & Durand-Guerrier, 2011), we hypothesize that this could be a benefit of the course. Nevertheless, difficulties remain: we have observed that for some students, the manipulation of formal statements remains at a syntactic level. They seem to use routines without any semantic controls. In the case where familiar mathematical objects (such as real numbers) are involved, the related difficulties are limited, while these difficulties seriously increased when knowledge of less familiar objects is required. A second result concerns the logical complexity of the tasks that the students had to achieve, complexity that had been partly underestimated when designing the assessment. We mention in particular those difficulties related to the negation of statements involving bounded quantifiers, so that difficulties related with the dialectic between syntax and semantics, in particular in cases unfamiliar mathematical objects are being considered. The next step of this on-going study is to conduct interviews in order to better understand the way students deal with this type of tasks, for which mathematical and logical aspects are closely intertwined.

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ENDNOTES

1. A mathematical organisation of a subject to be taught consists of teachers' or institutionnal's choices among « a set of mathematical elements » (types of problems, techniques, notions, properties, results, etc.) (for developments in English see, for example, Barbé et al. 2005).

2. LMD is the acronym for the three degrees of Higher Education in the Francophone area: Licence (Bachelor's degree), Master (Master's degree), Doctorat (Doctorate).

3. These concepts are taught in the course named « Mathématiques pour l'informatique 2 », organised during semester 2.

4. In the course, the coma is used to indicate that the scope of the universal quantifier is the conditional statement, so that the statement at stake is closed. In predicate calculus, we would write: $\forall x \in \mathbb{R}\ [(\forall \varepsilon > 0\ x \leq \frac{\varepsilon}{2}) \Rightarrow x \leq 0]$.

5. We notice an incorrect use of *thus* (in French *donc*), while what is expected here is *That is to say* (*c'est-à-dire* in French). Moreover, the statement provided by this students to formalise « *f* is not injective » is not correct.

Growth of mathematical knowledge for teaching – the case of long division

Jason Cooper

Weizmann Institute of Science, Rehovot, Israel, ntjason@weizmann.ac.il

When university mathematicians teach mathematics courses for non-mathematicians, there may be a discrepancy between the mathematics they aim to teach and the mathematics their students aim to learn. In this paper, I analyze a lesson on long division taught by a mathematics Ph.D. student, where the learners were in-service elementary school teachers. Taking a Commognitive approach, I describe some crucial differences in the teachers' and the mathematician's discourse on mathematics and on teaching, which created opportunities for mutual learning. Uncovering the affordances and limitations of this teaching/learning situation is expected to help mathematicians become more effective teachers of non-mathematicians in general, and of pre-service and in-service teachers in particular.

Keywords: Elementary school mathematics, university mathematics, professional development, Commognition.

INTRODUCTION

Researchers such as Nardi and colleagues (2014) are coming to view university mathematics as a discourse, conceived as accepted modes of communication in mathematics departments. However, other communities engage in their own mathematical discourses (physicists, chemists, mathematics teachers), which may be quite different from the discourse of mathematicians. What happens when mathematicians teach courses for non-mathematicians? What is the nature of productive learning in such situations? These are the questions that guide my investigation of an extreme case – a mathematics Ph.D. student teaching in-service elementary-school teachers a lesson on the long division algorithm (LDA). This mathematician may be an expert on abstract algebra, but what can he possibly know about division in elementary school, how it's taught, or what kind of difficulties students typically encounter? I show how the differences in mathematical discourses of the two parties created opportunities for mutual learning, and how this meeting of two communities of mathematics educators brought a rich perspective to the teaching of LDA, where pedagogical considerations of teaching and learning interacted with mathematical considerations. Understanding how this came about may guide mathematicians in teaching pre-university mathematics to non-mathematicians, particularly in the teaching of school teachers.

SETTING

The professional development (PD) under investigation was the initiative of a university professor of mathematics, and was taught by mathematics graduate students. Its declared goal was to broaden and deepen the teachers' understanding of the mathematics they teach. Approximately 90 teachers enrolled in the 2011–12 program, which consisted of ten 3-hour sessions taught in six groups. The data collected in this research project consists of audio recordings of all the sessions, interviews with the instructors, and teacher questionnaires – expectations at the outset and feedback after each session. In this paper, I analyze a lesson on LDA in which 15 grade 3–6 teachers participated. The instructor was a mathematics doctoral candidate.

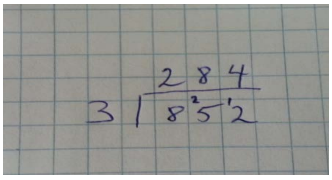
Here are some features of LDA that the instructor decided to attend to in this lesson:

- LDA is opaque – the underlying mathematical ideas of number decomposition, distributive rule, place value and regrouping are not salient.
- Treating the dividend as a sequence of digits discourages estimation.

— Answering "how many times does the divisor go into…" is difficult, since there is no margin for error – we must find the *greatest* multiple that "goes in".

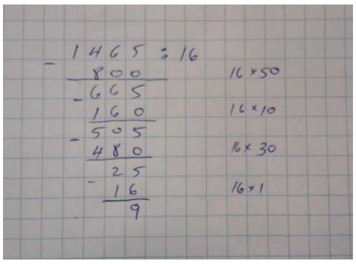
The lesson proceeded as follows in Table 1.

The Short Division Algorithm (SDA) discussed in lesson segment B is a variant of LDA for cases where the divisor has a single digit. Remainders are calculated mentally and written between the digits of the dividend (Figure 1). The sequence of LD problems (segment C) focused first on place-value decomposition: , then on decomposition induced by LDA: , connected with regrouping and the distributive property. Two alternative division algorithms were presented (segment D), neither of which requires the performance of division operations: Division by approximation: choose an *easy* multiple of the divisor, subtract it from the dividend, and repeat. This algorithm does not have a single correct implementation; you are free to choose any multiple of the divisor you are comfortable with (Figure 2). Division in parts: pre-calculate the divisor multiplied by 1, 2, 4 and 8 by repeatedly multiplying by 2, and use these multiples (possibly with added zeros) to *divide by approximation* as described above. This algorithm, unlike division by approximation, has a single correct implementation. You are not free to choose any multiple of the divisor, you should always subtract the largest one from the pre-calculated multiples (Figure 3).



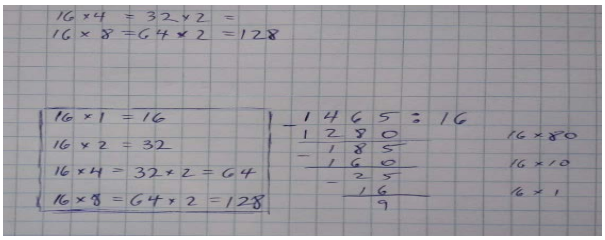
3 goes into 8 2 times. The remainder (2) is calculated mentally and written to the left of the 5, which is now read as 25. 3 goes into 25 8 times. The remainder is written to the left of the 2. Finally 3 goes into 12 4 times.

Figure 1: Short Division Algorithm (SDA)



"16×50=800 is easy for me. I subtract from 1465 to get 665. Subtracting 16×10 is convenient, leaving 505. I know 16×30=480. Subtracting leaves 25. Subtracting 16×1 leaves 9. 50+10+30+1=91. The solution is 91 (9)".

Figure 2: Division by approximation



Four multiples of 16 are pre-calculated. The algorithm proceeds as in Division by Approximation, except that each stage is determined – we must use the *greatest* multiple from the table, adding zeros where appropriate.

Figure 3: Division by parts

	Utterances	Duration	What was going on
A	88–195	11 min.	Introducing the lesson's topic and motivation
B	196–389	13 min.	LDA and SDA exemplified and compared
C	390–862	30 min.	Sequence of LD problems that focus on mathematical ideas one at a time.
D	863–1308	23 min.	Two alternate division algorithms

Table 1: Overview of transcript data

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

The framework of Mathematical Discourse for Teaching (MDT) (Cooper, 2014) takes inspiration from Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008) – in viewing mathematics *for teaching* as special and different from mathematics for other purposes. Epistemologically, MDT endorses tenets of the commognitive framework (Sfard, 2008), seeing fields of human knowledge, such as mathematical knowledge for teaching, as well defined forms of communication, and thus communication and cognition as aspects of a single entity termed discourse. Hence, MKT's distinction between Subject Matter Content Knowledge and Pedagogical Content Knowledge is mirrored as a distinction between Subject Matter Content *Discourse* (SMCD) and Pedagogical Content *Discourse* (PCD). Similarly, each of the six subcategories of MKT has its discursive counterpart. Discourses are associated with communities, thus university mathematicians' MDT and elementary school teachers' MDT are expected to be different. In the Commognitive framework, learning is conceived as changes in one's discourse, which typically start out as discourse-for-others – imitation, possibly thoughtful, of a leader's discourse – but may evolve into discourse-for-oneself – the discourse with which one communicates with oneself in problem solving (i.e. thinking).

Research questions

1) In what ways are the significance and the role of long division different for the instructor (in his university MDT) and for the teachers (in their elementary school MDT)?

2) What opportunities for learning emerged from these differences?

   a) How did the meeting of two MDTs create opportunities for learning?

   b) What learning actually took place in this lesson, on the part of the teachers and the instructor?

METHOD AND DATA

This paper draws on the instructor's lesson plan, a 77 minute audio recording of the lesson – fully transcribed and selectively translated, and an unstructured interview with the instructor five days after the lesson.

Commognitive methods of analysis focus on four interrelated characteristic features of discourse: *keywords*, *visual mediators*, distinctive *routines*, and generally *endorsed narratives*. Differences in interlocutors' discourse (e.g. differences in the ways they use keywords, in their attitudes to visual mediators, in the routines they typically engage in, or in the narratives they endorse), may present opportunities for learning. *Crisis points* in communication – "signs indicating that something has gone wrong in the interaction" (Jorgensen & Phillips, 2002, p. 125) – are a natural place to look for differences in interlocutors' discourse, and for learning taking place.

LONG DIVISION IN SUBJECT MATTER CONTENT DISCOURSE (SMCD)

Understanding LDA as process, concept and visual mediator

The teachers and the instructor agreed that understanding LDA is important, but the keyword *understanding* is used differently in their respective MDTs. For the teachers, understanding is strongly linked to the algorithm process, as is evident in a teacher's comment after a detailed review of a LD problem: *Here there's awareness of the process, it's not automatic. <u>This</u> is your understanding*. For the instructor, understanding LDA has to do with making connections between the related mathematical ideas – decomposition, the distributive rule and place value: *using the distributive rule and convenient decompositions (of the dividend) can be done without the algorithm... is very helpful for understanding the algorithm... sharpens the understanding of the distributive property in the context of division... understanding something we already know how to do*. Here are some additional examples of the procedural nature of the teachers' MDT versus the instructor's more conceptual nature: The keyword DMSB (acronym for divide, multiply, subtract, bring down) is central in teachers' discourse. Where the teachers speak of *remainder* in LDA ("in the LDA for 693÷3 there's no *remainder*"), the instructor replies "you mean there's no *regrouping*". Both keywords refer to the LDA procedure, but *regrouping* is the conceptual counterpart of *bring down*.

Another difference in the MDT of the two parties is in the significance attributed to the visual mediation of processes. Comparing SDA to LDA, the instructor notes: *we perform exactly the same operations in the same order... the difference is... [in LDA] we write a lot, and we've agreed that it's a big mess*. This does not do justice to the visual aspects of the algorithms. There are rules regarding how LDA and SDA are arranged on the page (see Figure 1), and these rules interact with the process and with the conceptual underpinnings, as is evident in some teachers' comments: *The bringing down is difficult... especially without grid paper* (a difficulty that is avoided in SDA where digits are not brought down); *[in SD the kids] immediately notice the remainder ... [because] they need to write it*; There are other examples of the teachers attending to visual mediation where the instructor seems to see it as secondary to the underlying mathematics: as the instructor walks through an alternate procedure, a teacher asks for a visual mediator: *don't you write down the [interim] result?* His response – *I'm trying to tell you how I'm thinking about it, not how I'd explain it to students* – implies that he views visual mediation as a teaching tool, having little to do with cognition. However, issues of visual mediation may be closely related to mathematical concepts. For example, working through using the approximation algorithm, the instructor mediates his actions by writing . A teacher says she would have written . Other teachers justify the division notation in many ways: *it's easier*; *it's confusing otherwise*; *in a classroom with students struggling with division we need to get them accustomed to thinking division*. For this last teacher, the visual mediation is entwined with a mathematical issue concerning the relationship between division and multiplication. In university mathematics there is no independent division operation, instead there is multiplication by inverse. This attitude to division is particularly evident in the alternate division algorithms, where division problems are solved without performing any division operations, only multiplications and subtractions! According to the instructor, the benefit is in avoiding the most difficult aspect of LDA – finding the *greatest* multiple of the divisor that goes into the dividend with no margin for error, but it is conceivable that he was influenced by his university conception of division as an unprivileged operation.

LONG DIVISION IN PEDAGOGICAL CONTENT DISCOURSE (PCD)

Teaching and learning long division

The instructor's Pedagogical Content Discourse pertains both to teacher education and to elementary school pedagogy. For him they are connected; in his words he aimed to *give the angle that will connect [the PD] to what goes on in the classroom*. One such connection would be the teachers using PD activities in their classrooms, possibly with minor modifications. This is something the teachers also hoped for, based on questionnaires that explored their expectations. However, the teachers deemed much of the LD lesson unteachable. Regarding the alternate division algorithms responses included: *[Division by parts is] not a method you can teach a class*; *[it's] explanations for good students*. The instructor appealed to other modes of relevance, suggesting the method as an aid for struggling students, but this too was rejected by a chorus of teachers: *it would confuse them so badly; it's difficult; no way*. Finally the instructor suggested yet another role for these algorithms, as a means for independent checking of standard LDA results. The teachers were not explicit about why they rejected methods that the instructor considered useful, but I offer some speculations, supported by what I have shown regarding the MDT of the two parties:

In the teachers' MDT, an alternate algorithm is yet another procedure that would need to be mastered, i.e. memorized. For the instructor these algorithms make so much sense that they should not need to be memorized.

The instructor considers the flexibility of the approximation algorithm a strength: *By using approximations and working with multiples that we're comfortable with, we're converting the problem to an easier problem*. However teachers may be wondering how to teach an idiosyncratic algorithm, which each student may solve differently. And what about students who are not comfortable with *any* multiples of the divisor?

Visual mediation may also be an issue. The instructor's focus was on the mathematics involved in each of the algorithms, but a teacher commented that *[we] need to remember [the multiplier] at each stage*, apparently attending to the lack of well-defined rules for organizing the solution visually. This is backed by teacher comments throughout the activities such as:

I'd have organized [the decomposition] in a [place value] chart; why don't you do [by parts algorithm] in the table?

WHAT ARE THE PARTIES LEARNING?

In commognitive terms, the differences in the MDTs of the teachers and the instructor present opportunities for mutual learning. Although the goal of PD was for *teachers* to learn – primarily mathematical content – learning on the part of instructor is crucial for the design and implementation of activities that will be relevant for teachers. In this section I show some examples of both kinds of learning taking place.

The teachers, in their active engagement in the various division algorithms – LDA, SDA, by approximation and by parts – were exploring connections between mathematical objects such as division, multiplication, decomposition, place value, estimation. They had accepted the goal of understanding LD and its entailments, as is evident in this teacher's comment, which followed the decomposition activity: *When we eventually reach long division, everything we did in this activity [decompositions of the dividend], the understandings, they disappear… The question is how to achieve understanding [for our students].* I now analyze some transcript excerpts to show the kind of learning that was taking place.

Excerpt 1: Developing Specialized Content Discourse, decomposing 852÷3

574 I: What I think can help prepare for LDA is doing the division first without mentioning the algorithm. Let's write it as a word problem. What do we get from dividing 8 hundreds, 5 tens and 2 units into 3 equal groups?
578 I: First we divide what we can…six-hundred… I'm left with 2 hundreds… not 2.
585 T1: Which are in fact 20 tens
596 I: [We now have 25 tens.] How many tens can be divided?
597 T1: 240
598 I: 24 tens…
599 T2: You can do 240. Why 24?
601 I: Oh, here I wrote six-hundred…
610 T3: Why convert to hundreds? … 6 instead of six-hundred… in Hebrew six…
613 I: Exactly. When I said six hundreds, I automatically thought six-hundred

Perhaps the most salient aspect of this excerpt is that teachers engaged in explorative discourse around a division procedure which is not part of the curriculum. I would like to draw attention to what may appear to be a rather trivial slip on the part of the instructor, saying and writing *six-hundred* (600) instead of *six hundreds*. This is not at all trivial. Three related but very different division algorithms are being considered, each with its own language usage. In LDA the symbol 6 in the hundreds place represents 600, in the precursor algorithm under discussion in the excerpt 6 hundreds need to be divided equally (explicitly compared by the instructor to the problem of equally dividing 6 melons), and in the alternate algorithms (by approximation and in parts) the number six-hundred needs to be divided as a number, not a quantity. These subtle differences are at the heart of the instructor's design. The alternate algorithms, in referring to the number, support estimation strategies. LDA makes sophisticated but opaque use of the principle of place value. The procedure in the excerpt subtly bridges the two; procedurally it follows LDA (hundreds, tens, units) while keeping track of the dividend as a quantity and not just a sequence of symbols. T1 and T2 are not yet fully aware of these subtleties, but T3, in catching the instructor's slip, appears to be on the way to making these distinctions part of her own discourse.

Excerpt 2: Specialized Content Discourse – endorsing an algorithm

Please refer to Figure 3 to make sense of this excerpt. T3 makes a "mistake", subtracting 128 from 185, the instructor goes along with it, T4 catches the mistake.

1125 T1: I'd start with 128
1127 T2: 1280
1128 I: Alright? 128, but times 10 is 1280. 1465 less 1280 is … 185
1175 T3: Less 128
1177 I: 128
1178 T4: But why did you do 128? You can do 160… It's much easier
1180 T3: Yes, 160 is preferable
1189 I: It's in the table, I forgot. I have 160 here, you're right.
1190 T5: Because in your table, instead of 8 you can do 10… 10 times 16.

Here again a number of teachers are actively exploring a division algorithm, and again are correcting the instructor's authentic error. An important aspect of this algorithm is that it has a unique "correct" move at each stage. By contrast, division by approximation is idiosyncratic – looking for "convenient" multiples of the divisor and subtracting them from the dividend. In division by parts we look for the unique *greatest* multiple from the table, where the instructor, using a procedural discourse, stated that *obviously we can add zeros*. T5 did not adopt this procedural language, preferring the more conceptual *10 times*. Furthermore, T3 and T4 seem to have appropriated something from the approximation algorithm, since they do not consider the instructor's slip an error, rather 160 would have been *easier* or *preferable*. This is not a minor point. LDA has a single correct flow; procedures that can correctly proceed in different paths are from quite a different discourse.

Excerpt 3: Specialized Content Discourse – mediating division and multiplication

The instructor commented that *how many times does 3 go into 8* is an instance of the measurement model of division. But *times* suggests multiplication at least as much as it suggests division. With this in mind, consider the following exchange:

1000 I: I think about it this way: I want to know how many times 16 goes into 1220.
1001 T1: So, times...
1002 T2: times...
1003 T1: 50 times
1004 I: Yes. Ok, times. So we found 50 times 16. It sounds a bit strange doesn't it?
1009 T3: Because you're asking how many times 16 there are in...
1011 T4: goes into 1220? It goes in 50 times.

The instructor invested some effort in mapping out connections between division and multiplication prior to the lesson. The discussion evolved in ways he could not have anticipated or prepared for. The teachers' testimonies regarding students' strategies for answering *how many times it goes in* (skip counting and repeated subtraction) revealed some such connections, but in this excerpt we see something different – the parties are listening to each other carefully and jointly exploring the role of the word *times* in mediating meanings of division and multiplication. We have "how many times 16 *goes in*" (division), "50 times 16" (multiplication), and "how many times 16 *there are* in 1220", which can be seen as bridging the two preceding meanings. This is a case of a joint object-level learning in the realm of Specialized Content Discourse, with obvious implication for Discourse of Content and Teaching.

Excerpt 4: Discourse of Content and Teaching – is the algorithm teachable?

From the teachers' participation throughout the activities it is clear that by and large they mastered the suggested algorithms, yet they were convinced that their students would not, whereas the instructor believed that they would. This discrepancy is due to the rules by which the parties endorse narratives about students and teaching – the teachers based on their experience and the instructor based on an analysis of the mathematics. The teachers' experience is a valuable resource and should not be taken lightly, and indeed I have shown why the alternate division algorithms might be difficult to teach, however, the teachers' don't have any direct experience regarding what they have *not* taught. Four teachers came to realize this towards the end of the lesson, where there were a total of 9 utterances to this effect, for example:

1237 T1: Could be, I haven't tried it. Could be that if you do one or two lessons this way... they'd understand the meaning of decomposition.
1240 T2: Exactly.
1244 T1: Not necessarily after LDA. I'm saying this cautiously since I haven't tried...
1248 T3: Not in order to know *how* to do it, rather to understand the meaning
1264 T4: Theoretically. We should try it some time.

Even as these teachers entertained the thought of teaching this algorithm, the principle by which it will (or not) be endorsed remains reliant on teaching experience.

Specialized Content Discourse – appreciating the role of visual mediation

I have shown that there is less attention to visual mediation in the instructor's discourse than in the teachers', however, the instructor was attentive to the teachers' comments and suggestions. He accepted two suggestions: mediating division by approximation in terms of division instead of multiplication; and, keeping track of the stages of division by parts in a table. Furthermore, in the discussion about SDA he realized that writing the remainder between the dividend's digits addresses the common error of skipping digits in the *bring down* stage. He also noted that the alternate algorithms generate solutions that *look completely different* than the LDA solution, even though they make use of similar mathematics.

SUMMARY

I have shown that the teachers tended to see LDA as a procedure that needs to be mastered and understood, whereas the mathematician saw it as an opportunity for deepening understanding by connecting a number of different topics. This is just one of the many ways in which their MDTs differed. In the face of such differences, the lesson could have followed various different paths: The instructor could have insisted on his agenda, alienating the teachers, or he could have adopted the teachers' point of view, setting aside his own agenda. Neither of these is what actually took place. The instructor was true to his mathematical agenda, but made a genuine attempt to appropriate the teachers' discourse. This can be seen both in his preparation of the lesson and in the way it played out. Why did he present two similar algorithms – approximation and in parts? The underlying mathematical principle is the same – incrementally decomposing the dividend into multiples of the divisor. I suggest that the instructor was developing sensitivity to the teachers' Discourse of Content and Teaching, and, realizing how difficult it would be to teach an idiosyncratic algorithm, suggested the deterministic version as an alternative. Furthermore, the motivation he gave for these algorithms was from the teachers' Discourse of Content and Students – they avoid the aspect of LDA that he considers most difficult for children (division).

The lesson on LDA may be considered productive in the sense that both the teachers and the instructor were enriching their MDT. It is interesting to note that this learning, when it involved changes in the rules of the discourse, did not follow the pattern described by Ben Zvi & Sfard (2007); there was no agreement on the leading discourse, the roles of the interlocutors, or the course of discursive change. Expertise was shared by teachers and mathematicians, who were all in the position of learners.

My aim in this paper was to point out *opportunities* for learning in the meeting of two MDTs. I have claimed that, in some cases, learning was in fact taking place, in the sense that the parties – teachers and instructor – were not superficially adopting aspects of an unfamiliar discourse. Rather they were in the process of transforming this discourse into *discourse-for-oneself, that is, into the type of communication in which the person is likely to engage of her own accord, while trying to solve her own problems* (Sfard, 2008, p. 285). This was evident on the part of the teachers; the mathematical activities stretched their Specialized Content Discourse, yet all the while they were considering implications for their teaching. In this sense the teachers were *constructing* new knowledge not through experience (teaching) but rather through discursive interactions, transforming mathematical ideas into ideas for teaching. How these discursive shifts subsequently influenced their teaching (if at all) is an important question that will be addressed in future research.

The instructor extended his own SCD while unpacking LDA in preparation for the lesson. The lesson itself presented opportunities for learning, but it is difficult to make claims regarding the nature of the instructor's learning based on the lesson transcript. For example, when some teachers suggested 30÷3=10 instead of 3×10=30, he responded: *Ok, but I'll tell you why I did the multiplication.* If he is thoughtfully considering the teachers' discourse and is on the way to transforming it to discourse-for-himself, there are no indications of it in the transcript. Nonetheless, such learning is crucial for mathematicians to be relevant for the education of teachers. Based on an interview following the lesson, the instructor was thoughtfully exploring ways in which his teaching might be *relevant* for the teachers. It is not clear if the instructor's learning would have been as productive in the absence of a researcher. However I believe that exposing mathematicians to these research findings is a crucial step for supporting their sensitivity towards teachers and their learning in similar situations.

ACKNOWLEDGEMENT

This paper was supported by the Weizmann Institute of Science and the Israel Science Foundation (grant No. 615/13). The author is grateful to Abraham Arcavi and Ronnie Karsenty for many helpful comments.

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Differential participation in formative assessment and achievement in introductory calculus

Rebecca-Anne Dibbs

Texas A&M University-Commerce, Commerce, USA

Prior formative assessment research has shown positive achievement gains when classes using formative assessment are compared to classes that do not. However, little is known about what, if any, benefits of formative assessment occur within a class. The purpose of this study was to investigate the achievement of the students in introductory calculus using formative assessment at the two different participation levels observed in class. Although there was no significant difference on any demographic variable other than gender and no significant difference in any achievement predictive variables between the groups of students at the different participation levels, regular participation in formative assessment was the most significant predictor of achievement in the hierarchical linear model.

Keywords: Approximation framework, calculus, formative assessment, hierarchical linear model, participation.

INTRODUCTION

Introductory calculus is one of the largest choke points for prospective undergraduates who wish to pursue STEM (Science, Technology, Engineering and Mathematics) careers. Students who leave STEM majors are very likely to do so during or immediately after the first semester of calculus (Ellis, Kelton, & Rasmussen, 2014; Rasmussen, Ellis, & Zazkis, 2014). There are several reasons why introductory calculus is a particularly difficult course. The majority of students enrolled in calculus are first-time freshmen, and mathematics and science classes are where students transitioning to higher education are most likely to struggle (Cabrera, Miner, & Milem, 2013; Waterson, Browne, & Carnegie, 2013). Furthermore, the students most likely to leave STEM majors after calculus are from groups that are underrepresented in STEM areas: women, first generation college students, English-language-learning students, and students from underfunded urban and rural high schools (Waterson, Browne, & Carnegie, 2013).

One study found that switchers were less likely to feel a sense of connection with their instructors (Ellis, Kelton, & Rasmussen, 2014), which suggests that the increased use of formative assessment, low stakes assignments for instructional planning purposes, may help to increase the number of prospective students in STEM majors past the first semester. The use of formative assessment with undergraduates appears to increase students' perception of a positive relationship with their instructor, make students more likely to seek help, and allows instructors to make data-based decisions on how much review instruction can/should be incorporated into a particular unit (Black & Wiliam, 2009; Dibbs, 2014). For the purposes of this study, formative assessments are defined to be written assignments graded on completion for the purposes of instructor planning.

Regardless of the content area or age of participants, the effect size on most quantitative formative assessment studies is around 0.5 (Karpinski & D'Agostino, 2013). These studies show that classes where formative assessment is used do better on average on common summative assessments than those classes where no formative assessment is used; however, even in classes where formative assessment is used, not all students will regularly complete the formative assignments. The purpose of this study was to investigate the influence of participation on students' growth trajectories on calculus labs designed to develop systematic understanding of limit concepts. Growth can be measured in either student achievement or increases in students' conceptual understanding. The analysis was delimited to the achievement definition of growth, though qualitative investigations of students' conceptual growth of the approximation framework also showed that regular participants appropriated nearly all limit concepts embedded in the approximation framework while the irregular participants showed little conceptual acquisition beyond procedural fluency (Dibbs, 2014). For this paper, I will distinguish between two different participation levels: regular and irregular. Students regularly participating in the formative assessments missed no more than two of 12 formative assessments during the semester. Although the regular participants earned significantly higher grades on the calculus limit labs, there were students that earned every possible final grade in each participation level.

METHODS

This study is part of a larger QUAL-quan mixed methods case study (Dibbs, 2014). Participants were recruited from two introductory calculus courses taught using the approximation framework at a midsized doctoral granting institution in the Rocky Mountain region. The students enrolled in introductory calculus are most commonly chemistry, science education, mathematics education, or mathematics major, and 35% of the students at the University are first generation college students. There were three sources of data collected: students' assignments, classroom observations, and interviews. The qualitative analysis consisted of daily classroom observations and student interviews. During classroom observations of labs, three of the eight groups in each class were closely observed for peer and instructor interaction: three groups of regular participants, two mixed groups, and one group of irregular participants. During non-lab class days, the instructor's interaction with the class and students' behavior was observed, with particular attention paid to the students observed during labs. After each lab, nine students (both of the mixed participation groups and one regular participation group) were interviewed about their lab write-ups using a cognitive think aloud technique. Students were given a clean copy of their lab and asked to explain each of their answers, who if anyone helped them figure out that particular portion of the lab, and if they would change their answer now; these qualitative data were analyzed to understand students' conceptual growth throughout the semester (Dibbs, 2014). The observations and interviews showed that there was some peer instruction during the labs, but most of the time, irregular participants only consistently understood the procedural computations portion of the lab. All assignments generated by each participant were collected for the quantitative analysis.

Introductory calculus is a four credit course that met Monday, Tuesday, Wednesday and Friday. The course begins with a brief pre-calculus review and ends with The Fundamental Theorem of Calculus and u-substitution. Students' final grade in the course was determined by online homework (10%), labs (20%), formative assessments (5%), three in-class exams (15% each), and a final exam (20%). Every week followed the same general schedule. On Monday, there was a new section of material introduced and students were given a prelab. Students were asked to complete the Unknown Value portion of the approximation framework (Figure 1) and identify a quantity with which to approximate the unknown value. Students were asked to complete the prelab before class on Tuesday; the prelab was graded on completion at the beginning of class. During class on Tuesday, students worked in groups of three or four on their assigned lab context. After class, students completed a postlab using the online course management software. Each postlab asked students to summarize what their groups did, evaluate how well they understood the material, perform a computation similar to the ones expected on the lab, and identify which portions of the lab they still needed help on. The post-lab was graded on completion, and instructors used students' answers to plan a 20 minute discussion about the lab to begin that class Wednesday.

Although the postlab completion grade was 5% of the students' final grades, the primary purpose of the assignment was to evaluate students' current understanding and plan the next class effectively; in that sense the postlab was primarily a formative assessment. Students were provided automated feedback through the CMS, and it took an average of 15 minutes/week to evaluate a classroom set of postlabs and plan the next class. The remainder of the week was spent on concepts from the textbook. For the derivatives and definite integral labs, the next week would be a repeat of the first; all of the other labs proceeded directly to the regrouping described next. On the third week, students would be placed in new groups, where they were responsible for teaching their context to their new group members; this type of presentation

Fluid traveling at a velocity v across a surface area A produces a flow rate of $F = vA$. Poiseuille's law says that in a pipe of radius R, the viscosity of a fluid causes the velocity to decrease from a maximum at the center ($r = 0$) to zero at the sides ($r = R$) according to the function $v = v_{\max}(1 - \frac{r^2}{R^2})$. In this activity you will approximate the rate that water flows in a 4-inch diameter pipe if $v_{\max} = 4.44$				
	Contextual	Graphical	Algebraic	Numerical
Unknown Value				
Approximation				
Error				
Error Bound				
Desired Accuracy				

Figure 1: Definite Integral Lab task and approximation framework

is called a Jigsaw presentation because each student is responsible for one piece of a larger idea. After this Jigsaw presentation, students were expected to write up their individual answers to the 20 parts of the approximation framework; this assignment was the summative assessment of each lab. Each lab had one formative prelab and two or three postlabs associate with the summative lab writeup.

The approximation framework is built upon developing systematic reasoning about conceptually accessible approximations and error analyses but mirroring the rigorous structure of formal limit definitions and arguments (Oehrtman, 2008). Approximation is the most common of the seven informal understandings of the definition of a limit; by incorporating the approximation labs into the curriculum, all students are trained to conceptualize limits in the same manner; this makes the transition for formal calculus topics easier for both students and instructors (Oehrtman, 2008).

For each calculus concept, students are asked to identify the unknown value that cannot be solved with algebra, an algebraic technique for approximating the unknown, quantify the error, bound the error, and describe how an approximation can be computed to any desired accuracy. Although the context (and calculus context) changed with each lab, students were asked the same questions on each lab; hence, the labs may be considered repeated assessments on the approximation framework concepts since the process was identical on each lab once the appropriate approximation had been determined. Students were asked to represent these five components of the approximation framework contextually (words and pictures), graphically, algebraically, and numerically. Each lab had three or four different contexts; one of which was more challenging and intended for students that had seen calculus before. While there is a lab every week, this research was delimited to the approximation framework labs dealing with limits, derivatives, and definite integrals. These labs accounted for 10/14 lab sessions in the semester and contained topics common to all introductory calculus courses. The remaining labs dealt with practice on applications of derivatives: Taylor polynomials, optimization, and related rates. There was also a lab before limits on quantitative reasoning that was not considered due to its low reliability.

The courses were taught at the same time and on the same schedule by two equally experienced instructors. All of the lab questions were scored dichotomously so the inter-rater reliability of the lab write-ups was not a concern. The content validity of the assessments was checked by the course coordinator and an additional expert on the approximation framework. Since the labs were scored dichotomously, KR-20 was used to calculate reliability, and all assessments had reliabilities within acceptable levels (Gall, Gall, & Borg, 2007): the limit, derivative, and definite integral labs had KR-20 values of 0.83, 0.72, and 0.78 respectively.

In addition to participants' lab write-ups, grade predicative variables and demographic information were collected from each participant. There were no significant differences between the sporadic and non-participation groups on all but one of the demographic or grade predictive variables tested ($p > 0.25$) [1]. Female students were significantly more likely to be regular participants in formative assessment ($p = .03$). Since asynchronous formative assessment, like the ones used in this study, require a greater level of organization and engagement, these assignments tend to slightly favor female students (DiPrete, 2013). Despite the selection bias inherent in the participation levels, there was no significant difference on any measurement of prior knowledge taken at the beginning of the study. These measures of students' prior knowledge, which all indicated students that chose to participate or not participate in the formative assessments did not have significantly different levels of prior knowledge were not included in the model.

There were 66 students that consented to participate in the study; 13 of the students were removed from the sample because they had prior exposure to the labs that could confound the results. Of the 53 students that were new to the approximation framework labs, only seven had no prior exposure to limit concepts in a prior course, and 27 of the students had AP Calculus [2] in high school. Students needed to have completed at least 10/12 formative assessments to be classified as a regular participant. There were 30 students classified as irregular participants; the remaining 23 students participated regularly in the formative assessments. Only students that completed at least two of the four approximation labs were included in the analysis and are included in the table.

The initial analysis used Bonferroni corrected t-tests found that there were significant differences in the mean number of questions answered correctly by students in each participation level on the three labs, which suggested a hierarchical analysis was most appropriate for the data (Raudenbush, Bryk, Cheong, Congdon, & Toit, 2004). After the null model showed significant differences in the intercept and slope between the two participation classifications, the final model was:

$$SCORE_{ij} = \gamma_{00} + \gamma_{01}{*}REGULAR_j + \gamma_{10}{*}LAB_{ij} + \gamma_{11}{*}REGULAR_j{*}LAB_{ij} + u_{0j} + r_{ij}$$

The dependant variable is the number of questions a student answered correctly; regular and is a dummy coded variables at the student and time level. Gender, ACT Math score (a standardized exam students take their third year of high school in preparation for applying to college), ethnicity, year in school, native language, major, and the Calculus Readiness Test (a standardized exam administered to students on the second day of classes), did not explain significantly more variance when included, and were all discarded in order to retain the most parsimoniousness model.

RESULTS

There were students that earned every possible final grade in both participation levels, but on the labs, the students that regularly participated in the formative assessments that followed answered more items correctly on the lab write ups than the irregular participants. Table 1 summarizes the results of the t test assuming unequal variances. The *p*-values have all been multiplied by a factor of three to account for multiple hypothesis tests. Despite the correction, the regular participants in formative assessment have a significantly higher mean score than those not participating regularly. While these two groups were not significantly different on any of the grade-predictive measures available at the beginning of the semester, the students that were irregular participants in the formative postlabs actually had slightly higher average scores than the regular participants.

Given that there appeared to be differences in both the initial level of performance and the rate of change in

	Irregular Mean	Regular Mean	*p*-value
Limits	8.48	13	0.006
Derivatives (Final)	5.29	16.52	<0.001
Definite Integrals	5.51	17.34	<0.001

Table 1: *t*-test results for mean number of items answered correctly on each lab by participation level

Random Effect	Standard Deviation	Variance Component	d.f	X^2	*p*-value
INTRCPT1, u_0	4.005	16.040	42	189.1522	<0.001
Level-1, r	3.97055	15.765			

Table 2: Null growth model results

score from lab to lab, an analysis that accounted for time was more appropriate to explore this phenomenon further. To confirm growth modelling was the appropriate choice of statistic, a null model [3] was run; this model was significant (Table 2), confirming that multilevel modelling was required; the unexplained variance was 0.504 [4].

The only measurement that resulted in a significant reduction of ICC was participation. When included as a Level-2 variable, the dummy code for regular participation reduced the unexplained variance to 0.363, a reduction of 0.141. The final variance component summary is given in Table 3.

The final estimation of the growth model fixed effects showed that students who regularly participated in the formative prelabs and postlabs were able to answer an average of 5.44 more questions correctly when compared to a student of similar ability who did not regularly participate in the formative assignments (Table 4). The maximum likelihood estimation of the number of significant parameters on the intercept was two, but it was not any of the measures collected as part of the set.

Regular participation in formative assessments also had a significant influence on the rate at which students improved on their lab writeups (Table 5). Given the pronounced ceiling effect of the regular participants' scores, it is not surprising that the slope is relatively small; the regular participants started with relatively high scores and had little room for improvement. However, the near-zero slope value for the non-regular participants does not imply that the irregular participants made no learning gains throughout the semester; rather by the end of the semester these students were able to answer the same questions correctly at the end of the semester on integration that they were able to answer correctly about removable discontinuities in the limits lab.

Although the score on the rewrite after accounting for initial score between the two participation groups was not significantly different (p=0.0501), the irregular participants' regression in the subsequent lab indicated that they were not able to apply all of the instructor's feedback in a new context. There is a marked ceiling effect on the final derivative lab and the integration lab for regular participants (Figure 2), which interview data (Dibbs, 2014) indicated was due to nearly complete conceptual acquisition of the limit concepts embedded in the labs.

Random Effect	Standard Deviation	Variance Component	*d.f.*	χ^2	p-value
INTRCPT1, u_0	2.99581	8.97489	41	124.9583	<0.001
level-1, r	3.96119	15.69102			

Table 3: Growth Model (participation) results

Fixed Effect	Coefficient	Standard error	t-ratio	Approx. *d.f.*	p-value
For INTRCPT1, β_0					
INTRCPT2, γ_{00}	9.504938	0.963904	9.861	41	<0.001
REGULAR, γ_{01}	5.447198	1.139113	4.782	41	<0.001

Table 4: Final estimation of growth model fixed effects

Fixed Effect	Coefficient	Standard error	t-ratio	Approx. *d.f.*	p-value
For SLOPE, β_1					
INTRCPT2, γ_{10}	0.064472	0.446211	0.144	110	0.885
REGULAR, γ_{11}	1.586417	.521549	3.02	110	0.003

Table 5: Final estimation of growth model slope effects

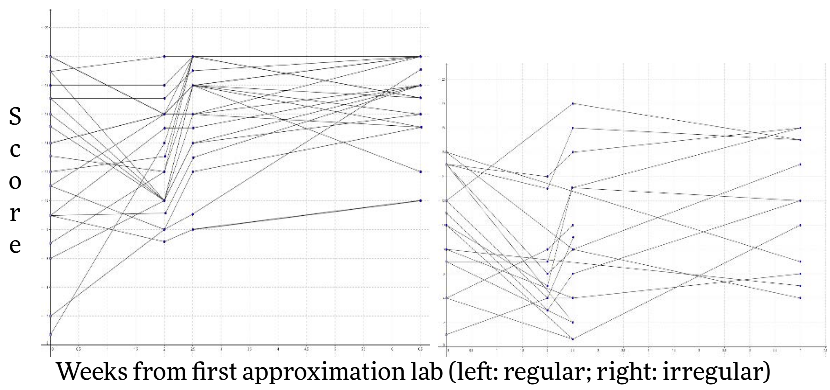

Figure 2: Growth trajectories by participation

DISCUSSION

The results of the study indicate that the students not participating in formative assessments were able to answer fewer questions on average than those students that did participate in the formative assessments, and improved at a slower rate throughout the semester. This is surprising because the students that did not complete any formative assessments attended class for the post-lab based instruction on the day after lab, and the students that did no formative assessments did not have significantly lower levels of prerequisites knowledge than those students participating in the formative prelabs and postlabs. Although this study has a relatively small sample size, this analysis showed that regular participation accounts for 28% of the interclass correlation.

While these results indicated that there were measurable achievement differences between the growth trajectory of those students who participated regularly in formative assessments and those that did not, the analysis also suggested that there were two significant student level factors in the data. Since there were no significant differences on any academic preparation measures for the participants in this study, this suggests that the missing factor in this model is not prerequisite mathematical knowledge. However, given that prerequisite mathematical knowledge is almost always an important factor, further research on the inclusion of this variable or the use of propensity scores in the model is warranted.

One possible explanation is calibration differences between regular and irregular participants in formative assessment. Calibration is considered to be a general metacognitive skill; it is the ability of a learner to accurately assess what they do and do not know (Hacker, Dunlosky, & Graesser, 1998). In this study, the opportunity for calibration occurred on the limits, first derivative lab, and the definite lab, and it is plausible that the regular participants in the formative assessments are better at identifying the areas in which they need additional help.

There is some support for this supposition in the data. In every lab there was a set of questions that no student asked about on their postlabs. Since none of the students asked for help on the post-lab for these items, an item was considered to be well-calibrated if the student produced the correct solution. In the labs, the statistical evidence for differences in calibration is not clear. The regular participants did answer significantly more of these items correctly, but there were no formal investigations of calibration during this study. Whether pattern of responses is because completing formative assessments on a regular basis helped students maintain a high calibration level or if the formative assessments helped students improve their calibration throughout the semester is an area for future research.

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ENDNOTES

1. Age, race, native language, and year in school showed no significant difference on a Chi-Square test. GPA, Math GPA, ACT Math Score, Math GPA, the pre-calculus skills test administered the second day of class, and time elapsed since the previous mathematics course showed no significant differences using Mood's median test.

2. AP Calculus is a one year introductory calculus course generally taught in the last year of high school. The content of this course is the same as the content in the one semester introductory calculus course. Students may elect to take the AP exam at the end of the year. Passing the AP exam (a nationally administered standardized test) would allow students to earn college credit for introductory calculus. Students in introductory calculus who took AP Calculus in high school did not take/ did not pass this exam.

3. A Null Model assumes that the intercept and slope are both constant for all participants. Failure to reject this test indicates repeated measures ANOVA to be the appropriate test.

4. HLM reports unexplained variance rather than R^2. The unexplained variance is $1\text{-}R^2$.

Students' personal work in mathematics in French business school preparatory classes

Lynn Farah

Université Paris Diderot, Paris, France, lgf00@mail.aub.edu

This paper presents parts of a research study pertaining to students' personal work in the learning of mathematics at the undergraduate level. It focuses on the main results conveyed by data collected through a questionnaire completed by students enrolled in two different tracks of French business school preparatory classes, at the beginning and end of their first year of study. The approach adopted in the description and analysis of results in this paper focuses on the role of the institutional context and its impact on student work.

Keywords: Mathematics learning, student personal work, institution, CPGE.

CONTEXT

Students enrolled in preparatory classes in France (Classes Préparatoires aux Grandes Écoles referred to as CPGE) seem to achieve much better results in mathematics than those enrolled in regular universities (Castela, 2011), where student failure during the first years has long been a serious widespread problem. These specific French higher-education institutions prepare students over two academic years to enter the "Grandes Écoles", which are mainly business schools or engineering schools, by passing the "concours"[1]. In the French educational systems, the two preparatory years are equivalent to the first two years of undergraduate study at university. However, the CPGE differ from universities in many ways. They are known for their selectivity in recruiting good – if not the best – high school students who hold the French baccalaureate, as well as their supportive culture, which favours student collaboration and provides them with close follow-up by teachers, in a relatively rigid high-school-like system within stable moderate-sized classrooms. These institutions constitute a rich and interesting field of observation and study given both the resources they offer to students and the constraints they weigh on them.

Furthermore, research about student personal work is not very common in mathematics education, in France and elsewhere. There have been few studies in mathematics didactics that tackled this topic however mostly in a marginal manner. In addition, given the diversity of the situations in higher education in France, the number of studies relevant to each situation is very limited. Some studies in sociology and education have closely explored student personal work but without taking into account disciplinary specificities. Our work comes as a continuation of the research conducted by Castela (2004, 2009, 2011) who studied students' personal work in mathematics in high school (grade 11) and in higher education (comparing university and CPGE). Our study focuses on the personal work of students enrolled in two different tracks of business school preparatory classes, Scientific and Technological[2], during their first year of study. S track students hold a scientific baccalaureate and have a strong background in mathematics and sciences; whereas T students have had a teaching specialized in human resources, marketing, business and finance or information systems with little focus on mathematics[3]. Hence, our study adds to the existing research about CPGE by targeting an unprecedented population while emphasizing the diversity brought by the two tracks. Furthermore, our study explores specific aspects of the "enveloping" institutional func-

1 The "concours" are national competitive exams which students take by the end of the second preparatory year in order to enrol in the "Grandes Écoles". There are specific required written and oral exams for each type of school.

2 In what follows, we will designate the tracks by the letters S and T respectively.

3 The background difference is clearly reflected through the mathematics level of students in each track.

tioning of the CPGE analysed and evidenced through a sociological study (Darmon, 2013) from a transversal point of view, adding a disciplinary focus on mathematics, while examining the link between the personal organization and the institutional organization of study in CPGE. It also introduces methodological novelties to the existing studies about CPGE in terms of data collection, by combining quantitative methods with methods that give a closer access to students and teachers.

CONCEPTUAL FRAMEWORK

We borrow several constructs from French didactics of mathematics and sociology in order to build our conceptual framework. In what follows, we define the two main components of this framework which are addressed in this paper.

Firstly, the notion of institution is at the heart of this research. We use the definition of an institution given by Chevallard (2003) as our starting point. He describes an institution as a social system which allows and imposes on its subjects – that is people who occupy different positions within the institution – ways of doing and of thinking. In the broader sense of the term, we consider that the institution designates the CPGE. We adopt Darmon's perspective who describes the preparatory institution as an "enveloping" institution,

> ...powerful but not totalitarian, violent but concerned about the well-being of its members, it operates by individualizing to the extreme rather than homogenising, thus reinforcing its take over the individuals which are its members (2013, p. 28).

The other essential notion we consider is student personal work. We conducted an extensive review of literature about student personal work, in order to define the aspects we are concerned with in our study, and situate them with respect to other research tackling the same topic. We advocate that, in mathematics, students need to construct practical know-how in addition to the theoretical knowledge they acquire in order to solve mathematical problems. We refer to Castela's work (2011) on mathematical functioning and her contribution to the praxeological model. Our underlying hypothesis is that such practical knowledge is not explicitly taught nor institutionalized. Thus, students are required to engage in "autodidactical" personal work in order to extend and complete what has been initiated in the classroom. In our study, we are looking to define the nature of this autonomous study and the gestures involved in it, that is what students do in addition to solving problems in order to learn something in mathematics. On one hand, we focus on the disciplinary specificities of mathematics in student personal work. On the other hand, we perceive personal work as defined and influence by the institution and not as an isolated individual endeavour.

RESEARCH DESIGN

Our target group includes first year students who come from preparatory classes in three different Parisian schools[4] of both tracks, where three volunteer mathematics teachers have accepted to cooperate for the research and allowed us into their classrooms over two consecutive academic years (2011–2012 and 2012–12013).

The study uses a combination of qualitative and quantitative methods in order to answer the research questions; the following questions are at least partially addressed within the scope of this paper:

— How does students' personal work evolve throughout a preparatory year, in terms of quantity and forms of study?

— What are the forms of study that students exhibit on their own initiative, in addition to those prescribed and supervised by teachers?

— What forms of study are exhibited by "good" students as opposed to those who are "weak"?

— How do social relationships promote student work, in particular the relationships that are established between the students and those built with the teachers?

Several instruments have been used to collect data: informal discussions with students and teachers, email exchanges with few volunteer students, samples of student notes and documents, student questionnaires,

4 Given that it is not possible to disclose the school names, we refer to them by their initials: D and K from the S track, B from the T track.

teacher questionnaires, interviews with few students, and interviews with the three teachers. All data collected through qualitative methods is intended to provide explanatory elements and validation for the hypotheses and conclusions brought to light through the student questionnaire, which is the main tool of our study and the focus of this paper.

Through this questionnaire, we seek to identify ways of working which are common to students in general, as well as those that differentiate the "good" students from the "weak" ones. We determine a student's achievement level (good, average or weak) solely according to his/her end of year mathematics grade. In fact, the grade is the only criterion used in the "concours" to evaluate students' success and rank them for admission to the "Grandes Écoles". We also look to establish comparisons between the schools and/or the tracks, and examine how the ways of studying evolve throughout the first preparatory year, while considering the influence of the particular institutional context of the CPGE and the social relationships on students' work.

This pre/post questionnaire explores the ways students work for the mathematics course at two moments of their educational path, at the end of grade 12 and at the end of the first preparatory year. It was inspired from several previous similar questionnaires used in studies related to our topic (Adangnikou, 2007; Castela, 2004; Najar, 2010) and was designed to match our research goals and conceptual framework. It includes 55 items from five categories: general work habits (including problems encountered by the students), in class (following, taking notes), between two sessions (studying the lesson, solving exercises, making summary sheets, preparing the "colles"[5]), when reviewing before an exam (the resources, the way of working, the exercises), and self-evaluation of performance and results. The two versions of the questionnaire (pre/post) are almost identical in terms of the questions asked, but each focuses on a different moment of the student path. Most of the items are four-level Likert items (the ordered responses are "never", "sometimes", "often" , "always", or equivalent statements for few specific items), in addition to two multiple choice items, five yes/no answer items, and two open-ended questions (the latter are only found in the post version of the questionnaire since they pertain to the "colles"). The questionnaire was filled out by students of the three schools involved in our study, respectively at the beginning and at the end of the first preparatory year, over two consecutive academic years (82 students, then 97 students).

We used SPSS in order to conduct descriptive statistics analysis and hypothesis testing for the data gathered through student responses. For each item and for both moments of the study, the frequencies of responses were first calculated for the whole sample, then for subgroups of students created according to school (B, D or K), track (S or T) and level of students (good, average, weak) respectively. McNemar tests were used to verify the significance of the evolution of frequencies between the beginning and end of year for both academic years. Chi-square tests were used to check dependence relations between each item responses and the different subgroup modalities. Next, the items were crossed two-by-two in order to look for significant dependence relations using Chi-square tests.

FINDINGS

In this section, we present some of the main findings of the questionnaire data analysis which was structured around eight themes: 1. collaboration between the students, 2. student difficulties, 3. taking notes, 4. managing work and revisions, 5.between two sessions, 6. resources, 7. before an exam, 8. colles[6]. We give few examples of items that differentiate between the two tracks on one hand and between the good and weak students on the other hand.

5 A "colle" is an evaluation tool specific to preparatory classes. It classically takes the form of a one-hour oral examination by groups of three students working individually but simultaneously on the classroom board, answering lesson questions and/or solving problems given by the teacher who is present to supervise and grade the work. In mathematics, a student is subjected to a "colle" every two to three weeks. The conditions and functioning of a "colle" may vary from one school to another. The questions related to the "colles" (4 Likert items and 2 open-ended questions) are only found in the end of preparatory year questionnaire given that this tool is not used in grade 12.

6 Given the specificity of the "colles" to the CPGE context, it is difficult to fully understand their characteristics without a detailed description. Hence, we omit the results pertaining to this theme given the space limitations of this paper.

Collaboration between the students

The data analysis shows that collaboration with classmates is highly valued by students of preparatory classes, even though it doesn't always take the form of group-work. Group-work seems to be a relatively common practice, with an average proportion of students who report working in groups often or always around 50%. The other half – students who never or only sometimes work in groups – can be at least partly accounted for by the fact that students in preparatory schools come from different areas outside Paris, hence they do not live close to school and choose to work most often at home rather than in school or at a classmate's. In addition, group-work is more widespread among students of the S track on one hand, and among the good and average rather than the weak students of both tracks on the other hand. Collaboration also takes the form of a solidarity bond between students who seem to rely on each other for moral support and encouragement for non-academic purposes. In fact, more than 80% of the students totally agree with the fact that mutual support between classmates is as determining as one's personal work for success. This disproves common stereotypes which advertise harsh competition in preparatory classes. In fact, teachers encourage student collaboration by allowing it in their classrooms for specific purposes. Irrespective of the form it takes, we believe that collaboration between classmates has a positive impact on students' personal work.

Student difficulties

Many students in preparatory classes have time management and concentration difficulties when studying at home. The proportions of students who say they often or always have difficulties by the end of the preparatory year are higher than those of grade 12, exceeding 60% in several cases. This is not surprising given the demanding requirements of these classes and the long intense school days. Many students also find it hard to stay focused and follow in class, the lesson rhythm being too fast for them. This problem is more frequent among students of the T track, particularly the weak ones, for whom the situation is aggravated compared to grade 12, whereas students of the S track seem to become less distracted by the end of the first preparatory year. One hypothesis which can partly explain these different behaviours is related to how students of each track perceive the importance of mathematics. This derives from the fact that mathematics is the main subject in the S track and plays a crucial role in the "concours" and recruitment process, while it can be counterbalanced by other subjects in the T track. Despite the many problems they face, students seek less help from others (teacher, parents, friends…) than they used to in grade 12. In fact, the number of students who solicit the teacher's help when they don't understand something in class radically drops by the end of the first preparatory year. This practice remains more common among the good students. Likewise, although the number of those who get help from others such as parents or friends varies from one class to another, the average proportion does not exceed 50%, which shows that students tend to handle their difficulties on their own. This can be interpreted as a statement of independence or a lack of confidence in others. Some students, mostly weak ones, seem to completely shut themselves off from any external assistance, including discussions with classmates, which could suggest they have given up on mathematics.

Taking notes

We explored the way students take notes during the lesson and what they add to those notes. More than 90% of the S track students copy everything the teacher writes on the board, while the proportion drops to around 73% for those of the T track. On the other hand, fewer students take notes based on the teacher's oral comments. The proportion of those who do differs from one class to another, but it doesn't exceed 60%. This seems to be related to the students' level in mathematics and their difficulties while following the course pace. As for student contributions to the notes, many claim that they add personal comments and signs, especially good students, but very few indicate the things they did not understand while taking notes. These behaviours are analyzed and interpreted in light of the differences between the teachers' lessons, the content of the sheets they distribute to the students, the ratio of written and oral comments they add, as well as the students' level and difficulties.

Managing work and revisions

We tried to establish patterns in the way students organize and schedule their regular work and exam[7]

7 In CPGE, mathematics exams are not typical university exams, but are rather similar to high school exams in format and content. They take place regularly every four to five weeks on a Saturday, are usually four hours long, and consist of exercises related to the last chapter or two chapters covered in class.

revisions for the mathematics course. Despite the differences between classes, it is possible to observe that, compared to grade 12, more students use a tentative work plan to organize their work. It appears that this practice is more common among good students of the T track. Additionally, the number of students who wait for exam periods to review their lessons and work decreases by the end of the first preparatory year, particularly in the S track, where many students become more systematic in their work. This can be partly attributed to the "colles" which require students to learn their lesson fortnightly. Similarly, very few students start exam revisions at the last minute or the day before the exam, while more students begin their revisions two days prior to or at least one week before the exam. This shows that students in general become more organized, regular and anticipatory than in grade 12, although some differences and exceptions can be noted. We can also find extreme cases of both very studious students and very careless students. No statistically significant conclusion can be drawn about the differences between good and weak students with respect to ordinary study habits or exam revisions.

Between two sessions

We first consider what students do with the lessons covered in class by focusing on the main three actions of mathematics learning: read, understand, and learn. Between two mathematics sessions, around 35% of the students on average read everything that has been done in class, while almost 45% of the students go over the things they didn't understand in class, a practice mostly common among good students. In addition, less than half the students learn the lesson (theorems, definition, formulas, proofs). These numbers lead us to believe that only some students work on a regular basis between sessions, probably in order to be prepared for the "colles", while others tend to keep most of the work for exam revisions. As for the exercises, few students in schools B and D (around 28% on average) solve the exercises assigned by the teacher for the next session, as opposed to more than 60% of the students of school K. Likewise, very few students in schools B and D complete the exercises which the teacher didn't finish in class, while this seems to be more common in school K. These results begin to reveal a particular attitude among the students of school K with respect to exercises, which is confirmed through the analysis of further items.

Resources

Next we consider the resources which the students have at their disposal in order to prepare for exams. First, we examine summary sheets, self-produced resources that students create using lesson notes and/or exercises. Around 40% of students create summary sheets, mostly by selecting and copying important elements from the lesson and to a lesser extent from exercises. As for the resources provided by the teacher, most students (more than 80%) say they are satisfied with the lesson which they find complete and sufficient for them to succeed, except for those of school D in 2011–2012 where the rate is 40%. Furthermore, around 60% of the students of schools B and K study the comments written by their teacher on their previous exams or graded homework, as opposed to only 45% of those of school D. Finally, less than half the students of schools B and K use resources other than their teacher's lesson, such as books or online references, as opposed to more than 70% at school D. In fact teachers discourage such practice. These numbers, in particular those of the class of 2011–2012 at school D[8], strongly suggest that special consideration should be given to the teacher role while interpreting the data.

Before an exam

In this section, we explore the way students review for an exam. To start with, we analyze the way students study the lesson. It comes as no surprise that students give high importance to learning formulas and their application conditions by heart, and to a lesser extent to learning definitions and theorems. In fact, between 65% and 80% of students verify that they know by heart the different lesson components when preparing for an exam. This could compensate for the fact that more than half the students do not learn their lesson between two mathematics sessions as said above. These practices are slightly more common among students of the S track on one hand and among good students on the other hand. Moreover, 60% of students of the T track on average read and try to un-

8 In this school, two different teachers taught mathematics over the two years of our study. A detailed analysis of several items pertaining to the teacher role as well as information gathered from the interviews and discussions with students and teachers indicate that the relationship between the first teacher (class of 2011–2012) and the students was problematic, while things were smoother for the new teacher despite some minor issues. Being aware of this delicate situation allows us to explain some of the numbers we see, such as the lack of appreciation of the first teacher's lesson.

derstand proofs, and 40% try to re-do proofs as part of exam preparations, while the respective average proportions for students of the S track are 50% and 15%. Hence, it seems that proofs play an important role in the T track exams, whereas the S students encounter them mostly in the "colles". Despite these differences, good students in general and in the T track in particular seem to pay more attention than others to studying proofs before exams. Lastly, 75% of students of the S track declare that they try to extract ideas (examples, methods, tricks) to remember when studying before and exam, while only 45% of students of the T track do so. These numbers underline differences between the two tracks which can be partly attributed to the nature and content of exams for each track. In fact, the latter are aligned with the "concours" objectives and requirements which are not the same for both tracks.

Lastly, we consider the way students handle solving exercises before an exam. To the multiple-choice question "the most important thing to do in order to succeed in mathematics when solving exercises", we provided four suggestions: 1. "being able to solve the exercises assigned by the teacher", 2. "practicing by solving other problems than the ones assigned by the teacher", 3. "identifying standard problems and knowing the methods and tricks to solve them", 4. "other (to be specified)". The majority of S students chose the third option, while the most common choice for T students was the first option. This difference between students of the two tracks can be interpreted using Castela's (2004) work styles conceptions: the choice of T students matches the "reproduction conception", which is also that of successful university students, while the style of S students tallies more with the "transfer conception", given that they tend to look for tricks and methods which can be applied to other problems. Another main difference between students of both tracks is relevant to the way they handle exercises that had been previously solved in class. It appears that, on average, twice as many T students as S students say they only read the correction of an exercise and try to understand it, instead of actually trying to redo the exercise either mentally or by writing. Furthermore, weak students of both tracks tend to avoid redoing the exercises by writing, and settle for either solving mentally or reading the correction, but no such general observation can be made regarding good students. We can conjecture that while redoing exercises by writing is necessary in some cases, it is not always obligatory. However, in order to get something out of these previously solved exercises, it is inevitable to redo them at least mentally, since mere reading does not seem to contribute towards success in mathematics. The difference between the two tracks is also underlined by the fact that more S students solve exercises of different types than T students. For example, 80% of S students on average solve exercises similar to the ones that are most likely to be given in exams, while less than 65% of T students report doing so. Similarly, solving exercises which have not been prepared prior to exam revisions is a common practice for 65% of the S students as opposed to less than 40% of the T students. It is important to note that we also find differences within the S track between the students of schools D and K, and between the two classes of each school. Hence, it is hard to formulate comparisons between students of different levels. Yet we can summarize the main commonalities as follows: it seems that good students are more selective with respect to the type of exercises they choose to work on before an exam; they tend to tackle the long and difficult ones rather than the simple ones or those they previously managed to solve.

DISCUSSION AND CONCLUDING REMARKS

Through our approach, we first consider the CPGE institution as a whole, and then we focus on the functioning of each classroom considered as an institution on its own in order to interpret the results. We examine each classroom as an institution whose stability allows the transmission of norms (Monfort, 2000), despite the different students and in some cases different teachers over the years. Moreover, through its dual role of subjugating its members while providing them with the necessary resources, the institution is responsible for transforming and producing particular student aptitudes. Thus, we investigate the causes behind the observed phenomena at the level of the institution rather than the individual.

It is very difficult to sum up our results given the extensive data and the different levels of analysis involved in the study. In fact, what we present in this paper are only some of the main findings which are accessible to the reader who is not exposed to all the details of the work. For example, we have omitted most of the results of items that distinguish between the three different schools and/or the two years of study. In addition, some items do not suggest any consistent behaviour across schools, classes or years, but instead either produce non-results or reflect special cases.

Nevertheless, we can formulate three main conclusions which are repeatedly conveyed by the results of several items. The first one pertains to the class of school B in 2012–2013. Students of this class appear to be much less studious and less diligent than those of the other classes, in particular when compared to their predecessors of the class of school B in 2011–2012. In fact, their responses suggest that they work significantly less than the others in general as well as before exams. The second conclusion pertains to the class of school D in 2011–2012 which seems to be facing a problematic situation with the teacher. Some repercussions of this situation are still visible in 2012–2013 despite the change of teacher, but to a lesser extent. The third conclusion concerns the class of school K in 2012–2013. Students of this class seem to share some common study gestures which differ from those of other classes, in particular those of the S track, especially with regard to exercises. This is partially ascribed to the fact that two-thirds of its students have an average level in mathematics by the end of the year, as opposed to other classes where more than half the students are weak.

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Teacher management of learning calculus: The case of sequences in the first year of university mathematics studies

Imène Ghedamsi

Tunis University, IPEIT, Tunis, Tunisia, ighedamsi@yahoo.fr

In this paper, I present my methodological tool for analyzing regular mathematics courses on calculus *and an application of this tool in the transition from secondary school to university. The tool is based upon the Theory of Didactic Situations (TDS), and especially on the constructs of the "didactic contract" and the "milieu". The data are taken from the transcription of a regular lesson on sequence convergence which took place during a first year university course. The aim is to investigate in what ways university calculus teachers attend to students' prior knowledge in their teaching. The results are not surprising but the used tool does suggest a method of analyzing university teaching and its affordances or limitations for bridging the gap between secondary school and university.*

Keywords: Transition, sequence convergence, university teaching and learning, didactic contract, milieu.

AIM OF THE PAPER

It is widely acknowledged that the transition from secondary school to university on calculus requires students to move from a problem-solving orientation to a formal orientation. A previous study (Bloch & Ghedamsi, 2004), focussed on crucial differences between the secondary school mathematics contents and the university one on calculus, led to the categorization and the formalization of many important changes that should occur in the way students are required to work at the first year of their university studies. We deploy the TDS construct of didactical variables (Brousseau, 1996), which are defined as parameters that influence the mathematics students' work, to characterize these changes. Here are three main relevant didactical variables:

— The use of proof setting: At the first year of the university, the mathematics organisation on calculus is based on Bourbaki's rules; as a result students have to deal with proof by using formal definitions, theorems, logical operations such as negation of quantified predicates, reductio ad absurdum, etc. At the end of secondary school, calculus contents focus on graphic or numerical proofs and indeed on algebraic proofs. At most, students use general statements concerning the operations on the limits of convergent or divergent sequences to "calculate" a limit of a sequence given by its general term.

— The use of technical methods: At the first year of the university to solve calculus tasks an amalgam of technical methods is introduced, whereas, at upper secondary school, a few methods are very well identified and many exercises allow a work on each of them. Students then are never surprised by the work they have to achieve. Yet, at the university they have the responsibility of choosing the adequate technical methods. For instance, to check the convergence or the divergence of a sequence at the university, one can identify adjacent sequences, use sub-sequences, use Cauchy theorem, use De l' Hospital's rule, etc.

— The use of conversion between semiotic settings: At the end of secondary school, the tasks emphasize a fruitful conversion between the setting of algebraic semiotic representatives and the graphic one. These tasks become rather common and helpful for students' work. However, students have almost no possibility to take the initiative of using a graph in a heuristic way since these conversions are generally explicitly enunciated. At the first year of the university, there are no more

> graphs. Students have the responsibility to draw a diagram or a graph and exploit by themselves their potentialities as a heuristic support during a phase of a control or an exploration.

The modifications of the values of didactical variables in the transition from secondary school to university suppose changes in the didactic contract, which is "the implicit set of expectations that teacher and students have of each other regarding mathematical knowledge and regarding the distribution of responsibilities during the teaching and learning processes." (González-Martin et al., 2014, p. 119). In the case of the study above, the values given to the didactical variables are mutually exclusive which may lead to an alteration of major rules of the didactic contract.

According to these results, it is important to investigate the reality of the work during regular mathematics courses, especially the role of the teacher to manage such crucial changes into the students' work at the university. The research questions of this paper therefore are: To what extent does university teaching support the students' shift to formal calculus? How does university teaching help students learn through adjusting previous knowledge? Finally – and this is the main question explored here – how can we model teaching and learning processes in order to allow the assessment of both the students' actual work and the teacher's management of these processes?

For this, I demonstrate my methodological tool for analyzing teaching and learning processes in a regular lesson on calculus, with a particular focus on the transition from secondary school to university. Then, I apply the tool to analyze a regular lesson which took place at the first year of the university, on sequence convergence.

METHODOLOGICAL TOOL – ASPECTS OF MAIN TDS CONSTRUCTS

The central object of TDS is the notion of Situation which is "defined as the ideal model of the system of relationships between students, a teacher, and a mathematical milieu." (González-Martin et al., 2014, p. 117). The learning process is highlighted through the interactions taking place within such system.

In the Situation, the students' work is modelled at several levels with a main focus: on the action "knowing appears as means for action through models that can remain implicit" (p. 119); on the formulation "knowing develops through the building of an appropriate language" (p. 119) and on the validation "knowing becomes part of a fully coherent body of knowledge" (p. 119). The students' work grows up within a milieu "namely the set of material objects, knowledge available, and interactions with others" (p. 119) including the interactions with the teacher.

The foundations of TDS constructs focus on the optimization of interactions taking place within the system mentioned above, "in ways that maximize the students' responsibility for producing knowledge". The use of TDS at the university level compels researchers to reconceptualize the "maximal responsibility" (p. 121) given to the students and leads to an adjustment of the role of the teacher, especially in helping students overcome the new requirements at the first year of the university.

In this sense, in the transition between secondary school and university in calculus, teacher's interventions should not be neglected since he/she has the responsibility to manage students' evolution from problem-solving skills to formal calculus. These interventions should enrich the students' work and its evolution within and against a mathematical milieu during the phases of action, formulation and validation. González-Martin and colleagues (2014) illustrate the potency of TDS to design and to experiment Situations at university level, and demonstrate its application in three recent studies related to calculus and proof.

In regular (non-experimental) mathematics courses, the interactions taking place within the system formed by the teacher, the students and the milieu are governed by the actual didactic contract and evolve according to its nature. As a result, the quest for optimizing the interactions taking place within such system, as stressed in the TDS constructs, has to be the essence of methodological tool that will be used to analyze a regular lesson and that will allow the assessment of the students' actual work and of the teacher management. The emphasis on the phases of action, formulation and validation in the students' work materialize this quest. This should be done with taking into account teacher's interventions to manage these interactions.

Taking these considerations, the methodological tool for analyzing a regular lesson introduced in this paper focuses on two categories of students' utterances that deal with the phases of action, formulation and validation in the students' work; and three categories of teacher utterances, one related to managing interactions and two related to managing phases that support learning. The teacher categories are divided into subcategories that cater for the particularities of paradigmatic examples of didactic contracts (Brousseau, 1996). The definition of these subcategories is helped by the use of Robert's studies (2003; 2007))concerning teacher practices in order to achieve better meaning of transition phenomena. In particular, Robert (2003) attaches great importance to the organization of knowledge as a condition of learning and argues for the "comparison of several methods and the simultaneous operations of several properties at once, including old and new." (p. 70). The subcategories referring to students' work are outlined according to the structuring of the milieu. In the following, I set out the methodological tool with more explanations for each subcategory.

Teacher management

1) Management of interactions

   MI1: Initiate discussion by asking questions about specific knowledge in relation with the aimed one.

   MI2: Leave openings that help students to make a choice, to ask questions and to organize knowledge (Robert 2003; 2007).

   MI3: Abbreviate students' work, including questions.

   MI4: Splitting tasks into elementary subtasks, or specify technical methods to use. In this case, it imports to clarify whether the teacher limits the students' work to an application of juxtaposed knowledge (Robert, 2007).

   MI5: Guide students to take distance from what is happening and to work at the meta-cognitive level (Robert, 2007).

2) Management of action and formulation

   MAF1: Treat examples and counterexamples.

   MAF2: Support students' formulations by providing them with opportunities "to make conjectures, to experiment with heuristic solution, and search for adequate means of reasoning." (González-Martin et al., 2014, p. 122).

   MAF3: Foster the changing of knowledge context by developing an operational status of the notions, if any, by emphasizing relationships among notions, by changing the setting of semiotic representations, etc.

3) Management of validation

   MV1: Enunciate statements about knowledge

   MV2: Argue by using formal proof.

   MV3: Argue by using formulations, explanations, and by changing semiotic settings in a relevant way.

   MV4: Exemplify general statements and discuss the implications of these statements on a certain class of notions (functions, sequences, sets of real numbers, etc.).

   MV5: Make assessments of knowledge (local or global syntheses, including those relating to the use of formal rules of calculation).

Among some of these subcategories, the teacher's interventions do not enable students to undertake efficient interactions within the milieu and to progress in the learning process; this is the case of MI3, MI4, MV1 and MV2. The remaining subcategories emphasize the role that the teacher can play to enrich students' work especially in the case of MI2, MI5, MAF3, MV4 and MV5.

Students' work

1) Action and formulation

   WAF1: Formulate questions concerning specific knowledge in relation with the aimed one.

   WAF2: Express spontaneously knowledge by changing semiotic setting, by making examples and counterexamples, by linking several notions, etc.

WAF3: Formulate views on knowledge.

2) Validation

WV1: Indicate technical methods.

WV2: Perform on validation.

WV3: Discuss validation patterns proposed by peers or by the teacher.

When students demonstrated evidence of maladjusted knowledge, it is important to stress this in the analysis of the given lesson.

EXPERIMENTAL SETTING

Data collection

In Tunisia, mathematics courses at university level are organized into lectures and tutorials. The tutorial constitutes a setting to apply definitions and theorems already studied in the lecture. The lesson I focus on in this study functions as a tutorial, and concerns the applications of the main theorems of sequence convergence studied at the first year of the university. The sequence convergence is a concept met by the students from the third year of secondary school (scientific direction), which lasts four years in Tunisia, and which makes it possible to build a very rich and diverse corpus of knowledge.

This paper draws on the tasks planned by the teacher and the transcription of the whole lesson translated verbatim from French. The lesson lasted 2 hours in which approximately 30 students participated.

Mathematical tasks

Three mathematical tasks were planned by the teacher. The first one is related to the convergence of geometric sequences, the second one is related to the study of several sequences given by their general terms and the third one focuses on the use of Cauchy's theorem. In the following, I present succinct, *a priori* analyses of tasks in order to identify the mathematical milieu, namely the targeted knowledge, the students' previous knowledge and the elements that may optimize learning.

Task 1: General statements for geometric sequences

Let $(u_n)_n$ a sequence defined by $u_n = a^n$, $a \in \in \mathbb{R}$

1) *What is the nature* of the sequence if a = 0, 1 and -1?*

2) *We suppose $a \in]-1, 1[$ et $a \neq 0$. Prove that the sequence converges to 0.*

3) *We suppose $a \notin [-1, 1]$. Prove that the sequence diverges.*

* Across the text 'what is the nature of a sequence' is meant as 'Study this sequence in terms of its convergence or divergence"

At the end of secondary school, the theorem on the limit of geometric sequence is stated only for the case of]-1, 1[. The validation is based on the use of both the graphic semiotic setting and the numerical one. This is done by plotting and discussing geometric sequences graphs, or by computing terms of larger orders. Both cases show how the sequence tends to a specific value.

For this task, students have to pick out, among several university technical methods used to prove convergence or divergence, the relevant one in the case of such sequences. Some of the most useful methods to prove divergence at the first year of the university are: use formal definition; use Cauchy's theorem; prove that the sequence is not bounded; find two subsequences which don't behave the same way; etc.

Task 2: Sequences defined by general terms

What's the nature of these sequences?

$$u_n = (-1)^{\frac{n(n+1)}{2}}\left(1 + \frac{1}{n}\right);\ v_n = \frac{\cos(2n^3+1)}{n+1};\ w_n = \sqrt{n^2+1} - n;$$

$$t_n = \sin\left(\frac{n\pi}{2}\right);\ s_n = \frac{C_n^p}{n^p},\ p \text{ a natural number} \neq 0.$$

The study of $(v_n)_n$ and $(w_n)_n$ requires routine methods from the secondary school which correspond to the use of cosines properties and algebraic operations. The sequences $(t_n)_n$ and $(u_n)_n$ diverge, numerical calculation of some terms permit to identify subsequences to prove the non-convergence. The general term of $(s_n)_n$ is not familiar but its algebraic transformation permits to deduce that the sequence converges to $\frac{1}{p!}$.

Task 3: Convergence and Cauchy's theorem

1) *Let $u_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$, $n \geq 1$.*
Prove that $u_{2n} - u_n \geq \frac{1}{2}$. Deduce the nature of this sequence.

2) *Let* $v_n = 1 - \frac{1}{2} + \ldots + \frac{(-1)^{n-1}}{n}$, $n \geq 1$.
Prove that $v_n - v_m \leq \frac{1}{n+1}$, $0 < n < m$. *Deduce the nature of this sequence.*

This task requires more formal methods referring to logical operations such as negation of quantified predicates – which is nevertheless rather implicit – necessary condition, and sufficient condition. The intermediate subtasks allow students to use Cauchy theorem in order to conclude. Studying such sequences refers in reality to the study of $\sum_n \frac{1}{n}$ and $\sum_n \frac{(-1)^{n-1}}{n}$.

Under these conditions, the task is widely different from what is common at the end of secondary school. As said by González-Martin and colleagues (2014), "The role of the teacher becomes essential in helping students overcome their difficulties and fully grasp the subtleties they are confronted with." (p. 130).

Overview of transcript data according to each task

A global description of transcripts shows few interactions between teacher and students. Teacher utterances are generally isolated from those of the students. The students' utterances are short and the interactions among peers are non-existent. This compelled me to organize the analysis of the teacher and students utterances separately, and to do it in tandem when possible.

The lesson proceeded as follows in Table 1.

DATA ANALYSIS

Teacher management

The details of teacher utterances are as follows in Table 2.

About 26% of teacher utterances concern the argumentation by using classical formal proofs (MV2), but no interactions were observed with the students. Likewise, the utterances relating to the enunciation of statements about knowledge (MV1) aren't correlated to students' work and refer to the definition of convergence, theorems on the convergence, the definition of Cauchy sequence and its negation, etc. For instance, this is the case of the utterances (35 and 53) bellow:

35 Teacher: If $(a_n)_n$ tends to 0 and $(b_n)_n$ is bounded then $(a_n b_n)_n$ tends to 0.

The second utterance is the only one (from MV1) that is preceded by a student utterance formulated as a question:

23 Student: What's the negation of a Cauchy sequence?

53 Teacher: $(u_n)_n$ isn't a Cauchy one $\Leftrightarrow \exists \varepsilon > 0$, $\forall n_0 \in IN$, $\exists n \geq n_0$, $\exists m \geq n_0$; $|u_m - u_n| \geq \varepsilon$.

The teacher intervened only five times to argue by using formulation and explanation (MV3). Among these interventions (20, 26, 31, 34 and 39), one of them carries some students' knowledge:

6 Student: [talking about $\forall n \in IN$, $n \leq \frac{\log|M|}{\log|a|}$] This statement isn't true because IN isn't bounded!

20 Teacher: Yes, it'is absurd... this means that the set of integers is finished!

The rest of these utterances are isolated from students' work as shown below:

26 Teacher: [talking about $u_n = (-1)^{\frac{n(n+1)}{2}}(1 + \frac{1}{n})$] There's no problem for $\frac{1}{n}$ which tends to 0, then $1 + \frac{1}{n}$ tends to 1.

Task	Teacher utterances	Student utterances	Duration (approximately)
Task 1	1 – 24	1 – 7	35 min.
Task 2	25 – 45	8 – 22	50 min.
Task 3	46 – 57	23 – 24	20 min.

Table 1: Overview of transcript data

	MI_1	MI_2	MI_3	MI_4	MI_5	MAF_1	MAF_2	MAF_3	MV_1	MV_2	MV_3	MV_4	MV_1
Occurrence	7	0	4	10	0	3	3	0	10	15	5	0	0

Table 2: Details of teacher utterances

34 Teacher: [talking about $v_n = \frac{cos(2n^3+1)}{n+1}$] Cosine of any number is between -1 and 1.

39 Teacher: [talking about $0 < w_n \leq \frac{1}{n}$ and $\frac{1}{n} \underset{n \to +\infty}{\to} 0$] This statement isn't available for 0... there is no problem as we search limit on $+\infty$.

However, more than 60% of the total interventions concerning interactions (MI) don't contribute to lead students to make links between knowledge and especially their previous ones (MI3 and MI4). More precisely, in the utterances concerning (MI4), the teacher specifies university technical methods to use; this way of doing limits students' work to an application of juxtaposed knowledge, such as the use of sub-sequences, Cauchy theorem and the theorem on bounded sequence. This is the case of utterances (9, 14, 16, 36 and 40) highly linked to students' work:

2 Student: $|u_n| = |a^n| \prec \varepsilon \Leftrightarrow |a|^n \prec \varepsilon$

9 Teacher: We can apply logarithm for the two members of the equality! [to prove that geometric sequence is convergent given $a \in]-1, 1[$ and $a \neq 0$]

In the utterances 14 and 16, teacher seems to limit the choices of the students by imposing over and over again his technical methods:

4 Student: We can use the subsequences! [to prove that geometric sequence is divergent given $|a| > 1$]

14 Teacher: Perhaps, we also can use a reductio ad absurdum.

15 Teacher: What's the main property of a convergent sequence seen in the lecture?

5 Student: A convergent sequence is a Cauchy one.

16 Teacher: Otherwise, it's bounded and then we suppose it and we conclude that it is absurd.

The teacher reacted in the same way for the utterance (36) that follows:

16 Student: $w_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}$.

36 Teacher: We can put $\frac{1}{\sqrt{n^2+1}+n}$ between two members that converge to zero.

The teacher kept on imposing his technical method for the utterance (40):

20 Student: $t_n = \sin(n\frac{\pi}{2})$, $t_0 = 0$, $t_1 = 1$, $t_2 = 0$, $t_3 = -1$, $t_4 = 0$.

40 Teacher: It's clear that the sequence diverges. You can easily find subsequences which converge to different values.

Among teacher utterances which explicitly abbreviate students' work (MI3), the following one prevents student to use his/her own method to prove that the sequence converges (for the others, 37, 38 and 56 see students' work section):

7 Student: In this case we can use subsequences to prove that the sequence diverges.

22 Teacher: This is not fast. [meaning the method]

Finally, only 10% of the teacher utterances could help students' work in the phases of action and formulation (MAF). These interventions focus on the potentialities of the graphic setting or the numeric one to make conjectures, as well as, permit to investigate some examples and counterexamples related to aimed knowledge. Nevertheless, these interventions are initiated by the teacher and isolated from students' work.

Students' work

The details of students' utterances are anonymized and are as follows in Table 3.

Among the few questions asked by students (WAF1), three of them (17, 19 and 24) were shortened by the teacher (students who intervened in the exchange below aren't the same):

17 Student: Can't we directly calculate its limit? [talking about the sequence $w_n = \frac{1}{\sqrt{n^2+1}+n}$]

37 Teacher: Well! You're used to do this at the secondary school. Now I apply the theorem on sequence bounded.

	WAF_1	WAF_2	WAF_3	WV_1	WV_2	WV_3
Occurrence	4	7	0	3	10	0

Table 3: Details of students' work

18 Student: $0 < w_n \leq \frac{1}{n}$ and $\frac{1}{n} \underset{n \to +\infty}{\to} 0$ then $\lim\limits_{n \to +\infty} w_n = 0$.
19 Student: What's the theorem on sequence bounded?
38 Teacher: You saw it in the lecture.

A little further:

24 Student: Why n is an integer different of zero? It isn't mentioned in the definition of a sequence?
56 Teacher: It depends on the sequence.

However, the few students' interventions which express spontaneous knowledge (WAF2) emerged in response to questions posed by the teacher (MI1). Nevertheless, both the teacher questions and the students' responses are basic and are not significant of real requirements on convergence.

Finally, students' work in the phase of validation (WV) is rather thin. In this phase, students employed technical methods widely used at the secondary school; some of their interventions are expressed in tandem with the teacher ones.

RESULTS

In this paper, I present my methodological tool, based on TDS constructs, for analyzing a regular lesson on calculus. The application of the tool for analyzing a regular lesson on convergence sequence at the first year of the university allows a global illustration of teacher management and its implications for the learning process, as well as a more local description of effective learning about the convergence of sequences, if any. In this situation, the teacher seems not to care much about students' work as he doesn't intervene to enrich this work by emphasizing relationships among notions, by changing the setting of semiotic representations, by allowing openings to organize knowledge, by making assessments of knowledge, etc. The interventions of the teacher failed to enable students to undertake efficient interactions with the mathematical milieu. For instance, he limits the choices of the students by imposing over and over again university technical methods, as well as, by abbreviating efficient students' interventions including questions that may contribute to make links between knowledge. Furthermore, the few teacher interventions that focus on the potentialities of secondary semiotic tools are initiated by him and isolated from students' work. The teacher's use of the formal semiotic tool is done in an "ostensive way" (Brousseau, 1996, p. 45), which is defined as the act of the teacher who "shows" a mathematical object under the illusion that the students see the object that this "showing" aims to achieve. As a result, students' work is not consistent. The students express basic knowledge with little appropriate knowledge on convergence at the university level. During the whole lesson, the students use methods from secondary school and do not succeed to shift to the use of methods expected at university level.

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Mathematics in Engineering: The professors' vision

Gisela Hernandes Gomes and Alejandro S. González-Martín

Université de Montréal, Département de Didactique, Montréal, Canada, gisela.hernandes@umontreal.ca

The debate surrounding the teaching of mathematics in Engineering courses has inspired research for more than a century. However, Engineering students still have difficulties in recognizing the mathematics required to solve problems in non-mathematical contexts. In order to study the vision of mathematics adopted by Engineering teachers and understand its role in shaping teaching practices, we interviewed two Engineering teachers with different backgrounds: one with BSc in Mathematics and MSc and PhD in Engineering, and another with BSc, MSc and PhD in Engineering. Our data reveal differences in the way these teachers approach topics such as mathematical rigor and approximation, and identify challenges faced by their students when working with modelling.

Keywords: Engineering, teachers' background, rigor and approximation.

INTRODUCTION

Mathematics is an important domain in science and technology, and is taught in a variety of university programs such as Administration, Economics, Computation, Engineering, and many other scientific fields of study. In Engineering, the interpretation and solution of certain problems require the direct application of mathematical models. To understand and analyse these mathematical models, it is often necessary to use elements of statistics, linear algebra, or differential and integral calculus. This is one reason why the teaching of mathematics in Engineering courses has been addressed in studies as far back as the early twentieth century (Howson et al., 1986, p. 159). Some authors, such as Murakami (1988), have suggested that mathematics should be taught to Engineering students by mathematicians with the help of engineers, with the latter selecting suitable exercises, to create a learning environment more in tune with students' professional realities.

Cardella (2006) sought to better understand how Engineering students use mathematics. She also identified the mathematics Engineering students use, and consequently, the brand of mathematics that should be taught in Engineering programs. Her results show that Engineering students use mathematical thinking in various ways to solve problems while designing a project (they apply informal and intuitive knowledge of a given mathematical field; employ facts, definitions, algorithmic processes, etc.). She noted the importance of these various types of mathematical thinking, indicating that professors must carefully address students' mistakes throughout their learning process to ensure they understand what is being taught. Regarding teachers, Maaß and Gurlitt (2009) noted that teachers' knowledge of and beliefs on a given topic influence the way they plan, select, implement, and assess tasks. Later, and Maaß (2011) agreed with Schoenfeld (1992), who states that if we know teachers' resources, goals, and orientations — including their beliefs — we can better explain their actions. In this same vein, the study of Clark and colleagues (2014) provides a framework that divides into four categories those elements that can influence teachers' beliefs: "(a) teachers' professional background and experiences, (b) teacher knowledge, (c) teaching contexts, and (d) students' experiences" (p. 251). The conjecture that different teachers will give different lectures on the same topic on the basis of their knowledge and beliefs was confirmed at the university level by Pinto (2013). Pinto analysed the way two different university instructors implemented an identical course plan (both were capable teachers with a solid background in mathematics). He showed that the two instructors held different beliefs, attitudes and objectives and demonstrated varying degrees of confidence in different resources. As a result they offered two substantially different lessons.

Regarding the teaching of mathematics through applications and modelling in the context of Engineering education, Cardella (2006) drew on the work of Schoenfeld (1992) and Doeer (2007) in recognizing the need for mathematics instructors in Engineering programs to better understand how mathematics and mathematical thinking are used in Engineering. She admits that instructors should be familiar with the mathematical content necessary for engineering, and should consider the importance of problem solving, resources, attitudes, practices and of the learning environment.

In this vein, we are interested in identifying and discussing the attitudes and beliefs of instructors with different academic backgrounds to pinpoint how these beliefs influence their teaching of mathematical elements in Engineering programs. To do so, we follow Schoenfeld's approach to mathematical thinking.

THEORETICAL FRAMEWORK

To analyse the mathematical elements preferred by Engineering teachers with different backgrounds, we draw on Schoenfeld's (1992) proposal of five aspects of mathematical thinking: 1) knowledge base, 2) problem solving strategies, 3) monitoring and control, 4) beliefs and affects, and 5) practices. Of these five elements, we focus on the fourth, because we are interested in analysing teachers' beliefs and whether their different backgrounds (as mathematicians, physicians, or engineers) can influence their attitudes and their teaching practices. At this stage of our research, we do not draw on Schoenfeld's later work (1998), which provides a "theoretical account of how and why teachers do what they do [...] why they are engaged in the act of teaching" (p. 1); instead, we draw on his early proposal (1992), since we base our research on Cardella (2006), who in turn based her own work on this early proposal, which discusses "what it means to think mathematically" (p. 334), including the dimension of beliefs.

Using Schoenfeld's work as a basis, Cardella (2006) refers to beliefs and affects in the following way: "an individual's beliefs about and feelings towards mathematics influence how and when the problem solver uses mathematics. These beliefs and affects may be cultural beliefs or affects, or they may be particular to the individual problem solver. Additionally, an instructor's beliefs and affects may affect the student's practices." (p. 26). Cardella concluded that one of the most frequent beliefs observed in the execution of Engineering projects was the "Need to be Precise" (p. 119) promoted by teachers. For her, "this belief impacted the way that the team [working on a project] dealt with Uncertainty and also affected their use of Estimation." (p. 119).

According to Schoenfeld (1998), the "definition of thinking mathematically includes having a solid knowledge base, but it also includes knowing a wide range of problem solving strategies, having modelling skills, metacognitive skills, productive beliefs, and more" (p. 79). This vision of *thinking mathematically* is broad and goes beyond mathematical content knowledge and its relationships; other elements are assigned importance as well, and beliefs about mathematics and mathematical activity are also a part of this expanded vision of *thinking mathematically*. For Schoenfeld (1992), teachers' beliefs about mathematics determine the characteristics of their classroom environment. In turn, this environment "shapes students' beliefs about the nature of mathematics" (p. 359). According to Schoenfeld, "whether acknowledged or not, whether conscious or not, beliefs shape mathematical behavior. Beliefs are abstracted from one's experiences and from the culture in which one is embedded. This leads to the consideration of mathematical practice" (p. 360). Certain practices identified by Cardella (2006) in Schoenfeld's work point to elements in the practice of Engineering teachers, such as: "making multiple conjectures, coming to grips with uncertainty, defending claims mathematically, engaging in a science of patterns, extracting tools from the solution of complex problems, having a mathematical point of view, mathematical sense-making, using symbolic representations and manipulating symbols" (pp. 18–19).

Cardella (2006) followed this approach to study the beliefs of Engineering students' that emerge through their work on projects. Her research influences ours, in which we seek to analyse the beliefs of Engineering teachers with different backgrounds and education.

METHODOLOGY

To pinpoint the different beliefs regarding mathematical elements that emerge from distinct teaching practices, we interviewed two teachers (T1 and T2) from the Engineering school in a private university in São Paulo (Brazil). T1 and T2 have different backgrounds: T1 is a female teacher with BSc of Mathematics, MSc of Space Engineering and Technology, and PhD in Mechanical Engineering. T2 is a male teacher with BSc, MSc and PhD in Mechanical Engineering. T1 has taught at the university for seven years, giving courses in Differential and Integral Calculus, Analytic Geometry, and Linear Algebra. T2 has been teaching at the university for six years. His courses include Introduction to Computational Science, Mechanics of Solids, and Resistance of Materials. In addition to the aforementioned courses, both teachers supervise students' capstone projects in the Production Engineering (T1) and the Mechanical Engineering (T2) programs. Because student supervision is common to both teachers, and considering Cardella's work, which focused on capstone design projects and the use of mathematics in a design problem, we centred our research on the teachers' supervision of student projects to identify implicit beliefs concerning the role and use of mathematics in Engineering programs. The interviews addressed the instructors' teaching practices, in particular their approach to the supervision of students completing their capstone projects. The teachers were interviewed together on two occasions to initiate dialogue between them and draw out contrasts between their different visions and beliefs. The first interview centred on the teachers' supervision of capstone projects in general and their approach to this supervision. The second interview focused more on the supervision process for two specific capstone projects that the teachers singled out during the first interview: Artificial Neural Networks Applied to the Prediction of Values in Sao Paulo's Stock Market, the project supervised by T1, and Software of Simulation Applied to Mechanical Engineering, supervised by T2. Students are required to develop their own capstone project over one and a half years (the last three semesters of the Engineering program). During the first six months, students must select a topic and a supervisor, learn the norms of scientific writing, and begin creating a review of literature in a course entitled Scientific Methodology. In the second semester, supervisors begin meeting with students weekly or biweekly to oversee the project's design, data collection, and the students' first analysis. In the third and final semester, students must finish their project and present it to a panel comprised of two teachers and their supervisor.

The interviews were semi-structured and addressed a variety of topics to extract as much information as possible on the supervision process for capstone projects. In this paper, we focus only on those elements that emerged from the interviews pertaining to the teachers' beliefs, in line with the elements related to beliefs identified by Schoenfeld (1992) and Cardella (2006). These elements illuminate the teachers' approach to supervision, which may influence the way their students use mathematics or how they design their projects. The topics originally proposed for the interviews were: 1) the academic and professional background of each teacher; 2) the supervision of the capstone projects and how this supervision was conducted; 3) the use in these projects of content studied in the Engineering program; 4) difficulties the students encountered while working on the projects; and 5) the resources proposed by the teachers and used by students in completing their projects.

DATA ANALYSIS

After transcribing the two interviews, we created codes based on the teachers' responses and discussions, which allowed us to categorise thematically our data. This codification process was not linear, necessitating multiple readings of the interviews to draft the first codes, which were subsequently divided into subcategories (these subcategories were created because some categories allowed for more than one possible response). The codes were then recombined according to their conceptual similarities or differences, to define the final categories and subcategories (Figure 1).

Both teachers started the interview by providing details of their academic education and their experience in teaching courses and supervising projects in their university's Engineering program. T1, who has a background in Mathematics as well as Engineering, raised the importance of addressing the students' needs:

T1: [...] I came from the Polytechnic School, where I did my PhD working with Artificial Intelligence, neural networks, and genetic algorithms. What could I offer to the student in Production

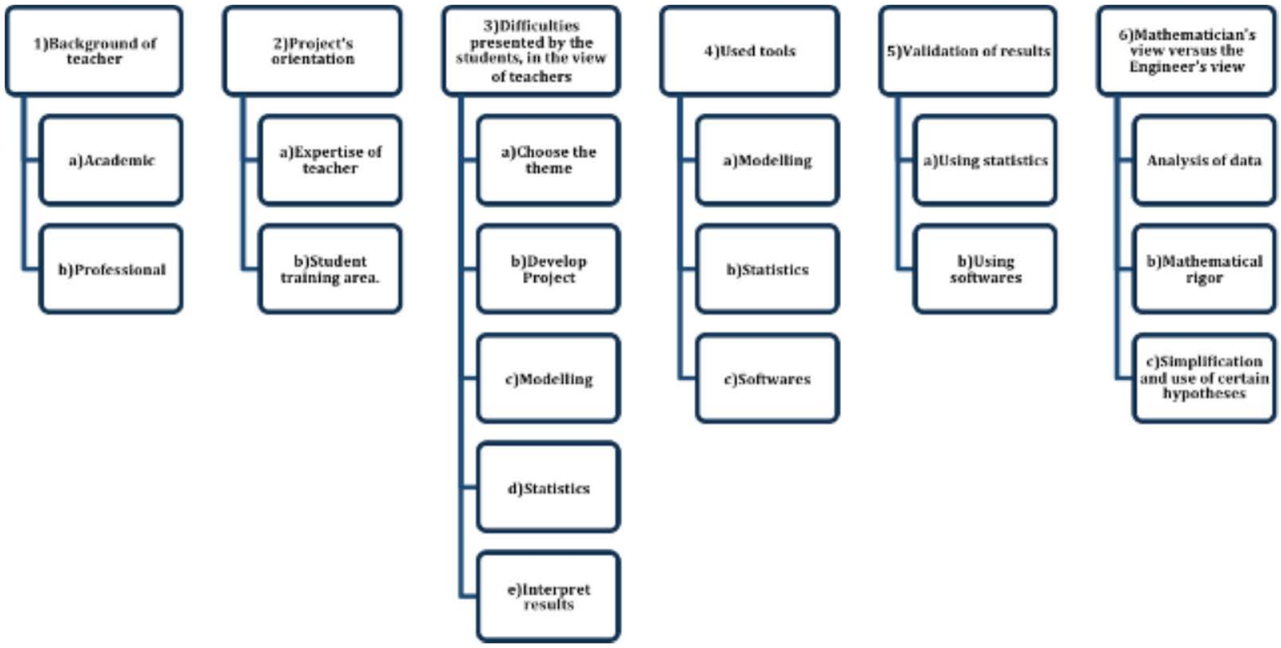

Figure 1: Final categories and subcategories

Engineering? For me to feel reassured, since I was also starting to supervise.

One of the elements mentioned by Schoenfeld (1992), *developing competence with the tools of the trade*, figures prominently in T1's response. In discussing her first experiences supervising students for their capstone projects, she revealed concerns about her ability to adapt her knowledge, which she considered atypical for Production Engineering; we observed that she felt it important to integrate her knowledge in a way that is useful to her students.

Unlike T1, T2's background is in Mechanical Engineering exclusively. However, like T1, his students study topics outside the scope of his education. His courses include Computing and Programming for Materials Engineering, a course in Electric Engineering and another in Mechanical Engineering. On supervision, he noted:

T2: I supervise students in Mechanical Engineering for a simple reason: because I'm a mechanical engineer and within my field of expertise it's more advantageous [...]

We observed a tendency in T2 to supervise students whose capstone projects relate to his field of research, resulting from a belief that it is easier to supervise projects that fall within one's area of expertise, where one masters *the tools of the trade* (Schoenfeld, 1992). Later in the interview, when discussing a particular Mechanical Engineering student's project involving the measurement of a frame, T2 stated:

T2: [...] with some measurement instruments, you measure the deformations of this frame and obviously it's subject to a static charge, which doesn't happen in reality.[...] And then, as you make a static trial, you affect that result with a coefficient, making the effect bigger. And there you judge, as if it were subject to a real condition, what is in fact an approximation. You can also do it with more sophisticated experimental equipment [...] All of that is experimental. You can also make a mathematical simulation, using computer programs, which are professional, commercial, available in the big firms, in the general market.

With statements such as "which doesn't happen in reality" and "what is in fact an approximation", T2 revealed some of his beliefs about mathematics, as an engineer who deals with approximations of real situations solved using computer simulations. This supports Cardella's findings (2006), who pointed out that engineers, when designing a project, work with approximate dimensions and make comparisons with already existing data to verify whether the ideas they propose fall within acceptable limits. According to Cardella (2006), this aspect of an engineer's way of thinking could be classified as a mathematical practice used by engineers. Cardella (2006) refers to Gainsburg (2003), who observed a tendency towards conservatism in structural engineers: "to understand a value or procedure as conservative is to attribute to it a broadly general and widely applicable quality, defined informally as 'safe'." (p. 9). Cardella observed that engineers adopt this attitude to ensure they meet minimum criteria, and that for more precise solutions, they reduce the margin of conservatism when it is justifiable to do so. She also found that despite the lack of precision inherent in "conservatism," engineers still use " = " rather than inequalities such as " > " or " < " (Cardella, 2006, p.20).

T2 also discussed software used to model certain structures. However, he stressed that students encounter difficulties using the software and when modelling problems. Many of his responses indicate that software and modelling are typically used by engineers to solve problems – in agreement with Cardella's (2006) model concerning required knowledge for teachers of mathematics in Engineering – and they revealed T2's awareness that students have difficulty with both.

T1 discussed the case of a Production Engineering student interested in economy and the stock market. T1 provided the student with data sources to allow the student to build an artificial neural network, and after a few simulations, they achieved satisfactory results: "the network had worked, the results were perfect regarding what we had seen in the site". However, for T1, it was still necessary to validate the results:

T1: What else can be done for some work, to give it a better basis, to really being able to say: "Ah, this work doesn't have any problem. Oh, let's do the statistical analysis of this data". And then, [the student] performed all the parts regarding regression.

In this excerpt, we can observe elements arising from a belief that only mathematics can be used to prove the validity of a solution found using other tools (in this case, she used statistical analysis), even when the results are satisfactory as a whole. The use of statistics is also identified as a way to validate knowledge obtained through results analysis. Regarding this last point, T1 added:

T1: [...] the analysis performed by a mathematician regarding that data is different from that performed by an engineer.

At this point, T1 was asked to define the difference, and an interesting exchange between T1 and T2 ensued:

T1: The difference I see: within Engineering, we don't need the exact, exact, exact value, as it is treated with all that...
T2: Rigor.
T1: The rigor with which it's treated in mathematics.
T2: The engineer... let's say he makes some hypotheses to simplify.
T1: Exactly. And she [the student] started from those hypotheses.

To further this discussion, the teachers were asked to elaborate their ideas of what constitutes mathematical rigor. T2 provided an example concerning the resistance of materials, called bending displacement:

T2: Among the different methods to determine the bending displacement, there's [one] called the method of the differential equation of the elastic line of the beam. The elastic line is that configuration the beam acquires when it's deformed because of the load. The equation from which all the textbooks start off, it's a second degree, second order differential equation. A very complex equation. And engineers, based on the fact that in a project – in the majority of projects, to speak carefully – we assume that in the angular displacements the slope is very small, the tangent to the angle, when the angle is very small it's almost the very same angle. And there, a first derivative which makes that differential equation harder for us... and which is to the square, we say it's so small, so small, that it's not considered...

T2 identified elements of mathematical practises that engineers use to solve complex problems, and noted their use of certain hypotheses to simplify mathematical calculations. T2 suggested that these practices are not well regarded by mathematicians:

T2: And there, mathematicians get... crazy. They say no, that's not the exact solution. That "we" [the engineers] are lying. And no. I'm just saying that the dislocation, the angle is so, so, so, so, so small, that I'm not gonna worry, and actually in reality it doesn't make a difference. I just simplify...

We see that T2 uses hypotheses to simplify calculations. Cardella (2006) identified this as a solving strategy used by students, which she termed "problem transformation" (p. 88). This strategy seeks both to simplify problems and transform them into familiar problems. Furthermore, we observe in T2 a way of speaking which seems characteristic of an engineer, using a language different from that of a mathematician (Nicol, 2002). Authors as Godfrey (2013) have labelled this as "An engineering way of thinking".

In addressing the notion of rigor, T1 came back to the work of her student in Production Engineering:

T1: In my specific case, when I speak about this mathematical rigor, when you look at a residual graph and you see the dots disperse or not, that interpretation is very subjective. Then, you take a graph [...] and you go validate with others. And there, I think they look alike, she can think they don't, but what was really important for us was to verify whether the system which had been modelled was giving us a prevision as close to reality as possible, which were the data we originally had. And she got to 98.6% of accuracy. [...] Furthermore, that rigor in Engineering is not considered, especially taking into account that the aim was actually to have a prevision of the data, using a computer tool, which is what was done. We wanted to use an artificial intelligence tool to know whether that tool was capable of doing the prevision. That was done, it was shown and with efficacy.

Again, here we can observe her belief that rigor in mathematics and rigor in Engineering are not the same. This aspect of her mathematical thinking (belief) may have a strong impact on others (monitoring and control, problem solving strategies, practices), and could influence her approach to the teaching of mathematics. Elsewhere in the interview, when discussing her first-year Engineering courses in Differential and Integral Calculus and Linear Algebra, T1 stated:

T1: For me it's extremely important to have that mathematical rigor [...] at the beginning. Mathematics is an extremely important science; [...] even Engineering can't live without those calculations, although in some cases you have: "Ah, it's negligible" or "You don't need a rigor as mathematical as when a mathematician does mathematics". That's one point. Another point is: [the student] needs to know the bond between mathematics, which he has in the first courses, and what he uses in the technical courses. Actually, we try to do that [...], trying to show him that he's gonna use that tool later, the link between all those calculations and the technical courses.

Her response reveals her desire to broaden her students' ability to *think mathematically*, and her beliefs appear to influence her approach to teaching. In the case of T1, we observe a more explicit concern for rigor, and an awareness of the dichotomy between what a mathematician would do and what an engineer would do. According to Schoenfeld (1992, p. 359), these beliefs may determine the way she teaches and, as a consequence, shape her students' beliefs about the nature of mathematics.

FINAL REMARKS

The aspects identified in this study, although taken from interviews with just two teachers, are in keeping with the literature review and the theoretical framework, and they inspire reflection about mathematical practices in Engineering courses.

One issue identified in the teachers' responses concerns mathematical rigor versus approximation. On the one hand, the teachers require students to know (T1) or at least follow (T2) formulas with all the rigor characteristic of mathematics. On the other hand, in applying mathematics to the practice of Engineering and in resolving problems, certain assumptions are made in order to simplify calculations (T2), which corroborates Cardella's research (2006) on the practices of Engineering professors. When results obtained through this process are close to those anticipated, based on the literature, they are deemed satisfactory by engineers. Regarding modelling, both T1 and T2 identified some difficulties related to 'approximation' and data analysis, which is in accordance with Doerr (2007) with respect to the challenges faced in teaching.

Our analyses have identified certain mathematical elements that could be classified according to the elements of mathematical thinking of Schoenfeld (1992) and in keeping with Cardella's (2006) arguments. However, we believe more research is needed in this area. Specifically, more observation of teaching practices is required, particularly of teachers with different backgrounds. Future research could look at the connections between teachers' beliefs, backgrounds, teaching practices and choices, and identify possible consequences for their students' learning. We intend to explore these aspects further in our future research.

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Engineering students in Spain and Germany – varying and uniform learning strategies

Inés Mª Gómez-Chacón[1], Birgit Griese[2], Bettina Rösken-Winter[3] and Carlos Gónzalez-Guillén[4]

1 Universidad Complutense de Madrid, Madrid, Spain, igomezchacon@mat.ucm.es

2 Ruhr-Universität Bochum, Bochum, Germany, birgit.griese@rub.de

3 Humboldt-Universität zu Berlin, Berlin, Germany

4 Universidad Politécnica en Madrid, Madrid, Spain

It is a pattern common to many countries that engineering students have trouble passing mathematics. The manifold reasons so far explored mirror various perspectives on the transition from secondary to tertiary education. Focusing on learning strategies presents a promising supplement to this range, as they have the potential to account for its cognitive complexity and affective considerations. By means of the LIST questionnaire, we explored learning strategies for two samples of 113 Spanish and 159 German engineering students. The findings show that both samples differ regarding their scoring on the scales Organizing, Elaboration, Repeating, and Metacognition. Finally, five clusters were used to group students according to their similarities, supporting the decisive role of metacognitive skills.

Keywords: Engineering, learning strategies, secondary-tertiary transition.

INTRODUCTION

For many engineering students, learning mathematics in tertiary education is a critical issue. They encounter epistemological/cognitive, sociological/cultural and didactical obstacles (cf. Guzman, Hodgson, Robert, & Villani, 1998) as mathematics at university differs significantly from school mathematics. Some authors even label the transition "abstraction shock" since university mathematics adds a formal world to the mathematics encountered at school (Artigue, Batanero, & Kent, 2007; Tall, 2004). Besides these studies that focus on cognitive perspectives, other researchers additionally identified epistemological, conceptual, social-cultural, motivational as well as metacognitive and affective variables having an influence on students' performance. Some authors explicitly elaborate on specific learning strategies students may not have developed throughout school time (Rach & Heinze, 2011). These learning strategies address a combination of both skills and attitudes such as self-organisation, perseverance and frustration tolerance (cf. Pintrich, Smith, Garcia, & McKeachie, 1991; Weinstein & Palmer, 2002; Wild & Schiefele, 1994). In particular, investigating the role of learning strategies allows for revealing the cognitive dispositions as well as affective barriers and pathways, and studying the interrelations between them (Wild, 2005).

Both Spain and Germany face alarmingly high numbers of students giving up studying due to mathematical problems. In engineering courses in Spain, students were usually enrolled for 52.4 credits, but only succeeded passing 31.8 credits, mathematics being one of the subjects with the highest fail rate (MEC, 2013; Rodríguez Muñiz, 2011). The situation is comparable in Germany as 48% of engineering students fail in their first year university studies (Heublein, Richter, Schmelzer, & Sommer, 2012). In this paper, we explore for the two countries which cognitive dispositions and beliefs of students seem advantageous to successfully continue their studies.

THEORETICAL FRAMEWORK

Studies exploring the transition from school to tertiary education mostly concentrate on cognitive aspects when it comes to question challenges in mathematics. These studies elaborate on cognitive difficulties and conceptual obstacles that students experience in how mathematics is communicated to them (Artigue, Batanero, & Kent, 2007), particularly referring to the formal level of university mathematics and the prevalent role of proofs within (Selden & Selden, 2005).

Cognitive processes involve affective stances that moderate the tension between modes of intuitive and analytical thinking (e.g., Fischbein, 1987; Stavy & Tirosh, 2000). In particular, the theory of dual processes in cognitive psychology has been adapted to mathematics education, and the role of affective variables has been pointed out in this context (e.g., Gómez-Chacón, García-Madruga, Vila, Elosúa, & Rodríguez, 2014). Some studies additionally investigate the connection between affective variables and student performances. Findings reveal that students' cognitive reflection, as a metacognitive variable, their beliefs about mathematics, and their self-efficacy, are all correlated positively and significantly with mathematical achievement (Gómez-Chacón, García-Madruga, Rodríguez, Vila, & Elosúa, 2011; Gómez-Chacón et al., 2014). There is also evidence that metacognition impacts positively on learning strategies which in turn influences achievement (Griese, Glasmachers, Härterich, Kallweit, & Rösken, 2011).

An instrument focusing on cognitive, metacognitive and resource-related strategies is presented by the LIST questionnaire comprising 13 dimensions of learning strategies grouped accordingly. The LIST questionnaire (Wild & Schiefele, 1994) for measuring learning strategies in academic studies was first compiled in the 1990s and has since been modified and tested several times. It encompasses general items that can be applied to all kinds of subjects (for examples, see Table 1 below) and uses Likert scales. One root of the LIST questionnaire is the Motivated Strategies for Learning Questionnaire (MSLQ) which measures college undergraduates' motivation and self-regulated learning relating to a special course (Pintrich, Smith, Garcia, & McKeachie, 1993). Apart from *Motivation*, the scales from LIST are derived directly from MSLQ, although the number of items varies. The main difference between the two questionnaires is that MSLQ puts more emphasis on including different aspects of motivation as *Goal Orientation*, or *Control of Learning Beliefs*. Another essential study influencing the LIST questionnaire is the Learning and Study Strategies Inventory (LASSI) by Weinstein and Palmer (2002) which also separates cognitive aspects. LASSI scales partly cover the same contents as LIST though holding different names. As there are no analogous German / Spanish questionnaires on learning strategies, our study opted for the LIST questionnaire, thus hoping for the further asset of a parallel instrument for both countries.

RESEARCH QUESTIONS

In both countries engineering students struggle with mathematics, and in a first attempt to capture differences or commonalities, we used parts of the LIST questionnaire to explore students' learning strategies with respect to *Organizing, Elaborating, Repeating* and *Metacognition.* We therefore translated the German LIST questionnaire into English and then into Spanish (and back into German for additional dependability) and checked for scale reliability. Finally, we investigated if students can be grouped based on their ratings of the different dimensions of the questionnaire. In our comparative study, we particularly pursued the following research questions:

Research question 1a: Does the Spanish translation of the LIST questionnaire yield sufficient scale reliability?

Research question 1b: How do Spanish and German engineering students' differ with respect to their learning strategies?

Research question 2: How are the different learning strategies of students in both countries correlated?

Research questions 3: How can students from both countries be classified with respect to their learning strategies?

METHODOLOGY

Participants and instrument

The two samples from Spain and Germany are comparable in terms of age, gender percentage, and academic year in engineering studies. Moreover, their academic courses are similarly organised.

113 (71.7% male) Spanish undergraduate students attending the first academic year of the *Industrial Engineering Degree* participated in this study. Students enrolling in this course must have obtained high scores in the test of university entrance, even higher than for other engineering degrees. The mathematics module consists of 200 minutes of traditional lectures per week, with optional tutorials and digitalized learning material. The examination is a written test with a focus on calculation and normally without proofs. If students fail, they must retake the course.

Dimension	# Items	Example Item
Organizing	8	I go over my notes and structure the most important points.
Elaborating	8	In my mind I try to connect newly learnt facts to what I already know.
Repeating	7	I learn the subject matter by heart using scripts or other notes.
Metacognition (Planning)	11	Before starting on an area of expertise, I reflect upon how to work most efficiently.

Table 1: LIST dimensions and example items

159 German students (72.3% male) were selected from a larger sample from an ongoing research project to match the Spanish data. The German students had enrolled in different kinds of engineering courses at a university (meaning a slightly more challenging course, compared to a technical college) all starting with near identical mathematics lectures in traditional format, lasting 180 minutes per week, with optional tutorials and digitalized learning materials. As in Spain, there is a written test with focus on calculations. If students fail, they must retake the course, multiple fails result in expulsion. In both groups of students, attendance of lectures is optional and often low.

Originally, the LIST questionnaire comprises 13 dimensions of learning strategies, grouped into cognitive, metacognitive, and resource-related strategies. They each contain between three and eight items. For the study at hand we concentrate on cognitive and metacognitive strategies; an overview on example items is provided in Table 1.

The *Metacognition* scale contains the three subscales *Planning*, *Monitoring* and *Regulating*.

Data analysis

In both cases (Spain and Germany) the data was analyzed by computing the means, standard deviation and internal consistency (Cronbach's α) for each of these scales of the survey (based on Likert scales, from 1 to 5 respectively from 1 to 4); the correlation between scales; the factor pattern matrix; and clusters. Factor analysis was conducted using the extraction method of *Principal Component Analysis* and the rotation method of *Varimax* with Kaiser normalization. For rescaling the data, we calculated $2.5[1/n(x_1+...+x_n)-1]$ for the 5-point Likert scales respectively $100/3[1/n(x_1+...+x_n)-1]$ for 4 points and were thus able to correct the effect of the different numbers of items in different scales. This finally yielded scores from 0 to 100 (scale scores under 25 describing rare use, between 25 and under 50 infrequent use, between 50 and under 75 regular use, 75 or more continual use of the learning strategies).

Regarding clustering the data, the most common partitioning method is the k-means cluster analysis. Conceptually, the k-means algorithm follows the following process: it selects k centroids (k rows chosen at random), assigns each data point to its closest centroid (determined by the Euclidean distance), recalculates the centroids as the average of all data points in a cluster (i.e. the centroids are p-length mean vectors, where p is the number of variables) and assigns data points to their closest centroids. Steps 3 and 4 are repeated until the observations are not reassigned or the maximum number of iterations is reached. The distances are reported in Table 2.

The results obtained show the closest clusters are 1 and 5 or 1 and 3. For the hypothesis contrast, we obtain that the clusters represent data in variables of *Organizing*, *Elaborating*, *Repeating*, *Planning*, *Monitoring* and *Regulating* because of having $p<0.0001$ for values of the p-values. We note that for mathematic academic performance the centroid of this variable is 0 in some clusters and we have significance $p<0.02$ for mathematic academic performance.

Clusters	2	3	4	5
1	56.14	44.00	55.01	39.45
2		61.91	95.47	46.06
3			52.09	44.50
4				54.85

Table 2: Distances between the final centroids

RESULTS

Research question 1a. We calculated scale reliability for the factors *Organizing*, *Elaboration*, *Repeating*, and *Metacognition*. All Cronbach's alpha values are higher than 0.7, except for *Metacognition* in the Spanish study, see Table 3. In sum, the results show sufficient reliability between the different items. The comparative analysis between both countries indicates that there are no significant differences, except for *Elaborating* strategies. For this dimension, Cronbach's α is slightly higher in the Spanish study.

Research question 1b. The results of the factor analysis let to the four main dimensions *Organizing*, *Elaboration*, *Repeating*, and *Metacognition* and the three subscales *Planning*, *Monitoring* and *Elaborating* for the *Metacognition* scale. The variance explained by this factor structure is 54.7 % for the Spanish data and 43.92 % for the German data. In Table 4 we provide an overview on how the students rated in the different dimensions. Students' learning strategies differ significantly across the two countries. In all factors of the LIST questionnaire, except for *Monitoring*, we noted distinctly higher values for the mean in the Spanish than in the German results. For this metacognitive variable there are no significant differences between both means. In terms of standard deviations, Spanish students on the whole produce lower values.

Regarding the maximum and minimum scores, they are the exactly the same for *Organizing* and *Regulating*. For *Elaborating*, *Planning*, *Monitoring*, and *Metacognition* as a whole, the maximum value is higher in Germany, for *Repeating* it is lower in Germany. For the minimum values, we obtain significant differences between all variables but *Organizing*. The results show that the Spanish minimum values for these variables are always higher than the German ones.

Research question 2. In addition to comparing the means for the Spanish and the German data, we explored the correlations among the factors. An overview is provided in Table 5, where the additional variable *Mathematics Academic Performance* is coded as 1 for pass and 0 for failing the exam. It is worth to point out the highly significant strong correlation of the variable *Organizing* (defined as the ability to structure and restructure matter) with the three variables of metacognitive skills: *Planning* (r_S=.45 for Spain, r_G=.48 for Germany), *Monitoring* (r_S=.51 respectively r_G=.48) and *Regulating* (r_S=.46; r_G=.34). In the German data, there are significant and strong positive correlations between *Organizing* and *Repeating* (r_G=.60), *Elaborating* and *Monitoring* (r_G=.52) and *Repeating* and *Monitoring* (r_G=.57). For the mathematic academic performance, results show a negative (though not significant) correlation with *Repeating* for the German data (r_G=-.47) which cannot be found in the Spanish sample.

Research question 3. We realized a k-means cluster analysis with the 272 participants from both countries (113 Spanish (81 male and 32 female) and 159 German

Scale/Country	Spain	Germany
Organizing	.84	.82
Elaboration	.83	.74
Repeating	.73	.73
Metacognition	.65	.73

Table 3: Scale reliabilities for the Spanish and German data

	Mean		Std. Deviation		Maximum		Minimum	
	S	G	S	G	S	G	S	G
Organizing	61.92	47.87	20.67	22.16	100	100	0	0
Elaborating	59.21	48.76	16.58	17.47	90.63	95.83	6.25	0
Repeating	50.60	40.34	15.44	18.73	92.86	85.71	17.86	0
Meta – Planning	57.80	47.78	16.90	21.78	93.75	100	6.25	0
Meta – Monitoring	47.01	46.27	18.53	21.16	87.5	100	6.25	0
Meta – Regulating	72.35	65.15	15.48	20.58	100	100	25	0

Table 4: Descriptive statistics for Spanish and German students

(115 male and 44 female)) according to the variables *Organizing*, *Elaborating*, *Repeating*, *Planning*, *Monitoring* and *Regulating*, using the k-means method. In this case, we present the results using five clusters (Table 6).

The interesting part is how to describe the clusters in reference to the learning strategies employed by the respective students. Table 6 shows the number of participants that belong to each cluster in the last row. *Cluster 1* has high values (>55) in *Planning* and *Regulating* and a comparatively low value in *Monitoring* (<40). *Organizing*, *Elaborating*, and *Repeating* score medium. In this cluster there are 42 students (15.44%), of them 15 (35.72%, 13 male and 2 female) are Spanish and 28 (64.28%, 22 male and 6 female) are German. *Cluster 2* has high values (>55) in *Organizing*, *Elaborating* and *Repeating* strategies and *Metacognition*. In this cluster there are 95 students (34.93%), of them 61 (64.21%, 37 male and 24 female) are Spanish and 34 (35.79%, 20 male and 14 female) are German. *Cluster 3* has relatively high values in *Planning* and *Organizing* and medium values in *Monitoring* and *Regulating* but low values in *Elaborating* and *Repeating*. In this cluster there are 28 students (10.29%), of them 6 (21.43%, 3 male and 3 female) are Spanish, and 22 (78.57%, 15 male and 7 female) are German. *Cluster 4* has low values in all variables except medium values in *Regulating*. In this cluster there are 39 students (14.34%). Very few (2) are Spanish (5.13%, 1 male and 1 female). The rest (37) are German (94.87%, 23 male and 14 female). *Cluster 5* has high values (>55) in *Regulating* and *Elaborating* and medium in *Organizing*, *Repeating*, *Planning* and in *Monitoring*. In this cluster there are 68 students (25%), of them 29 (42.65%, 26 male and 3 female) are Spanish and 39 (57.35%, 27 male and 12 female) are German.

In summary, the data shows that 54.93% of students are concentrated in cluster 2 and cluster 5, whose students show medium and high levels in their learning strategies. However, there are differences between countries. 79.65% of the Spanish students are in these clusters, contrasting with only 45.91% of the Germans.

	E		R		MP		MM		MR		MC		MA
	S	G	S	G	S	G	S	G	S	G	S	G	S
O	.20*	.29**	.32**	.60**	.45**	.48**	.51**	.48**	.46**	.34**	.66**	.57**	.04
E	1	1	.16*	.32**	.11	.29**	.41**	.52**	.44**	.26**	.43**	.47**	.19*
R			1	1	.28**	.44**	.33**	.57**	.12	.24**	.36**	.58**	.06
MP					1	1	.32**	.33**	.06	.31**	.69**	.78	.16
MM							1	1	.40**	.27**	.84**	.75**	.07
MR									1	1	.59**	.64**	.10
MC											1	1	.08

(O = *Organizing*, E = *Elaboration*, R = *Repeating*, MP = *Metacognition-Planning*, MM = *Metacognition-Monitoring*, MR = *Metacognition-Regulating*, MC = *Metacognition*, MA = *Mathematics Academic Performance*)

Table 5: Pearson correlation (1-tailed, *p<0.05, **p<0.001) for the factors

Factors/Cluster	1	2	3	4	5
Organizing	39.14	72.35	61.94	19.31	42.85
Elaborating	40.35	63.97	31.66	23.37	57.17
Repeating	42.52	55.69	23.51	21.80	39.22
Metacognition – Planning	66.12	64.47	50.67	26.71	38.51
Metacognition – Monitoring	25.05	59.63	40.77	25.27	47.09
Metacognition – Regulating	69.25	80.18	47.02	50.21	70.06
Students (total = 272)	**42**	**95**	**28**	**39**	**68**

Table 6: Final centroids of the cluster analysis and number of students

Regarding German students, it is remarkable that there is a group of 23.27% with low levels of learning strategies (cluster 4).

CONCLUSIONS AND DISCUSSION

In our exploration of two samples, we were able to detect commonalities between first-year engineering students of both countries: As much as three quarters of engineering students are male, only one quarter are female. That may seem unbalanced, but it describes a steady growth towards equality over the last decades. Our comparisons cannot be generalized to universal statements about two societies, but provide interesting insights into students' learning behaviour against the background of different economic conditions (where the unemployment rate of 25% in Spain raises higher demands than Germany's 6%).

We found that the questionnaire employed works well in both countries, despite the initial double translation, backed up by the fact that the retranslated items correspond well to the original ones. The differences in learning strategies between the two countries can be condensed in the fact that the German engineering students showed more variation and often scored lower, meaning that the Spanish students tended to state desired behaviour, i.e. diligent learning activities. Both groups score equally low on *Monitoring* skills, which can be interpreted as a teaching perspective.

For both countries, the interrelations between the variables (apart from forming a complex pattern) stress the fact that metacognitive skills are at the core of learning behaviour, and can be viewed as an effective lever by which to influence learning strategies and thus learning success. However, there are no clear indications as for which learning strategies support examination success.

As our sample consisted of 41.5% Spanish and 58.5% German students, we would expect this distribution to reflect on the different clusters as well. That is more or less the case for all clusters but one: Cluster 4 contains almost exclusively German students and can be described as incorporating students who generally score very low, i.e. they do not report to work very ex- or intensively for their studies. This might be traced back to the fact that in Spain, you cannot enter a university course in *Industrial Engineering* without proving your commitment, motivation and capability in an exacting university entrance test. In Germany, there is restriction to university education, too, but it is less strict (meaning they can be sidestepped by time or space). These conditions may have influenced the pattern on cluster 3 as well, where Germans are overrepresented. This cluster contains students with high scores on learning strategies that reflect good intentions (*Planning* and *Organizing*), but low scores on actually realizing these in tedious day-to-day work (*Elaborating* and *Repeating*). Identifying clusters of students with homogeneous learning behaviour implies offering customized courses fostering specific deficiencies.

As a final outcome, our interest in describing, developing and evaluating metacognitive strategies with respect to short- and long-term achievement in mathematics has been kindled. We expect to learn more from future investigations, particularly from a comparative exploration of Rasch analyses of the two surveys. Additionally, we would like to conduct qualitative research which can help to enlighten the quantitative data we already have.

ACKNOWLEDGMENT

The paper' elaboration has been supported by special action grant from Instituto de Matemática Interdisciplinar (Spain) (IMI-2014-Actions Research Committee Grant UCM).

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The use of textbooks by pre-university teachers: An example with infinite series of real numbers

Alejandro S. González-Martín

Université de Montréal, Département Didactique, Montréal, Canada, a.gonzalez-martin@umontreal.ca

Textbooks are central to the teaching process at all levels, including the tertiary level. However, the phenomenon of textbook use in higher education has not been extensively studied. In this paper, we analyse textbook use in the teaching of infinite series of real numbers at the pre-university level in Quebec. We interviewed five teachers about their textbook use in order to investigate similarities between their personal relationship with series and the institutional relationship with series transmitted through textbooks. Our results show that the teachers' courses generally follow the textbook's presentation, and that their documentation system is reduced to almost a single textbook, to which they adhere almost exclusively. We also compare our results with those of Mesa and Griffiths (2012) at the tertiary level.

Keywords: Textbook use, pre-university, personal relationship, documentation system.

INTRODUCTION AND BACKGROUND

In this paper, we analyse textbook use by pre-university teachers in Quebec (Canada) in the teaching of infinite series of real numbers (*series* hereinafter). Textbooks play a crucial role in the school environment and "have always played a major role in mathematics education" (Sträßer, 2009, p. 70). However, the vision of mathematics transmitted by textbooks can shape what teachers teach and what students learn. Textbooks can also influence students' attitude toward the discipline of mathematics and affect their self-perception as learners (Raman, 2004). Moreover, textbooks can mould students' beliefs about what they can learn and how they can access and use knowledge (Mesa & Griffiths, 2012). For instance, González-Martín, Giraldo and Souto (2013) recently analysed a sample of secondary textbooks to identify how real and irrational numbers are introduced. They showed that the approaches used in textbooks could have major consequences for students' learning, influencing their vision of mathematics (e.g., 'it is possible to show that a statement is true by giving some examples'), and affecting their subsequent learning of other Calculus topics.

In addition, textbooks seem to play an important role in the teaching of tertiary mathematics (Mesa & Griffiths, 2012, p. 85). Although one might expect such textbooks to meet more rigorous academic standards appropriate for university classrooms, research seems to contradict this. For instance, in her study on continuity in Precalculus, Calculus, and Real Analysis courses, Raman (2004) concluded that textbooks send conflicting messages on the status and purpose of mathematical definitions. These results agree with those of Giraldo, González-Martín and Santos (2009), who also found that the presentation of content related to continuity in undergraduate textbooks sometimes conjures erroneous images that can impede the learning of derivatives and integrals. Lithner (2004) analysed the types of reasoning that Calculus textbook exercises can encourage and instil in students, and demonstrated that exercises in which students merely need to reproduce a given example predominate. Regarding the topic of series, González-Martín, Nardi and Biza (2011) also established a preponderance of algorithmic exercises in pre-university and university textbooks, as well as a lack of conceptually-driven tasks; we come back to this last work later. Because most of these studies on the introduction of Calculus reveal problems with the way post-secondary textbooks address mathematical topics, we decided to examine how post-secondary instructors use textbooks in preparing and teaching their courses.

Research on tertiary textbook analysis is rapidly evolving, but the phenomenon of instructors' textbook use in higher education has not been extensively studied (Mesa & Griffiths, 2012, p. 85). Mesa and Griffiths (2012) addressed this issue and found that tertiary instructors use textbooks for different purposes: generating the syllabus, preparing classes, and designing homework. The various ways textbooks were used included: using the same information contained in the textbook (*offloading*), supplementing it with alternative examples designed by the instructors themselves or culled from other textbooks (*adapting*), or changing the presentation altogether, including using different notation (a form of *improvising*). For all the tertiary instructors in their sample, the textbook appeared to be a crucial artefact in the instruction preparation process. However, the more frequently an instructor gave the same course, the less he or she relied on the textbook due to familiarity with its contents and with what it did and did not offer.

Another important observation was that instructors saw the textbooks they used as written *for* students, and not as a tool from which the instructors themselves could learn (e.g., they did not mine textbooks for new ways of understanding certain topics, nor did they draw inspiration from textbooks to vary their teaching methods). Instructors also did not see textbooks as tools that could help them select problems or decide how to sequence topics in constructing their syllabi. In fact, the textbook features that were perceived to be the most helpful for improving the instructors' teaching were problems and examples.

The research presented in this paper seeks to provide more information on this phenomenon, particularly with regard to the teaching of a very specific topic: series. Series are a key notion in mathematics: already present in early Greek mathematics, they were crucial in the development of Calculus. They have many applications within mathematics (such as the calculation of areas by means of rectangles) as well as outside mathematics (including the modelling of situations such as the distribution of atmospheric pollutants). These factors may partially explain why the study of series is included in introductory Calculus courses in many countries.

This is the case in Canada, where each province develops its own official curricula. In the province of Quebec, compulsory education ends at age 16 and students who wish to attend university must first complete two years of pre-university studies (called *collégial* —other countries, such as Spain, follow similar systems). Students pursuing scientific or technical careers are introduced to Calculus during their *collégial* studies, where series first appear. It is in this context that our research, like Mesa's and Griffiths' (2012), seeks to better understand the phenomenon of textbook use. However, while Mesa and Griffiths looked at general textbook use in universities, our work centres on preparatory courses at the *collégial* level and focuses on a topic that is introduced in university in many countries. Moreover, studying the *collégial* experience may help pinpoint gaps and continuities between textbook use at the *collégial* and university levels that could affect students' transition. This could open the door to further research on similarities and differences in practices with regard to a specific topic of study.

Before beginning our investigation of teachers' use of textbooks in the teaching of series, we first developed an analysis of how series are presented in *collégial* textbooks, following an anthropological approach. We also identified some possible consequences of this presentation for students' learning (González-Martín et al., 2011). Our sample consisted of 17 textbooks used in *collégial* studies in Quebec from 1993 to 2008 and our main conclusions can be summarised in four main results:

R1: Series are usually introduced through organisations that do not lead students to question their application or importance (*raison d'être*).

R2: Organisations tend to introduce series as a tool to later introduce functional series, but the inherent importance of series is not usually discussed.

R3: These organisations tend to ignore some of the main difficulties in learning series identified by research.

R4: The vast majority of tasks concerning series are related to the application of convergence criteria, or to the application of algorithmic procedures.

Having identified how *collégial* textbooks introduce series, the next stage of the research consisted of analysing *collégial* teachers' practices and their use of textbooks (González-Martín, 2010). In this paper, we discuss how *collégial* teachers use and view their textbooks, specifically in relation to the topic of series.

THEORETICAL FRAMEWORK

We are interested in two main issues: determining how *collégial* teachers use their textbooks, and defining the relationship between teachers and textbooks.

To study how teachers interact with a range of resources, and how these interactions are central to their professional activity, we followed the documentational approach (Gueudet, 2014; Gueudet, Buteau, Mesa, & Misfeldt, 2014). In this approach, a resource is anything that can possibly intervene with the activity of a subject, including artefacts or even a discussion with a colleague. In the case of teachers, they may select, combine, and design their own resources. They may use resources in class, modify them (on the spot or afterwards), or share them. All this constitutes the teacher's *documentation work* (Gueudet et al., p. 142) and, as a consequence, the teacher develops a structured *documentation system*. Teachers, through their use of given resources in pursuit of a teaching objective, develop a *document*: "a mixed entity, associating resources and utilization schemes of these resources" (Gueudet, 2014, p. 2336); this process is called a *documentational genesis*.

The knowledge involved in developing schemes is professional knowledge and can concern a given resource ('this exercise is a good one to trigger an interesting discussion to start this chapter') or the mathematical content to be taught ('I have to work on the idea of slope of functions before introducing derivatives'). The *resource system* is the part of the *documentation system* that refers only to the resources used. This approach therefore considers the professional activity of a teacher in its entirety, both in and out of class.

All the processes studied through the documentational approach are developed in an institutional environment, which establishes (and sometimes imposes) a set of conditions and constraints. Chevallard's anthropological theory provides tools that allow a better understanding of choices made by an institution in organising the teaching of mathematical concepts, as well as the possible consequences of these choices on an individual's practices. A fundamental aspect of this theory is the notion of *institution*. An institution I is defined as a social organisation that allows, and imposes on its *subjects* (every person x who occupies any of the possible positions p offered by I), the development of *ways of doing and of thinking proper to* I (Chevallard, 2003, p. 82). For instance, a classroom is an *institution* (with two main positions: teacher and student), as is a school (consisting of several more positions: teachers of various disciplines, students in different grades, the principal, course coordinators, etc.), or an educational system.

To analyse how an *institution* approaches notions, further definitions are required. An *object* is any entity, material or immaterial, that exists for at least one individual; in particular, any intentional product of human activity is an *object*. Every *subject* x has a *personal relationship* with any object o, denoted as $R(x, o)$, as a product of all the interactions that x can have with o (using it, manipulating it, speaking of it, etc.). This *personal relationship* is created, or modified, by coming in contact with o as it is presented in different *institutions* I, where x occupies a given position p. From this *personal relationship*, an individual will be endowed with what could be designated as 'knowledge', 'know-how', 'conceptions', 'competencies', 'mastery', 'mental images', 'representations' and 'attitudes' (Chevallard, 1989, p. 227).

This notion of *relationship* is also applicable to *institutions*: given an object o, an *institution* I, and a position p in I, we define the *institutional relationship* with o in position p, $R_I(p, o)$, as the relationship with the object o, which should ideally be that of the subjects in position p within I (Chevallard, 2003, p. 82); this is, 'what is done with o within I' (Chevallard, 1989, p. 213) for any *subject* in position p. By becoming a *subject* of I in position p, an individual x is subjected to the *institutional relationships* $R_I(p, o)$, which in turn will re-model his or her own *personal relationships*. For our research, we consider as institution the system of mathematics teaching at the *collégial* level (MTCL). The *institutional relationship* of MTCL with series is mainly determined by official programmes and by textbooks that develop the contents required by these programmes.

In the case of teachers, their *personal relationships* will be affected by the *institutional relationships*, which impose constraints on what to teach and how to teach it (for instance, through textbooks). This *personal relationship* can be seen as an element of the schemes developed as a part of a *document*. In this sense, the *document* is constructed by taking into account a number of resources, as well as the *personal relationship* of the teacher with the topic being taught, which has a strong influence on his or her view of what should be taught and how this should be done, and guides the teacher in selecting which resources to use.

We can now state the main objectives of the research presented in this paper. We are interested in: 1) analysing *collégial* teachers' *personal relationship* with series and seeing how it relates to the *institutional relationship* promoted by textbooks; 2) analysing *collégial* teachers' *documentation work* concerning textbook use in preparing for the teaching of series.

METHODOLOGY

The research reported here is a part of a larger project aiming at identifying *collégial* teachers' practices regarding series (González-Martín, 2010), guided mainly by our results on the introduction of series in textbooks (González-Martín et al., 2011). To achieve the objectives of this larger project, and guided by the results of our analysis of textbooks, we constructed a protocol for semi-structured interviews. These included questions about the textbook used by the teachers, their opinion on the adequacy of this textbook for the students and for the teacher, the number and type of different representations used, the number and type of examples and applications used to teach series, their opinions on the most important tasks for students to perform during the learning of series, and their awareness of the main difficulties in learning series, among others. For this paper, we focus on questions concerning the textbook and its use as a resource in preparing lessons about series. The interviews were conducted from June to November 2009, with an average length of 45 minutes. They were videotaped and later transcribed for further analysis. Once transcribed, the data was organised into clusters of different topics, with special attention paid to keywords that could serve as indicators of the teachers' *personal relationship* with series and of their approach to preparing and organising their textbook use.

In order to cover a wider variety of practices for the teaching of series, we selected five teachers from various *collégial* establishments in Montreal, the biggest city in the province of Quebec. We thereby avoided interviewing teachers working in the same establishment, who tend to organise their teaching in similar fashions. These teachers, designated T1, T2, T3, T4, T5, had varying levels of teaching experience at the *collégial* level and varying levels of experience teaching series (Table 1).

It is important to clarify that because we were interested in the use and role of the textbook within the *resource system* of our teachers, we did not collect all the resources of the teachers (see Gueudet, 2014); we had previously analysed these teachers' textbook (which shared the characteristics *R1* to *R4* described in the Introduction) and inquired about their vision and use of it. The study of the *resource system* as a whole will be the focus of future research. In the following section, we present the main results derived from the interviews.

SOME RESULTS

Coincidentally, at the time of the interview our five teachers were using the same textbook for their courses. This textbook was part of our earlier study's sample, and, as mentioned above, had been analysed by us previously. It displays the general characteristics *R1* to *R4* of the textbooks within the sample. The teachers' reasons for choosing it were varied (all quotes have been translated from the original French):

T1: I find that it is better than the others. [...] All textbooks are basically similar, but the order and the way they present content, I think that this one is good.

T2: Because two teachers from here wrote it [...].

	T1	T2	T3	T4	T5
Experience teaching in *collégial*	5 years	20 years (mathematics and informatics)	32 years	6 years	7 years
Experience teaching series	5 years	4 years	More than 20 years	4 years	2 years

Table 1: Teachers' experience at the *collégial* level and experience with teaching series

T3: [...] for many reasons. The simplest reason is that [it] covers the course content so it's a good work tool [for] students [and] the teacher.

T4: [...] I found that it was better where integration techniques are concerned. Especially the way the exercises are grouped together. [...] At the beginning, the drill exercises are grouped together so that one can associate a concept with many examples [...] So, I find that it is an appropriate learning sequence. [...] I believe a lot in drill exercises. I try to create a balance, but there are many textbooks that I do not like [...].

T5: Well, it's the one that was used before I started teaching... [...] it is good, it has theory, [...] it has a lot of exercises... it's [...] good enough for the students because [...] it helps the student a lot, [it also helps] the teacher because... it's what we're asked to teach. Meaning that, the parts... I mean, the proofs... [...] here they're not in the textbooks [...] so in that way we know that we don't have to teach those proofs... and [...] it's a well enough organised book... [...] Yes, we use the textbook, but the teachers' syllabi are also useful in order to know how... [...] which is the best way to introduce a concept.

We see here the different purposes of textbook use identified by Mesa and Griffiths (2012): generating the syllabus, preparing classes, and designing homework (when they speak about the exercises). However, the most prevalent use seems to be *offloading,* with the exception of cases where some portions are removed with no alternative information added (which we do not consider to be *adapting*). T2 and T5 seem to be subject to particular restrictions, indicating that they were not entirely free to select the textbook ('*two teachers from here wrote it*' and '*it's the one that was used before I started teaching*'), although T5 adds information concerning his opinion on content and structure. We can observe a phenomenon that was also detected by Mesa and Griffiths: the textbook seems to be perceived as written *for* the student. Moreover, aside from the syllabus, the *resource system* appears, so far, reduced to the use of a single textbook, which suggests that professional knowledge is underdeveloped with regard to the teaching of series in MTCL. We did not see any of the teachers use more than one textbook in preparing their courses.

We were able to confirm later that the most prevalent way of using the textbook was through *offloading*. When asked whether they follow the order of the textbook when they teach series, T1 and T4 said yes (T4 said '*I think so*'), whereas T2, T3 and T5 (transcribed in his previous response) indicated that they '*almost*' do:

T3: Certain sections are set aside and left out, but on the whole we almost follow the order.

T4: Yes, but the textbook contains much more than the course content. So you have to make a selection. For example, the convergence criteria in the textbook — almost 10 criteria are presented. In fact, we only have time to get the students to master perhaps half of the criteria.

In each case, it seems that the teacher's *personal relationship* with series does not diverge significantly from what is presented in the textbooks; the only concern seems to be that there is more material in the textbooks than necessary, which results in some sections being set aside. It is possible that this similarity between their *personal relationship* with series and the *institutional relationship* transmitted by the textbooks is at the origin of the teacher's decision not to enrich their *documentation system* with different sources. This could lead the teachers to develop schemes for the teaching of series that privilege the presentation of routine tasks. This became more evident when we asked them whether they thought the textbook they use meets the needs of the students and the instructor:

T1: Needs of the instructor, yes... I think that it's complete enough when it comes to sequences and series... [...] Meeting the students' needs, well, that's the eternal question of why do we teach that [...].

T2: Hmm, we are currently re-evaluating what we are doing about sequences and series [...] I would say that for the moment, there are not any textbooks that really correspond to what we would like to do. Especially because we are not certain yet of what we would like to do [...]

As long as we don't know what we want to do, we will not find the appropriate textbook…

T3: The one that we use, yes. It's not the only one […]. But that one, yes.

T4: The one that I use does, yes.

T5: Hmm… yes, I would say for the teacher, yes… hmm… for the student […] well […] we do not have a lot of time to make the student practise. They are the ones who have to practise. So the textbooks… […] there are no parts where they can do activities or experiment. Even the teacher does not have a lot of time to experiment with students on the concept of series and organise activities for them. So, time is short for that… and the students… all they have to do is practise with the exercises […]

These teachers seem to see the textbook as being *for* the student and they suggest that it meets students' needs in their learning of series. T2 was the only teacher who seems to question how series should be taught (or *what* should be taught), but he ultimately bends to the pressure of the *institutional relationship* and uses the textbook exclusively. On the other hand, T1 also raises the fact that the reason for learning series may not be clear to students, but he does not seem to *adapt* the textbook and provide extra information to clarify this issue. It seems that the weight of the *institutional relationship* with series deters him from taking initiative and using other resources to enrich his *documentation system* and address these issues.

The teachers' adherence to the presentation and organisation of the textbook seems absolute, and they generally insist on the importance of performing the exercises, which was identified as the most helpful textbook feature by Mesa and Griffiths (2012). In fact, the tasks that the teachers consider to be most crucial in learning series were all taken from the textbook (González-Martín, 2010). As in Mesa's and Griffiths's research, no teacher identified the textbook as a tool from which they could learn or that could help them decide how to sequence the topics.

FINAL REMARKS

We cannot present more excerpts from the interviews here, which would allow us to gain better insight into the use of the textbook by our teachers and paint a clearer picture of their *personal relationship* with series and their apparent compliance with the *institutional relationship*. This will be the subject of future papers.

However, the data presented here allows us to observe a kind of *yielding* to the institution's approach to introducing series (through textbooks). There appears to be a similarity between the teachers' *personal relationship* and the *institutional relationship*, in spite of the weaknesses on the way series are introduced by textbooks (González-Martín et al., 2011). Perhaps the fact that the teachers' *resource system* seems so restricted can be explained largely by this *yielding*: our teachers may not see any reason to seek out complementary resources in preparing their lessons. Our data indicates that the *resource system* of our teachers is by and large reduced to the use of a single textbook that the teachers do not question, and the choice of this single resource as core of the *resource system* appears to be guided by the *personal relationship* of these teachers with series, which appears to be quite close to the *institutional relationship* transmitted in the textbooks.

In general, our results echo those of Mesa and Griffiths (2012). However, textbook use at this school level seems to be more restricted to *offloading* (which in this case includes using material from the textbook while setting some parts aside). *Offloading* is the primary way textbooks were used by our interview subjects. Also, textbook use is quite prevalent, which is common in *collégial* studies in Quebec although it diminishes at the university level. In spite of their varied experience, the instructors relied on the textbook to an equal degree, and in this way they seemed to differ from the tertiary instructors interviewed by Mesa and Griffiths.

The conclusions of the study conducted by Mesa and Griffiths, as well as our own research, reveal a number of important tendencies related to textbook use that may have a strong impact on students' learning at the post-secondary level, especially in light of research results on post-secondary textbook content. We are aware that our sample is quite small, and that we only focus on a specific mathematical topic; however, the similarities between our and Mesa's and Griffiths's results lead us to identify some possible concerns. The need for more research on textbook content and textbook use at the post-secondary level is therefore crucial.

ACKNOWLEDGEMENT

This project has been funded by the *Fonds Québécois de Recherche Société et Culture* (FQRSC – 128796) and by the *Social Sciences and Humanities Research Council* (SSHRC – 195873).

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Media and milieus for complex numbers: An experiment with Maple based text

Niels Grønbæk[1] *and Carl Winsløw*[2]

1 University of Copenhagen, Department of Mathematical Sciences, Copenhagen, Denmark, gronbaek@math.ku.dk

2 University of Copenhagen, Department of Science Education, Copenhagen, Denmark

Based on the notion of institutionally conditioned relationship to an organisation of knowledge and practice, we present our first design and implementation of a self-study module on complex numbers in an introductory mathematics course for non-mathematics majors. The basic idea is to develop an "interactive text" in the computer algebra system "Maple", designed to create a reasonably self-sustaining dialectics of media and milieus for students to learn about and work with complex numbers. We discuss some of the obstacles and constraints met in a first implementation of such a text, as well as hypothesis for future implementations.

Keywords: Media-milieus, Maple, self-study, interactive text.

INTRODUCTION

Mathematics departments and auditoria are among the last strongholds for chalkboards. Sfard (2014) reflects on the reasons why lectures continue to be a common teaching format in undergraduate mathematics education and hypothesizes that "watching a mathematician in action and imitating his moves while also trying to figure out the reasons for the strange things he is doing may be the only way to come to grips with [*mathematical objects*]" (p. 202). She also points out that this "coming to grips" can be initiated but by no means accomplished through watching; it requires solitary "thinking" (or "self-communication", in the terminology of Sfard). However, the importance of individual study in advanced mathematics is indeed upstream our current "world of incessant chatter, where everybody talks to everybody else and where educators preach collaborative learning" (p. 202).

This necessity of individual work could, in turn, be related to and in part explained by the primacy of *written discourse* in post-elementary mathematics, a phenomenon which appears already in the context of school algebra and which breaks with the usual role of written text as merely a formalized version of spoken language:

> It is important to notice that in the algebraic symbol manipulations, this relationship between oral and written work is reversed: writing comes first and orality is just a "secondary" accompaniment of the written algebraic formulations, which are furthermore not always easy to "oralize". Contrary to our mental habits, written algebraic symbolism is not a derivation of oral language: it is the source, the manifestation and the touchstone of algebraic "thinking". (Bosch, 2012, p. 7)

Certainly, the primacy of individual written work is fully compatible with the ostensive functions of chalkboard lectures and also with other common formats of university teaching such as exercise tutorials. It does not imply that other more interactive forms of teaching cannot be very useful. But given the crucial role of individual writing in advanced mathematics, the proliferation of online courses, and the increasing pressure on universities to deliver cost effective teaching, the following question is increasingly urgent for university mathematics education: to what extent can cost-intensive face-to-face teaching be dispensed with? What kinds of "real time" interaction with teachers and fellow students are necessary, if any?

These questions arose in a very practical and rather abrupt form for the first author, as he prepared to teach a first year course on calculus for a mixed public of science students: due to a mismatch of administrative rules and the calendar around Christmas, the course had to do with 8 instead of the usual 9 weeks of teaching. Given the density of topics to be covered in the course, spanning from review of secondary level calculus over linear differential equations to surface and multiple integrals, the decision was made to select one week of the normal teaching programme, in which complex numbers and basic complex functions are introduced – and replace it with "self-study". That did not mean that complex numbers should be any less part of the course as they continued to appear in other topics, in exam questions etc. It also did not mean that no support was to be given for students work with complex numbers; only would normal teaching time not be set aside for it, neither in lectures taught by the first author, or in class sessions taught by instructors.

Our basic idea for dealing with this situation was to create an "interactive" text based on the computer algebra system *Maple 17*. In this paper, we present our first designs and observations, as well as a theoretical framework, which helps to shape our work with the task and to situate it in the wider problem area, which was outlined in the first paragraphs above. We stress that this is on-going work, and we present neither a fully developed design, nor a systematic empirical study of its functioning. Thus, the points of the paper are, mainly, a set of theoretically sharpened design ideas.

THEORETICAL BACKGROUND: MEDIA AND MILIEUS

To render the quandaries outlined in the introduction more precise, we have found it useful to model them within the framework of the anthropological theory of the didactic (see, for instance, Chevallard, 1999; Winsløw, 2011, for more detailed introductions). In the most general formulation, we consider an organization O of mathematical practice and knowledge, a generic student x within an institution I, and we are interested in the conditions and constraints for establishing a given relationship, denoted $R_I(x,O)$, between x and O. Among the most immediate conditions are the *media* and *milieus* (cf. Chevallard, 2009) made available by the institution to establish and develop $R_I(x,O)$. According to the original definitions,

> The word *media* designates any system of representation of a part of the natural or social world in view of a certain public: the lecture of a professor of mathematics, a treatise on chemistry, a televised news programme, a regional or national newspaper, an Internet site, etc., are in this sense media systems. A *milieu* is understood in a sense close to that of *adidactic* in the theory of didactic situations. In fact, we designate as *milieu* any system that can be regarded as *devoid of intention* in terms of the answers it can bring, implicitly or explicitly, to a given question. By contrast (...) media are in general motivated by certain intentions, for instance, the intention to "inform". Naturally, a media can also, with regard to some particular question, be considered a milieu and used as such. (Chevallard, 2009, p. 344).

Medias and milieus in the university context

Adidactical situations in the university context do not always take the form of teacher initiated student work within a classroom setting, as in the primary school settings investigated by Brousseau (1997). The individual work of students referred to in the introduction must be considered an important form of *adidactical situation* if this notion should apply to central learning situations in the university setting. Thus, when students study a textbook and encounter an inference they do not follow, they are supposed to consider the text as a milieu that resources and constrains their efforts to fill in the gap. Similarly, an exercise to be solved is part of a milieu to which the students can apply and adapt their own relevant knowledge, and thus develop their relationship with the mathematical organizations to be acquired.

Consider a generic student x of the mathematics course *MatIntro* at the University of Copenhagen (I) whose relation to the mathematical practices and theories related to complex numbers ($O_\mathbb{C}$) we wish to study, and which of course have to be specified further. Without further assumption about x, we may have reason to stipulate a number of conditions and constraints related to x, in view of developing $R_I(x,O_\mathbb{C})$ according to the aims of I; in particular, we may assume that the students has established certain relations $R_I(x,O_i)$ to other mathematical organizations O_i in some way related to $O_\mathbb{C}$, as a result of being a student in secondary school (I') and university (I). In the example, such assumptions concern the detailed practice and theory previously encountered by students in relation to relevant domain such as arithmetic and basic algebra, including polynomial equations, vectors in the plane and exponential functions. Such assumptions can be based on a study of media and milieus through which many students are likely to have established their relation to O_i – for instance textbooks (media), tasks devolved to students (part of milieu) and computer algebra systems (which can function as kind of a milieu for students).

Chevallard (2009, p. 345) insists that "the existence of a vigorous (and rigorous) dialectics between media and milieus is a crucial condition to avoid that the study process is reduced to an uncritical copying of elements of answers which are scattered in institutions and society". In fact, even in the situation of lectures referred to in the introduction, this dialectics is possible: while the lecturer is of course, basically, acting as a medium, he may use the blackboard to create a milieu and let the students observe how he interacts with it, as when he says: "Let's see what happens if we replace z by $re^{i\theta}$ in these identities", and subsequently calculates in *real time* (with the possibility of committing errors, of volunteering students contributing at least orally, etc.). This way, students may observe "the mathematician in action" against a milieu, and students should try to follow and even anticipate the moves, which bring about the solution to whatever problem is at stake – but in the secure position of individual "thinking", without being exposed to the responsibility of completing all required actions in public, as the lecturer.

Even in the most traditional university teaching of mathematics, one can find rigorous systems of media and milieus. The main problem is that of their being vigorous: to what extent do students develop a critical and autonomous relationship to the "answers" found in the media proposed by the institution? Do students interact with milieus in which these answers are related to meaningful questions?

CONTEXT AND DESIGN

We now return to the somewhat special task of "teaching" first year students the basics of complex numbers without disposing of any regular teaching time. Focusing on this context as a case, the aim of this paper is to discuss design principles for creating a *rigorous and vigorous system of media and milieus for students' self-study*, using a computer algebra system, i.e. a mono-media rather than a multi-media design. And, since *vigour* is also an empirical quality, we end by a few observations of the first experiments with our design. We first specify some conditions and constraints, which the design was based on.

Conditions and constraints from the context

The course *MatIntro* caters to a number of different BSc study programmes in science, including pure and applied mathematics, and its contents are thus the result of adapting to the needs of these. The reasons for sharing this and other first year courses among study programmes are in part financial, but also the flexibility it gives for students to change programme without having to take new basic courses. On the other hand, this arrangement leaves little room for the teachers in charge to change the contents of the course. The "self-study" material developed for complex numbers thus had to be relatively neutral in relation to its specific use in science disciplines.

The design was experimented in a run of a version of the course with about 250 students from biochemistry, chemistry and nano-science. There was no face-to-face teaching assigned to the design, and only very limited human resources available for interacting in other ways with the students, for instance providing feedback to exercise work. The only exception was what the lecturer (the first author) could use his own time during the course period. The design should thus provide students with media, which they could access on their own, and milieus in which they could both acquire and validate adequate relationships to the subject.

Another important condition for the solution to be developed is that *MatIntro* uses the advanced computer algebra system (CAS) Maple substantially, including weekly "labs". Maple functions as a learning resource in the course and as a tool supplementing and enhancing paper and pencil techniques. The focus is on insight rather than on computing results. The course grade is based on weekly hand-ins requiring Maple and two multiple-choice tests, in which no electronic tools are allowed.

As with many other CAS, work in *Maple* takes place in a window where input and output appear in consecutive lines, much as in a word processor window with the crucial exception that the latter has *only* input. The contents of the window can be saved as a file (called a *Maple sheet*), for later use and development. Students less familiar with *Maple* often use *Maple* sheets found on the Internet (including the manufacturer's own support pages) in order to get ideas for how to solve a given problem. In *MatIntro* and similar courses, teachers regularly publish Maple sheets on the course web page, to demonstrate *Maple* techniques and other mathematical points to students. Such sheets are typically short and sketchy, and the idea is that students may use them (or parts of them) in their work with related tasks.

Design

The basic idea of our design was to create an interactive introduction to complex numbers as a set of Maple sheets to be used for self-study by the students. 'Interactive' means that the document serves both as media and milieu and 'self-study' means that Sfard's 'observing a mathematician in action' (e.g., in a lecture) is replaced by 'observing Maple in action' through embedded input-output turns.

The choice of a course theme (complex numbers) for self-study was motivated by two considerations. Firstly, this subject is somewhat isolated in the course syllabus, which makes the experiment less risky for the course as such. Secondly, it seems particularly suited to exhibit *Maple in action*. The course goals focus on a small number of connected techniques within complex arithmetic, algebra of low degree polynomials, and basic properties of the complex exponential function. These are linked only to relatively simple Maple routines; moreover, complex numbers is a new area to most students and as already said, is relatively isolated in the course. So it could be conceivably developed as a kind of "Maple world".

Traditional use of Maple in teaching is as a *milieu* in which certain computations, visualisations and experiments can be performed using an input-output scheme. This milieu becomes operational only as a supplement to media and other elements of milieus that students interact with during lectures, exercise solving etc. The students must navigate in this juxtaposition of different media and milieus to build a set of relationships $R_I(x,O)$ as demanded from I. The common formats of teaching gives various support to the student, and in particular the 'mathematician in action' is not just part of a media system for mathematical content but also demonstrates how to use milieus (which could be Maple experiments) to take on various concrete tasks. If "live" (blackboard or otherwise) manipulations of mathematical objects are to be replaced by a simple Maple sheet, it must reproduce some of the same experience of purposeful, reasoned instruction, complemented with work in explorative milieus. One added potential (difficult to realize!) is that students could take on a more active role as compared with the common situation in lectures.

In our design, we used Maple to create a media-milieu dialectics in three ways: as a generator of *dynamic text* (media with embedded milieu), as a generator of *drills for techniques to solve specific tasks* with *feedback to students' solution proposals* (milieu with embedded media), and to *explore and exhibit phenomena* (media with embedded milieu). In order to explain these options, we name the main components of a Maple sheet:

- *Maple text*: non-executable text, essentially as in a word processor;
- *Maple input* : executable text;
- *Maple output* : results of executions;
- *Maple Components*: programmable interactive components (code is not shown) including sliders, buttons, Math Containers for in- and output, and other items.

The user or author of a sheet may hide or show Maple input, to produce different appearances of the worksheet. With the input hidden, the worksheet may read as a simple textbook (media) with its mixture of text, formulae, calculations and graphics; it is invisible that the latter are in fact Maple output. When the Maple input is revealed one sees the embedded milieu, which can be acted on.

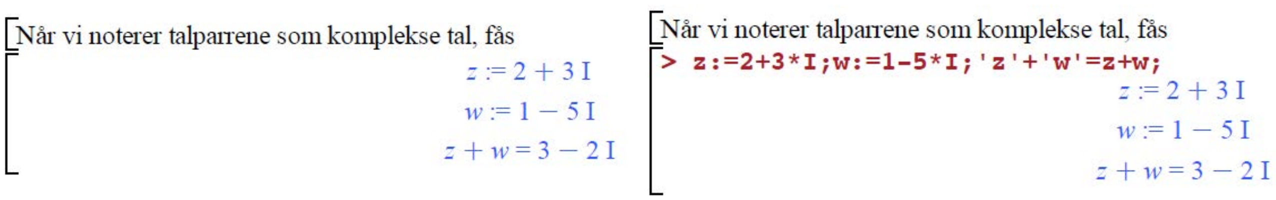

Figure 1: An excerpt of a Maple sheet with text, input (hidden in left version) and output

Here is a simple example (cf. Figure 1 which shows a short excerpt of an introductory Maple sheet for the course): if we have defined two complex numbers by `>z:=2+3*I;w:=1-5*I;` a statement of their sum should be coded `>'z'+'w'=z+w;` rather than `>'z'+'w'=3-2*I;`. In both cases the output as part of the text reads $z+w=3-2\,I$, but the first option will preserve the correctness of the second part, even if changes are effected in the first. Thus, '`=z+w;`' is "Maple in action", while '`=3-2*I;`' is just communicating the sheet author's knowledge.

The example illustrates how media-milieu dialectics is used explicitly in the design. When Maple is the primary source of mathematical content the Maple in- and outputs are a-didactical and the dialectics is rigorous so that the user may act in this milieu (here, mainly re-assign *z* and *w* or change the operation). Conversely, no non-Maple manipulation, such as '`=3-2*I;`' can be negotiated by the user and therefore may lead to imitation rather than construction of knowledge.

Milieus are embedded more explicitly into the sheet by using Maple applets. The most basic way is a *drill*, aimed at education a specific technique. Computation of reciprocals of complex numbers serves as an example. It consists of six components (see Figure 2): a button "new number", an output container exhibiting a complex number, an input container to be filled by the student, an erase button, a true-false button, and an output container displaying either "true" or "false".

When "new number" is clicked, Maple exhibits a complex number randomly chosen from a pre-set range (with more than 400 different numbers). Thus Maple serves two purposes. It generates a variation of tasks of identical type and provides feedback to the student's performance with a technique to solve the task.

Math containers are also used to let the students investigate phenomena from a more theoretical point. The geometry of complex multiplication by a real factor is explored by means of a "new number" button, a slider to choose the real factor and a plot container exhibiting the geometric effect of multiplication. Similar milieus are offered to explore the geometric meaning of multiplication by purely imaginary factors and the general case. The student is asked to insert personal descriptions in terms of modulus and argument in the text (for this, no feedback is available). The full Maple material can be consulted at the web address http://www.math.ku.dk/kurser/2013-14/blok2/matintrokem/selvstudium/, where the reader may explore the examples mentioned above, and more.

The functioning of the sheets produced so far is incomplete in at least two respects. The plainest is that feedback, which can be produced with Maple, is very limited and (as with most software) merciless on syntax and other formal errors, which are naturally common for beginners. Also, institutionalization (in the sense of Brousseau, 1997, p. 215) is independent of what students have actually achieved in a milieu.

To address these problems in part a discussion forum dedicated to the self-study part of the course was set up in the on-line platform of the course. Here students and teachers could post and answer any question or comment. The intention was that teachers monitored the discussion forum in a stipulated weekly time slot, and students were promised answers and rectifications to questions and problems, if possible immediately but else within a deadline.

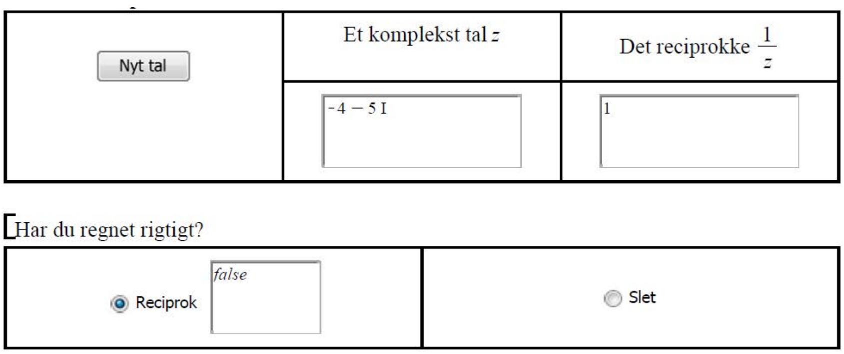

Figure 2: A simple drill applet to work with a technique for taking reciprocals

SOME OBSERVATIONS FROM A FIRST EXPERIMENTATION

Students were free to choose when to work with the material, and to work individually or with others. In their rather tight schedule only Friday afternoons were available to all students. Collaboration was encouraged in this slot. We reserved three afternoons for the monitoring of the discussion forum, corresponding to the three sub-modules into which the Maple sheets for the full self-study, corresponding to one week of teaching, were organized: (i) arithmetic and geometry of the complex plane, (ii) the quadratic equation, the exponential function, (iii) review of main points. Very few used the Friday slots and the discussion forum was practically unused (6 questions altogether). We have no solid evidence of the reason for lack of use of the discussion forum, but from focus group interviews, it appears that students preferred to confer with fellow students and instructor, as well as more informal channels such as *Facebook*.

A focus group test and interview

Following the first module a focus group interview was set up with four volunteer students commencing with four written tasks very similar to the drills of the first module sheets. Having collected the students' individual solutions to these tasks, we asked a number of questions about their experience with the sheets. The students reported to have worked with the sheets, in a combination of individual work and conferences with study groups. None reported difficulties; some even said the exercises were too easy. However, our written tasks suggested severe difficulties; for instance, not one of them had been able to compute $(1+2I)(3-I)$, and in at least two of the cases, this appeared to be related to an inappropriate mastery of the distributive law. From this experience at least two observations can be made.

The first is methodological: what students *say* can be strongly misleading as to what they are actually able to do. The reasons could be reluctance to admit difficulties to fellow students and lecturers as well as self-deception. The latter might in part be due to insufficient feedback from the sheets (in fact, one student noted that the applets may only let you know *that* you are wrong, not why).

The second point is specific to students' insufficient relationship with the distributive law (and other organisations of knowledge in the borderland between arithmetic and algebra). Whether or not it is rooted in CAS-use in secondary school, it becomes an obstacle to using the theoretical definition of operations in the setting of complex numbers. We had not anticipated such obstacles in our design.

Students' results

The most important measure of $R_I(x,O)$, from the viewpoint of students and institution (the university) alike, is that of summative assessments. In *MatIntro* naturally some of the items of the multiple choice tests are on complex numbers. Each test had one such task: in test 1, to calculate $\frac{(2+3I)(1-I)}{1+I}$; in test 2 to find a polynomial with roots $1+7I$ and $1-7I$. Texts and notes, but no computers, were allowed at the test. Students performed roughly 20% less well on these items than on the tests as a whole, but this is insignificant as the relative difficulty of items cannot be assumed to be uniform. However, the 250 students were organized in nine classes (for exercise sessions with instructors), and we observed significant variations among classes as concerns the ratio of *class average score on complex numbers items* and *average score on all items*. We suspected these could be related to class instructors' own initiatives to include complex number tasks in their teaching, even if they were not asked to. So, a survey on this was sent to them after the completion of the course. Indeed, it turned out that one instructor (whose class performed very well on complex numbers) had written and used a hand-out on complex numbers, based on the Maple sheets and providing an overview of most formulae and results on complex numbers to know about; these were also used by the instructors of a few other well performing classes. One could thus suspect that parts of the variations could be ascribed to these initiatives and the differences of focus. We do not have firm evidence to support our hypothesis that test oriented teaching caused the variations observed, but we can confirm that the "self-study of Maple sheets" were, in the end, far from the only source of media and milieus for students to learn about complex numbers.

Students' opinions

Most of our other evidence is gathered indirectly on students' impressions and opinions. These are in particular expressed in the anonymous on-line evaluation of the course, which is usually not very positive with students from the study programmes concerned. In fact, 71 respondents give mostly negative comments about the course as a whole, and also about the self-study (complex number) part. They criticize the lack of feedback, and some say directly that there should be accompanying lectures or videos of lectures. The idea of drills (built into the Maple sheets) seems, however, well accepted by those who mention them. Regular meetings between the lecturer and student representatives confirm these trends.

PERSPECTIVES AND FUTURE EXPERIMENT

The experiment with an "island" of self-study was imposed from the outside as an unexpected addition to the other constraints on this tightly packed introductory course for non-mathematics majors. The reaction of students and teachers alike seem to suggest that a satisfactory form of self-study is very hard to realize in these circumstances. In the absence of familiar teaching formats, they tend to replace a coherent organisation $O_{\mathbb{C}}$ of theory and practice related to complex numbers (as developed in the Maple sheets and foreseen in the course description) with a rather minimal set $o_{\mathbb{C}}$ of disconnected practices which appear in the most rigorously administered parts of the summative assessment (the multiple choice tests). This way, the institution as a whole seems to develop $R_I(x,o_{\mathbb{C}})$ rather than $R_I(x,O_{\mathbb{C}})$, as its way to measure $R_I(x,O_{\mathbb{C}})$ is in fact rather a minimal measure of $R_I(x,o_{\mathbb{C}})$.

One tempting way to proceed (under similar conditions) may thus be to accept that a generic student x in this institutional context I cannot be expected to develop more than $R_I(x,o_{\mathbb{C}})$. The design could be adapted to this situation by replacing most of the text with succinct expositions of task types and techniques, enriched with many more interactive drill items, examples and warnings on typical errors, and so on. This kind of approach seems to be endorsed, at least in deed, by many students and instructors.

This, however, would not be acceptable from the institutional point of view, for the reasons outlined in our exposition of "Conditions and constraints from the context". One could then try to pursue the (quite plausible) claim that the course is much too heavily packed with content, given the course time allotted. This makes the course very vulnerable to incidences as the one we described (one week disappearing), and one may also suspect that whether one accept it or not, outcomes of type $R_I(x,o_{\mathbb{C}})$ rather than $R_I(x,O_{\mathbb{C}})$ will be common unless the size of the total content organisation is reduced and adjusted more exactly to the students' programme. This, however, takes us away from the level of course design to the level of institutional politics.

Indeed, we are certain that the designed Maple sheets reflect the course goals adequately. They have been used with good results in two high school classes, in the spring of 2014 (with much more direct instruction on the use the sheets). This only strengthens our hypothesis that the framework (at the level of pedagogy) for their first implementation in *MatIntro* was inadequate, and that an island of unaccompanied "self-study" in such a course is very likely to result in side effects as those observed.

A third way out, which we shall try to pursue in the next run of the course, is therefore to make use of a new concession of the institution, allowing lectures on some of the Friday afternoons. They will be used to introduce the students to the sheets, and in particular how to make use of them (both as media and milieus). The class instructors will be asked to align with these lectures and use the course material rather than self-made cookbooks on how to pass the tests; in turn these latter will be amended to correspond more fully to both practical and theoretical levels of $O_{\mathbb{C}}$.

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University teachers' resources and documentation work

Ghislaine Gueudet

University of Brest, ESPE Bretagne, CREAD, Brest, France, ghislaine.gueudet@espe-bretagne.fr

In this paper, I investigate university teachers' documentation work: their interactions with resources for preparing and delivering their teaching, and the consequences of these interactions. I have interviewed and collected the material resources used by six university teachers working in France. Analysing the data collected, I observe specific features of their documentation systems, concerning crucial resources, professional development, and the place of digital resources.

Keywords: Documentation systems, documentational geneses, professional development, resources.

INTRODUCTION: STUDYING TEACHERS' WORK WITH RESOURCES AT UNIVERSITY

The work presented in this paper belongs to the growing field of research concerning university teachers' practice (e.g., Nardi, Jaworski, & Hegedus, 2005), with a specific feature: I consider the teacher's work in class, but also out-of-class. I investigate indeed university teachers' work with resources, with a documentational approach perspective (Gueudet, Pepin, & Trouche, 2012). Previous steps have been done at CERME8, with a study concerning the detailed case of one teacher, working in a technological institute (Gueudet, 2014). The theoretical aspects of such a study and the specific features from university concerning teachers' documentation work have been deepened in Gueudet, Buteau, Mesa and Misfeldt (2014), within the RME special issue concerning "Institutional, sociocultural and discursive approaches to research in university mathematics education".

I firstly briefly recall the main elements of the theoretical approach. Then I present the methods, organised around the cases of six teachers with different profiles. I present the results obtained by analysing interviews with these teachers and the material resources they use, focusing on the documentation systems structure, on crucial resources, on professional development and on the place of technology.

THEORETICAL FRAME, CONTEXT AND METHODS

I retain in this work the theoretical frame provided by the documentational approach of didactics (Gueudet et al., 2012). According to this approach, teachers interact in their work with a variety of *resources*. The concept of resource is considered here with the meaning introduced by Adler (2000): a resource can be a textbook, but also a symbol, or more generally anything likely to re-source the teacher's professional activity. Teachers look for resources, sometimes they meet resources that they were not looking for (discussing with a colleague around the coffee machine, for example). They associate these resources, modify them, conceive their own resources and use them with students. All this activity is called *the documentation work* of the teacher (Gueudet et al., 2012). During this documentation work, interactions take place between the teacher and the resources; and these interactions contribute to teachers' professional development. Drawing on the instrumental approach (Rabardel, 2002/1995; Guin, Ruthven, & Trouche, 2005), the documentational approach considers that teachers are involved with sets of resources in a goal-oriented activity. Along this activity and for a given goal, they develop a *document*: the association of resources and of a scheme of use (Vergnaud, 1998) of these resources. Schemes of use encompass three ingredients: the objective of the activity; rules of action (a usual way to act for this objective); operational invariants, which are here professional beliefs. This process (development of a document) is called a *documentational genesis*. Multiple documentational geneses occur along the teacher's work for various goals; they contribute to produce the *documentation system* of the teacher, which is the structured set of all the documents he/she develops. With this perspective and concepts, the central research questions that I address in this paper are:

What is the content and structure of university teachers' documentation systems? Which are the evolutions of these systems, and how are these evolutions linked with teachers' professional development?

In Gueudet and colleagues (2014), I have discussed possible specific features of teachers' documentation work, in the context of university. One such feature is the possible central role of the work with resources for teachers' professional development, since in many countries teacher education is limited at university. Such a role has already been identified in the case of textbooks by Mesa & Griffiths (2012), using an instrumental approach: they have identified different schemes of use, shaping the teachers' practice. Moreover, many digital resources are available for university; and the online platforms permit to develop distant work with students. Are these digital resources present in the teachers' documentation systems, where do they intervene? In Gueudet (2014), I have investigated the case of a single teacher, who taught in a technological institute and was not involved in research. He used many technological resources, but he worked in a specific context. The place of technology in the documentation systems of university teachers in more "ordinary" contexts still has to be investigated.

I retain these foci, for the results I present here: the link between documentation systems and professional development, in particular the intervention of research in the development of documents and the place of technology in university teachers' documentation systems.

In this article, I study the cases of six colleagues, all of them working in France in the same middle-size university. These colleagues have been chosen to represent a variety of conditions that can influence their documentation work: experience, research domain, studies in France or abroad, position, gender. Table 1 below summarizes the six cases, according to these factors.

They also teach in a variety of "teaching units", concerning calculus, linear algebra, number theory, probability, numerical analysis in the first or second year of university. A teaching unit lasts 12 weeks and can comprise between four and six hours a week, with generally half of the time for the lectures and half of the time for tutorials. Most of these teaching units concern the "Mathematics, Computer Science, Economy, Electronics" degree (*MIEE*; 270 students in first year, amongst them 60 specialized in mathematics). Some of these teaching units concern all the MIEE students; some concern only one or two options, i.e. "economy and computer science". Most lectures are given in an amphitheatre for a maximum of 150 students. In fact for most teaching units, there is a single amphitheatre, with two exceptions: the first year calculus teaching is organized in small groups, like secondary school classes and in the first year linear algebra teaching, two lecturers work in parallel in different amphitheatres.

I met each of these colleagues for an individual interview (see the interview guidelines in the appendix) and collected all the resources they mentioned. The interviews were recorded and transcribed. I noted in each interview the kind of activities mentioned by the teacher, the resources cited, how many times each of them is cited, and in connection with which activity(ies). I also noted the collective work (with who, for which objective), the beliefs expressed about mathematics or about teaching issues. I then connected teachers' declarations and the content of the resources he/she uses or designs. For space limitation reasons I cannot present here a detailed analysis of each case,

	Experience	*Country of the studies, position*	*Research domain*
Bob (M)	7 years	France, lecturer	Numerical analysis
Doris (F)	17 years	France, lecturer	Symbolic computation
Nadia (F)	24 years	Italy, lecturer	Partial differential equations
Bill (M)	13 years	Germany and UK, lecturer	Geometric theory of groups
Mary (F)	2 years	France, PhD student	Geometric theory of groups
John (M)	1 year	France, PhD student	Spectral theory

Table 1: Profiles of the university teachers interviewed

I only present the main results from the whole group of participants.

RESULTS

General structure of documentation systems for teaching

Documentation systems are individual constructs. A detailed investigation of these systems, for each teacher participated, reveals specific features. Nevertheless here one of my aims is to identify common aspects in the structure and content of these systems. Naturally, a first general description leads to identify two potential systems: one for teaching and one for research. I did not ask the teachers about their resources and documents in the context of their research activity. I focus here only on the documentation system for teaching – but I try at the same time to identify in it documents which can also belong to the documentation system for research.

"Teaching" here refers to a large range of activities, with different aims: preparing a lecture, giving a tutorial, correcting worksheets, answering students' e-mails, etc.

So the documentation system for teaching is itself composed of different subsystems. For the teachers I interviewed and the teaching units concerned, except in special systems which I discuss later, the documentation systems for teaching comprise four subsystems, linked to different activities: (preparing and giving) lectures; tutorials; assessments; and communication.

Starting from this general structure, I first try to identify crucial resources, defined as those that are present in the intersection of several subsystems. These resources intervene in documents for different aims of the activity. As a consequence, their features are linked with schemes and thus beliefs concerning mathematics and their teaching which have an important influence on the teacher's practice.

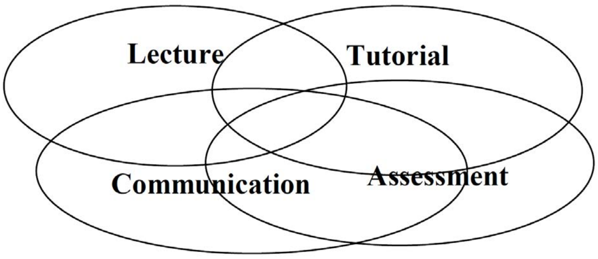

Figure 1: General structure of the documentation systems for teaching

The analysis of the interviews indicates that three kinds of resources appear as crucial: the "polycopie"; the exercises sheets; and the texts of previous exams.

THE CRUCIAL RESOURCES AND CORRESPONDING DOCUMENTS

The "polycopie"

The polycopie is a text, corresponding more of less to the content of the lecture, which is available for the students on the lecturer's webpage as a pdf file, from the beginning of the teaching. Teachers and students at university in France do not use textbooks. This polycopie is used by the lecturer, for the preparation of his/her lecture and by the teachers offering the tutorial, to be informed about the content of the lecture and prepare the tutorial. It also intervenes in the communication between teachers and between teachers and students but not in the preparation of the assessments. The documents developed from the resource polycopie are not the same, when the aim is to prepare a lecture, or to prepare a tutorial. The documents associate indeed resources and schemes. The schemes can incorporate the same beliefs (e.g. "When a usual function is a bijection on a given interval, its reciprocal is also a usual function whose properties must be learned"), with different objectives and rules of action. More importantly, the teachers have different beliefs concerning directly the content and the role of the polycopie. Some of them (Bob, Bill, John) think that students need to have access to the precise text that has been written on the blackboard (the most usual practice in this university, only one colleague amongst the six I interviewed used projects slides). The content of the polycopie should correspond to this content: if a student took incorrect notes, or missed a course, this is not a problem since the polycopie contains the reference text. Others (Doris, Nadia) think that the polycopie must only be a summary of the course, containing the most important results ("for theorems I do not write the proof on the poly", Doris): the students' notes are the reference text for their learning of the course. And one of the teachers I interviewed, Mary, thinks that the polycopie must complement the course by giving additional details, examples, worked exercises ("students must find additional information in the poly"), in particular for high-achieving students. As a result, from one teaching unit to another, the content of the polycopie can be linked to the content of the lecture in different manners.

The exercises list and the previous assessments texts

The exercises lists and previous assessment texts are crucial resources, both for the tutorials and the assessments (and they naturally also intervene in communication). Similarly to the polycopie, the exercises list and the previous assessments texts are given to the students from the beginning of the year, available on the lecturer's webpage. For most teaching units, the exercises list exists for a long time, amongst the teachers I interviewed, only two of them have been involved in the design of such a list. Nevertheless, the teachers develop from this list different documents. For the same objective, these documents encompass different rules of action, linked with different beliefs.

> Nadia: Each year the first thing to do is to investigate who your students are, and adapt the content and expectations to these students *(rule of action).*

Nadia believes that, for the first year students who have a limited mathematical background, it is not necessary to ask for proofs in the tutorials.

> Mary: I always start the tutorial by a summary of the content of the polycopie *(rule of action),* because they do not learn the course before the tutorial, some of them even do not attend the course *(operational invariant).*

Some of these beliefs, present in the documents developed for the tutorial, are linked with the teacher's research activity:

> Nadia: Those who can say – well, I do not know this problem, but I will try to do something with my hands – this is a very important attitude for research.

The preparation of the exam and intermediate assessments texts is always a collective work. It can be shared in the group of teachers for a given teaching unit (in each teaching unit the students take between 2 and 6 exams and intermediate assessments), or proposed by the lecturer(s). This particular documentation work: "writing an assessment text" has been studied by Lebaud (2009). The data confirm the results of the study by Lebaud. The document developed by the six teachers comprises the list of exercises, the previous assessments texts, and beliefs as: "the exercises for the assessment must be similar to those done in the tutorials"; "the assessment must cover all the content of the course". Participating in this collective documentation work contributes to the development by novice teachers of beliefs shared with their colleagues. This issue can be further studied in terms of communities of practice (CoP, Wenger 1998): novice students progressively become member of the CoP of mathematics lecturers, by sharing the same resources and practice.

Professional development and development of the documentation system

The university where the study takes place provides no professional development for the teachers. The progressive integration of novice teachers is managed by giving them gradually increasing responsibilities in the teaching, which I interpret here as: the professional development for teaching of novice university teachers corresponds to a development of the documentation system.

The PhD students I interviewed (similarly to all PhD students in this university) only offer tutorials. This means that they are given the polycopie, the exercises list and previous assessment texts at the beginning of the year. Their responsibility is to choose exercises from the list to work with their students during the tutorial. They could naturally choose other exercises, for the tutorial or for homework, but they do not do it, because the list of exercises is sufficient. Their first opportunity to design themselves exercises is provided by the writing of assessment texts, with other colleagues, as described above. So the professional development corresponds to the development of the documentation system: from a documentation system reduced to the subsystems for tutorials, assessment and communication to a documentation system incorporating also the subsystem for lectures.

In the case of a new lecture, the colleagues declare that they use books. In some cases these books are mathematics books addressed to higher level students: about numerical analysis (Bob), or symbolic computation (Doris), for example. The teacher then makes an important didactical transposition work

Lecture *Polycopie + Books (from higher level, or other countries). Sometimes: lecture notes of a colleague.* *Discussions with students at the end of lectures, evaluation of the teaching unit by students*	**Communication** e-mails sent by students, by colleagues website of the university, of the mathematics department *For a lecturer: Uploading files on his/her webpage to inform the colleagues and the students; discussing the case of specific students with colleagues.*
Tutorials Polycopie+ Exercises sheets + previous assessment texts + e-mails from lecturer + his/her webpage + discussion with the students Students' productions (homework and assessments) + corrections	**Assessment** Exercises sheets + previous control texts + e-mail discussion with colleagues + students productions

Table 2: Evolution of the documentation system: from tutorials to lectures as an enrichment of the documentation system (in italics the lecturer additional part)

(Chevallard, 1985), to produce content accessible for first or second year students.

Digital resources: An increasing, yet limited use

Proposing the polycopie, the exercises sheets, the previous exam texts on his/her professional webpage is done by 3 of the 4 lecturers. The lecturer's webpage is a central resource for the communication from the lecturer to his/her colleagues and his/her students. In general, the use of digital resources for the communication between colleagues, or with the students is a very important evolution, observed by each colleague. Information about the students is available on the university Virtual Learning Environment. Complementary information, such as the official curriculum, students' photos, is provided by the mathematics department website. The discussions within the team of teachers of a given teaching unit, the information on the course progress by the lecturer(s) for the teachers of tutorials are made via e-mail. Some students also write e-mails to teachers, but the in person discussion at the end of the lecture or tutorial remains the main communication mode, between students and teachers.

Nevertheless the use of digital resources remains limited, compared for example with secondary school (Gueudet *et al.*, 2012). The six colleagues declare that they do not search for Internet resources to prepare their courses, considering that they will not find something corresponding to their precise teaching objective. The use of the calculator by students is allowed during tutorials, but forbidden for the assessments. For four out of the six teachers I interviewed, the use of technology seems to be restricted to their documentation system for communication.

I observe a different situation for two colleagues, contributing to two special teaching units: one about symbolic computation (Doris) and the other about numerical analysis (Bob). These teaching units encompass in particular, aside the tutorials, "practical works" in a computer lab. I consider here more precisely the case of Doris. Doris uses Maple for all the aspects of the teaching: lecture, tutorials, practical work – except for the assessment. She has developed different documents involving Maple, and has in particular a strong belief about the link between writing algorithms, programming them, and learning mathematics:

> Doris: They work with algorithms by programming them. This way they can evaluate their efficiency, and see if they are useful.

She uses also Maple in her own research. For this special course, the resources for teaching and the resources for research have a significant intersection: Maple, and associated computer programs. The same situation happens for Bob in numerical analysis, with Scilab. In these cases, the course is designed for mathematics majors, and concerns a topic linked with the teacher's research; there is an associated link between the resource system for teaching and for research.

CONCLUSION AND PERSPECTIVES

The documentation systems of the six university teachers interviewed could be classified in three categories: Lecturer, for a "usual" teaching unit; Lecturer, for a teaching unit involving a specific software; Novice teacher (here PhD student), giving only tutorials in a "usual" teaching unit. The novice teachers' systems encompass only three subsystems: tutorials, assessments, communication; while the lecturers' systems also have a subsystem for lectures, and a more developed communication subsystem (see Table 2). Naturally other kinds of documentation systems exist, for other cases. Nevertheless these six systems have interesting common features and differences.

For all of them, the polycopie, the exercises list, and the previous assessments texts are crucial resources. None of them incorporates resources (exercises, mathematical texts) found on the Internet. This situation is completely different from the documentation work of secondary school teachers in France (Gueudet et al., 2012). Searching for resources on the Internet is a very usual practice in this context, in particular for the choice of "introductory activities" (problem texts, aiming at the introduction of a new concept or method). In fact such introductory activities do not exist, in the teaching at this university. The new concepts and methods are presented in the lecture, and then applied during the tutorials.

Novice teachers progressively develop their documentation systems. They firstly develop documents for the tutorials; then for the assessment, being involved in the collective preparation of assessments' texts. Along this work, they develop schemes which incorporate beliefs shared with their more experienced colleagues. Starting to give lectures is another step, which can lead to the development of more personal beliefs, in particular for a new lecture with no previous polycopie.

In the documentation systems for teaching units involving a specific software, this software is a crucial resource. Otherwise, only technologies for communication intervene, with the lecturer's webpage playing a central role.

Naturally this study remains limited, since I only met six colleagues, all of them working at the same university. Other universities can have different teaching strategies, kinds of resources, different organisations, or local projects that shape the documentation work of teachers. I intend to extend this work, nationally and internationally. Another perspective is also to deepen this work by meeting again the six teachers interviewed, and by observing their courses to confront their actual practice with their declarations.

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APPENDIX: INTERVIEW GUIDELINES

This interview concerns resources (mostly material resources), intervening in your teaching for the first or second university year. Our aim is to understand which resources you use, which resources you design for your students etc.

Years of experience in teaching: Research domain:

1) Let us consider a teaching you did this year, for example « linear algebra in year 1 ». Which resources did you use, and design, for this teaching? For the lectures, if you gave lectures; For the tutorials or practical works; For the preparation of the intermediate assessments and exams texts.

2) About digital resources: do you use a professional webpage, a virtual learning environment, specific software? Do you use online resources to prepare your courses, do you project slides during your courses?

3) About collective work: do you work with colleagues to prepare your teaching? Which kind of work do you make for your teaching with colleagues?

4) For experienced teachers: which evolutions do you retain in the last 10 years, concerning the resources you use and design for your courses? For novice teachers: do intend to modify your teaching next year, how and why?

5) Link with research: are there resources that you use both for your research and for your teaching? Or other links, between your teaching and your research?

6) Did I forget to mention important resources, or something else that you consider important concerning your teaching?

Abstract algebra, mathematical structuralism and semiotics

Thomas Hausberger

University of Montpellier, Montpellier, France, thomas.hausberger@univ-montp2.fr

I report in this paper on my attempt to help students reflect on the axiomatic method and structuralist thinking in mathematics through a didactically-engineered activity (the theory of banquets, an invented structure simpler than group theory, but still quite rich semantically), as a lever to tackle the issue of the learning of abstract algebra. It sheds light into the cognitive processes involved in the conceptualization of an abstract algebraic structure, which are discussed within a semiotics framework. Empirical data show an insufficient syntax-semantic dialectic and mental processes based on the recognition of (visual) patterns.

Keywords: Abstract algebra, mathematical structuralism, didactics and epistemology of mathematics, semiotics, syntax and semantics.

INTRODUCTION

This article focuses on the teaching and learning of abstract algebra (the discipline dedicated to the study of algebraic structures, that is, the investigation of logical consequences of specific systems of axioms involving composition laws, and the relationships among them) which is taught at Montpellier University at the third-year university level. The difficulties are acknowledged by several authors (Leron & Dubinsky, 1995; Nardi, 2000; Hausberger, 2013) and reflect a "transition problem" (Gueudet, 2008) which, in the present case, occurs inside the university curriculum.

The epistemological analysis presented in Hausberger (2013) allowed a connection with the following epistemological transitions: "the systematization of the axiomatic method, after Hilbert, and the transition, after Noether, from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms". Indeed, as emphasized by Cory (2007):

> This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure.

In other words, this epistemological gap leads to the *vanishing of concrete mathematical objects in favor of abstract structures*. This induces the following didactical problems: the teaching of abstract algebra tends to present a *semantic deficiency* regarding mathematical structures, which are defined by abstract axiomatic systems and whose syntactic aspects prevail. How does the learner build an "abstract group concept"? Indeed, what kind of representations can he rely on to do so when the purpose is to discard the particular nature of elements, in other words the mathematical context? Moreover, the investigation of the didactic transposition of the notion of structure shows that it is a *meta-concept* that is never mathematically defined in any course or textbook (and cannot be so):

> As a consequence, students are supposed to learn by themselves and by the examples what is meant by a structure whereas sentences like "a homomorphism is a structure-preserving function" is supposed to help them make sense of a homomorphism (Hausberger, 2013).

As announced in loc. cit., I have engineered an activity for students to reflect on the axiomatic method and structuralist thinking in a simple context (simpler than group theory): the *theory of banquets*, an invented structure. It aims at operating the fundamental *concrete-abstract and syntax-semantic dialectics* (see below) and at clarifying the concept of mathematical structure using the meta lever (Dorier et al., 2000), that is "the use, in teaching, of information or knowledge *about* mathematics. [...]. This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics".

The purpose of this article is to present a few results that were obtained as I experimented with this activity. It tackles the following questions: what kind of cognitive processes and reasoning do students use to make sense of an axiomatically-presented structure such as the banquet structure? How do they engage in the task of *classifying models* of the axiomatic system (and interpret the task: for instance, what kind of representations do they use, and do they formalize a concept of isomorphism of banquets)? What kind of abstract banquet structure concept do they build through the completion of such a task? Similarly as in the context of classical algebra in secondary education, semiotics will give interesting tools to answer these questions. Still, some adaptation needs to be made to reflect the context of abstract algebra, since structures represent a higher level of organization compared to the classical mathematical objects that they formalize, generalize and unify.

EPISTEMOLOGICAL AND DIDACTICAL FRAMEWORKS

Abstraction

The French verb *abstraire* has three different meanings: 1. to discard ("faire abstraction de") 2. to isolate (from a context) 3. to construct (a concept). Although these are three different actions, they may take place in order to reach a common goal as is the case in abstract algebra: mathematicians disregard the particular nature of elements and isolate relations to build the structure as an abstract concept.

The "principle of abstraction" as a process to create concepts has been used by Frege (1884) to define cardinal numbers. To introduce the reader to this revolutionary idea, Frege gives the enlightening example of the direction of a line which is defined as the class of all lines that are parallel to the given line. The principle was formalized later on by Russell (1903): to say that "things are equal because they have some property in common" and reduce a class to a single element, the relation that traduces this property should be symmetric and transitive (an equivalence relation).

Semiotics

Just as language is compulsory to express any idea, mathematical objects are accessed through mathematical signs. Frege's semiotics will be used in this paper, thus making the distinction between the sense and the denotation of a sign (Frege, 1892). The denotation of the sign is the object it refers to whereas the sense is related to the "mode of presentation" of the object. Mathematical signs are often polysemic but the context is meant to determine the reference uniquely. Conversely, different signs may represent the same object, thus having a different sense but a common denotation. In this way, different representations may bring to light different aspects of an object; they are acknowledged as denoting one and the same object through the realization that a particular processing or conversion of semiotic register of representation (Duval, 1995) allows to transform one representation into the other, and reciprocally. In other words, as stated and illustrated by Winsløw (2004, p. 4), one may "think of objects as *signs modulo object preserving transformations* (OPT)":

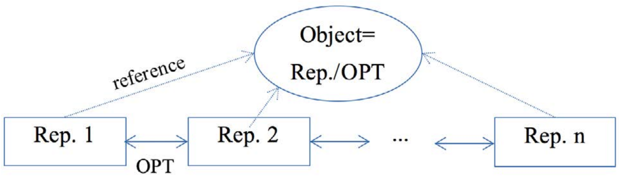

Figure 1

Syntax and semantic

Mathematical signs are organized within sentences and formulae that are built according to strict syntactic rules. From a logical point of view, a definition is an "open sentence" that may be satisfied or not when the variables of the sentence are assigned in a suitable universe of discourse: this is the semantic conception of truth introduced by Tarski (1944); see also Durand-Guerrier (2003) for a more detailed account and didactical applications. In this respect, a piece of data that satisfies the definition of a mathematical structure (which involves a set of axioms that forms its syntactic content) may be called a *model* of the structure (in the given universe of discourse or hosting theory). The

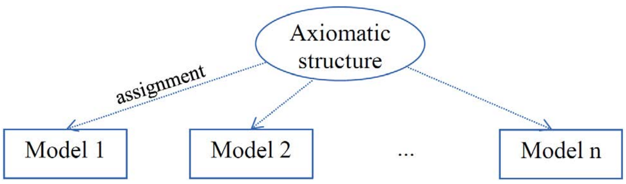

Figure 2

models will be regarded as the semantic content of the axiomatically-defined mathematical structure, its extension as a concept.

Tarski also defined the notion of logical consequence from a semantic point of view. It will be used below in order to show that a given axiom A_1 cannot be deduced from other axioms A_i: it amounts to showing the existence of a model satisfying the A_i's but not A_1. This contrasts with syntactic methods which consist of deduction by application of valid rules of inference.

Structural objects

In a famous dispute with Hilbert, Frege argues against the legitimacy of abstract definitions by systems of axioms. One argument concerns the intrinsic polysemy of such definitions: in semiotic terms, an axiomatic definition as a sign has multiple references, the models of the axiomatic system. In abstract definitions, the context doesn't inform on the denotation simply because it is abstracted (in meaning 1 of the verb).

In order to build an abstract structure (group, ring, banquet, etc.) concept, and therefore give a more adequate (still polysemic) semantic meaning to the set of axioms as a sign, one needs to use structure preserving applications (SPT), the so-called isomorphisms, which are defined as relation-preserving bijections (all models may be viewed as sets endowed with additional data which define relations and satisfy the axioms). This allows us to associate to an axiomatic structure its "structural objects" (our terminology), the isomorphism classes of models or models modulo SPT, in the same manner as mathematical objects were built from representations modulo OPT, by means of the principle of abstraction.

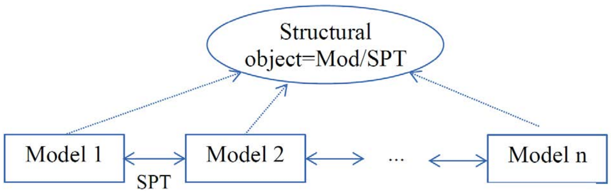

Figure 3

It should be pointed out that, compared to Winsløw's diagram, dotted arrows do not represent the denotation of a sign but only "quotient maps". Since models are accessed through signs, the preceding diagram should in fact be reprinted, to reflect semiotic views, replacing each model by one of its representation and SPT by its semiotic version SOPT (structural object preserving transformations). Dotted arrows may then represent denotation when the context indicates a structural perspective: for instance, Z/2Zx Z/2Z and "the symmetry group of a rectangle" may both refer to the Klein 4-group V_4, as an abstract group concept. One may also write $V_4 = \langle a,b; a^2=b^2=(ab)^2=1\rangle$ for a more syntactical description. Nevertheless, since mathematicians take care in emphasizing the difference between a class and one representative, many authors would prefer to use the sign V_4 to denote the group Z/2Zx Z/2Z and say that it is isomorphic to the symmetry group of a rectangle or to the quotient of the free group on two generators by the relations $a^2=b^2=(ab)^2=1$. The idea behind structural objects is, following Sfard, that some kind of reification must occur for concept building: "Reification is defined as an ontological shift – a sudden ability to see something familiar in a totally new light" (Sfard, 1991). For this to happen, a plurality of models should be needed, borrowed from different mathematical domains and represented in different semiotic registers. Similarly as in Winslow's context, the coordination of these representations (through SOPT) should be crucial to obtain a conceptual schema of the structural object. It should open the possibility to abstract from "templates" a "pattern" (Resnik, 1997). Nevertheless, unlike in Winsløw's context, a representation of a model as a sign may now refer to both the model and the structural object (in a context where both appear), whereas a mathematical distinction must be kept. Solving this issue would require a more direct mediation of the structural object by a new adequate (to be specified) sign.

THE THEORY OF BANQUETS

As a piece of didactical engineering, the theory of banquets was built on the basis of an epistemological analysis previously presented (Hausberger, 2013) in order to cover the three usage contexts of the meta-concept "structure": 1. the structure as defined by a system of axioms 2. the abstract structure (of a given group or banquet) 3. a 'structure-theorem' (which describes the way an object can be reconstructed from simpler objects of the same type). It is filled with meta-discourse, as is already visible in the worksheet title: "The theory of banquets: a mini-theory to reflect on structuralist thinking". The interested reader is requested to email the author for a copy of the complete worksheet.

The activity is divided into three parts: 1. logical investigation of the axiomatic system and classification of models 2. elaboration of an abstract theory of tables (this is the other way round: students are asked to formalize the disposition of guests around a round table) and structure-theorem for banquets (a banquet is the disjoint union of tables) 3. connection with the theory of permutations (a reinterpretation of the banquet theory that permits you to see the structure-theorem as a direct consequence of the well-known theorem of canonical cycle-decomposition of a permutation).

Part 1 and 2 clearly bring-in a concrete-abstract dialectic. A top-bottom approach has been chosen in part 1 for two reasons: as this is the standard strategy in textbooks, it is interesting to inquire how students will make sense of such a definition; moreover, part 1 will suitably enrich the didactical *milieu* for students to be able to model the situation given in part 2. Nevertheless, part 1 is already dialectical in itself: it amounts to making learners move on from the still abstract and syntactical conception of a structure exemplified by Figure 2 to the more concrete and semantic conception of Figure 3 (with several structural objects). The expected result of the abstract-concrete and syntax-semantic dialectics is a formulation of an abstract and syntactic characterization of structural objects (question 2 d of the worksheet, see below).

The definition of a banquet is as follows: it is a set E (the objects) endowed with a binary relation R (encoding the relations between objects) which satisfies the following axioms: A1. No object fulfills xRx A2. If xRy and xRz then $y=z$ A3. If yRx and zRx then $y=z$. A4. For all x, there exists at least one y such that xRy.

In part 1, students were asked the following questions:

1 a. Coherence: is it a valid mathematical theory, that is, are the axioms non-contradictory? In other words, does there exist a model?

1 b. Independence: is one of the axioms a logical consequence of others or are all axioms mutually independent?

2 a. Classify all banquets of order $n \leq 3$

2 b. Classify banquets of order 4

2 c. What can you say about Z/4Z endowed with?

2 d. How to characterize abstractly the preceding banquet (meaning its abstract structure of banquet among all the different classes of banquet, in fact how to characterize its class)?

Solving these questions amount to solving the following tasks and sub-tasks:

T1. Construct a model by suitable assignment of variables

T2. Classify banquets of a given order:

ST2a. Define a notion of isomorphism

ST2b. Give a list that covers representatives of all possible classes

ST2c. Show that two elements of the list are non-isomorphic

T3. Show that 2 models are isomorphic by explicit construction of an isomorphism

T4. Characterize abstractly an isomorphism class

Note that answering question 1 b amounts to solving T1 from the semantic point of view of logical consequence (see above) and negation of an axiom. In doing so, the boundaries of the banquet concept will be marked out. In the sequel, it will be necessary to focus first on T1 and give a list of available domains of interpretation for the axiomatic system and corresponding semiotic registers, since available representations greatly impact the other tasks.

Empirical interpretation: the name banquet may suggest by itself (or by reading the entire worksheet) guests around tables, so one defines xRy if and only if x sits on the right of y. Note that proving that this universe of discourse can serve to interpret the whole banquet theory reduces to proving the structure-theorem. One could also imagine a rectangle table and pick up guest sitting face to face, as a particular model.

Set theory: the set E is described by naming its elements and the binary relation is represented by its graph inside E^2. This straightforward representation is not very interesting since it doesn't “encode” much structure.

Matrix theory: a binary relation may be seen as a function from E^2 to {0,1} (true/false), and therefore be represented by a double-entry table, in other words a matrix. In this interpretation, the axioms say that the diagonal contains only zeros, that there is exactly one '1' in each row and at least one in each column. In finite dimension, one can easily prove that there is exactly one '1' per row and column, hence it is a permutation matrix.

Graph theory: xRy if and only if vertices x and y are connected by an edge directed from x to y. The axioms say that from a vertex there originates exactly one edge and terminates at most one; therefore, unlike in general graph theory, is it easy to see when two drawings define the same graph.

Function theory: According to axioms A2 and A4, defines a function f and the other axioms say that it is injective and has no fixed point. When the set E is finite, then f is a permutation without fixed points and one may use the standard semiotic representations for these (including cycle-decomposition).

It is in fact quite amazing to see the diversity of interpretations and models, which certainly reflects the unity and creativity of mathematics. Models may be represented in a mixed or purely symbolic register. When the graphical register is used, it may be a personal idealization of people around tables or an institutional representation borrowed from graph theory. Of course, one cannot expect students to connect to all these theories: for, instance, we bet that students won't translate the problem fully in the function setting and identify the connection with permutation theory (which would ruin part 3 of the activity). But one can wonder about the importance they may give to representations connected to everyday life (empirical setting). Moreover, models may be more or less "generic": compare (E,f), E=N and $f(x)=x+1$, with a matrix that may serve to represent any binary operation. Students may also think that a model should be given by a mathematical formula (like the example given in question 2 c), and restrict themselves to concrete examples in function theory, whereas a generic representation of R is necessary to complete the other tasks.

To give an idea about processes and conversions from one setting to the other, one should notice that a representation such as points marked on circles (empirical) is easily transformed into a graph by adding arrows clockwise between points; one may then associate to the graph its adjacency matrix, from which the function is soon reconstructed by reading the positions of the ones. When E is finite, the algorithm of cycle-decomposition of the permutation gives the tables (one per cycle) and the length of the cycle gives the number of people around each table, thus coming back to the empirical setting.

The pertinence of the setting (choice of a domain of interpretation) depends on the task: graph theory may easily suggests a model that verifies all the axioms except A2; matrix theory is quite pertinent for ST2b (a complete list of all possible relations), still, graph theory again (or even better, a cyclic representation as obtained in the empirical setting) is best to decide if two banquets are isomorphic, as it gives a visual representation that makes common pattern visible and illustrates the etymology of isomorphism as a form-preserving mapping.

I will now present some students' productions. Tasks 2 to 4 will be discussed in greater detail while analyzing these.

EMPIRICAL DATA

The full banquet activity has been tried out during the academic year 2013–14 with third year university students with a background in group theory before teaching ring and field theory. They worked in small groups of 4–5 students and were asked to keep a research notebook that was collected before each phase of institutionalization (Brousseau, 1997). In parallel, in laboratory sessions, I videotaped two pairs of more advanced students (having a master's degree). The interviewer (myself) intervened only at the end of part 1 in order to discuss with the students their answers and their conceptions regarding the classification task. Due to lack of space, I will only give an account of the laboratory session with one of the pairs. Nevertheless, this will already be enough to give an idea of some interesting phenomena that could be observed by using our theoretical framework and in particular Duval's *semiosis*, the apprehension or production of semiotic representations (Duval, 1995).

The pair of students tried to recognize the banquet structure as a *pattern* in known mathematical objects and theories ("what is it, what's this structure?"). Unlike in the classroom experiment, they didn't bring

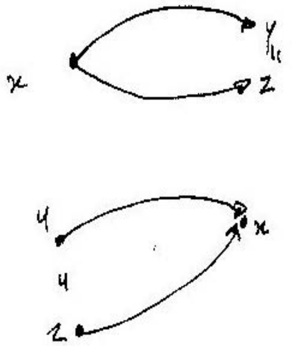

Figure 4a

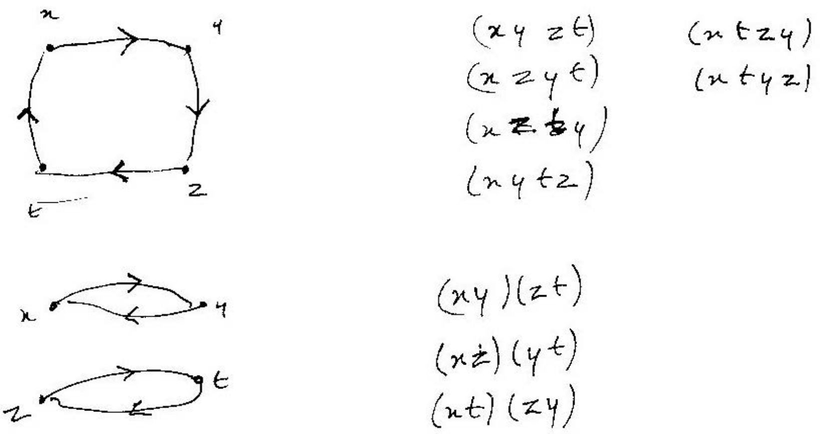

Figure 4b

in wedding banquets; they first thought about the order relation, then analyzed the example B_4=(Z/4Z,*R*) of question 2c as a "kind of a shift" and generalized it (*E*=Z, f(x)=x+1 or x-1). *Semiotic representations* of the *semantic* meaning of axioms A2 and A3 in the *graphical register* (Figure 4a) led them to build *models in graph theory* which they used for tasks T1b and T2. Recognition of *cyclic patterns suggested permutations*, as a common representation: they performed conversions of registers (but didn't connect to the function setting), producing the following classification which comprises 9 banquets of order 4 (Figure 4b).

Student A: Here, we are doing with what we know, but we speak about a structure
Student B: Wait, we can always number the elements [...]
Interviewer: For you, this is an abstract classification because you didn't consider particular relations and you can always rename elements x,y,z,t
A: So there would be 2 classes up to isomorphism?
B: Here Z/4Z and there Z/2Z x Z/2Z
I: You are thinking about the classification of groups [...] So there are 2 types of objects and (x y z t) and (x z y t) would be the same?
B: Not the same, of the same type

The student B couldn't define what he meant by a type, he just made a connection between the word used by the interviewer and the notion of type of a permutation. The word bijection finally appeared but students found it difficult to define what "structure-preserving" meant. They drew the graph for (x z y t) but obtained crossing edges which confused them even more (both are identical as graphs but not as drawings). On the contrary, converting to a graph the example B_4 allowed connection to (x y z t) (obvious congruence of drawings). They didn't realize that abstracting the nature of elements simply meant forgetting letters, a mental process that makes the recognition of isomorphism classes in the representation as cycle products automatic.

CONCLUSIONS AND PERSPECTIVES

This study, through its theoretical framework and the analysis of the presented data, contributes to the recognition of the influence of semiotic representations in cognitive activities dedicated to the learning of abstract algebra. I have discussed the hypothesis that a logical investigation of an axiomatic system and the classification of its models up to isomorphism, in the paradigm of the "theory of banquets" which connects to group-theory, is a cognitive activity that could bring good conditions for learners to develop an appropriate conceptualization of an abstract structure, and in particular access what I called "structural objects". Empirical data show mental processes based on the recognition of (visual) patterns. Conversions of registers were operated on by the two students in order to realize that two objects are isomorphic, a strategy which is successful when the congruence of representations is obvious, but the students couldn't handle the treatments inside a register since they couldn't rely on a formal definition of an isomorphism or make this definition functional, which is evidence of insufficient syntax-semantic dialectic. This also suggests an incomplete understanding of abstraction as a process that leads to structural objects. Finally, the two students tried to work out the analogy with group theory from which they borrowed directly or tried to adapt representations and concepts (see transcript above). As stated by Winsløw (2004), "mathematical concepts are not learned one by one but as coherent patterns or structures", and this also happens at the level of structures themselves, thus gaining access to what I called level-2 unification (Hausberger, 2012).

The analysis of empirical data will be pursued in greater details in an expanded version of this article. It is expected that these investigations and refinements of the semiotic tools will lead to a better understanding of students' difficulties in abstract algebra which are inherent in structuralist thinking, from a cognitive point of view.

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The transition from higher education to the world of work: Measuring student teachers' beliefs and practices for purposeful sample selection

Eivind Kaspersen[1], Birgit Pepin[2] and Svein Arne Sikko[1]

1 Sør-Trøndelag University College, Trondheim, Norway, eivind.kaspersen@hist.no

2 Technische Universiteit Eindhoven, Eindhoven, The Netherlands

This paper reports on methodological results from an on-going project investigating student transition from teacher education to the world of work. We argue that research benefits from purposeful sampling, and we present two Rasch-calibrated instruments that aim at finding participants with particular characteristics. With items from existing instruments, one practice instrument is calibrated on Norwegian student teachers. Furthermore, these items are rephrased to fit a second instrument measuring beliefs about teaching mathematics. Finally, 'virtual equating' is used to align items so that measures can be compared across instruments.

Keywords: Student-centredness, measurement, Rasch modeling, sample selection.

INTRODUCTION

A considerable amount of research exists that sheds light on student teachers' transition from higher education to the world of work. Most studies on novice teachers' experiences show that the transition from teacher education to work (as a teacher) is problematic. For instance, many studies describe a gap between higher education and work (e.g., weak relationships between courses and field experiences) (Feiman-Nemser, 2001), and that what students learn in school is not adequately linked to their future practices as teachers (e.g., Liston, Whitcomb, & Borko, 2006). Other studies describe new teachers' first period in work as a time where the workplace communities expect new employees to be able to teach like experienced teachers (Worthy, 2005), a period with high emotional intensity (Flores & Day, 2006) and a period where there is a gap between new teachers' actual identities and the 'designated identities' shaped by the workplace communities (Haggarty, Postlethwaite, Diment, & Ellins, 2011). In most studies, however, the sample selections do not seem to be based on pre-determined criteria (or these are not made explicit).

To build further on existing research, in this paper, we argue that Rasch-calibrated instruments can facilitate the selection of persons with certain characteristics. For instance, some studies would benefit from selecting persons who identify with reform-minded practices, traditional practices, or persons who follow certain trajectories in the transition from education to work (e.g. resisting change; complying with traditional practices; coming to identify with reform-minded practices) etc. Thus, our research question in this paper is: *how can Rasch-calibrated instruments inform the sample selection in studies on student teachers' transition from higher education to the world of work/school?*

In the paper, we report on results from an on-going study where the overall aim is to understand how identities are negotiated in the transition from higher education to the world of work, including those of mathematics teachers in schools. The study follows on from previous studies of the TransMaths project (www.transmaths.org) (e.g., Pampaka et al., 2012; Pepin, Lysø, & Sikko, 2012), and we will present two Rasch-calibrated instruments that measure persons' practices and beliefs about ideal practice. Finally, we discuss how these instruments can inform sample selection.

THEORETICAL BACKGROUND

The literature suggests different ways of characterising teachers. As a frame for our instrument, we build on the notion of student-centredness, defined by Stephan (2014) as

> (...) an approach to mathematics instruction that places heavy emphasis on the students taking responsibility for problem solving and inquiry. The teacher is viewed as a facilitator by posing problems and guiding students as they work with partners toward creating a solution (p. 338).

The notion of student-centred teaching, both in practice and research, has grown in many different directions, and it is impossible to describe one single approach. Nevertheless, Stephan (2014) listed five characteristics of student-centred teaching: problem-solving; collaboration; mathematical discourse; tools/manipulations; and classroom environment (pp. 340–342). Regarding classroom environment, Stephan (2014) emphasized four social norms, documented by Yackel and Cobb (1996), which are supporting student-centred teaching. That is, students are expected to: explain/justify solutions; attempt to make sense of others' explanations; indicate agreement/disagreement; and, ask clarifying questions (p. 340).

The rationale for choosing student/teacher-centredness as a frame is that it is uni-dimensional, and thus meets one basic requirement for Rasch analysis. The construct was used by Pampaka and colleagues (2012) earlier in the TransMaths project, when they used Rasch analysis to construct an instrument to capture teachers' self-reported pedagogies. Their instrument was based upon Swan's (2006) practice-questionnaire, which in turn was based on three teacher orientations: *transmission*, *discovery*, and *connectionist* (Askew, Rhodes, Brown, Johnson, & Wiliam, 1997).

In order to measure the relationship between practice and belief about ideal practice, we have extended the practice instrument (Pampaka et al., 2012) with a 'belief-dimension'. We argue that there are three reasons for including a belief-dimension to the original instrument. First, several studies suggest that teachers' practices are influenced by workplace norms (e.g., Haggarty et al., 2011). Gresalfi and Cobb (2011) distinguish between three ways in which teachers can relate to these workplace norms: *consent and identify*; *consent and comply*; or, *resist*. As such, knowledge about teachers' practices alone does not distinguish between those who identify with their own practice and those who merely comply with a workplace norm (or are constrained by other contextual influences). Second, knowledge about teachers' beliefs alone does not inform the researcher about how central those beliefs are. That is, two persons that express the same beliefs about teaching mathematics can hold those beliefs with different strengths. Green (1971) identified 'the degree of conviction' as one of the three dimensions in belief-systems. That is, beliefs can be central or peripheral, where central beliefs are more strongly held than peripheral beliefs. If the researcher wants to locate persons who hold certain beliefs strongly, we argue that persons who can relate those espoused beliefs to actual practice are more likely to meet this criteria than persons with different espoused beliefs and practices. Third, in longitudinal studies, knowledge about both practice and belief can provide information about participants' trajectories in terms of the three categories presented by Gresalfi and Cobb (2011).

In sum, we claim that knowledge about persons' beliefs and practices can help researchers in making well-targeted sample selections, and that this is pertinent to studies conducted in the context of higher education.

DATA COLLECTION AND ANALYSIS

In our study items in the instruments were influenced by the original items in the 'practice instrument' (Swan, 2006). For the belief instrument, these practice items were translated into belief items. For instance, the practice item: "Students work with tasks with a clear answer" was translated into a belief item: "Students should work with tasks with a clear answer." The items and the response categories were then discussed at a Ph.D. seminar in mathematics education. From this discussion, 27 practice items and 27 belief items were piloted on 42 Norwegian student teachers in their second and third year of education, in addition to 9 teacher educators at the same institution. The items were discussed briefly with all participants. The items were then revised and piloted again on 36 student teachers in their second year of education. After a final revision, 32 items were assigned to a convenience sample of 83 student teachers in their fourth (and for many, final) year of education. As the original practice-instrument identified some problems with the use of five response categories, four response categories were chosen in our study ('in none/almost none of the lessons'; 'in some of the lessons'; 'in most of the lessons'; and 'in all/almost all of the lessons').

In the analysis, the Rasch-Andrich Rating-Scale Model and the WINSTEPS software were used to construct *one practice scale* and *one belief scale*. The Rasch model turns categorical data into interval measurements. Moreover, the model assumes an underlying trait (e.g., teacher-centredness) and is based upon the idea that persons with high measures (e.g., highly teacher-centred) are more likely to agree with the items that define the trait than persons with low measures (e.g. highly student-centred). Similarly, each person is more likely to agree on items with low measures than on items with high measures. A key feature of the Rasch model is that persons and items are not discriminated, which means that they can be measured on the same scale (Wright & Stone, 1979).

Since the purpose of the instruments was to detect persons with particular characteristics (e.g., persons that identify with their practice, or persons with a more teacher-centred practice than belief), we pursued equally scaled instruments. That is, if a person identified with her practice, then her practice-measure should, ideally, be equal to her belief-measure. Moreover, if her practice was more teacher-centred than her belief, then her practice-measure should be larger than her belief-measure. Thus, we conducted four steps for “virtual equating” (Luppescu, 2005): 1) identified pairs of items with, possibly, similar ‘difficulties’; 2) cross-plotted the pairs of items; 3) removed pairs of items that were not close (within a .95 confidence bound); and, 4) rescaled the measures on the beliefs test to compensate for different item spacing. In rescaling the belief-instrument, we used two raw scores at each end of the practice-scale (20 and 45), and their corresponding measures, and computed new UIMEAN (mean of the item difficulty) and USCALE (the user-scale value for 1 logit) for the belief-instrument, so that the measures of the raw scores 20 and 45 were equal in both instruments. This is in principle similar to rescaling a Fahrenheit-instrument to a Celsius-instrument by defining two values (e.g. points of freezing and boiling).

To ensure further validity, we used guidelines presented by Wolfe and Smith Jr (2006) which extends Messick’s (1995) validation framework with two aspects of evidence put forth in The Medical Outcomes Trust (MOT). To summarize, validity is viewed as a unified concept. That is, there are not different kinds of validity, rather, different kinds of evidence that support validity. Accordingly, Messick (1995) presents six different aspects of validity where evidence can be found: the *content, substantive, structural, generalizability, external,* and *consequential* aspects. Furthermore, the MOT presents two aspects not mentioned by Messick: *Responsiveness* and *interpretability*.

RESULTS

We now present the final instruments and corresponding validity arguments. Both instruments consisted of 12 items, whereas item 3 (marked with an [x]) was reversely coded due to negative point measure correlation earlier in the analysis. To find evidence for the *content* aspect of validity, the technical quality of items has been evaluated. Mean squared fit statistics are chi-squared statistics divided by their degrees of freedom (and hence, have an expected value of 1). OUTFIT is outlier sensitive fit, and INFIT is information-weighted fit. Linacre (2002) suggests values between .5 and 1.5 as productive for measurement, and all items in both tests were within this interval, with belief item 3 having the largest misfit (Figure 2). Furthermore, person reliability values, analogous to Cronbach’s alpha, were .87 (practice) and .78 (belief),

Item STATISTICS: PRACTICE MEASURE

ENTRY NUMBER	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PTMEASURE-A CORR.	PTMEASURE-A EXP.	Item
10	4.35	.20	1.19	1.2	1.16	1.0	.69	.71	I make sure that all use the same methods
11	2.41	.20	.77	−1.5	.76	−1.6	.67	.68	I teach each subject separately
2	2.13	.20	.82	−1.2	.80	−1.3	.79	.69	Students work with tasks with a clear answer
3	2.04	.20	1.18	1.1	1.32	1.9	.51	.68	[x]Students work with mathematics in small groups
8	1.95	.20	.86	−.9	.85	−.9	.80	.69	I explain for the students how to solve different tasks
4	1.79	.20	1.00	.1	.99	.0	.62	.68	Students start with easy tasks
1	1.58	.20	1.10	.7	1.12	.8	.67	.67	Students work with tasks from the textbook
9	1.53	.20	.96	−.2	.94	−.3	.71	.67	Students work with tasks individually
5	.95	.21	.98	−.1	.94	−.2	.69	.65	I explain methods for the whole class
7	.42	.21	.92	−.4	.88	−.6	.65	.63	Students work with tasks where they use methods they have learned
13	−.86	.23	.99	.0	.99	.1	.54	.56	Most students work with the same subject
6	−1.97	.28	1.25	1.3	1.02	.2	.38	.47	I know before class which subjects the students will work on
MEAN	1.36	.21	1.00	.0	.98	−.1			
S.D.	1.55	.02	.15	.9	.15	.9			

Person: REAL SEP.: 2.57 RELIABILITY: .87 Item: REAL SEP.: 6.96 RELIABILITY: .98

Figure 1: Final practice items

Item STATISTICS: BELIEF MEASURE

ENTRY NUMBER	MEASURE	MODEL S.E.	INFIT MNSQ	INFIT ZSTD	OUTFIT MNSQ	OUTFIT ZSTD	PTMEASURE-A CORR.	PTMEASURE-A EXP.	Item
10	3.82	.20	.99	.0	1.06	.4	.55	.51	The teacher should make sure that all use the same methods
2	2.53	.19	.76	-1.5	.74	-1.5	.61	.56	Students should work with tasks with a clear answer
11	2.44	.19	1.00	.1	1.01	.1	.58	.56	The teacher should teach each subject separately
3	2.34	.19	1.35	1.9	1.39	1.9	.49	.54	[x]Students should work with mathematics in small groups
8	2.10	.19	.80	-1.1	.83	-.9	.67	.57	The teacher should explain for the students how to solve different tasks
1	2.06	.19	.72	-1.7	.72	-1.6	.62	.57	Students should work with tasks from the textbook
9	1.24	.17	.90	-.6	.88	-.7	.56	.58	Students should work with tasks individually
4	1.22	.18	1.15	.9	1.10	.6	.59	.57	Students should start with easy tasks
5	1.07	.17	1.13	.9	1.13	.8	.56	.59	The teacher should explain methods for the whole class
7	.13	.16	.86	-1.0	.81	-1.3	.65	.60	Students should work with tasks where they use methods they have learned
13	-.30	.16	1.11	.8	1.09	.6	.52	.60	Most students should work with the same subject
6	-1.44	.17	1.18	1.2	1.20	1.0	.51	.57	The teacher should know before class which subjects the students will work on
MEAN	1.43	.18	1.00	.0	1.00	.0			
S.D.	1.38	.01	.19	1.1	.20	1.1			

Person: REAL SEP.: 1.90 RELIABILITY: .78 Item: REAL SEP.: 7.33 RELIABILITY: .98

Figure 2: Final belief items

and item reliability was .98 on both tests, an indicator that the sample was big enough to provide information for item calibration.

When practice items were cross-plotted against belief items (Figure 3), most items were within a .95 confidence bound. An exception was one border-line item (item 4), but the effect on person measures, when this item was removed, was negligible. Thus we have decided to keep the item in the analysis.

Ideally, we would want each pair of items to have equal measures. The DTF-analysis, however, showed that measures were close but for most items not equal (Figure 3). The next step, then, was to see if the measures were 'close enough'. And indeed, when person measures were compared with the ideal situation (anchoring practice items to be equal to the belief items), only small differences in person measures could be found (.26 logits at the most with r^2=1.00).

Moreover, rating scale analysis has been conducted for both instruments, to find evidence for *substantive* validity. None of the four guidelines suggested by Wolfe and Smith Jr (2006, p. 210) were violated: 1) each rating scale category contained more than 10 observations (65 observations in the first category in the practice instrument, being the least); 2) the shape of each rating scale distribution was smooth and unimodal; 3) the average respondent measure associated with each category increased with the values of the categories; and, 4) the unweighted mean-squared fit statistics were all less than 2.0 (1.25 at the most).

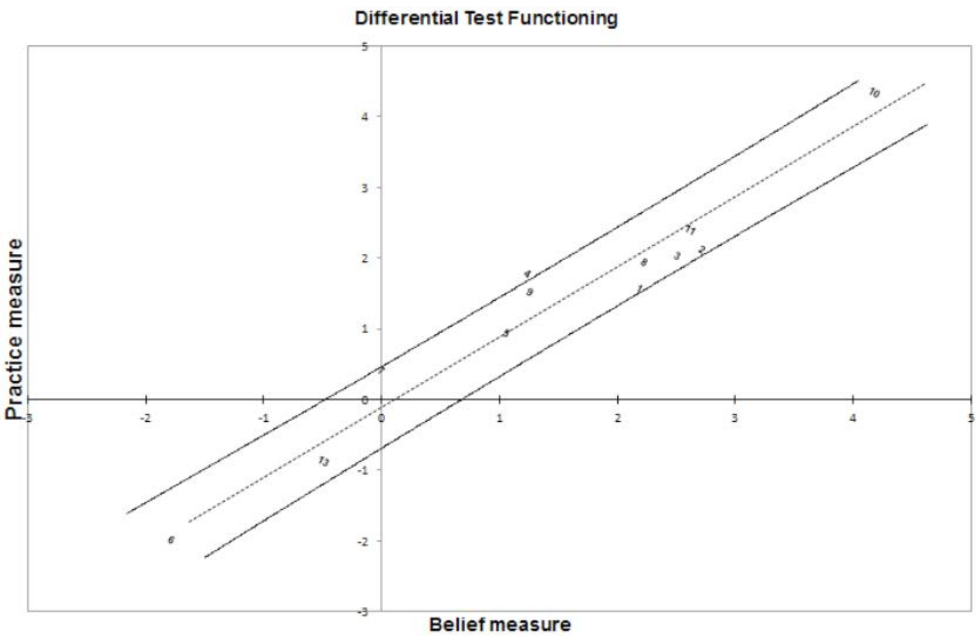
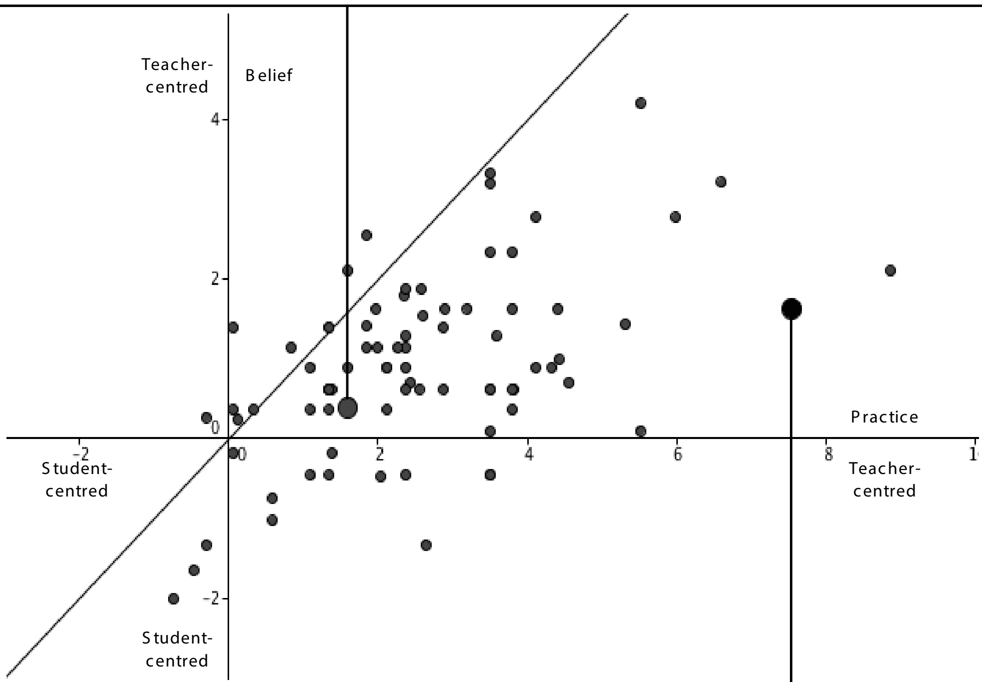


Figure 3: Differential Test Functioning (DTF)

[…] So, we used envelopes and paperclips. Eh, and that X was the envelope, and the number of paperclips in the envelope was the value of X. So, by taking away and adding on both sides of a line, they could solve equations. Eh, and it worked very well.

[…] A good task. Hmm… […] that there is, kind of, a possibility for individual interpretation and exploration. Eh, that it is, it is open. That it is possible to do it in your own way. That it is not constrained by, by the method you are supposed to use, for instance.



[…] it was jus a recap of, they were having a test. And, from one chapter in the book. And then, I read in the book and made my own summary. Formula booklet summary. And then I went to the classroom. And then I went through all I had written down in advance.

[…] that I want, I don't want it to be loose and floating, and, now you are going outside to think about this, and now you are going to play with these bricks. I want some structure, and I want it to be like, ok, now we are going to do this, why do we do it? And I want some rules and stuff. […] But, I think it is also important that you should explore and work with things you know. Perhaps go outside, in mathematics in particular, I think it is important that mathematics is not only abstract, but that you have to work with concretes, real life situations.

Figure 4: The relationship between Norwegian teacher students' practice and espoused beliefs about ideal practice

Uni-dimensionality is a basic assumption in Rasch analysis (Bond & Fox, 2003, p. 32). Thus, we have evaluated dimensionality (*structural* validity) using principal component analysis of the standardized residuals after the Rasch dimension was extracted. Among the practice items, a second dimension could explain 1.9 (in Eigenvalue units) (6.5 %) of the unexplained variance, and among the belief items, a second dimension could explain 1.9 (6.8 %) of the unexplained variance. By default, WINSTEPS stratifies items in three clusters for each contrast. The dis-attenuated correlations between person measures in these clusters were close to 1 on the belief instrument, and .85 in the practice instrument. Thus, we treated the second dimensions as strands (like addition and subtraction on a mathematics test), and not as dimensions that needed separate instruments.

To find evidence for the *generalizability* aspect of validity, we have used Differential Item Functioning (DIF): *the loss of invariance of item estimates across testing occasions* (Bond & Fox, 2003, p. 309). Item calibrations have been compared between genders, classes, and high/low-measured persons. The DIF (Rasch-Welch) t-value was less than 2.0 in all cases, where the belief item 5 had the most misfit between males and females (DIF-contrast = .78 with p = .07).

To look for *external* validity, we have compared the results from our study with the literature on the relationship between beliefs and practices. The results in Figure 4 show that there was only a moderate correlation ($r^2 = .28$) between students' practice and espoused beliefs, consistent with the existing literature (Liljedahl, 2009). However, the responses lie heavily on one side of the identity line; inconsistency was more evident for those who held student-centred beliefs. Although it has not been expressed explicitly, we assert that traces of this relationship can also be found in the literature. Even if different notions are being used, inconsistency is mostly described in situations where participants express reform-minded beliefs (e.g., Kesler, 1985; Vacc & Bright, 1999).

Evidence for *responsiveness* validity can be found in the Person-Item Map (Figure 5). Marks on the right represent item measures, and marks on the left represent person measures. From this, we can see that when person measures exceeded -2 to 4 logits, we can expect that measures were being less accurate. However, since we have used the Rasch-Andrich Rating Scale Model, each item covered more of the trait than a dichotomous item. Thus, the 'lowest' and the 'highest' item measures were not to be considered the 'floor' and the 'ceiling'.

The *interpretability* aspect of validity is the degree to which qualitative meaning can be assigned to quantitative measures (Wolfe & Smith Jr, 2006, p. 227). Thus, excerpts from two interviewed cases are presented in Figure 4. In addition, nine teacher educators, at the same institution, were asked to respond to the belief-test. Other than providing construct validity (as teacher educators were expressing significantly more student-centred beliefs than the students), the instrument was rescaled, so that the mean of the teacher educators' measures was set to zero. This was then used for qualitative interpretation: values close to zero could be thought of as in accordance with the values of the educational institution.

CONCLUSION

In this paper, we have argued that research can benefit from purposeful sample selection, and we have presented two instruments for this purpose. From these instruments, certain 'kinds of persons' can be selected: persons who seem to practise and identify with institutional values (measures close to the origin); persons who seem to practise and identify with traditional values (high measures on both instruments); persons who seem to consent and com-

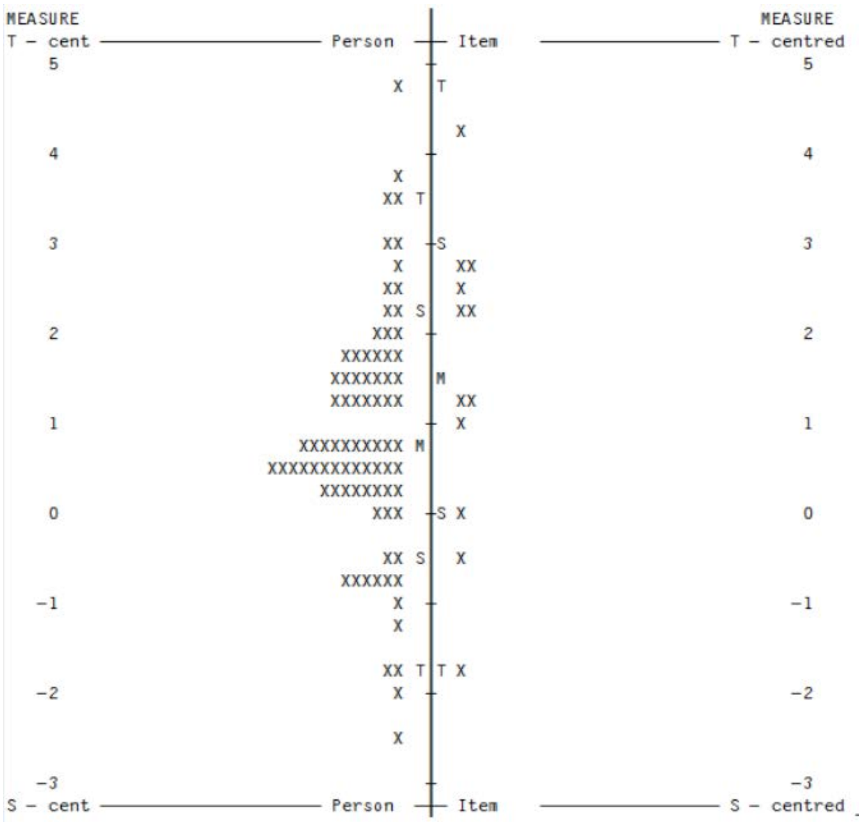


Figure 5: Person-Item Map (belief instrument). M=mean, S=one standard deviation from person or item mean, T=two standard deviations from person or item mean

ply to a workplace norm (having trajectories moving away from the identity line), etc. Our intention is to use these instruments for studying student teachers' transition from higher education to the world of work. However, these instruments can also be used in other kinds of research, we argue. For instance, research on teaching and learning of mathematics in higher education can benefit from selecting particular kinds of participants (e.g. lecturers with a certain practice).

Although we assert that our and similar instruments can be helpful tools for sample selection, we acknowledge their limitations. In our case, we have reduced the practice (and belief) of teaching mathematics to one dimension. This was done due to the statistical benefits, but we emphasize that teaching is clearly multidimensional. Thus, persons with similar measures might, and are likely to have, different practices/ beliefs, even when they are measured with reliable instruments. All we can say is that they probably have some characteristics in common. Nevertheless, we conclude that sample selection is, in many cases, better when it is well-targeted rather than opportunistic.

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On advanced mathematical methods and more elementary ideas met (or not) before

Margo Kondratieva

Memorial University, Faculty of Education and Department of Mathematics and Statistics, St. John's, Canada, mkondra@mun.ca

Some mathematical problems, which could be solved by a general approach, also have different and often original solutions that appeal to less advanced mathematics. For example, drawing a tangent line to a parabola can be done by methods of differential calculus, and by exclusively using methods of Euclidean geometry. A project that I conducted with students who recently completed a Bachelor degree focusing in mathematics revealed that they were familiar with advanced methods but lacked more elementary views. I argue that unfamiliarity with alternative elementary solutions hinders for students the opportunities both to build mathematical connections and to appreciate the groundwork of related advanced methods.

Keywords: Problems with multiple solutions, connectivity of mathematics, parabola, Euclidean geometry, calculus.

INTRODUCTION

Some problems employed in teaching to illuminate the essence of a mathematical method appear to be universally useful in a variety of courses. In these cases students can compare how different ideas and techniques are applied to address the same mathematical question. The practice of identification of problems useful for systematic use in various university level courses is discussed in the literature. For example, Mingus (2002) referred to "calculation of n^{th} roots of unity" as a problem that "encourages students to see connections between geometry, vectors, group theory, algebra and long division" (p. 32). Further discussion revealed, "proving identities involving the Fibonacci numbers provide a solid connection between linear algebra, discrete mathematics, number theory and abstract algebra" (ibid, p. 32). Winsløw (2013) referred to several ways to approach constructions of a^x for $a > 0$ and x real, which are based on either "direct" extension of the domain from natural to real numbers, or the inverse function to $\log_e(x) = \int_1^z dt/t$, or the initial value problem $dy/dx = y$, $y(0) = 1$, or the functional equation $f(x + y) = f(x) \cdot f(y)$, or the Maclaurin power series (p. 2481). Sun and Chan (2009) discussed nine proofs of the "Mid-Point theorem of triangles". The fact that "the sum of the interior angles in a plane triangle is 180°" can be shown in eight different ways (Tall et al., 2012, p. 35). In my view, these are examples of *interconnecting* problems, which have the following characteristics: they allow various solutions at more elementary and more advanced levels; they can be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions; and they are used in different courses and can be understood in various contexts (Kondratieva, 2011a, 2011b).

When students familiar with a problem from their prior experiences use their intuition to support more elaborate techniques applicable to a problem, they also have a chance to perceive mathematics as a consistent subject (Kondratieva, 2011a). By means of investigation of such problems in different courses "students were able to review concepts from previous courses and improve their understanding of the old and new concepts" (Mingus, 2002, p. 32). Knowledge of multiple ways to treat a mathematical object can strengthen the relation of a learner to the object "in the sense of providing extensions or alternatives to standard presentations" (Winsløw, 2013, p. 2483). Leikin and her collaborators extensively studied "tasks that contain an explicit requirement for solving the problem in multiple ways" (Leikin & Levav-Waynberg, 2008, p. 234), particularly in the context of the development of mathematics teachers' knowledge, and for an examination of mathematical creativity (Leikin & Lev, 2007). When used in mathematics teachers' education, interconnecting problems foster teachers' ability to link elementary ideas with advanced techniques (Kondratieva, 2013), which might contribute to the construction of their horizon content knowledge (Ball, Thames, & Phelps, 2008), that is, an awareness of how mathematical topics are related in the span of the entire curriculum. While being familiar with some individual examples of the use of interconnecting problems in teaching mathematics, I do not know to what extent students in general are exposed to teaching practices that encourage them explicitly to make connections between advanced and more elementary mathematics. According to Winsløw (2013), "there remains a practical need for systematic didactical research on how standard undergraduate mathematics is, or could be, developed in view of facilitating its use by students in inquiries related directly to high school mathematics." (p. 2477)

My research question is to what extent students who have completed a Bachelor degree with a focus in mathematics are familiar with both advanced mathematical methods and more elementary ideas related to them. In order to address this question I developed a research instrument in the form of a handout and questionnaire, which is described in this paper. The handout includes an interconnecting problem, which is typical and familiar for students studying undergraduate mathematics. This problem can be solved by a standard method taught at the university level and also has a more elementary treatment, which reveals some insightful ideas. This research instrument, along with the theoretical framework and results of testing in a small group of students, are discussed in the following sections.

THEORETICAL CONSIDERATIONS

Modern curriculum at all levels is moving from a formal approach to a more inquiry-based study of mathematics, focusing on genuine understanding and connecting various concepts and methods. House and Coxford (1995) argued that presenting mathematics as a "woven fabric rather than a patchwork of discrete topics" is one of the most important outcomes of mathematics education. The goal of mathematical instruction consists of helping a dedicated learner go beyond instrumental understanding secured by knowing mathematical procedures, and achieve relational understanding between different mathematical topics (Skemp, 1987), which assumes connections of various mathematical ideas. "An ability to establish and use a wide range of connections offers students alternative paths to the solution" (Hodgson, 1995, p. 19). While making connections and multiple representations of ideas are recognized among the primary processes in learning mathematics (NCTM, 2000), there is also a need for teaching strategies "for engaging students in exploring the connectedness of mathematics" (House & Coxford, 1995, p. vii).

One possible way to address this need is to use problems which allow multiple solutions, or specifically, interconnecting problems. In the latter approach (Kondratieva, 2011a), students encounter an interconnecting problem several times as they progress throughout their education, each time learning a new aspect of the same problem and building their understanding on "supportive met-befores" (Tall, 2013, p.15). Rephrasing Watson and Mason's (2005) description of reference examples, an interconnecting problem is "the one that becomes extremely familiar and is used to test out conjectures, to illustrate the meaning of theorems" (p. 7). Indeed, problems that have a range of solutions not only can help learners to move from elementary to advanced understanding, but they also may be used to exemplify advanced methods in elementary terms or to come up with an alternative and more elementary explanation of results found in a different way (Kondratieva, 2013).

In order to collect and analyze results presented in this paper, I employ Tall's notion of crystalline concept. Formation of a crystalline concept in a learner's mind refers to a phenomenon where an object of mathematical study "which originally was a single gestalt with many simultaneous properties, and was then defined using a single specific definition – now matures into a fully unified concept, with many properties linked together by a network of relationships based on deductions" (Tall et al., 2012, p. 20). The crystalline concept of an object combines all prior experiences of a learner in relation to this object, which include perceiving, acting upon, describing in natural or symbolic language, further formalizing, theorizing, and organizing knowledge about the object in a compressed way. Perceiving the object in various contexts, recognizing its multiple representations, and establishing equivalence relations between its various properties are important steps in the cognitive development of a learner towards building a corresponding crystalline concept. Eventually, "equivalent concepts may be grasped as a single crystalline concept that has all the requisite properties blended together within a single entity. Powerful mathematical thinking at the highest level involves the external relationship between, and the internal relationship within, crystalline concepts" (Tall, 2013, p. 403).

The notion of crystallization comprises various frameworks and theories of knowledge compression including the Structure of Observed Learning Outcomes (SOLO) Taxonomy (Biggs & Collis, 1982). The SOLO Taxonomy describes the progression of a learner's development using the following stages: (0) pre-structural, when the learner demonstrates a very limited understanding of a problem; (1) uni-structural, when the learner uses only one aspect of a concept and follows a single procedure to solve a problem; (2) multi-structural, when the learner refers to several aspects and is able to carry out several procedures to solve a problem; (3) relational, when the learner relates several aspects together or sees the equivalence of different procedures; (4) extended abstract, when the learner grasps the concept so well that they can apply it outside of the problem's domain. Thus, crystallization requires both the familiarity with multiple aspects of a concept and relational unification of them.

Further analysis of the development of mathematical thoughts at both the historical and individual level suggests that while mathematical arguments become more sophisticated and formal, "true mathematical thinking should become not only more powerful but more simple" (Tall, 2013, p. 19). According to Atiyah, "not only mathematics but science as a whole, only progresses if you can understand things... Its aim is to produce ideas and explain things in simple terms" (see Tall, 2013, p. 400). Similarly, Polya (1945/2004) suggested to always look back at your solution and ask yourself "Can you derive your result differently? Can you see it at a glance?" However, the simplicity and transparency of mathematical arguments produced by students often depends upon their prior exposure to basic but enlightening ideas related to more advanced methods (Kondratieva, 2014), as well as on whether they possess proper crystalline concepts. The aim of this paper is to delve on this issue using an example from undergraduate mathematics.

RESEARCH INSTRUMENT AND ITS PRELIMINARY TESTING

A group of 16 students who had completed their undergraduate degree participated in the project. These students were enrolled in my methods course for pre-service teachers whose teachable areas included mathematics and another subject (most commonly science). Each of these students had taken at least eight mathematical courses including at least three courses at the 3rd or 4th undergraduate level (with average mark above 70%). While the project was conducted during a regular class time, the participation was optional and no mark was assigned to this work. The students could have chosen to perform an alternative practice assignment; however, everyone in class agreed to participate in the project. The students worked in pairs. They were asked to develop a theoretical solution and implement it using technology. They were presented with the problem and questionnaire.

Problem: Use dynamic geometry software to draw a parabola and its tangent line at a point using tools of your preference.

Through other assignments in my course the students were familiar with dynamic geometry software such as GeoGebra capable of drawing points, segments, parallel and perpendicular lines, circles etc., as well as graphs of functions given by their equations in the form $y = f(x)$, and manipulating these objects. The tool of drawing a tangent line to a given curve was not amongst the available options.

Questionnaire:

(1) Have you seen this or similar problems before?

(2) If yes, identify the course title or corresponding mathematical context appropriate to treat this problem (e.g. linear algebra, geometry, combinatorics, pre-calculus, differential or integral calculus).

(3) List all methods you know to approach the problem. Do you see any connections between them?

(4) If you have never seen this problem before, state another problem of which this problem reminds you, and which you can solve.

When students had completed the first four questions (normally within 20 minutes or less) the last question was given to them.

(5) Read the following sample solutions and choose the most appropriate description: (a) I have found the same solution; (b) This approach is familiar but I

forgot the details; (c) I have never seen this method before but I can understand and explain the ideas used in this approach; (d) I have never seen anything like that and I do not understand it.

The students who chose answer (b) or (c) were asked to explain the corresponding solution in writing. Then, we had a whole class discussion where students could reflect on the solutions and connections between them.

Sample solution 1: Use the standard equation of the parabola in the Cartesian coordinates and Differential Calculus.

Let the parabola be given by the equation $y = ax^2$. The derivative of the function $f(x) = ax^2$ is $f'(x) = 2ax$. The tangent line at a point (x_0, ax_0^2) has the form $(y - ax_0^2) = 2ax_0(x - x_0)$ or after a simplification $y = (2ax_0)x - ax_0^2$. Now we can draw the parabola and the line in the coordinate plane using the tool "graphing functions given by their equations $y = f(x)$" for some values of $a \neq 0$ and x_0.

Sample solution 2: Produce this construction by means of Euclidean geometry given the focus F and directrix l of the parabola.

Geometrically, a parabola is the locus of points equidistant from the given point F and the given line l not passing through F. This definition enables the following construction (see Figure 1, left): (1) Drop the perpendicular from F onto l with foot at A; the midpoint V of the segment FA is the vertex of the parabola; (2) Pick any point B on l; join F and B and draw the perpendicular bisector l' to FB; draw l'' perpendicular to l at B; let l' and l'' meet at C. As point B moves along line l, point C traces a parabola (Figure 1, right). Then the line l' is tangent to the parabola with C being the point of tangency.

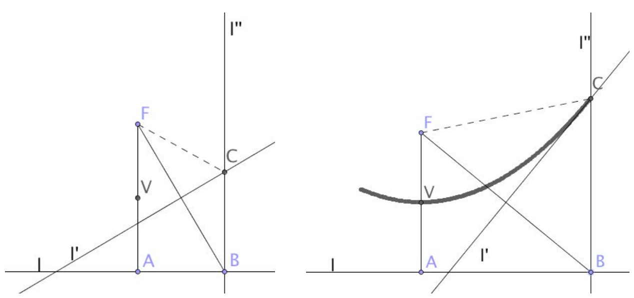

Figure 1: Construction of a parabola given its focus *F* and directrix *l*

RESULTS AND DISCUSSION

The problem that was given to the students is a typical problem studied in Calculus courses, where students are taught that the derivative $f'(x_0)$ represents the slope of the line tangent to the curve $y = f(x)$ at a given point $x = x_0$. They are also supposed to know that the equation of a straight line passing through a given point (a, b) with a given slope k is given by $(y - b) = k(x - a)$. Therefore, Sample solution 1 is a universal approach to this type of problem and all of the 16 participants recognized it as a familiar method to apply. However, none of them mentioned the second method as a possibility to solve this problem and only one participant recognized familiar ideas in the method. The majority of participants (15) answered "I have never seen this method before but I can understand and explain the ideas used in this approach". The problem was that the majority of students did not know the definition of a parabola in terms of its directrix and focus.

Nevertheless, once students read the geometric definition of the parabola all of them were able to understand and explain the construction presented in Figure 1: since C lies on the perpendicular bisector l' of FB, the triangle FCB is isosceles and hence C is equidistant from F and l. In terms of the SOLO Taxonomy described above this signifies passage to the second stage, namely to the multi-structural understanding of the problem by the students.

I refer to the second solution as being more elementary because it uses mathematical ideas less advanced compared to those needed to develop calculus. However, students may have a different point of view. They may perceive the algorithmic approach studied in Calculus as being easier than using Euclidean geometry, which they experienced to a lesser degree.

Knowing two approaches to solve the problem is obviously progress. However, in order to develop the crystalline concept it is also necessary for students to see a relation between the two approaches. Indeed, the students were able to establish a connection to the formulas given in the Sample solution 1 by introducing Cartesian coordinates so that l has equation $y = -p$, and the coordinates of points are $F(0, p)$, $V(0, 0)$ and $C(x, y)$. Without loss of generality we assume that $p > 0$. Then, by the Pythagorean theorem, $\overline{FC}^2 = x^2 + (y - p)^2$. But, $\overline{FC} = \overline{CB} = y + p$, and after simplifications one gets the equation $y = \frac{x^2}{4p}$. At the same time the slope

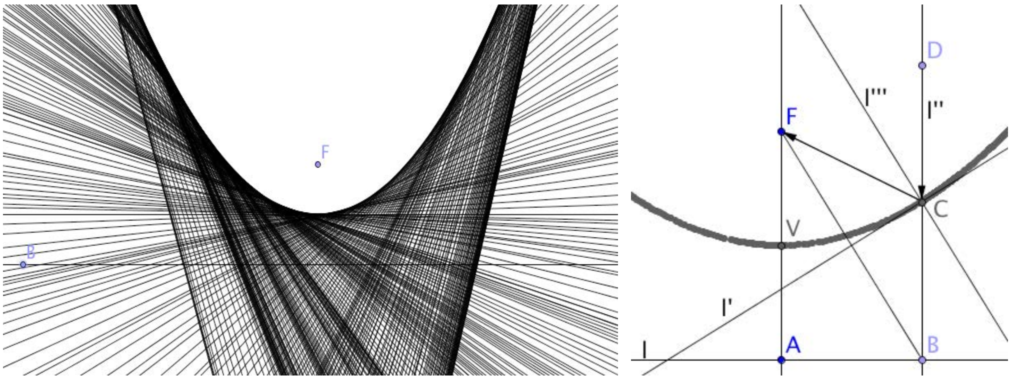

Figure 2: A family of tangent lines to the parabola (left) and reflection property (right)

of FB is $\frac{-p}{x}$, so the slope of l' is $\frac{x}{2p}$. This result agrees with the Calculus approach, by which the derivative is $\frac{d}{dx}(\frac{x^2}{4p}) = \frac{x}{2p}$. Finally, the equation of the tangent line at point $(x_0, \frac{x_0^2}{4p})$ is $y = f(x) = \frac{x_0}{2p}x - \frac{x_0^2}{4p}$. One can also see that $f(x_0) = \frac{x_0^2}{4p}$ and for any $x_1 \neq x_0$ we have $f(x_1) = \frac{2x_0x_1 - x_0^2}{4p} < \frac{x_1^2}{4p}$ since $x_0^2 - 2x_0x_1 + x_1^2 = (x_0 - x_1)^2 > 0$ and $p > 0$. The inequality demonstrates that the points of the parabola $y = \frac{x^2}{4p}$, $p > 0$, lie above any tangent line so that the family of tangent lines form the envelope of the curve (Figure 2, left).

The above derivation, along with reflection upon its results, is an example of constructing the crystalline concept of the parabola. In this project, I was able to observe that my students had adequate knowledge in order to move from the first to the third stage of SOLO Taxonomy in understanding and conceptualization of the parabola within only one lesson period. I did not conduct any activities aiming at identification of the fourth, extended abstract, stage. Nevertheless, I will comment on some opportunities emerging from this lesson. Clearly, a learner familiar with the geometric definition of parabola may find it satisfactory to obtain the same result by the more universal method learned in Calculus. By using dynamic geometry software and moving the point B along the line l, one can see that the point C traces a curve to which the line l' is tangent (Figure 1, right). A learner equipped with such experiences will develop a more comprehensive understanding of the object called parabola and might be able to use alternative representations and properties of a parabola and its tangent line depending on the problems they need to solve. By developing representational flexibility of an object (e.g., the parabola) students become better prepared for solving non-routine problems (see also Bergsten, 2015).

The geometrical view allows students to make closer connections of mathematics with physical phenomena, such as reflection of light from a parabolic mirror. Indeed, since $\angle FCB$ and $\angle FCD$ are supplementary angles, their angular bisectors l' and l''' are orthogonal to each other (Figure 2, right). The parabolic mirror near point C can be 'replaced' by the tangent line l'. This implies that the ray DC parallel to the axis FA of the parabola will be reflected at the point C towards the focus F. Such observations and insights are especially important for future teachers of mathematics and science because it will allow them to enrich discussions in their classrooms.

Students' unfamiliarity with the geometric definition of the parabola prior to my lesson can be explained by recent changes in the mathematical curriculum. Bergsten (2015) observed that in Sweden, while in the 1960s the study of the parabola was embedded in a local mathematical organization of analytic and Euclidean geometry, since the 1980s it became embedded in a local mathematical organization of functions. Similarly, in Canada the geometrical definition of conics no longer has a place in the secondary school mathematics. At the university level, this definition is supposed to appear in the Calculus stream, for example, when equations of conics in polar coordinates are introduced. But at that point there is no time to study the geometry of the parabola in any detail because the focus of the course is on different methods. Thus, university graduates with a mathematical degree may actually never have seen a discussion of a parabola as a geometrical object even though they may know properties of geometrical objects (isosceles triangle, perpendicular bisector, etc.) necessary for understanding how and why the Sample solution 2 works, as was the case in my project.

CONCLUSION

The fact that completion of a bachelor degree in Mathematics does not automatically ensure *deep* knowledge of elementary mathematics is not new (see discussion in Winsløw & Grønbæk, 2014) and references there). Indeed, even if students can apply an advanced approach or formula familiar to them, they are not always able to elaborate or explain why that formula works through connecting more advanced ideas with elementary facts. I conjecture that in some cases the students are simply not aware of relevant elementary ideas, definitions, and interpretations.

To verify my conjecture I developed an instrument that can identify the degree to which students are familiar with various solutions of a standard problem in Calculus. A preliminary study revealed that while the students felt comfortable applying the standard method, most of them were not familiar with the ideas which are more elementary compared to the considered standard method. I am planning to conduct a larger study in order to validate these findings.

Meantime, I suggest that while it is important for students to be exposed to standard and advanced approaches through university courses, their education should also include experiences highlighting other approaches and more elementary mathematical ideas related to these advanced methods (especially if the students already have necessary background to understand and elaborate on these ideas). Otherwise, they will tend to stick to procedural and formal methods and will not be able to fully appreciate the results of advanced approaches because the connection to more basic but enlightening mathematics will be lost.

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The Rechenbrücke – A project in the introductory phase of studies

Ronja Kürten and Gilbert Greefrath

University of Münster, Münster, Germany, ronja.kuerten@uni-muenster.de

In the transition from secondary to tertiary mathematics education, major difficulties occur for many students. To respond to this situation, the University of Applied Sciences Münster started the Rechenbrücke project with the object to alleviate the transition from school to university for engineering students through various support measures in the area of mathematics. In this article, we present the preliminary course which is part of the support measures of the Rechenbrücke and aims at refreshing basic calculating skills. The initial results of the pilot study presented indicate the potentially positive effect of these measures.

Keywords: Secondary-tertiary transition, preliminary courses, bridging courses.

INTRODUCTION

The transition from school to university brings along many changes which take part over a vast period of time starting at secondary school and ending at some point during the course of studies. Differences between school and university can be found in thought structures and knowledge organisation, argumentation, and mathematical communication, as well as teaching methods and the expectations of the institution (Gueudet, 2008). Bridging or preliminary courses cover one part of that period and as they are currently common measures to alleviate transition from school to university they present an important factor in the transition issue.

In Germany there are at least two types of tertiary education institutes. Although the differences were reduced in the last years, there are still some important distinctions between universities and universities of applied sciences. While the first ones are "normally strongly research-oriented and typically offer a wide range of subjects" (Hochschulrektorenkonferenz, n.d.) the latter emphasise practical work and application and "usually offer a narrower range of subjects [with focus] on fields of engineering, business, and social sciences" (ibid). Furthermore students at universities of applied sciences are even more heterogeneous with often considerably lower mathematical skills than those at universities as you can start your studies without an A-level school leaving certificate (Abitur) but with an advanced technical certificate (Fachhochschulreife) or even with a finished apprenticeship and some work experience. Therefore many students spend some years between school and university working or in an apprenticeship.

The secondary-tertiary transition is experienced as difficult by many new students who take mathematics courses at tertiary level (De Guzman, Hodgson, Robert, & Villani, 1998). High drop-out rates and low satisfaction with the chosen study course are the results. In universities of applied sciences the drop-out rate of engineering students has been at 31% in the last years (Heublein, 2014, p. 501). While it is difficult to interpret the concrete percentage due to varying definitions of drop-out, the drop out quote of engineers is the second highest (after *Mathematics and Natural Sciences* with 34%) and is notably higher than the average drop-out of 23% (ibid). The difficulties described above also occur for engineering students at University of Applied Sciences Münster, for which reason the Rechenbrücke project has been started in order to alleviate the transition from school to university. The Rechenbrücke is a cooperation project between five of the engineering faculties of University of Applied Sciences Münster and the Institute for Education in Mathematics and Computer Science of the University of Münster [1]. In this project, various mathematics support measures for prospective and first-year engineering students at University of Applied Sciences Münster are developed and evaluated.

THEORETICAL FRAMEWORK

In many countries students entering higher education lack proficiency in basic mathematical concepts and therefore are not able to develop a sound understanding of those key mathematical ideas needed at universities (e.g., ACME, 2011, Biehler et al., 2014). The situation is even worse at universities of applied sciences as there students with many different learning backgrounds enter studies. The Standing Conference of the Ministers of Education and Cultural Affairs of the Länder in the Federal Republic of Germany (Kultusministerkonferenz) adopted educational standards in mathematics in 2003/04. These standards describe subject-based competencies to be acquired by students by certain stages of their school career. The competence K5 "Mit symbolischen, formalen und technischen Elementen der Mathematik umgehen" (KMK, 2012, pp. 8f.) which includes e.g., working with variables, terms, equations, functions etc. is needed by the other competencies as a tool (ibid). Therefore lacking mastery of K5 often results in lacking mastery of the other competencies.

Preliminary courses and other support measures have been created in many countries to alleviate the secondary-tertiary transition. As such courses are often designed by single universities, there are many different approaches. Decisions to be made in designing preliminary courses are discussed in Kürten, Greefrath, Harth, & Pott-Langemeyer (2014): Framework conditions, objectives and contents, and competencies to be taught have to be clarified. Framework conditions vary from e-learning to blended-learning to classroom courses, from some weeks to two semesters, with voluntary or mandatory participation. Course objectives range from compensation of mathematical deficits to repetition of school mathematics to development of skills used at tertiary education. The contents taught vary from specific mathematical contents (from lower secondary to tertiary level) to mathematical ways of thinking to metacognitive competencies (Biehler et al., 2014). Despite this richness in course conceptions, research on these measures is still at its beginnings. Questions like "How can we measure the effects of preliminary courses?" or "What are the effects of preliminary courses?" have yet to be answered (Biehler et al., 2014).

Many support measures share the aim to enhance success rates by reducing student drop-out, but drop-out quotes can only be directly measured after the students of a cohort have left university. Therefore the effects of support measures have to be measured by other variables which can be measured shortly after realization of the support measure and are predictors for the drop-out quote. Heublein (2014, p. 504) characterises the drop-out as result of a complex inter-relationship between individual and institutional factors. Performance in studies is just one factor in that process but on the other hand it is a factor which can lead to drop-out enforced by university. Therefore, and as it can be measured by exam grades, in this paper, we will focus on effects the preliminary course has on the students' performance in studies. A similar approach is used at the University of Applied Sciences Aachen were first results show, that students that attended the preliminary course get significant better results in their mathematics exams than students that did not attend (Greefrath, Hoever, Kürten, & Neugebauer, 2015).

INITIAL SITUATION

In individual interviews with the responsible mathematics lecturers of University of Applied Sciences Münster an estimation of the difficulties, students faced at the beginning of their studies, was ascertained. All the academic staff asked stated a lack of proficiency from school mathematics, especially from lower secondary level. Namely they claim that students fail in written exams in mathematics due to missing skills in term conversion (a common mistake is the linearization of all kinds of terms as in $(a + b)^2 = a^2 + b^2$) and fraction arithmetic (e.g. a common mistake is the addition of numerator and denominator: $\frac{a}{b} + \frac{b}{d} = \frac{a+b}{c+d}$). These basic skills from lower secondary level are required for studying engineering at University of Applied Sciences Münster and therefore are not treated explicitly in the lectures. Nevertheless they are needed for solving of various problems in calculus and linear algebra. In her study regarding the British education system, Kitchen (1999) also determined deficits in the prior knowledge required of new students by universities. The answers from the mathematics lecturers point out numerous parallels to the results of De Guzman and colleagues (1998). They describe a lack of interest in mathematics itself, of necessary prior knowledge, of an appropriate thought structure, independence, learning methods and organisational skills. Therefore the situation at German universities of applied sciences is not unique in Europe. Thus answers to the question whether basic calculation skill training can alleviate transitions problems (especially for students with low mathematical skills) should be of broad interest.

CONCEPTUALISATION OF SUPPORT MEASURES

Within the scope of the project, various measures have been designed which should alleviate the transition to university in the area of mathematics. As the target group contains many students with assumed low mathematical skills, focus was laid on competence K5 to create a basis on which further mathematical understanding could be built as described above. In an initial step, a collective minimum requirements catalogue for mathematics was created. Based on this, a modular preliminary course was developed that encompasses an upstream diagnostic pre-test. Different scientific evaluation measures (e.g., a post-test and evaluation sheet subsequent to the preliminary course) are used to examine the effectiveness and feasibility of these offers.

The minimum requirements catalogue

The minimum requirements catalogue of mathematical prerequisites for students in engineering courses of study is based on the mathematics minimum requirements catalogue which the cooperation team school – university (cosh) in Baden-Württemberg developed as a consensus of schools and universities. As a result, it can provide an expedient basis for the selection of mathematical contents in a preliminary course or a mathematics test at the beginning of studies (Dürrschnabel et al., 2013). In the project this catalogue was adapted and defined in coordination with the mathematics lecturers at University of Applied Sciences Münster. To improve the transparency of the requirements, the catalogue has been published for all interested parties on the homepage of the University of Applied Sciences [2].

The mathematics preliminary course at University of Applied Sciences Münster

The concept for the mathematics preliminary course was chosen to account for three aims: As much facets as possible mentioned above should be considered, it should be possible to recapitulate most of the topics posed in the minimum requirements catalogue by visiting the preliminary course and the expectations of the teaching staff in mathematics should be satisfied as far as possible. The result was a modularly structured preliminary course concept created in the first half of 2013. In the course, contents of school mathematics from the lower and upper secondary levels are addressed in ten modules developed as a consensus of the participating staff of which the following arose from the initial interviews: *Basic calculation methods and fractions*; *terms*; *equations with one unknown* and *functions (part I)*. The minimum requirements catalogue was taken as a basis for the modules that complement the preliminary course: *Percentage calculation, power and square root calculation*; *inequations with one variable*; *functions (part II)*; *differential calculus*; *integral calculus* and *simultaneous equations, vectors and matrices*. Besides post-processing of school mathematics, preparation for the requirements of the study course is also featured. Two modules address interdisciplinary mathematical and course-relevant aspects such as notation of statements and equations, argumentation, systematic approaches, lecture post-processing, examination preparation, etc. Here preparation refers to metaknowledge, interdisciplinary methodology, and organisational information (Hoffkamp, Schnieder, & Paravicini, 2013). The preliminary course takes place as a classroom course composed of lectures and tutorials, which is supported by E-learning contents. Amongst other things, this includes parts of scripts, exercises and short videos regarding individual subjects. Due to the large number of new students, the course is conducted in two parallel courses divided according to faculty.

Based on an entry test, students can decide which modules of the preliminary course they wish to process. Subsequent to conducting the test, students receive feedback as to which tasks they have solved correctly or wrongly. For every wrongly answered task, they receive the additional urgent recommendation to visit the corresponding module of the preliminary course. In this way, the target group should be sensitised to the support offer. Given that the clientele of University of Applied Sciences Münster comprises of full-time, parallel-to-profession and dual system students, the online materials should simultaneously enable preparation for the study course. Furthermore, students can decide which module they wish to process in the classroom course or online. Modularisation and flexibility should furthermore enable preparation for new students for their study course, adapted to their personal proficiencies.

RESEARCH QUESTIONS AND METHODS

Given the initial situation described above and the project aim to alleviate secondary-tertiary transition for engineering students with heterogeneous and partly low mathematical skills, three research questions were designed.

1) What are the difficulties experienced by new engineering students when trying to use school mathematics?

2) Can a preliminary course reduce these difficulties and if yes, to which extend?

3) To which extend do these difficulties influence the outcomes of written exams in mathematics?

Within the scope of the project, data from mathematics tests and exam results are collected. In addition to these quantitative tests, semi-structured interviews are conducted with students and the students' evaluations regarding the conducted measures are gathered with the aid of a questionnaire in order to ascertain the effects of the measures in a differentiated manner. In the following, we address the stated quantitative research methods and initial results.

In the project a pre- and post-test are used in connection with the preliminary course. The pre-test should not merely act as a self-diagnostic instrument providing the students with feedback regarding their existing deficits, but moreover, in combination with the post-test, should serve as an evaluation of the preliminary course. Based on these objectives, the test has become mandatory for all who wish to take part in the preliminary course, and is set up voluntarily for all others. Two dates were selected for implementation – prior to and shortly after the preliminary course. The test procedure is computer-based and, for the objectivity of test procedure, takes place centrally in PC pools at the university. The tests have been integrated into the learning platform, ILIAS, used throughout the university. The evaluation requires automatically correctable task formats such as single choice and multiple choice tasks and tasks with short answers. Given that fundamental proficiencies are queried in these tasks, the test procedure was designed as assistance tool-free. The test comprises of 13 items which are associated with the contents of the preliminary course modules (see Table 1). The following items are two of those used in the pre-test 2013 (translation into English).

Task 2: Calculate and write as an irreducible fraction $\frac{a}{b}$.

a) $\frac{\frac{1}{3}+\frac{3}{5}}{\frac{7}{5}}=\frac{a}{b}$ $a =$ ____ $b =$ ____

Task 10: We are seeking an equation of the line with the following characteristics: intersects the x-axis in and has the slope . The equation should be stated in the form $y = m \cdot x + b$.
$m =$ ____ $b =$ ____

In addition, „Type of school qualification", „Time since the end of school", „Average mark in school leaving qualification", „Final mark in the subject mathematics", and „Use of calculators in upper secondary level" were ascertained. After the preliminary course, the students take part in a parallel-designed post-test. This takes place at the beginning of the lecture period. The first implementation of this test took place as a pilot study in winter semester 2013/14. For organisational reasons, the pre-test was carried out by students on home PCs and the post-test as a paper test during Mathematics I lecture at the beginning of the semester. In addition to the information ascertained in the two tests, the students' results in the written mathematics exams are ascertained after the first (Mathematics I) and second (Mathematics II) semester. An example for items used in the exams is shown in Figure 1.

The written exam marks of the students are anonymously assigned to the corresponding data gathered in the tests. A possible correlation between a preliminary course attendance and the results in written exams should be investigated using this data.

RESULTS

689 test persons took part in the pre-test, and of these, 603 completed the test and provided evaluable results. Of initial interest for the Rechenbrücke project was

1. b) Calculate the real and imaginary part of $z = \left(\frac{i+1}{\sqrt{2}}\right)^{14} \cdot i^3((i+1)^2 + i)$.

Figure 1: Item of the Mathematics I exam in mechanical engineering (translation into English)

the question of whether the contents of the preliminary course were expediently selected, i.e. whether room for improvement actually exists for these topics. The solution rates and topics of the individual tasks are depicted in Table 1. Even the task regarding basic calculation methods, with the highest solution rate, could only be solved by less than three quarters of the test people. Task 4 (power and square root calculation) had the lowest solution rate with only 21% correct answers. The solution rates of the items indicate that addressing these contents in the preliminary course is sensible.

Due to the non-objective test procedure in the pilot study, a comparison of the solution rates between pre- and post-test is only conditionally possible. For a small portion of the random sample (N = 209), the results from the pre- and post-test could be correlated. In the course of this, significant changes with a small effect size (|Cohens d| > 0.2) were indeed determined for several tasks (improvement with tasks 1, 4 and 6 and deterioration for task 7), however these results serve as an indicator at the most and will be checked in the upcoming run in the actual investigation. The

Task and Topic		Solution rate
1	Basic arithmetical operations involving brackets in Z	73%
2	Basic arithmetical operations on fractions, cancelling	61%
3	Percentage calculation: markup problem	67%
4	Solving of a fractional equation where numerator and denominator involve powers and square roots	21%
5	Term conversion using binomial formulas	52%
6	Calculation of inverse proportionality using the rule of three	69%
7	Solving of a fractional equation	70%
8	Solving of a quadratic equation	36%
9	Solving of a linear inequality	63%
10	Setting up the equation of a linear function	53%
11	Offsetting of a parabola	31%
12	Solving of a simple logarithm equation	33%
13	Trigonometry: definition of sine, cosine	53%

Table 1: Solution rate for the individual tasks in the pre-test (N = 603)

reliability of the tests was determined by means of Cronbach's Alpha. For the pre-test, a value of 0.72 (N = 603) resulted. For the post-test, the value lies at 0.73 (N = 805). Therefore the internal reliability of the test is acceptable.

Based on the data of the tests and written exams, the correlations between written exam results in Mathematics I (the first lecture on mathematics in engineering studies at University of Applied Sciences Münster) and potential influencing factors have been investigated. The variables investigated were „Average mark of school leaving qualification", „Final mark in mathematics", „Number of points in the pre-test" and „Number of points in the post-test". Statistically significant correlations ($p < 0.01$) were determined for all variables which lie between 0.37 and 0.45. The differences between the correlation coefficients were tested with Fisher's z-transformation and are not significant.

Furthermore, it was investigated whether the results in the written mathematics exams of students who have or have not visited the preliminary course differ. Looking at the overall random sample (N = 280), no significant deviation could be initially determined. However, if the results are divided according to the preliminary course groups, then for group B (Faculties of Chemical Engineering, Mechanical Engineering and Physical Technology, N = 176) a significant deviation ($p = 0.037$) is obtained with the Mann-Whitney U-test. In these faculties, the written exam results of the students who have taken part in the preliminary course are significantly better than those of the students who have not taken part in the preliminary course. In the other preliminary course group A (Faculties of Electrical Engineering and Computer Science and Energy – Buildings – Environment, N = 104) no significant deviation of distribution could be determined. A comparison of the average mark of school leaving qualifications showed no significant difference between the students either in group A or group B who took part in the preliminary course or not.

DISCUSSION

In winter semester 2013/14, the first measures of the Rechenbrücke project started. In the course of the investigation conducted, initial data was ascertained in pre- and post-tests and evaluated.

The difficulties experienced by new engineering students determined in the pre-test already confirm that the repetition of elementary calculation skills from mathematics lessons is expedient. Corresponding results are also indicated in the tests conducted by University of Applied Sciences Aachen. For each initial test, the students there attained an average solution rate of 40–50%, depending on the prerequisites (Greefrath et al., 2015). The solution rate of the entrance test which has been conducted for more than ten years at the Universities of Applied Sciences in North Rhine-Westphalia lies even below this (Knospe, 2011). A possible reason for this lack in basic skills might be the task of solving arithmetic problems without using a calculator which is unusual for students in North Rhine-Westphalia.

At this point of the project a definite answer to the second research question cannot be given as the results of the pre- and post-tests are not comparable due to the non-standardised procedure. The determined tendencies of a slight improvement for some tasks could however be an indicator for the success of the preliminary course. The cause of the deterioration for the result of task 7 must be further investigated. The introduction of prerequisites which must be observed prior to solving a fraction equation possibly had a negative influence on the previous, direct calculation of the solution. The positive correlation determined between preliminary course participation and performance in the Mathematics I written exam in some faculties can have diverse reasons, such as a higher level of motivation and willingness to learn among students who take part in a preliminary course. The quantitative numbers presented here are underpinned by the answers given in the questionnaire: Students state, that they feel better prepared for mathematics lectures after visiting the preliminary course. The evidence presented above gives reason to state the hypothesis that the preliminary course can reduce the difficulties in school mathematics of new students. Further investigations will test the hypothesis to answer the second research question.

During the search for causes for different effects in both of the preliminary course groups, differences in prior knowledge were sought and measured on the basis of the school leaving qualification marks. However, the research has not yet revealed any significant differences between the different groups. Both preliminary course groups were conducted by the same lecturer in the same way; however, the subsequent mathematics activities in the various faculties are different in their content and in the way they are conducted. For this reason, the match between the preliminary course concept and the lectures is possibly not given to the same extent in all faculties. In the following execution of the preliminary course, audio recordings or written minutes could improve the monitoring of the course implementation.

Concerning the outcomes of written exams in mathematics in this investigation correlations between all variables and the Mathematics I exam have been found. Correlations between pre-test respective post-test and the Mathematics I exam can be explained by the basic skills which are needed to solve the items (e.g., that in Figure 1) used in the exams. An interesting result is that the final mark in mathematics seems to be no better predictor for exam results than the average mark from school leaving qualification or the points achieved in the post-test.

The modular preliminary course concept with the possibility of an individual compilation of preliminary course topics by the students provides great flexibility. To what extent this flexibility is utilised by the students and which effects this has on performance in the written exams will be investigated in the main study. Furthermore, the Rechenbrücke supports the first semester students for the transition from school to university through further, in part freely selectable measures. The results from the pilot study presented in this article indicate promising findings from the performance of the main study in the coming years.

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ENDNOTES

1. The Rechenbrücke has a run time from January 2013 until July 2016 and is financed by the “Bund-Länder-Programm für bessere Studienbedingungen und mehr Qualität in der Lehre” (Federal and state programme for better study conditions and quality in teaching).

2. The catalogue is available for download here: https://www.fh-muenster.de/rechenbruecke (downloaded on January 29, 2014).

Use of mathematics in engineering contexts: An empirical study on problem solving competencies

Malte Lehmann, Bettina Roesken-Winter and Sven Schueler

Humboldt-Universität zu Berlin, Berlin, Germany, malte.lehmann@hu-berlin.de

The research field of problem solving in mathematics is highly relevant in mathematics education. There are manifold approaches to understand the process of problem solving like Polya's phase model or Bruder and Collet's heuristic methods. In two studies, we investigate how engineering students' performances in higher mathematics and technical mechanics are connected (study 1, n=37) and what role their problem solving competencies play (study 2, n=8). In the first study, we ascertained that mathematical competencies and beliefs about physics are substantial for success in technical mechanics. In the second study, students had to complete sequences of tasks and their usage of heuristics was investigated. The results show that successful students' heuristic tools and strategies are more elaborated.

Keywords: Problem solving, engineering education, technical mechanics, qualitative content analysis.

INTRODUCTION

Mathematics is an important subject in engineering education. The first courses of study are characterized by a high usage of mathematics, be it in mathematics, in physics or further engineering lectures. Besides continuously improving their declarative and procedural knowledge in the respective fields, it is important for students to develop their problem solving competencies as this is one of eight competencies that engineering students need to learn according to the SEFI[1] (2013) framework: *thinking mathematically, reasoning mathematically, posing and solving mathematical problems, modelling mathematically, representing mathematical entities, handling mathematical symbols and formalism, communicating in, with and about mathematics* and *making use of aids and tools*. In accordance with findings in mathematics education (cf. Törner, Schoenfeld, & Reiss, 2007), in SEFI (2013) problem solving as competency for engineering students is characterized as follows:

> This competency [mathematical problem solving] includes on the one hand the ability to identify and specify mathematical problems [...] and on the other hand the ability to solve mathematical problems (including knowledge of the adequate algorithms). What really constitutes a problem is not well defined and it depends on personal capabilities whether or not a question is considered as a problem. (p. 13)

The formation of these competencies is, however, often hampered by an asynchronicity of mathematical and engineering education. The overarching aim of the project KoM@ING[2] is to measure mathematical competencies of engineering students and the relations between the different lectures by combining a quantitative and a qualitative perspective. In the quantitative approach (project partners from IPN[3] Kiel and University Stuttgart), IRT-based measures for higher mathematics and technical mechanics are developed to capture students' development in their first year of study (Nickolaus, Behrendt, Dammann, Ştefănică, & Heinze, 2013). Thus, individual competencies are measured reliably and validly, but no insight is provided into the students' actual problem solving processes. This paper reports on the work of the qualitative project that scrutinizes these processes.

1 European Society for Engineering Education

2 German acronym for "Modelling Competences of Engineering Students", funded by BMBF (Ministry of Education and Research)

3 Leibniz Institute for Science and Mathematics Education

THEORETICAL FRAMEWORK

Mathematical problem solving is an important field in mathematics even at school and at university (cf. Halmos, 1980). There are many different approaches for research on this topic (cf. Schoenfeld, 1985; Chinnappan & Lawson, 1996; Rott, 2013). One perspective focuses on the *inner structure* of problem solving processes considering heuristics and beliefs, another perspective elaborates on the *outer structure* in terms of timing and organizing of processes. In our study, we focus on both aspects by investigating phases of the problem solving process (cf. Polya, 1945) and the use of heuristics (cf. Bruder & Collet, 2011). In his seminal work, Polya (1945) differentiates *understanding the problem, devising a plan, carrying out the plan,* and *looking back* as essential phases of any problem solving cycle. Schoenfeld (1985) extends the model of Polya by adding an *exploration* phase which interacts closely with the *planning* phase, but also allows a throwback to the first phase. Likewise, Chinnappan and Lawson (1996) stress the importance of the first two phases: "[...] the planning process forced the solver to make optimum use of information that was identified and information that was generated" (p. 13). In addition, Chinnappan and Lawson (1996) could show that not only the single use of heuristics is important, but also a training in management strategies that allows for effectively coordinating different phases. However, Rott (2013) could show that completely linear models are not always suitable to describe problem-solving processes and provides empirical evidence for applying a more flexible model. Bruder and Collet's (2011) work on heuristics places the methods in the center and explains the process of problem solving by heuristic tools (e.g., *informative figure*), heuristic strategies (e.g., *using analogies*) and heuristic principles (e.g., *symmetry principle*). In addition, the problem solving process is influenced by the interplay of mathematics and physics as described in the framework by Tuminaro and Redish (2007) that elaborates on so-called *epistemic games*. These *epistemic games* like, for instance, *mapping mathematics to meaning,* allow the description of how students make the transition from novices to experts, and contribute lenses to elaborate on either *individual-related* or *task-related* characteristics. In order to solve problems with physical contexts, students need adequate beliefs about the physical concepts involved. From beliefs research we know that pre-service teachers "who approach learning physics with a more favourable belief structure are more likely to achieve higher learning gains" (Mistades, 2007, p. 185). An instrument to reveal students' beliefs about the force concept is provided by Hestenes, Wells, and Swackhamer (1992) which proved to be useful in several studies. Based on these theoretical foundations, we investigate how students' performances in higher mathematics and technical mechanics are connected and what role their problem solving skills play:

Research question 1: Do students' performances at school, their mathematical skills, and their beliefs about physics concepts (related to the force concept) predict their achievement in technical mechanics?

Research question 2: Can task difficulties in higher mathematics and technical mechanics be described through analysing occurrences of Polya's phases and students' use of different heuristics?

Research question 3: Does working on task variations in technical mechanics that imply increasing difficulty requires using more and different heuristics?

The first and second research question will be answered using the results from study 1. Research question 3 is answered by referring to the results of study 2.

METHODOLOGY

The research comprises two different, but related surveys. The first one is taken from our larger study to investigating the development of students' problem solving competencies in their first year at university. That is, students work on the IRT-scaled test for higher mathematics (HM) and technical mechanics (TM) from the quantitative project. The other survey is task-related and investigates four sequences of tasks about statics and the different usage of problem solving competencies. Here, students are observed while working on and discussing tasks of increasing difficulty.

Our project partners developed *TM/HM pre-tests* delivered at the beginning of students' studies. The pre-tests consist of 28 (TM) and 36 (HM) items with a combination of closed (multiple choice, multiple true-false) and (semi-) open questions. The TM pre-test covers items on statics, kinematics, kinetics, energy and momentum, oscillations and basic concepts, all on a higher school level. For example, one task is:

Time	Content
09/2013	TM/HM pre-test, IQ-test, FCI; Video: easy/difficult tasks (pre-test)
04/2014	Video: easy/difficult tasks (pre-test)
09/2014	TM/HM post-test, FCI; Video: easy/average/difficult tasks (post-test)

Table 1: Overview study 1

Draw a possible amplitude curve of a damped oscillation in the diagram.

The HM pre-test, based on work by Hauck (2012), asks questions on mathematical basics, calculus and geometry. One example is:

Given is a plane E: $6x_1+6x_2-3x_3= -12$ and the point A (3/6/4)

a) Determine the distance d of the point from the plane.

b) Calculate the coordinates of A' (A mirrored at the plane).

These tests are IRT-scaled (pilot study with N=1069 students) and thus enable us to categorize our participants in terms of their mathematical, physical and technical competencies at the beginning of their university course. Additionally, the scaling affords us selecting the five easiest and most difficult tasks.

The Force Concept Inventory (FCI; Hestenes, Wells, & Swackhamer, 1992) investigates students' beliefs about the concept of force which is an important part of competences in mechanics. It consists of 30 items, each of them describing a physical situation. One must choose among five possible answers, one relating to the correct Newtonian concept and four relating to alternative concepts. Opportunities to analyse the scores vary. Either the raw score of correct answers can be used or special attention can be paid to how the wrong answers relate to misconceptions. We decided to use the raw score to investigate students' changes during their first year (cf. Hestenes & Halloun, 1995), because we have been interested in analysing the changes in students' beliefs during the first year.

Design of study 1: Development of problem solving competencies

In the first study data was collected at three points (Table 1). In September 2013, the students (n=37; male=27, female=10) worked on four tests: the TM pre-test, the HM pre-test, an intelligence test (CFT-3, Culture Fair Intelligence Test) and the FCI. In this paper, we only focus on this first measurement point, but provide an overview on the whole study in Table 1. The longitudinal design was chosen to capture students' development of their mathematical and physical skills, and problem solving competencies. These results will be presented elsewhere.

In addition, the students worked in groups of two or three on the five easiest and five most difficult tasks of the two pre-tests (Table 2).

The group work was recorded on video for later analysis. The *thinking aloud*-method (Ericsson & Simon, 1984) rendered thoughts and ideas observable. In order to analyse the video data and the students' work, we developed a category system based on the theoretical framework described above (Table 3).

The category system was carefully tested and optimized in a pilot study. As a result we arrived at 38 categories as an assessable selection with reasonable frequencies.

HM-Topic	Difficulty	TM-Topic	Difficulty
Equation	-3.71	Kinetic energy	-1.98
Number line	-3.39	Deflection curve	-1.89
Reflection at a plane (see above)	2.80	Trajectory	1.80
Exponential function – Extremum	3.22	Load	3.17

Table 2: Overview on topics of the chosen tasks (in excerpts: 2 easy/2 difficult tasks)

Structure	Category	Sub-categories
Inner Structure	Heuristic tools	Informative figure, Table, Equation
Outer Structure	Polya's phases	Understanding the problem, devising a plan, Carrying out the plan, Looking back

Table 3: Category system (in excerpts)

Design of study 2: Task-related problem solving competencies

The second study investigates characteristics of tasks by special sets of TM tasks at two measurement points (cf. Table 4). The set consists of two to four tasks with increasing difficulty for the four topics of gravity, reaction forces, displacement and deformation, free body diagrams and force systems (cf. Figure 1). Four students handled every sequence. These tasks are taken from the piloting phase of the TM post-tests where they showed inadequate model fit and therefore were rejected in the IRT-model. In our study we investigate how slightly changing task features influence students' problem solving behaviour.

Time	Content
05/2014	Video: easy tasks
06/2014	Video: more difficult tasks; FCI

Table 4: Overview study 2

To control for students' competences we considered their first term grades (calculus, linear algebra and mechanics) as indicators. The participants (n=8, male=6, female=2) worked alone on the tasks and received in advance a detailed introduction into the *thinking aloud*-method (Ericsson & Simon, 1984), including a preceding practice task. After that, students chose a topic, were given two to four tasks accordingly (cf. Figure 1), and processed the tasks in ascending difficulty. Finally, all students were interviewed and asked to reflect their courses of action. The video data was analysed by using the category system (see study 1). While students' work was analysed as in study 1, the interview data was also checked for task characteristics and the conscious use of problem solving strategies.

RESULTS

Study 1

As mentioned earlier, only the findings from the first measurement point can be reported at the moment, as the data from the second and third survey have not yet been completely analysed. Our participants are comparable with the students from the larger pilot study conducted by the quantitative project. In detail, they received the following mean personal ability estimates and results: M_{HM}=-0.11 (*SD*=1.38), M_{TM}=-0.86 (1.01), FCI=11.35 (5.78) out of 30 points. To clarify which factors affect students' performance in the processing of tasks due to physical contexts, several multiple linear regressions and the corresponding correlation matrices were calculated. For the correlations of pre-test results and prior knowledge, represented

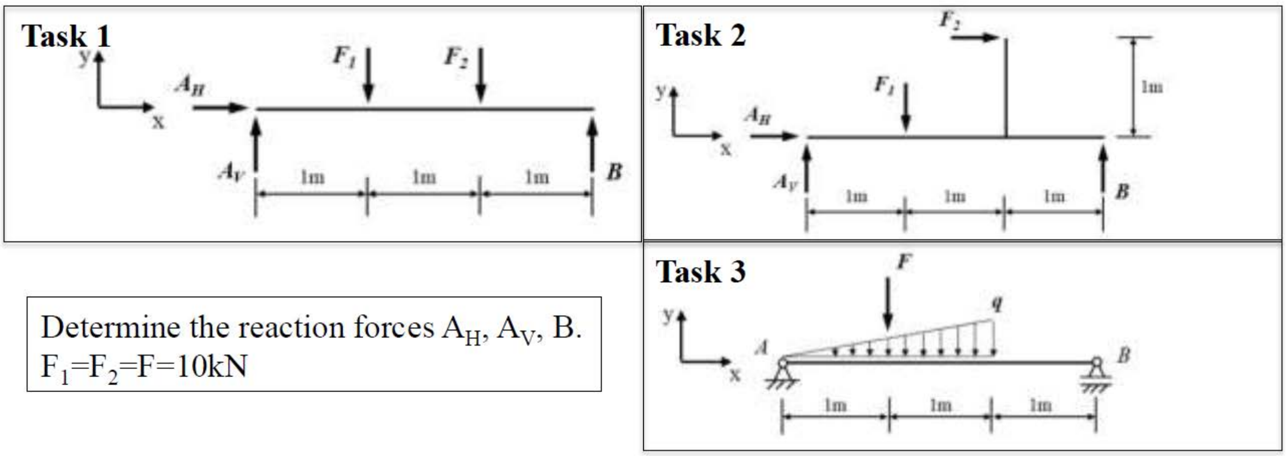

Figure 1: Task sequence: "reaction forces"

	Results TM pre-test	Results HM pre-test	Results FCI	Grade School Certificate
Results TM pre-test	1			
Results HM pre-test	.676**	1		
Results FCI	.744**	.696**	1	
Grade School Certificate	-.382*	-.612*	-.372*	1

**$p < .01$, *$p < .05$

Table 5: Correlation matrix (first measurement point)

Predictor	b_i	β_i	*Sig.*
(Constant)	-1.883		$p = .00$
Results HM pre-test	.236	.306	$p = .05$
Results FCI	.093	.531	$p < .01$
R^2=0.602			

Table 6: Multiple linear regression

	Easy tasks	Difficult tasks
Heuristic Tools (HM/TM pre-test)	31%	69%
Heuristic Strategies (HM pre-test)	37%	63%
Heuristic Principles (HM/TM pre-test)	0%	100%

Table 7: Distribution of heuristics

by the final grade in school certificate, the results are provided in Table 5.

Collinearity analysis detected that the factor *Grade School Certificate* contains the same information as the factor *result HM pre-test*, it is therefore not important for further calculations (cf. Table 6).

The model explains 60% of the variance in the TM pre-test results. In particular, beliefs about physics have a great influence on these results. In contrast, only 20% of all observed uses of heuristic strategies occurred when students worked on physics problems. Thus, the mathematical problem solving approaches, such as problem solving strategies and Polya's phase model, are only partly suitable for describing processes in solving physics problems. Here analyses by guide of the epistemic games framework, which are reviewed in a forthcoming analysing step, will probably deliver further explanations. However, when just looking at the use of heuristic tools and heuristic principles, we can conclude that these are used in a similar way when solving both mathematical problems and physics tasks. Table 7 shows the distribution of heuristics for the processing of easy and difficult tasks.

The results show that the students are increasingly using both tools and principles for difficult tasks. The situation is similar for mathematical tasks when using heuristic strategies. Overall, the students used heuristics rather rarely. Only in slightly more than half of the task processing of the difficult tasks, the use of heuristics could be observed (tools: 54%, strategies: 55%, principles: 27% of all difficult tasks). When solving the easy tasks, heuristics were less observed (tools: 30%, strategies: 45%, principles: 0% of all easy tasks).

Applying Polya's phase model to analyse the mathematics tasks, it becomes clear that for difficult tasks, students worked much more in cycles (for example, between *carrying out a plan* and *devising a plan*) in their solution process than for easy tasks, where they tend to go through the problem solving process linearly.

Study 2

For the sake of brevity we limit ourselves to presenting only the results of the sequence of tasks to reaction forces (see Figure 1). All students successfully solved the first task, using one specific algorithm. At the end of their solving process, only two students recognized

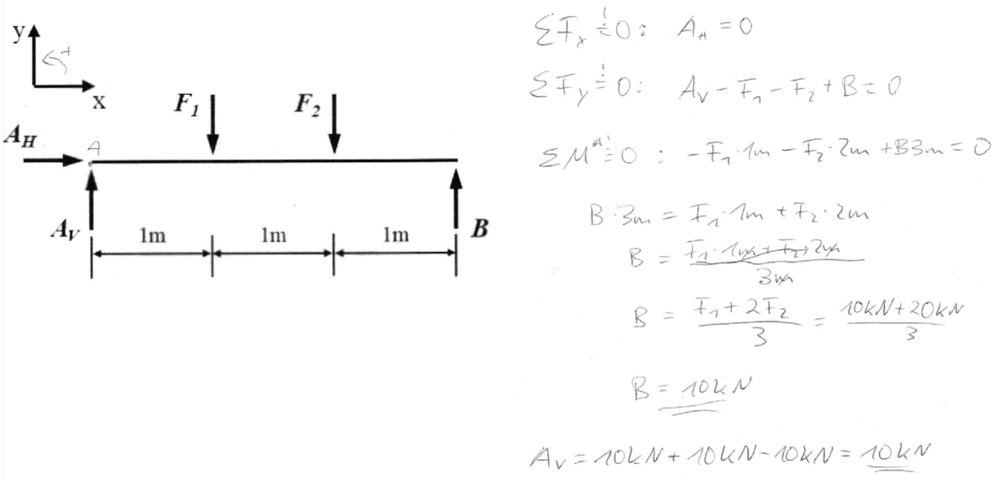

Figure 2: Student D's work on task 1

the symmetry aspect of the task. Students' work differed only slightly concerning the amount of the heuristic tools and strategies used. It is remarkable that the students who had mastered the more difficult tasks of the sequence, switched more frequently between the heuristic tools used when solving the simple task (here *informative figure* and *equations*). Table 8 summarizes the results that the four students received. As it is shown in Figure 1 and Figure 2, students were asked to compute the three forces A_H, A_V and B in each task; yielding the correct result was coded 1, otherwise 0 (see Table 8).

The following transcript of Denise's work shows the alternating use of the two heuristic tools mentioned above. She was able to determine the first two equations for calculating the x- and y- directions, and thus A_H, and started to calculate the third equation for the momentum.

From her think aloud protocol we gain the following explanations:

> So, for example, the sum of all momentums, what will be best? So you have to have any equation with only one unknown that you are looking for, yes. Then you could take, for example, the point (showing on the point, A_V and A_H point to), because the two go through that, then they do not have to be calculated. [The] Sum of all momentums in point A which is here (draws a point to the place to which A_V and A_H point and labels it with A) must be 0. So the positive direction of rotation is this (draws a curved arrow). F_1 (redraws arrow F_1) rotates negatively with the lever arm 1m, meaning minus F_1 times 1m, F_2 negative with 2m, and B (indicates B) is positive with 3m equals 0. F_1 and F_2 on the other side mean B times 3m is equal to F_1 times 1m minus F, plus F_2 times 2m. Then, you need to convey the 3m here (indicates the side of the equation with F_1 and F_2), by dividing, and then we have B alone in the end.

This procedure of combining heuristics is also seen in the processing of the two other tasks. The two other students (Alice, Bob) do not connect the *informative figure* and the *equations* in a similar way. They fail to solve the third task, since they apparently cannot transfer all the information from the figure correctly into their equations.

DISCUSSION

We were able to provide evidence for the expected influence of mathematical skills and beliefs about physics on the achievements in technical mechanics even in our small sample. In view of the above-mentioned asynchronicity of engineering education, this

	Alice[1]	Bob	Chris	Denise
Task 1 (A_H/A_V/B)	1/1/1	1/1/1	1/1/1	1/1/1
Task 2 (A_H/A_V/B)	1/0/0	1/1/1	1/1/1	1/0/0
Task 3 (A_H/A_V/B)	1/0/0	1/0/0	1/1/1	1/0/1

Table 8: Results of the task sequence "reaction force"

means that students first need to master the mathematical basics to be able to successfully work on physics problems and not, as is often the case, learn them at the same time or after encountering the physics contents. The results about problem solving in our first study suggest two conclusions. On average, more difficult tasks require a greater use of heuristic tools, strategies and principles. In addition, we noticed that in some tasks which are assigned a high difficulty estimate by the IRT scaling, the students' use of heuristics is comparable to their processing of the easy tasks. Overall, the students rarely use heuristics in solving the tasks, both in the easy and in the difficult tasks. We note that only in just over half of the difficult tasks the students use tools and strategies when working on them. The cause may lie in a low consideration of problem-solving tasks in mathematics education at school.

The results of the second study clearly show that it is not enough to simply master heuristics, although the use of individual heuristics helps solving tasks, as can be seen in the first item of the task sequence "reaction forces". Only when these heuristics are meaningfully connected, in a way that they support and enhance each other, students are able to solve challenging tasks successfully. An exemplary situation is the following one: first, a student used the diagram to understand the situation. The situation requires a certain algorithm which the student formulated in a general form. Then the components were adjusted using the diagram in the given situation. After that the result was checked for plausibility referring to the diagram. Therefore learning heuristics should attend to two competencies: Students must be able to use some heuristics deliberately, and they need to be able to effectively combine tools, strategies and principles.

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ENDNOTE

1. All students' names are changed to preserve anonymity.

Perceived autonomy in the first semester of mathematics studies

Michael Liebendörfer and Reinhard Hochmuth

Leibniz Universität Hannover, Hannover, Germany
Leuphana Universität Lüneburg, Lüneburg, Germany, liebendoerfer@khdm.de

We focus on the perceived autonomy of mathematics students in their first semester at university. According to self-determination theory by Deci and Ryan (1985), students have to satisfy their need for autonomy in order to develop intrinsic motivation. Using two facets of autonomy, we analyse interview data to explore which situations foster or hinder the students' perceived autonomy. The main factors affecting students' autonomy are briefly discussed.

Keywords: Self-determination theory, autonomy, first study year.

INTRODUCTION

Students' interest is an important factor in learning mathematics, especially for deep understanding (Köller, Baumert, & Schnabel, 2001) end of Grade 10, and middle of Grade 12--in order to investigate the relationships between academic interest and achievement in mathematics. In addition, sex differences in achievement, interest, and course selection were analyzed. At the end of Grade 10, students opted for either a basic or an advanced mathematics course. Data analyses revealed sex differences in favor of boys in mathematics achievement, interest, and opting for an advanced mathematics course. Further analyses by means of structural equation modeling show that interest had no significant effect on learning from Grade 7 to Grade 10, but did affect course selection--that is, highly interested students were more likely to choose an advanced course. Furthermore, interest at the end of Grade 10 had a direct and an indirect effect (via course selection. However, in their first semester at university, mathematics students often experience motivational problems (e.g., Daskalogianni & Simpson, 2002), which form – at least in Germany – one major reason for drop-out (Heublein, Hutzsch, Schreiber, Sommer, & Besuch, 2009). Self-determination theory (SDT) by Deci and Ryan (1985; Ryan & Deci, 2002) postulates three basic psychological needs which are central to the support of interest: the needs for perceived competence, autonomy and social relatedness. In this paper, we focus on autonomy, investigating how German first-semester mathematics students experience the satisfaction of their need for autonomy at university. We have chosen autonomy because in our data, many students have a problematic autonomy experience despite possible positive influences at university (they study a topic of their own choice in an institution that gives many freedoms, e.g. minimal attendance requirements). After an explanation of the concept of the need for autonomy we state the research goals and describe the methods we have used. We then present empirical findings and finally discuss factors influencing students' autonomy.

This research is part of a PhD-project focussing on students' interest development.

THEORETICAL BACKGROUND AND RESEARCH GOALS

In SDT, the role of the basic psychological needs for the development of motivation compares to the role of basic physiological needs (food, water) for the development of our body: need satisfaction is necessary to thrive. A major difference is, however, that the satisfaction of psychological needs is a personal perception. Thus, even in the same situation, different persons may experience need satisfaction very differently. The term autonomy as it is used in SDT should not be confused with different usages like independence or influence on one's learning. The need for autonomy is described as referring "to being the perceived origin or source of one's own behaviour" and "concerns acting from interest and integrated values" (Ryan & Deci, 2002). Following Lewalter (2005), we distinguish two different facets of autonomy of Ryan and Deci's description. The first one is the *perceived locus of causality* (PLOC) people feel when they initiate and control their actions (but not necessarily the outcomes, however). The second one refers to their *personal goals and values* (PGV). For example, one might compose music and feel a PLOC concerning the emerging ideas. In contrast, one might play music in an orchestra and give the initiative to the conductor but might still feel autonomous in the PGV sense if one's goal is to play music in an orchestra and one sees being conducted a valuable way to do so. Since studying mathematics includes external causations (e.g. choice of content, exams) one might or might not agree with, we expect the usage of the two concepts to give more detailed results. Based on SDT, Reeve (2002) stresses the importance of autonomy in educational settings and gives descriptions of autonomy-supporting behaviour like being responsive, supportive and flexible (e.g., giving students time to work in their own way). In contrary, using directives, evaluating students and motivating them through pressure hinders their perception of autonomy. Since motivational problems are known in the first semester, it would be interesting to see if and how the basic needs are satisfied. However, we could not find descriptions of students' autonomy experience in university mathematics courses.

Our first goal is to identify typical ways to have autonomy-related experience in the first semester. Our second goal is to find out in how far autonomy in studying mathematics may be perceived very differently across students. Since for the first two aspects, we distinguish two facets of autonomy (PLOC and PGV), our third goal is to analyse the relation between both facets.

DATA COLLECTION AND ANALYSIS

Our data is formed by 17 semi-structured interviews with first-semester students. Some were enrolled in a secondary teacher programme, others studied for a mathematics degree, but they all attended the same lecture on real analysis. The professor held the course in a rather abstract way and included only few numerical examples. The setting was typical for German mathematics or higher secondary teacher programmes: About 100 students attended the lectures where they had to hand-in a task sheet every week. The sheet was marked and then returned in a weekly tutorial where the solutions were presented and discussed. Only students who got at least 50 % of the maximum score were allowed to take the exam. Attendance was neither required in the lectures, nor in the tutorials. The lecture was based on definition and proof and also many tasks included proofs. Many students also attended a course on linear algebra in their first semester, which was more calculation-based in comparison to the analysis course. All students were asked to come for an interview in the lecture. They agreed in the scientific use of their data and were given the possibility to discontinue the interview or delete passages from the tape at any time.

The students were interviewed in the third or fourth week of their first semester and were asked to broadly describe their experience and learning behaviour at university. Subsequently, they were more specifically asked for their satisfaction of the basic needs including autonomy (e.g., referring to their narrative, the interviewer could ask for situations which were similar or very different from those which affected students' PLOC and for issues they did or did not agree with). The students were asked to include the linear algebra course in case they attended it, but happened to mostly speak about their analysis course. The interviews lasted 30 to 60 minutes and were taped and transcribed. To illustrate the very personal nature of need satisfaction, we picked out two pre-service teachers, whom we call Betty and Chris. We included one more interview with each of them which was conducted at the end of the semester in the same manner. They were chosen to maximise the contrast of autonomy experience: Betty experienced very little autonomy and Chris quite a lot.

Since need satisfaction is usually connected to emotions and may thus be remembered quite well, we assume that the students recalled large parts of their important experiences. We thus expect to cover a very broad range of need satisfaction. Students' statements in the transcripts were coded for PLOC and PGV concerning university mathematics anywhere in the interview, not only when the students had been asked for autonomy. As the interviews included both positive and negative experiences relating to need satisfaction, we distinguished positive from negative in our coding depending on students' reported emotions or evaluations. PLOC was coded when students either referred to themselves as source of their actions or described situations where they were far from being the origin of their behaviour (having no idea what to do, feeling desperate). PGV was coded when students reported that issues were in accordance or in conflict with their goals and values. Coding one passage for both categories was also possible. We want to give some examples of our coding:

Positive PLOC

> For example today, I could calculate some things because I knew how to rearrange them. And that felt great, I was really proud since I knew, it was right. And it was my thought.

Negative PLOC

> I found the worst thing which frustrated me very much and still does: You shall prove or justify something but no one tells you how.

Positive PGV

> I find it a great thing, that you simply have a task sheet every week. For this reason I am confident that if I engage with the sheet, then my studies will work.

Negative PGV

> For me, it isn't right, that the lecture is so fast and you really can't follow and I find this really bad.

The codings of each category were then grouped into typical situations by the first author in a rather exploratory way. We believe the method to be sufficient to identify ways to experience or hinder autonomy, although there is no methodological control.

RESULTS

In order to answer the first question of how autonomy typically arises or is hindered, we present the main clusters according to the two theoretical facets.

For the PLOC-facet, 156 out of 204 codings were negative, which corresponds to most students' general experience of some frustrating first weeks.

- Each student felt pressure to hand-in the task sheets in time and get a sufficient grading, which was a rather permanent experience.
- On a situational level, they lost PLOC when they couldn't (immediately) understand the lectures and the pace was too fast for them, so they could only take lecture notes.
- The students also lost their PLOC when they did not know if they were mathematically right or wrong (e.g. if a proof was correct or complete, which argumentation was allowed) or missed explanations of mathematical objects.
- The most serious restrictions were experienced when the students worked on the task sheets. There is a frustrating feeling of being stuck which occurred in three different types of situations:
  - when students did not understand the task itself,
  - when they had no idea how to tackle the problem
  - and when they had an idea, but did not know how to write it down.
- Another autonomy restriction concerns the evaluation of the students' work. The marking on the task sheets, sometimes just the score, and the tutorials did often not provide the students sufficient information to understand in which parts and why they had been right or wrong. Sometimes, students even got different grades for the same solution, so they felt treated arbitrarily.
- When students positively experienced a PLOC, they mostly referred to situations contrary to the above mentioned ones (e.g. having ideas for the tasks). This especially related to calculation tasks.
- In addition, managing their resources like books, the internet or peers gave them feelings of initiative on an organizational level and thus autonomy.

For PGV, negative codings amounted to 65 out of 113 codings.

- Students did not agree with the way mathematics was presented (too few explanations and examples, the high pace in the lectures)
- and in general with a system that is so demanding that many students struggle.
- For the future teachers, another conflict was that the subject matter was perceived to be not appropriate for their future work.
- Positive PGV-experiences were reported with regard to the general aim of deeper understanding
- and in particular concerning the weekly task sheets, which made students work harder than they would have done otherwise and thus learn more.

In order to illustrate personal differences in need satisfaction, we now focus on the two students Betty and Chris who were enrolled in the teacher programme.

The case of Betty

After school, Betty did not immediately go to university but completed a vocational training as an educator and then decided to become a teacher. Her autonomy experience at university was mainly negative. On her first task sheets, she received far less than 50 % and thus reported enormous pressure and lack of time. Her experience covered examples for every negative PLOC category mentioned above. Especially, she did not understand why it had to be so hard. She could work on proof tasks only when the tasks were about proving that something satisfies a given definition. Once, she handed-in a solution copied from a book which proved an implication, where the task had asked for a proof of equivalence. She felt treated unfair since even the book's solution got a bad grading (affecting both PLOC and PGV). Asked for autonomy concerning her studies she confirmed that she knew she had autonomy, "but the feeling didn't show up". Despite her previous aims to fully understand the subject matter, she soon focussed on the minimal requirements and therefore merely worked on the task sheets without having time to review the lecture notes. Betty experienced autonomy using her resources like peers, books and the internet where she tried to get hints or solutions from. In the second interview she said that she had started copying homework although she did not like it. She was torn in her view on university mathematics. On the one hand, she characterised mathematics as "explaining simple things in complicated ways" and saw no use in it, especially in proofs since they would not have any application. On the other hand, she did not want to call university mathematics useless and gave it some importance since teachers should know what is behind the results. Her only autonomy experiences related to connections to school mathematics (PGV), managing resources (PLOC) and few numerical calculation tasks or examples (PLOC and PGV).

The case of Chris

Chris directly entered university after school, where he had reached the highest grades in mathematics. He felt pressure after his first task sheet was graded less than 50 %. He sometimes did not know how to tackle a task or write it down, but also solved some tasks so that positive and negative experiences balanced. Due to his partial success, he believed he could reach the necessary score so that fear and insecurity soon started fading out. He felt his autonomy restricted by having no guidance for proof tasks, getting no informing feedback on the marked sheets and having to do a lot for the tasks so he had less time for self-directed learning. All three aspects refer to both his PLOC and PGV. However, in principal he appreciated the task sheets since they make people work hard. Chris also readjusted his aims from getting good grades to simply passing and started using books and internet resources.

In the second interview, Chris still described some tasks as frustrating but he received sufficient scores and had developed confidence in eventually meeting this criterion. So although he still tried to solve every task, he did not feel pressure to do so. Chris also mentioned that he usually had several ideas and strategies for the sheets, including reviewing suitable parts of the lecture notes. He also highlighted his own evaluation of his solutions and his understanding in addition to the grading. Chris found having his own ideas was much more motivating than collecting information from different sources. Nevertheless, he sometimes searched for hint on the internet. When he found other interesting things there, he followed them for his own interest (PLOC).

Personal differences in autonomy experiences

It is clear, that different situations may lead to different autonomy experiences. Betty perceived very little autonomy when she copied solutions for the tasks, whereas Chris could experience autonomy since he solved the tasks on his own. It is also clear that in similar situations, students may have similar experiences. Betty and Chris both feel pressure from the tasks sheets (PLOC) which they generally agree with (PGV), because they see the sheets as a good measure for their learning. They both appreciate that university mathematics aims at deeper understanding and they both like calculation tasks because they usually know how to tackle them. However, Betty and Chris also had very different autonomy experiences in similar situations:

They both believed that they would not need the university mathematics in their future work at school, but related this to their goals and values very differently. Betty said "Why? I do not want to become a mathematician but a teacher". Chris' reaction was rather opposite: "What else should we do then? They can't tell us again what we already learned in school!" In addition, they both had been looking for solutions in internet forums, but all they had found were hints. Betty disliked such posts and expected others to present the solution. Chris appreciated such posts as protecting him from copying something he would not understand (PGV). Differences also appeared for the PLOC-facet. When working on the tasks, Betty had mostly no idea what to do next whereas Chris often had several ideas and could then choose. In addition, Betty's only reason to work on the tasks was the score, since "I need the crap 50%". Chris, in contrast, saw things differently: "if I did it just for the external pressure, then I would tackle it very differently".

The relation between PLOC and PGV

Although the two facets of PLOC and PGV are connected, they turn out to contradict sometimes. While there was no situation where students had a PLOC but did not experience PGV, it happened that students' experienced no PLOC but felt their PGV were respected. Like Betty and Chris, all students (partially) lost their PLOC due to the pressure from the task sheets. However, this was mostly according to their PGV. A similar situation (no PLOC, but accordance with PGV) occurred when Chris could not find a solution for a task on the internet, but only hints. Concerning the content matter, Betty had very few PLOC experiences and was torn, struggling to connect the content to her goals and values, whereas Chris experienced both facets positively.

DISCUSSION

In our study, we analysed interview data concerning experiences of autonomy using the two different facets of PLOC and PGV. We could reach our first goal by describing typical situations of need satisfaction or dissatisfaction. We could see that studying mathematics at university provides rich opportunities to experience need satisfaction but also dissatisfaction which may cause serious problems for students' intrinsic motivation. Many typical situations could possibly also appear in other study subjects. However, especially those which refer to the task sheets (not understanding the task, having no idea how to tackle it or how to write down a solution) and proof (when is a proof correct and complete?) seem to be typical for university mathematics. Concerning the PLOC concept, autonomy seems to not only have heteronomy as a counterpart, but also feelings of being stuck and having no idea what to do next.

For our second goal, we compared the two cases of Betty and Chris, which illustrated that autonomy experiences strongly depend on both the person and the environment. Similar situations may foster the autonomy experience of some students and hinder the autonomy experience of others.

Concerning the third goal, we could see different, sometimes contradicting experiences for PLOC and PGV, although both facets origin in the same framework. Distinguishing the two facets helped analyse students' autonomy by revealing hidden tensions, where students feel no PLOC and yet experience need satisfaction in terms of PGV. In such a situation, the students may sustain their intrinsic motivation.

General issues affecting perceived autonomy

The *duty to get good grades* on the weekly task sheets clearly affects students' perceived autonomy, as it puts much pressure on them, violating Reeve's (2002) criteria for autonomy support mentioned above. However, to some extend most of the students appreciated the pressure which would eventually make them learn more. In addition, the case of Chris shows that students do not necessarily experience this duty as pressure if they are confident in meeting the criteria. However, like Betty, many students focussed mainly on the tasks and put their learning goals on hold.

Although the students did not express it, this might not meet their PGV.

Another major background is the *personal competence*. We could see that competence affects autonomy on different levels: First, content knowledge affects students' opportunities to tackle the tasks (PLOC) and also to evaluate their own mistakes (PLOC). As Chris' case shows, this ability can also be important to assess the personal work independent of external judgements like grades. Second, mathematical language is needed to even understand the task itself and to write down solutions (PLOC). Language can also affect the satisfaction of PGV, like in the situation where Betty misunderstood the task. A third level of mathematical competence refers to insecurities like how detailed proof needs to be or which arguments will be accepted. They may be described in terms of sociomathematical norms (see, e.g., Yackel, Rasmussen, & King, 2000) as normative understandings which are negotiated in the social context. It is clear to see that calculation tasks, which all students liked, do not address these problems apart from including difficult content. In turn, students may experience a PLOC based on their competence, e.g. seeing different ways to tackle a task or having their own ideas for solving problems.

Proof as a new concept fitted many students' PGV of heading for deeper mathematical understanding. However, students may have trouble to connect proof to their PGV because they do not see an application. In cases of proving obvious statements, they might see no use at all. Here, one decisive aspect is whether students see building a mathematical theory as part of their goals or restrict them to calculations and applications. In addition, *proving* as an activity may involve all competence-based problems mentioned above.

A very interesting point is the *questioning of the need of university mathematics for teaching at school*. The question itself is part of an open debate (e.g., Davis & Simmt, 2006) so there is no clear and simple answer. The students usually do not have teaching practice and see teaching often as "explaining". Taking this as basis, they need to trust in the university doing what is necessary for them. In our study, this did not always happen. We could see that students felt a conflict with their PGV only if they also had a PLOC-conflict. However, it is also possible that some students do actually not agree with the content but do not start to question it unless they start struggling. What we could observe is that the conflict reduced when connections to school were made explicit. Recent projects successfully provided such connections (Leufer & Prediger, 2007). In addition, the compulsory task sheets show that students may value things although they negatively affect their PLOC.

Unlike in school, university requires learning activities which are not explicitly assessed such as reviewing the lecture notes, monitoring the personal progress, searching for alternative explanations and examples or generating them. Students may not be aware of the new rules and experience these changes as "bad and too few explanations" affecting both PLOC and PGV. Such a mismatch of expectations can be described in terms of the *didactical contract* (Brousseau, 1984) which was highlighted by Hourigan & O'Donoghue (2007)the consequences of the 'Mathematics problem' are a source of concern for the education sector and governments alike. Growing consensus exists that the inability of students to successfully make the transition to tertiary level mathematics education lies in the substantial mismatch between the nature of entrants' pre-tertiary mathematical experiences and subsequent tertiary level mathematics-intensive courses. This paper reports on an Irish study that focuses on the pre-tertiary mathematics experience of entering students and examined its influence on students' ability to make a successful transition to tertiary level mathematics. Brousseau's 'didactical contract' is used as an intellectual tool to uncover and describe the contract that exists in two case mathematics classrooms in Irish upper secondary schools (Senior Cycle for the secondary-tertiary transition.

Some findings parallel the work of Sierpinska, Bobos, & Knipping (2008), who investigated "students' frustration" (which can be seen as contrary to autonomy) in prerequisite mathematics courses from an institutional point of view. Especially since both of us found critical elements like techniques of problem solving and learning strategies, rules and norms as well as the didactic contract, we hypothesise that these elements are generally important for autonomy in the learning of mathematics at university level and thus for students' motivation.

Implications

It seems that in general some conflicts with students' autonomy are inevitable. However, at some points help seems accessible, especially concerning aspects which are typical for university mathematics. For instance, students expected to be guided more in their learning process. Making explicit new demands in strategies for learning and problem solving, sociomathematical norms and the mathematical language could possibly help students to experience more autonomy. Since many students seek for suitable resources, books addressing study skills (e.g., Alcock, 2013) and mathematical language (e.g., Beutelspacher, 2004) could be a good start. In addition, making connections to school mathematics more visible would help pre-service teachers to connect their courses to their PGV.

The German community should also question the task sheet system. For instance, students could have to hand-in documents of their learning process (e.g. a learning diary) rather than worked-out solutions only. They could then also be admitted to the exam for learning activities of their choice like engagement with the lecture notes.

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Characterising university mathematics teaching

Angeliki Mali

Loughborough University, Mathematics Education Centre, Loughborough, UK, a.mali@lboro.ac.uk

This paper reports early findings from university mathematics teaching in the tutorial setting. The study distils characteristics of two tutors' mathematics teaching and through interviews, their underlying considerations. Analysis of a teaching episode from each tutor illuminates their different teaching approaches and suggests ways in which approaches are linked to students' meaning making.

Keywords: University mathematics teaching, small group tutorials, meaning making.

INTRODUCTION

University mathematics teaching is of major significance in mathematics education. Some of the numerous benefits from research in this area of study are the production of professional development resources for novice and experienced university teachers as well as support to those teachers to create rich learning opportunities (Speer, Smith, & Horvath, 2010). However, the benefits from research go beyond university mathematics teaching and learning. Speer and Wagner (2009, p. 537) stressed that by studying the practice of teachers with strong mathematical knowledge, teacher educators "are better able to detect directions for growth in other areas of knowledge" such as "knowledge of typical ways students think (correctly and incorrectly) about the task or content in question" (p. 558), which teachers at all levels may need.

This report is based on the analyses from my ongoing doctoral project that examines university mathematics teaching through an exploration of tutors' teaching practice with first year undergraduate mathematics small-group tutorials. The analysis suggests that teaching practice can be construed in terms of three elements *tools*, *strategies* and *characteristics* of teaching (Mali, 2014). I focus here only on *characteristics* of teaching, which are patterns in the ways that tutors teach in the tutorials. From observations of tutors' teaching and interviews with tutors about their teaching, the aim of the doctoral project is to identify aspects of teaching practice and knowledge and connect them with students' meaning making in mathematics. Tutorials are studied since they provide opportunities for teacher-student dialogue and interaction through which meaning making can be discerned.

The focus of this report is on two tutors' teaching approaches. It is significant to juxtapose *characteristics* which look similar but are used in different ways by the different tutors to promote student meanings. So, the three research questions are: What are the *characteristics* of the two tutors' teaching? In what ways do two different tutors implement similar *characteristics* of teaching? What are the tutors' actions that encourage students to make meaning (promoting meaning making.) and what do tutors do to find out what meanings have been made (discerning meaning making)?

THEORETICAL BACKGROUND AND LITERATURE REVIEW

University mathematics teaching is an area of interest where research is still rather limited. Speer, Smith and Horvath (2010), after conducting a systematic literature review in university mathematics teaching, reported that there is no systematic data collection and analysis focusing on teachers and teaching. As to the small group tutorial setting, the number of studies is far fewer. Certain studies (including this one) focus on the nature of tutor's teaching and the knowledge that frames it. For example, Jaworski and Didis (2014) introduced the *questioning approach* to teaching and suggested tutor's awareness about her teaching as the basis of *knowledge in practice* which informs future action. Mali, Biza and Jaworski (2014) focused on *characteristics* of university mathematics teaching, such as the use of generic examples, and suggest that the research practices of the tutor (mathematician) influence her teaching practices; an influence which accords with findings in the format of lectures (Petropoulou, Potari, & Zachariades, 2011). Jaworski (2002) distinguished tutors' exposition patterns (*tutor explanation, tutor as expert* and forms of *tutor questioning*) as the main teaching aspect in the context of tutorials. She also stressed that teaching/learning was idiosyncratic to the tutor and to some extent to the particular students. These studies give insight into elements of tutors' teaching practices and their reflective thinking. Gaining access to students' meaning in relation to teaching provides significant information about how tutors encourage students to make meaning; for example, Jaworski and Didis (2014) relate students' meaning making to the *why questions* of the tutor. In our study reported here, we investigate, through a sociocultural perspective, teaching practices of selected tutors that encourage students to make meaning.

The socio-cultural paradigm, rooted in Vygotskian psychology, considers the overall social and cultural context, which frames mathematics teaching and learning in its complexity. Concepts and meanings are experienced and understood in the social and cultural small group tutorial practices (e.g., engagement, participation and interaction). Meaning, thinking and reasoning are products of social activity and take place first on the social plane. Tutors' teaching mediates students' mathematical meaning making by using material (e.g., textbooks, problem sheets) or intellectual tools (e.g., exposition).

In this paper, I embed tutors' *characteristics* of teaching in the socio-cultural tutorial practices of tutor-student interaction and participation relating them to students' meaning making. For the purposes of the analysis, I draw on the literature which considers meaning making in terms of making connections within mathematics through different representations, such as symbols, diagrams, pictures (Haylock, 1982); and between mathematics and "other aspects of the world" (Ormell, 1974, p. 13), such as real world situations. I interpret this collective mathematical meaning making through observing and analysing tutors' and students' actions in the classroom. The tutor's actions relate to the nature of teaching and the approach, and accord with what the tutor says (I can read in transcripts); does (gestures, body language) and intentions (I can ask in interviews or hear in the classroom). The students' actions are what they say and do during the tutorials.

THE CONTEXT OF THE STUDY

The study is being conducted at an English University, where students are in their first year of a straight or a joint programme in Mathematics. They are expected to attend lectures (in analysis modules and linear algebra) and a small group tutorial of 5 to 8 students. Tutorials are 50 minutes weekly sessions and work is on the material of the lectures (lecture notes, problem sheets, coursework and exams). Students are expected to work on the material of the lectures beforehand and bring their questions to the tutorial. The tutors are lecturers in modules offered by the mathematics department and conduct research in mathematics or mathematics education. Phanes and Alex are experienced lecturers as well as researchers. Phanes holds a doctorate in mathematics and Alex holds a doctorate in mathematics education.

THE METHOD OF THE STUDY

This study is part of a doctoral project, which has analysed data from one tutorial from each of 26 tutors, as a basis for conceptualisation of teaching. This has been followed by a systematic study of the teaching of three of the 26 tutors for more than one semester. Phanes and Alex are two of the three tutors. Data consists of observation notes and transcriptions of their audio-recorded tutorials and follow up interviews. The interviews are discussions with them about their thinking behind the teaching actions in these tutorials. A grounded analytical approach is taken to the data in which aspects of tutors' actions that seemed to be informed by their teaching knowledge are coded. Analysis is based on the identification and grounded study of teaching episodes; there are several cycles of interpretation: from initial ones using open coding to more advanced ones creating categories. *Characteristics* of teaching have been identified repeatedly throughout the analysis of each tutor's teaching, emerging from this as a category in the nature of teaching. Examples are provided in the accounts below.

RESULTS

Both Phanes and Alex work on a number of questions/mathematical issues in their tutorials, which is a fre-

	Grounded characteristics of teaching	Phanes	Alex
1	Use of graphs, diagrams or gestures to provide a visual intuition for formal representations	✓	✓
2	Know-how exposition about procedures or techniques for the work on mathematics	✓	×
3	Use of problem-solving techniques	✓	✓
4	Use of kind(s) of/multiple examples	✓	✓
5	Use of different mathematical representations/notation	✓	✓
6	Explanation/revision of theory from special to general cases devising less to more complicated examples	✓	×
7	Provision of time to students to work on their scripts while tutor is circulating and supporting	✓	✓
8	Request to students to devise an example	✓	✓
9	Tutor's intuitive or formal explanation of concepts	✓	✓
10	Use of funneling	✓	✓
11	Invitation to specific students to answer	×	✓
12	Request to students to find definitions in lecture notes	×	✓

Table 1: Characteristics of the two tutors' teaching

quent general practice at tutorials. As preparation for the tutorial they look at the lecture materials, including problem sheets, a few minutes before the tutorial. In the following Table, I present *characteristics* of teaching, identified so far, from both tutors. These characteristics emerged in the process of data analysis after the tutorial observations.

In order to scrutinise the different ways tutors implement common *characteristics* and the different issues that are raised, I offer a teaching episode from each tutor. These episodes are paradigmatic cases of the tutors' teaching in terms of manifesting a number of *characteristics* of their teaching.

Phanes' approach

This episode is situated in the second tutorial of the year and concerns work on an exercise from the first problem sheet in analysis:

"Rewrite $||x|-1|$ without modulus signs, using several cases where necessary."

Reading the exercise, Phanes suggests: "we can just sketch the graph of the function". He uses *exposition about know-how to* get rid of the modulus sign (*characteristic* 2/Table 1): "You see, to get rid of the modulus sign of $|x|$, you need to know that x is positive or negative. You have to consider cases. But there is another outer modulus. It's external. Again, to get rid of it, you need to either consider the case whether the expression inside it is positive or not." He offers *a less complicated example to reveal the work on modulus signs* (*characteristic* 3/Table 1); he constructs on the board the graph of $|x^3|$ reflecting the negative part of the graph of x^3 about the x-axis. Then he requests students to work on their scripts for $||x|-1|$ (*characteristics* 7/Table 1), after which the following episode occurs:

Phanes: So, how do I solve this problem? I'll show you. I saw correct pictures; all of you had correct pictures. So, what am I going to do? I will do it step-by-step. First, I will construct $|x|$, right? $|x|$ is this. [Phanes sketches the graph of Figure 1.] Ok? Then, we do $|x|-1$. $|x|-1$ means that you take $|x|$ and you shift it down by 1. This means -1, right? So, it gives you this [g in Figure 1]. These points are 1 and -1. And this point is -1. This is the expression under the modulus sign. And then, you take the modulus of this function and it means that you reflect this negative bit about the x axis, right? And you get this function. Ok? This is the graph of the function. Now, we have to write down the equations for this. You can see that it's given by different functions on different intervals. For instance, this expression is what [f in Figure 1]? This was $y=x$ [e in Figure 1], and then, you shift it by 1, so this is $x-1$ [f in Figure 1]. Is this clear? Please stop me if something is unclear. So, this is $x-1$ [f in Figure 1]. So, what is this [c in Figure 1]? What is this – this bit [c in Figure 1]? It has the same slope as $x-1$ but it's shifted it up.

St2: It's $x+1$.

Phanes: It's $x+1$. So, this bit is $x+1$ [c in Figure 1]. Now, what is this [a in Figure 1]? This graph is $y=-x$ [b in Figure 1], and we shift it down, so it's $-x-1$. So, this thing is $-x-1$. And what is this [d in Figure 1]?

St4: $-x+1$.

Phanes: $-x+1$. $-x+1$. So, what can we now say about this function $||x|-1|$? It equals. Now, it depends on where x is, right? So, we know for this function that on this [Phanes points to interval $[1,+\infty)$], it's $x-1$ if x is greater than or equal to 1. Agree? It is $-x+1$, $-x+1$, if x belongs to $(0, 1)$. It is $x+1$, $x+1$, if x belongs to $(-1, 0)$. And finally, it's $-x-1$ if x belongs to $(-\infty, -1)$.

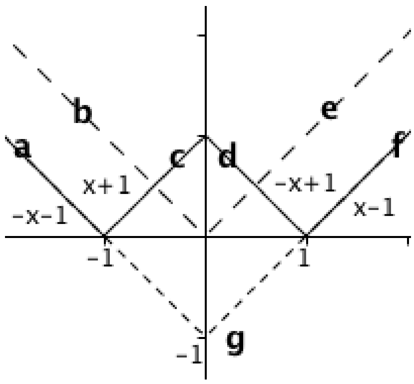

Figure 1: Graph on board, episode 1

In the above episode, Phanes uses the *graph of* $||x|-1|$ *to provide a visual intuition for rewriting the algebraic expression* $||x|-1|$ *without modulus signs* (*characteristic* 1/ Table 1). He first constructs the graph "step-by-step" and then the equations; in this way, he *divides the mathematical task into steps* (*characteristic* 3/ Table 1) and *uses geometric and algebraic representations* (*characteristic* 5/ Table 1). For the construction of the graph, he offers *know-how exposition for the work on modulus signs* (*characteristic* 2/Table 1). Furthermore, for the construction of the equations, he shows how to find $x-1$ and $-x-1$ asking students for $x+1$ and $-x+1$ respectively (*characteristics* 4/ Table 1).

The above *characteristics* (1–5/ Table 1) are within Phanes' thinking on the mathematics. Phanes uses the graph of $||x|-1|$ as a tool to think on the mathematics; he adjusts basic graphs (i.e. $|x|$, x, $-x$) to construct $||x|-1|$ and from that, he extracts the essential information (i.e. equations and intervals) for the solution of the specific exercise. He uses the graphical representation $||x|-1|$ and visual intuition of the equations as a problem solving technique (*characteristic* 3/Table 1), thereby negotiating different contexts (geometric and algebraic) of the concept of modulus sign. Connecting the two contexts/representations, he promotes students' meaning making of the modulus sign. To this end, he also uses know-how exposition, problem-solving techniques, multiple examples of expressing equations and provision of time to students for individual work (*characteristics* 2–4, 7/Table 1).

In the beginning of the episode, Phanes comments that "I saw correct pictures; all of you had correct pictures." This suggests that while circulating and supporting students (*characteristic* 7/Table 1), he also made some judgements about their meaning making of the modulus sign. These judgements arise from his assessment of the students' scripts and indicate that he used *characteristic* 7 to discern their meaning making; not only to promote it. Phanes can also use the multiple examples of equations (*characteristic* 4/Table 1) to discern students' meaning making of the graph by assessing their correct answers for $x+1$ and $-x+1$.

The use of visual intuition of the equations on the graph does not provide enough insight into the intervals. After the episode, Phanes stresses to students that the function is continuous, so "it doesn't matter" if the endpoint is included in the interval; he says to them "strictly speaking, you should include it". In our discussion and in response why he chose a geometric solution when some mathematicians avoid choosing them, Phanes connected his choice with mathematicians' research practices.

> It depends on your research area. If you are a geometer [Phanes is a geometer], you are happy with geometric solutions; it depends on your background I think. [...] You see to me it is easier to see the graph. [...] For instance if you are a programmer writing computer programs, then it is more convenient to you to give an algorithm.

Phanes approaches mathematics teaching putting emphasis on the mathematics and geometric thinking, whereby he relates geometric solutions to his research area. From this thinking on mathematics, he draws out his teaching practice which I recognise through his actions (*characteristics* 1–5, 7/Table 1) to promote and/or discern students' meaning making. In this episode, Phanes presents the ways he is working through the graphs and symbols dissecting the mathematical task to make its aspects more visible to students. He thus works within his thinking about the graphs in *characteristics* 1–5. *Characteristic 7* is different in nature from the others since it can be used in the teaching of other subjects as well as mathematics.

Alex's approach

In his third tutorial of the year, Alex used Venn diagrams to explain the definition of injectivity (*characteristic* 1/Table 1) as well as examples and non-examples of the concept (*characteristic* 4/Table 1). In discussion after the third tutorial, he reflected:

> By the reaction I got when I asked for the definition [of injectivity] the student couldn't even say what the symbols were there. So, I had to repeat it for him. There was not so much meaning making there. So, that's why I decided to use examples, use the Venn diagrams for the sets and what exactly it means to be injective and surjective. [...] If the students get it, I am not sure about that, because after that they still have the face of 'what are you talking about?' So, at that point you say 'Mmm if I carry on with more examples, eventually they will get it', because I don't have any other didactical instrument to make it even clearer for them. Ah in fact when I was preparing my module for another lecture, I thought of a very *good example of the function*. When you go to the supermarket and I am going to say to them next time [...] to explain what an injective and a surjective function is. [...] And I think that's more near the experience of the students, so that they can say "ah yes, I get it now".

Alex implemented the "good example of the function" in the fourth tutorial. He said to the students: "A function is a relationship between a set of inputs, in this case the products in the supermarket, a loaf of bread, and the set of permissible outputs, in this case the prices. So it relates each product to the one, the only one price, it cannot be related to two." In the following episode, we see Alex's implementation of the example for the concept of injectivity. Before the start of the episode, Alex asked the students to express injectivity in the context of his example. As a response to their inability to do so, he asked them to find the definition of injectivity [$\forall x,y \in Dom(f), f(x)=f(y) \Rightarrow x=y$] in their lecture notes (*characteristic* 12/ Table 1). He then wrote the definition of injectivity on the board.

Alex:	How would you read that [the definition of injectivity] in the supermarket example? Which are the x's? What's the domain of the function? St3, what would the x's be in this example in Tesco [i.e. supermarket]?
St3:	Products. Products..
Alex:	The products, exactly. So for all the products in Tesco
St3:	They should be x and y.
Alex:	So why would it be x and y?
St3:	Because it's product and price; x is product, y is price.
Alex:	x could be read, y could be milk, mm? So what would this mean, this then? [Alex points to $f(x)=f(y)$.]
St2:	The same price. [St2's voice is almost inaudible.]
Alex:	What would that [Alex points to $f(x)=f(y)$] mean in the example? I want you to contextualise a very abstract formal definition so we do an everyday job that you can understand; that you give some meaning to those things. Try to think on the example of the supermarket, what would $f(x)$ equal, what would $f(y)$ equal, what would x be, what would y be?
St5:	$f(x)$ would be prices.
Alex:	Yes, the prices, OK. So it says if the prices are equal, let's say 99p, what has to happen to x and y? [Alex sketches Figure 2 on the board.] Let's say x is bread and y is milk, OK? And I notice that the price of the bread and the price of the milk are the same, they are both 99p. Yes? If this function was injective, then the bread would have to be milk, well that's impossible isn't it? [Alex deletes milk on Figure 2.] In other words, I cannot have the price of 99p that belongs to two products, two different products, mm, in the abstract definition, there is no way that 99p comes from bread and milk. Does that make sense or not? Say no if ... well your faces say no.

St1: No, I'd say no.
Alex: OK. Can you think of another example? [...] Do you play a sport?

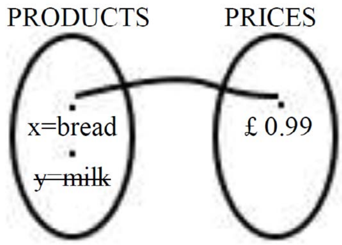

Figure 2: Diagram on board, episode 2

St1 mentioned hockey and Alex devised another example regarding a function that relates hockey players with their scores (*characteristic* 4/Table 1). Despite the real world context of the example of function in the supermarket (a product cannot be related simultaneously to two final prices), a function that relates products/players with their prices/scores is not injective in real life since, there, two different products/players can have the same price/score.

In the above episode, Alex devises a real word example regarding a supermarket (*characteristic* 4/Table 1) to promote students' meaning making of the concept of injectivity: "I want you to contextualise a very abstract formal definition so we do an everyday job that you can understand; that you give some meaning to those things." He uses funnelling (*characteristic* 10, Table 1) by asking what the x, y, $f(x)$ and $f(y)$ are and invites st3 to answer (*characteristic* 11/Table 1); then, he uses the Venn diagram of Figure 2 to provide a visual intuition for the definition of injectivity in the context of the supermarket example (*characteristics* 1, 5/Table 1). In the interview excerpt, Alex stresses that in order to promote meaning making he uses multiple examples (*characteristic* 4/Table 1) and Venn diagrams (*characteristic* 1/Table 1). In discussion with Alex about the use of real world examples, he connects it with research in mathematics education.

> By making it [the example] nearer to the students' experience; that comes from mathematics education. [...] Because you need to make connections in order to make meaning. To understand something you need to make the appropriate connections from your own experiences.

Alex discerns students' meaning making from their faces: "Does that make sense or not? Say no if... well your faces say no" (episode excerpt) and "If the students get it, I am not sure about that, because after that they still have the face of 'what are you talking about?'" (interview excerpt1). In the episode, when st1 answers he doesn't make sense of the example, Alex asks him to devise an example close to his interests (*characteristics* 8, 4/Table 1). After the fifth tutorial, Alex reflects:

> I thought it went a bit better last time when I asked st1: "What do you do in your life?" I play hockey he said. And it went well I thought; at least they said: "Oh yeah I understand now what you mean." That's the design at least to connect with what they do outside.

He also discerns meaning making by the reaction he gets from students.

Alex approaches mathematics teaching bringing in awareness from research in mathematics education; he connects mathematics with students' everyday experiences for meaning making (e.g., Ormell, 1974). A number of his actions (*characteristics* 8, 10, 11, 12/Table 1) to promote and/or discern students' meaning making relates to students' participation, and can be used in the teaching of other subjects as well as mathematics. In this episode, Alex steps out of mathematics, goes into the context of the students and chooses examples there that he can use to parallel injectivity (*characteristic* 4/Table 1). So, Alex roots the abstract mathematics in examples of an everyday nature; starts in the abstract mode through symbols; discerns that students do not make meaning of them; and then brings in a diagram as an alternative way of representing injectivity. He uses this diagram as a tool to explain the mathematics to students; it constitutes another representation of the formal definition which he enriches with explanatory exposition.

CONCLUSIONS

In this paper, I presented two different approaches to teaching, where both tutors put a considerable effort so that students make meaning of the mathematics of the lectures. I related this effort to their actions to promote and/or discern students' meaning making coded in *characteristics* of teaching. I thus looked at the tutor's perspective for students' meaning making, acknowledging that there is no right or better approach. Zooming in each tutor's approach to teaching, they deploy different ways to implement common characteristics; for instance the *characteristic use of graphical representations to provide visual intuition for formal representations*. Phanes uses the graph to make fundamentally mathematical ways of thinking transparent to students, whereas Alex uses it as an alternative to explain the mathematics. In order to promote students' meaning making of how a process works for the modulus function, Phanes uses exposition, problem-solving techniques, explanation, examples and representations all within his thinking of the mathematics. By his exemplification of the process, students can potentially gain mathematical expertise (i.e., use of graphs and visual intuition) to apply in other problems thereby being enculturated into mathematical practices. Alex's real world example is localised around the concept of function and its properties (e.g., injectivity), which is fundamental in mathematics. In order to promote students' meaning making of the concept of injectivity, he uses examples, representations, funneling and invitations as well as requests to students. Evaluation of students' scripts, responses and facial expressions are ways tutors discern students' meaning making. However, promoting and discerning meaning making are two processes that cannot be separated in some cases; for instance, while the tutor *provides time to students to work on their scripts*, he both supports and evaluates them. In future studies, I will analyse data from the other tutors and juxtapose their characteristics of teaching in order to identify aspects of teaching practice and knowledge and ultimately connect these aspects with students' mathematical meaning making.

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Faculty knowledge of teaching in inquiry-based learning mathematics

Vilma Mesa and Anne Cawley

University of Michigan, Ann Arbor, USA, vmesa@umich.edu

In the context of using Inquiry-Based Learning (IBL) for teaching, 74 instructors teaching a wide range of university mathematics courses and with different levels of familiarity with IBL filled out bi-weekly logs about the challenges they faced and the solutions they found. The analysis of pairs of concerns and solutions expressed over the three-year study reveals that faculty may be drawing from different domains of teacher knowledge for teaching to solve concerns teaching with IBL. Level of familiarity with IBL did not seem to play a role in the types of solutions proposed, except when faculty reported not having a solution. We conjecture that the method may induce faculty's pedagogical awareness, independently of how experienced they are with teaching university mathematics.

Keywords: Inquiry-based learning, teacher knowledge, university mathematics.

Originally proposed as a way to understand how faculty learn to teach mathematics courses using Inquiry-Based Learning (IBL) approaches, the *Learning to Teach Mathematics with IBL Project* followed a group of instructors new to using IBL over a period of three years and documented their experience using IBL. Prior work interviewing a handful of instructors who were new to the method at our university suggested that at the beginning of the semester, their concerns centred on spatial and logistical aspects of managing the class: where do I stand in the room? How much time is it OK to wait before answering a question? As the instructors gained more experience with IBL their concerns shifted towards an interest in understanding students' thinking (Mesa & Cheng, 2008). Such dramatic change over the course of one semester countered some literature available on instructors' knowledge, more notably work done by Nyquist and Sprague (1998), whom having worked with teaching assistants (TAs), identified three distinct stages of concerns: a self-centred stage in which beginner TAs worried about themselves and students' perceptions of them (am I knowledgeable? do they like me?); a management-centred stage, in which more advanced TAs worried about managing the logistics and the different aspects of the classrooms (making sure students are not engaging in disrespectful behaviours); and a student-learning stage, in which the expert TAs worried mainly about whether students were understanding the material. These are described as stages in the development of TAs' expertise in teaching. Because our own experience did not suggest these stages with the faculty on our campus (Mesa & Cheng, 2008), we proposed this study. The design of this study includes faculty who stated their familiarity with IBL as beginners, novices, advanced, or experts; its longitudinal nature allows faculty to participate two to six consecutive semesters, with some participating only once. New faculty were added each year. We focused on the concerns faculty described and the solutions they proposed to those concerns when they engaged in four areas of teaching: sending students to the board, designing and managing group work, designing assessments, and using assessments. Specifically we investigate the following two research questions: What are the solutions that faculty propose to address concerns in each area of teaching? Does the level of familiarity with the method determine differences in the solutions used? The next section briefly reviews the literature that informs and situates this study; we follow with a brief description of methods. We then present the main findings and their discussion.

LITERATURE REVIEW

We conceive of *teaching* as the interactions between teachers, students, and the mathematical content and embedded within a particular environment (Cohen, Raudenbush, & Ball, 2003). Strategies are defined here as those activities that teachers organize to facilitate some of the interactions of teaching (e.g., discussion with other students, technology or manipulative use, group work, etc.). Laursen, Hassi, Kogan, and Weston (2014) refer to IBL as a student-centred approach that "invites students to work out ill-structured but meaningful problems… [and] construct, analyse, and critique arguments… present and discuss solutions alone at the board or via structured small-group work, while instructors guide and monitor this process" (p. 407). Initial research has documented the positive trends on self-reported student gains in the cognitive and social domains (see, e.g., Laursen et al., 2014). Laursen and colleagues (2014) indicate that students in IBL courses report higher cognitive gains than students in non-IBL courses, in terms of mathematical thinking, understanding of concepts, and application of mathematical knowledge. Students in IBL courses, and in particular future teachers, reported higher cognitive gains. In addition, students in IBL courses reported higher gains in terms of confidence, positive attitude, persistence, independence, and collaboration, than students in non-IBL courses. Finally, women in IBL courses reported higher gains than women in non-IBL courses. Thus, if teaching with IBL methods is related to these gains, assisting faculty as they learn to use these methods is crucial. As important as students' gains are, an important aspect of the process is the teacher and his or her teaching.

The literature on teacher knowledge as it pertains to teaching is extensive at the K-12 level but is sparser in the post-secondary level. Nardi, Jaworski, and Hegedus (2005) interviewed tutors at the University of Oxford over an 8-week period that prompted them to reflect on aspects of their teaching. The researchers identified four types of pedagogical awareness—naïve and dismissive, intuitive and questioning, reflective and analytic, and confident and articulate (p. 293)—which, they propose, reveal a spectrum of awareness about students' difficulties, strategies to overcome those difficulties, and self-reflection about teaching practices. These researchers claim that instructor awareness can feed into other teaching formats (p. 293), which may suggest a similar categorization for instructors engaged in using an instructional method such as IBL. Other accounts of teaching with inquiry-oriented curriculum (e.g., Speer & Hald, 2008; Stephan & Rasmussen, 2002) point at specific dilemmas that instructors face, in particular navigating the need to stay away from lecturing and moving into more discussion based classes. This literature is informative and allows us to think that there might be common concerns that faculty have when they teach with IBL methods, revealing key dilemmas that mathematics teachers face in teaching (e.g., Chazan & Ball, 1999). Simultaneously, we think that instructors bring with them knowledge that helps them deal with many of these dilemmas, and that they can actually be very resourceful in solving some of those challenges on their own, depending on the type of work they are called to do, even if their main source of information is their own experience with lecturing (the predominant pedagogical strategy in university mathematics classrooms, Blair, Kirkman, & Maxwell, 2013). This study, which started with an investigation of faculty concerns, allowed us to also investigate a possible association with the familiarity using this method and the solutions proposed to deal with concerns in teaching.

THEORETICAL FRAMEWORK

We conceive of the work reported here as being under the umbrella of practice-based theorizations of teacher knowledge, as proposed by Ball and colleagues (Ball, Thames, & Phelps, 2008). Their theory starts from observation of practice to derive the types of tasks that teachers frequently engage in when teaching and it seeks to understand the interaction between those tasks of teaching and the specific subject matter one is teaching. It is a further elaboration of Shulman's (1986) idea that there was a kind of specialized knowledge that was needed for teaching. Ball and colleagues' theory includes six areas: Common Content Knowledge, Specialized Content Knowledge, Horizon Knowledge, Knowledge of Content and Students, Knowledge of Content and Teaching, and Knowledge of Content and Curriculum [1]. The last three areas are a further categorization of what Shulman called Pedagogical Content Knowledge. Knowledge of Content and Students, "(KCS), is knowledge that combines knowing about students and knowing about mathematics. Teachers must anticipate what students are likely to think and what they will find confusing" (p. 402). Knowledge of Content and Teaching, (KCT) is knowledge that "combines knowing about teaching and knowing about mathematics" (p. 402), exhibited when teachers decide which tasks or representations to use. Knowledge of Content and Curriculum (KCC) refers to knowledge of the place in the larger curriculum of the various elements of content to teach. Our focus in this paper is on this further elaboration of pedagogical content knowledge as revealed through teachers' practice with a novel method, IBL, and gleaned through two pieces of information, what instructors find difficult to handle—which signals an area of knowledge needed—and what they propose as a solution for that need, which signals the use of one or various domains of knowledge. Within this theory, we also contend that experience with teaching in general and with IBL in particular informs these domains and that differences in how faculty report concerns and solutions signal the use of those domains.

METHODS

Faculty were recruited from the networks associated with the Academies of Inquiry Based Learning and the R. L. Moore conferences and workshops in the United States [2]. Over three years, 74 instructors from 30 different states took part of the study. We collected two types of data, bi-weekly logs and interviews. The bi-weekly logs asked faculty to log into a secure system and document concerns that they experienced over the prior two weeks on 10 different areas (preparing class, designing assessments, using homework, using quizzes/tests/exams, pacing, lecturing, having students do presentations, group work, large group discussion, and mathematics) and the solutions they had found or ideas of what could help them if no solutions were available. In all we collected information on 171 different courses [3] through 943 log entries. In addition to the logs, we selected a purposeful sample of faculty for conducting in-depth interviews. The interviews covered two main areas, instructors' understanding of IBL and the implementation of IBL in the most recent course they had used to fill the bi-weekly logs [4]. We conducted a total of 30 interviews.

In this paper, we discuss faculty concerns and solutions about two IBL practices, group work and student presentations, and two assessment activities, design and use of quizzes, tests, and exams as revealed through the bi-weekly logs. Table 1 shows the number of instructors recruited who logged a concern in each of the areas included in this analysis, separated by their level of familiarity with IBL [5]. Group work and student presentations are two strategies that depart significantly from lecturing and constitute key features of an IBL course. We focus on assessment because our analysis of interviews suggests that this is a major concern for faculty, as they struggle to fit what they understand the role of assessment is with the goals they want to promote by using IBL (Whittemore & Mesa, 2014). In addition, assessment is a practice that teachers, independently of the method, need to manage. We conjectured that the faculty would therefore draw from their prior knowledge more readily when dealing with concerns emanating from these areas than from the other two, therefore providing an important contrast.

We went through several iterations to build a coding system, using constant comparative methods (Corbin & Strauss, 2008), focusing at times on concerns and at times on solutions. Our intention was to create an all-encompassing, parsimonious system for coding concerns and solutions across all areas of teaching. We identified 14 solutions that describe what faculty proposed to solve the challenges they faced over the course of the term when teaching with IBL. The analytical strategy to answer our first question is qualitative in nature, describing the kinds of actions teachers propose, which will be used to connect to the types of knowledge they draw from. To answer our second research question, we model the likelihood that a solution will be chosen to answer concerns in each of these areas (each is coded as 1 or 0 depending on whether there is a concern in that area), controlling for instruc-

	Number of Instructors (L/M/H)[a]	Number of Logs (L/M/H)
IBL Practices		
Group Work	33 (20/12/4)[b]	80 (45/25/10)
Student Presentations	60 (31/24/7)	172 (75/80/17)
Assessment Activities		
Design	38 (21/13/5)	78 (38/28/12)
Use	38 (21/15/5)	79 (45/28/5)

Notes: a. Level of familiarity with IBL, L: low, M: medium, H: High. b. Numbers do not add up because some teachers who participated more than one semester could change their level of familiarity after each semester.

Table 1: Number of instructors reporting solutions to selected areas of concerns and number of logs teachers used for reporting concerns by level of familiarity with IBL

tors' familiarity with IBL and type of course. More specifically, we estimated generalized linear logistic regression models with a binary outcome (proposing any given solution or not) for all the areas of concern.

FINDINGS

In this paper, we describe the four solutions most frequently coded: Provide Direction, Adjust/Clarify Expectations, Prepare, and No Solution. We first describe the qualitative nature of these solutions and then present the analysis by level of familiarity.

Provide Direction referred to solutions in which the instructor guided students along a desired path while refraining from assuming the role of lecturer. The instructor may provide hints, ask students to break procedures down into smaller steps, or pose questions in general with the intent to incite comprehension/reflection or metacognition:

> I try to get the class to realize the problems. Sometimes I jump in, if it involves writing. Sometimes I give the class hints until they get it. I think this is the most challenging part of the experience, and it takes a lot of energy! (Log2.1_I57_M_LD) [6]

Adjust/Clarify Expectations referred to solutions in which the instructor reminded students of course expectations, resources available to them, and the motivation and purpose for success within the course. The solution may involve relaxing or tightening classroom procedures, minimizing or adding to assignments/coursework, and adjusting the intensity of adherence to classroom procedures:

> For the weak students, I have asked students to come and see me prior to class and work with group members outside of the class. I have asked students to go to the numerous daily tutorial sessions. (Log6.5_I58_L_LD)

Prepare referred to solutions in which instructors reread course notes and lecture topics, and put together materials for classes in general. This solution describes instructors desiring to remain vigilant and flexible to the evolving nature of the unfolding work in the classes and ideas of actions they intend to take in the next class. It also included tasks that the instructor had found or created, plans to use those in class, or reports that the students had used those tasks:

> I have been leaning towards teaching IBL style for the last few semesters, so I am adapting a lot of what I've previously used. I've also found some resources by looking on the Internet and emailing people if I saw something interesting. Since this class is really more of a hybrid IBL and not "pure" IBL and we don't do a lot of theory-related things, some [of] the things I have found (JIBLM) don't quite work. (Log6.3_I45_L_UD)

The No Solution code was assigned when the comments indicated that the instructors were unsure about solutions to their concerns, that they had an idea for a solution but that the solution was not working, or that they were "at a loss" about what would help:

> I know I should have the students presenting more proofs in class, but somehow I am not successful this semester at getting proofs presented by the students in class. Most are not even trying the proofs outside of class. Even the examples are not being tried by most of the students. I'm at a loss even to the type of resources I might need. (Log7.4_I62_L_UD)

The frequencies with which these solutions were assigned to the logs by each area of concern are given in Table 2.

	IBL Practices		Assessment	
	Group Work (n = 80)	Student Presentations (n = 172)	Design (n = 78)	Use (n = 79)
Provide Direction	21%	24%	8%	13%
Adj./Clarify Exp.	30%	33%	18%	31%
Prepare	15%	20%	35%	17%
No Solution	24%	7%	13%	17%

Table 2: Frequency of Solutions Proposed by Faculty to concerns on IBL Practices and Assessments (N=409 logs)

We run three separate logistic regression models that included all the areas of concern (not only the ones that are the focus of this paper), the course type, and the level of familiarity with IBL as independent variables and the solution as the dependent variable (using it or not, one model per solution: Provide Direction, Adj./Clarify Expectations, and Prepare). The regression coefficient for level of familiarity with IBL was not significant in any of these models; that is, when controlling for the area of concerns and the type of course (upper division, lower division, or future teachers), the probability of using a particular solution was not significantly different for different levels of familiarity. For simplicity in presentation Table 3 shows only the odds ratios estimate, standard errors, and significance of the corresponding coefficient for the three models for the four focal areas of concern.

We highlight four points from Table 3. First, relative to teachers who report that they have concerns with group work, teachers who do not have concerns are 2.5 (1/0.393) times less likely to use prepare as a solution. This suggests, perhaps unsurprisingly, that when solving problems related to organizing group work in the classroom, faculty are likely to expect to spend time in advance, anticipating ways to handle those problems. Second and third, relative to teachers who report that they have concerns with designing assessments, teachers who do not have concerns in this area are 3.2 times less likely to use provide direction as a solution (1/0.309) and 3 times less likely to adjust or clarify expectations as a solution (1/0.332). This suggests, that in solving problems related to designing assessments for IBL work, faculty will resort to modify the assessment or provide better explanations for the requirements or for the work, in a way suggesting that they need to be flexible in order to account for the special nature of assessments in this environment. Naturally, this is a plausible interpretation that would need to be corroborated with a different design in which instructors describe problems preparing assessments in non-IBL courses. Finally, relative to teachers who report that they have concerns with using assessments, teachers who do not have concerns in this area are 2.3 times less likely to use prepare as a solution. Thus, similar to concerns faculty face with group work, faculty with concerns with the use of assessments for students will spend time thinking through those assessments based on the feedback they gather from how things work during class. When assessments are used they do provide direct information about students' learning, which may trigger realizations for teachers of the need to learn to anticipate issues better.

We obtained an intriguing result when modelling No Solution. The odds ratio coefficient was significant for faculty identified as having low level of familiarity with the method (Odds Ratio, Low Level of Familiarity = 0.296, $SE = 0.603$, $p < .05$) relative to faculty having high level of familiarity. This means that relative to faculty who had the highest level of familiarity, faculty with low level of familiarity were 3.4 times less likely to report that they had no solution for their concerns.

DISCUSSION

Our analysis of the solutions that faculty proposed to the concerns they had with teaching with IBL methods reveals the ways in which faculty use the knowledge they have to manage the situations. Of the three solutions most frequently assigned, Provide Direction, Adjust/Clarify Expectations and Prepare, it seems that the first two need to be deployed by calling upon teachers' appraisal of the instructional situation, namely the way in which students and the material are interacting in the moment and the possible ways in which teachers can provide feedback. In Provide Direction, the teacher refrains from giving a full explanation to the students to answer a particular question. This requires an in-the-moment assessment

Area of Concern	Provide Direction	Adj./Clarify Exp.	Prepare
Group Work[a]	1.137 (0.525)	0.781 (0.458)	0.393 (0.584)†
Student Presentation	1.140 (0.480)	0.835 (0.416)	0.586 (0.598)
Design Assessments	0.309 (0.617)†	0.332 (0.484)*	1.233 (0.617)
Use Assessments	0.595 (0.558)	0.789 (0.454)	0.424 (0.595) †

Notes: a. The reference category for each coefficient is presence of the concern. The odds ratio is the exponential of the coefficient modelled for the cases in which there is no concern reported. The reciprocal of the value gives the relative ratio for the reference category. †: $p < .10$; * $p < .05$.

Table 3: Odds ratios and Standard Errors by Solution and Area of Concern

of students' thinking paths vis a vis the mathematics at stake and a decision-making process regarding the mathematical idea or question that may nudge the student in a different direction. We see this as evidence of teachers drawing from their Knowledge of Content and Students. In contrast, when using Adjust/Clarify Expectations, teachers are taking a larger view of the constraints imposed by using the method; the actions taken refer to larger organizational aspects of the course that impinge on the quality of students' work: more resources, tighter or more relaxed requirements. In order to make such decisions, instructors account for the contextual situation in which students are working and how that affects the pacing or other goals of the course; they need to keep in mind that they might need to adjust further the class organization so that the course maintains its coherence. We see this as evidence of drawing on their Knowledge of Content and Teaching. When using Prepare, on the other hand, instructors bring to bear knowledge in both domains. They need to take into account what the students have done in any given class, and decide on a path that will satisfy the goals of course and the demands students have for understanding the content. It also involves the generation of documents informed by such thinking processes.

The level of familiarity did not play a role in the types of solutions proposed, except in the case in which faculty reported having no solution. We take this result as an indication that the method itself makes similar demands on teachers' knowledge independently of the experience they have with the method. This result does not mean that the method does not get easier with time. Recall that we are talking about solutions, that is, about the types of knowledge that could be drawn from to solve problems of practice. Theoretically, there is no reason to believe that instructors of different levels of familiarity with the method will draw from different knowledge sources. This way of teaching, by making students visible, forces faculty to navigate through the spectrum of awareness described by Nardi and colleagues (2005) and puts all the teachers at the same stage in Nyquist and Sprague's (1998) model of teacher development, in which they have to deal with student thinking in every lesson.

This work implies that teachers use their knowledge of teaching when teaching with a new method. While we did see a large amount of no solutions to individual concerns, we see that faculty draw on their experience and knowledge to deal with the concerns that they face in IBL practice. Because teaching is in the moment, instructors must find a solution to concerns as they emerge in order to manage instruction in the classroom, and working on augmenting their mathematical knowledge for teaching could be a fruitful strategy for sustaining this way of teaching. While there are many outside resources to help teachers using IBL, it seems that for the instructors in this study, given their knowledge, the instructors themselves are their most valuable resource.

ACKNOWLEDGMENT

We thank Inah Ko, Kyle LaFollette, and Tim Whittemore for assistance in earlier analyses of the data. This work has been supported by the Educational Advancement Foundation and the Undergraduate Research Opportunity Program at the University of Michigan.

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ENDNOTES

1. Due to space considerations we only describe the domains that are relevant to our study. We refer the reader to Ball and colleagues (2008) for full definitions of the other domains.

2. See http://legacyrlmoore.org and http://www.inquirybasedlearning.org for details.

3. Lower division courses were intended for students in their first two years of college (e.g., 100–200 level courses: Calculus I, II, Discrete mathematics, Cryptology); Upper division courses were intended for students in their last two years of college (e.g., 300–400 level courses: intro to proof, modern or abstract algebra, topology, real analysis, etc.); Future teacher courses were those intended for future teachers and could be of any level (e.g., geometry for teachers, math for elementary teachers, etc.)

4. The protocols for the logs and the interviews are available from the authors.

5. Familiarity with IBL was divided into three levels, Low, Medium, and High, depending on instructors' described experience teaching with the method (e.g., # of IBL courses), the use of own generated notes, and the amount of mentoring they provide, among other dimensions.

6. The identifier Log2.1_I55_L_LD indicates the semester when the data was collected (second), the number of the log (first in the term), the instructor (#I55), the level of familiarity (Low, Medium, High), and the level of course for which the log was being recorded, (Lower Division, Upper Division, or Future Teachers).

Students' understanding of marginal change in the context of cost, revenue, and profit

Thembinkosi P. Mkhatshwa and Helen M. Doerr

Syracuse University, Syracuse, USA, tpmkhats@syr.edu

This paper describes how eight undergraduate students majoring in economics and business studies reasoned about marginal change (marginal cost, marginal revenue, and marginal profit) in the process of deciding how they would advise the management team of an airline about an economic decision involving the addition of another jet plane. To elicit students' understanding of marginal change in an economic context, pairs of students were engaged in a task-based interview. Nearly all of the students were able to reason correctly about marginal change within the immediate context of the task, while four of the students also did so beyond the context presented in the task. Only one student considered the marginal change information in the task as a rate of change.

Keywords: Marginal change, rates of change, business calculus, undergraduate mathematics education, economic decision making.

INTRODUCTION

The role of context in the way students reason about rate of change (average rate of change and instantaneous rate of change) has received considerable attention from researchers interested in the learning and teaching of this concept and students' interpretations of rates of change in various contexts. In particular, there is a large body of research literature on students' understanding of rate of change in a motion context (Beichner, 1994; Bery & Nyman, 2003; Monk, 1992; Nemirovsky, Tierney, & Wright, 1998). Research exists on students' understanding of rate of change in non-motion contexts such as fluid flow, heat flow, temperature, discharging capacitors, and light intensity (Bingolbali & Monaghan, 2008; Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Doerr, Ärlebäck, & O'Neil, 2013; Johnson, 2012; Marrongelle, 2004). However, there is little research on the context of economic change, which is the motivation for this study.

THEORETICAL FRAMEWORK

This study draws on a multiple representations theoretical framework (Davis, 2007) to examine how students reason about the context of cost, revenue, and profit within a real-world context representation and across other representations. Davis' multiple representation framework contains five representations: (1) tables, (2) algebraic, (3) graphs, (4) spoken language, and (5) real-world contexts. At the center of Davis's framework are real-world contexts. As Davis puts it, "students' investigations are dominated by real-world contexts and students are frequently translating between tables, graphs, and equations, and vice versa" (p. 391). The framework is an adaptation of Lesh's (1979) multiple representation framework. The current study is part of a larger study that used three tasks that situated the context of cost, revenue, and profit in multiple representations, namely graph, table, and text. The current study reports on what students' reasoning about the context of economic decision making, presented as text, revealed about students' understanding of marginal change. Marginal cost refers to the cost per additional unit produced, marginal revenue refers to the revenue generated per additional unit sold, and marginal profit refers to the profit per additional unit produced and sold. Mathematically, marginal change can be calculated using instantaneous rate of change which can be approximated using average rate of change.

LITERATURE REVIEW

A review of the research literature on students' understanding of rate of change in context reveals several things. First, even high achieving students in calculus have difficulties understanding and interpreting rate information (points of inflection and concavity) in mathematical tasks that are situated in a non-motion context (Carlson et al., 2002). Second, the use of physical models such as the sliding ladder used by Monk (1992) and technology in the form of motion detectors and graphing calculators can enhance students' understanding of rate of change in a motion context (Monk, 1992; Nemirovsky, Tierney, & Wright, 1998). Finally, the research literature reveals that a good understanding of a motion context in physics could enhance students' ability to reason about rate of change when solving calculus problems that have been stripped of context (Marongelle, 2004).

However, research on students' understanding of rate of change in a business and/or economics context is lacking. To the knowledge of the authors, only one study by Wilhelm and Confrey (2003) investigated students' reasoning about rates of change in an economic context. These researchers studied algebra I students' ability to project their understanding of average "rate of change in the context of motion onto the context of money" (p. 887). Wilhelm and Confrey found that some of their participants were able to project their understanding of average rate of change from a motion context to a banking context. However, their study did not examine what students' reasoning about the motion and banking context revealed about students' understanding of instantaneous rate of change and marginal change. The current study seeks to address the gap about what it is that students' reasoning about the context of cost, revenue, and profit reveals about their understanding of marginal change.

Conflating function output values with the average rate of change values for the function considered over subintervals of the function's domain is a well documented difficulty that students have when reasoning about rate of change (Carlson et al., 2002; Monk, 1992; Prince, Vigeant, & Nottis, 2012). It is also known that students have difficulty distinguishing between the amount by which a function changes and the rate at which the function is changing over unit subintervals (Confrey & Smith, 1994; Cooney, Beckmann, & Lloyd, 2010). Research also indicates that students' "understandings of rate in one representation or context are not necessarily transferred to another" (Herbert & Pierce, 2012, p. 455). This study is part of a larger study that investigated students' reasoning about marginal change in three representations, namely graph, table, and text. This paper reports on students' reasoning about marginal change in the latter representation.

METHODOLOGY

This qualitative case study used task-based interviews (Goldin, 2000) with eight undergraduate students currently enrolled in business calculus. Four pairs of students were engaged in the following non-routine task that was designed to elicit their reasoning about the marginal change information rooted in the task, adapted from Hughes-Hallet and colleagues (2006):

> JetBlue is a major airline that currently operates 195 jet planes. The airline serves 84 destinations in 24 states and 12 countries in the Caribbean, South America, and Latin America. The airline is trying to decide whether to add an additional jet plane. The choice that the airline has is between adding this jet plane and leaving things the way they are. The airline's decision is to be made purely on financial grounds.
>
> How should the airline decide on whether or not to add the 196^{th} jet plane?

Setting and respondents

The study was conducted on the campus of a medium sized research university located in the north-eastern part of the United States. The respondents were eight undergraduate students, six females and two males, from the department of economics and the business school who had recently completed a business calculus course as a prerequisite for other required courses in their programs of study. Four of these students were sophomores, one student was a freshman, two students were seniors, and the other student was a junior. Students taking this course are familiar with average rate of change, instantaneous rate of change, and the context of cost, revenue, and profit, hence the reason for recruiting them to participate in the study. Data were collected during a regular semester and the summer following that semester. Three of the four interviews were both audio and video-recorded; one interview was only audio-recorded. Each interview lasted for about 75 minutes. The interview data was transcribed for analysis. Work written by students during the interview was also collected as part of the data.

Data analysis

Transcripts were coded for students' understanding of the marginal change information rooted in the task. In particular, the transcripts were analyzed for students' abilities: (1) to identify and interpret the marginal change information embedded in the task, and (2) to give reasonable advice on how the airline should decide on whether or not to add another jet. Reasonable advice was considered to be one which takes into consideration a comparison of the company's marginal cost, marginal revenue, and marginal profit associated with the jet in question in the process of deciding for or against the addition of another jet plane.

RESULTS

Seven of the eight students who attempted this task were able to reason about marginal change in the context of making an economic decision, that is, advising the management of an airline on whether or not to add another jet plane. Only one student considered the marginal change information rooted in the task as a rate of change. Two of the four pairs of students reasoned beyond the immediate context presented in the task. Following is a discussion of students' reasoning about the marginal change information rooted in the task, first within the immediate context of the problem, and then beyond the immediate context of the problem.

Reasoning about marginal change within the context presented in the task

Two of the four pairs of students reasoned about the marginal cost, marginal revenue, and marginal profit ideas within the context of the problem. The following excerpt illustrates Isabel and Sally's initial response to the question: How should the airline decide on whether or not to add the 196th jet plane?

Isabel: If the rate of change increases with the more jets that they have they should add another jet but if the rate of change is decreasing they should not add another jet.

Researcher: What do you mean by the rate of change?

Isabel: Like say if they had a graph, if the graph shows their financial cost would look like that [drawing the graph on the left in Figure 1] which means that they should add…and then if they have another graph [drawing the graph on the right in Figure 1] where it is going more like this, then they should not add.

Researcher: What is the curve [asking about the graphs she drew]? What does it represent?

Isabel: Their financial, like how much they are making.

In stating that the airline should add another jet if "the rate of change increases", it appears that Isabel is referring to the increasing profit that the addition of this jet would bring to the airline. It also appears that she is working on the assumption that whenever the airline added a jet in the past their profit always increased (graph on the left in Figure 1). In the graph on the right in Figure 1 which illustrates when the airline should decide against adding another jet plane, Isabel appears to be referring to the decreasing profit that would continue if the jet is added. One may argue that Isabel's focus on the graph shifted her attention away from the context. Isabel, however, did not make any reference to marginal cost, marginal revenue, and marginal profit while talking about her advice to the management. Isabel was the only student to spontaneously talk about rate of change.

In talking about her decision, Sally's reasoning was based on comparing the marginal cost and marginal revenue associated with the additional jet as the following excerpt illustrates.

Figure 1: Isabel's graphical illustration of how the management of JetBlue Airline should decide on whether or not to add the 196th jet plane

Researcher: Sally, what's your own view? How should the airline decide on whether or not to add the 196th jet plane?

Sally: I think the profit [referring to Isabel's graphs] works but it would be easier if you do a graph like it was before [referring to a graph from an earlier task that had two curves: a non-linear cost function and a linear revenue function] with the marginal revenue and like the marginal cost. So if this [referring to the graph on the left in Figure 2] I would say 195 just to make it easy, then you would not add with that because for that it would make the revenue smaller than the cost.

Researcher: What is the line?

Sally: It's the marginal revenue.

Researcher: So in that case they should not add?

Sally: Right, but then in that case [referring to the graph on the right in Figure 2] they should add.

Sally appeared to be comparing the marginal cost and marginal revenue associated with the addition of the 196th jet plane. Her left graph in Figure 2 shows a realistic situation for when the jet should not be added: the marginal cost is greater than the marginal revenue generated for any jet plane added beyond the 195th plane. Sally's decision for adding the 196th jet plane in the case of the situation depicted by the graph on the right in Figure 2 is also quite reasonable in that it shows that adding a jet beyond the 195th increases the airline's marginal revenue while adding a jet beyond the 195th decreases the airline's marginal costs and hence it would be in the best interest of the airline to add the 196th jet. The following excerpt illustrates Isabel's reasoning about Sally's graphs (Figure 2) and their final thoughts on the advice they would give to the management of the airline.

Isabel: I see them showing the same thing, the only thing is that you would never know this [pointing at the MR line and MC curve extended beyond 195 in Sally's left graph in Figure 1] because we don't have more than 195 planes, so this is, I don't know but that [pointing at the MR line and MC curve extended beyond 195 in Sally's left and right graphs in Figure 1] wouldn't exist on the graph anyway.

Researcher: Putting your ideas together, you want to give an advice to the management of this airline on what they should do. How can you present your advice?

Sally: If the marginal revenue is higher than the marginal cost for the 196th plane you should purchase it but if it's the other way round you should not.

Researcher: Isabel, how could you present your advice?

Isabel: I agree with her.

Researcher: Would you use exactly the same words she used or?

Isabel: Basically yeah. If it's increasing the profit then purchase it, if it is lessening the gap between the cost and the revenue, then purchase it as well but if it's increasing the gap and the profit is less then don't purchase it.

Sally's advice to the management is based on comparing the marginal cost and marginal revenue. This suggests that Sally is indirectly looking at the marginal profit that adding of the 196th jet would bring to the airline. Isabel, on the other hand, made her decision based on the increasing profit associated with the addition of the 196th jet. When talking about "the gap between the cost and the revenue," Isabel appears to be drawing on Sally's reasoning about marginal change.

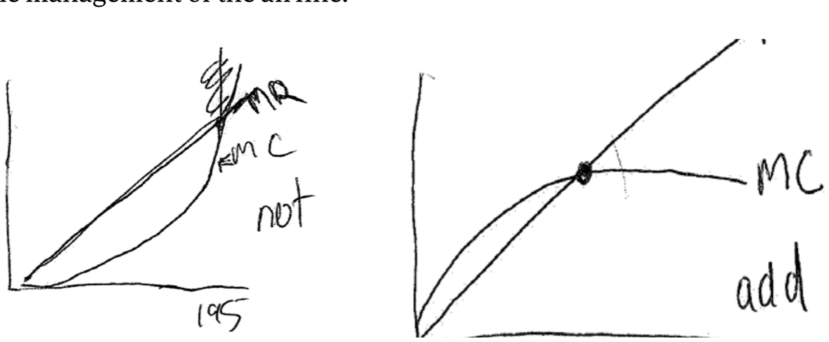

Figure 2: Sally's graphical illustration of how the management of JetBlue Airline should decide on whether or not to add the 196th jet plane

A second pair of students, Paige and Yolanda, reasoned similarly to Sally. Paige said that "they need to see what the potential revenue is from that one plane and then compare with the cost of that plane and see if that would give them more money". Paige's advice is based on the marginal cost, marginal revenue, and marginal profit (which she referred to as "more money") associated with adding the 196th jet. Yolanda said that "they need to see how the profit is at 195 and see how much more one plane makes," thus reasoning about the marginal profit associated with the addition of the 196th jet.

Reasoning about marginal change beyond the context presented in the task

The other two pairs of students who attempted this task reasoned about the marginal cost, marginal revenue, and marginal profit ideas beyond the context of the problem. In particular, the students' thinking showed a consideration of the broader economic issues that needs to be considered in trying to decide whether or not another jet should be added. In the case of Beth and Mary, Beth said that the airline's decision should be "based on how much more profit they will make versus how much they will pay to make the plane". Beth reasoned about marginal change and in particular she reasoned about the marginal cost and marginal profit associated with the jet in question.

Mary, on the other hand, said that "if the quantity of people that are flying increases then it would be acceptable to get another jet plane but if it's decreasing there is no need for another one." After Mary's response, Beth added that "you should always add the plane because that way you could even add more destinations." In giving her advice, Beth initially took marginal profit and marginal cost (cost of making the 196th jet plane) into consideration but then she also thought about the expansion of the airline in terms of adding more destinations. Mary's reasoning was based on the increase or decrease in demand, that is, number of people flying with this airline. Mary's thinking, however, does not show any evidence of considering the marginal change information rooted in the problem.

Noel and Paul's advice to the management of the airline on whether or not to add another jet is another example of reasoning beyond the immediate context. Noel said that "if it adds incrementally to the profit of the airline, then truly based on financial grounds you would add that 196th jet." He went on to add that "they need to know whether there would be capacity on that plane, actual demand, and cost of other things like fuel and union contracts." Noel's decision not only takes into consideration the marginal profit associated with adding another jet, but also several other important economic factors.

Paul, Noel's partner, said that "they should look at the cost of the plane and the marginal revenue they would get." He went on to add that "they should look at the cost per average passenger and average revenue per passenger." Later in the interview, Paul said that "if the marginal profit equals their marginal revenue minus marginal cost equals zero, it makes sense they are still in a capacity to add this jet, if this [the difference between marginal revenue and marginal cost] is negative then it doesn't make sense to add another jet." In this situation, Paul showed an understanding of marginal profit as it relates to the marginal cost and marginal revue associated with adding another jet, thus using his understanding of the mathematical content knowledge (marginal change) to understand the context of the problem.

Summary of students' reasoning about marginal change in context

In summary, the results from this study revealed that: (1) students can correctly reason about marginal change when presented with a problem that is situated in a context [air travel] that is familiar and meaningful to the students, and (2) students are able to reason beyond the immediate information given in a task when it is presented in a familiar situation. A key finding of this study is that students understood marginal change as an amount of change (the difference) and not as a rate of change (the difference quotient). This was supported by the fact that in another task during the interview (not reported in this paper) all but one of the students indicated that the units of marginal change would be dollars instead of dollars per unit.

DISCUSSION AND CONCLUSION

This study investigated what students' reasoning about a real-world problem in the context of cost, revenue, and profit reveals about their understanding of marginal change. Nearly all the students were able to correctly talk about the marginal change information rooted in the task. However, only one student (Isabel) used the language of rate of change to give her advice to the management.

A major finding of this study is that students reasoned about marginal change as an amount of change and not as a rate of change. This finding would appear to be consistent with results from other studies (Confrey & Smith, 1994; Cooney, Beckmann, & Lloyd, 2012). Distinguishing between the amount of change (a difference) and marginal change (a rate of change over a subinterval of unit length) would appear to be especially difficult for students.

Even at the end of a business calculus course students do not speak of marginal change as a rate of change as it is presented in their calculus course and their textbook (Haeussler, Paul, & Wood, 2010). This is problematic because one major goal of a business calculus course is to help students move from understanding marginal change as an amount of change to understanding it as a rate of change. It might be important for future research to consider using modelling tasks (Doerr & English, 2003; Lesh et al., 2003) to support the development of students' understanding of marginal change as a rate of change.

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Challenging the mathematician's 'ultimate substantiator' role in a low lecture innovation

Elena Nardi[1] *and Bill Barton*[2]

1 University of East Anglia, Norwich, UK, e.nardi@uea.ac.uk

2 University of Auckland, Auckland, New Zealand

In this paper, we draw on our experiences as member of the International Advisory Board and principal investigator of a research project on undergraduate mathematics teaching and learning to comment on the study of university mathematics as a process of enculturation into new mathematical practices and new ways of constructing and conveying mathematical meaning. We see this enculturation as the adaptation of different ways to act and communicate mathematically. We take a discursive perspective and we treat the changes to the mathematical and pedagogical perspectives of those who act – students and lecturers – as discursive shifts (Sfard, 2008). Our particular focus is on the shifts concerning the 'ultimate substantiator' role typically attributed to the lecturer.

Keywords: Low lecture, 'ultimate substantiator', commognition, enculturation.

UNIVERSITY MATHEMATICS: AN ENCULTURATION PERSPECTIVE

Mathematics undergraduates, and their lecturers, often describe university mathematics as a process of enculturation into new mathematical practices and new ways of constructing and conveying mathematical meaning (Nardi, 1996). As often described in the literature (e.g., Artigue, Kent & Batanero, 2007), what characterises the breadth and intensity of this enculturation varies according to factors that include: student background and preparedness for university level studies of mathematics; the aims and scope of each of the courses that the students take at university; how distant the pedagogical approaches taken in these courses are from those taken in the secondary schools that the students come from; the students' affective dispositions towards the subject and their expectations for what role mathematics is expected to play in their professional life. On their part, lecturers' views on their pedagogical role (e.g., Nardi, 2008) may also vary according to factors such as: length of teaching experience; type of courses (pure, applied, optional, compulsory, etc.) they teach; perceptions of the goals of university mathematics teaching (such as to facilitate access to the widest possible population of participants or select those likely to push the frontiers of the discipline); and, crucially, institutional access to innovative practices (Skovsmose, Valero, & Christensen, 2009).

Here we draw on our experiences, respectively, as member of the International Advisory Board (Nardi, 2014) and principal investigator of the LUMOS project (*Learning in Undergraduate Mathematics: Output Spectrum*; Barton & Paterson[1], 2013) to comment on aforementioned student enculturation, particularly with regard to how students and lecturers experience the innovations introduced in the project. We first outline the project.

LUMOS AND THE LOW LECTURE INNOVATION

LUMOS is a two-year project funded by Ako Aotearoa, the New Zealand government body that distributes national research grants for tertiary education research, as well as the New Zealand Teaching & Learning Research Initiative (TLRI). Its main aim is to understand how course delivery at class level can achieve a range of desired learning outcomes for undergraduate mathematics that includes content and skill related outcomes as well as outcomes related to the processes of mathematics, affect, and broader graduate issues. It is expected that the project will generate evidence that different types of courses contribute to student learning in different ways. Therefore developing a variety of pedagogical practices is part of the project. Three innovations are currently under trial: *team-based learning*, *intensive technology* and *low lecture*. The third of these, *low lecture*, is the focus of this paper.

There are three key assumptions behind the *low lecture* innovation. First, lectures are not necessarily the best means of imparting information or developing skills. They are however useful for material overviews, demonstrating model ways of communicating mathematical ideas and enthusing newcomers with the skill and fluency that can often be found in the communicational practices of old-timers – thus one per week is sufficient. Second, responsibility for learning content and acquiring skills is handed back to students using specific guides of what they are expected to learn and where to find print and online resources, and with regular self- and lecturer-monitoring of progress. Third, learning about, and induction into, the processes of being mathematical are absent from most undergraduate courses, hence the time saved from lecturing is spent in small group sessions of semi-authentic mathematical experiences free from content-learning requirements.

The *Low lecture* innovation was trialled for the first time in 2013, with 14 MATHS108 students. MATHS108 is a Year 1 course for non-mathematical majors that covers: linear functions, linear equations and matrices; functions, equations and inequalities; limits and continuity; differential calculus (one/two variables); and, integral calculus (one variable). Faculty members, as members of the LUMOS team, run the trial on an extra-to-load basis. The trial consists of one lecture per week for the duration of the semester and three 2-hour *engagement sessions* which students need to prepare for in advance, as well as write up a report for afterwards. These reports substitute assignments. The remaining parts of MATHS108 (tutorials, tests and final written examination) stay the same.

The discussion we present here was initiated by the first author's account (Nardi, 2014) of her experience of observing an *engagement session* and the discussions that followed this observation. Our account adopts the *commognitive* perspective (Sfard, 2008). *Commognitive* terms in it are in *italics* and used as defined in the abridged presentation of the framework in (Nardi, 2014, p. 5–6) and (Nardi, Ryve, Stadler, & Viirman, 2014, p. 183–5). We conclude the paper with a consideration of the shifts in the lecturer's role as experienced by the observed lecturer (second author).

As outlined in (Nardi, 2014) the observed *engagement session* was part of the *low lecture* MATHS108 course. Five students (thereafter Students B, N, J, D and A) participated in the session which was their first *engagement session* and took place in the early weeks of the first semester. The session was run by the second author, leading member of the LUMOS team (thereafter Lecturer L). In the account that follows we outline what unfolded in the session and then present the discussions between the observer (first author) and L (the lecturer and second author) that followed.

In presenting this account we are driven by the following questions:

— What were these 'newcomers' to the practices of university mathematics to make of the open task set to them (see below)?

— What were their expectations of the 'old-timer' who led the session?

— In return, what were the 'old-timer''s expectations of the students?

— And, finally, what kind of bearing, if any, did the slightly unexpected nature of the task have on the session and its aftermath?

OBSERVING AN ENGAGEMENT SESSION OF A LOW LECTURE COURSE

The five MATHS108 students arrived in the small, cosy meeting room where their first experience of an *engagement session* was about to kick off. Their preparation for the session consisted of engaging with an open task, sent to them a week prior to the session: exploring functions from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$ – see an outline of the task in Figure 1. The students expected to be invited to share their explorations with the lecturer and the group. We note the deliberately unexpected nature of the task: these students were so far accustomed to working with functions from $\mathbb{R}$ to $\mathbb{R}$ and may have had a general awareness of functions from $\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$. L comments on the spirit of the task as follows:

> "Engagement Session situations are intended to be open-ended mathematically, both conceptually and procedurally. That is, they are intended to ask students questions about mathematical concepts that they have not encountered before,

although they may be related to the work in the course. Additionally, there are not only many "take-off" points (places where students can start working), but also several, different ways of developing their work.

Furthermore, there is no presumed "correct" process or result. [...] What is important is what they then do, mathematically. [...] Students are not given marks for "correctness". They are marked on "mathematical thinking" in whatever form it is exhibited."

We return to the two omitted ([...]) parts from the above L quotation later in the paper when we examine a little more closely some of the student productions in preparation, during and after the session.

The students and the lecturer were seated around a rotund table, arranged in the middle of a small meeting room. The students arrived with their preparatory

Engagement Situation #1: Functions from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$

Most functions we have been using map a Real Number onto a Real Number.

We write $f : \mathbb{R} \rightarrow \mathbb{R}$ and we say "f maps $\mathbb{R}$ onto $\mathbb{R}$".

But functions can be about any numbers, not necessarily the Real Numbers. That is why we have to specify the domain when we define a function. In fact, a function can map anything onto anything, vectors or matrices, for example.

Not only that, we can define functions that map TWO numbers onto one number. You will learn more about such functions later. An example of such a function is

$$f(x,y) = 3x - y^2$$

We write $f : \mathbb{R}\times\mathbb{R} \rightarrow \mathbb{R}$ and we say "f maps $\mathbb{R}$ cross $\mathbb{R}$ onto $\mathbb{R}$".

What about a function that works the other way? It starts with a Real Number but produces TWO Real Numbers. That is $f : \mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$.

Our first problem is to find a suitable notation. Let's take an example. We start with a function f and a variable x. Let the first number created by the function be x^2, and the second number be $(1/x)$. Thus $f(2)$ is 4 and ½.

1) Devise a suitable notation for this.

2) Devise a new function $h : \mathbb{R} \rightarrow \mathbb{R}\times\mathbb{R}$. Make up your own rules for h. Explore some values of h. Check: is h a function? That is, will each separate input x give a unique output pair?

3) Can you find a way of graphing h ? This will need to be a new kind of graph.

4) What can you say about the values of h for different inputs? E.g. what happens to $h(x)$ when x is close to zero, when x gets very large, when x is negative? Find some other things to investigate about $h(x)$.

5) Can you find another function, $j(x)$, which behaves differently? Will your graphing and notation scheme work for $j(x)$?

Figure 1: An outline of the *Engagement Situation* task pre-distributed to the students

work in hand. One – N, the only female in the group – also had her laptop with online access, which she used often during the session. The ambience was convivial and highly respectful of all. The students granted permission to the observer (first author) to join the session and seemed comfortable with her presence. The account that follows – described in (Nardi, 2014) as a sequence of episodes that evidence a substantial shakeup of the *learning-teaching agreement* – is based on notes jotted down during and right after the session. The account is written from a *commognitive perspective* and aims broadly at addressing the questions listed at the end of the previous section.

Shakeup of the learning-teaching agreement in a low lecture session: evidence

At the very start of the session L reminds the students that its overall aim is set out in the preparation sheet (Figure 1). L had set two tasks for this exploration: first, propose a notation for this type of function; second, devise a relationship of this type and explore how we would secure that it is a function, what its range of values would be, what its graph would look like and what its behaviour would be for very small or very large values of *x*. The preparation sheet ends with a request to devise a second function of this type and repeat the exploration with a view to comparing with the first. The students are also reminded that they will be expected to communicate the outcomes of their exploration and that some aids to doing so will be available in the room for them to this purpose. As the session starts, L reminds them that they are ultimately expected to produce a four-page report consisting of: an account of their pre-session efforts (on the first page), their take on the exchanges during the session (on the second and third pages) and their further explorations soon after (on the fourth and final page).

The final words on the preparation sheet were 'happy mathematising and they encapsulate explicitly the *discursive object* of the activity that the students are invited to participate in. L's overall demeanour and utterances throughout the session also convey exactly that: this session is about engaging with the *routines* of a mathematician (he lists several of these in at least two occasions, including hypothesising, justifying, proving, visualising, extrapolating etc.). The students' responses to these meta-discursive utterances by L – particularly when L asks them to cease activity for a moment to heed what they are doing, and how – is rather mute: they seem keen and confident to act but perhaps less so to take up this invitation for reflective distancing from the action. In fact it takes no more than a few seconds for them to return to the vicarious discussion of their exploratory work.

On the grounds of this discussion – which we sample selectively in what follows – there was little doubt that the students' take on the purpose of the session was essentially congruent to that of L. Sfard (2008, pp. 223 onwards) speaks of mathematical *routines* in terms of *deeds*, *rituals* and *explorations* and it would be hard to perceive what was happening in the session as anything other than evidence of *exploration*. L's recollection of the students' work substantiates this claim further:

> [...] in this situation, while students may graph their functions as lines in 3-space, other alternatives are acceptable. For example, students have used the first element created to define a new (curved) axis, on which the second element is plotted; others have used the first element to define a line in 2-space as in a conventional graph, and the second element to determine the width of the line, hence creating a ribbon. [...] For example, the ribbon is not a function, as it is not 1 to 1. However an attempt to redefine "1 to 1" for this context would be an entirely acceptable process.

Let us now consider two aspects of the students' activity that relate to the questions we listed earlier: first, some *features of the students' exploratory work*, particularly in relation to the slightly unexpected nature of the task (from '$\mathbb{R}\times\mathbb{R}$ to $\mathbb{R}$' to '$\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$' functions); then, some evidence of the students' – and L's – *perceptions of the learning-teaching agreement* that sessions such as this may bring to question.

With regard to the first (students' *exploratory* work), the session was marked by the high likelihood on several occasions of *commognitive conflict*, emerging from the students' *word use* and form of *visual mediation*. Throughout the session the students' standard approach to *substantiation* was to *endorse or reject a narrative* about the objects at stake through indications in favour of – or against – a claim as evident on a screen, or on roughly produced drawings on paper. Combined with their generally non-standard use of *symbolic realizations* (notation, graphs and related terms), the ingredients seemed to be there for *com-*

mognitive conflict. According to the task set by L in the preparation sheet (Figure 1), the students were expected to consider how a graph of a function from ℝ to ℝ×ℝ would look. However, on various occasions, their utterances, and scribbles produced during the session, seemed to concern functions that looked more like $f + g$, fg, $f \circ g$, rather than $f : \mathbb{R} \to \mathbb{R}\times\mathbb{R}$. In this sense the question 'what does a function from ℝ to ℝ×ℝ look like?' – central in the preparation sheet – was not pursued as directly as L might have expected.

L's recollects some of the student productions (not only the five observed in this session) as follows:

> In their preparatory work most students defined two functions, e.g. $g(x) = x^2$; $h(x) = \sin(x)$, and then, to draw the graph, composed them in some way to graph the equivalent of $g+h$, gh, or $h\circ g$.
>
> When it was pointed out that they had essentially created a function ℝ to ℝ, this led to other suggestions such as using the first element to define the axis for the second element, or the first element to create a conventional graph that then got altered by the second element to create some kind of ribbon (2D) or envelope (3D).

Throughout L's contribution was to point out anomalies in such a way that further mathematical invention or adjustments could be attempted, to ask for more exact formulations of what was intended, or formulations using known conventional terms. We remind the reader that the aim was to trigger mathematical actions from the students, not "correct" objects.

In the *commognitive* perspective, one way to evaluate whether the focus and object of the exchanges amongst interlocutors (here L and the five students) are well-coordinated is to examine the forms of *word use* evident in these exchanges. Sfard (2008, pp. 181–2) distinguishes between *passive*, *routine-driven*, *phrase-driven* and *object-driven word use* – and systematic scrutiny of the exchanges can reveal the type of *word use.* In sessions such as the one we are discussing here there is plenty of deictic language, aimed at screen or paper, and this renders such scrutiny more difficult. Audio or video recording of the sessions (not done for the session we discuss here) is then crucial and this is a methodological decision that the LUMOS team might consider (taking account of the intended non-intrusiveness of the innovations).

A similar observation to the one made above considering how the students' engagement met L's expectations applies to the students' loose, non-standard deployments of notation. During the session L seems also alarmed by this and on several occasions he draws on his *ultimate substantiator* (Sfard, 2008, p. 234) status to alert the students to the precariousness of such loose use of notation (see later in the paper one such occasion concerning the use of the expression 'cos(10x) on x^2'). There was one occasion, initiated by Student A, who proposed the introduction of the notation $t \to (\mathrm{f}(t) \cdot g(t))$, which came closest to a standard notational *realization* of the type of function that the preparation sheet invited the students to consider. We elaborate some repercussions of not pursuing this in the session towards the end of the paper.

Further, while the confidence with which the students deployed online software to generate complex and attractive *visual realizations* of their suggestions – often gazed at from all angles and bringing home the potentiality of speedy, intuition-friendly resources – was impressive, it was also notable that these visual awe-inspiring moments were hardly interpreted or explicitly connected to the task set by L in the preparation sheet.

With regard to the second aspect we wish to examine in this account (the students' and L's perceptions of the *learning-teaching agreement*), our account is far less hesitant: simply put, these 'newcomers'' expectations of the 'old-timer', L, who led the session, were very open. It is in fact this openness which brought about the use of 'shakeup' in the title of (Nardi, 2014), the first account of these observations.

Certainly the ethics requirement of the *learning-teaching agreement* for 'tolerance and solidarity' (Sfard, 2008, p. 287) was amply met. One incident illustrates what we see as a substantial power-shifting observed in the session: the exchanges taking place in such a session will, in Sfard's terms, eventually result in conceding to one of the present discourses being ultimately accepted by the interlocutors as privileged and paradigmatic. In a more conventional setting this would most likely be L's discourse. In the observed session this conceding did occur – but on the discursive path proposed by one of the students, not L. This was Student D, who proposed an innovative elaboration of the graph of a function from ℝ to ℝ×ℝ: The student defined two functions, the first was drawn in the conventional manner, and then the second was drawn

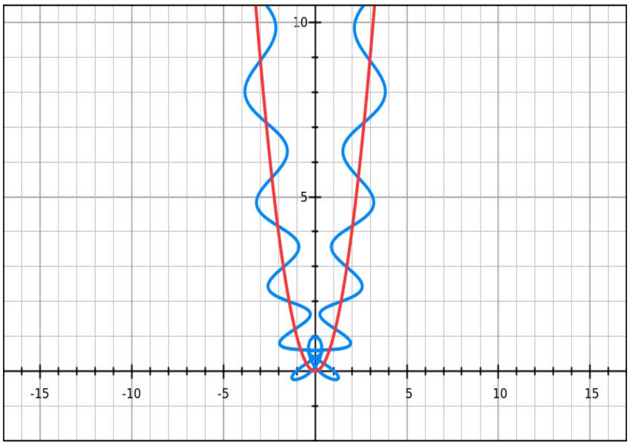

Figure 2: Student D's "$f(x) = \cos(10x)$ on x^2" production. He defines $f: \mathbb{R} \to \mathbb{R}\times\mathbb{R}$, $f(t) = (x,y)$, where $x=t-\cos(10t)\sin(\operatorname{atan}(10t))$ and $y= t^2+\cos(10t)\cos(\operatorname{atan}(10t))$

using the graph of the first as the independent axis with scales along this graph, and perpendicular to it, being the same as the originals. A short time after the session Student D had worked out how to use a computer graphing package to handle drawing such a function and offered the following (Figure 2):

In all this, L coordinated the intense exchanges with explicit and deliberate distancing, in fact with minimal use of his ultimate substantiator status.

It is in this ease with conceding this status that, in our view, the grandest element of the aforementioned 'shakeup' lies: L seemed uniformly open to the *narratives* proposed by the students; he seemed to actively hold back from encouraging their endorsement or rejection by the group. He seemed to sustain a mental list of proposed *narratives* that there had been no time to pursue, such as Student A's (see earlier in this section). In the pragmatic context of limited time – and Student D's more vocal presence attracting perhaps more attention than Student A's – this is not unlikely to happen during teaching. The observations of the session suggest that Student N appeared to experience the most obvious discovery moments. Student J's gestural language and body positioning also suggested so, particularly when 3D images started appearing on the screen of Student N's laptop. Only Student B appeared minimally participant, and quietly perplexed.

The session had buzz and warmth – but also left a slightly anxious sense of unfinished business about not having worked on Student A's proposed *narrative*. We note however that the events that followed on the same evening of the session to some extent appeased that anxiety: Student A wrote to L with an imaginative account of Student D's idea (omitted here due to limitations of space). He had nobly conceded to the temporary dominance of another student's proposed *narrative* in the session but made the most of it ...afterwards. There is at least one implication of this turn of events (and we do say this in full awareness of the modesty of a claim based on evidence from a single observation of a LUMOS innovation session):

> for at least the two hours of this *engagement session* these 'newcomers' slipped comfortably into the shoes of the 'old-timers', with all the fallibility and excitement that walking in these shoes entails. For that alone, surely this is an innovative path worth treading. (Nardi, 2014, p. 10)

A COMMOGNITIVE TRACING OF DESIRED LEARNING OUTCOMES?

Several questions emerge from our account of the 'shakeup' of the *learning-teaching agreement* in the observed session: Is this 'shakeup' liberating, perplexing to the students, both? How does it sit alongside the rest of these students' experiences at this university? They seem comfortable with it but will they stay so throughout? When, if at all, will they demand a reinstatement of L's *ultimate substantiator* status in the form of a demand for (say) specific assessment of their proposed *narratives* (on functions from $\mathbb{R}$ to $\mathbb{R}\times\mathbb{R}$, and beyond)? We conclude with tentative responses to these questions, based on the written reflections of L (second author).

L notes that the 'conceding of much of [his] status' we evidence here does not refer to his 'administrative status, nor [his] professional status' but his 'status as 'old-timer''. He prefers, however, the phrasing 'controller of content of discussion':

> 'I still controlled the direction quite a lot, although I used their prompts, choosing between them for (hidden) pedagogical and mathematical reasons. I believe that I can remember making both pedagogical and mathematical decisions at such moments.'

While on that point he stresses the pressures of 'running one of these sessions':

> '[it] is exhausting for the lecturer because of the constant attending to the direction of the conversation and evaluating it for potential mathemati-

cal (content and process) and pedagogical value. It is why, when a mathematician first watched me run a session [...], then tried it himself, he said that it was much, much harder than it looked.'

As to whether the experience is liberating or perplexing to students, he estimates that 'about half' 'find it liberating' and recalls students talking about 're-finding the creativity in mathematics' and 'expressing their pleasure at the sessions'. For 'about a quarter' though 'it is perplexing – they just do not seem to get what it is about' and for 'another quarter it is a mix between the two – interesting but they feel a bit out of their depth'. These estimates are his 'subjective judgement' and he highlights that 'these groups are not at all related to the students' mathematical ability'.

In relation to how the *low lecture* experience sits alongside the rest of the students' experiences L notes that his institution is 'reasonably liberal' and that it would not be unusual to find lecturers who are willing to 'cede some of their status'. Also many of these students 'will have had a similar sort of experience at times in their final year of school' where they are likely to have been 'treated quite respectfully as mature learners'. While 'probably unusual at first year' this respectful treatment in the *low lecture* innovation would then be 'not so strange'. Other factors, such as the presence of mature students in the group, may also reduce the 'strangeness' of the experience and make the students' commitment to this approach more resilient too:

> 'I've not seen any students in any session get MORE perplexed or uncomfortable, I've seen some get less and some stay the same. For those who were comfortable with it, a few grow into it significantly quite quickly.' [L's emphasis]

As an example, L returns to Student A's 'radical' follow up (see earlier) of the discussion in the observed session: 'He was checking with me that [his ideas] were ok, but he had really taken on the idea that the mathematics was there to be played with'. And while the students 'do check things out' with him (L), 'they have never seen this as "assessment" in the formal way (is it right or wrong)' but rather:

> '[they] seem to have caught on to the fact that this is exploration, and anything goes in some respects – it is what you do with it that counts, not what it is you are working with. I take this as a huge endorsement of the idea of the *engagement sessions* – that they are not about content but about process. I did not expect that most of the students would "get" this so quickly, although I did reiterate it often both in writing and verbally.'

Here we sampled towards deploying the analytical potential of the *commognitive* approach. Analyses from the implementation of the LUMOS innovations are ongoing.

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ENDNOTES

We dedicate this paper to our beloved LUMOS colleague Judy Paterson who passed away in January 2015. She will be sorely missed by all in this project and beyond.

A model of mathematicians' approach to the defining processes

Cécile Ouvrier-Buffet

Université de Reims Champagne-Ardenne, CEREP EA 4692, Reims, France, cecile.ouvrier-buffet@univ-reims.fr

This paper presents a modeling of the defining process used by mathematicians. This modeling has a strong epistemological background and has didactical outcomes. The aim here is to propose tools in view of studying the ways to implement a mathematical activity (close to the mathematicians' one) at the university level.

Keywords: Mathematicians, epistemology, conceptions, definitions, processes.

INTRODUCTION – ON THE IMPORTANCE OF STUDYING THE DEFINING PROCESSES OF THE MATHEMATICIANS

The study of defining activities is a discreet but constant didactical topic of research in mathematics education since the 90s. Mariotti & Fischbein (1997) have underscored the importance of such research: "(…) learning to define is a basic problem of mathematical education." (Mariotti & Fischbein, 1997, p. 219). Characterizing the defining processes and the defining activities is a new way, in mathematics education, to explore the mathematical concepts, their learning and their teaching. To place the definitions in the core of the mathematical activity (i.e. an activity that builds new knowledge, brings new proofs and theories), actually reveals an epistemological interest and a didactical interest: besides, the construction of definitions is a component of the research process of the mathematicians. Some researchers in mathematics education have characterized heuristics and behaviors of mathematicians (e.g., Schoenfeld, 1985, Burton, 2004; Carlson & Bloom, 2005; Gardes, 2013), but little research propose a modeling of the defining processes used by the mathematicians. Moreover, the modeling of defining processes in mathematics and the characterization of problems that allow a defining activity should be fruitful from a didactical point of view. Indeed, it brings a new way to analyze the processes of constructing new concepts, new proofs and new theories; then, the analysis and the guidance of the mathematical activity of students becomes feasible.

My research, i.e. the modeling of the defining processes in mathematics, fits into a wider field in sciences education that is the inquiry-based learning. The study of the implementation of mathematical and scientific processes in the classrooms (for mathematics, it means every process at stake in a mathematical inquiry of a mathematical problem) is a crucial point for the research in mathematics education. Besides, the question of the closeness between the results of the on-going mathematical research and the contents of university mathematics has to be studied both by mathematics educators and mathematicians.

In this paper, I propose first to synthesize the research in mathematics education that deal with the activity of mathematicians, with a short focus on the defining processes and the defining activities. I have developed in my research (Ouvrier-Buffet, 2013) a complete modeling of the mathematical defining processes. This model is based upon an epistemological background with a didactical efficiency. In this paper, I have chosen to present the results of interviews with professional mathematicians regarding their defining processes: these interviews bring concrete features of the professional defining activity. They also confirm and enrich the epistemological choices I have made for the design of my model (see Ouvrier-Buffet, 2006, 2011 and 2013 for the whole epistemological component). The last part of this paper opens new perspectives for university mathematics education.

STUDYING THE PRACTICE OF MATHEMATICIANS: AN OVERVIEW

In mathematics education, a recent kind of research deals with the practice of mathematicians. The focus is often made on proof and proving processes, which is legitimated by the fact that proofs are the holder of the mathematical knowledge (Rav, 1999) among other reasons. Hanna & Barbeau (2008) extend this point of view and underscore that the proofs can have other functions such as introducing new methods, tools and strategies to deal with new problems. It fits with Weber's conclusions: Weber (2011) shows that one of the reasons that mathematicians read the proofs of colleagues is to transpose ideas and techniques which may be useful in their own research. Wilkerson-Jerde & Wilensky (2011) analyze the reasoning of mathematicians and students dealing with an unfamiliar proof. They show several processes such as: the use of previous knowledge, the construction of examples, the deconstruction of a concept or an idea into subcomponents in order to explore the different components of a concept, the tests and the explorations of definitions, the attempts to connect definitions. As for Shriki (2010), he considers the creativity in the construction of knowledge. He shows that teachers reckon that mathematics can be taught as a reconstruction: the same kind of defining activities (reconstruction of geometrical concepts) as in the research of Larsen & Zandieh (2008) and Zandieh & Rasmussen (2010) is at stake. The place of the study of the construction and the use of examples in mathematics is also important. The focus on the examples that can be produced by students when they try to understand new concepts and when they want to illustrate mathematical ideas and properties, a focus which has been made by Watson, Mason and other colleagues, leads now to the analysis of the construction and the use of examples in proof processes (Watson & Mason, 2002; Sandefur et al., 2013). Recently, Gardes (2013) models the concept of "gesture" and defines it in a new way based upon the contemporary epistemology in order to analyze the practices of mathematicians. This concept appears relevant (for further research) to analyze the processes of mathematicians and of students during a research, and to consider the question of the transposition of the work of mathematicians to the classroom. In fact, in mathematics education, the defining activities are usually evoked during the study of proofs and of problem solving processes, but are not much studied for themselves. Indeed, it is commonly accepted that a proof can imply the necessity of the reconstruction of a definition: in this case, it concerns the exploration of the meanings of a definition (it can lead to the need of a better definition) and/or the study of the consequences of an assumption (Hanna, 2000). Besides, the way the students learn new concepts can be described with the enrichment of their concept images (Tall, 1991; Vinner, 1991). In fact, in these examples of the emergence of a defining activity, the emphasis is made on a part of the mathematical activity only, and not on the whole process that deals with definitions in the research activity of mathematicians. From an epistemological point of view, the work of Lakatos (1961, 1976) gives a unique example in the literature where the defining process and the proving process interact.

My research (see Ouvrier-Buffet, 2013 for the outcomes) proposes a reference epistemological modeling of the mathematical defining activity, taking into account the dialectic between defining and proving. Theoretical frameworks from the didactic of mathematics (the model of conceptions (Balacheff, 2013) and mathematics (complexity theory, Garey & Johnson, 1979) are called upon. This modeling is also based upon several experiments (at secondary and university levels) and upon interviews with mathematicians. The first level of my research was to identify emblematic epistemological conceptions which can characterize the mathematical defining processes: I have taken on the Lakatosian, Aristotelician, and Popperian conceptions (see Ouvrier-Buffet, 2006, 2011). Yet, these conceptions were not enough to describe the defining processes because some cognitive aspects, for instance, were not taken into account (the didactical experiments have shown this aspect). Besides, the characterization of problems which can lead to a defining activity had not been carried out. Moreover, it was difficult to understand how the epistemological conceptions coexist and interact. Then, I have enriched the epistemological component and I have conducted interviews with mathematicians to propose a complete overview of the defining processes. I have then design four main components to define my modeling of defining activities and processes: the characterization of three main epistemological conceptions regarding defining processes (the Lakatosian, Aristotelician, and Popperian conceptions; Ouvrier-Buffet, 2013, p. 67); the definition of problems that can lead to a defining activity (Ouvrier-Buffet, 2013, p. 72); the emphasis of four moments of work involving definitions (this part highlights the role and the place of the epistemological conceptions and the cognitive aspects; Ouvrier-Buffet, 2013, p. 69); and a didactical methodology to build, to analyze and to guide defining processes in classroom situations (Ouvrier-Buffet, 2013, p. 76).

DEFINING PROCESSES OF MATHEMATICIANS

Research questions and methodology

The aim of the interviews with mathematicians was twofold: firstly, I wanted to enrich my model of the defining activity based upon three epistemological conceptions, and intrinsically to validate it. Secondly, I was searching for a way to complete the aforementioned conceptions and to grasp the whole defining activity of mathematicians. Therefore, my underlying research questions were: how do the conceptions interact? Can one identify different moments when the mathematicians work on definitions, and are the epistemological conceptions operational? What are the types of definitions which the mathematicians use?

Regarding my previous epistemological research, I have defined *six lines* to analyze the data. They deal with the different kinds of definitions which the mathematicians talked about; the features of their defining processes (a focus is made on the interplay between defining and proving); the reasons of the evolution of a definition during a research; the ways the mathematicians valid a definition; the view they have on the defining activity in university mathematics classes; the identification of the moments when the mathematicians work on definitions, and then of their moments of work dealing with defining processes. These features and the elements of the epistemological conceptions gave me a grid with several components to analyze the interviews. I have also focused on the actions described by the mathematicians in order *to connect them to the epistemological conceptions*. In this article, the aim is to provide concrete examples to give an overview of a professional defining activity (following the six lines above-mentioned) and to highlight the characterization of four moments of the work on definitions in particular.

Data for this research

Eight professional mathematicians participated in this study. They come from different French universities and different fields of mathematics. Semi-structured interviews (1 hour or less long) were audiotaped and then transcribed. The questions were oriented towards four elements: their professional profiles, their practices of mathematics, their practices of the definitions and of the interplay between definitions and proofs, their representations of teaching at university level (do they think that implementing defining activities at university level can be useful and relevant?). The description of these interviews is available in Ouvrier-Buffet (2013).

Several kinds of definitions

The interviewed researchers actually identify several kinds of definitions:

— the discrimination between the definitions one knows beforehand and the definitions one can deduce from other results;

— the distinction between the definitions which remain and will belong to the public domain and the local definitions which are used to shorten a talk.

— the working definitions: one starts from the intuition that one gets from objects and problems. With this kind of definitions, one can work with the mathematical objects, and the statement of these definitions can be put off.

These kinds of definitions are linked to different aspects of mathematics and then several moments of the mathematical activity already appear: the moment when the intuition of objects and problems catalyses a research; the moment when working definitions (which can be local ones) give legitimacy to a new object which becomes worthy of interest; and the moment when the theoretical, formal and logical definitions are at stake (here the axiomatic theory is concerned).

The defining process and its interplay with proof

It is clear that the formalized definitions come "after" during a constructing process. The processes involved in the construction of a theory are not really dealt with by the mathematicians. By "defining process", they circumscribe a heuristic domain, from an intuitive exploration of mathematical objects and problem to the validation of a result. The insight of the results is often present. The defining processes specific to the construction of a new theory with axiomatic, logical and linguistic constraints are not much developed. For the mathematicians, to define is motivated by different needs: to have a better understanding of a concept or a problem, to simplify, to generalize, to explore different linked frames or connected fields than the first one, to communicate. The mathematicians consider that the proof is the master and that the proof can imply an evolution of the definitions. When they define or re-define objects during a proof, it is first to continue the research process, then to determine the domain of applicability of an idea or a proof, and finally to study more general cases or more particular cases. The definition of a really new concept (i.e. without an insight of it during a first exploration of a problem and/or a proof) can emerge during a proof. The proof process can have a local or a global impact: the significance of the new built concept will be proved later. Therefore it implies a long-term study of this new concept.

The reasons of the evolution of a definition

The "communication" dimension is important for the mathematicians: they explain that a definition will evolve when they will communicate their results. It can be in different institutions (seminary, talk, publications, university textbook) and the context will lead to one definition in particular or another. The mathematicians underscore the difficult use of the examples and the counter-examples in research. Some mathematical fields, where the concrete and discrete dimensions are involved (such as discrete mathematics), seem to be more suitable for building and using examples and counter-examples. Some defining problems (i.e. problems that can lead to a defining process) are pointed out by the mathematicians and are shortly described through research questions such as: to search if the problem can become a more general one, to search analogies with linked mathematical fields or with other concepts including a similar structure, to define the dual concept (it also implies the construction of new problems).

The validation of a definition

It occurs in different levels and places. The validation (i.e. when one considers that a definition is "correct") can be an individual and personal process (it is a self-validation). The colleagues play the part of the validation too, as well as the mathematical community. The fact that the proof works is also an element for the validation of a definition. In the same way, when a conjecture is "almost proved", or when there are no more counter-examples, or when the validity of a statement is controlled by the use of examples, the mathematicians consider that a definition can be valid. And a definition can also be agreed upon when it has a good strength during mathematical natural transformations or the implementation in different mathematical frames or structures. This being said, a mathematician indicates that for some cases, one cannot bring a validation, in particular when one does not know if the research process is ending (this is consistent with the Lakatosian view). These elements go in the same directions as Weber's results (2008) regarding the validation of a proof by the mathematicians.

A MODEL OF HOW CONTEMPORARY MATHEMATICIANS DEFINE

A basis: Three epistemological conceptions

I have previously presented an epistemological framework taking into account several conceptions: the Aristotelian one, the Popperian one, and the Lakatosian one (Ouvrier-Buffet, 2006, 2011). I have shown the ability of students to build definitions and to make a working definition evolve with the lack of counterexamples and with the reinvestment in a proof. I have also shown that the three aforesaid conceptions are useful to describe the students' defining processes, but not enough to grasp the whole processes, in particular the intuition and the in-action processes. I have then reintegrated the in-action dimension (this cognitive feature was missing in the model) with the in-action definitions and extended the modeling (Ouvrier-Buffet, 2013).

Mathematicians dealing with defining processes: Four moments of work

Four moments of work have been characterized during the interviews with the mathematicians. These moments do not describe a linear activity, but they are connected: they give a dynamic overall view of the defining activity in the mathematical research which also integrates the epistemological conceptions and underscores the different kinds of definitions. The names that I have chosen for these four moments of work are directly connected to the different kinds of definitions which exist in the speeches of the interviewed mathematicians.

The "in-action" moment of work

This moment of work deals with the intuition of mathematical objects, ideas, and results. The mathematical activity is here mainly an exploration and an impregnation of one or several problems and of objects in order to know them better (the use of examples, non-examples, counter-examples is here at stake). Analogies with other close mathematical fields can be used and new weak problems are stated. The Lakatosian conception is operational, with several operators: the statements of problems, the construction of examples and counter-examples, and the change of mathematical framework. In this "in-action" moment of work appear "in-action definitions" (Ouvrier-Buffet, 2011) and concept images (Vinner, 1991). An "in-action definition" is a statement used as a tool (not an object) that enables students to be operational without an explicit definition.

A transitional moment of work between "in-action" and "zero" – A potential link with the "axiomatic" moment of work

Two processes characterize this moment of work: to construct a first classification of mathematical objects and to re-use existing classifications; and to try to use analogies with existing concepts and theories. Here, the classifying, the categorizing activities and the denomination process of objects constitute the defining process. The Aristotelician and Lakatosian conceptions can be mobilized. If a link is made with the "axiomatic" moment of work, the Popperian conception can be used.

The "zero" moment of work

This is the place of the "zero-definitions" (Lakatos, 1961; Ouvrier-Buffet, 2011, 2013), but also of definitions that have a local impact. A "zero-definition" marks the beginning of the research process. A zero-definition can be modified in order to protect a primitive conjecture from a "monster" or because the concept is altered by the presentation of a proof. The Lakatosian operators (to use and to build examples and counter-examples, to use the method of monster-barring for instance) and other processes can be mobilized such as: to do false things, to reach an idea of the proof (the proof constraints the concepts and their definitions, quoting the mathematicians). Then, the zero-definitions and other local definitions have different functions: to denominate, to bring up several ways to grasp a concept, to work on a proof, to delimit the range of use of an idea or of a conjecture or of a proof, to communicate. The Lakatosian operator "to use another mathematical framework" can also be used and a link with the "axiomatic" moment of work made (in particular when a local or global theory pre-exists).

The "formalized" moment of work

I would like to underscore the "communicational" aspect, which appears during both a heuristic research, and a need of formalization, then I use the term "formalized" for this moment of work of mathematicians. Mathematicians can have to communicate (local) results during seminary, prepublications, talks etc. or to write a more formalized paper. In these both cases, there is a gap – an abstraction jump – compare to the "zero" moment of work. During the "formalized" moment of work, the mathematical activity concerns the use of some Lakatosian controls such as: the end of counter-examples implies that definitions, conjectures and/or proofs are solid. Here, "proof-generated definitions" can emerge. The proof and the proof-generated definition work together. A proof-generated definition originates from a proof while stemming from the development of the potential of a zero-definition. The catalysis of a proof-generated definition is impossible without the proof idea. Other operators take part in this moment of work, such as Popperian ones dealing with the construction of local (or even global) axiomatic theories. The writing of formalized and successfully completed definitions can be partially described with the Aristotelician conception. This moment interact with the concept definition (Vinner, 1991). Besides, the statement of new problems (a Lakatosian operator) allows the continuation of the mathematical research. The understanding of the new built concepts, and the generalization and the use of definitions, problems and results can lead new questionings. The exploration of neighboring concepts leads to a new "in-action" moment of work.

The "axiomatic" moment of work

The construction of a new theory (which can be momentarily local) and the construction of new concepts inscribed in this theory are at stake in this moment of work. I choose to call the definitions which are built during this moment "theoretical definitions" in order to underscore that there are inscribed in a theory. The Popperian conception is here clearly useful to characterize the research process of the "axiomatic" moment of work. In particular, the construction of the involved mathematical theory implies the search of the minimal number of rules (axioms) in order to generate results with a wide range. The axiomatic process can also unify concepts (see, for instance, the case of linear algebra, Dorier et al., 2000). The transposition of concepts to other mathematical fields brings opening questions for the research too.

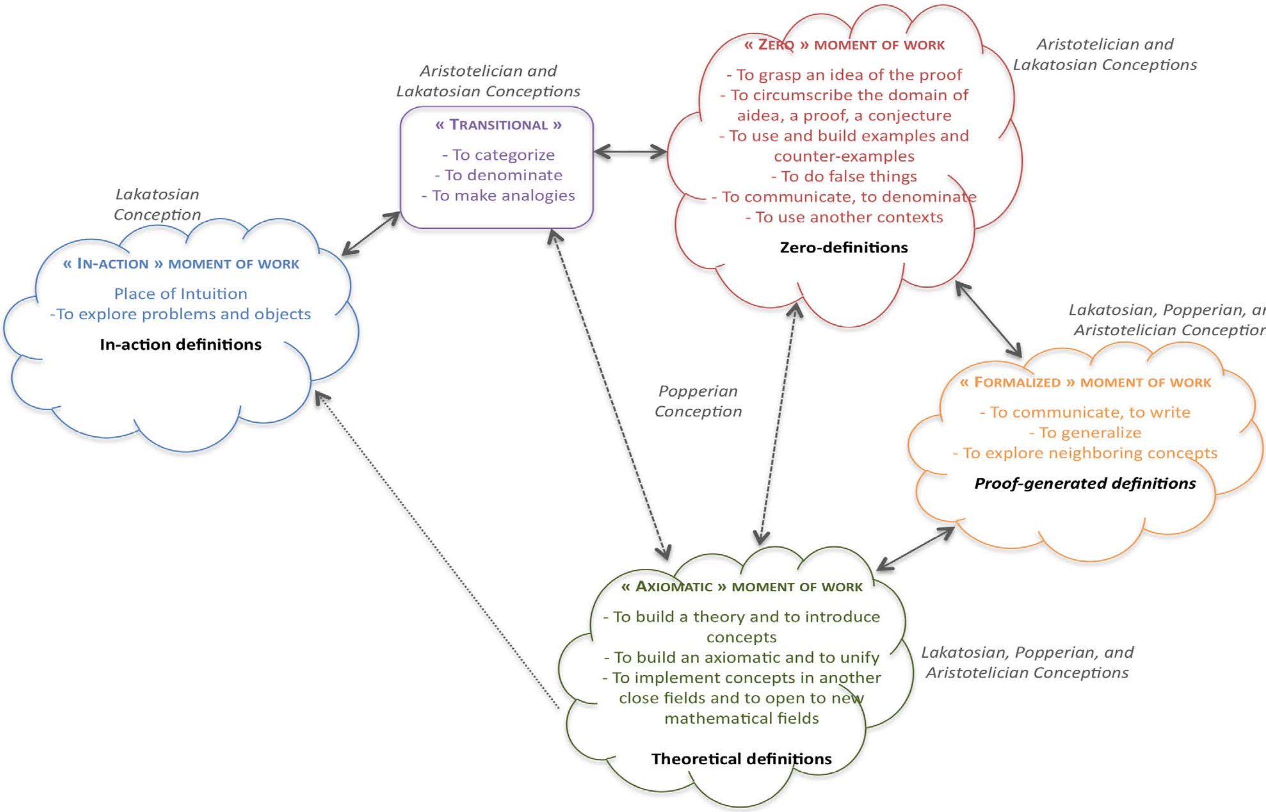

Figure 1: Overview of the defining process in mathematical research

The epistemological conceptions are operational in the moments of work

In the different moments of work on the definitions, as well as in the transition from one moment to another, the mathematicians mobilize one ore more conceptions (mainly, they express in words different elements which fit with operators and control structures of the epistemological conceptions). Then, Figure 1 synthesizes the features of each moment (in the coloured clouds), and shows the places where the epistemological conceptions are operational (in a cloud or in a transition between two clouds). The arrows mark the transition between two moments and sometimes the efficiency of a conception in such a transition. According to the interviewed researchers, one researcher cannot deal with all these moments, except for brilliant mathematicians. We can continue this research with new interviews, taking into account the different fields of mathematics.

OPENINGS

Ouvrier-Buffet (2011) has shown how a mathematical experience with a problem involving a defining activity can be conducted at university level. Such experiments also bring opportunities to reinvest the constructed concepts and the ways of reasoning in other mathematical fields. Then the question of the in-service teacher education (of university mathematics teachers) becomes crucial, as well as the definition of the contents of the university mathematics i.e. the concepts and the processes involving a real mathematical activity (and not only proof). There are clearly needs to engage collaborative research between university mathematics teachers and researchers in mathematics education (following Nardi, 2008). The interviewed mathematicians think suitable to implement defining activities at the university level, but they cannot conceptualize the way it can be implemented with students. They are very interested in the didactical research that can lead to new situations for the university. One can also extend this idea to sciences, especially with the study of the inquiry-based learning.

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How do research mathematicians teach Calculus?

Georgia Petropoulou[1], Barbara Jaworski[2], Despina Potari[3] and Theodossios Zachariades[3]

1 University of Athens, Department of Mathematics, Section of Mathematics Education, Athens, Greece, enaprosena@yahoo.gr

2 University of Loughborough, Loughborough, UK

3 University of Athens, Athens, Greece

We investigate Calculus teaching at university mathematics departments and in particular research mathematicians' teaching practice in the context of lectures. We are interested in how lecturers draw mathematics students into mathematical culture. In this paper, we focus on the teaching of a lecturer of a large cohort of students that we analyse using grounded techniques and the Teaching Triad construct (Jaworski, 1994). In spite of the lecture format, the analysis suggests that this lecturer's teaching is characterized by the way he supports students' engagement in the lecture and the way he familiarises them with mathematical production. The use of the Teaching Triad brings to our insight what sensitivity to students could mean in this traditional setting and what mathematical challenge could be at the university level.

Keywords: University teaching, sensitivity to students, mathematical challenge.

INTRODUCTION

Teaching at the University level deals commonly with large groups of students and the lecture is the instructional activity within which teaching usually takes place. Arguably the lecture instruction has received a strong criticism and has been widely maligned by mathematicians and mathematics educators alike (Weber, 2004). It seems that lecture is considered as a unified, single teaching paradigm and thus a small number of studies investigated mathematics teaching in this context (Speer, Smith, & Horvath, 2010). For example, Weber (2004) addressed the nature of lecturing showing that it is not as a single teaching paradigm as it thought to be and Jaworski, Treffert-Thomas and Bartsch (2009) pointed out the tensions that a lecturer experiences in satisfying student needs and mathematical values. Yet, Speer and her colleagues (2010) in their systematic literature review argued that university mathematics teaching is "an unexamined practice" and drew attention to the need for empirical research that examines and describes the work of teaching university mathematics in detail.

Seeking to readdress this scarcity of empirical research and to gain better understandings of the actual teaching at this level, our study aims to explore characteristics of first year university mathematics teaching in Greek mathematics departments. The topic in focus is *Calculus* which is a first year compulsory course taught in a lecture format. Research concerning the learning of Calculus at the university level has shown that students experience difficulties in aligning with advanced mathematical processes and concepts when they enter into university (Artigue, Batanero, & Kent, 2007; Nardi, Biza, & González-Martin, 2009). A question is how teaching at this level and in the particular context of a lecture could deal with the above difficulties while drawing mathematics students into mathematical culture. In this respect, there is a growing body of research seeking to characterize elements of teaching practice that takes students into account (some examples will be discussed below in the theoretical background section) but not at the university level. Being aware of the complexity of teaching that has been identified by many researchers into secondary level teaching (e.g., Potari & Jaworski, 2002) and does not end with the transition to the university level, we attempt to investigate first year Calculus teaching in the, most usual for this level, format of a lecture. In particular, we focus on one lecturer's teaching actions and the rationale behind these actions to identify the characteristics of his teaching and we attempt to realise the nature of the identified characteristics in this particular context.

THE THEORETICAL BACKGROUND

Exploring characteristics of teaching deals inevitably with the fundamental question "what is teaching" addressed by some philosophical studies (e.g., Hirst, 1971). Some other studies refer to the debate whether teaching is a practice or a means to introduce students to another practice – the mathematical practice (e.g., Noddings, 2003). We see teaching as an activity following Pring (2000) who claims that: "An action might be described as 'teaching' if, first, it aims to bring about learning, second, it takes account of where the learner is at, and, third it has regard for the nature of what has to be learnt" (p. 23). Adopting a sociocultural perspective, teaching is considered not only as a product of the constructive activity of the individual teacher but also as a social practice; a complex nexus of social inter-relationships (Jaworski, 2002). Learning mathematics at this level is seen as enculturation in advanced mathematical practices (Artigue, Batanero, & Kent, 2007). Enculturation, in the sense described by Bishop (1991), is an interpersonal process so the role of people who have special responsibility for this process is emphasized. This "cultural group" of people are for us the mathematicians who teach at university level. Bishop (1991) quoted Wilder (p. 6) who wrote for the mathematicians: "Those people who do mathematics – the 'mathematicians' – are not only the possessors of the cultural element known as mathematics but, when taken as a group in their own right, so to speak, can be considered as the bearers of a culture, in this case mathematics". This view offers us a base to interpret lecturer's attempts to draw students into mathematical culture and characterize his practice since "introducing children into the culture of a mathematical practice is basically a social process" (Van Oers, 2001, p. 73).

Speer and colleagues (2010) made a distinction between instructional activities and teaching practice. According to this distinction the lecture, the context of our study, is an instructional activity while teaching practice concerns what teachers do when they are planning, teaching and reflecting on their lesson. Thus, lecture is the usual instructional activity at university level yet in lecturing different teaching practices may take place. We investigate a lecturer's teaching practice i.e. his/her teaching actions (what s/he does intentionally) and the rationale behind these actions, seeking to characterize this practice and gain deeper insights into university teaching.

Our research tool in the endeavour to interpret the nature of teaching characteristics is the Teaching Triad (TT). TT is an analytic framework that emerged from an ethnographic study of investigative mathematics teaching at secondary level (Jaworski, 1994). Its main goal was to capture essential elements of the complexity of mathematics teaching by analyzing classroom interactions. Jaworski (2002) describes that the triad consists of three "domains" of activity in which teachers engage: *management of learning* (ML), *sensitivity to students* (SS) and *mathematical challenge* (MC). ML describes how the teacher organizes the classroom learning environment (e.g., groupings, planning of tasks, setting norms). SS describes teacher knowledge of students and attention to their needs and in particular the ways that he/she interacts with individual students and guides group interactions. Sensitivity to students has been shown to relate to both affective (e.g., offering praise, encouraging students to participate) (SSA) and cognitive (e.g., judging appropriate questions, inviting explanation) (SSC) domains. MC describes the challenges offered to students to engender mathematical thinking and activity. This includes tasks set, questions posed, and emphasis on metacognitive processing. The above elements are closely interrelated as the study of Potari and Jaworski (2002) indicated. The authors claim that a balance between sensitivity to students (in both cognitive and affective domains) and mathematical challenge is an indicator of effective mathematics teaching in the sense that students can be involved in rich and meaningful mathematical activity. The triad has also been used to characterize teaching for example in a study of interactions in university mathematics tutorials (Jaworski, 2002, Nardi, Jaworski, & Hegedus, 2005) but it has not been used to characterize lecturing so far. It is a question for example, what sensitivity to students could mean in lecturing large groups of students or what mathematical challenge describes in this context. It is in our interests to reassess the elements of the Triad and define the potential meaning they gain at this level.

Nardi and colleagues (2005) also characterized teaching approaches in small group tutorials from the perspective of the tutor offering a theoretical perspective on the links between mathematics and pedagogy. Lobato, Clarke & Ellis (2005) developed a theoretical reformulation of telling, characterizing teaching actions according to their function and Anghileri (2006) characterized teaching by identifying a hierarchy of interactions that relate to teaching practices that can enhance mathematics learning. Grandi and Rowland (2013) analyzed the function of teacher interventions while Drageset (2014) characterized teachers' comments. The above studies characterized approaches to teaching and gave us the theoretical underpinnings to formulate specific criteria for coding the lecturer's teaching actions.

METHODOLOGICAL ISSUES: DATA AND ANALYSIS

This paper is a part of an ongoing study with aim to investigate the first year Calculus lecturing in two mathematics departments. Six lecturers participated in the wider study. Calculus is a proof-based, first year course in both departments and it is taught in parallel in two or three classes of approximately 100 students each (4 hours for theory and 2 hours for solving exercises per week for one semester – all in a lecture format). In this paper, we analyse data from one lecturer who is an experienced active research mathematician and university teacher, with the ultimate goal to draw students into mathematical culture. Data for this lecturer were collected through observations (3 two-hour lectures), interviews (3 interviews right after each lecture) and group discussions (5 three-hour group discussions). During the observation of the lectures, field notes were also kept and after class interviews with the lecturer conducted by the first author. Moreover, in group discussions among some of the 6 participants of the wider study and mathematics education researchers, more general issues about university teaching were discussed. The lecturer of this study participated in all group discussions. All lectures, interviews and discussions were audio-recorded (one of the lectures was also video-taped) and transcribed.

In data analysis, grounded approaches (Strauss & Cobin, 1998) and the Teaching Triad (Jaworski, 1994) were used. The analysis was conducted in three phases. In the first phase, each two-hour lecture was divided into episodes according to the accomplishment of teaching a theorem. Each episode was coded with descriptive codes according to what the lecturer did during this episode and from these descriptive codes teaching actions were identified. Codes for teaching actions were merged or refined after continuous comings and goings through the whole data and were supplemented by categories taken from Anghileri (2006), Drageset (2014) and Grandi & Rowland (2013). This process enabled us to form criteria for characterising the codes of the teaching actions. Finally, the analysis resulted in 12 codes. The rationale of teaching has been investigated through the analysis of interviews and discussions which were also coded according to the subject under discussion and in this way the lecturer's teaching practice (i.e. teaching actions and rationale) has been identified. In the second phase, teaching actions were grouped into categories and in this way the characteristics of this lecturer's teaching practice were identified i.e. characteristics are categories of teaching actions and their rationale. In the third phase we used the Teaching Triad to gain deeper insight into the identified characteristics. Thus characteristics of teaching were analysed further by using the TT to explore potentials of TT's elements in this level and in this way we gained insights into the nature of these characteristics per se.

In the next section, first we give an example of analysis to illustrate the analytical process and the related emerging issues and then we give a brief account of the results from the analysis of the whole data for this lecturer.

RESULTS

A teaching episode that appears to be typical of the way the lecturer interacts with students follows as an example of analysis. In this episode, which is about the convergence of series of real numbers, the lecturer tries to facilitate students to arrive at a conjecture.

An example of analysis: A teaching episode and its analysis

At the beginning of the lecture after the definition of a convergent series, the lecturer gave the geometrical series

$$\sum_{k=0}^{+\infty} x^k = \frac{1}{1-x},\ |x| < 1 \text{ (S1)}$$

as an example of a convergent series (a familiar example from secondary school). Subsequently, he gave the example of divergent series:

$$\sum_{k=1}^{+\infty} (-1)^k \text{ (S2)}$$

Then the dialogue below followed in Table 1.

An episode (The lecturer facilitates students to arrive at a conjecture)	Descriptive Codes	Teaching actions
L: Now, can you hypothesize, when a series may converge?	Challenging students to be engaged in mathematical exploration	Exploration
You can base on the series S1 and S2.	Prompting students to focus on the examples [(S1) and (S2)]	Focusing
(No response) The lecturer wrote again the two examples (S1 and S2) on the board (one next to the other) with the words "converges" and "does not converge" respectively next to each example.		
L: Can you see any differentiation in these two cases S1 and S2]? They are two specific cases of course but... is there any difference between them?	Asking a more focused question to compare example's characteristics	Focusing
S1: Yes. The base [of the *v*] in the first case (S1) is a positive number.	Gaining insight from a student's response	Evaluating
L: Not necessarily. If x is a negative number the sign [of x^k] changes. (he moves his hand left and right showing the alternation of the signs over an imaginary number line).	Offering a counterexample	
S2: Counting starts from different numbers	Gaining insight from another student's response	
L: No, it has nothing to do with... We could start from the same number. (He changed (in S1) the counting from $k = 0$ to $k = 1$ on the board). Now on the board the two series were: $\sum_{k=1}^{+\infty} x^k$ (S1a) and $\sum_{k=1}^{+\infty}(-1)^k$ (S2)).	Modifying the example to avoid irrelevant details	Simplification
L2: Here you are! Now the series ((S1a) and (S2)) seem comparable. Finite number of terms at the beginning of a series does not have any influence on its behaviour [related to its convergence].	Providing a reason for the modification	Explaining
S3: Is the difference that the 1st series (S1a) has a variable?	Gaining insight from a third student's response	
L: It's not a matter of variable. OK. If I set here... I don't want to confuse you. Let me take a specific geometrical series. (He writes the series $\sum_{k=1}^{+\infty} \frac{(-1)^k}{10^k}$ (S1b)). I took units everywhere. This number... I made it to look about the same.	Specifying the example	Simplification
S1: Is the difference that the 1st case (S1a) has the condition $\|x\|<1$?	Having relevant responses	
S4: Is the difference that in the 2nd case (S2) the absolute value is 1 ($\|-1\|^k = 1$)?		
L: This observation (of S4) may imply something. It's a correct observation.	Confirming	Evaluating
Your fellow student said we should take these terms (the sequence in S2) and we'd consider their absolute value. This absolute value will always be 1. This might be a difference. Whilst in this case (S1b) the absolute value of the sequence in the series is $1/10^k$. And?	Rephrasing student's talk	Drawing on students' experiences

An episode (The lecturer facilitates students to arrive at a conjecture)	**Descriptive Codes**	**Teaching actions**
S5: ... and this absolute value is less than 1.		
L: Yes. And?		
(No response)		
L: If k tends to infinity, where does $1/10^k$ tend to?	Providing a hint	Directing discussion
S1: To 0.		
Then the lecturer summarizes the result of the investigation and formulates the property.	Recapping and formulating the conjecture	Summarizing Explaining

Table 1: A teaching episode and its analysis (Translated from Greek)

In the above example we see some of typical teaching actions of this lecturer's teaching. In particular, the lecturer mainly poses a problem; calls students to express their observations; uses examples familiar to them ; evaluates students' observations by giving comments and by simplifying the examples; asks more focused questions when students do not follow; uses students' relevant observations as a basis to direct discussion and summarizes the main results.

In terms of the TT, we see expressions of his sensitivity to students and the mathematical challenge he offers to them. For example, he tries to motivate students to participate by asking questions and reassures students that their contributions are acceptable (SSA); highlights and builds on their ideas and comments (SSC). He also addresses mathematical important questions; encourages students to think deeper to identify relations (e.g. by reflecting on familiar examples) and engenders students' interpretations of relationships and representations (MC). Moreover, he organizes the content of the lecture to support students' reflections (e.g. he brings an example and later he uses it as a basis for exploring relationships) and he establishes norms of a working group (e.g. he always uses "we" in his talk) (ML). Analysing all the data from this lecturer, two specific goals were identified. The first goal was mainly affective: the lecturer tried to stimulate students to become confident in their engagement with the advanced mathematical content:

"What I do consciously is that I try to tone up students psychologically ... Speaking emphatically, I believe that, in order for someone to be engaged in the learning process he must have the appropriate psychology. I think that the 50% of learning is the learner's psychology." (Translated from Greek)

The second goal was mainly mathematical: he tried to initiate students into advanced mathematical thinking and mathematical production: "What I try to do is to teach students how to think mathematically... And I have to make them understand how someone thinks and produces mathematics." (Translated from Greek).

These goals were carried out with specific teaching actions. These teaching actions were related and grouped forming the characteristics of this lecturer's teaching which have been analysed further with the TT. Table 2 presents how all teaching actions are grouped into characteristics and how the characteristics are related to the three domains of the Teaching Triad.

We related the actions "explaining", "directing discussion" and "summarizing" because they characterise lecturer's guidance to students by informing or providing suggestions to them. In terms of the TT, we see this guidance as an indication of ML and SS as the lecturer gives explanations, suggestions or points out an idea with sensitivity to students' needs.

We related the actions "evaluating", "drawing on students' experiences" and "checking for consensus" because they characterise lecturer's supporting of students engagement. The lecturer evaluates students' contributions in a way that rewards their involvement even if the contribution is invalid and he is not moving on without their consensus. In terms of the TT, we see supporting students' engagement as an expression of lecturer's SS as the lecturer opens up students' ideas, builds on them and confirms that these ideas contribute in the learning process.

Teaching actions	Characteristics
Explaining	Guidance of students (ML and SS)
Directing discussion	
Summarizing	
Evaluating	Support of students' engagement (SS)
Drawing on students' experiences	
Checking for consensus	
Focusing	Discard any irrelevant features (MC and SS)
Simplification	
Exploration	Use of problem-solving techniques (MC)
Posing a problem	
Relating mathematical ideas	
Translating mathematical ideas	

Table 2: Teaching actions and characteristics of teaching

We related the actions "focusing" and "simplification" because they both characterise discarding any irrelevant features. The lecturer simplifies examples a strategy which he uses in his research: "This is the way I produce work." So, discarding any irrelevant features is a process of mathematical production at least for him. As he states in an interview simplifying also allows him "to include more students in the lecture", so to be more in accord with students' needs. Thus, in terms of the TT, we see discarding any irrelevant features as an expression of MC which, in this case, is related with SS.

We finally related all the actions through which the lecturer introduces students to problem-solving techniques by creating conditions for investigations ("posing a problem" and "exploring") and by encouraging connections ("relating" and "translating mathematical ideas"). In terms of the TT, we see that using problem-solving techniques is a characteristic of MC as the lecturer challenges students to be engaged in mathematical exploration and to relate and represent mathematical ideas – both important processes of mathematical thinking and production.

Summing up, we could claim that this lecturer's teaching is characterized by the way he takes students into account showing sensitivity to their needs and by the way he tries to draw them into mathematical production offering mathematical challenge.

DISCUSSION

In this paper, we studied the teaching practice of a lecturer teaching large cohorts of students and we identified characteristics of his teaching which we interpreted in terms of sensitivity to students, mathematical challenge and management of students' learning. In particular, this specific lecturer supported students to participate in the processes of advanced mathematical thinking and production bringing experiences from his research activity into his teaching.

In terms of the Teaching Triad, we see that in this case, mathematical challenge at the university level is directly related with the mathematical production. This is a different dimension of mathematical challenge than the ones may be found in secondary level and we believe that it is worth to explore what other dimensions mathematical challenge could receive at this level. We also see that in spite of the lecture format, this lecturer's teaching is characterized by his sensitivity to students since he builds on their ideas to support their engagement in the learning process. This is far from obvious in lecturing large group of students.

We tried to shed light on what takes place during the dominant instructional activity of the lecture. Our results showed that the lecture is not a unified teaching paradigm. This lecturer's teaching practice contrasts with the practices thought of as usually adopted in Calculus lectures and offers evidence that teaching can exist in a lecture format that is sensitive to the students and resembles mathematical production.

Collegiate teachers seem to work in different ways in their classrooms under the instructional context of a lecture. For this reason, as Speer and colleagues (2010) state their practice and reasoning is worthy of study because it can help others (teachers and researchers) understand how and why teaching happens in certain ways.

Obviously, many lectures at the university level are still taught in a transmissive way in the sense that the lecturer conveys information and the students listen and passively take notes. However, this study provides evidence that there are alternative ways for teaching in the instructional context of the lecture and might be used as a lecturers' self-awareness springboard towards improving teaching.

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Students' concept images of inverse functions

Sinéad Breen[1], Niclas Larson[3], Ann O'Shea[2] and Kerstin Pettersson[3]

1 St Patrick's College, Drumcondra, Ireland

2 National University of Ireland Maynooth, Maynooth, Ireland

3 Stockholm University, Stockholm, Sweden, kerstin.pettersson@mnd.su.se

We analyse data from two studies in Ireland and Sweden relating to the concept of inverse function. In particular, we consider components of the participants' evoked concept images of this topic when answering open questionnaire questions. The results show that the students' concept images contain algebraic, geometric and more formal components: both Irish and Swedish students describe inverse functions as swapping x and y, as a reflection or a reversal. How various components may or may not enrich students' conceptual understanding of inverse functions is discussed.

Keywords: Inverse function, concept image, university mathematics education.

INTRODUCTION

In this paper, we aim to identify elements of undergraduate students' concept images of inverse functions. The concept of inverse function is usually covered in introductory calculus courses either at school or at university, with a well-developed conception of 'function' necessary for understanding 'inverse function' and deciding when an inverse function exists. However, as Pettersson, Stadler and Tambour (2013) argue, the function concept is a troublesome one for students and Carlson, Oehrtman and Engelke (2010) point out that students who are unable to conceive of a function as a process (rather than taking an 'action' view) have difficulties inverting functions. The concepts of function and inverses are essential for representing and interpreting the changing nature of a wide array of situations (Carlson & Oehrtman, 2005) and to describe the relationships between logarithms and exponentials for example. In this study, to enrich and broaden the data, we considered data from two projects in two countries (Ireland and Sweden). Both projects were focussed on the development of conceptual understanding of the function concept, and both involved collecting data from first year undergraduate students. The two studies included questions on the inverse function concept and we will present an analysis of these data here.

Despite the volume of research into the concept of function (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992), there does not seem to have been much work which concentrates on the topic of inverse function alone. One such paper is that of Even (1992) where prospective secondary mathematics teachers' knowledge and understanding of inverse function were investigated. In an open-ended questionnaire the participants were asked to find**.** Both the function () and the inverse were given. The answer is straightforward if one uses the idea of an inverse as undoing. However, the results showed that several students did not draw on their conceptual knowledge of the inverse property of undoing and instead used a chain of calculations to get the answer. This tendency to calculate instead of using the conceptual meaning of inverse function may be related to weak conceptual knowledge. However, Even (1992, p. 561) concluded that "a solid understanding of the concept of inverse function cannot be limited to an immature conceptual understanding of 'undoing'", which she claimed may result in incorrect conclusions, e.g. that all functions have inverses.

The conception of undoing is not the only way to look upon inverse functions. Vidakovic (1996) also placed importance both on composing the function and the inverse to get the identity, and on the action of swapping variables. Carlson and Oehrtman (2005) categorise three different conceptions of inverse function: inverse as algebra (swap x and y and solve for y), inverse as geometry (the reflection in the line $y = x$) and inverse as a reversal process (the process of 'undoing'). Carlson and colleagues (2010) showed that students who conceived of inverses as reverse processes were able to answer a wide variety of questions about inverses.

The algebraic and geometric views are considered in a paper from Wilson, Adamson, Cox and O'Bryan (2011). They argue that the common procedure of swapping x and y to find the inverse is confusing for students and can lead to the meaning of the result being obscured, especially for contextual or real-world problems. They contend that both swapping the variables and drawing the graph as a reflection in the line $y = x$ do not take into account the important aspect of the domain of the inverse function being the range of the function and vice versa. In particular this causes problems when the dependent and the independent variable of the function are in different units. Wilson and colleagues (2011) instead proposed the approach of solving for the dependent variable, rather than literally swapping x and y, to reduce confusion and enhance students' conceptual understanding of inverse functions. Attorps, Björk, Radic and Viirman (2013) commented on the geometric view, using GeoGebra to teach inverse functions. The results of their study indicated that several students showed an intuitive conception of inverse functions as some kind of reflection, but lacked the full comprehension of why and where the reflection should be performed.

Bayazit and Gray (2004) reported on teaching inverse functions to Turkish high school students and they observed that the students who showed a conceptual understanding of the inverse function put particular emphasis on the '1–1 and onto' conditions. They suggested that teaching should link the inverse function more explicitly to the concept of '1–1 and onto' as well as to the concept of function itself.

ANALYTICAL FRAMEWORK AND RESEARCH QUESTION

One way of studying students' conceptions is to use the theory of *concept image* (Tall & Vinner, 1981). This theory has, for a number of decades, proved to be a useful tool in analysing undergraduate students' conceptions of mathematical concepts (e.g., Bingolbali & Monaghan, 2008; Wawro, Sweeney, & Rabin, 2011). A concept image is defined to be the cognitive structure associated to a concept and includes the individual's interpretations of characteristics and processes that the individual connects to the concept. It also includes examples, intuitive ideas, mental images and, if known, formal definitions and theorems. The cognitive structure is built up successively through the individual's meetings with the concept. When meeting tasks involving the concept different parts of the concept image can be activated; the part activated at a particular time is called the *evoked concept image*.

In this study we are looking for components in the students' evoked concept images. Our research question is: What characteristic elements can be found in the evoked concept image of inverse functions of first-year university students?

METHODOLOGY

The Irish study

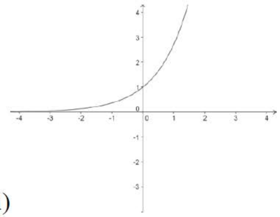
The data from Ireland involved students' responses to one of twelve questions on a concept inventory instrument designed to investigate undergraduate students' understanding of the concept of function (the full instrument can be found at http://staff.spd.dcu.ie/breens/documents/ConceptInventoryforFunction.pdf). First year Humanities, Education, and Finance students taking calculus modules (taught by the first and third authors) in two Irish universities were asked to voluntarily complete the inventory at the end of their module. 100 students took the test, 65 of whom answered at least part of Question I (see Figure 1).

The second level syllabus followed by these students mentioned inverse functions solely in the context of inverse trigonometric functions and the textbooks did not contain formal definitions or geometric representations of inverses. Inverse functions were initially discussed in both university modules (recall these were taught by the first and third authors) as reverse processes, and the role of bijectivity in determining whether an inverse exists was identified. A formal definition of inverse (for all x in the domain of f and all y in the range of f) was presented, and means of adjusting the domain or codomain of a function to make it bijective and thus invertible was discussed. The graphs of a function and its inverse as mirror images of each other in the line $y = x$ were explored, while the algebraic method of finding an inverse was acknowledged when presented by students.

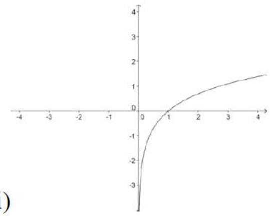
The Swedish study

The Swedish data were collected as part of a project aiming to explore students' development of their un-

I. (a) Look at the graphs (i) and (ii) below. The claim is that graph (ii) represents a function which is the inverse of the function represented by graph (i). Explain what this means and state whether you think the claim is true.

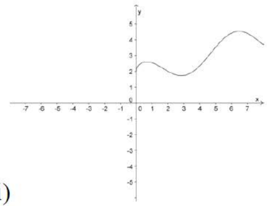


(i)



(ii)

(b) Look at the graph (iii) given below and draw, on the same axes, a graph to represent its inverse if you think there is an inverse. Explain clearly why you think there is an inverse or not an inverse.



(iii)

Figure 1: The questions in the Irish study (I)

derstanding of a threshold concept (Pettersson et al., 2013). A study group, in total 18 prospective secondary teachers who were enrolled in courses in mathematics, was observed during their second semester of teacher education. In the second level syllabus in Sweden inverses are not explicitly included although they might be mentioned. At university these students were introduced to inverse functions in the course 'Vectors and Functions'. Because of lecture observations we know that the inverse function was defined by . Both algebraic and geometric aspects were mentioned and also that the function needs to be 1–1 for an inverse to exist. In a subsequent calculus course inverse trigonometric functions were included and the need to restrict the domain of the function to ensure it is 1–1 was discussed.

The Swedish students participated voluntarily in three questionnaires relating to their understanding of the concept of function (Pettersson et al., 2013). For the present paper, focussing on the concept of inverse function, the answers on three of the questionnaire tasks were analysed (see Figure 2). In a questionnaire at the end of the course on vectors and functions, the students were given the question S1. In the questionnaire given at the end of the semester, not far from the end of the calculus course, the students were asked two questions on inverse functions, S2 and S3.

S1: Make a mind-map or a concept map with the word inverse function as starting point. Bring in other words/concepts that you can associate with inverse function. Please also indicate how you think that the words/concepts are interrelated.

S2: What is required for a function to have an inverse?

S3: In the picture (see below) you see two graphs of two different functions. What can you, based on the picture, tell about these functions?

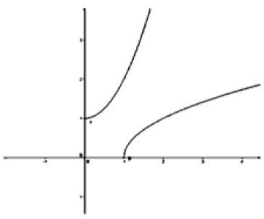


Figure 2: The questions in the Swedish study (S1, S2 and S3)

Coding

In each country the students' responses were coded using a grounded theory approach: that is the students' responses were read multiple times, codes were assigned, and these codes were then grouped into categories. At the coding stage, the responses to the tasks above were initially coded by one of the researchers, and then checked by another before the agreed codes were grouped into categories. The two sets of categories which emerged were then compared to check for consistency.

RESULTS

Results from the Irish study

We first considered the students' answers to I(a) above, i.e. the students' explanations as to what it means to say one function is the inverse of another. Only 4 students explained the concept correctly, 50 students gave an explanation which contained errors or was incomplete, while 11 answered true or false with no explanation. An example of a correct answer was:

> Let be the function in (i). Let be the function in (ii). The claim states that . It claims that is a reflection of through the line $y = x$. I agree with the claim. (Cathy)

The categories of components that arise in the evoked concept images of inverse function are shown in Table 1. Note that some students are counted more than once here if their answer referred to two or more of the concept image components identified. For example one student's answer was coded using 'opposite' and 'reflection' and so now appears under both 'Opposite' and 'Reflection' in Table 1.

We can see that the most frequent conception is 'reflection'. This category also includes responses from students who used the term 'mirror'. Its close associates of 'opposite' and 'symmetry' are also frequent. Seeing the inverse as a reverse process is common, while only 5 students gave anything resembling a concept definition of an inverse in answer to this question. There were 13 responses categorised in the 'other' category, these include: 6 responses which refer to a feature of the given graphs, 2 responses which mention 'folding', 1 response for 'domain and range interchanged', and 1 'example'. The remaining responses in the 'other' category are not mathematically relevant. We have included both correct and incorrect notions within each component of the concept image in Table 1; for instance, although 33 students used 'reflection' or 'symmetry' in their explanations, only 9 students correctly described the line of reflection or symmetry as the line $y = x$. Two referred specifically to reflection in the origin and 3 to reflection in the x-axis, while a further 6 spoke of a reflection without being specific. Students used 'opposite' when describing their concept image of an inverse in different ways: 6 of them used the word in a way that suggested reflection, and 3 mentioned graphs; 7 talked about functions being the opposite of each other; 2 spoke about opposite values. Some examples of responses in different categories are given in Table 2 below.

When asked if the claim given in I(a) was true, 41 of the 58 students who answered this question were able

Conception	Reflection	Opposite	Reverse	Symmetry	Definition	$1/f$	Swap x and y	Other
Total	27	17	10	6	5	3	1	13

Table 1: Conceptions emerging in response to I(a)

Conception	**Sample explanations given by students**
Reflection	Inverse is a reflection of the function. The inverse is that function mirrored through the line $y = x$.
Opposite	Inverse means they are exact opposites of each other, they cancel each other out.
Reverse	It is the function in reverse.
Symmetry	The inverse of a function is the function given through symmetry in the line $y = x$.
Swap x and y	When the x and y coordinates swap, e.g. here the point (1, 2) becomes (2, 1).

Table 2: Sample responses from Irish students

to correctly identify that the graphs shown did indeed represent inverse functions.

In response to I(b), 47 students attempted to draw an inverse function. 45 of these students sketched a reflection of some sort; Table 3 shows the distribution of these attempts.

Task I(b) was answered correctly by 11 students, that is, they were able to say that the function did not have an inverse and were able to give a reason for their answer. These reasons were: fails the horizontal line test (6), not 1–1 (6), inverse would not be a function (1) – illustrating a further component of the concept image held by these students. Note that two students said the function was not 1–1 and also illustrated using the horizontal line test which accounts for the numbers adding to 13. A further 4 students correctly stated that the function had no inverse but did not give a reason. Two students said that the function did not have an inverse but did not give a complete explanation, for example one of them said that "it is not just a line/angled line". The students who said that the function did have an inverse gave a variety of explanations for their answer: for instance, one said that every function has an inverse, while another said that the function was "defined for its whole domain". We saw that the conception of inverse function as a reflection in the line $y = x$ could be misleading with four students making remarks such as "There is an inverse as possible to draw line $y = x$ and reflex (sic!) images".

Students who gave correct and complete answers on I(b) offered a variety of answers to I(a) illustrating a variety of elements of concept images associated to inverse functions. These were: definition (2), reflection (3), reverse (1), opposite (2), no answer (3).

Results from the Swedish study

In the analysis of the survey, we first categorised the components of the evoked concept images that were exposed in the answers to task S1 (11 answers) and S3 (12 answers), see Table 4. The categories included also incorrect answers. Answers that contained more than one component were counted into more than one category.

To illustrate the components that arise in the students' concept images of inverse function we have picked answers from three frequent categories. In the category 'reflection' a correct answer to S3 was given by Anna: "They [and] are each other's reflections in the line ." Another student, Peter, gave through the mind map in S1 an explanation which is partly correct: "Reflection of a function in a line gives us an inverse function." Like several students Peter omitted the line in which the reflection takes place. Helena wrote correctly: "A kind of reflection of the function. Sometimes the reflection is not a function and then there is no inverse. [...] The reflection is performed in " But Helena also wrote: "A function can also be reflected in other ways, e.g. in or in the x-axis." That is of course true if you just talk about the graph, but it is irrelevant for inverse functions. Helena was the only student mentioning reflections in lines other than

The following excerpt from Anna, exposes a component of concept image categorised as 'example': " and is an example of inverse functions." Anna also drew a graph of the two functions and the line . Furthermore, Anna suggested that has no inverse. Other students also mentioned these functions and some students commented that if we restrict to , then the inverse exists. Answers which can be seen as exposing a component of concept image categorised as 'swap x and y' were given by e.g. Anna and Bob, who wrote more or less the same phrase "change places for x and y".

Type of reflection	In (0, 0)	In x-axis	In y-axis	In $y = x$	In $y = -x$	In $y = 2$
Total	22	13	3	3	1	3

Table 3: Answers to I(b)

Conception	Reflection	Example	Reverse	Graphical features	Swap x and y	Opposite	Definition	Other
Total	16	10	6	5	4	2	1	3

Table 4: Conceptions emerging in response to S1 and S3

Helena gave a more detailed version: "To get the inverse to , we switch x and y ."

The students often gave answers with several parts and each part was connected to one of the categories. To give an idea of a complete answer we present the answer from Frida, who gave a comprehensive explanation:

> The property for can be transformed into , e.g. ; . To 'raise' has the inverse process 'cube root'. is a reflection of in the line . when (inverse processes). There is a mapping of x such that the obtained value gives the original value, ; .

The analysis of task S2 (10 answers) was done with the same procedure as the other tasks. The categories of components of concept image evoked were: 1–1 (6 answers), continuous (3), reverse (2) and graphical features (1). For example, Dora stated "the function needs to be able to run backwards" (categorised as 'reverse') and continued with the following description, which was categorised as '1–1':

> A 'regular' function needs only to satisfy the requirement that each x is mapped to one y. But several x can have images at the same y. To have an inverse function it must additionally meet the requirement that each y can be traced back to only one initial x.
>
> Like most students Dora did not use any mathematical word for injectivity but our interpretation is that she understood that the function must be 1–1 and had grasped the difference between the definition of a function and of an injective function.

DISCUSSION

The data presented here come from studies in two different educational systems and as such are quite rich. The tasks given in the two studies touched on the same content but differed in several ways. In spite of that, the answers revealed similar components of the concept images evoked; in particular, the notions of reflection, reversal and injectivity were found to be important. We saw that students' concept images contain algebraic, geometric, as well as more formal components. However, very few students in either study gave a comprehensive explanation (similar to Frida's above) or attempted a formal definition of an inverse function (such as Cathy as seen previously) in response to the tasks assigned.

The intuitive conception about inverse functions as reflections noted by Attorps and colleagues (2013) emerged also in this study: 33 of 58 Irish students who stated if the claim in I(a) is true or not mentioned reflection or symmetry, and 16 instances of reflection were observed in the Swedish students' answers for task S1 and S3. In keeping with Attorps and colleagues' findings, many of the Irish participants also failed to correctly describe the reflection with only 9 students identifying it as being in the line $y = x$. It could be argued that the nature of the Irish question and, in particular, the graphs shown, prompted a reflection conception of inverses. However, it is surprising then that many students inaccurately commented on reflection in the x- or y-axis. The Swedish students showed a clearer understanding on this point and the students who mentioned a line of reflection correctly gave the line $y = x$.

None of the Irish students mentioned the necessity of 1–1 or onto properties of a function when explaining what an inverse is. But in answering task I(b), 10 of the 11 students who answered correctly were able to identify the lack of injectivity of the function (articulated as 'not 1–1' or 'fails the horizontal line test') as a reason for an inverse not to exist. However, only 2 of these 11 students had given correct and complete answers to I(a), while 6 of the students gave answers to I(a) that contained errors or were incomplete and 3 failed to give any explanation. Thus, it does not seem reasonable to argue that the students who emphasised the 1–1 property in relation to inverses had a more robust concept image or a showed a greater conceptual understanding of inverses as Bayazit and Gray (2004) may have assumed. For the Swedish data, when asked specifically what is required for a function to have an inverse, 6 students explained the necessity of a function being 1–1. We found evidence that at least Dora, despite not using any mathematical words for injectivity, understood that the function must be 1–1 and had grasped the difference between the definition of a function and of an injective function. From our experience it seems that many students find this distinction difficult; it may be that study of the inverse function concept could be used to reinforce students' understanding of function itself.

We did not find evidence in the Irish data to support Even's (1992) claim that a naïve conception of inverse functions as 'undoing' may result in incorrect conclusions, such as that all functions have inverses. Only one student revealed this particular misconception and that student did not mention reverse processes at all. On the contrary, four of the students, whose conception of an inverse was as a reflection, believed that the ability to reflect the graph of a function in a certain way confirmed it was invertible. However, Dora who used the words "be able to run backwards" gave additional requirements for a function to have an inverse as mentioned above.

The studies have, despite different tasks given to the students, evoked similar components of students' concept images. The students in both studies were in the beginning of studying mathematics at university level. Breidenbach and colleagues (1992, p. 251) remark that progress in cognitive transitions "is rarely in a single direction", thus it is not surprising that the concept images that emerged are complicated, with many overlaps between categories and variations within categories. Indeed, Bingolbali and Monaghan (2008) found that students' concept images of derivatives evolved over the course of a semester and were influenced by both the lecturer and the students' area of primary study. Learning more about students' progress in developing their concept images could inform our teaching preparation and help us to provide greater opportunities for students to gain valuable insights into the concepts of function and inverse function. At the very least, being aware of the concept images that students may hold and their likely consequences could inform teaching; for example lecturers could refer to the definition at different stages during the course as a means to develop intuition (Wawro et al., 2011), and engineer cognitive conflicts in order to give students opportunities to refine their concept images and deepen their understanding.

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Trustworthiness of information about students' competencies in fundamental concepts in calculus provided by written examination

Kristina Raen

University of Agder, Kristiansand, Norway, kristina.raen@uia.no

The research reported in this paper aims to explore how students' competencies in calculus are exposed when being assessed. Several different competencies are required to achieve proficiency in mathematics. However, one problem is when the main focus of the student lies within procedural fluency because this is what pays off in the final examination. In this paper, written examination and task-based interview are used to find information about one student's competence in using formulae, fluency in written procedures, and strategies used for solving tasks. It is argued that the information about the student's competencies seen through a task-based interview is different from information gained after analysing the written final examination.

Keywords: Assessment, mathematical proficiency, higher education.

INTRODUCTION

The purpose of this paper is to provide a report from an on-going study that explores how students' competencies in calculus are exposed in a setting of assessment. The study is set in the context of a mathematics course taught as part of the first year of an engineering degree programme. Advanced engineering students are required to have knowledge of basic calculus in order to focus on their subject while using mathematics as a tool. Students in higher education tend to learn what is needed to get a good grade in the final examination, what the teacher focuses on in the lectures has less impact (Entwistle & Entwistle, 1992). Because of this, what is being measured in the examination should be of great concern, and effort should be taken in striving to measure the competence that students are required to develop.

At the University of Agder, one calculus course is compulsory for all engineering students in their first semester. This course is offered each autumn semester and has a large number of students, with around 400 to 500 students enrolled every year. This course comprises six hours of lectures and two hours with task based sessions in smaller groups each week. The assessment in this course consists of four compulsory assignments, voluntary home-work-tests every week, and the final examination. All external resources except communication are allowed because this is what is closest to what the students meet in real life, so this is an open book examination. Three of the compulsory assignments have to be approved before the students can take the final examination. The compulsory assignments and the home-work-tests are conducted using a software package called MyMathLab Global (MyMathLab, 2014) which comes with the textbook (Adams & Essex, 2010) used in the course. The students claims that they like to use MyMathlab because they get immediate feedback whether their answer was correct or not (Brekke & Hogstad, 2010).

This paper reports on the learning of one student, here named John. I explore his competencies as exposed through a task based interview and the final examination. The research question guiding my research is: What information about students' competencies in calculus is available through assessment?

REVIEW OF LITERATURE

Kilpatrick, Swafford and Findell (2001) argue that there are five strands of competence that are needed to achieve mathematical proficiency. They are interwoven and interdependent, and these skills should be a goal for the teacher when teaching in mathematics. The strands are:

conceptual understanding – comprehension of mathematical concepts, operations, and relations

procedural fluency – skill in carrying out procedures flexibly, accurately, efficiently, and appropriately

strategic competence – ability to formulate, represent, and solve mathematical problems

adaptive reasoning – capacity for logical thought, reflection, explanation, and justification

productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (Kilpatrick et al., 2001, p. 116)

The authors focus on children from kindergarten to eighth grade, however, I find these strands useful when it comes to being proficient in mathematics in higher education as well.

For Kilpatrick and colleagues (2001), every strand is an important competence that is necessary in achieving proficiency in mathematics. However, one problem is when the main focus of a student lies within procedural fluency because this is what pays off in the final examination. "This suggests a tension in the student's mind between learning the subject and passing the examination, with rather different strategies being involved for each" (Entwistle & Entwistle, 1992, p. 2). Brown, Bull and Pendlebury (2013) observe that the assessment defines what students regard as important. The response to this has to be that the examination measures what the student needs to know to be proficient in mathematics.

I adopt the definition of assessment used by Niss (1992): "Assessment in mathematics education is taken to concern the judging of the mathematical capability, performance, and achievement…" (p. 3). Moore and Carlson (2012) used what they called structured task-based interviews to gain insight into undergraduate pre-calculus students' difficulties experienced while attempting to solve novel applied problems in functions and graphs. In my PhD-project I explored the mathematical capability, performance and achievement of twelve students through such an interview.

METHODOLOGY AND ANALYSIS

Research design and methodology

In 2013, about 400 students participated in the calculus course for engineering students at the University of Agder. Their final examination followed a traditional form of open book examination, and was carried out at the end of the semester in December. John, who is the informant for this paper, is part of a larger study of twelve students that volunteered for the research after receiving an e-mail from the lecturer. The students were interviewed in January, as soon as possible after the examination so that their competence about these subjects would be as close as possible to their competence at the time of the examination. I did not interview them before the examination to avoid affecting their performance in the examination. Even though the students did not prepare for this interview the same way they did for the examination, I will argue that their skills were about the same. Since the communication went back and forth in the interview, I doubt quite strongly that the students knew things on the examination that he or she did not remember after some discussion in the interview. During the interview, I asked for their approval to gain access to their examination papers. The aim of this study is to make a judgment about the competence shown through the formal examination and compare this with the competence I can expose during a structured, task-based interview.

In order to explore the student's competence in calculus I used interviews which followed the methodology described by Goldin's principles of structured, task-based interviews (2000). The students were asked to think aloud when solving a mathematical task, in addition to writing. When they came no further, guiding questions were asked that would lead them into the path they needed to solve it.

Analysis

The interviews were audio recorded, and in the analysing process I have the access to the paper on which the students wrote during the interview and the papers the students handed in at the final examination. I follow Clement's (2000) methodology of analysing clinical interviews and my study is a Grounded Model Construction study. This means I have an interpretive analysis and that there are no predetermined categories into which the data will be coded.

At the end of the analysing process, the focus was about finding the mathematical competence of the student. This was also the case when analysing the written examination. As I now have two different representations of the student's competence, the discussion will be about comparing the two.

FINDINGS

This paper sets out to demonstrate that there is information about students' mathematical competence in the examination papers. To demonstrate the trustworthiness of this information I compare it to the information gained through the interview. In this paper, I will view this by focusing on three factors. The first two are about what the student can do and the last focuses on how he does it. This is only a small part of what is seen; nevertheless, I believe that the features of these factors provide a good indication of the differences of the two ways of evaluating the student. In Table 1 below I will sum up my main findings of what competencies that are exposed through the examination and through the interview, and in the following sections, I will report using excerpts of my data and some analytical arguments for my interpretation.

In the final examination I will use two part tasks, one in which John is asked to evaluate an indefinite integral, the other to solve a definite double integral, to discuss the competence revealed. These tasks are shown in Figure 1. The examination consists of four tasks with subtasks to be solved in five hours. If the assumption is that the same amount of time should be spent on each subtask, the students should spend 40–45 minutes on these two tasks.

Competence in using formulae

In the final examination there are a lot of tasks where formulae are needed in order to solve them. In order to achieve a good result the students need to find the right formulae and they need to be able to use them. In the following examples, John uses the formulae for integration by parts and substitution.

According to what we can see in the final examination John displays knowledge as to which formula he should use where, as well as how to use it in a given task. Even though there are some calculation errors, throughout the examination he uses the correct formula and he replaces the variables in the formulae with the correct terms from the task. In the first task John makes clear which formula he uses as it is named; "integration by parts" and given in the form it is shown in the formula sheet with u and v. He does not show directly what he chooses for u and v, but by reading the first line after the framed formula it is evident that he does it correctly. There is not a framed formula in the second task, but here it is evident that he substitutes xy with u and then follows the rules for integration by substitution. This is done correctly, and he integrates and replaces u with xy again before inserting the limits. Even though the calculation is incorrect, these examples show that he finds the correct formulae and uses them correctly. From these tasks it is not possible to infer anything about whether he knows the formulae by heart, that he understands what they mean or how he works when choosing a formula.

In the interview I asked John to solve two tasks similar to the two discussed above; $\int(4x + 3)e^x \, dx$ and $\int_0^\pi \int_0^y 4y^2 \cos x \, dxdy$, similar to $\int(x^2 - 5x)e^x \, dx$ and $\int_0^1 \int_0^{y^2} 3y^3 e^{xy} \, dxdy$, only one step simpler in both. At this first task he starts by saying there is a formula for

	Examination	Interview
Remembering, finding and using formulae	John can choose the correct formulae and use these when calculating, it is not possible to determine whether he knows the formulae or the rationale for choosing them.	John does not have immediate recall of the formulae, however he does know how to use them in the tasks he solves.
Fluency in written procedures	John is not fluent in procedural skills	John is not fluent in procedural skills
Showing strategies	It is not possible to know how John chooses the strategies he uses to solve a task	Some information is available about how John chooses the strategies that he wants to use when solving the task.

Table 1

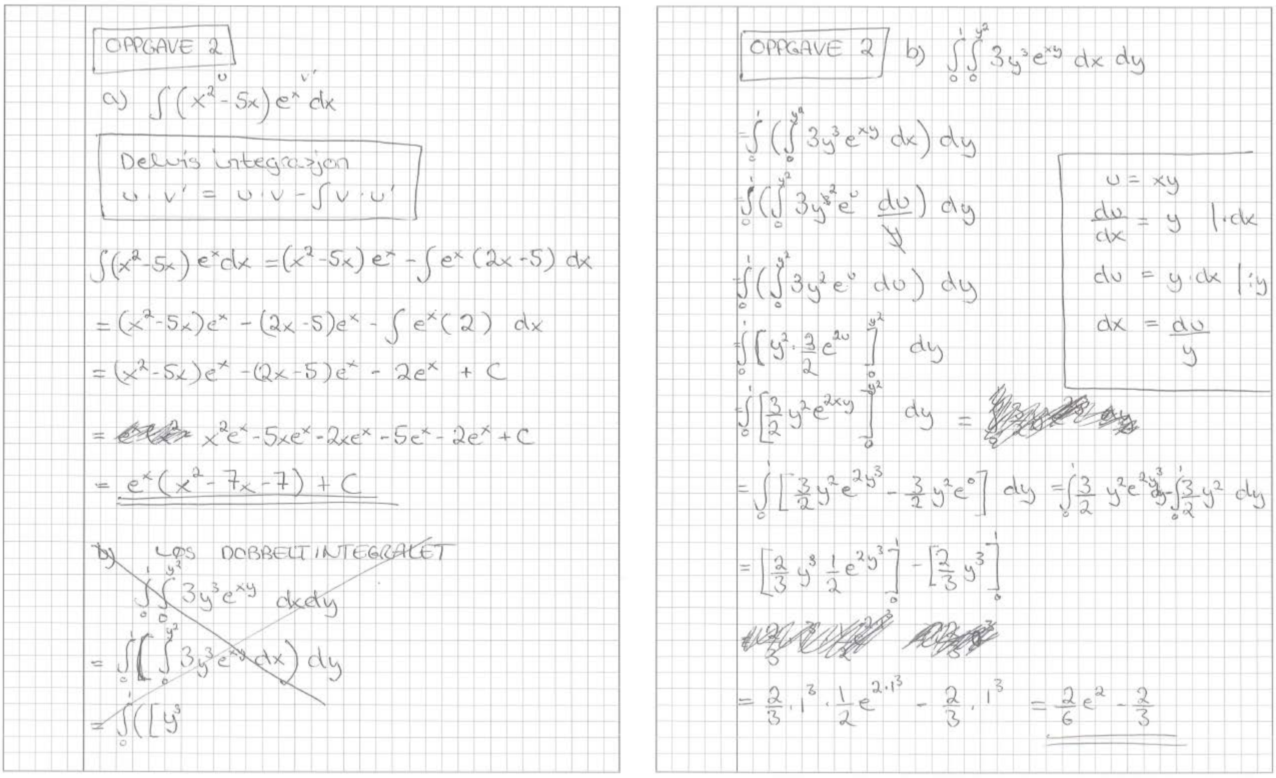

Figure 1

multiplication and division in integration and that he would attempt to use this. Below are two excerpts of the transcript from the interview where "I" is the interviewer (author) and "J" is the student. I ask how that formula looks, and his answer suggests that he is quite dependent upon the formula sheet, however, it seems that this formula is quite well known as it is almost correct.

> J: Yes, how does it look, it says in the formula sheet ... But I guess it is something like u dash v minus integral of v dash u, I think, without being able to remember it right now, I have not put too much effort into remembering any formulae, because everything is written somewhere.

This formula is almost identical to the correct formula, and I write that down. This was the formula he used himself in the examination. About picking u and v he says:

> J: And then the crucial thing is to pick more or less appropriate what should be u and v here. Because e to the power of x is the same whether you differentiate or integrate it, I would pick u as this, four x plus three, and this as v (points at e to the power of x)...

The first excerpt of the transcript is an example that shows that he does not know all the formulae by heart. Still, the fact that he says that he did not put too much effort into remembering the formula suggests that he trusts his competence to find the right one in his formula sheet when needing it, and use it correctly. In this interview I did not have the formula sheet available so I do not know how John works when finding a formula and thus how well he does it. The second excerpt shows how he substituted the letters in the formulae by the terms in the task and how he wanted to choose which factor fitted where, so by this I conclude that he knows how to use the formulae in tasks.

Procedural fluency

Procedural fluency is used as defined in the report from Mathematics Learning Study Comity, edited by Kilpatrick, Swafford and Findell; "Procedural fluency refers to knowledge of procedures, knowledge of when and how to use them appropriately, and skill in performing them flexibly, accurately, and efficiently" (2001, p. 121). To investigate procedural fluency showed in the examination, I will regard the same tasks as above. See Figure 1 for John's answers. In the first task there are two tables of errors and his answer is slightly different from what it should be, however, other parts of the examination shows correct sign after dissolving the parentheses.

The second task shows the solution of a double integral. John tries to do the inner integral by substitution even if he did not need to. Although the integration is done correctly in the first task it is not done correctly now. The limits are correctly inserted but the outer integral is incorrect. Table 2 shows excerpts of the two tasks, what is correct and what John wrote.

This indicates that John has some fluency albeit at a fairly low level when it comes to integrating the exponential function and polynomials. Even though the

Task	Correct	John
$\int e^x(2)\,dx$	$2\,e^x$	$2\,e^x$
$\int 3y^2e^u\,du$	$3y^2e^u$	$y^2 \cdot \frac{3}{2}e^{2u}$
$\int \frac{3}{2}y^2e^{2y^3}\,dy$	$\frac{1}{4}e^{2y^3}$	$\frac{3}{2}y^3\frac{1}{2}y^{2y^3}$
$\int \frac{3}{2}y^2\,dy$	y^3	$\frac{2}{3}y^2$

Table 2

two upper tasks are the same, the latter looks more complicated and it is therefore not surprising that the mistake happens in this task.

John also displays lack of fluency with respect to integration of exponential functions in the interview. Somewhere in the process of solving the task $\int(4x + 3)e^x\,dx$ he must solve $\int 4e^x\,dx$, and this gives him trouble. He says that $\int 4e^x\,dx = 4xe^x + C$:

> J: And then we can put the number four outside I think, and... no, we cannot... e to the power of becomes the same no matter how you twist and turn it, yes, it becomes four *x* e to the power of *x* then, if you integrate this, four *x* plus three minus four *x* e to the power of *x* plus C is what you get...

Here is evidence that he is unsure of the path he is allowed to take because he has to consider what to do about the number in front of e^x. After a couple of pauses he draws the wrong conclusion and ends up with a wrong answer. Nevertheless, he shows some degree of fluency because he considers the correct approach immediately, and when he realises that what he got was incorrect after a bit of prompting, the correct calculation was the only alternative left for him.

> I What would you do to check whether this is correct or not?
>
> J: Eh... differentiate it again, could you do... this is maybe not exactly right
>
> (...)
>
> J: Yes, because you only put the number four outside then?

John choosing strategies

When calculating a mathematical task, there is a need for choosing a strategy. In the final examination the students are asked to write all the calculations, and this is what provides the information as to which strategy the student might have chosen. If the sheet that the student hands in contains a crossed out section, this could indicate uncertainty in solving the task and if it is possible to read what was written before being crossed out, it might indicate which strategy was considered for solving the current task. Therefore, at best, the information available about which strategy has been used and possibly which have been considered lies in the paper, but there is nothing that indicates what influenced the student to choose that strategy or the thinking process.

In the interview the tasks were intended to facilitate John's explanation of what he normally did. In addition, my request to think aloud resulted in a certain insight into John's procedures when starting to solve a task. I have already argued that the student's competence is fairly close in the setting of the examination and the setting of the interview, and I will further argue that it is sufficiently close to say that more insight into the students' proficiency in one of the settings will tell me what is not being communicated through the other. Now returning to the task about the unlimited integral, the information made clear from John's strategy choice is from what he communicates to me when attempting to explain his thought process. He starts by saying that he wants to find a formula that suits the task.

> J: ... I would use some kind of formula for... you have this multiplication formula (...) I would try to use that... and see where I ended up

From the sections about competence in using formulae, the transcript excerpt indicates that he wants to look in the formula sheet to see if he can find a suitable formula. After examining the formula, he then proceeds to pick the appropriate *u* and *v* to fill into the formula. When looking at the transcript excerpts from the section of procedural fluency it is for instance evident that he is uncertain about which integration strategy to choose. The two alternatives he names are rules of integration which also could be located in the formula sheet. He is worried about what he is allowed to do and appears to focus on recalling it because he is not stating a reasoned cause for choosing one of them, nor does he spend time thinking it through. After realising his mistake, he asks whether the other alternative was the correct one and is satisfied with a yes or no. The information about how John chooses a formula and wants to "see where I ended up", and the fact that he expresses his uncertainty when using an integration rule yet still not explaining how he chose which rule to go for prompts my assertion that this interviewing method provides some information as to how the student chooses solving strategies.

DISCUSSION

When discussing the information about students' competence in calculus I will compare the findings from the interview and the examination. The strands needed for proficiency in mathematics are conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive disposition (Kilpatrick et al., 2001). When taking into consideration that procedural fluency can be "in addition to written procedures, mental methods for finding certain sums, differences, products, or quotients, as well as methods that use calculators, computers, or manipulative materials such as blocks, counters, or beads" (Kilpatrick et al., 2001, p. 121), I claim that finding the correct formula and using it correctly comes into this category. The final examination shows that John has procedural fluency in formulae within this mathematical concept and I see evidence of this being the case also from the interview. He does explain to me where he would look to find the formula and he shows me that he knows how to substitute the variables in the formulae by the terms in the expression. However, this is all that is being tested in this examination. In the setting of the interview it is also revealed that John has not committed all the formulae to memory. He admits that this is not something he prioritises, which is understandable since the examination is an open book examination. There is also information about which strategies he is using when working with formulae, which I intend to review below. The questions could have been changed to gain more information around this in both ways of evaluating the student but the information provided is based on the questions posed at the time.

I concluded that there was some, albeit fairly low, written procedural fluency. My experience was that approximately the same information about the student's overall fluency was gained from the two methods of evaluating the student. The crucial step for both the methods is to provide sufficient tasks in order to establish what the students *can* do, what they *cannot* do, and what they *sometimes can* do. Yet, there is a small advantage for the interviewing method in information about what parts of the calculation are the most difficult. During the interview I could see at what steps John used the longest time, and statements he made gave information about what parts he is confident of. One problem is however presented. In the interview it seems that he is quite certain about the integration of e^x. It is the part where this is multiplied with a constant that causes trouble. In the examination it seems that he does not recall what he later states in the interview and he mixes the rule for integrating e^x with the rule for integrating x^a, at least once. I see two possible reasons for this. The first is that John actually learned this after the examination. My other possible reason is that a researcher always has to bear in mind that a student can have a "bad day" when testing him only once and that this could be a subject for the consideration of data reliability.

The last aspect explored in this paper concerns the use of strategies. From Kilpatrick and colleagues' mathematical proficiency (Kilpatrick et al., 2001) I argue that this goes into strategic competence, and then mainly about representing a problem. That part is about representing the task accurately and mathematically, and in order to do so the student should have some degree of understanding of the problem. However, I also argue that a part of this belongs to the part of procedural fluency which is about methods for finding certain sums, differences, and so on, as well as methods that use calculators, computers, or manipulative materials. This applies for instance when the student searches a formula or wishes to use a formula, expressing that he has to see where it takes him. There is very little information about this in the examination. Sometimes we see which strategy John did choose, but quite often the only way to know which strategies a student chooses is to rely on common mistakes and misconceptions revealed in previous research and compare this with what the student answered. In the interview, when John was thinking aloud, it was quite often obvious which strategy he was using, which he was considering and which he was confident about. Still, if I wanted to know why he was choosing a strategy or what else he considered I would have to ask more questions.

CONCLUSION

By evaluating students in different ways, there is different information to extract about students' competencies in fundamental concepts in calculus. In order to gain mathematical proficiency there is a need for the five competence strands; conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition. Of the few issues addressed in this paper, the traditional open book examination that this course uses is quite good in testing written procedural fluency of the concepts on which there is preferably more than one task of different difficulty. The interviewing method which was used provided information as to how John's confidence was displayed in the different written procedures, how fluent he was in procedural methods, and which strategies he used when solving tasks. It turned out to be the discussions, after a student got stuck on a task that really revealed a lot about these last factors.

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Extreme Apprenticeship – Emphasising conceptual understanding in undergraduate mathematics

Johanna Rämö, Lotta Oinonen and Thomas Vikberg

University of Helsinki, Helsinki, Finland, johanna.ramo@helsinki.fi

Extreme Apprenticeship (XA) is an educational method that has been used in teaching undergraduate mathematics in the University of Helsinki. In this paper, we analyse the course assignments and exam questions of a certain lecture course that has recently been reformed to an XA-based course. The results show that the XA method has made it possible to move the emphasis from rote learning towards understanding the concepts behind the procedures.

Keywords: University mathematics education, Extreme Apprenticeship, conceptual knowledge, tasks, large classes.

INTRODUCTION

The pedagogical decisions in introductory mathematics courses at university-level are of great importance. On these courses the students form their first perceptions of what university mathematics is about, and most importantly, what studying mathematics in university will be like. Traditionally, the first year courses have concentrated on memorising procedures and algorithms. The procedure-centred approach can lead to problems since procedures that lack connections with conceptual knowledge may deteriorate quickly and do not transfer easily to new situations (Hiebert & Lefevre, 1986).

Over the past years many student-centred approaches have been used in mathematics teaching for facilitating conceptual understanding (Abdulwahed, Jaworski, & Crawford, 2012). These approaches include Inquiry-based learning (IBL) and Problem-based learning (PBL), in which learning revolves around real life based problems and questions that require critical thinking (Chang, 2011; Mokhtar, Tarmizi, Ayub, & Tarmizi, 2010; Retsek, 2013). Both IBL and PBL emphasise collaborative work, presentation of conclusions and development of learning skills. Another innovative and widely used approach in mathematics teaching is Flipped classroom or Inverted classroom (Jungić, Kaur, Mulholland, & Xin, 2014; Talbert, 2014). In this approach students familiarize themselves with the new information through online video resources outside of class, and class time is reserved for discussions and cooperative problem solving. Peer instruction (Lucas, 2009) and Just-in-Time Teaching (Natarajan & Bennett, 2014) are further examples of interactive teaching methods that allow teacher to adjust teaching to the needs of students. These can be used separately or with the Flipped classroom approach.

A new student-centred method for teaching large introductory courses, Extreme Apprenticeship (XA), has been adopted in the Department of Mathematics and Statistics in the University of Helsinki (Hautala, Romu, Rämö, & Vikberg, 2012; Vihavainen, Paksula, & Luukkainen, 2011). The XA method has similarities with the approaches mentioned above, as it promotes active engagement of the students and preparation prior to the class. However, the main idea of the XA method is to support students in becoming experts in their field by making them participate in meaningful activities, which resemble those carried out by professionals. This means there are neither video lectures nor screencasts. Instead, students read the course material with the help of the teaching assistants. Another difference between XA and the other novel approaches is that in XA the main method of teaching is personal instruction: students have to do a lot of work outside of the classroom, but they are offered guidance in a drop-in basis several hours a day.

Previously it has been shown that students find the XA method satisfactory, and the passing rates do not drop even though the workload is significantly increased and the requirement level raised (Hautala et al., 2012). Also, the XA method has increased student engagement and effort (Rämö & Vikberg, 2014).

In this paper, we study whether the XA method has shifted the emphasis towards tasks that promote the development of conceptual knowledge and the linking of procedural and conceptual knowledge. This is done by comparing the assignments and exams used on an XA-based course to those used traditionally. The course under study is the first year course "Linear algebra and matrices I". While this paper focuses on the linear algebra course, the XA method has been used on many undergraduate mathematics courses, including algebra, logic and probability, and our experience leads us to believe that the conclusions drawn in this paper apply more generally than just in linear algebra.

We use the definitions of conceptual and procedural knowledge by Hiebert and Lefevre (1986): Conceptual knowledge is characterized as knowledge that is rich in relationships of many kinds, whereas procedural knowledge is made up of two distinct parts: the formal language of mathematics and the algorithms for completing mathematical tasks.

EXTREME APPRENTICESHIP

The Extreme Apprenticeship (XA) method is an educational model for organising instruction in an effective and scalable manner. Its theoretical background is in situated view on learning and Cognitive Apprenticeship (Brown, Collins, & Duguid, 1989). The method was originally invented as an instrument for teaching university-level computer programming (Vihavainen et al., 2011), and later employed in mathematics courses (Rämö & Vikberg, 2014).

In XA, the amount of tasks is substantially larger and the number of lectures smaller than traditionally. Students learn skills and gain knowledge by doing tasks that offer them small and approachable goals (Vihavainen et al., 2011). Passive activities, such as sitting in the lectures, are reduced to the minimum and active work done by the students is emphasised.

The main method of teaching in XA is personal instruction, which is based on the concepts of coaching and instructional scaffolding in Cognitive Apprenticeship (Collins, Brown, & Holum, 1991; Lave & Wenger, 1991). Instructional scaffolding refers to temporary support given to students (Wood, Bruner, & Ross, 1976), and is interlinked with the concept of zone of proximal development introduced by Vygotsky (1978). Coaching refers to the broader perspective of regulating pace and difficulty of assignments in the course. A necessity for coaching students' progress is bi-directional feedback between the students and the teaching team (Kurhila & Vihavainen, 2011). The teaching team receives feedback on the progress of the students by evaluating their solutions during the course, and also from the conversations with the students during personal instruction. Students receive feedback on how they are performing, but also encouragement and support in completing the assignments.

EDUCATIONAL SETTING

University studies in Finland resulting in a Master's degree are intended to last five years with three years of Bachelor's studies and two years of Master's studies. There are no tuition fees. The students are selected by their performance in the upper secondary school matriculation examination, an entrance exam, or both.

A traditional lecture course

On a traditional lecture course, there are 4–5 hours of lectures per week, in which the lecturer covers all the theory of the course. Every week the students are given problem sheets consisting of 6–7 tasks they have to solve. The solutions to the tasks are discussed under the guidance of a teaching assistant in a group session that lasts for two hours. In each group there are approximately 20–30 students, who usually take turns in explaining their solutions to the problems on a blackboard.

Extreme Apprenticeship based course

On an XA-based course, the amount of tasks is substantially larger than traditionally, approximately 15–20 problems per week. There are relatively easy problems on new topics, but also more challenging tasks regarding more familiar concepts studied in the previous weeks. The tasks are designed to support the development of conceptual knowledge, and to aid the students in building relationships between procedural and conceptual knowledge.

Students are offered guidance by the teaching team to complete the assignments in drop-in sessions, approximately 20 hours per week. This one-on-one or small group instruction forms the main part of teaching. The purpose of the instruction is to lead the student subtly towards the discovery of a solution through a process of questioning and listening, instead of simply giving away the answers.

One or two of the tasks are selected for inspection each week. Students receive written feedback on their reasoning and also on the readability and language of the solution, and they are encouraged to improve their solutions when necessary.

New kinds of learning spaces have been created to encourage student collaboration. The main corridor of the department has become a huge drop-in class where the tables are arranged into groups and act as whiteboards, and the walls are covered with blackboards for the students to share their thoughts with each other and with the instructors.

The amount of lectures is significantly smaller than on traditional lecture courses, only 2–3 hours per week. As the assignments force the students to investigate the topics by reading the course material prior to the lectures, it is not necessary to deliver content or go through details in the lectures. Instead, it is possible for example to discuss the meaning and consequences of definitions and to address student misconceptions through various small group activities.

Linear algebra and matrices I

The course investigated in this study is Linear algebra and matrices I. It is a first year course, and for most students it is the first mathematics course they take in the university. Approximately half of the students on the course have mathematics as a major, and the rest study mathematics as a minor subject. Among these students the most common majors are computer science, physics and economy. The amount of students taking the course has increased over the last few years. In 2008, there were 394 students enrolled for the course, whereas in 2013 the number was 484.

The content of the course has varied slightly from year to year, but the main topics have remained the same. They are systems of linear equations, matrices, spanning sets, linear independence, basis and coordinates. The course lasts for 6 weeks. The workload of the course is approximately one third of the total workload of the students.

In this paper, we investigate the years 2008–2013. Two different lecturers taught the course as a traditional lecture course during the years 2008–2010, one of them in 2008 and 2010 and the other in 2009. In 2011, the course was transformed to an XA-based course (Hautala et al., 2012). Improving the implementation of the XA method continued in 2012–2013 (Rämö & Vikberg, 2014). The teacher responsible for the XA-based courses was the first author of this paper.

METHOD

The aim of this study is to compare the assignments and exams used on an XA-based course to those used traditionally. This was done by classifying the tasks using the classification scheme by Pointon and Sangwin (2003), which was modified slightly to fit the purposes of this study.

The classification of Pointon and Sangwin consists of 8 categories shown in Table 1 (categories 1–8). When the tasks of this study were analysed, it became clear that one category, namely category 9, had to be added.

The classification was executed as in the paper of Pointon and Sangwin: Each question was evaluated individually, and given equal value. If a question had multiple parts, it was classified by estimating the proportion of each category. The evaluation was done by the second author.

1. Factual recall
2. Carry out a routine calculation or algorithm
3. Classify some mathematical object
4. Interpret situation or answer
5. Proof, show, justify (general argument)
6. Extend a concept
7. Construct example/instance
8. Criticize a fallacy
9. Information transfer

Table 1: The task classification scheme

The categories are described briefly here, and more detailed descriptions with examples can be found in the paper by Pointon and Sangwin (2003).

1) Factual recall: A question that requires only the recall of some factual knowledge, usually verbatim.

2) Carry out a routine calculation or algorithm: A question that requires routine use of algebra, calculus or matrix operations. Often such tasks may be performed by a computer algebra system.

3) Classify some mathematical object: Solving the task requires recalling a definition and providing justification to show that some specific object satisfies the definition.

4) Interpret situation or answer: The task requires modelling of a physical situation or interpretation of a mathematical model.

5) Proof, show, justify (general argument): A question that requires a general argument involving abstract or general objects rather than specific examples.

6) Extend a concept: Students are asked to evaluate previously acquired knowledge in a new situation.

7) Construct example/instance: Students are required to provide an object satisfying certain mathematical properties.

8) Criticize a fallacy: Students are asked to find mistakes in supposed proofs, or criticize reasoning.

9) Information transfer: A question that requires transformation of information from one form to another, as well as processing this information. This category was added by the authors of this paper. It is explained below in detail.

Category 9: Information transfer

Category 9 did not occur in the original classification scheme of Pointon and Sangwin that consists of categories 1–8. A category bearing the same name, information transfer, can be found in the MATH Taxonomy proposed by Smith and colleagues (1996). Our category resembles theirs, but is only a subset of it. Introducing a new category was necessary, as many of the tasks did not fit in any of the categories 1–8. Questions in category 9 require transformation of information from one form to another, as well as processing this information. Typically, in these questions students are asked to draw pictures, interpret diagrams, explain something in their own words or draw concept maps.

Examples of category 9:

— Denote $v_1 = (-3, 4)$, $v_2 = (1, 1)$ and $v_3 = (\frac{2}{3}, -2)$. Draw pictures of the subspaces $span(v_1)$, $span(v_1, v_2)$ and $span(v_1, v_3)$. You do not need to justify your answer.

— The vector space $\mathbb{R}^2$ has a basis $B = ((1, 1), (2, 3))$. Determine, by drawing a picture, a vector $u \in \mathbb{R}^2$ whose coordinates with respect B to are 3 and −2.

— Explain in your own words why an elementary matrix always has an inverse matrix.

RESULTS

Course assignments

The weekly course assignments were evaluated using the classification described in the previous section. Table 2 shows that in category 2 (routine calculation), the proportion of tasks has decreased. In 2008–2010, when the course was a traditional lecture course, 36–46% of the assignments were from category 2. In 2011, when the XA method was introduced, the proportion was still high (43%), but it dropped to 28% in 2013.

In category 9 (information transfer), the proportion of tasks has risen. In 2008–2010, 1–5% of the assignments were from this category, whereas in 2011–2013, the proportion was 15–18%.

Exam tasks

Also the exam tasks were analysed in order to find out how much weight each of the categories had in the final exam. The tasks were divided into the nine categories, and the maximum score of the tasks in each category was calculated. Table 3 shows the proportions of maximum scores in each category.

It can be seen that the weight of category 2 (routine calculation) decreased when the XA method was introduced: in 2008–2010, the proportion of points that could be obtained from category 2 tasks was 42–54%, whereas in 2011–2013 when the XA method was used, the corresponding percentage was 0–25%.

The weight of category 5 (proof) has not decreased when using the XA method. A new category (7, construct example) has appeared with the introduction of XA.

Categories, %	1. Factual recall	2. Routine calculation	3. Classify object	4. Interpret	5. Proof	6. Extend a concept	7. Construct an example	8. Criticize a fallacy	9. Information transfer	No. of tasks
2008		41	22	4	15	3	12		3	30
2009		46	13	8	28		3		1	30
2010		36	22	8	15	3	12		5	30
2011	2	43	14	2	13	2	7		18	133
2012		29	21	4	21		10		15	89
2013		28	27	7	11		10		18	89

Table 2: Proportions of course assignments in different categories. In 2011 the XA method was introduced

Categories, %	1. Factual recall	2. Routine calculation	3. Classify object	4. Interpret	5. Proof	6. Extend a concept	7. Construct an example	8. Criticize a fallacy	9. Information transfer	No. of exam points
2008	8	42	25		25					24
2009	8	54		13	25					24
2010	17	42	42							24
2011	8	25	25		25		17			48
2012	6	8	33	25	27					48
2013	13		50		25		13			48

Table 3: Proportions of exams points in each category. The points are divided into nine categories according to the category of the task they are awarded for. In 2011 the XA method was introduced

	2008	2009	2010	2011	2012	2013
Number of students	307	262	280	324	345	383
Mean (%)	69	63	72	69	75	65
Standard deviation (%)	24	19	23	19	19	21
Lower quartile (%)	50	54	58	58	65	52
Median (%)	71	67	75	73	79	67
Upper quartile (%)	92	75	92	81	90	81

Table 4: The number of students taking the exam and statistical parameters of the exam scores

Exam performance

Information on the performance of the students was obtained by studying the exam scores. From Table 4 it can be seen that the average scores were 63–72% in traditional teaching and 65–75% in XA. There seems to be no distinctive change in the student performance since the XA method was introduced.

CONCLUSIONS

The aim of the XA method is to educate skilled professionals. As professional mathematicians need to understand the concepts they are working with, this should be emphasised also when teaching future mathematicians, even first year students.

The results show that when the XA method was introduced, the weight of routine calculations in the course assignments decreased and the weight of category 9 (information transfer) increased. It can be concluded that there was a change of focus from rote learning of routine procedures towards tasks that require also conceptual understanding. However, there are still categories that are almost non-existent, such as "extend a concept" or "criticize a fallacy".

Also in the course exams the weight of category 2 decreased when the XA method was introduced. Instead of routine calculations, the tasks required constructing examples or interpreting situations or answers. At the same time, there was no drastic change in the performance of the students. In the light of these results, we can draw the conclusion that the students have developed conceptual understanding. However, when interpreting the exam results, one should note that the level of difficulty of the exams may have varied slightly over the years.

Conceptual understanding is not developed at the cost of routine skills: the actual amount of course assignments in category 2 has not dropped. Since the number of tasks in XA is greater than in traditional teaching, the amount of category 2 tasks has actually risen from 12 tasks in 2008 to 25 tasks in 2013 (Table 3). This means that also in the XA method the students get plenty of practice in routine calculations.

There are many features in the XA method that facilitate emphasising conceptual knowledge. The teaching revolves around the tasks, and there are plenty of them for the students to work on. Therefore, it is easier to give students a wide range of diverse tasks. Because of the one-on-one instruction, also the weaker students have a change to fully work on the problems, and they do not need to give up if a task seems too difficult for them. The lectures support the development of conceptual understanding by focusing on motivating the concepts and discussing how they are linked together. However, not all the students take advantage of the instruction: less than half of the students who submit course work speak with the teaching assistants, and many of the students do not attend the lectures. Our next goal is to find ways to encourage students to take part in the instruction.

In this study, the tasks were categorised by the second author. The reliability of the study would be improved if the tasks were given to an independent evaluator who does not know which tasks are from which year. Also, a detailed statistical analysis would give more information about the changes that have taken place.

The tasks given to the students should be versatile and varied because mathematical competence involves knowledge of both concepts and procedures, as well as understanding the relations between them (Hiebert & Lefevre, 1986). Also, when students are offered diverse problems, they learn problem-solving strategies of experts (Collins et al., 1991). In this light, our results indicate that the Extreme Apprenticeship method is a step in the right direction.

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First steps in re-inventing Euler's method: A case for coordinating methodologies

Michal Tabach[1], Chris Rasmussen[2], Rina Hershkowitz[3] and Tommy Dreyfus[1]

1 Tel Aviv University, Tel Aviv, Israel, Tabach.family@gmail.com

2 San Diego State University, San Diego, USA

3 The Weizmann Institute, Rehovot, Israel

In this report, we highlight the epistemic actions and concomitant discursive shifts of four students as they reinvent the fundamental idea and technique in Euler's method. We use this case to further the theoretical and methodological coordination of the Abstraction in Context (AiC) approach, with its associated model commonly used for the analysis of processes of constructing knowledge by individuals, and small groups and the Documenting Collective Activity (DCA) approach, with its methodology commonly used for identifying normative ways of reasoning with groups of students. In this report, we show students' first steps towards re-inventing Euler's method and explicate the theoretical and methodological commonalities of AiC and DCA.

Keywords: Documenting collective activity, abstraction in context, networking theories, Euler's method.

INTRODUCTION

Research at the undergraduate level is moving beyond the documentation of student difficulties towards the design, implementation, and analysis of innovative learning environments where students reinvent important mathematical ideas and methods. For example, in differential equations, research has documented that students are able to reinvent, given appropriate task sequences and learning environments, Euler's method, bifurcation diagrams, and even an analytic approach for solving systems of linear differential equations (e.g., Rasmussen, 2007). Such reinventions are, from our perspective, both individual and collective accomplishments. Methodological approaches for analysing such accomplishments, however, are sorely needed. In this report, we highlight the epistemic actions and concomitant argumentation of four students as they reinvent the fundamental idea and technique in Euler's method. We use this case to further the theoretical and methodological coordination of the Abstraction in Context (AiC) approach and the Documenting Collective Activity (DCA) approach (see Hershkowitz et al., 2014; Tabach et al., 2014 for initial attempts at coordinating these two approaches). The two approaches have various theoretical and methodological commonalities that we will refer to as environmental and underlying ones; the analysis in the present paper led to the discovery of additional commonalities that we will refer to as environmental and underlying internal ones. We explicate these commonalities to set the stage for the analysis of student reinvention, but first begin with a brief summary of the AiC and DCA approaches.

ABSTRACTION IN CONTEXT AND THE RBC+C MODEL

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating abstract mathematical knowledge (Hershkowitz et al., 2001). Abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner. According to AiC, the genesis of an abstraction passes through three stages (ibid): (i) the arising of the need for a new construct, (ii) the emergence of the new construct, and (iii) the consolidation of that construct. AiC includes a theoretical/methodological model, according to which the description and analysis of the emergence of a new construct and its consolidation relies on a limited number of epistemic actions: Recognizing, Building-with, and Constructing (RBC).

These epistemic actions are often observable as they are expressed by learners verbally, graphically, or otherwise. Recognizing takes place when the learner recognizes a specific previous knowledge construct as relevant to the problem currently at hand. Building-with is an action comprising the combination of recognized constructs in order to achieve a localized goal, such as the actualization of a strategy or the solution of a problem. The model suggests Constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and interweaving previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner.

Recognizing actions are nested within building-with actions, and recognizing and building-with actions are nested within constructing actions. Moreover, constructing actions are at times nested within more holistic constructing actions. Therefore the model is called the nested epistemic actions model of abstraction in context, or simply the RBC+C model. The second "C" stands for Consolidation. The consolidation of a new construct is evidenced by students' ability to progressively recognize its relevance more readily and to use it more flexibly in further activity.

DOCUMENTING COLLECTIVE ACTIVITY OVERVIEW

The methodological approach of documenting collective activity (DCA) is theoretically grounded in the emergent perspective (Cobb & Yackel, 1996), a basic premise of which is that mathematical learning is a constructive process that occurs while participating in and contributing to the collective activity of the classroom. The collective activity of a class refers to the normative ways of reasoning that develop as students work together to solve problems, explain their thinking, represent their ideas, etc. These normative ways of reasoning can be used to describe the mathematical activity of a group and may or may not be appropriate descriptions of the characteristics of each individual student in the group. A mathematical idea or way of reasoning becomes normative when there is empirical evidence that it functions in the classroom as if it is shared. The empirical approach makes use of Toulmin's model of argumentation, the core of which consists of Data, Claim, and Warrant. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant. It is not uncommon, however, for rebuttals or qualifiers to arise once a claim, data, and warrant have been presented. Backing provides further support for the core of the argument.

The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off, 2) When certain parts of an argument (the warrant, claim, data, or backing) shift position within subsequent arguments, or 3) When a particular idea is repeatedly used as either data or warrant for different claims across multiple days. See Rasmussen and Stephan (2008) for an illustration of the first two criteria.

ENVIRONMENTAL COMMONALITIES

The use of both methodologies, AiC and DCA, requires very explicit classroom norms. First, they require classrooms in which students are routinely explaining their thinking, listening to and indicating agreement or disagreement with each other's reasoning, etc. If such norms are not in place, then evidence is unlikely to be found of challenges, rebuttals, and negotiations that lead to ideas where knowledge is constructed and starts functioning as if shared by the whole class. We call such classrooms "inquiry classrooms." Second, they require the intentional use of tasks that were purposefully designed to offer students opportunities for constructing new knowledge by engaging them in problem solving and reflective activities allowing for vertical mathematiziation.

Both methodologies focus on the ways in which mathematical progress is achieved and spreads in the classroom. RBC+C focuses on individuals or small groups working in the classroom and DCA focuses on group discussions. In this sense, the two methodologies complement each other in analyzing a sequence of lessons including individual and group work and in tracing how knowledge is constructed and becomes normative along this sequence.

UNDERLYING COMMONALITIES

Other characteristics of a classroom culture in which DCA and RBC+C methodologies might be enacted together are that the tasks are designed to afford inquiry and the emergence of new constructs by vertical mathematization from previous constructs; such learning materials allows for interweaving collaborative work in both small-group work and whole-class discussions, where the teacher adopts a role that encourages inquiry in the above sense.

Another underlying characteristic relates to the centrality of the shared knowledge. AiC defined shared knowledge as "a common basis of knowledge which allows the students in the group to continue together the construction of further knowledge in the same topic" (Hershkowitz et al., 2007, p. 42). This definition relates to cognitive aspects. We find its counterpart in sociological terms, in the phrase "function as if shared" used by the DCA approach. What is common between the two constructs is the point that each operationalizes when particular ideas or ways of reasoning are, from a researcher's viewpoint, beyond justification for participants. At the collective level, ideas or ways of reasoning that function as if shared have the status of accepted mathematical truths for the group. At the individual level, consolidation results in individuals accepting something as a mathematical truth.

FIRST STEPS TO REINVENTING EULER'S METHOD

We begin with the following excerpt, used also in Stephan and Rasmussen (2002) and in Tabach and colleagues (2014) but for different purposes. It is a discussion between Liz, Joe, Deb and Jeff, four students in a class of 29 STEM, first year undergraduate students, working on the following problem during group work on the first lesson:

Consider the following rate of change equation, where P(t) is the number of rabbits at time t (in years): dP/dt = 3P(t) or in shorthand notation dP/dt = 3P. Suppose that at time t = 0 we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits at t = 0.5 and t = 1.

Prior to this task students received no instruction on Euler's method, but the class did develop graphical depictions of what the exact solution should more or less look like (e.g., not linear but increasing at an increasing rate). The excerpt includes a DCA analysis and an RBC analysis. The DCA analysis classifies the shaded parts according to Toulmin's model as data [D], claim [C], warrant [W], backing [B], or qualifier [Q]. For example, D2 is the Data used for Claim 2. We indicate at the end of a turn if one of the three criteria has been met. The RBC analysis is based on an a priory analysis of the activity that yielded the following knowledge elements intended to be constructed: Csy – establishing connection between P and dP/dt (if you know P you can find dP/dt); Cpit – population iteration (given P and dP/dt at a moment in time allows one to find P at a later time); and Crit – rate of change iteration (applying Csy at that later time one can find the corresponding dP/dt); and finally Cit: Cpit and Crit can be combined into a repeating loop. We conjecture that in previous courses students constructed dP/dt as a ratio (Crat) and hence they can recognize and build-with this construct. To keep things transparent we omit mentioning previous constructs to which our analysis does not explicitly refer. RBC actions were italicized in students' talk and coded in the third column as recognising (R), building-with (B), constructing (C) or consolidating (CC). This side-by-side analysis was done to facilitate coordination between the RBC+C and DCA methodologies. This coordination is then helpful for analysing student's re-invention of Euler's method.

1	Liz	*I would plug in the population of rabbits for P to determine the rate of change initially. What is the rate of change when time equals zero* [W1]. *So if we had a graph, its kind of like what we were just talking about, we are trying to determine the rate of change when this time is equal to zero* [B1].	R B
2	Joe	Oh ok. *This is where 10 rabbits at zero* [D1].	R
3	Liz	What do you think?	
4	Deb	Oh ok, *so I get the rate of change at time initially the rate of change would be 3* [sic] [C1]. Did I multiply it right?	R B Csy
5	Liz	And then I guess the simple ...	
6	Joe	How did you do that?	
7	Liz	*Okay, well this* [D2] [differential equation] *is the change in the population over the change in time* [C2].	Rrat
8	Joe	Right.	
9	Liz	Okay, and *this 3 I'm taking as being the constant or whatever you call the growth rate. And this P of t is the population at any given point of time t, but this is just short hand notation for it . So I*	

thought, if we know the population is ten when our time equals zero [D1 & D2 elaborated], *can we plug in the P(t) population at time zero and find out what initially the rate of change is* [W1]? B Csy

10 Joe It would be 10 = 3...

11 Liz *Times 10* Csy

12 Jeff *Okay I see so it would be 30* [C1]. Csy

13 Liz 30, I mean does that,

14 Jeff Yeah that does make sense.

15 Joe Well, *wouldn't 10 = 3P(t)?* [C3] *At time zero we have 10 rabbits* [D3]. (Note that his claim is incorrect) R B

16 Liz Well 10 is actually the population [D4] so I'm taking that that has to actually be the population at time t. *I don't think it's telling us how the population is changing which would be dP/dt* [C4]. CCsy

17 Liz *So if we have that [initial rate of change is 30]* [D5], *the question is how can we use that to help us figure out the population after a half unit elapsed?* [32 sec pause] (identifies a need to construct Cpit) Rsy

18 Jeff How would we work time into the equation?

19 Liz *If we think of it right now as our time equals zero*, we could say... B

20 Deb *We have the 30* [D5]. Rsy

21 Liz *We have the 30 to work* [D5] *with, so couldn't we say we don't* [5 second pause]Bsy

22 Deb You said *the population is 10 right* [D5]? B

23 Liz um hm.

24 Deb So three times ten would give us our rate of change [D5]. *Say 0.5 years passes, this is our rate of change. Then we'll take that 0.5 times the rate of change* [W5] *which will give us what, the new amount of rabbits plus the old amount of rabbits*. [C5] [Criterion 2 met for Csy, see turn 12 where this was C1] Cpit

25 Liz So *the old amount of rabbits is ten* [D6] R

26 Deb Am I making sense?

27 Jeff I think so, so *that would be 25* [C5], is that what you're saying? Cpit

28 Liz Okay I think I get what you're saying. So we're *at time zero and we have 10 rabbits, and the rate of change is 30* [D6] *so its going to grow at a rate of 30 rabbits per year* [C6]? [Criterion 2 met for Csy, similar to turn 24 by Deb] Cpit

29 Deb Right. So we'll have 30 more rabbits.[D7]

30 Liz But we only want to go a half a year.

31 Deb So it'll be *0.5 times 30*,[W7] *which is 15* [C7]. [Criterion 2 met for part of Cpit (namely that 30 is also the change over one year), claim C6 is now D7] CCpit

32 Liz And so we're really not figuring out the rate of change we figuring...*Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year* [B5]. Cpit

33 Deb Well the new population...

34 Joe Well *if t is 0* [D8] *then we have 0* [C8]. But you said *when t is zero we have 10* [Rebuttal to Argument 1]. (note that his assertion is incorrect) Rsy

35 Liz I think it just means *initially we have 10* [Rebuttal to C8]. R

36 Joe Well according to this *when t is zero* [D8] *we would have zero rabbits. Or the rate of change would be 0* [C8]. B

37 Liz Well actually *we're going to multiply it by a half a year* [B5, continuation of turn 32]. Cpit

38 Deb This is what I did. First I looked at the fact that *this is a rate of change equation. So this is telling me how many rabbits are being produced every year* [W10]. *So If I know 3 times the original population is produced every year, then I have 3 times 10 is produced every year* [Criterion 3 met for Csy]. *But I want to know how many is produced in 0.5 years* [D10]. *So I know how many rabbits are produced per year, so if I multiply that by 0.5 then I'll know how many more rabbits have been produced. So I take that new number that I get and add it to the old population* [C10] CCpit

39 Deb Uh huh, so then *I find the one with my new rate of change* [W11], so *I just take that population and put it in for p* [D11]. *And that is 3 times whatever that is* [C11]. Bsy

40 Liz Do you get what Deb is saying?

41 Jeff Yeah *you get 25 and then you get 55* (sic) [W11]. Bsy

42 Deb I think we should make a chart like he did. [showing her paper to Jeff] But

this would be your equation. This would be your 0.5, and then rabbits per year, and that will be your new amount of rabbits that's been added, then you add that to your old amount of rabbits, and you'll get your new population [B11]. CCpit

43 Jeff I think *you can go dp/dt=30, actually your dt will be 0.5, and then you do it again for the next one* [C12]. [Criterion 1 met for Csy] Bsy

44 Liz What do you have right there?

45 Deb *You take your old rate of change which we already know is 30 rabbits per year, and how much time that has passed equals 0.5. So 0.5 times 30 will get me how many new rabbits I have* [D13]. *So I take the new amount of rabbits I have and add it to the old amount of rabbits I have and that will give me the new population. And once I know the new population I know the new rate of change because I know the rate of change is right here.* [C13] CCpit Crit

46 Liz And the reason for putting in the new population would be what? (identifying a need to build Cpit which Deb has already constructed)

47 Deb Because *now my population is larger and I know the population changes at a constant of 3 times whatever that population is* [W13]. CCrit

48 Liz Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that *we found our rate of change initially at time zero and that we are using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15*, [D14]. CCpit

49 Deb No no

50 Liz I mean *25 because 15 plus 10 is 25* [D14]. CCpit

51 Jeff Then *we have to do it again* [C14]. [turns 48–51 repeat with specific values Argument 13] Crit

52 Liz Then *you start over again* [C14], so *its kind of like our new initial population, so we could label it time equals zero if we wanted to* [B14]. Crit

Since space constraints prohibit a complete accounting of the individual constructions and normative ways of reasoning evidenced in this episode we only highlight individual constructing actions associated with Cpit, the method for computing the next population value. By recognizing and building-with previous constructs (e.g., turns 1, 4, 7, 9, 20) we see Deb first construct Cpit in turn 24, followed by Jeff in turn 27 and Liz in turns 28+32+37. Per the DCA methodology we see that "knowing P means you can find dP/dt" (Csy) functions as if shared at the collective level per Criterion 2 (in turns 24 and 31 this idea was Data whereas in turn 4 it was a Claim). Even in this brief analysis we see how the coordination of the RBC+C and DCA traces well the individual and collective processes in mathematical progress.

The epistemic actions and concomitant discursive shifts resulted in these students reinventing the core idea of Euler's method, namely Cit. One way to express this core idea is in the following algorithm: $P_{next} = P_{now} + ((dP/dt)|_{now})*0.5$. Indeed, this particular formulation of Euler's method would be a viable extension of students' natural language. In particular, in turns 25–32, three of the four students essentially co-create the first step of the iterative process and then in turn 45 Deb succinctly provides a verbal summary of the algorithm. In turns 51 and 52 Jeff and Liz respectively highlight the iterative nature of the algorithm ("we have to do it again" and "then you start over again"). The use of "next" and "now" in the algorithm closely resembles students' verbal description of the process. This however, is only the first step in developing a comprehensive understanding of Euler's method.

We now further the theoretical and methodological advance for analysing individual and collective mathematical progress that was started in Tabach and colleagues (2014) and Hershkowitz and colleagues (2014). In particular, we use the previous episode to show how the various individual epistemic actions are intertwined with the collective production of arguments. This intertwining reflects the internal commonalities between the RBC+C and DCA methodologies.

RBC+C AND DCA INTERNAL COMMONALITIES

We begin by relating each of the RBC constructs to the DCA approach and then we relate the three criteria of the DCA approach to consolidation.

Relationship between Recognizing and Data. Theoretically, we argue that Recognizing actions are largely associated with Data. One uses some piece of information as Data because that piece of information makes sense to him/her. Recognizing action means recognizing a piece of information as relevant as data. Empirically, in the above example, parts of students' talk which were coded as Data were also coded as Recognizing. However in some cases, when a construction takes place, it happens that part of the argument is coded as Data (e.g., turns 21–24). In the previous example, we see that Recognizing actions are primarily associated with Data. In some cases (e.g., turn 1), Recognizing actions can be associated with Warrants, which are at times difficult to disentangle from Data.

Relationship between Building-with and Warrants. Theoretically, Warrants establish a connection between data and claim; in order to establish such a connection, one needs to build-with what one has. In the example this commonality is largely the case. Sometimes Building-with is linked to Data, because oftentimes Warrants and data are interchangeable (e.g., turn 36). While the previous excerpt also shows some slight differences in the relationship between Building-with and Warrants (e.g., Building-with may be linked to Claims for which, by Criterion 1, the Data and/or Backing drops off (see turn 43), additional data sets are needed to empirically test the conjecture about the relationship between the two constructs.

The relationship between Constructing and Arguments as a whole. Constructing requires vertical mathematization. Constructing actions are more global than Recognizing or Building-with actions; they incorporate sequences of interweaving Recognizing and Building-with actions (plus the glue between them). Similarly, arguments interweave Data-Claims-Warrants and Backings as a whole. Hence, in a line by line coding it is not feasible to indicate the holistic nature of an argument and it is typically indicated after a line by line coding (see for example Tabach et al., 2014). Moreover, arguments are usually co-constructed by several participants over several turns. Such exchanges are similarly typical for constructing actions.

Consolidating and the three criteria for identifying function-as-if-shared ideas. In processes of consolidating as well as across the three criteria for identifying when an idea functions as if shared, there is a repetition, reuse, revisiting, or repurposing of earlier ideas. To clarify, in Criterion 1 there is a repetition, but the repetition is partial in the sense that some parts of the argument (Data, Warrants) cease to be explicitly stated. In Criterion 2 there is repurposing of previous part of an argument (e.g., Claim) as either Data or Warrant. In this sense there is a repeating and reusing, but for a different purpose. In Criterion 3 there is a revisiting of either Data or Warrants to establish new Claims. In consolidation, previous constructs are recognized as relevant (i.e., revisited), and then built-with (i.e. used, possibly repeatedly) for example for solving a problem, reflecting on a situation or result, or even in the framework and for the purpose of an additional constructing action (for example, in lines 19–23, Csy is built-with as part of constructing Cpit).

Further commonalities between consolidating and the three criteria can be seen by considering characteristics of consolidation: awareness, self-evidence, flexibility, immediacy, and confidence (Dreyfus & Tsamir, 2004). Self-evidence links to Criterion 1 – the evidence is the Data, which drops off in subsequent arguments. The subsequent argument also then relates to immediacy and confidence in the validity of the idea. Flexibility links to Criterion 2 – components of an argument are being reused and repurposed (as sign of flexibility) in subsequent arguments. Similarly, Criterion 3 relates to flexibility in a different way. Flexibility lies in the fact that one is able to use an idea (e.g, Build-with) as Data or Warrant for a variety of different Claims. Hence close relationships exists between the criteria and Consolidation characteristics

CONCLUSION

Students in undergraduate mathematics classrooms are increasingly experiencing inquiry based learning and research is pointing to the strong benefits on student success in terms of grades and subsequent coursework (Freeman et al., 2014; Kogan & Laursen, 2013)[2]. While broad measures of student success are needed, there is also a need for methodologies that provide a fine-grained analysis of the individual and collective processes that make inquiry learning possible, and may have the potential to explain at the micro-level how such learning works and why it is beneficial. This report makes a contribution in this direction. The DCA analysis helps illuminate what is happening on the social or discursive plane, while the

RBC+C analysis helps illuminate what is happening on the cognitive side.

In this report, we used the case of a group of students reinventing Euler's method and we used this case to explicate the environmental, underlying, and internal commonalities between the AiC and DCA approaches. This represents considerable progress toward the call for what Prediger and colleagues (2008) refer to as the local integration of different theoretical/methodological approaches as well as contributing to our understanding of how undergraduate students individually and collectively reinvent important mathematical ideas.

ACKNOWLEDGEMENT

This study was partially supported by the Israel Science Foundation (Grant No. 1057/12).

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ENDNOTE

1. While acknowledging the teacher's crucial role, we did not relate to it here, as this is the next step in our research plan.

Analysing university closed book examinations using two frameworks

Athina Thoma and Paola Iannone

University of East Anglia, Norwich, UK, a.thoma@uea.ac.uk

Assessment influences students' approaches to learning and conveys to the learners what the exam-setters value. Frameworks have been developed in order to understand and analyse the demands of the assessment tasks. In this paper, two frameworks are used to analyse one undergraduate closed book examination in abstract algebra. The analysis of the tasks resulting from the two frameworks are presented and discussed. Finally, some aspects regarding the applicability of the frameworks are highlighted and further steps are suggested.

Keywords: Undergraduate, closed book examinations, abstract algebra.

INTRODUCTION

Research highlights the strong relationship between assessment demands and students' approaches to learning (Ramsden, 1983; Trigwell & Prosser, 1991). Assessment also conveys what lecturers consider significant about their subject (Smith et al., 1996; van de Watering et al., 2008), thus it is important to examine and understand the demands of assessment.

In mathematics departments in the United Kingdom, the predominant method of summative assessment is the closed book examinations (Iannone & Simpson, 2011). This paper focuses on one closed book examination from a Year 2 course in pure mathematics. Three tasks from this examination are analysed using two frameworks: Mathematical Assessment Task Hierarchy (MATH) developed by Smith and colleagues (1996) and the framework introduced by Tang, Morgan and Sfard in 2012. The analysis of the tasks using both frameworks is presented and discussed. This will allow us to better understand the potential of each and to improve their use for specific research questions.

In what follows we first present the two frameworks and introduce the context of our study. Afterwards, we analyse in detail one of the examination tasks and offer an overview of the analysis of the rest. We then discuss the results, comment on the applicability of the frameworks and make some suggestions for further research.

THE FRAMEWORKS

Different frameworks exist offering ways of analysing the tasks used in assessment. One of the most common is Bloom's taxonomy of educational objectives (Bloom et al., 1956). This taxonomy examines the different educational objectives the educators set for their students and assists them in developing balanced assessments. Different adaptations of this taxonomy have been used in mathematics. One of those adaptations tailored specifically for undergraduate mathematics closed book examinations is offered by a team of mathematicians and mathematics educators (Smith et al., 1996). Smith and colleagues (1996) introduce the MATH taxonomy aiming to assist lecturers in constructing examinations demanding a range of knowledge and skills. They distinguish between eight categories of knowledge and skills and they propose three groups: A, B and C (Table 1). In solving Group A tasks, the students are asked to recall factual knowledge and fact systems, comprehend factual knowledge and be able to use basic procedures. Students have to display the ability to transfer information and apply information or methods in new situations when attempting to answer tasks belonging to Group B. Finally, in answering tasks from Group C students are asked to justify and interpret a result, offer conjectures and comparisons and evaluate results. Smith et al. argue that examinations should have items from all the groups and note that items from Group A could guide students to adopt surface learning approaches (Ramsden, 1992) whereas items from Group B and

Group A	Group B	Group C
Factual knowledge and fact systems Comprehension of factual knowledge Routine use of procedures	Information transfer Application to new situations	Justification and interpretation Implications, conjectures and comparisons Evaluation

Table 1: The MATH taxonomy

C might help them foster deep learning approaches (Ramsden, 1992).

While the MATH taxonomy was developed to assist in the creation of examinations assessing a range of knowledge and skills, the framework introduced by Tang, Morgan and Sfard (2012) was developed to characterise the discourse of school mathematics. Aiming to examine whether the nature of students' participation in the mathematical discourse changed in the last thirty years, they focused on analysing the public examinations in the UK taken at age 16 (GCSE – General Certificate of Secondary Education). This framework draws on Systemic Functional Linguistics (Halliday, 1978; Morgan, 2006) and Sfard's theory (2008) of commognition. The framework explores the mathematical discourse that the student engages with when reading and responding to an examination task. This analysis "allows a subtle characterisation of the nature of mathematics and of student mathematical activity construed through the forms of language used" (Morgan & Tang, 2012, p. 242). The framework has two components: mathematics and the student. The mathematics component characterises the mathematical discourse the student is expected to engage in when reading and solving the task. This is further distinguished using Sfard's theory (2008) in four categories: vocabulary and syntax, visual mediators, routines and endorsed narratives. The student component investigates the relation between the examination task and the student. More specifically, it examines the positioning of the student in relation to the exam-setter, the presence of human beings, in the task, engaged in everyday or mathematical activities, and finally the decisions and directions shaping the student's response.

METHODOLOGY

For the purpose of the paper, we analyse examination tasks from an abstract algebra course in a mathematics department in the UK. This is a compulsory course and focuses on linear algebra in the autumn term and on group and ring theory in the spring term. Our data consists of the coursework tasks, the examinations tasks focused on group and ring theory and their model solutions, produced for departmental use, and was collected for a doctoral study (Ioannou, 2012). The examination accounted for 80% of the course's final grade and selected coursework tasks accounted for the remaining 20%. This examination, which is the focus of our analysis, had six tasks: three on linear algebra and the remaining three on group and ring theory. The examination lasted three hours and the students had to respond to five of the tasks. Notes were not permitted in the examination and the students were told that they could use the general theorems without proof unless stated otherwise. Non-programmable calculators were permitted during the examinations. In what follows we present a detailed analysis of task 4 (Figure 1) followed by an overview of the analysis of the other tasks on group and ring theory (Figure 2).

4)

(1) Describe the group S of rotational symmetries of a solid cube in $\mathbb{R}^3$. List the possible axes of rotation and angles of rotation, and hence show that $|S| = 24$. Let l be an axis passing through the centres of a pair of opposite faces of the cube and T be the set of rotations in S which send l to itself. Prove that T is a subgroup of S and $|T| = 8$.

(2) Suppose G is a group and H a subgroup of G. Prove that the relation $\sim$ on G given by $g_1 \sim g_2$ if and only if $g_1^{-1}g_2 \in H$ is an *equivalence relation*, saying carefully what this means. In the case where G is a finite group, prove that all equivalence classes have $|H|$ elements.

(3) State Lagrange's Theorem, and use (2) to give a proof of this.

Figure 1: Examination task 4

ANALYSIS USING THE MATH TAXONOMY

Task 4 consists of three subtasks. In (4.1) the students have to describe the 24 elements of the group of rotational symmetries of a solid cube, listing all the possible axes and angles of rotation. This is classified as information transfer since the students have to visualize the cube, identify the axes and angles and describe the elements. Afterwards, in the same subtask, the students are asked to examine whether a specific set of rotational symmetries is a subgroup and to prove that the order of the subgroup is 8. Here, the students are asked to select from the 24 elements the ones satisfying the criteria of sending l to itself. This is categorised as evaluation.

In the second subtask (4.2) students have to examine whether the given relation is an equivalence relation. In the MATH taxonomy the process of deciding whether the conditions of a definition are satisfied belongs to different categories depending on the definition of the concept. If the definition is considered simple it belongs in the comprehension category and if "understanding [*the definition*] requires a significant change in the students' mode of thought or mathematical knowledge" (Smith et al., 1996, p. 69) it is considered a conceptual definition and belongs to the information transfer. Here, the concept of the equivalence relation is considered a conceptual definition and thus classified as information transfer. Then, the students need to comment on the form of the equivalence classes defined by this relation, which are actually the left cosets, and this is classified as comprehension. Finally, the subtask asks to prove that if the group G is a finite group then the equivalence classes formed from the relation have the same elements. The students have to examine whether these left cosets have the same order as the subgroup H by defining a bijective function. This is categorised as comprehension, as the students have to show understanding of the equivalence classes' concept and define the bijective function.

The students, at subtask (4.3), must state Lagrange's theorem and prove this theorem using the knowledge demonstrated previously in subtask (4.2). This subtask is classified as factual knowledge and fact systems and justifying and interpreting.

The skills and knowledge needed to respond to tasks 5 and 6 (Figure 2) are classified as factual knowledge

5)

(1) Suppose G is a group.

(a) What does it mean to say that a subgroup N of G is a *normal* subgroup? If N is a normal subgroup of G, explain how to make the set G/N of left cosets of N in G into a group.

(b) State the First Isomorphism Theorem for groups, defining the terms *kernel* and *image* in your statement.

(c) Suppose H is a cyclic group. By defining a suitable homomorphism $\phi : (\mathbb{Z}, +) \to H$, or otherwise, prove that $H \cong \mathbb{Z}/m\mathbb{Z}$ for some $m \in \mathbb{Z}$.

(2) Let R be the ring $\mathbb{Z}[\sqrt{-7}] = \{m + n\sqrt{-7} : m, n \in \mathbb{Z}\}$. In the following you may use the fact that the function $N : R \to \mathbb{Z}$ given by $N(m + n\sqrt{-7}) = m^2 + 7n^2$ satisfies $N(ab) = N(a)N(b)$ for all $a, b \in R$.

(a) Prove that the only units in R are ± 1.

(b) Give two different factorizations of 8 into irreducibles in R and deduce that R is not a unique factorization domain. You should justify carefully any assertions you make about irreducibility of various elements of R.

6)

(1) Suppose R is a commutative ring. What is meant by saying that a subset $I \subseteq R$ is an *ideal* of R? Suppose $f \in R$ and let $fR = \{fr : r \in R\}$ be the *principal ideal* generated by f. Prove that this *is* an ideal of R. Prove that, for $f, g \in R$

$$fR \subseteq gR \Leftrightarrow g \text{ divides } f \text{ in } R.$$

(2) Prove that the element $f = x^3 + x^2 + x - 1$ in the polynomial ring $R = \mathbb{F}_3[x]$ is an irreducible (where $\mathbb{F}_3$ denotes the field with 3 elements). Explain why the ideal $I = fR$ is a maximal ideal in R. What does this imply about the quotient ring R/I? Let $h = x^4 - x^2 + 1 \in R$. Is hR a maximal ideal in R? Justify your answer.

Figure 2: Examination tasks 5 and 6

and fact systems, information transfer, application to new situations and justification and interpretation.

ANALYSIS USING THE TANG, MORGAN AND SFARD FRAMEWORK

First, we present the analysis regarding the student component. Considering the student and exam-setter relationship, we examine whether the students are given commands or asked to examine mathematical questions. In this task, the students are given imperative instructions. Regarding the directions, the students are directed to present the group S in a specific way ("list…rotation"), they are also given directions, by implication ("and hence show") on how to prove that the order of the group is 24. Also, they are given directions, by instruction in subtask (4.3) as it specifically states that they should use the facts proved in (4.2) to prove Lagrange's theorem. Regarding the depth and accuracy of the expected response, the students are directed in (4.2) ("saying carefully what this means"). Examining the decisions the students have to make, we observe that in (4.1) the students can decide whether or not to provide visual representations of the rotational symmetries of the cube. Finally, in our analysis of these tasks we do not consider the presence of the human beings as the tasks do not offer descriptions of human beings engaged in mathematical or everyday activities.

In our analysis regarding the mathematics component we focus on the routines, due to space limitation. The routines discuss the patterns observed in the discursants' activity when attempting to construct and substantiate narratives endorsed by the mathematical community (Sfard, 2008). Here, according to the framework, we examine the form of student engagement and the areas of mathematics involved. The imperatives, present in the tasks, are analysed in order to examine the student's engagement which is distinguished in: engagement in material processes, construing the student's role as a 'scribbler'; and engagement in mental processes, construing the student as a 'thinker' (Rotman, 1988). Here, the students are asked to engage in material actions ("describe", "list", "state", "use") as well as in mental activity ("show", "let", "prove", "suppose"). Relating to the areas of mathematics involved we see that the students have to engage with concepts from set and group theory, but also to demonstrate knowledge from geometry when asked on the rotational symmetries of the cube. Finally, we offer a categorisation of the routines, using Sfard's theory, into: construction, resulting in new endorsable narratives; substantiation, assisting in the decision to endorse a previously constructed narrative; and recall, bringing to mind previously endorsed narratives (Sfard, 2008, p. 225). In this task, the students are requested to engage in a construction routine when asked to describe the elements of the group S and in a substantiation routine when examining which of the elements of S send l to itself. In (4.2), the students have to engage in a substantiation routine, as they have to verify the definition of the equivalence relation. Then they have to prove that all the equivalence classes have $|H|$ elements engaging in a construction routine in order to construct one of the classes and the mapping of the elements; and next in a substantiation routine where they have to examine that the mapping is bijective. In the last subtask, the students are required to state and prove Lagrange's theorem using (4.2). Consequently, the students have to engage in a recall and a substantiation routine.

In tasks 5 and 6 the students' actions are pre-shaped since they are given explicit or implicit directions regarding the presentation, the depth and accuracy of their response and the methods. There are only a few instances where the students can decide on the method and the degree of accuracy. The tasks involve number theory, set theory, group and ring theory. We also note that the student's role is construed both as scribbler and thinker. Finally, the students are asked to engage in construction, substantiation and recall routines.

DISCUSSION AND SUGGESTIONS FOR FURTHER RESEARCH

The aim of this paper was to uncover the potential of two frameworks in analysing examination tasks. To this aim we discuss the results obtained from the analysis of three examination tasks and we offer here some reflections on the application of the frameworks.

The MATH taxonomy highlights the nature of the skills needed to respond correctly to the task. The students are asked to demonstrate their knowledge of the basic concepts and theorems used in the course. They are required not only to remember but to show their understanding of them too. In order to solve these tasks the students have to demonstrate factual knowledge and fact systems, comprehension of factual knowledge, information transfer, application in new situations, justification and interpretation and evaluation. The analysis using the Tang and colleagues (2012) framework highlights that most of the students' actions are pre-shaped with the implicit or explicit directions given to them, allowing them to be autonomous in very few cases. Furthermore, the student's role is interpreted as both a scribbler and a thinker engaging with material actions and mental activity. The students are asked to engage with concepts from the following mathematical areas: geometry, number theory, set theory, group and ring theory. Finally, the students engage in recall, substantiation and construction routines in different parts of the tasks.

Both frameworks deal with the concept of familiarity through the categories of recall routine or factual knowledge and factual systems. We should point out that familiarity, a highly contextual and subjective concept, is not clearly defined in either one. A task may be considered as familiar to some students and thus require them to engage in a memory retrieval procedure, while the same task presented to students exposed to different teaching material might require them to engage in mathematical activities of a different nature. Also, familiarity can be different for the individual students belonging to the same teaching group as each one engages differently with the given material. In our analysis we classified a task as belonging to the categories above when it required stating a definition or a theorem. However, we should note that some parts of the tasks were given to the students as coursework although the model solutions of these tasks were not made available to them, as these were the ones assessed in the coursework. Here we should report that there is a framework which examines the concept of familiarity more rigorously. Bergqvist (2007) used a framework developed by Lithner (2008) regarding the reasoning expected of the students. She analysed examination tasks from a Swedish university and categorised them into tasks requiring imitative or creative reasoning. A task was classified as demanding imitative reasoning if it asked for a fact or a theory item, for which the students were clearly informed that it might be requested in the examinations; or the task occurred at least three times in the textbooks of the course. Note that the context of our study is different from the one in Sweden, as in the United Kingdom the students mostly rely on their lecture notes and not on textbooks.

In our analysis of the tasks using the MATH taxonomy we were unable to exclusively classify one task into one of the eight categories. Furthermore, we experienced difficulties when trying to position the tasks in some categories as we didn't have clear instructions regarding the effect of the background information on the classification of the tasks. However, both of these issues are related to the origin of the taxonomy. The taxonomy was developed to assist lecturers in creating balanced examinations and not as a research tool as illustrated in the quote below:

> [I]t is not our aim to be able to uniquely characterize every conceivable assessment task. Rather, the aim of the descriptors is to assist in writing examination questions, and to allow the examiner's judgement, objectives and experience to determine the final evaluation of an assessment task. (Smith et al., 1996, p. 68)

Some interesting aspects of the tasks are pointed out using the Tang and colleagues (2012) framework. More specifically by examining the areas of mathematics involved we gain information regarding students' engagement with mathematical concepts from other mathematical areas than the one that the course focuses on. This aspect is not highlighted in the MATH taxonomy as the focus is on the activity and not on the areas of mathematics. To be more specific, the tasks analysed here are assessment tasks from an abstract algebra course, but in order to solve them the students have to display knowledge of geometry (4.1), number theory (5 and 6) and set theory, which are not the focus of this course. This emphasises the nature of the mathematics involved; the prior knowledge expected from the students and also examines the students' ability to draw on different areas of mathematics.

The level of guidance given to the students and the degree of their autonomy when solving a task is also highlighted by the Tang and colleagues (2012) framework. As it examines the directions given to the student; and the complexity of the response expected of the students, namely the decisions they have to make. Examining a task in this respect might provide some information on the exam-setters' perceptions of their students, though this would need to be confirmed with interviews with the exam-setters. The explicit or implicit directions on the method may display what the exam-setters value or think that their students would be able to manage better. Similarly, by examining the directions regarding the presentation of the response we gain information on the exam-setters' perception of the depth and the accuracy of students' responses. For example we have three instances where the students are explicitly asked to provide a response with a certain degree of accuracy and depth ("saying carefully what this means" (4.2), "you should justify carefully" (5.2.b) and "justify your answer" (6.2)). On the other hand, we have the decisions the students had to make in these tasks on the degree of accuracy (5.1.a) or the presentation of their response (4.1). It would be interesting to examine how the accuracy and the depth of the students' responses in this case are assessed by their examiners. Finally, there were some decisions the students had to make regarding the method of solution. One of them was explicitly stated: "By defining ... or otherwise" (5.1.c), but having in mind the students' knowledge of the subject their methods of solution are limited to the specific methods they have encountered in the course.

The classification process of the examination tasks is highly subjective as a response to the task is taken into account. In order to position the task in a specific category in the MATH taxonomy and in order to examine the routines using Sfard's theory (2008) we have to consider a possible solution to the task. The final classification depends on the researchers' choice of response and their opinion of that response (Jolliffe & Ponsford, 1989). In our attempt to reduce the subjectivity of our classification we took into account the model solutions produced by the lecturer of the course for departmental use (Figure 3).

Choosing this response for our analysis we investigate the lecturer's expectation of the students' solution and not the actual solutions produced by the students. In order to examine the actual routines the students engage in, for the Tang and colleagues (2012) framework, and the skills and knowledge, for the MATH taxonomy, we need to examine the solutions produced by the individual students. Follow up interviews with the students are also necessary as different students can employ different routines and different skills and knowledge, depending on their background, to arrive to the same solution.

In conclusion, our analysis of the same examination tasks using two different frameworks highlights some interesting aspects regarding the two frameworks and their applicability. We should note that in our analysis of the Tang and colleagues (2012) framework from the mathematics component we took into account only the routines aspect. We intend to explore the results from the other aspects of the framework namely vocabulary and syntax, visual mediators and endorsed narratives. Finally, in the following stages of this research, we aim to seek the views of the lecturers, who

(1) S consists of the following rotations:
  (i) Axis of rotation through a pair of opposite vetrices; angle of rotation $\pm 2\pi/3$, number of this type: $4 * 2 = 8$.
  (ii) Axis of rotation through a pair of opposite faces; angle of rotation $\pi/2, \pi, 3\pi/2$, number of these: $3 * 3 = 9$.
  (iii) Axis of rotation through mid-point of a pair of opposite edges; angle of rotation π, number of these: 6.

  Together with the identity, this gives a total of $1 + 8 + 9 + 6 = 24$ rotational symmetries. T consists of 4 rotations (including the identity) of type (ii) with axis l, together with rotations about axes perpendicular to l through an angle π which 'invert' l. There are two of these of type (ii) and two of type (iii). Thus $|T| = 8$.

(2) Need to show that $\sim$ is:
  Symmetric: if $g_1 \sim g_2$ then ${g_1}^{-1}g_2 \in H$ so as $H \leq G$, $({g_1}^{-1}g_2)^{-1} = {g_2}^{-1}g_1 \in H$ i.e. $g_2 \sim g_1$.
  Reflexive: $g_1 \sim g_1$ as ${g_1}^{-1}g_1 = e \in H$.
  Transitive: if $g_1 \sim g_2$ and $g_2 \sim g_3$ then ${g_1}^{-1}g_2, {g_2}^{-1}g_3 \in H$. As H is a subgroup $({g_1}^{-1}g_2) * ({g_2}^{-1}g_3) = {g_1}^{-1}g_3 \in H$ so $g_1 \sim g_3$.
  An equivalence class is of the form $\{g_2 : g_1 \sim g_2\}$ for some $g_1 \in G$ and by definition this is $\{g_1 h : h \in H\}$
  The map $H \longrightarrow \{g_1 h : h \in H\}$, $h \longrightarrow g_1 h$ is bijective so the number of elements in the set is $|H|$.

(3) Lagrange's Theorem: If G is a finite group and H a subgroup of G then $|H|$ divides $|G|$. The equivalence classes in (2) partition G: every element of G lives in a unique equivalence class. All classes have $|H|$ elements, so $|G| = |H|x$ number of equivalence classes. Thus $|H|$ divides $|G|$.

Figure 3: Model solution of task 4

set the tasks, and the views of the students solving the tasks and relate these findings with the results from the frameworks.

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The constitution of the nature of mathematics in the lecturing practices of three university mathematics teachers

Olov Viirman

University of Agder, Faculty of Engineering and Science, Kristiansand, Norway, olov.viirman@uia.no

The study reported in this paper investigates how notions of the nature of mathematical knowledge and mathematical objects are articulated in the discursive practices of three university mathematics teachers at a Swedish university. The data consists of video recorded lectures, and the analyses were informed by classifications presented by Lerman (1990) and Davis and Hersh (1981). The results indicate that different epistemological and ontological positions are indeed constituted through the discourse. Although the discourse is generally highly objectified, the ways in which mathematical objects are introduced differ. Mostly the discourse was within an absolutist paradigm, but there were also examples of how the socio-historical nature of mathematical knowledge is emphasized.

Keywords: Tertiary mathematics, university teaching, epistemology of mathematics, ontology of mathematics, mathematical discourse.

INTRODUCTION

Recent years have seen an increasing research interest in university mathematics teaching, including a growing number of studies focusing on so-called traditional mathematics teaching (e.g., Gücler, 2013; Viirman, 2014b; Weber, 2004). One aspect of university mathematics teaching, which has so far received less attention, is how notions of the character of mathematics are constituted through the teaching practices in mathematics lectures. The present paper is an attempt at addressing this issue. There are many ways of characterizing mathematics (see, e.g., Devlin's (2000) four faces of mathematics), but in this short paper I have chosen to focus on the classic philosophical questions of epistemology and ontology, that is, on the nature of mathematical knowledge and mathematical objects, and how these notions are manifested in the teaching discourse.

THEORETICAL PERSPECTIVES

The study takes it theoretical starting point in the commognitive framework of Sfard (2008). In this theory, mathematics is viewed as a discursive activity, with mathematical discourse constituted by its use of *words*, *visual mediators*, *narratives* (sequences of utterances speaking of objects, relations between and/or processes upon objects, and subject to endorsement or rejection within the discourse) and *routines* (repetitive patterns characteristic of the discourse). In the present paper the focus will be mainly on narratives and substantiation routines, that is, routines aimed at deciding whether to endorse previously constructed narratives. Mathematical discourse is also characterized by its high degree of objectification, where words in the discourse are viewed as signifying independently existing objects (ibid, p. 300).

Concerning the questions of the epistemology and ontology of mathematics, these have been central topics within the philosophy of mathematics for hundreds of years, and a more thorough discussion is far beyond the scope of this paper. For the purposes of this study the distinctions made by Lerman (1990) and Davis and Hersh (1981) will suffice. Lerman, in a study investigating the relation between views of the nature of mathematics and teaching practice, distinguishes between absolutist and fallibilist epistemologies. The absolutist sees mathematical knowledge as certain, absolute and timeless, and views the history of mathematics as "a demonstration of the errors and mistakes along the way to certain knowledge" (Lerman, 1990, p. 54–55). Fallibilism (inspired by Wittgenstein and Lakatos) on the other hand, "sees the growth of mathematical knowledge as a process of conjectures, proofs and refutations, and accepts the uncertainty of mathematical knowledge as part of the nature of mathematics" (ibid, p. 54). A similar stance is taken by Davis and Hersh, (1981), who also consider mathematical ontology. They distinguish between Platonism, where mathematical objects are seen as real, having an objective existence outside of human experience, and an alternative position, later denoted as humanism (Hersh, 1997), where mathematical objects are seen as human creations, but still objective in the sense of being external to the consciousness of any single individual. Instead they belong to the social, non-material culture of mankind.

Thus, the focus of this paper is on how the epistemology and ontology of mathematics are expressed in the teaching discourse. The question that the paper aims at answering is the following: How are notions of the nature of mathematical knowledge, and of mathematical objects, articulated through the discursive practices of the teachers?

PREVIOUS RESEARCH

Traditionally, questions about the nature of mathematics in relation to teaching practices have been handled in the context of research on teacher beliefs. That is, the focus has been on what beliefs about the nature of mathematics teachers possess, and how these beliefs might impact on their teaching practice (for overviews of the early achievements in this field, see, e.g., Pajares, 1992; Thompson, 1992). The study by Ernest (1989) can serve as an example. Ernest identifies three categories for characterizing individuals' views of the nature of mathematics: Instrumentalist, Platonist, and Problem-Solving. However, Ernest's study, like most similar studies, focuses mostly on pre-service or practicing teachers in elementary or secondary school, although there are some examples of studies of university teachers' beliefs about mathematics (e.g., Mura, 1993; Speer, 2008). Also, studying teacher beliefs shifts the focus from the teaching to the teacher, which runs contrary to the aim of the present study. Furthermore, the very notion of teacher beliefs as a topic of research has been the subject of much criticism, from a methodological standpoint – beliefs being notoriously hard to define and gain access to (Speer, 2008; Skott, 2009) as well as on a purely conceptual level (Skott, 2013). Skott suggests that we move away from the objectified construct of beliefs, focusing instead on "patterns of participation" in social practices, that is, on the processes said to give rise to beliefs (Skott, 2013, p. 549). This approach is similar to Sfard's (2008) commognitive framework, with its focus on discursive practice as a patterned activity.

Unfortunately, studies looking at how notions of the nature of mathematics, as described for instance in the literature cited above, are expressed through teaching practice are rare. The previously mentioned study by Lerman (1990) could be said to be one, although the principal focus of the empirical analysis in his (mainly theoretical) paper is on what student teachers' interpretations of a teaching episode tells us about their views of mathematics. Another example, which also happens to concern university teaching, is the study by Österholm (2010), investigating "what types of epistemologies are conveyed through properties of mathematical discourse in two lectures" (p. 241). Despite *Österholm's* paper being framed in the language of belief research, the analyses in fact focus solely on the epistemological character of mathematics as conveyed through the discourse. To this end, Österholm considers the types of statements and the type of argumentations used by the teachers. Of particular relevance for the present study is the distinction between use-statements, related to procedural knowledge; and object-statements, related to conceptual knowledge. A dominance of object-statements can be seen in the calculus lecture, indicating a focus on conceptual knowledge.

METHOD

The analyses presented in this paper are based on data collected for my doctoral thesis (Viirman, 2014a). This data consists of video recordings of first-semester mathematics lectures by seven teachers at three different Swedish universities, approximately two hours of video for each teacher. The teachers were selected among those volunteering to participate in the study, aiming for variety both in teaching experience and in topics taught. In this study, however, I am using data from lectures by three of the teachers (denoted A1, A3 and A4 in what follows, in accordance with other publications arising from this data, e.g., Viirman, 2014a; 2014b). These were chosen since the analysis conducted for the doctoral thesis indicated that they were the richest and most varied with regard to the aims of the present study. All three teachers work at the same university, one of the largest and most well-established in Sweden, and are experienced teachers, having taught university mathematics for more than 10 years. Teacher A1 is female, while the other two are male. All three lectures were given in courses aimed at engineering students, and were taught in a traditional style, with the lecturer talking and writing on the board. The number of students ranged from about 50 (teacher A1) to about 150 (teachers A3 and A4). Teacher A1 taught an introductory course, preparatory for calculus; teacher A3 taught linear algebra; while teacher A4 taught single-variable calculus. The topic in all three lectures was various aspects of the function concept. For more detail on the process of data collection, see (Viirman, 2014a; 2014b).

As part of the work on the thesis, the video recorded lectures were transcribed verbatim, speech as well as the writing on the board. For the present study, the transcribed lectures given by teachers A1, A3 and A4 were then analysed, first separately and then in comparison, looking specifically at how mathematical objects and the nature of mathematical knowledge were expressed through the teachers' discourse, focusing on word use and narratives. Regarding the teachers' use of definitions, and how new mathematical objects were introduced I looked, for instance, at how notions of agency were expressed in the discourse – whether these new objects were spoken of as originating outside of the discourse or within it. Concerning means of substantiation I looked, for instance, for utterances suggesting change in such means over time. Throughout I used the categorisations of Lerman (1990) and Davis and Hersh (1981) to guide my analyses. I want to stress that this is not a study of teachers' beliefs. I make no claims as to whether the ways in which mathematical objects and mathematical knowledge are articulated in the discursive activity in the lectures have any bearing whatsoever on the views the teachers might be holding regarding these matters.

RESULTS

Considering first the question of mathematical ontology, the discourse documented in this study is typically mathematical in that it is generally highly objectified. The mathematical objects are spoken of as being independently existing [1]:

> Teacher A4 So, it's about continuity, and that is a property that functions can have.
>
> Teacher A3 There is a transformation that we call, I don't know, id for identity, that takes every vector to itself.

In fact, functions are so much like physical objects that they can be moved around:

> Teacher A1 That is a function; it is the function x^2 that I move one step to the right and two steps upwards.
>
> Teacher A4 What happens to this function when x is bigger than one? (...) It goes down, yes, and then it will wander here, and get bigger and bigger and bigger.

However, looking at how new mathematical objects are introduced, narratives are framed in different ways, suggesting different ontological positions. Consider the following example:

> Teacher A1 for us to know what we are talking about, we have to begin by saying exactly what we mean by a function. I think that most of you already have a feeling for what it is that a function is, but maybe you haven't seen exactly a definition. Because, you know, it is like this in mathematics that all words we use, we have to say exactly what we mean by them so that we are totally agreed, if I say that all functions have a certain property, then all have to agree with me what objects we are talking about. We have to agree about what we mean by the word function.

Here, even though the functions are explicitly spoken of as objects, it is still clear that regarding the properties of these objects it is up to the participants in the mathematical activity to decide what they are. But all participants have to agree in order to be able to use them meaningfully. This way of talking is very much in accordance with the humanist philosophy, as formulated by Davis and Hersh (1981). Later in the same lecture, we find the following example. The teacher is discussing the unit circle, an example drawn on the board.

> Teacher A1 We would perhaps want this to be a function (...) but then when we insert something which isn't one or minus one

then we get- there are two y-values that fit, so the function doesn't give us exactly one, it gives more than one, and then it isn't a function.

Hence, although we, the participants in the mathematical activity, have ourselves constructed this object called function, we are not free to do whatever we want with it. Once we have decided what to mean by the word, it comes equipped with properties that are not ours to decide over. It is not enough that we want something to be a function; it has to agree with what we decided that a function should be. Again this corresponds to the humanist position, where mathematical objects, although human creations, still have objective properties.

Yet another example from the same lecture: having introduced trigonometric functions through right triangles the teacher notes that these functions are only defined on the interval $(0,\pi/2)$, and continues:

Teacher A1 We would want these functions sine and cosine to be defined for all real numbers, we would like to exchange this little piece for the whole of *R* (...) then we have to figure out how to do this, and before we do this we have to say how we are going to measure angles.

Again, it is up to us as participants to decide how to define the sine and cosine functions outside of the interval $(0,\pi/2)$. But, it can't be done any way we want. It has to agree with what has already been decided, that is, with how the functions are defined for acute angles.

On the other hand, the introduction of mathematical objects can be done through narratives framed in a very different manner:

Teacher A3 A function from R^m to R^n is called a linear transformation if it is linear, that is, if it satisfies two conditions:

The teacher then goes on to describe the two linearity conditions. After giving, as an example, a characterization of all linear transformations from *R* to *R*, he concludes:

Teacher A3 this was a very small example, and an example that has completely negligible importance for the rest of the course, but anyway it says something about that linear transformations are very rare. In any case they are very important.

As presented here, there are some things called functions, and among these a small number satisfy certain conditions, and these are called linear transformations. Indeed, these are spoken of as rare, almost as if talking about rare birds or flowers. There is nothing in the way these objects are discussed suggesting that they are anything other than objectively existing, independently of our working with them. No reasons are given for the linearity conditions looking the way they do, for instance. Looking at the example given at the start of the section about the identity transformation, the transformation is there, we as practitioners of mathematics just give it a name. Similarly, concerning the relationship between matrices and linear transformations:

Teacher A3 Every linear transformation defines a matrix which we will call the standard matrix of the linear transformation.

Again, the transformation defines the matrix; we only have to give it a name. This way of speaking about mathematical objects fits with the Platonist position.

Considering epistemology, that is, the questions of mathematical knowledge and how it is obtained, much of the discourse documented in this study is consistent with an absolutist paradigm. Claims are mainly justified using traditionally mathematical means of substantiation, for instance through mathematical proof or by reference to already established mathematical facts (for more detail on the teachers' substantiation routines, see Viirman, 2014b). Some examples:

Teacher A4 It isn't obvious that it is like this, this is not a property that all functions have, but for us it was a consequence of what? Two things. The basic properties of limits (...) and the standard limits.

Teacher A3 Why is it linear? It is linear precisely because matrix multiplication works like that.

The mathematical substantiations are mostly done through algebraic and numerical reasoning, as in the following example, where the aim is calculating the range of the function $f(x) = (x-1)^2 + 2$:

> Teacher A1 In this case, there is an even square here, plus 2. Even squares can be zero or bigger. (...) Hence it can be 2 plus something positive, so it can be all numbers that are greater or equal to 2.

Or here, part of the process of calculating the value of $\sin(\pi/4)$ from a diagram of half a square:

> Teacher A1 $[x_0^2 + x_0^2 = 1^2 \Rightarrow x_0^2 = 1/2 \Rightarrow x_0 = 1/\sqrt{2}]$

There are also examples of using geometrical reasoning to substantiate claims:

> Teacher A3 That these two vectors are perpendicular, this you can see immediately geometrically (...) If we project something which is perpendicular to *v* on *v*, then we just get zero, right? [Draws a vector pointing straight upwards, draws another vector pointing straight to the right and marks it by *"v"*, marks the angle between the two vectors as right.] Here is *v* [points to the vector *v*], here is something which is perpendicular [points to the vertical vector], if we project it down here [points to the base of the vertical vector] then we get sort of nothing.

In a similar fashion, teacher A3 shows the linearity of rotations in the plane by geometrical arguments, drawing vectors and their images under the transformation and showing why the linearity conditions are satisfied.

There are also examples of substantiations emphasizing the cumulative nature of mathematical knowledge, referring back to previously known facts, as in the excerpt quoted above, as well as in the following examples:

> Teacher A4 If someone were to twist your arm and say: How do you know that? Then it is precisely from the good old rules of limits.
>
> Teacher A4 And now it's time to reconnect this part of the brain that you have neglected for some time, namely complex numbers. Because the fundamental theorem of algebra says that a third degree polynomial has three roots, in general complex.

This is still consistent with an absolutist paradigm. There are however some examples of substantiations consistent with a fallibilist epistemology, where the historical development of mathematics is explicitly used in the teaching, emphasising the socio-historical nature of mathematical knowledge. Some examples:

> Teacher A4 In fact, this is how one often defines continuity during the 18th century, this means that the graph of this function if we are to draw it [Draws a coordinate system, and draws a connected curve] it hangs together like this; I can draw it without lifting the chalk from the blackboard.
>
> Teacher A4 This is the beginning of a line of work that is quite important within analysis, and which gained momentum during the latter half of the 19th century, when the deal was to find really pathological functions, that test our understanding of the function concept and what we can assume (...) A lot of great mathematicians spent time on this, and some people thought that it was totally nuts, such things don't exist, they are totally insignificant, can't be used for anything, and that turned out to be totally wrong.
>
> Teacher A4 I just read an article by a German mathematician from the late 19th century (...) and he spent half a page in his article explaining what this meant, it wasn't established at that time, the terminology was vague, and it isn't totally trivial what it means.

This way of using historical examples indicates how mathematical definitions are subject to change over time, and how even what is to be counted as mathematics is the subject of disagreement and controversy.

DISCUSSION

The results of the study show how different philosophical positions on the nature of mathematical knowledge and mathematical objects are articulated in the teaching discourse. The study is thus an example of how notions traditionally studied within the context of research on teachers' beliefs can indeed be studied purely on the level of discourse.

One prominent characteristic of the discourse of all three lectures documented in this study was the use of a highly objectified language, something Österholm (2010) also notes in his study. Some of the differences that could be seen, for instance, regarding the type of substantiations used for claims made, can probably be explained by differences in the courses taught. For instance, in an introductory course such as the one taught by teacher A1 one does not expect substantiation through previously established facts to be very prominent.

Still, there are other differences, more directly related to notions of the nature of mathematics, which are not obviously explained by differences in topic or level of the courses taught. For instance, the results show clearly how two different, and indeed contrasting, positions concerning mathematical ontology can be seen in the teaching discourse: a humanist position, emphasizing how the definition is something we as participants in the mathematical activity have agreed upon, and a more depersonalized Platonist position where mathematics is presented as something that is discovered, appearing fully formed.

Concerning the epistemology of mathematics, although the discourse documented in the study is generally consistent with an absolutist paradigm of mathematical knowledge, such statements may be interpreted differently depending on other aspects of the discourse. For instance, statements suggesting an absolutist position may be interpreted differently in the light of whether emphasis is generally placed on the man-made character of mathematical objects or, on the other hand, on a more Platonist way of talking about mathematical objects. Similarly, an emphasis on the socio-historical development of mathematics (humanism position) makes an absolutist interpretation of the mathematical discourse less likely. One might point out here, however, that the use of historical examples does not in itself necessarily suggest a fallibilist epistemology. As Lerman (1990) indicates, "[w]hilst it may be generally accepted that, in its external history, it is influenced by cultural determinants and social factors, the prevailing view is that the mathematical knowledge that results is self-justificatory in terms of its truth" (p. 54). It might be possible to interpret some of the statements quoted above in this way. For instance, on the topic of the classic theorems on continuous functions (the maximum value theorem, the mean value theorem etc.) when the teacher notes that "it wasn't established at that time, the terminology was vague", this could be taken to mean that mathematicians have now established the true state of affairs. Although such an interpretation might be less likely, one would have to gather more data, both in the form of further recordings of his teaching, and data on how students interpret the use of the history of mathematics in teaching, to be able to draw more certain conclusions.

Indeed, data on the students' interpretations of the teaching practices would be useful, to establish whether the notions of mathematics constituted through the teachers' discursive practices actually have any effect on student learning. It has been claimed by many within the mathematics education community that students' beliefs about mathematics affect their learning (e.g., Pajares, 1992). Even if one accepts Skott's (2013) critique of the concept of belief, increased knowledge of how notions of the nature of mathematics are constituted through teaching practices, and of how students interpret these practices, would be useful to gain further insight into the relationship between teaching and learning mathematics.

More generally, although the present study is small, the conclusion that notions traditionally considered as belonging to the field of belief research can actually be studied as features of discourse could be of importance to any researcher interested in such aspects of mathematical teaching and learning.

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ENDNOTE

1. All excerpts have been translated from Swedish by the author. Text within [square brackets] indicates writing on the board.

TWG14

Posters

The transition from informal to formal understanding of the concept of order in abstract mathematics

Mustafa Akdemir, Serkan Narlı and Muhammet Kaşıkçı

Dokuz Eylül University, Faculty of Education, İzmir, Turkey, mustafa.akdemir@deu.edu.tr

This study presents an analysis of the answers given by prospective teachers to a questionnaire that consisted of 6 open ended questions. They answered the questions before meeting the formal definition of order relation and after getting familiar with the formal definition of order relation. In this context, the theoretical perspective of concept image and concept definition is used in the study.

Keywords: Concept image, concept definition, concept of order, order relation.

INTRODUCTION

Considering the theoretical perspective of concept image and concept definition by Tall and Vinner (1981), this study constitutes a detailed report regarding how and to what extent prospective teachers convert the informal information regarding order notion to formal information after the subject of order relations is taught in abstract mathematics.

RESEARCH QUESTIONS

RQ1: What are the concept images and informal definitions of undergraduate students related to the concept of order?

RQ2: After meeting the basic definitions of order relation, what type of changes happen in students' concept images?

CONTEXT OF THE STUDY

Order relation is one of the important concepts in abstract algebra. By using order relation, the elements of a set can be ordered and the maximum, minimum, supreme, minimum, maximal and minimal elements of this set can be determined. In daily life whenever a comparison is done such as big-small, thin-fat, richest-poorest etc., we can say that order relation is an intrinsic part of these comparisons. Nevertheless, it may be thought that students generally may not be aware of these associations between the daily usage and the concept of order relation which they learn in school (Narlı, 2013). In this context, it can be thought that the concept of order used in daily life informally can be associated with the concept of order relation in an academic setting.

In this qualitative study, using content analysis, a questionnaire – that aimed to inform us about students' concept images of order and was made up of 6 open ended questions – was applied to 25 first year students in the department of Elementary Mathematics Teaching at Dokuz Eylül University. The following six questions were asked to the students:

Q1.What comes to mind when you hear the phrase, "order notion?" Q2. Can the symbols "□,D,⋕,*" be ordered? If yes, show how you order them. Q3. May an infinite set be bounded? Explain. Q4. May a finite set be boundless? Explain. Q5. Is there any set where any two different elements can't be compared with each other in terms of greatness-smallness? Write your justification and exemplify it. Q6. What is order relation? Explain.

RESULTS AND CONCLUSION

Our analysis revealed some concept images (CI) as well as data that we were not able to classify as concept images (CD, as in "categorized data") – see Table 1.

If we consider our research from the theoretical perspective of concept image and concept definition, we

	Before meeting definition (with frequency)	After meeting definition
Q1	*CI-1*"Grouping of elements" *(5)* *CI-2*"Determining arbitrary criterion (rule)" *(11)* *CI-6*"Everything can be ordered" *(3)* *CI-4*"Only the numerical data can be ordered" *(6)*	*CD-2(22)*"The category points to undergraduate students who answered Q1 by associating the formal definition (order relation)", *CI-1(3)*
Q2	*CD-1(17)* "The answers that seem as unconcerned to our research questions and the answers that are left blank can be categorized as CD-1", *CI-2(8)*	*CD-2(21), CI-2(4)*
Q3	*CI-3*"Boundedness and boundlessness vs. finiteness and infiniteness" *(21), CD-1(4)*	*CI-3(15), CD-1(10)*
Q4	*CI-3 (15) CD-1(10)*	*CI-3(7) CD-1(3) CD-2(14)*
Q5	*CI-4(17)* *CI-5*"Elements should be of the same type" *(7)* *CI-7*"Infinite decimal numbers cannot be ordered"*(1)*	*CD-3(19)* "Some undergraduate students tried to answer the question by defining a mathematically convenient order relation", *CD-1(6)*
Q6	*CD-4*"this category points to the answers which have statements that are inconsistent with the formal definition" *(20)* *CD-1(5)*	*CD-5(21), "*students provided formal definition of order relation*" CD-1(4)*

Table1: Categories of concept images and categorized data

can say that concept images have a vital role in formation of the formal definitions. Prospective teachers' concept images need to be approached attentively in the instruction plan. This study may be carried forward with greater sample groups and different theoretical perspectives.

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Improper integrals in a CAS environment

Francisco Javier Cortés González and Hugo Rogelio Mejía Velasco

CINVESTAV, Ciudad de México, México, fjcortes@cinvestav.mx, hmejia@cinvestav.mx

With an instrumental approach in a CAS environment, our research is oriented towards the development of instrumented action schemes. Involving concepts related to improper integral and the use of the parameter, this paper presents the tasks that aim to identify the functional relationship that fits the graph of the points defined by the parameter values and the value of the improper integral convergent.

Keywords: Instrumental genesis, improper integral and instrumented action scheme.

This paper presents part of a teaching sequence based on the *instrumentalist theory* (Verillon & Rabardel, 1995), where an *artifact* and an *instrument* are differentiated. Trouche (2005) argues that the construct called *instrumental genesis* is a complex process linked to the characteristics of the *artifact* (its potentialities and constraints) and it can be explained by the *use of schemes* and *instrumented action schemes*.

An *instrumented action scheme* is a stable mental organization which includes technical skills, concepts and theorems, supporting a way to use a device, in order to perform a class of tasks (Drijvers & Gravemeijer, 2005). Drijvers and Gravemeijer (2005) use the notion of the *instrumented technique* developed by Lagrange (1999) and consider it the visible external part of the *instrumented action scheme*.

METHODOLOGY

The teaching sequence comprises two stages: one for training in a CAS environment and another where students perform the designed activities. The CAS environment where our research is based is the Derive software. Ten engineering students voluntarily participated in the experiment.

Following the line drawn by Drijvers and Gravemeijer (2005) for the identification of an *instrumented action scheme* as a list of key elements which includes technical and conceptual aspects for a specific task involving the use of a command from the CAS environment, the *instrumented action scheme* is established as *a priori* guidance on the design of activities to be implemented.

The given functions were , ; with, , and . The task consisted of exploring the "Product of the function and the function ". Using the "vector" command of the Derive software, the requested activities were to explain the behavior of the family of curves generated by the product where runs on the interval for possible values of the parameter ; to calculate the improper integral for some values of the parameter , to complete a table with the values of the parameter and the value of the improper integral when converges; to plot the points and to consider if there was any function or piece function that fits the plotted points above.

A training for using the "vector" command allows the student to use the potential in this facility. We expect the student to use the syntax and in order to analyze different behavior when runs between and with distinct *steps*.

RESULTS AND COMMENTS

The purpose of the activity related to the use of skilled and trained software Derive "vector" command is fulfilled by students. This is observed in actions undertaken with the software and the answers given in their worksheets. Intervention for students assigned a different "*step*" to an integer value in the corresponding syntax needed.

The students require intervention to identify the improper integral as , which may be finite, infinite or not exist. They also recognize the range of real numbers of the variable for the function has an asymptotic or exponential behavior.

The students explained the asymptotic or exponential behavior of the generated family of curves, use the "vector" command and graphics software resources. They also employed the "vector" command to calculate the requested improper integral and to identify the functional relationship which conforms the graphical representation of for values of the parameter when the improper integral is convergent. The students developed the *use schemes* for CAS commands. However, those are the conceptual aspects of the commands that require intervention. We can see this in the difficulty to identify the parameter as a real number that infinitely takes many values – and thus the potential use of the "vector" command – as well as in the difficulty for the students to properly identify when an improper integral converges or diverges. The differentiation of the technical and conceptual aspects from the *scheme instrumented action* allow timely intervention on the weaknesses shown by students.

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Revival of classical topics in differential geometry

Thierry (Noah) Dana-Picard and Nurit Zehavi

Jerusalem College of Technology, Weizmann Institute, Jerusalem, Israel

We present a reflection on the usage of modern algorithms and technology, in order to revive topics of classical Differential Geometry. This enables students to acquire new mathematical knowledge and provides new skills for applied science.

Keywords: Differential geometry, 1-parameter families of curves, envelopes, technology.

MOTIVATIONS

Classical topics in Differential Geometry like the study of 1-parameter families of curves and their envelopes have been abandoned in the past for various reasons: the classical theory may not be rigorous enough, it has too many particular cases, it may be so rich that it cannot be forced into a rigorous pedagogy, nothing ensures that all the "pathological" cases have been included in a catalogue, etc. (Thom, 1962).

Nevertheless, this topic has a great interest, both in mathematics and in modern applied science. It can be studied via a blended activity, i.e. pencil and paper work together with CAS work. Moreover it has many applications in science and engineering – caustics and wave fronts, i.e., Geometrical Optics and Theory of Singularities (Arnold, 1976), robotics and kinematics, rigid motion in 2-space and in 3-space, collision avoidance, etc. (Pottman & Peternell, 2000). This topic is also connected to the theory of billiards.

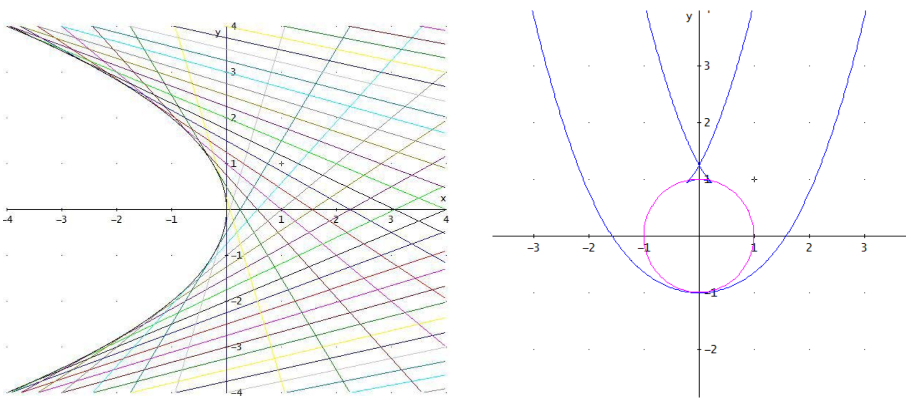
EXAMPLES

Let be given a family of plane curves by an equation $f(x, y, c) = 0$, where c is a real parameter. An envelope of the family, if it exists, is a curve tangent to every curve in the family. It can be shown that this envelope is the solution set of the system of equations $f(x, y, c) = 0$, $\frac{\partial f}{\partial c}(x, y, c) = 0$. Figure 1a shows the envelope of the family of lines given by the equation $x + cy = c^2$ (it is the parabola whose equation is $x = -y^2/4$. Figure 1b shows the envelope of the family of circles with radius 1 and centre on the parabola whose equation is $y = x^2$ (here the result has two components).

STRUCTURE OF A WORKING SESSION

We use infinitesimal methods for exploration and derivation of the equations, then algorithms based on Gröbner bases computations, first to solve the system of equations, and then to transform the obtained parametric representation into an implicit one.

This is a core issue in other fields (Pech, 2007). A similar, but non identical, scheme gives the mathematical educational frame for other topics in Differential



(a) (b)

Figure 1: Exploration of envelopes

Geometry, giving an extension of the curriculum (Dana-Picard, Mann, & Zehavi, 2014).

CORE ISSUES

Such a study provides an opportunity to discover new topics beyond the scope of the regular curriculum, sometimes together with applications to practical situations. New computation skills with technology may be developed, in particular for the experimental aspect of the work (e.g., exploring the existence of cusps, as in Figure 1b). For this, the availability in the software of a slider bar is a central issue. Moreover, ability to switch between different registers of representation may be improved, within mathematics itself (parametric vs implicit) and with the computer (algebraic, graphical, etc.). An interesting instrumental genesis may appear (Artigue, 2002) and mind-and-machine interaction is a central feature of the work.

An envelope may not exist (e.g., for a family of lines where the coefficients of the equations are affine functions of the parameter). These issues have been observed by the authors in sessions for in-service teachers at the Weizmann Institute of Science.

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Tutorial teaching to enable undergraduate students to make meaning with mathematics

M. Gozde Didis[1] and Barbara Jaworski[2]

1 Gazıosmanpasa University, Faculty of Education, Tokat, Turkey, gozde.didis@gop.edu.tr

2 Loughborough University, Loughborough, United Kingdom

The poster reports from a study of teaching in one small group tutorial with first year mathematics students in a UK university. The aim was to characterise teaching that was designed to support students' meaning making in mathematics. The research was developmental in that it contributed to the development of the teaching, as well as the tutor's knowledge in teaching. Two researchers, one who was also the tutor, collected and analysed data in the form of recordings of each tutorial and reflections of the tutor. Analysis was grounded in the data and a theoretical construct, the Teaching Triad, was used to support analysis. Findings showed the tutor taking a questioning approach, seeking to probe students' meanings, but needing to prompt frequently due to students very short and tentative responses. The study points to the difficulties in encouraging students to articulate their mathematical understandings.

Keywords: University mathematics teaching in small group tutorials; teaching triad; teacher questioning; difficulty in students' articulation of mathematics.

INTRODUCTION

In this paper, we focus on teaching in a small group tutorial in the first year mathematics programme at a UK university. We studied 10 tutorials in all and focus here on one tutorial which addressed questions in linear algebra. We are interested in studying relationships between the planning for teaching and teaching approaches, and the responses of students in so far as they gave access to the students' meaning making in mathematics (Jaworski & Didis, 2014). Our three basic research questions are:

1) What is the nature of the teaching manifested in the tutorials?
2) What student meanings can we discern and in what ways?
3) In what ways can we link (1) and (2) and what issues does this raise?

We took a sociocultural approach toward the research, recognising the many factors that underpinned activity in a tutorial, contributed to the interactions between students and tutor, and to mathematical meaning-making by the students. We see that, making connections to the worlds of mathematics and beyond, and processes of socialisation into culture and values are all central to how mathematics is taught and, associated with this, how students make meaning in mathematics.

APPROACHES TO OUR RESEARCH

The tutor (the second author of this paper) planned the tutorials and worked with the students. Her co-researcher (the first author) observed and audio-recorded tutorials and transcribed the recordings. Our conversations we sat together after tutorials were also audio-recorded; capturing the tutor's teaching reflections and our analytical discussion as two researchers. In working on our data we combined a grounded approach with the application of the 'Teaching Triad', a theoretical model which has been used extensively for analyses of teaching (e.g., Potari & Jaworski, 2002).

We read and reread the data, developing a coding scheme and seeking to make sense of the data in relation to our research questions. In addition we explored the nature of the teaching using the Teaching Triad. These forms of analysis were inter-woven to provide a rich characterisation of teaching practice. Our approach was two-fold: (a) reflections on the tutor's discerning of meaning-making in the tutorial in order to guide the teaching approach; (b) the discerning of meaning making through analysis of tutorial dialogue in order to link teaching with learning.

RESULTS AND CONCLUSION

Analysis revealed many aspects of teaching and ways in which the teaching approaches "in practice" related to the approaches planned. A finer grain of analysis showed that the codes relating to "tutor questioning" were the most prevalent. Through questions, the tutor not only invited and encouraged students' participation, but also tried to control the mathematical focus of the tutorial. She asked *prompting questions* to invite students to respond and articulate students' meaning, trying to promote both individual meaning and collaborative meaning of students. The following dialogue illustrates an example of the tutor's approach.

Tutor:	Now, Julia, what is the standard basis in R^3? *[Prompting-Q]*
S: (Julia)	A matrix *[SR-short/hesitant]*
Tutor:	Is a basis a matrix? *[Prompting & Probing-Q] [Mathematical challenge]*
S: (Julia)	No *[SR]*
Tutor:	Ok, what is a difference? *[Probing-Q] [Mathematical challenge]*
S: (Julia)	Vectors (*student smiles*) *[SR-short/tentative]*
Tutor:	If I asked to write down the standard basis in R^3, what would you actually write? *[Prompting-Q]*

Students' responses were mostly short, tentative and hesitant. This led us to deduce that discerning meaning making was difficult and time consuming. University cultures and practices, in which students are rarely expected to speak their mathematical thoughts or engage in discussion, result in such engagement being uncommon. Therefore, in relating the teaching approach to students' meaning making with mathematics, we need to address further: (i) what students expect from these tutorials and what their readiness is to deal with the planned teaching; (ii) what the tutor expects from the tutorial and wants to see from students, and how this can be achieved; and, (iii) what time factors influence the depth of students' meaning making.

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A geometric approach in teaching differential equations

Younes Karimi Fardinpour

Islamic Azad University, Ahar Branch, Department of Mathematics, Ahar, Iran, y-k-fardinpour@iau-ahar.ac.ir.

INTRODUCTION

This study aims at exploring the challenges that students encounter when they learn about differential equations. The study was conducted as a one-semester project with second-year students at one Iranian university. Students received nearly 9 hours of instruction in a differential equation course. In the task "What does the differential equation $dy/dt = y^2(4 - y^2)$ predict $y(t)$ for (a) $y(0) = 3$, (b) $y(0) = -1$ and (c) $y(0) = -2$" one of the students, Rogayeh, figured out the correct answers to parts (a) and (b) but she was unable to address the part (c). Why would she be able to do parts (a) and (b), but fail to part (c)? I had the opportunity to become aware of some challenges that students encounter when they learn about differential equations by conducting an in-depth one-on-one task-based interview with Rogayeh. She had arrived at correct conclusions, yet underneath lay erroneous ideas and conceptual gaps.

RESULTS AND DISCUSSION

As shown in Figure 1, Rogayeh does not plot the equilibrium solution curves. I asked "What is the limit of $y(t)$ for (a) $y(0) = 3$, (b) $y(0) = -1$ and (c) $y(0) = -2$?" She was unable to address the last of these questions, although she answered part (a) and (b) correctly as 2 and 0. So I asked "may you show me the equilibrium solutions in your solution curves?" She couldn't. So I changed my question to "What is an equilibrium solution?" She answered "… equilibrium points are −2, 0 and 2. … they are functions whit fixed values." Hopefully this answer was helpful, I continued "Excellent. Well, now where are these fixed value functions?" But she couldn't answer.

After some more conversation she said "If you ask me about $y(0) = 2$ then the answer is 2. But in $y(0) = -2$, solution curves are not convergent." By her description, I realized why she couldn't answer part (c). Because she saw the solution curves related to the $y(0) = -2$ were not converging (see Figure 1). This is a challenge that students encounter when they learn about differential equations. As a matter of fact, she thought that the differential equation $dy/dt = y^2(4 - y^2)$ predict $y(t) = 2$ for $y(0) = 2$ because solution curves are convergent. It shows that behind the students' correct answers there often lay an incorrect conception of equilibrium solution (Rasmussen, 2001).

Then it was a good time to ask "What do you predict if the $y(t)$ starts off at exactly zero?" Immediately she answered "zero" and explained that "by putting a zero instead of y in $y^2(4 - y^2)$, the answer is zero". I would expect Rogayeh's notion of equilibrium solution in zero to be a subset of her notion of a fix value function as she mentioned before. But this did not appear to be the case. Consistent with Zandieh & McDonald's (1999) findings, this incident suggests a conceptual challenge that may lie beneath a correct answer.

When I asked "What is the relationship between solution curves and the graph of the $f(y)$?" Rogayeh did not answer. So I asked "How do you see the relationship between solution curves and the phase line?" She answered "… solution curves are increasing or decreasing according as arrows in the phase line … arrows on the phase line are according as $f(y)$ sign…" Although she links the phase line and the solution curves, as well the sign of $f(y)$ in the sign chart and the phase line. But she is not able to connect the graph of $f(y)$ and solution curves.

This study is yielding new insights into the challenges that students encounter when they learn about differential equations. It seems more research into these challenges is needed so that new pedagogical strategies can be created that will help students overcome these challenges.

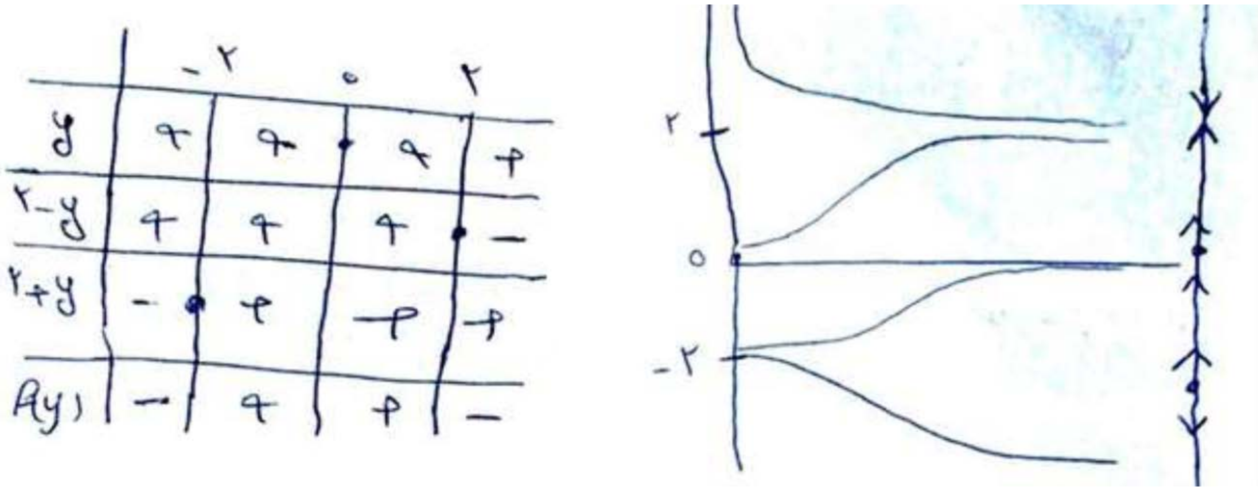

Figure 1: Sign-chart, solution curves and phase line in the task $dy/dt = y^2(4 - y^2)$

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Aspects obstructing or facilitating examination success for first year engineering students

Birgit Griese[1], Malte Lehmann[2] and Bettina Roesken-Winter[2]

1 Ruhr-Universität Bochum, Bochum, Germany, birgit.griese@rub.de

2 Humboldt-Universität zu Berlin, Berlin, Germany

When analyzing influences on examination success in reference to learning strategies, we found that Effort played an important role. This supports the importance of affective and motivational aspects in the teaching of mathematics.

Keywords: Learning strategies, effort, affect, motivation.

Engineering students often have trouble passing mathematics examinations. Some approaches to remedy this relate to cognition, others to affect. Another one, improving learning strategies, is promising as it relates to both aspects *and* provides a perspective for interventions.

THEORETICAL CONSIDERATIONS

The obstacles students are facing when starting a course in mathematics at university can be categorized in different ways (cf. de Guzmán, Hodgson, Robert, & Villani, 1998). When the emphasis is on learning strategies, interventions can address cognitive aspects, learning resources or the metacognitive level. The design research (van den Akker, Gravemeijer, McKenney, & Nieveen, 2006) project MP²-Math/Plus is based on the hypothesis that learning strategies are the key to influencing learning behaviour (cf. Dehling, Glasmachers, Griese, Härterich, & Kallweit, 2014). Apart from providing a selected group of students with special coaching, MP²-Math/Plus researches learning behaviour and collects data on personal circumstances and examination success. In order to cover different aspects of learning behaviour, the LIST questionnaire (Wild & Schiefele, 1996) was employed. This questionnaire is divided into eleven scales covering different cognitive, metacognitive and resource-related learning strategies. Thus it is appropriate for answering the research question: *What patterns are emerging when analyzing examination success in terms of these three measures: learning strategies, participation in a project on learning strategies, and gender?*

SELECTED METHODOLOGICAL DECISIONS

LIST factor scores were calculated (N=653), yielding values between 0 and 100. Mutliple linear regression identified factors with relevant influence. Descriptive statistics were computed (N>1750), differentiating between project participation, examination success, and gender.

RESULTS AND IMPLICATIONS

Internal reliability of the scales was satisfactory ($\alpha > 0.7$). The correlations between the factors were below 0.63 (highest between *Effort* and *Metacognition*), allowing for multiple linear regression with LIST factors as predictors and the examination result as outcome. This explained 23% of variance and highlighted the importance of *Effort* ($p = 0.000$), *Using Reference* ($p < 0.001$), and *Time Management* ($p < 0.01$), with standardized $\beta_{\text{Effort}} = -0.50$, $\beta_{\text{Reference}} = 0.24$ and $\beta_{\text{Time}} = 0.18$. Interestingly, all these are resource-related learning strategies, and among these three, only *Effort* improves examination achievement. Other supportive (though not significant) influences were found in *Organizing* (structuring and summarizing subject matter) and *Learning Environment*. The hypothesized relevance of *Metacognition* is not directly mirrored in the results, but becomes evident through its relatively high correlation to *Effort*.

Gender had no detectable influence on any kind of learning behaviour, although females participating in MP²-Math/Plus achieved significantly higher pass rates than their male counterparts, e.g. in 2013, 79.31% of female but only 68.29% of male participants passed the examination. On the whole, project participation helped examination success, with some variance among project years, the most substantial success being a pass rate of 72.86% among participants, compared to 57.97% for non-participants in 2013. Moreover, achievement was significantly higher ($p = 0.007$, $t(89.928) = -2.125$, $r = 0.219$), although the MP^2-Math/Plus concept postulates deliberately choosing students with below average prior performance.

These results indicate the importance of fostering aspects covered by the *Effort* scale. These items (e.g., *I do not give up even though the subject matter is very difficult and complex* or *Whenever I have planned a certain workload, I make an effort to master it*) relate to perseverance and motivation, describing attitude rather than aptitude. It seems advisable to continue focusing on affective and motivational aspects and not relying on cognition alone. In order to gain detailed insight, the next step will be qualitative analysis, e.g. through interviews.

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ENDNOTE

The poster presented and discussed at the conference can be found at http://www.ruhr-uni-bochum.de/ffm/Lehrstuehle/stochastik/griese.html.

Language and students' conceptions of logic in undergraduate mathematics

Mercy Kazima, Levis Eneya and Patrick Sawerengera

University of Malawi, Zomba, Malawi, mkazima@cc.ac.mw

Issues of learning mathematics through a foreign language have been discussed by many researchers, however, the focus has mostly been on primary and secondary schools and not university. This poster presents findings from a study that focused on university and investigated undergraduate students' meanings for mathematical terms used in logic. The sample was 89 second year university students that had studied the topic of logic in mathematics in Malawi where English is the language of instruction but not the students' mother tongue. The students were asked to write down what they understood by a list of mathematical statements such as: if and only if, implies, and, or, and not. Findings suggest that even at undergraduate level, the mother tongue influences students' conceptions of mathematical terms. The poster presents some examples of students' responses and discusses possible reasons for the responses in relation to the mother tongue. Furthermore, the poster discusses some implications for teaching logic specifically and teaching mathematics in general, to undergraduate students' learning mathematics in English as foreign language.

Key words: Language, logic, Malawi, mathematical terms, undergraduate.

INTRODUCTION

This paper reports a study that investigated students' conceptions of meanings of mathematical terms used in Logic, which are also used and applied throughout the undergraduate mathematics courses, and their meanings are often assumed to be understood by the students. The paper focuses on four terms: *or, implies*, *if and only if*, and *there exists*. The study took a constructivist view of learning that meaning is not transferred from teachers to learners but that meaning is constructed by the learners. This construction of meaning is influenced and affected by the learners' prior knowledge (Vygotsky, 1962). From previous studies, we expected that ordinary English and the students' local languages might influence their conceptions of the mathematical terms (Kazima, 2006). Furthermore, it is well known that students face difficulties with mathematical terms that have different meanings in ordinary English from mathematical English (Orton, 1992).

METHODOLOGY

Fifty (50) undergraduate students that have studied the topic of Logic in English, which is not their mother tongue, were asked to write down what they understood by a list of mathematical statements including the following four: (i) *or*, (ii) *implies*, (iii) *if and only if*, and (iv) *there exists*. The students were also given a short test where the statements were applied, These included *True* or *False* statements where they had to provide reasons for their answers, for example, '8 = 27 *if and only if* 2 = 3' Students responses were coded

	Term	Percentage of students that gave valid meaning	Percentage of students that gave valid example
1	or	14	12
2	implies	56	42
3	If and only if	4	8
4	There exists	38	36

Table 1: Percentage of students that gave valid meanings and examples

	Term	Common meaning	Examples
1	Or	One or the other	A baby can be a girl or boy
		Not sure	I am not sure if I should go or not
2	Implies	As a result	I cook implies I eat
3	If and only if	The only way	if and only if it stops raining then I will go to town
		Only condition	I will have money if and only if my mother comes
4	There exists	Something is discovered	There exists some numbers

Table 2: Some common responses for meaning of terms

and analysed using SPSS, and common responses for each of the terms were recorded.

FINDINGS

Table 1 shows the percentages of students that gave valid meanings of each of the terms and percentages that gave valid examples for the terms. Table 2 shows some of the common responses.

DISCUSSION AND CONCLUSION

The findings suggest that even at undergraduate level, language is an issue and influences students' conceptions of mathematical terms. This is important because it affects the students' understanding of Logic as a topic in mathematics and their ability to apply logical statements in mathematics in general.

The common responses that *or* is a 'choice' from two things that cannot occur together, and that *if and only if* is emphasis on the condition for something to happen may not be new, but worth noting because these terms are used throughout the undergraduate mathematics courses, and therefore important for students to have precise meanings.

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What does pedagogical content knowledge (PCK) in the context of university mathematics teaching mean?

Azimehsadat Khakbaz

Bu Ali Sina University, Hamedan, Iran, azimehkhakbaz@ymail.com

Although university mathematics teachers have an influential role on developing culture of teaching-learning mathematics in society, little is known about which kinds of teaching knowledge a future university mathematics teacher should have. Pedagogical content knowledge (PCK) provided a suitable theoretical base to study university mathematics teaching knowledge. This study aimed to find out what pedagogical content knowledge (PCK) in context of university mathematics teaching means. To answer this question, this research has done through grounded theory. Data were gathered by interviews from 27 university mathematics teachers (8 mathematics faculty members and 19 PhD students as future university mathematics teachers) and analysed through coding and categorizing. Data analysis revealed a detailed model which described PCK in context of university mathematics teaching.

Keywords: University mathematics teaching, university mathematics teacher, teaching knowledge development, pedagogical content knowledge (PCK).

RESEARCH PROBLEM

PCK was first introduced by Lee Shulman as a "missing paradigm" in teaching in the 1980s. Shulman (1987) identified seven domains of teacher knowledge, one of which is PCK: content knowledge; general pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners; knowledge of educational context and knowledge of educational ends. He described PCK as a special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding. He also noted that PCK itself includes knowledge of learners, knowledge of educational context, and knowledge of instructional materials.

One of the most important conceptualizations of PCK was proposed by Hill, Ball and Schilling (2008) and was labelled mathematical knowledge for teaching (MKT). These authors divided MKT into two categories: subject matter knowledge (SMK) and PCK. SMK was described as consisting of three domains of knowledge: common content knowledge (CCK), specialized content knowledge (SCK) and horizon content knowledge (HCK). Knowledge of content and students (KCS), knowledge of content and teaching (KCT) and knowledge of content and curriculum (KCC) are defined as being included in PCK.

The MKT model has provided a useful foundation in mathematics education at lower grade. Speer and King (2009) analyzed CCK, SCK and PCK in mathematics teaching at higher levels. Their research concluded with a set of questions that they believe are important for the research community to consider as investigations of teachers' knowledge more broadly and beyond school contexts.

Speer and colleagues (2015) demonstrate that the nature of SMK, especially CCK and SCK, for university teachers are different from that of school teachers. Therefore, conceptualization of PCK in university mathematics teaching needs more study.

RESEARCH METHODS

This research was conducted through the grounded theory approach. The study took place in Iran, a country with a centralized higher education system. Two universities were selected for gathering data. In each university, participants were 8 mathematics faculty members and 19 PhD students as future university mathematics teachers. Data of this study were gathered mainly through semi-structured interviews of the participants' experiences of teaching and learning mathematics at university. Interviews were transcribed and analyzed through coding and categorizing.

FINDINGS

Data analysis revealed that we could conceptualize PCK for teaching university mathematics through relationships between four main categories of knowledge:

- Knowledge about the context of mathematical concepts
- Knowledge about students' understanding of mathematics
- Knowledge about mathematics curriculum planning
- Knowledge about creating an influential mathematics teaching-learning environment

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Tackling the difficulties of the transition from school to university mathematics

Eirini Kouvela[1], Irene Biza[2] and Theodossios Zachariades[3]

1 Loughborough University, Loughborough, UK, e.kouvela@lboro.ac.uk

2 University of East Anglia, Norwich, UK

3 University of Athens, Athens, Greece

This paper reports on a study that investigates the difficulties encountered by first year undergraduate mathematics students during their transition from school to university mathematics. The research is a case study of five students who were followed regularly from the beginning until the end of the semester through questionnaires and interviews while they were attending a Calculus course. The results of the research revealed difficulties faced by undergraduate students regarding the content of the module and the social and academic context. In addition, the research suggests a categorization according to the approaches that students adopted in order to tackle these problems.

Key words: *Transition, university mathematics, first year undergraduates, calculus.*

This paper reports on a study that investigates the difficulties first year undergraduate students of a Greek department of Mathematics who attend a Calculus course face in their transition from secondary to tertiary education and how these students deal with these difficulties. The main research questions are: (a) Which difficulties do first year mathematics students face in their learning of Calculus during their transition from school to university? (b) To what extent do the social and academic environment affect the transition? (c) How do students deal with these difficulties? The results we report in this paper regard mainly the third question.

Transition problems from secondary school to university mathematics have been a recurrent issue which varies across different educational systems. Researchers note that there is a gap between school and university mathematics. School students study mathematics in a different way than the one required in university and as Clark and Lovric (2008) have indicated, mathematics courses at university focus on conceptual understanding whereas school mathematics tends to have more procedural characteristics. Advanced mathematical thinking requires students to develop working techniques that are necessary to understand and apply mathematical notions, definitions, theorems and proofs and this is a challenge for first-year students entering tertiary education (Hoffkamp, Schnieder, & Paravicini, 2013). Apart from factors related to mathematical content, other factors are also important when students are entering university. These include their mathematical background, the academic and social environment, and their study habits (Pongboriboon, 1992).

In the department of mathematics where this study was conducted, Calculus is offered in the autumn semester of the first year with four hours of lectures and two hours of tutorials. Attendance is not obligatory and students have the option to sit the exams just after the end of the teaching period in January and resit in July or September, if they fail, or transfer their assessment to the resit periods directly. Also, students have the option to drop out of the module and register in it again the following year.

The research is a case study of five students who were followed from the beginning until the end of the semester. Data collection was separated in two phases. In the first phase a questionnaire was handed out in a Calculus course consisting of general questions about students' profile (e.g., gender, grades in secondary school, etc.) and some mathematical tasks. Then five students were selected according to the criteria of gender, grades and responses to the tasks and were interviewed. The second phase of the research included three successive interviews that were spread across the first semester and were conducted by the first author. During the first two interviews students talked about their experience in the university, responded to a set of mathematical tasks and commented on their responses to these tasks. In the last interview students were asked about their overall experience in the Calculus module and their new life in university.

The results of this research revealed that students face various difficulties during their transition from secondary to tertiary education. Some of these difficulties concern the content of the taught module, in our case Calculus, and some others to the new social and academic environment that students have to adjust to. Analysis suggested three categories of students in terms of the approaches they used to tackle these difficulties. In all categories students begin their studies with great interest in mathematics. In the first category, students face difficulties mainly with the mathematical content. They consider the Calculus module very difficult. They become disappointed in the first weeks and they decide to transfer the module to the following year and pay attention to other modules which are easier to them: "It is not going very well because I am left behind, I said I would study on my own because I could not understand the first things we have done and then I just quitted and I thought of studying another module." In the second category students realize from the beginning of the semester the differences between school and university, such as the new environment, the content of the module, the way the course is taught and the study habits. Despite the difficulties, they study hard: "[...] I started dealing with it [module] more seriously, studying more and considering the whole situation in a more mature way". However this is not always effective due to their lack of mathematical understanding. In the third category, similarly to the second one, students face difficulties but mostly to the social and academic environment of university. The autonomy of university work challenges them mostly but gradually they adopt a new way of thinking. In the end they find a way to adjust to the new institution: "There are differences [from school] that make you feel insecure, it is a new environment that you have to integrate into, find your own path [...] in the beginning you feel like you are lost, but once you find your way everything works better."

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Typology of incoherence due to the didactic transposition in Quebec textbooks introducing abstract algebra

Ismaïl Régis Mili[1] and Isabelle Ascah-Coallier[2]

1 University of Montréal, Department of Didactic, Montréal, Canada, ismail.mili@umontreal.ca

2 University of Montréal, Department of Mathematics, Montréal, Canada ascahc@dms.umontreal.ca

INTRODUCTION

As shown in the Quebec Ministry of Education curriculum, the first notions of abstract algebra are presented in the linear algebra class, at CEGEP level (students of 17 to 19 years old). After a census of the 13 French CEGEP of Montreal region, which offer pre-university preparation for mathematics, we found out that four textbooks (regularly re-edited) are used by the majority of teachers as guidance for this class (Ouellet, 2002; Amyotte, 2009; Charron & Parent, 2005; Papillon, 1993, 2012). Unlike high schools, CEGEP are free to choose the didactic material they propose to their students; textbooks do not have to be approved by a Ministerial commission. However, for the sake of consistency, editors try to respect ministerial guidance and organise the textbooks following objectives given by the ministry. Notably, amongst those programme objectives (MELS, 2010, p. 5), we found that the most relevant to the class of linear algebra students were:

- identify a certain amount of ideas in link with the subject to compare, classify and evaluate them;
- link pertinent ideas in a logical order;
- building a coherent argument / reasoning / proof (loose translation).

Goal of the research

It therefore seems relevant for us to inspect to what extent the organisation of didactic sequences of these textbooks permits the attainment of the ministerial objectives. In reference to the aforementioned ministerial objective, we also want to study what are the valid domains of reasoning in the different textbooks. We propose the outline of a typology of incoherence revealed following the textbook analysis.

FIRST ANALYSIS AND RESULTS

Our analysis revealed that, in terms of style in which linear algebra is taught (Dorier, 1997), the way usually chosen by Quebec authors consists of building on previous notions, already introduced to the students, to present their generalisations [1]. However, when the notion with a high level of abstraction is presented, very few references are shown with a low level abstraction. We can observe here a first breach in the presentation since these references could serve as examples and these passages could allow to work on the comparison and classification of ideas as mentioned in the ministerial objectives. More generally, by making a crossing between the typology of abstraction levels and the notion of didactic transposition (as in Chevallard, 1985), three categories of incoherence were identified:

1) Breach in the generalisation process: This includes elements such as the circular generalisations (lack of distinction between the didactic transposition levels) and the lack of examples (either prior to introduce generalisation, or to verify the coherence of the previous knowledge of low level of abstraction in this generalised theory). We also include here breaches in the deductive structure (presentation order of notions.)

2) Unnecessary generalisation and presentation of mathematical concepts of low relevance within the deductive sequence: This includes elements such as the introduction of concepts which are never revisited or in only in an artificial and

technical way, solely to achieve technical ends. This category can also generate breach in the deductive structure.

3) False concept and creation of didactic obstacles (in the sense defined by Brousseau, 1998): This includes elements such as the introduction of mathematical concepts by failing to mention their range of validity. These concepts turn out to be incorrect once they are evaluated out of this implicit range. For example, to consider a vector and a scalar quotient only makes sense in real vectorial spaces.

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ENDNOTE

1. In reference to a term used by Aristotle (and notice by Cleary, 1985), notions from low level of abstraction are generally used as foundation for the presentation of high level abstraction notions.

An international comparison between final secondary assessments: Detected differences through an a priori analysis of tasks

Charlotte Derouet[1], Carolina Henríquez[2], Romina Menares[2] and Monica Panero[3]

1 Université Paris Diderot, LDAR, Paris, France

2 Pontificia Universidad Católica de Valparaíso, Valparaíso, Chile

3 Università degli Studi di Torino, Dipartimento di Matematica, Torino, Italy, monica.panero@unito.it

This paper presents a comparative study of the final secondary assessments in France, Italy and Chile, through the a priori analysis of tasks involving functions. We detect particularities in each expected resolution process, and we focus more specifically on the "availability" level of a piece of knowledge, which corresponds to activating it without any help or indication. At the core of the comparison, we distinguish between two degrees of availability: as "object" and as "tool".

Keywords: Function, international comparison, transition high school/university.

CONTEXT OF THE STUDY

The aim of this paper is to have insight into what is expected from students at the end of secondary school. This can be a way to study the prerequisites "at the entrance" to university courses in mathematics. Analysing the tasks proposed in the secondary school final assessment seems to be a first possible approach. In this study, we want to present a comparison between three countries: Chile, France and Italy. France and Italy are two European countries with similar school systems, which have already been the object of a comparative study about the teaching of functions (Derouet & Panero, 2014). Including Chile allows us to enlarge and to enrich the comparison by considering a country with a different educational system. The Chilean secondary school ends at grade 12. The final assessment, called "Prueba de Selección Universitaria" (PSU), is a test that ranks students for accessing to university. However, it does not evaluate all the notions studied at secondary school. The PSU test in mathematics is a multiple-choice test. In France, the secondary school goes from grade 10 to grade 12. At the end of this period, students have to pass an exam, called "Baccalauréat", which is compulsory to enter university. The "Baccalauréat" varies according to section. In the scientific section, the exam in mathematics is composed of four exercises, with detailed questions. Finally, the Italian secondary school ends at grade 13 (so it lasts one year more than the Chilean and the French secondary school). The final exam is called "Maturità" and, as in France, it is necessary to access to university. In scientific section, the exam in mathematics consists of two problems, of which only one has to be solved, and 10 questions (the candidate chooses and solves 5 of them).

Clearly, preparing students for the final assessment represents one of the main aims of the last year of secondary school in each country. In our study, we focus on the tasks involving functions. And we wonder what mathematical activity is expected from students at the end of the secondary school in Chile, France and Italy.

A PRIORI ANALYSIS OF TASKS

We focus on one representative task on functions for each of the three countries and, through an *a priori* analysis, we try to detect particularities in each resolution process. We partially refer to the methodology of analysis of tasks introduced by Aline Robert (1998). Specifically, we wonder if the question is open or closed, we focus on the activated frames (Douady, 1986), working frames and registers (Duval, 1995). Moreover, we consider the adaptations to do (introducing steps, choosing a method, recognising the modality of application) as well as the expected level of activation of knowledge (Robert, 1998). We focus on the "availability" level, which corresponds to activating knowledge without any indication in the statement. We can distinguish two degrees of availability. On the one hand, a certain notion/property can be recalled and employed as "object": for example, the memorisation of a formula to directly work on the involved notion. We call it "availability as object". On the other hand, a notion/property can be recalled and introduced by the students themselves as "tool", to solve a question that does not involve directly the notion. We call it "availability as tool". The degree of availability of knowledge is at the core of our comparison.

CONCLUSIONS

This analysis allows us to notice some remarkable differences between the assessments of the three countries. Our main result is the observation of a great dissimilarity at the level of availability and of autonomy expected from the students, linked to the degree to which the tasks are guided. Chilean students appear to be required to activate knowledge at a high level of availability as object. French students seem to be expected to mobilise some pieces of knowledge at the level of availability as tool, but the availability as object prevails, with little space left to autonomy. Italian students appear to be given more autonomy in solving tasks and in mobilising knowledge at a high level of availability as tool.

ACKNOWLEDGEMENT

This contribution is part of the research funded by Project ECOS-Sud C13H03.

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Student noticing of exponential and power functions in university financial mathematics

Natalie E. Selinski

Hochschule für Wirtschaft und Umwelt Nürtingen-Geislingen, Nürtingen, Germany
University of Kassel, Nürtingen, Germany, nselinski@hotmail.com

Exponential functions are critical for university study of financial mathematics, yet they persist as a challenging mathematical topic within finance and in the mathematics classroom. My research examines students' initial conceptions of exponential functions and student noticing regarding exponential functions in an inquiry-oriented instructional sequence taght in a university mathematical bridging course. My analysis identifies a strong tendency for students to initially not distinguish between power functions and exponential function, but, with carefully designed tasks, students were able to identify the characteristics of exponential and power functions that make these types of functions distinct.

Keywords: Exponential functions, student mathematical noticing, university students.

INTRODUCTION

Exponential functions are critical for the university study of financial mathematics, yet they persist as a challenging mathematical topic within finance (e.g., Stango & Zinman, 2009) and in the mathematics classroom (e.g., Strom, 2008). While researchers and educators alike traditionally perceive exponential functions as a remedial topic for university students, the multitude of courses covering "remedial" topics suggest students are still actively developing their conceptions of these topics during their university studies (Bausch, Biehler, Bruder, Fischer, Hochmuth, Koepf, Schreiber, & Wassong, 2014). In this paper, I add to the literature regarding exponential functions, by addressing two research questions. First, in the context of university financial mathematics, what are students' initial conceptions of exponential functions? And during the instructional sequence I developed, what do students notice about exponential functions?

SETTING AND DATA SOURCES

Data was collected in the summer semester of 2013 at a university in southern Germany during a voluntary remedial mathematics course running parallel to the students' required first semester financial mathematics studies. In particular, data was drawn from two sources: (a) a written survey on students' initial conceptions of exponential functions prior to the instructional sequence (16 participants), and (b) fully transcribed and translated video recordings of the remedial classroom (2–8 participants). The analysis in this paper focuses on one day of the instructional sequence, in which students explored the distinction between exponential and power functions. Only two students, Alisa and Heidi, were present in the class on this day.

RESULTS

An initial survey was used to identify students' initial conceptions of exponential functions. When asked "What is an exponential function?" 10 (of 15) students claimed it was function containing an exponent. When asked about their conceptions of exponential functions as used in financial mathematics, 7 (of 14) students stated that exponential functions are used "to construct cost, revenue, and profit functions," often giving as examples polynomials such as .

Following the results of the survey, I developed an instructional sequence that included tasks guiding students in distinguishing between exponential functions and power functions. Specifically students were prompted to model two scenarios for borrowing 100€, one with a fixed interest rate i and variable time n

(known as the Brother's Situation, represented by the formula $K_n = 100 \cdot 2^n$) and another situation with a variable interest rate i and fixed time n (known as the Cousin's Situation, represented by the formula $K_n = 100 \cdot (1+i)^2$).

Analysis of the transcript from the day that the instructional sequence was conducted drew on the Student Mathematical Noticing Framework (Lobato, Hohensee, & Rhodehamel, 2013). Lobato *et al.* define noticing as "selecting, interpreting, and working with particular mathematical features or regularities when multiple sources of information compete for one's attention" (p. 809). For the purpose of this paper, I worked with only two primary components of the Student Mathematical Noticing Framework: centers of focus, which are the properties, features, regularities or conceptual objects which students notice, and focusing interactions, which are the discourse practices including diagrams and talk that give rise to the centers of focus.

Three centers of focus were identified:

1) Positive input values lead to similar graphs showing increasing growth;

2) Including negative input values lead to dissimilar graphs, a parabola for the Cousin's Situation and an exponential curve for the Brother's Situation;

3) Differences exist in parameters and variables in algebraic representation when comparing the Brother's Situation and the Cousin's Situation.

This last center of focus was best summarized by Alisa: "For parabolas [points to Cousin's function]... the exponent is always known and for [points to Brother's function]... there it is unknown, the, the exponent." It was by identifying the difference of a "known" parameter in the exponent versus an "unknown" variable that permitted Alisa to distinguish between the Brother's exponential function and the Cousin's power function. This finding suggests that there is room, and necessity, for an expansion of Strom's (2008) Exponential Functions Framework, particularly to include how distinguishing from non-linear functions, especially power functions, is critical to students' conceptions of exponential functions.

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Mathematical literacy of students in teaching mathematics in the first year of studies

Soňa Švecová[1], Iveta Kohanová[2] and Janka Drábeková[3]

1 Slovak University of Agriculture in Nitra, Faculty of Economics and Management, Nitra, Slovak Republic, svecovas@uniag.sk

2 Comenius University in Bratislava, Bratislava, Slovak Republic

3 Slovak University of Agriculture in Nitra, Nitra, Slovak Republic

The aim of the project outlined in this paper is to launch a new performance tracker of students' mathematical performance. Our project consists of two parts – data collection and analytical assessment – both of which will happen over three academic years. The data collection will result in a detailed database of students' math grades. This will provide us with a solid basis for understanding the overall performance of our students and any links between students' secondary education background and their university grades. More broadly, we aim to gain more insight into approaches to students' motivation, and, over time, we aspire to track the impact of any changes to lecturing materials or assessment framework on students' performance. As lecturers this will also add an indirect feedback loop for the quality of our teaching. for students from various types of schools. We aim to publish our findings by 2017.

Keywords: Student performance tracker, mathematical literacy, motivation.

INTRODUCTION

The aim of the project outlined in this paper is to launch a new performance tracker of students' mathematical performance. This database will serve as a platform for analysis and understanding of determinants of student performance, their motivation, and background and indirectly we can quantify the impact of changes of syllabus. Whilst the tracking of student performance is common in Slovak pedagogical universities, we are the first group to research and propose a performance measurement tool in a setting of an economics faculty. The project proposal is focused on university education. We aspire to factor the relevant information into our ongoing live project. Our project consists of two parts – data collection and analytical assessment, both of which will happen over three academic years. The data collection will result in a detailed database of students' math grades. This will provide us with a solid basis for understanding the overall performance of our students and any links between students' secondary education background and their university grades. More broadly, we aim to gain more insight into approaches to students' motivation, and, over time, we aspire to track the impact of any changes to lecturing materials or assessment framework on students' performance. As lecturers this will also add an indirect feedback loop for the quality of our teaching.

RESEARCH PLAN

Our envisaged broader plan of action is the following: Our data collection begins in the current academic year of 2014/15, where we are able to track and store mid-term student scores, end of term exam grades as well as secondary education information. At the end of the current academic year, we will have sufficient data to draw simplified conclusions. We are particularly interested in the link between students' secondary school mathematics performance and their exam scores in our faculty. We are also keen on exploring the progression determinants of the students' first year mathematical performance: starting from mathematics admission test at our faculty, through any mid-terms test scores up to the final exam grade.

We have developed theoretical hypotheses about determinants of students' university performance using a variety of available literature sources.

Our statistical analysis of the collected data will provide a test for validity of these assumptions in the context of a Slovak university setting.

We also aspire to be able to draw broader practical conclusions about student motivation based on factors such as composition of tutor groups, student geographic background and the type of secondary school and the graduation score from mathematics. We would like to identify the motivation of students by short pre- and post- questionnaires or interview from a random sample of students. Research (Anderson, 2007; Rock, Gregg, Ellis, & Gable, 2008;) suggests that, if we would like to support differentiated classroom practices, we will have to continually assess, reflect and adjust content, process, and product to meet students' needs. On a more personal note, our own competence as lecturers can be greatly improved by having more data points and statistical evaluation at hand.

In the academic year 2015/16, we will use the same data collection and analytical approach and this time with a new set of first year undergraduates and the same process will be repeated in 2016/17 as well. Of note, the students at our faculty only have first year undergraduate mathematics; because after this first year of study they have only optional mathematics and statistics is provided by the department of statistics at our university. Hence our data set is different each academic year.

RESEARCH EXPECTED OUTPUTS

With a three-year data set, we will then be ready to draw more robust conclusions and identify the causality between mathematical performance and prior education background as well as the teaching framework at our faculty. We have to be very careful when working with data. For example, we need to be attentive to the complexities generated by the type of secondary school students come from. However, we would propose creative uses of pedagogical propositions that accommodate for differentiated learning. Tomlinson (1999) offers a caveat: "For all its promise . . . effective differentiation is complex to use and thus difficult to promote in schools. Moving toward differentiation is a long-term change process" (p. 6). It would be simpler and less invasive form of introduction of specialized mathematical seminars for students from various types of schools.

We aim to publish our findings by 2017, with a publication that highlights the determinants of the quality of education in our university.

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Student understanding of linear independence of functions

Megan Wawro and David Plaxco

Virginia Tech in Blacksburg, Blacksburg, USA, mwawro@vt.edu

We present findings regarding student understanding of linear independence of vector-valued functions, a concept common in linear algebra and differential equations. Data were collected from written homework questionnaires and a paired interview, prior to any formal instruction about the definition of linear independence of functions. Grounded theory was used to categorize student approaches, resulting in five main categories: Function Combination, Focus on t Values, Focus on Graphs, Focus on Scalars, and Previous Rule. Results suggest that students' transition from linear independence of vectors in Euclidean space to linear independence of functions in function space is not trivial and that fostering an object view of function may facilitate students more readily operating with functions as elements of a vector space.

Keywords : Linear algebra, linear independence, funcion space, Euclidean space.

Linear algebra and differential equations are important courses for mathematics and engineering students, and research shows that students tend to struggle with these courses (e.g., Dorier, 2000). While a growing body of literature exists about student understanding of linear independence of vectors in $\mathbb{R}^n$, few empirical studies report on student understanding of linear independence of functions. In one report, Harel (2000) contended that students might struggle to determine A = $\{x, x^2, x^3, x^4\}$ is linearly independent if they "have not formed the concept of function as a mathematical object, as an entity in a vector space" (p. 181). The focus of this study is on how students make sense of linear independence of vector-valued functions, a concept common in linear algebra and differential equations. In particular, we investigate students' initial notions of how linear independence in Euclidean space might extend to linear independence in function spaces.

THEORETICAL FRAMEWORK AND METHOD

This study is framed by Cobb and Yackel's (1996) emergent perspective, which coordinates the individual cognitive perspective of constructivism and a sociocultural perspective based on symbolic interactionism. Within our analysis, we assume that learners acquire knowledge from their daily experiences, that prior conceptions affect interaction with new ideas, and that knowledge structures are contextually dependent.

Data were collected during the Fall 2012 semester in an honors linear algebra and differential equations course for first-year mathematics or engineering students. Data sources include a semi-structured pairwise interview and written responses to two homework surveys collected prior to students' encounter with the formal definition of linear independence of functions. This timing allowed us to capitalize on a unique opportunity to investigate how students initially make sense of linear independence of vector-valued functions with only their prior understanding of vectors in $\mathbb{R}^n$. Questions analyzed in this study, written exactly as they were presented to students, are given in Figure 1. Grounded theory (Strauss & Corbin, 1990) was used to analyze student responses, which led to five categories of student approaches.

SUMMARY OF RESEARCH RESULTS

We analysed two aspects of student responses: their problem solving approach and their answer for if the given functions were linearly independent (LI), linearly dependent (LD), or something else. For the first aspect, we found student approaches could be sorted into five main categories: *Function Combination, Focus on t Values, Focus on Graphs, Focus on Scalars,* and *Previous Rule*. The *Function Combination* approach, for example, which is typified by a primary focus

1. Consider the functions $f(t) = 2$ and $g(t) = \sin t$ with $f\colon \mathbb{R} \to \mathbb{R}$ and $g\colon \mathbb{R} \to \mathbb{R}$. Would you say these functions are linearly dependent or independent for all $t \in \mathbb{R}$? Explain.
2. Consider the functions $F(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $G(t) = \begin{bmatrix} \sin t \\ 0 \end{bmatrix}$ with $F\colon \mathbb{R} \to \mathbb{R}^2$ and $G\colon \mathbb{R} \to \mathbb{R}^2$. Would you say these functions are linearly dependent or independent for all $t \in \mathbb{R}$? Explain.
3. Consider the functions $F(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$, $G(t) = \begin{bmatrix} t^2 \\ 2 \end{bmatrix}$, and $H(t) = \begin{bmatrix} t^3 \\ 0 \end{bmatrix}$ with F: $\mathbb{R} \to \mathbb{R}^2$, G: $\mathbb{R} \to \mathbb{R}^2$, and H: $\mathbb{R} \to \mathbb{R}^2$. Would you say these functions are linearly dependent for all $t \in \mathbb{R}$? Explain your reasoning.

Figure 1: Questions analysed about linear independence of vector-valued functions

on whether or not the functions could be written as a linear combination of one another, supported the correct answer (LI) across the three questions. The *Focus on t Values* approach, which never supported a correct answer in this data, is characterized by first evaluating the functions at specific t values and then making conclusions about vectors in $\mathbb{R}^n$ rather than about functions in a function space. For the second aspect, we categorized student answers for each question. For instance, for Question 2, twelve students stated correctly LI, six stated LD, and four stated LI at some t values and LD at others. A description of all results, with examples, can be found in the poster presented and discussed at the conference at meganwawro.com/presentations.

We acknowledge that the students' ways of reasoning captured in both their approach and final answer may have been influenced by the formulation of the questions in Figure 1. For instance, the phrase "for all $t \in \mathbb{R}$" was included to align with the formal definition for linear independence of functions the instructor had used; however, this may have led to more approaches aligning with the *Focus on t Values* category than if the questions had not included that phrase. Similarly, students who stated linearly independent at some discrete t values but dependent at others may have been influenced by the inclusion of that phrase.

This preliminary work illustrates that students' transition from linear independence of vectors in $\mathbb{R}^n$ to linear independence of functions in function spaces is not trivial. The *Function Combination* approach, however, not only led to correct answers but also seemed within reach for these novice students; as such, it deserves consideration as a fruitful approach to be leveraged as students encounter linear independence of functions. This further suggests an importance of fostering an object view of function so that students can more readily operate with them as elements in a vector space.

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TWG15
Teaching mathematics with resources and technology

Introduction to the papers of TWG15: Teaching mathematics with resources and technology

Jana Trgalová[1], Alison Clark-Wilson[2], Iveta Kohanová[3] and Mirko Maracci[4]

1 University of Lyon 1, Lyon, France, jana.trgalova@univ-lyon1.fr

2 University College London, Institute of Education, London, UK

3 Comenius University, Faculty of Mathematics, Physics and Computer Science, Bratislava, Slovakia

4 University of Pavia, Pavia, Italy

The technology working group continues to increase in size since its inclusion at the first CERME congress in 1999. During CERME 9, for the first time the group was divided from the beginning, giving birth to two separate groups: TWG15 focusing on issues related to the teaching mathematics and teacher education and professional development, and TWG16 focusing on students' learning with technologies and software and task design issues (see Weigand, Lokar, Robutti, & Sinclair, 2015).

TWG15 builds on the legacy of the group work at previous conferences. A number of important issues related to technologies and resources and their use by teachers and teacher educators emerged from the group discussions, such as the need to elaborate specific methodologies for analysing and evaluating the efficiency of teacher education programs, or the construction of models that facilitate analyses of the evolution of teachers' practices related to their ICT use. The research presented in the group contributions tended to be: local, focusing on a particular aspect of teaching mathematics; short-term; and often conducted in controlled laboratory conditions, which prevented general conclusions being drawn about the benefits of ICT in mathematics education. The group concluded that it was necessary not only to learn more about "real" uses of ICT in classrooms and beyond, but also to understand why ICT is not used and to conduct long-term studies with "ordinary" teachers in "ordinary" classes in order to explore the impact of the ICT on students' learning and on teachers' practices (Trgalová, Maracci, Psycharis, & Weigand, 2013).

The call for contributions thus proposed to deepen the community's understanding of these issues by addressing themes such as: the specific knowledge and skills required for an efficient use of ICT; teacher education programs embedding these knowledge and skills and assessment of their impact on teachers' practices; theoretical and methodological approaches to the evaluation of the evolution of teachers' practices, "best practices" with using technologies, among others.

WORKING GROUP IN A FEW NUMBERS

The group involved 31 participants from 13 countries: Cyprus, Czech Republic, France, Germany, Greece, Israel, Italy, Poland, Portugal, Slovakia, Sweden, Turkey, and United Kingdom. 17 papers and 6 posters were presented and discussed during the working group sessions.

REPORT OF THE WORKING GROUP DISCUSSIONS

For each session, the group Co-leaders defined a particular theme that was discussed in relation to a group of a few papers and posters. The discussions were framed by two or three questions raised after the paper and poster presentations. In what follows, we give a brief overview of the themes discussed and the main outcomes. We conclude with an outline of some emerging perspectives for consideration at the next conference.

Teacher professional knowledge and frameworks for its analysis

Various frameworks are used to analyse teachers' practices with technology and digital resources in math classrooms: *Bozkurt and Ruthven* refer to the Structuring Features of Classroom Practice (SFCP) (Ruthven, 2009) that "identifies key aspects of the craft of teaching that indicate the corresponding professional reasoning and craft knowledge that teachers must develop about these aspects in order to successfully incorporate new technologies". *Robová and Vondrová* draw on the technological pedagogical content knowledge (TPACK) framework (Mishra & Koehler, 2006) to analyse teachers' specific skills needed for work with dynamic geometry software. *Rocha* introduces a new framework, Knowledge for Teaching Mathematics with Technology (KTMT) that she applies to analyse teachers' practices with the use of graphing calculators.

The participants were invited to discuss the specificities of these frameworks and their usefulness both for the observation of teachers' practice and knowledge and for structuring teacher education programs.

Fostering creativity in mathematics

Creativity in mathematics is a new issue raised within three contributions to TWG15. *Papadopoulos Papadopoulos, Barquero, Richter, Daskolia, Barajas and Kynigos* raise the issue of the design of resources fostering the development of students' creative mathematical thinking (CMT) based on teachers' representations of CMT. *Kynigos and Kalogeria* propose an analysis of a collaborative design of resources for CMT within a specific technological environment. *Jančařík and Novotná* discuss scaffolding strategies in an e-learning mathematics course attended by gifted students and their effectiveness in supporting the students' problem solving.

These contributions promoted the issue of how ICT supports creativity in the students. Two main aspects have been highlighted in this respect: encountering different registers of semiotic representations through technology, and linking communication about mathematics between teachers, students and students and teachers.

Design, appropriation, orchestration of teaching situations

Contributions to these issues include both studies on local, short-term projects focusing on particular aspects of teaching mathematics alongside studies involving teacher development within large-scale projects concerning mathematics teaching with technology. *Benacka and Ceretkova* present the results of a survey evaluation of a course involving 28 pre-service mathematics teachers centred on the use of spreadsheets. *Sollervall and de la Iglesia* investigate how a co-design methodology can support teacher's orchestration of a didactical situation aimed at fostering so-called "logos-oriented discussions" among students and between students and teachers. *Turgut* presents the results of a pilot study of a larger research project involving the design and orchestration of teaching interventions within a linear algebra course with the use of ICT. *Clark-Wilson, Hoyles and Noss* elaborate a conceptual framework and methodological approach for research aiming at evaluating the success of the professional development part of a large-scale intervention. *Fahlgren*'s study aims at identifying those elements of activities involving students in scaling coordinates system in dynamic software that can affect the process of instrumental genesis. *Lavicza, Juhos, Koren, Fenyvesi, Csapodi, Kis and Mantecón* outline the theoretical framework, the different stages and highlight initial results from the first phase of a Hungarian national project promoting technology integration into Hungarian schools.

The theme of design, appropriation and orchestration of teaching situations appeared a really multi-faceted one; hence the issues proposed for the discussion concerned a lot of different aspects: the educational objective which can be pursued through the use of ICT, and how the choice of educational objectives influences the teacher's use of ICT in her/his practice; the principles which can inform the design of tasks or teaching situations and their "teach-ability"; what is needed from theories to inform the design of teaching activities, situations or sequences centred on the use of ICT; and the conditions under which short, local experimentations could be scaled, and the methodology through which scaling can be monitored and evaluated.

Assessment issues

Within the context of TWG15, the term assessment, and the role of technology within assessment, incorporates two perspectives: the assessment of teachers' individual knowledge and practice concerning their classroom technology use; and technological tools and approaches to facilitate the summative and formative assessment of students' learning. From the perspective of teachers, the paper by *Karatas and Tutak* describes a Turkish study that adopts the TPACK M Scale by Handal, Campbell, Cavanagh, Petocz, and Kelly (2013) to assess a group of 138 secondary teachers' technological, pedagogic and content knowledge, raising important methodological questions concerning the reliability and validity of such scales. The remaining papers (and poster) focus on the assessment of students' learning. *Chenevotot-Quentin Chenevotot-Quentin, Grugeon-Allys, Pilet, Delozanne and Prévit* describe the design and uses of a diagnostic assessment tool *Pépite* for the assessment of algebra at different grade levels in France. At the classroom level, *Aldon* focuses on how *critical incidents* arising from the students' uses of *TI-Nspire* technology provide the context for teachers' formative assessment of students' mathematical conceptions. In her poster contribution, *Juskowiak's* study concerns the assessment of students' mathematical outcomes with non-standard problem solving tasks involving the use of graphic calculators.

The discussions that arose from these contributions prompted questions concerning the different approaches to the assessment of students' learning with technology and the role of technology in supporting teachers to assess students' mathematical learning and, ultimately make decisions based on this assessment.

Teachers' support of students' conceptualisation in mathematics

Contributions to this issue explore the ways in which instrumented activities, orchestrated by a teacher, can benefit students in meaning-making of mathematics concepts. *Psycharis* presents two computational systems and discusses their potential in helping students conceptualize the notion of functions, based on both an *a priori* analysis of the systems affordances and on empirical studies of students' interactions with them. Likewise, *Stoppel* shows how the use of different media, providing different functionalities and command syntax, may lead the students to applying different methods in a problem solving activities. *Diamantidis, Economakou, Kaitsoti, Kynigos and Moustaki* explore the emergence of meaning of a concept of angle in 3D space in students working collaboratively in computer-based environment. Their study highlights the importance of technology supporting communication, collaboration and joint mathematical thinking. The learning potential of a video with mathematical content shared on the web is studied by *Palatnik*. His study evidences that Web2.0 resources can become a new source for generating interest in mathematics amongst a public audience. *Bagdat*'s study highlights the role of a teacher in helping students make sense of the concept of variable while working with spreadsheets.

The discussions of issues related to this theme brought forward two main ideas: the ability to assess affordances of technological tools as one of the basic skills teachers need to develop in order to successfully integrate technology in their practices, and the awareness of the importance of social learning, either in classrooms or informally through open Web2.0 resources.

CONCLUDING REMARKS AND PERSPECTIVES

The papers and posters presented and discussed within the group encompass a wide variety of research topics. Several convergent concerns emerged from the exchanges among the participants. First, an important role of teachers in the design of appropriate instrumented tasks was acknowledged, as the expression from one of the participants "*teachers as designers of students' learning with technology*" documents. A need was expressed to redefine teachers' skills enabling them to support students' collaborative learning not only in a classroom, but also while working online or offline, and subsequently to elaborate education programs embedding these skills. The discussions brought forward a complexity of meaning-making processes in mathematics, which requires from teachers the ability of analysing affordances of computer-based tools in order to avoid situations in which the students can succeed to solve a given problem without understanding the underlying mathematics. Also, articulating syntax-related issues inherent to digital tools with mathematics appears as an important aspect in teaching and learning with technologies, which is in line with the theory of semiotic mediation (Bartolini Bussi & Mariotti, 2008).

What perspectives can be outlined for the next CERME10 congress? There is still a need to develop a more comprehensive theoretical framework to address "old" but still topical themes, such as task design and methods for large-scale dissemination of "good practices" with digital technologies use. The role of technologies in formative assessment, of networked classroom technologies and e-learning appear among the emergent issues that require further theoretical and methodological development. Finally, some topics, which are under-represented, would deserve researchers' interest: touch technology, 3D technology, including 3D printers, virtual reality in mathematics education, Massive Online Open Courses (MOOCs) and web2.0 or web3.0 environments, and technology for educational special needs.

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TWG15

Research papers

The parable of the broken pencil or syntactic incidents and their consequences

Gilles Aldon

ENS de Lyon, EducTice-S2HEP, Lyon, France, gilles.aldon@ens-lyon.fr

When using technology, the translation of a mathematical concept into a particular technology brings syntactic difficulties that may lead to problems that can either be an excuse to withdraw from knowledge construction or a starting point for a mathematical reflexion. The boundary between the two attitudes is directly linked to the situation and to the integration of technology within the classroom. In this paper, I'll present the notion of syntactic incident and show in a particular class situations how students react in front of the consequences of such an incident.

Keywords: Technology, didactical incidents, formative assessment, theory of didactic situations.

INTRODUCTION

When Evelyne decided to begin her novel after a long reflexion about the characters she wanted to introduce, the context in which they will evolve and the general scenario of the story she wanted to share with the world, she sat down in front of her desk, laid down a sheet of white paper, verified that her eraser was on the desk and took a pencil. "Once upon a ti..." she wrote. And her pencil broke. She first searched for a pencil sharpener but she didn't find one (she remembered having given it to her nephew for the beginning of the school year); after a thundering cry of anger, she decided:

1) to write with a pen instead of a pencil even if it will not be possible to erase her writing,

2) to take a knife and to try to sharpen her pencil even if the lead will not be sharp enough for her writing,

3) to stop writing of the first page of her novel and to go shopping (buy a pencil sharpener, she thinks)

4) to sit down in her chair thinking about the incipit of her novel and remembering the most famous incipits of the literature... "Longtemps, je me suis couché de bonne heure...", "Call me Ishmael.", "The sun shone, having no alternative, on the nothing new", "I have never begun a novel with more misgiving"... [1]

5) to call a friend asking for a pencil sharpener,

6) to abandon this adventure which is not for her, even the things are against her,

7) to go and buy a computer, it will surely be safer to write her novel and she has always thought that it is high time to understand computer science,

8) ...

An incident occurred thwarting her intentions and this incident is directly linked with the tool she wanted to use with precise intentions: she wrote with a pencil because she wanted to erase her clumsiness and to keep a clean manuscript. The next episode of this story may be very different regarding the different attitude she would adopt.

In the first case, a direct consequence will be that she will not be able to erase the first draft. If she wants to keep the idea of a clean manuscript, she'll have to think more carefully to the sentences she'll write, otherwise she'll have to strike through her draft but, in doing so, she'll keep in memory her different trials. The choice of the tool she'll use has consequences on the organization of the content. In the same time she loose properties of the first tool, she gains new ways of writing.

The second attitude will modify the potentialities of her tool: she chose a pencil instead of a pen or a quill for some reasons, rational or irrational. But she surely had built for a while schemes of utilization of this particular tool and the combination of an unprecedented sharpness of detail and smoothly flowing movement will disappear.

In the third case, there is a break in the continuity of her work. She perhaps will come back to this particular state of mind that allows to begin the writing of a story, but for the moment, the incident stopped her progression. This important disturbance modifies deeply the next step of the story: instead of writing, she goes shopping and perhaps, she may come back to her task later.

The fourth case is also a break in the progression of her writing, but in that case, the new direction she takes, brings her in a deep reflexion about her own writing in the literature's world. She stays in the general context of her task but still work on the first sentence in another way she did initially.

In the fifth case, there is an externalisation of the procedure. The solitude of the writer is broken by an external component. She will have to explain and perhaps to justify her will of writing a novel, she'll surely have to summarize her first ideas, to introduce the theme of her novel... The first environment that she designed is deeply modified.

This sixth case is surely the most radical case where the incidents lean to an abandonment of the realization of the task. It is certainly because the project was not deep enough within the writer's mind, and using didactical words, that the devolution of the task is not made.

The seventh case is also a very radical transformation of the writing conditions. A result of the incident is to consider that the tool is not appropriated for the task and that the learning of a new way of writing is a precondition to complete the project successfully.

There are surely other developments of this story, but the lesson of this parable is that, regarding the conditions, the actors, the environment, a same event can lead to different scenarii that may modify the continuity of a story. Taking into account this parable in the field of mathematics learning can give interesting tools to analyse the mathematical activity of students in front of a specific task.

More theoretically, when students are learning in a digital environment that is the result of a construction of the teacher who has specific intentions and of students' knowledge regarding both technology and mathematics, the questions relative to the syntactic knowledge and interpretation are crucial for the construction of mathematical knowledge at stake.

More precisely, a question that can be addressed and that will be developed in the paper concerns the conditions that allow students to overcome syntactic incidents in order to transform them into mathematical questions and lean to knowledge construction.

THEORETICAL CONSIDERATIONS

In order to answer such questions, the first approach is to precise in which context the answers will be searched. In our case, the methodology is based on the theory of didactic incidents (Aldon, 2011) which took its founding principles in the theories of didactic situations (Brousseau, 2004) and of instrumental genesis (Rabardel, 1995; Artigue, 1997; Trouche, 2004; Drijvers & Trouche, 2008).

The theory of didactic situations takes, as a starting point, the relationship between an interaction of a player in a particular game with his/her *milieu*, and knowledge. The didactic situation is for an observer the modeling of the environment of the game, and is the game itself for the student. From the point of view of students, the environment of the game is integrated within the game and knowledge construction results from the interaction of the player with the entire environment including his/her own knowledge, mathematical situation given through a specific wording, the interactions with the teacher and available artefacts at this moment. Interactions of the player with the environment produce knowledge through the experiences build with the different parts of this environment. An important point is to consider in this environment the different available artefacts and the process of transformation of these artefacts into instruments useful for winning the game. The instrumental genesis theory that initially comes from ergonomic studies considers the artefact as a thing without any intentions. The use of the artefact in specific context transforms it slowly into an instrument that can be considered as the combination of an artefact and schemes of utilization. The integration of technology into the classroom is of the same nature and can be considered as a slow process in which the given technology (the artefact) becomes an instrument through the double movement of instrumentation and instrumentalization. The instrumentation is the process where the artefact modifies the subject's activity and the instrumentalization is the process where the subject modifies the artefact for her own use.

A didactic incident is defined as "an event of the didactical system that occurs sporadically, that is unforeseen, and that requires an appropriate answer of the actors" [2] (Aldon, 2011, p. 26). A didactical system is the implementation of a didactic situation in a particular context. In the different didactic incidents that have been picked out, some of them are directly linked to the digital environment of the didactic situation. A syntactic incident is a problem that occurs in the conversion from a register of representation in another. The term *syntax* refers to the units that make up rules to accomplish an action. For example, the drawing of a line in the "language" of paper and pencil can be done using a ruler and a pencil (place the ruler on the paper, place the pencil along the ruler and follow the ruler with the pencil) and, this same drawing in the language of a GDS will be to choose the menu create a point, to show with the mouse the place of the point, to click with the right button of the mouse, to choose the menu line, etc.

Especially, when there is a translation into a digital representation, syntactic incidents can *a priori* be triggered by two factors: the operation is not foreseen by the software and must be built or the operation is provided but doesn't work due to a misunderstanding of the syntax of the command or, is not known by the operator, in a particular environment. The perturbations that follow can be short in the case of an assistance provided by an actor of the situation (teacher, other student, element of the *milieu*...) or by understanding of the phenomenon by the subject (which is part of his/her instrumentation). But, they may also have long-term consequences, as shown in the parable of the writer who breaks her pencil: a disengagement of the subject locally (3) or globally (6) leading to a new instrumental genesis, a loss of devolution of the situation (3, 6, 7) that may lead to a disengagement of the student, a questioning of the relevance of the artefact in achieving the task (1, 2, 7) that may come out to a reflection about the pertinence of a tool relatively to a mathematical task or a mathematical concept, a modification of the working environment (5) and a reorganization of the way to approach the problem (4) whose consequence could be either the beginning of a new learning different from the teacher's intention or a new approach allowing knowledge construction.

We took the opportunity of a wide introduction of handheld technology in different classes to study the impact of such a technology on teaching and learning of mathematics. The example that is developed in the next paragraph comes from a class which is equipped with TI Nspire calculators. The teacher (J.L. in the following) worked in this class in a perspective of inquiry-based learning and proposed to his students (16 years old, in a scientific major) problems that allow a personal and collective reflection. The example of this paper has been designed, observed and discussed in the context of the European project EdUmatics [3] and illustrate a syntactic incident leading to different perturbations.

EXPERIMENTATIONS

The context of the experimentation is a mathematical problem that is proposed to the students. The general context of this problem is the study of the relationship between the distance of a walker to a given point according to his/her position on a path. In this study, we follow students who are searching the following problem:

Pjotr moves at constant speed along a square $ABCD$ with center O (intersection of the two diagonals) and given side L, starting from vertex point A.

Pjotr wants to describe how his distance from the center O of the square changes while he is moving along the square. How can you help him/her? (If you wish, you can choose any positive real number as the length L of the side of the square).

The observed group worked either on their own calculators or on a computer with the software TI Nspire. The different illustrations of this paper show the camera viewpoint for this group of four students who are working both with their own calculator and with the software available on computer. The first analyzed episode comes after a first episode of work with paper and pencil that leads students to explicitly calculate the relationship of the distance that Pjotr covered along the first side of the square and his distance to the center as shown on the Figure 1.

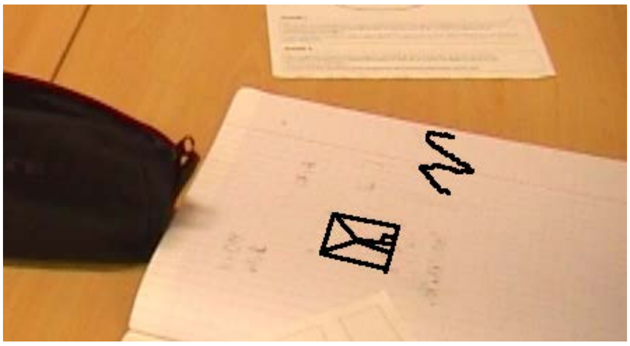

Figure 1: The first approach of the problem with paper and pencil

It is interesting to notice, that a freehand drawing of the behaviour of the function (which has been stressed on the Figure 1) has been made on the paper before any calculation: "*it'll be like that, it'll make a wave*" a student said drawing the freehand drawing. The second episode of work consists in a transposition of the mathematical situation to the software. The idea developed by the four students was to capture in the spreadsheet the values of the distance of *P* (the position of Pjotr on the square) to the centre of the square and to represent on a graphic these values according to the covered distance of point *P* from his starting point *A*. The syntactic incident comes from the translation of this idea in the language of the software. It is indeed possible to measure the distance of two points but there is no menu giving the distance from a point to another on a given path (here the square *ABCD*). The first trial was indeed to consider the abscissa of the graphic representation as the distance of *A* to *P*. The resulting graphics appeared to be in contradiction with the idea expressed previously and drawn on the paper. In that case, the syntactic incident associated with the previous mathematical reflexion led one of the students (S1) to think differently the parametrization of the point *P* in this task. Students are here in the fourth situation of the parable: the incident is the starting point of a new mathematical reflexion that comes to a definition of a piecewise function:

J.L.: And how did you do to obtain the second arch?

S1: In the next drawing, I made BP but each time I added six because six is the length AB...

In the same time and in front of the same situation, the second student (S2) working on his handheld tried to solve the same problem. However, his mastery of the technology or the mathematical reflexion are not sufficient to allow a transformation of his research strategy. In the contrary, most of his time was spent to try to solve the syntactic problem using the function "distance of two points" and without referring to the mathematical situation. The result was in contradiction with the former reflexion but this student tried to solve the problem without the good tool. He was exactly in the situation of the second choice of the parable: the tool is not adapted to a precise goal but can give the illusion to work well. The result of the work is illustrated on the handheld screen of the Figure 2. In the didactic situation, there was, in that case, what Margolinas (2004) called a didactic bifurcation where the problem that try to solve the student is no more the problem that the teacher wanted to be solved. It is also an illustration of the seventh choice of the parable: there is a knowledge construction or a trial of knowledge construction but out of the intentions of the teacher. Even if this knowledge participates to the instrumental genesis of the student, the lack of institutionalization leads the student to consider this trial as a failure. It's not sure that the different trials and errors done by this student lead to a better understanding of the technology and, in the contrary it could be a pretext to abandon this technology which is "*a waste of time*" as expressed by another student in an interview:

Interviewer: [...] and do you remember the time you said, oh no, I do not want any more this calculator?

Student: It was very early, yes because we had to go to the menu, go to this location there, finally, Click everywhere, we had quite a

Figure 2: Two different consequences of the same syntactic incident

> journey to make a calculation you could do very easily with our old calculator, in fact faster. [4]

In this paper, we only study one example of syntactic incident in a particular episode, but, it is by studying different observations that we have been able to determine the types of behaviour described in the parable of the introduction.

MAIN FINDINGS AND CONCLUSION

One syntactic incident and its consequences on two different students has been detailed in the previous section showing that the perturbations following an incident may differ regarding the choice that follows the incident. The question that we posed in the introduction was to analyze the reasons why one or another consequences occur and more precisely why and when an incident is a starting point of a re-organization of knowledge or not. Answers to these questions are interesting for the student as well as for the teacher in a perspective of formative assessment, defined by Bell and Cowie (2001, p. 536) as "the process used by teachers and students to recognize and respond to student learning in order to enhance that learning, during the learning". The European FP7 project FaSMEd [5] "aims to research the use of technology in formative assessment classroom practices in ways that allow teachers to respond to the emerging needs of low achieving learners in mathematics and science so that they are better motivated in their learning of these important subjects." The incident analysis is part of the toolkit that allows to understand better the behavior of students in a digital learning environment.

In the different observations in teaching and learning digital environment, the analysis of syntactic incidents and the perturbations that follow show different cause that can be clues for students and for teachers in order to understand when and why knowledge at stake in a particular mathematical situation is not reached.

Incidents lead to perturbations that prevent knowledge construction when:

- Technology is external to mathematics, that is to say technology is not included in the set of mathematics tools useful in the resolution of a mathematical problem for a given student at a certain moment.
- The knowledge of the syntax overcomes mathematical knowledge: in that case learning the syntax (in the sense defined in the second section) adds technical or conceptual difficulties that lead students to forget the mathematical notions at stake.
- The technological knowledge is more difficult that math knowledge at stake in a given situation: in that case, it is important to think about the adapted technology.
- Technology that is used doesn't supplies potentialities that are necessary in the mathematical situation most of the time because of a bad initial choice of technology.

In the contrary, incidents lead to construction of knowledge when:

- The class culture takes into account the experimental part of mathematics, and technology is used internally in different mathematical situations.
- The instrumental genesis is sufficient to give adapted technological skills in the learning situation.
- The knowledge of the syntax of a particular technology is either sufficiently natural or trained before having to use it in a complex mathematical situation

A continuation of this study will be to make this kind of analysis operational both for teachers in a perspective of formative assessment and for students in a perspective of auto-evaluation. This work is part of the FaSMEd project.

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ENDNOTES

1. Proust, *A la recherche du temps perdu*, Melville, *Moby-dick*, Beckett, *Murphy*, Somerset Maugham, *The razor's edge*.

2. "un événement du système didactique qui se produit de manière irrégulière, non prévu, nécessitant des acteurs une réponse appropriée" (Translated by us).

3. 50324-UK-2009-COMENIUS-CMP; European Development for the Use of Mathematics Technology in Classrooms, http://www.edumatics.eu

4. I: et c'est à quel moment, vous vous souvenez le moment où vous avez dit, ah non, je ne veux pas de cette calculatrice ?

E: C'était très rapidement, oui parce qu'il fallait aller dans le menu, aller dans cet endroit-là, enfin cliquer de partout, on avait pas mal de cheminement pour faire un calcul qu'on pouvait très bien faire avec notre calculette, plus rapidement en fait. (Aldon, 2011, p. 652) (Translated by us).

5. Grant agreement no: 612337, Improving progress for lower achievers through Formative Assessment in Science and Mathematics Education. http://research.ncl.ac.uk/fasmed/

Graphing functions and solving equations, inequalities and linear systems with pre-service teachers in Excel

Ján Beňačka and Soňa Čeretková

Constantine the Philosopher University in Nitra, Nitra, Slovakia, jbenacka@ukf.sk

The article presents the results of a survey on using Excel for graphing functions, solving equations and inequalities without and with a parameter, and solving systems of linear equations with pre-service mathematics teachers. The experimental group were 28 master students of teaching mathematics. The aim was to ascertain that the methodology can benefit teaching the topics and make mathematics more interesting.

Keywords: Spreadsheets, modelling, constructivism.

INTRODUCTION

The spreadsheet is a tool that enables access to ideas and concepts through a computational experiment without any need for programming. It allows using inquiry and heuristic methods and the immediate feedback provokes into experimenting and discovering.

The use of spreadsheets in mathematics education has been researched for decades, however, mainly on primary and lower secondary level (Healy & Sutherland, 1990; Rojano & Sutherland, 1993; Hošpesová, 2002ab; Haspekian, 2005, 2011, 2014; Ainley, Bills, & Wilson, 2005; O'Reilly, 2006; Wilson, 2006; Tabach, Arcavi, & Hershkowitz, 2008; Tabach & Friedlander, 2008; Tabach, Hershkowitz, & Arcavi, 2008; Drake, Wake, & Noyes, 2012; González-Calero, Arnau, & Puig, 2013; Watson & Callingham, 2013; Geiger, Goos, & Dole, 2014). There is a considerably smaller number of articles written on the use of spreadsheets in mathematics education at upper secondary level (Molyneux-Hodgson, Rojano, Sutherland, & Ursini, 1999; Sivasubramaniama, 2000; Neurath & Stephens, 2006; Forster, 2007; Topcu, 2011; Benacka & Ceretkova, 2013).

Research of Molyneux-Hodgson and colleagues (1999) with 16 to 18 years old students from England and Mexico showed that the mathematical culture in the country affected the choice of the means when the students solved tasks with spreadsheets. Research of Sivasubramaniam (2000) with student of age 14 and 15 showed that the use of spreadsheets had a positive impact on understanding Cartesian graphs. Forster (2007) researched a group of 17 and 18 years old girls on the use of technology when investigating the trend of a set of data from jewellery. Projecting graphs created in spreadsheets had a positive effect. Topcu's (2011) research with 16 years old boys showed that the boys who used spreadsheets to solve algebra tasks were considerably more confident as they were aware of the possibility of checking and correcting errors. Neurath and Stephens (2006) researched the effect of integrating Excel into teaching of algebra with 14 to 17 years old students. The students' opinion of the lesson was surveyed by a questionnaire. The result was that 69% were interested in solving tasks with Excel and understood algebra better, and 77% of the students enjoyed the course because solving tasks with Excel made algebra more interesting. Just 8% disapproved in both cases. Benacka and Ceretkova (2013) gave account on an experiment in which 16 to 19 years old students developed Excel applications to graph functions, find extremes, solve systems of linear equations, calculate the area of a planar figure by the rectangle method and Monte Carlo method, and simulate motion of a projectile in a vacuum. The students' opinion of the lesson was surveyed by a questionnaire. The result was that 100% of the students found the lessons interesting, 35% understood all and 65% understand most of the mathematics.

This article presents the results of a survey on using Excel in mathematics teachers education. The experimental group were 28 first year master students of

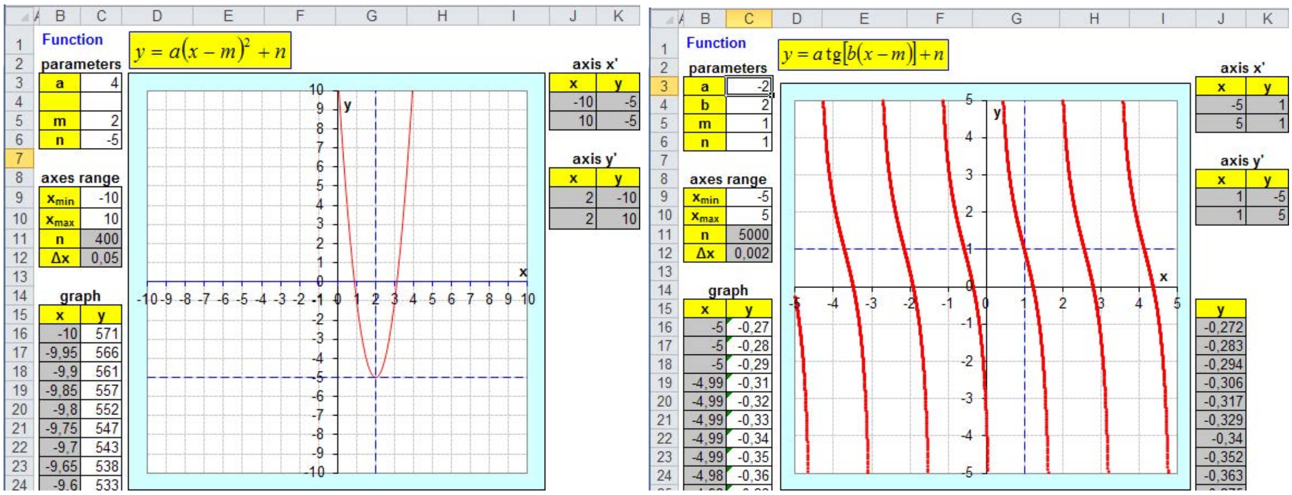

Figure 1: Graph of a function with definition domain R (left) and not R (right)

teaching mathematics. There were 6 men and 22 women in the group. The topics were Graphing functions, Solving inequalities, Solving equations with a parameter, and Solving systems of three linear equations with three variables. Aplications developed by the first author were used. The aim was to ascertain that the applications, the tasks and the presented teaching methodology can make teaching mathematics at upper secondary level attractive, benefit the teaching of the topics and contribute to the technological and pedagogical knowledge (TPACK) of the pre-service teachers. Each topic took a 90 minute lesson. A questionnaire survey with the following questions was carried out at the end of each lesson:

A) *The lesson was (1 = very; 2 = quite; 3 = little; 4 = not) interesting.*

B) *I understood (1 = all; 2 = most; 3 = little; 4 = nothing) of the mathematics involved in the activity.*

C) *The lesson contributed to my TPACK*

D) *(1 = a lot; 2 = quite a lot; 3 = little; 4 = not at all).*

E) *The applications benefit the teaching of the topic*

F) *(1 = a lot; 2 = quite a lot; 3 = little; 4 = not at all).*

G) *Developing the applications helps comprehend the core of the topic*

H) *(1 = a lot; 2 = quite a lot; 3 = little; 4 = not at all).*

I) *I am a man (1 = yes; 2 = no).*

Answers 1 and 2 in questions A–E were considered positive. Altogether, 96 questionnaires were evaluated. The survey summary is in the last section.

GRAPHING FUNCTIONS

The application on the left side of Figure 1 graphs functions if the definition domain is R. The graph is produced from 100 points in range B16:C116. Cell C12 contains the formula =(C10-C9)/C11. Cell B16 contains =C9, Cell B17 contains the formula =B16+$C$12. Cell C16 contains the formula =$C$3*(B16-$C$5)^2+$C$6. The two formulas are copied down as far as row 116. The application on the right side of Figure 1 graphs functions if the definition domain is not R. The graph is made over 5000 points in range B16:B5016 and J16:J5016. Cell C16 contains the formula =$C$3*TAN($C$4*(B16-$C$5))+$C$6. Cell M17 contains the formula =IF(ISERROR(C16);NA();C16). The two formulas are copied down as far as row 5016. The points that are out of the definition domain are skipped by function NA().

Lesson and survey: There were 28 students in the lesson, 6 men and 22 women. The students downloaded the template (Figure 1 with empty white and grey cells) from a website and developed the application with the teacher's help. The shape of the graph, definition domain, range and symmetry of elementary functions $y = x$, $y = x^2$, $y = x^3$, $y = 1/x$, $y = 1/x^2$, $y = \sqrt{x}$, $y = \sin x$, $y = \cos x$, $y = 2^x$, $y = \log_2 x$, $y = \tan x$ and $y = \cot x$, was discussed and visualized by the applications. The effect of the sign and absolute value of the parameters on the orientation, steepness, period and shift of the graph were checked. The following algorithm for graphing functions $y = af(a - m) + n$ or trigonometric

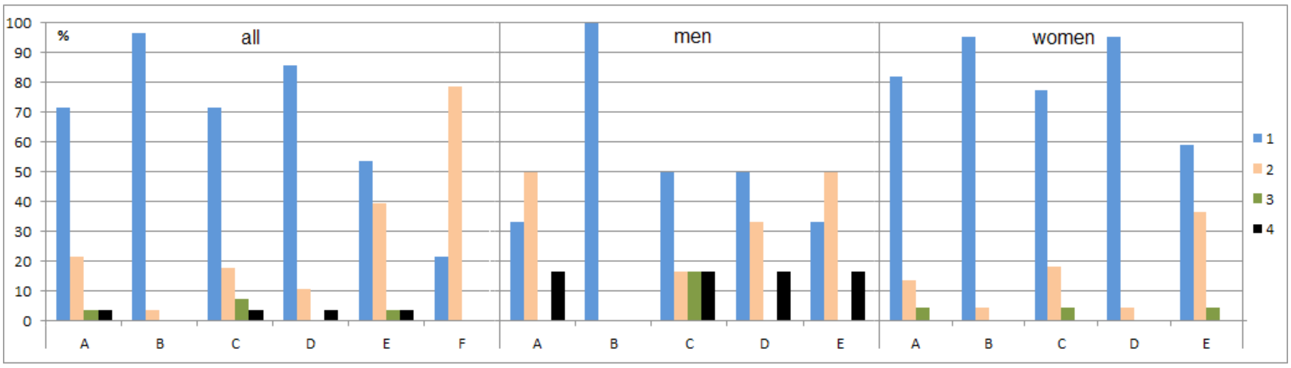

Figure 2: Relative frequency of the answers

$y = af(b(a - m)) + n$ by hand was deduced and exemplified by graphing the functions $y = (2x - 4)^2 - 5$ and $y = 2\tan(2 - 2x) + 1$ (Figure 1). At the end of the lesson, the students answered the questionnaire. The result is graphed in Figure 2.

The result is: (A) 92% found the lesson interesting (71% very, 21% quite); (B) 100% understood the topic (96% all, 4% most); (C) 89% had the feeling that the lesson contributed to his/her TPACK (71% a lot, 18% quite a lot); (D) 97% found the applications to be benefitting (86% a lot, 11 % quite a lot). (E) 93% were of the opinion that the developing helped comprehend the core (54% a lot, 39% quite a lot). The women found the lesson more interesting and benefitting than the men.

SOLVING INEQUALITIES

Solving an inequality analytically may be an intricate process that takes many minutes. With Excel, the solution is quick, transparent and credible. The inequality $\frac{|3-5x|}{x-2} \geq -x - 2$ is resolved by using the application in Figure 3 and 4. The solution is clear from the position of the graphs.

Lesson and survey: There were 21 students, 5 men and 16 women. The analytical solution to the inequality was quickly gone through. Then the students developed the application from the application on the right side of Figure 1 and resolved the inequality in two ways. First, the intersection point was found by iteration (Figure 4). The maximum and minimum of axis x (cells C9:C10) were set up to be close to the intersection point from both sides. Then, the x coordinate of the intersection was found in the first column of range B16:D5016 in the row in which the values in the second and third column were equal (cell B3090). The other way of solving was by using Goal Seek (Figure 3, range K12:N13). The solution is $x \in [-0.19, 1.14] \cup (2, \infty)$. The exact solution is $x \in [\frac{5-\sqrt{29}}{x-2}, \frac{-5+\sqrt{53}}{x-2}] \cup (2, \infty)$; however, the bounds have to be converted into decimal numbers for practical use, which is the form shown above. At the end, the students answered the questionnaire. The result is graphed in Figure 5.

The result is: (A) 96% found the lesson interesting (67% very, 29% quite); (B) 100% understood (90% all, 10% most); (C) 96% had the feeling that the lesson contributed to his/her TPACK (67% a lot, 29% quite a lot); (D) 100% found the applications to be benefitting (71% a lot, 29% quite a lot). (E) 100% were of the opinion that the developing helped comprehend the core (81% a lot,

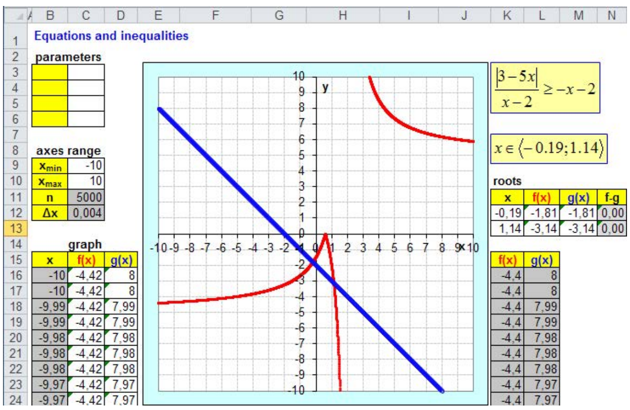

Figure 3: Solution with Goal Seek

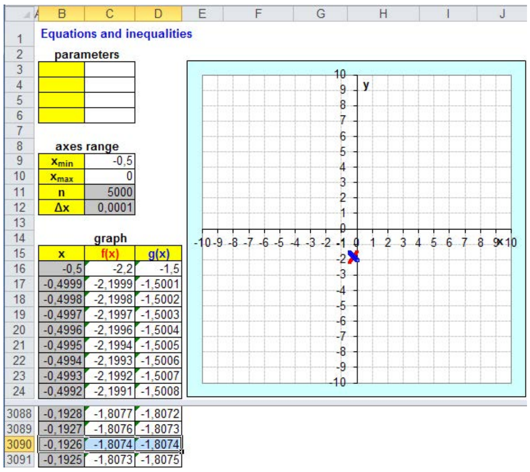

Figure 4: Solution by iteration

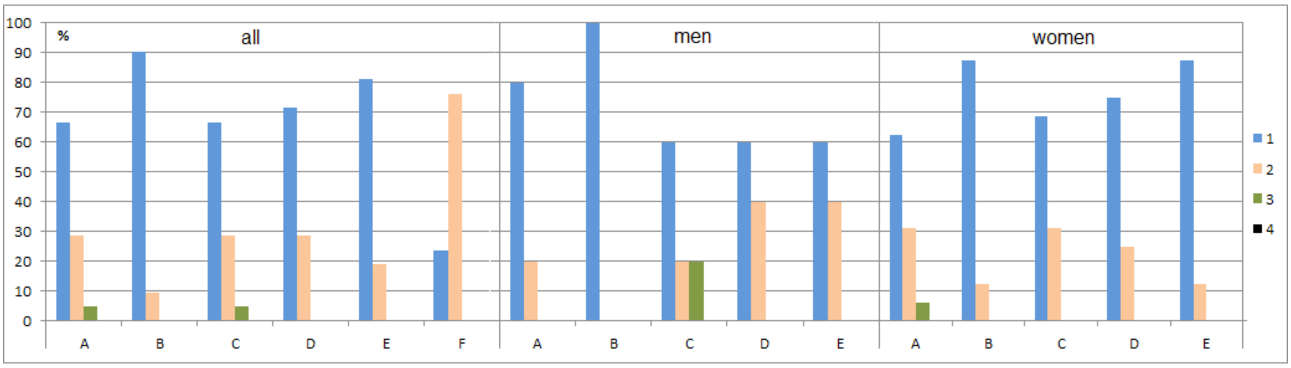

Figure 5: Relative frequency of the answers

19% quite a lot). The men found the lesson more interesting while the women found it more benefitting.

SOLVING EQUATIONS WITH A PARAMETER BY USING ANIMATION

Solving an equation with a parameter is a hard task at upper secondary level. The graphical interpretation helps. If the parameter is changed quickly, the relation between the parameter and the solution can be easily revealed. The left side of the equation $(p + 1)x^2 + px + |p| = 0$ is graphed in the application in Figure 6. Parameter p is controlled by a spinbutton. Clicking it and holding down animates the graph. The positions at other values of parameter p are depicted in Figure 7. The solution is:

$$p \in (-\infty, -1) \cup (-1, -0.8): \quad x = \frac{-p \pm \sqrt{p^2 - 4(p+1)|p|}}{2(p+1)}$$

$$p = -1: \quad x = 1$$

$$p = -0.8: \quad x = 2$$

$$p = 0: \quad x = 0$$

$$p \in (-0.8, 0) \cup (0, \infty): \quad x \in \{\}$$

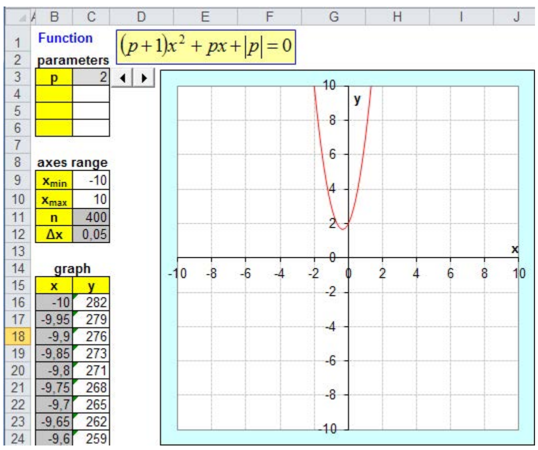

Figure 6: Position of the graph at p = 2

Lesson and survey: There were 23 students, 4 men and 19 women. The analytical solution to the equation was quickly gone through. The students developed the application from the application on the left side of Figure 1. They found that the graph is in a special position if p = 0, −0.8 or −1. The solution at these boundary values was obtained by substitution. The solution for the values inside the intervals given by the boundary values was obtained by applying the quadratic equation. At the end, they answered the questionnaire. The result is graphed in Figure 8.

The result is: (A) 87% found the lesson interesting (61% very, 26% quite); (B) 96% understood the topic (83% all, 13% most); (C) 83% had the feeling that the lesson contributed to his/her TPACK (57% a lot, 26% quite a lot); (D) 83% found the applications to be benefitting (65% a lot, 18% quite a lot). (E) 87% were of the opinion that the developing helped comprehend the core (65% a lot, 22% quite a lot). The women found the lesson more interesting and benefitting than the men.

INTERACTIVE SOLUTION TO A QUADRATIC EQUATION AND TO A SYSTEM OF THREE LINEAR EQUATIONS WITH THREE VARIABLES

Solving quadratic equations and systems of three linear equations with three variables are skills that are often applied in solving tasks at upper secondary level. The following task requires applying both skills:

A projectile was fired in a vacuum. Three points of the trajectory were detected by radar. The trajectory is a parabola. Find the points of shot and impact.

Substitution of the coordinates of the points in the equation $y = ax^2 + bx + c$ yields a system of three linear equations with variables a, b, and c. In Figure 9, the system is solved by using the Gaussian elimination method (GEM).

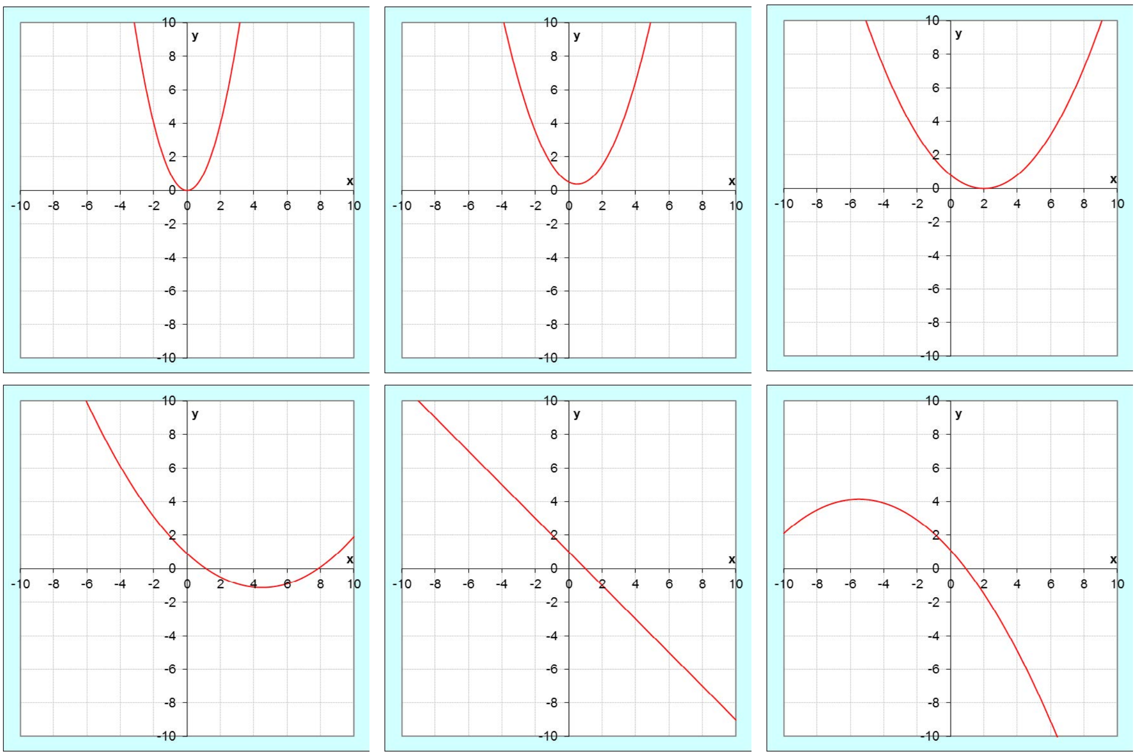

Figure 7: Positions at p = 0; -0.5; -0.8; -0.9; -1; -1.1 (left to right by rows)

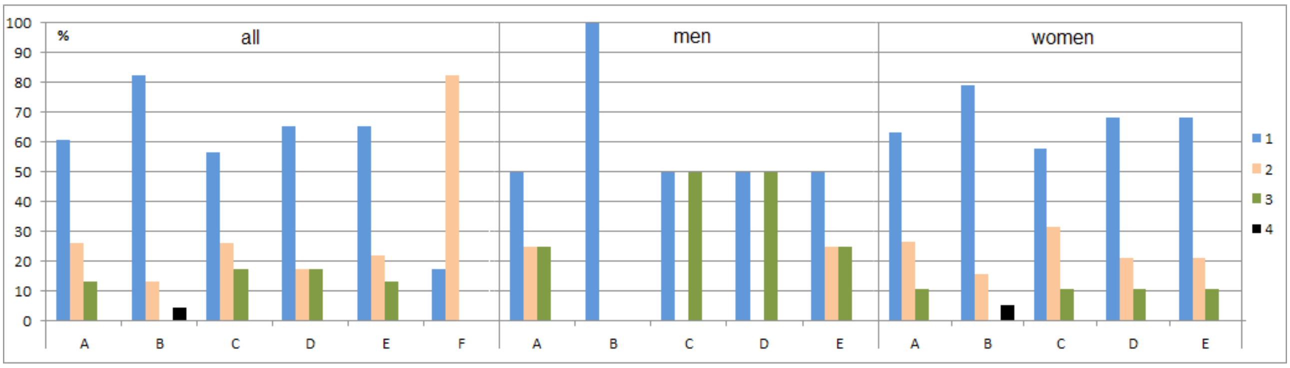

Figure 8: Relative frequency of the answers

The matrix is in range N5:Q7. The upper triangular matrix is obtained in four steps over ranges N9:Q11, N13:Q15, N17:Q19 and N21:Q23. Variables a, b and c are calculated in range C3:C5. The system always has a unique solution, which stems from the physics of the task. The discriminant of equation $ax^2 + bx + c = 0$ is in cell L10. It is always positive.The roots are in cells L13 a L14. The vertex coordinates are in cells K18 a L18. The application is interactive – if the coordinates of the points in range K5:L7 change, the point of shot and impact (L13, L14) are recalculated.

Lesson and survey: There were 24 students, 4 men and 20 women. The solution to the systems of linear equations by GEM was gone through. The students developed the application in Fig 9 from the application on the left side of Figure 1 and resolved the task. At the end, they answered the questions. The result is in Figure 10.

The result is: (A) 92% found the lesson interesting (54% very, 38% quite); (B) 100% understood the topic (83% all, 17% most); (C) 92% had the feeling that the lesson contributed to his/her TPACK (54% a lot, 38% quite a lot); (D) 92% found the applications benefitting (63% a lot, 29% quite a lot); (E) 96% were of the opinion that the developing helped comprehend the core (50% a lot, 46% quite a lot). The men found the lesson more interesting while the women found it more benefitting.

SURVEY SUMMARY

Altogether 96 questionnaires were answered, 19 by men and 77 by women. The result is graphed in Figure 11.

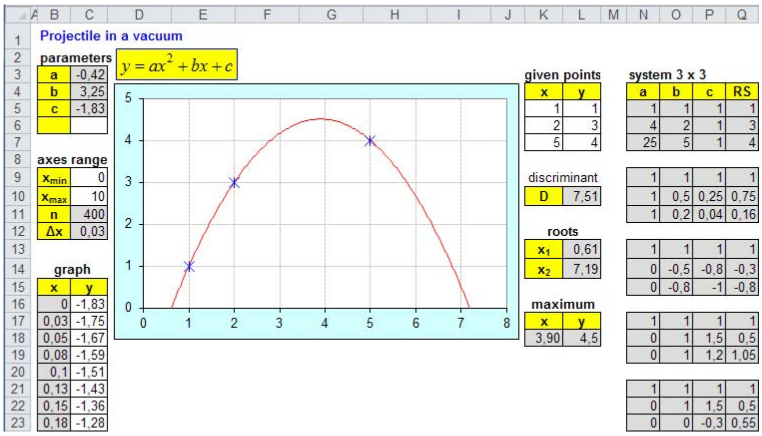

Figure 9: Trajectory of a projectile in a vacuum

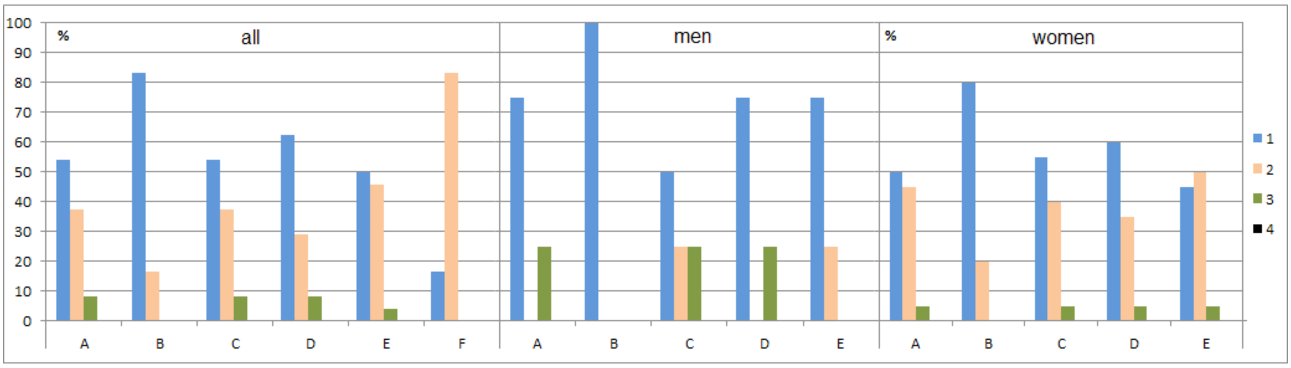

Figure 10: Relative frequency of the answers

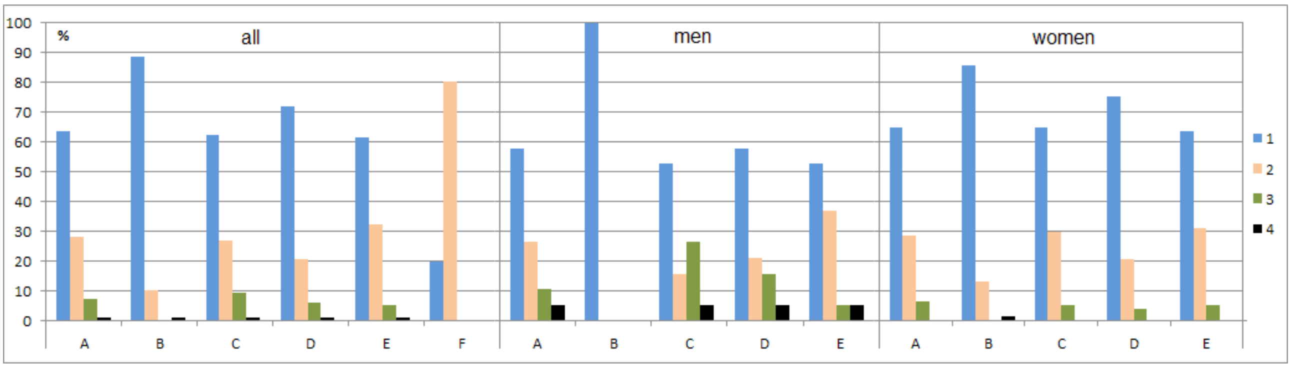

Figure 11: Relative frequency of the answers in total

The result is: (A) 92% found the lesson interesting (64% very, 28% quite); (B) 99% understood the topic (89% all, 10% most); (C) 90% had the feeling that the lesson contributed to his/her TPACK (63% a lot, 27% quite a lot); (D) 93% found the applications to be benefitting (72% a lot, 21% quite a lot). (E) 94% were of the opinion that the developing helped comprehend the core (62% a lot, 32% quite a lot).

The outcome for all, men and women was:

A) *The lesson was interesting: 92%, 84 %, 94 %*

B) *I understood of the mathematics involved: 99%, 100 %, 99 %*

C) *The lesson contributed to my TPACK: 90%, 69 %, 95 %*

D) *The applications benefit the teaching of the topic: 93%, 79 %, 96 %*

E) *Developing the applications helps comprehend the core: 94%, 90 %, 95 %*

F) *I am a man: 20%*

The lessons contributed considerably more to the TPACK of the women than men (difference of 26%). The women found the applications more benefitting the teaching of the topics and more interesting than the men (difference of 17% and 9%). The women were stronger of the opinion that developing the applications helps comprehend the core of the topic (difference of 5%).

The result implies that the presented method of teaching mathematics through developing spreadsheet applications that model and solve tasks is of interest to pre-service teachers, benefits the teaching of the topics and contributes to the technological and pedagogical knowledge of the teachers, mainly women. The outcomes correspond well with the result of the authors' research with high school students (Benacka and Ceretkova, 2013). That suggests that "learning by doing" with spreadsheets has a potential in promoting mathematics at upper secondary level.

ACKNOWLEDGEMENT

The authors are members of the team of LLP Comenius project 539234-LLP-1-2013-1-AT-COMENIUS-CAM.

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Expert and novice teachers' classroom practices in a technological environment

Gulay Bozkurt and Kenneth Ruthven

University of Cambridge, Faculty of Education, Cambridge, UK, gb408@cam.ac.uk

This case study compares the teaching practices and craft knowledge of two secondary mathematics teachers using GeoGebra to teach the same geometrical topic. It forms part of a programme of research aimed at developing a more comprehensive understanding of technology integration in classrooms, by providing a model of key structuring features of classroom practice (Ruthven, 2009) which shape the use of technology in lessons and the kinds of professional knowledge required. In accordance with this conceptual framework, the classroom practices of an expert technology-using teacher are analysed in contrast to those of an experienced but technologically novice teacher, so shedding light on the character of expertise and the process through which it develops.

Keywords: Classroom practice, mathematics teaching, dynamic mathematics software, technology use, teacher knowledge and thinking.

INTRODUCTION

There has been considerable recent interest in issues of the integration of new technologies into ordinary classrooms in which the particular focus is on the teacher dimension. Researchers have suggested that teachers are faced with the need to consider a range of classroom management issues. They need not only to develop new types of interactions (Monaghan, 2004; Drijvers et al., 2010) but also to manage different kinds of time in their classrooms (Assude, 2005). They must establish a relationship between a technological environment and a paper and pencil environment in order to "build connections with the official mathematics outside the microworld" (Guin & Trouche, 1998, p. 200). Furthermore, some researchers (Monaghan, 2004; Lagrange, Dedeoglu, & Erdogan, 2006) have shown how technology can affect the emergent goals of the teacher during the lesson. When a teacher integrates Dynamic Geometry Software (DGS) in order to make students engage with the task, s/he must consider not only a mathematical learning trajectory but also the instrumental aspect of this including the operation of the tool (Lagrange et al., 2006). In this light, it is not surprising that the process of orchestrating technology-integrated mathematics learning is not straightforward, in particular for teachers who are novice in technology-use, since the use of digital resources calls for "change in teachers' professional knowledge and development" (Gueudet & Trouche, 2009, p. 199). In this respect, Tabach (2012) showed that classroom practices of teachers in relation to technology-use influence their knowledge and thinking and vice versa. Similarly, a number of studies (e.g., Monaghan, 2004; Drijvers, 2012) have shown that, over time, teachers' practices with technology generate professional growth as teachers revise and adapt their teaching in relation to technology through feedback from their experiences. In other words, teachers build up knowledge from their reflection on their own classroom experiences and strategies they develop when, in the course of using the technological tools available, unexpected issues occur in the classroom.

Two fundamental questions, then, need to be addressed: "What aspects of classroom teaching shape the ways in which teachers integrate new technologies?" and "What kinds of professional knowledge do teachers need to teach with them?"

THEORETICAL FRAMEWORK

The recently conceived Structuring Features of Classroom Practice (SFCP) (Ruthven, 2009) was the framework chosen for this research on teachers' classroom practices with the use of new technologies. This framework takes a naturalistic approach, focusing on constructs that are directly related to teachers' classroom practices. Drawing from prior research on teaching and schooling in general, it identifies five constructs which have already been developed to analyse what happens within classrooms and which bear crucially on incorporation of technology within classroom practice. These are working environment (e.g., Horne-Martin, 2002), resource system (e.g., Cohen, Raudenbush, & Ball, 2002), activity format (e.g., Burns, 1984), curriculum script (e.g., Putnam, 1987; Leinhardt, Putnam, Stein, & Baxter, 1991), and time economy (e.g., Assude, 2005).

Working Environment: Using new technologies typically requires adaptation of the working environment, either through moving teaching to a specialised computer laboratory or re-organising the ordinary classroom.

Resource System: New technologies increase the range of tools and resources available in classrooms, and pose a challenge for teachers to create a coherent resource system.

Activity Format: Technology-based lessons may require the adaptation of existing activity formats from non-technology based lessons or the development of some new format which "structures the activity and provides the organizational means by which learning tasks will be accomplished" (Burns, 1984, p. 103). Grouping arrangements, delivery system and interaction patterns define the activity format.

Curriculum Script: Mathematics teachers who are integrating new technologies in their classrooms need to develop their curriculum script in order to have a structure for planning and interpreting classroom teaching of a topic with technology. Curriculum script was developed by Putnam (1987) and defined as "a loosely ordered but well defined set of skills and concepts students were expected to learn, along with the activities and strategies for teaching this material" (p. 13).

Time Economy: Assude (2005) argues that time related issues in classrooms stem from time management difficulties in the classroom. Following Assude, Ruthven (2009) uses the concept of time economy to denote "how teachers seek to manage the 'rate' at which the physical time available for classroom activity is converted into a 'didactic time' measured in terms of the advance of knowledge" (p. 14).

The SFCP framework identifies key aspects of the craft of teaching that indicate the corresponding professional reasoning and craft knowledge that teachers must develop about these aspects in order to successfully incorporate new technologies. Ruthven (2009) explained the interaction of these aspects in teaching decisions and activities: for example, "Each of these modifications of an established activity format calls for the establishment of new classroom norms for participation, and of classroom routines to support smooth functioning" (p. 10). Employing this framework, the research reported here studies teachers' use of technology in classrooms and the associated expertise teachers use in their everyday practice.

RESEARCH CONTEXT

This study examines classroom practices of two secondary math teachers' use of GeoGebra (GGB) for math teaching, working within the English school system. GGB was of particular interest due to its accessibility. Although there is a lot of interest in GGB, difficulties arose when looking for suitable lessons, which indicated that in practice its use still remains quite rare. The most popular topics are concerned with graphing functions, geometrical transformations and angle properties in a circle.

After teachers agreed to participate in the research, the first author visited their schools to discuss their timetable and to find out when they were planning to make some significant use of this technology. Observations took place at a time agreed in advance with each teacher. Thus, this study involved no attempt to control or influence teachers' lessons. In particular, the researchers did not participate in the lessons planning, although it is clearly possible that a teacher's planning was influenced by the knowledge that a lesson would be observed. To try to forestall this, the research protocol made clear to teachers that they themselves should choose the topic, using GGB however they saw fit, and in any manner they wished.

Both teachers in this study chose to teach the topic referred to in the English curriculum as "Circle Theorems". One (pseudonym Chris) is an Advanced Skills Teacher (a recognised grade of classroom teacher within the English school system, also taking special responsibility for leading professional development). He has around twenty years of teaching experience and is an expert technology user who utilises new technologies in a progressive way in math instruction. He taught the topic over a series of four lessons to a top set Year 11 class. The other teacher (pseudonym Susan) also has around twenty years teaching experience, but is a novice technology user. Although Susan was experienced in teaching Circle Theorems, it was the first time she had integrated GGB into her teaching of this topic to this extent. She taught the topic over two lessons to a lower set Year 10 class.

Teacher Interviews: Semi-structured post-lesson interviews were conducted in order to clarify the professional thinking behind the observed lessons according to the SFCP framework key themes. Interview questions mainly focused on how using technology in the lesson might make it rather different to organise and run from a similar lesson without technology. These interviews were audio-recorded.

Classroom Observations: A semi-structured, non-participant observation approach was adopted for which the SFCP framework as an interpretative lens provided guidelines. In this regard, observation data aimed to provide evidence about teachers' classroom practices in relation to use of technology, with a focus on pre-specified aspects of the SFCP framework. All observed lessons were recorded and transcribed verbatim. To do this the teachers were asked to wear a microphone during lessons to capture all speech during individual teacher-student interactions.

RESULTS

The data acquired from this study are presented in detail due to space constraints. Instead some of the main findings are outlined and compared with other research.

Working environment

Chris's lessons took place in a Computer Lab where the computers for students were arranged in a U shape around the back and sidewalls, so that students working at them were facing away from the front of the room. There was also sufficient seating in the centre of the room to accommodate the entire class being taught as a group facing the front. At the front of the room there was a computer connected to a data projector for the teacher, and a Smart-board. Susan's lessons took place in a normal classroom where the seating consisted of rows of tables with chairs. There was a Smart-board as well as an ordinary whiteboard at the front. The teacher provided students with laptops on which GGB was downloaded, which they used at their normal tables in the classroom. In order to ensure that students would have a functioning computer system she spent a considerable amount of time in preparation.

Chris was satisfied with the working environment since the layout allowed him the most convenient method of monitoring students, which in turn supported interaction between teacher and students.

> I can see all of their screens. So if I stand in the middle and turn my head I can see everybody's screen which means that I know immediately if somebody isn't doing what they should.

He also appreciated the availability of software that allowed the teacher to control students' screens.

Although Susan made an active choice to use laptops because her class had too many students to fit into the available computer room, she described her ideal layout as being a bigger U-shaped computer room where she could monitor students in a more straightforward way.

> I'd rather have a bigger room. My ideal would have been a big U shaped computer room with big screens that they can see and that I can walk around easily. You can position yourself to see 3 quarters of the screen to scan the room. And I suppose this is where for me I am always thinking about the classroom management issues particularly in these lessons where I'm a bit out comfort zone.

Resource system

Chris as an expert technology user had a full command of the software. He was aware of difficulties that might arise with students' use of the resource and preempted these through use of technical demo (Drijvers et al., 2010). For instance, he made sure that students all knew how to measure an angle with GGB. This was because, over the years, he had noticed that students often "measure" without clicking in the anti-clockwise order (a convention embedded in GGB).

> For example measuring angles, you can click on the three points and depending on which order you click. If you click one two three, then it will

> measure the interior angle (he shows on the paper). If you click them the other way round it measures the reverse angle. That was just a matter of try clicking in a different order.

However, Susan was a novice in using GGB and did not have a full knowledge of it. So she was learning some aspects of the software during the lessons. While she was doing a technical demo at the beginning of the first lesson she wanted to show how to measure an angle on GGB; however she was not aware that GGB measures anti-clock wise and she got the measure of an obtuse angle a number of times while wanting to measure an acute angle. But during students' work on laptops, a student told her about clicking in the anti-clockwise direction.

> I was learning on the hoof this bit where it measures it anti clockwise; I had not really tweaked that. As I say Mike *(a student)* helped me with that one. I mean, yeah yes I did learn during the lesson. So then I could show the students. But the only way I have got now that in my head is by going through that pain and then teaching the students.

Similarly, she did not know how to set GGB to measure to the nearest whole number, which she thought created some confusion for students in this lower set since they had difficulties in seeing the relationships between angles.

Both teachers allowed students to use GGB by themselves for around half of each lesson as their aim was for students to discover some CT rules (Susan) and to explore some mathematical ideas, i.e., proof through CT (Chris). During students' use, teachers' role was to walk around and make sure that students engaged with their task. The teachers also used GGB for whole class teaching to explain or discuss problems. However, the two teachers had prepared rather different sets of tasks for students. Chris used more open-ended tasks, which aimed to encourage students to use GGB as tools or representations to help thinking. With this particular class, which is a high ability group of students, Chris pointed out that they have established this way of working over several years. His aim was to show students that there were many paths to reach the same mathematical solution; in particular he saw the topic of Circle Theorems as a vehicle for developing ideas about mathematical proof by focusing on different ways of proving. He also thinks that students have to learn to interpret the diagrams through building them up by themselves. On the other hand, Susan asked students to set up the diagrams shown on the worksheet by using GGB and then to explore triangles inscribed into circles and try to formulate four different CT rules through using dragging techniques.

> I expected them to discover the circle theorems. That was the aim of it. I wanted them to have that moment of awe and wonder, when they notice that special things happen in the diagrams. And I think the GGB lent itself to that.

During her reflections, Susan indicated that she used GGB to add 'fun' to lessons and to help students engage with more examples through using dragging techniques and to facilitate noticing the relationships between angles in circles. In comparison with undertaking the same worksheet on CT by hand, she considered that GGB assisted investigating CT through its accuracy, speed and manipulative ease.

Activity format

SFCP does not suggest specific activity formats, however Instrumental Orchestration (e.g., Drijvers et al., 2010; Drijvers, 2012), which also focuses on teachers' classroom practices in technological environments, has particularly focused on how activity structures relate to particular ways of making use of the technological component of the resource system, and has provided operational descriptions based on a combination of data- and theory-driven analysis (Ruthven, 2014). We will make use of the six activity format types identified for Instrumental Orchestration research; Technical-demo, Explain-the-screen, Discuss-the-screen, Spot-and-show, Work-and-walk-by and Sherpa-at-work (Trouche, 2004; Drijvers et al., 2010; Drijvers, 2012)

Chris' lessons broadly broke down into 3 segments. Initially the teacher introduced lessons in a whole class when he gave information about the lesson agenda and projected student work from a previous lesson to remind students what they had done and what direction to take from there. In the second segment, students went onto the computers and worked to discover for themselves in groups of at least two while the teacher walked around to monitor their progress and guide when necessary. Work-and-walk-by activity format comprised about 50% of a lesson. This format gauged students' involvement and engagement with the task, and also determined whether they had any difficulties during tasks. The teacher also guided the students to think in different ways during this format. Therefore, elements of Discuss-the-screen and Explain-the-screen were apparent in the teacher-students interaction during Work-and-walk-by. Finally, the teacher gathered the students in the middle of the room in order to discuss important points of the lesson primarily to create collective knowledge. Chris often made use of a Spot-and-show activity format during each segment of his lessons perceiving it as a means to enhance student involvement and discussion. He spotted students' examples, in particular different approaches, and showed them to the whole class in order to make pupils think of different ways of doing things. Another activity format that Chris made use of was Sherpa-at-work (Trouche, 2004): he spotted an example while circulating in the classroom, and stopped the students to project this spotted example on the IWB for whole class discussion. However the difference from the Spot-and-show format was that the teacher let the student, whose example he spotted, run the example from her computer and at the same time he was explaining/discussing the screen of what was going on. In other words, the Sherpa-student was the owner of the spotted example and she was in charge of the technology. Predict-and-test was another activity format used during the middle segment. The teacher encouraged students to make their own conjectures and then test them out on the computer.

The activity formats for Susan's lessons consisted of whole class teaching as well as student work at computers with the teacher moving around to monitor and provide support. The former was mostly to demonstrate some tool techniques, and to outline what the lesson was about. Since she was teaching this topic with GGB for the first time and most of the students had never used GGB, she spent half of the first lesson using a Technical-demo format to show some tool techniques that they could use for angle properties. She demoed briefly what she expected students to do, and to exemplify she used GGB in modelling on the IWB how to tackle the first question from the printed worksheet. During this, she showed the specific tools in GGB that students would need for constructing the shapes shown on the worksheet. Susan's activity formats were still influenced by non-technology patterns, and not fully instrumented by the technology. For instance, when she made use of Spot-and-show, rather than using GGB on the IWB she would hold up a student's worksheet in order to show others how this student had done the question.

Curriculum script

Examining the tasks sequences used in these cases provided a starting point for looking at how technology featured in the curriculum scripts (CS) of the two teachers, and how these scripts were developing in response to using technology. In particular, post lesson interviews focused on aspects where teachers reported change/development in their thinking about teaching the topic and/or structuring the lesson related to using GGB. Striking contrasts appear, with the CS of the technologically expert teacher being richer and more detailed in this respect than that of the technologically novice teacher. On that note, it was much easier to elucidate information from more experienced teacher about his CS. This richness was also evidenced by the differing nature of, and the degree of emphasis on, CS against resource system. The novice teacher's focus tended not to be on the CS, but was more about developing instrumental knowledge to make it part of the resource system, first for herself personally and then also for her students. For instance, Susan's reflections on what issues came up in lessons that she did not expect focused more on her lack of knowledge of GGB such as measuring acute angles and rounding measurements. These issues have already been raised in the section on resource system and they are more about GGB not yet being part of a functioning personal resource system for the teacher. In that sense she did not have a well-developed aspect of her CS for the topic that is specific to technology, which combines content and technology knowledge. Her CS consisted of following the sequence of her worksheet (prepared for teaching this topic in a non-technological environment) and she was at the stage of learning and adapting. However, Chris, who had been teaching with GGB for many years, had layers of accumulated knowledge about how to teach this topic with GGB.

Time economy

It became clear in post lesson interviews how the use of DGS contributed to time efficiency in lessons. Both teachers indicated that GGB – in particular the angle-measuring tool and the edit/undo option – enabled students to progress faster compared to measuring/calculating angles by hand.

> It will let us do lots of measuring without needing to do measuring or calculating so it will measure without us needing to do those by hand. That is a huge time saver. That saves a lot of time over drawing them out. (Chris)

This speed helped the students to produce and engage with more examples without spending as much time capital, therefore, maximised the didactic time.

> The speed is that they can see lots of versions of it and all the weird versions. (Susan)

As an expert technology user, Chris's use of working environment and available resources such as control software for showing examples led to efficient use of instructional time. Also, instrumented use of Spot-and-show during students' work at computers provided 'intermediate syntheses' (Assude, 2005), which helped him save time capital. However, the fact that Susan was not aware of some tool techniques introduced some time diseconomy in her lessons.

While students worked at computers in groups, both teachers walked amongst students and guided them through making 'authoritative contributions' (Assude, 2005, p. 201) and posing leading questions in order to increase the didactic time during activity by avoiding students dissipating the limited time capital available.

CONCLUSION

As a novice technology-user, Susan believed that students need more monitoring during technology lessons, and that some room arrangements offer better scope for monitoring and interaction. Both teachers were very conscious of the degree to which GGB formed part of a functioning resource system for their class: in Susan's case, that she was just embarking on establishing such a system; in Chris's case, that he could count on such a system having already been established. In terms of activity formats, most of the orchestration types characterised in Drijvers and colleagues (2010) and Drijvers (2012) studies featured in this study but with more variants. However, an additional type of activity format, Predict-and-test, was identified in my study (resembling an activity format already noted in Ruthven, Deaney, & Hennessy, 2009). The interaction becomes between the teacher, class and computer, with the students making a prediction and then testing it out on the computer rather than teacher validating – or invalidating – it. That shifts the role of the teacher towards becoming an organizer/observer of this process. Also, this activity format appears to be specific to technology use, depending crucially on its use. Therefore, it could be said that the results of this study also contribute to extending the Instrumental Orchestration framework to some extent. There was a notable difference in the way in which teachers instrumented activity formats through the use of technology. The novice teacher tended to customize existing (and more generic) formats for implementing activity. The expert teacher believed that Sherpa-at-work was distinctive to technology use where the involvement of the computer uniquely influenced the nature of the interaction-taking place.

This study makes a threefold contribution. Firstly, it provides a further empirically-based test of the usability and usefulness of the SFCP conceptual framework (Ruthven, 2009) as a tool for advancing research on technology integration. Secondly, in focusing on the teacher dimension (Lagrange et al., 2003) this case study provides an illuminating comparison which math teachers and teacher educators engaged in professional development on the issue of the integration of technological tools could employ to build a holistic understanding of the conditions of technology in ordinary classrooms and shed light on how teachers can adapt new technologies to their teaching. Finally, there might be a third type of contribution through presenting the report of these teachers' teaching with technology in ways that could be directly accessible and useful to other teachers, by reporting practical solutions to concrete problems that a teacher commencing integrating technology into their teaching would encounter.

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The diagnostic assessment Pépite and the question of its transfer at different school levels

Françoise Chenevotot-Quentin[1], Brigitte Grugeon-Allys[1], Julia Pilet[1], Elisabeth Delozanne[2] and Dominique Prévit[2]

1 Université Paris Diderot, Laboratoire de Didactique André Revuz (LDAR), Paris, France, chenevotot.francoise@neuf.fr

2 Université Pierre et Marie Curie, Laboratoire L'UTES, Paris, France

This paper deals with the question of the transfer of the diagnostic assessment Pépite at different grade levels. First, we present the theoretical foundations and the modalities of the diagnostic assessment Pépite. Then, we characterize the model of the diagnostic test and we study how this model is compatible according to different grade levels. We detail the response analysis and also we explain how to transfer it to different grade levels.

Keywords: Diagnostic assessment, Information and Communication Technology (ICT), elementary algebra, student's profile, teaching suggestions.

CONTEXT

Teachers are looking for tools in order to help their students. In fact, to allow each student academic progress, teachers need detailed diagnosis about individual student's learning. However, teachers also need to manage the whole class by proposing differentiated activities which are adapted to groups of learners with close competences or who require the same teaching strategy.

This paper addresses the TWG15 "Teaching mathematics with resources and technology". Our research concerns the development and the use of online resources for diagnosis and differentiated learning in the field of elementary algebra. It takes place into the *Pépite* project whose objective is to design and implement a web-based application to support mathematics teachers in managing the cognitive diversity of their students in school algebra classes (Delozanne, Prévit, Grugeon-Allys, & Chenevotot-Quentin, 2010).

Since 2011, we spread our research tools on *LaboMep* (Pilet, Chenevotot, Grugeon, El-Kechaï, & Delozanne, 2013), the online databank developed by *Sésamath*, a French mathematics teachers association. The success of the *LaboMep* platform shows us that such online resources may answer to the teachers' needs (Artigue & Gueudet, 2008): they are looking for valuable information about consistency and misconceptions in student's algebraic activity. First, we implemented the diagnostic assessment *Pépite* for $9^{th}/10^{th}$ grade students (15–16 years old). Second, we implemented a tool providing automatically teaching suggestions according to the learning objectives aimed at by the teacher and adapted to students' diagnostic assessment (Pilet et al., 2013).

This paper deals with the question of the transfer of the diagnostic assessment *Pépite* at different grade levels. First, we present the modalities of the diagnostic assessment *Pépite* and the theoretical elements mobilized. Then, we characterize the model of the diagnostic test and we study how this model is compatible according to different grade levels. We detail the response analysis and we also explain how to transfer it to different grade levels. Last, we conclude with some research perspectives.

THE DIAGNOSTIC ASSESSMENT PEPITE

The theoretical foundations

The modeling presented here is not based on a psychometric approach. The *Pépite* diagnostic assessment is based firstly on an epistemological and anthropological approach and secondly on a cognitive approach of elementary algebra in order to define a reference (Artigue, Grugeon, Assude, & Lenfant, 2001).

In its *tool* dimension (Douady, 1986), the algebra field covers traditional arithmetic problems, problems of generalisation and proof, problems where algebra appears as a modelling tool, problems to put into equation, algebraic and functional problems (Chevallard, 1989). In its *object* dimension, algebra is a structured set of objects with specific properties, semiotic representations and treatment modes taking into account both their semantics and their syntax (Kieran, 2007; Vergnaud, Cortes, & Favre-Artigue, 1988). The diagnosis *Pépite* relies on a multidimensional analysis of the algebraic activity (Grugeon, 1997; Kieran, 2007 [1]) which allows identifying consistency in student's algebraic activity and following its evolution.

According to the anthropological approach, mathematical knowledge strongly depends on the institutions where it has to live, where it has to be learnt or to be taught. Mathematical objects do not exist per se but emerge from practices, which are different from one institution to another one. Chevallard (1999) analyses them in terms of praxeology, i.e., in terms of type of tasks, techniques used to solve these tasks (praxis), technological discourse developed to explain and justify particular techniques, and last, theories which structure the discourse (logos). Here, the diagnostic assessment *Pépite* depends on the curriculum at the end of compulsory education. At each grade level, diagnostic tasks are characterized by a type of tasks, the complexity of algebraic objects involved and the level of involvement of tasks in the resolution.

Pépite diagnostic tasks

The *Pépite* test is composed of ten diagnostic tasks that cover the algebraic field grouped into four sets of types of tasks: calculus (developing or factoring algebraic expressions, solving equations), production (of expressions, formulas or equations), translation or recognition of mathematical relationships from one register of representation to another, solving problems in different mathematical frameworks (numeric, algebraic, geometric, functional) with algebra in order to generalize, prove properties, model or put into equation. The diagnostic tasks may be multiple-choice questions or open responses with multistep reasoning.

The conceptual IT model of classes of tasks developed by Prévit (Delozanne, Prévit, Grugeon, & Chenevotot, 2008) allows characterizing equivalent tasks on a diagnosis point of view. Prévit developed *PépiGen*, software that automatically generates the tasks and their analysis. It uses *Pépinière* software that generates anticipated student's correct or incorrect responses, according to a priori analysis of the tasks.

We work on two main points to transfer *Pépite* assessment at several grade levels. On the didactic modelling side, we have to define a set of tasks that cover the mathematical field at the grade level considered and the associated didactical variables. On the IT modelling side, we have to build generic tests in order to have a same framework for each test.

The response analysis

The diagnostic assessment *Pépite* includes three stages:

- The *local diagnosis* (on a single exercise) analyses each student's response on several dimensions and not only in terms of correct/incorrect and the diagnostic system provides a set of codes that characterize this response according to types of anticipated responses;

- The *individual global diagnosis* (on a set of exercises) collects similar codes on different exercises to build the student's cognitive profile expressed by a level at three scales of skills, success rates and personal features (relative strengths and limitations, false rules and correct rules);

- The *collective global diagnosis* defines groups of students who have close cognitive profiles.

In order to transfer the response analysis, for every task, we need to anticipate the different types of responses and their different forms and study if the computer algebra system *Pépinière* allows automated analysis. At last, we have to build the algorithm to calculate the student's profile.

We now expose the issue of the transfer of the diagnostic assessment *Pépite* at the $7^{th}/8^{th}$ grade level and define some conditions to succeed.

THE TRANSFER OF THE DIAGNOSTIC TASKS

The first step of the transfer of *Pépite* concerns the design of diagnostic tasks with two necessary conditions: theoretical foundations and institutional constraints.

Types of tasks	Number of items	Test item
Calculus	4 / 27	5.1 / 5.2 / 5.3 / 5.4
Producing algebraic expressions	6 / 27	3.1 / 8.1 / 8.2 / 8.3 / 9 / 10.2
Translation or recognition	16 / 27	1.1 / 1.2 / 1.3 / 1.4 / 2.1 / 2.2 / 2.3 / 3.2 / 4.1 / 4.2 / 4.3 / 4.4 / 4.5 / 6 / 7 / 10.1
Problem solving in different mathematics frameworks	3 / 27	8.3 / 9 / 10.3

Table 1: Organization of the 9th/10th grade level test in terms of types of tasks

Types of tasks	Number of items	Test item
Calculus	4 / 22	7.1 / 7.2 / 8.1 / 8.2
Producing numerical expressions	1 / 22	5
Producing algebraic expressions	3 / 22	3.1 / 5 / 6
Translation or recognition	14 / 22	1.1 / 1.2 / 1.3 / 2.1 / 2.2 / 2.3 / 3.2 / 4.1 / 4.2 / 9.1 / 9.2 / 9.3 / 9.4 / 10
Problem solving in different mathematics frameworks	1 / 22	6

Table 2: Organization of the 7th/8th grade level test in terms of types of tasks

A tasks transfer that ensures the mathematical coverage area

To ensure that the test takes into account all types of tasks involved in the algebraic field, we characterize each item of a diagnostic task by one or more of these types of tasks (Chenevotot-Quentin, Grugeon, & Delozanne, 2011). We consider that the diagnostic test covers the types of tasks in the field if all types of tasks are involved. As shown in Table 1, the ten diagnostic tasks of the initial test (9th/10th grade level) cover the algebraic field. The test for the 7th/8th grade level is also composed of ten diagnostic tasks. The Table 2 shows that they are spread across all types of tasks. As at this grade level, numeric skills play an important role in the entry into the algebraic activity, we add a new type of tasks to our list: *Produce a numerical expression*. We assume that the question of whether a student may or may not produce a numerical online expression with correct parentheses is an important indicator of its interpretation of algebraic expressions. We illustrate this task in the following text.

A transfer by adapting existing tasks or adding new tasks

To comply with the curriculum experienced by students in grade 7th in France, the transfer of the test

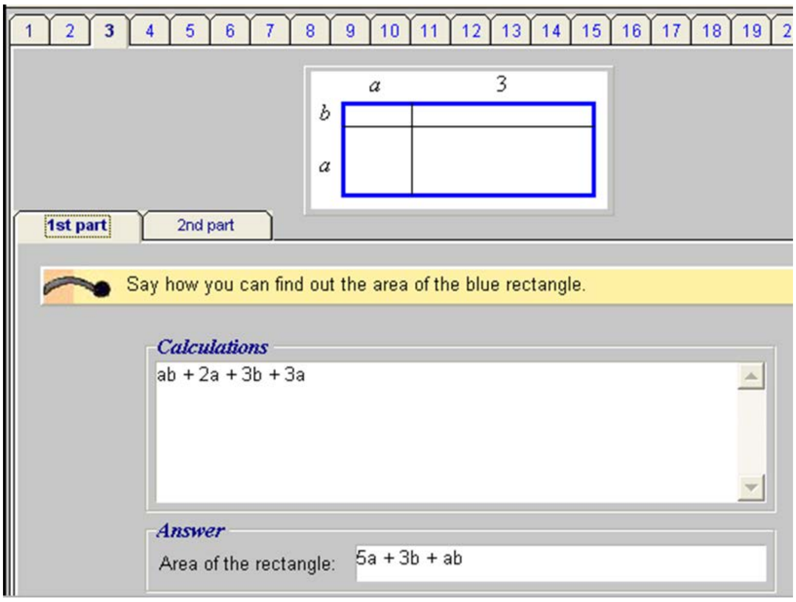
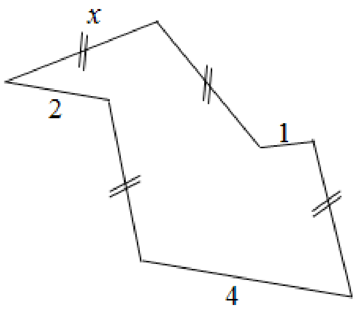


Figure 1: Item 3.1 of the initial test



Shows how to calculate the perimeter of the figure	
Draft for calculations	Perimeter of the figure

Figure 2: Item 3.1 of the 7th/8th grade level test: an example of an adapting task

13 girls and 15 boys go to the movies. Each pays his ticket € 6.80, buys one soda 3 €, popcorn € 3.20 and also one glass € 2.50. Write one line expression to find the amount spent by the group without doing calculation.
Expression of the amount spent by the group

Figure 3: Item 5 of the 7th/8th grade level test: an example of a new task

requires a specific work on tasks. We distinguish, on the one hand, characteristic tasks of the algebraic field that are transferred with adaptations from the 9th/10th grade to a lower grade, and, on the other hand, tasks of the numerical field, specifically designed for 7th/8th grade level. We adapt the tasks of the algebraic field by adjusting the values of didactical variables such as the structure of algebraic expressions or the choice of numbers. These adaptations are justified by both the curriculum and the algebraic activity expected at this grade level. We present an example of adapting item: adapting the item 3.1 of the initial test (Figure 1) for the 7th/8th grade level test (Figure 2). In both cases, the type of tasks is *Produce an algebraic expression in the geometric setting*. In the initial (resp. new) test, the task relates to the area (resp. perimeter) of a rectangle (resp. figure). The area of the rectangle is a second-degree expression and the structure is too complex for the 7th/8th level. That is why we have chosen for this grade level a figure leading to a first-degree expression. Moreover, this choice allows identifying students that concatenate all terms (11x response) or those who are still in a repeated addition (x + x + x + x + 7 response).

To take into account the numerical skills of students who just discover algebra, we design new tasks. Figure 3 shows one of these tasks which belongs to the type of task: *Produce a numerical expression*. This task assesses if students can correctly produce one numerical expression with parentheses or if "step by step" reasoning persists.

THE TRANSFER OF THE RESPONSE ANALYSIS

We expose the issue of the transfer of the response analysis in the diagnostic assessment *Pépite* by successively visiting the three stages of the process: local diagnosis, individual diagnosis and collective global diagnosis.

First stage: Local diagnosis

The whole process of the diagnostic assessment *Pépite* relies on the quality of this first step: assessment of each student's responses. Students' responses are not only evaluated in terms of correct/incorrect. They are also coded in terms of consistency in the student's algebraic activity, determined by *a priori* analysis (skills, recurring errors). We define six analysis codes found-

Tick the correct equation			
$\frac{1}{2}+\frac{1}{3}=\frac{3}{2}$	$\frac{1}{2}+\frac{1}{3}=\frac{2}{5}$	$\frac{1}{2}+\frac{1}{3}=\frac{2}{6}$	$\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$

Choice	A priori analysis	Code
1	Incorrect calculation based on the cross product	V3 EA5
2	Addition of numerators and of denominators	V3 EA42
3	Addition of numerators and product of denominators	V3 EA33
4	Correct	V1 EA1

Figure 4: Local diagnosis of the item 1.4 for the 9th/10th grade level test

Tick the correct equation			
$\frac{2}{3}+\frac{1}{6}=\frac{3}{6}$	$\frac{2}{3}+\frac{1}{6}=\frac{3}{9}$	$\frac{2}{3}+\frac{1}{6}=\frac{3}{18}$	$\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$

Choice	A priori analysis	Code
1	Addition of numerators without putting the same denominator	V3 EN33
2	Addition of numerators and of denominators	V3 EN42
3	Addition of numerators and product of denominators	V3 EN33
4	Correct	V1 EN1

Figure 5: Local diagnosis of the item 1.1 for the 7th/8th grade level test

ed on theoretical study: validity of the response (V), meaning of the equal sign (E), use of letters as variables (L), algebraic writings produced during symbolic transformations (EA), representations used during translating a problem (T) and level of justification (J). Figure 4 and 5 show the responses of a 9th/10th and 7th/8th grades student to the task exposed in Figure 1 and 2 and their analysis.

Are these six analysis codes adequate to transfer the assessment to 7th/8th grade student? It is necessary to complete the six previous codes by adding two new codes to study the numerical writings produced during symbolic transformations (EN) and the skills with negative and decimal numbers (N). The EN code is created to take into account that the algebraic skills are built from numerical skills. As the EA and the EN codes have a close structure, we do not distinguish EA and EN in the initial test.

The local diagnosis is based on the anticipated responses obtained both by a didactical analysis and a corpus of responses; it produces the analysis grid. This method is adapted to the IT designing and computer programming. The two pieces of software *PépiGen* and *Pépinière* automatically generate tasks, their analysis and anticipated students' responses. The computer programming of the multiple-choice questions is easier than the one line open responses. In this case, the analysis is effective and generic. Around 10 to 15% of the open responses are still not analysed because of the complexity of the algebraic reasoning, sometimes written with a French text. The analysis of open responses needs specific treatments.

Second stage: Individual global diagnosis

The individual global diagnosis operates on the set of the ten exercises. The system analyses student's responses and calculates the student's cognitive profile through a transversal analysis of the codes of all their responses.

To build the student's cognitive profile, we define a scale of skills with three components founded on theoretical study: Use of Algebra for solving problems (coded UA); flexibility in translating different types of representations (geometric figures, graphical representations, natural language) into algebraic expressions and vice versa (coded TA); ability and adaptability in various uses of algebraic calculation (coded CA). For each of the three components, we identify a scale with different modes, and appropriate criteria for each (Delozanne, Vincent, Grugeon, Gélis, Rogalski, & Coulange, 2005). Figure 6 shows the individual global diagnosis for a 9th grade student with CA3-UA3-TA3 (Figure 6). This student does not give much sense to algebraic activity and does not use it as a tool for solving problems.

To transfer assessment to other grade levels, we also need to complete the algorithm to compute the profile by adding a fourth component to evaluate the various

Components	Personal features	Level
Algebraic Calculation With few signification	Success rate for the technical questions	2 / 12
	Success rate on the meaning of the algebraic expressions	7 / 23
	Mastery of the algebraical calculus	Failing
	Mastery of the rules	Failing
	Interpretation of the expressions	Failing
Usage of Algebra Not motivated and not understood	Success rate for the mathematisation questions	1 / 9
	Mastery of the algebraical tool	Failing
	Type of justification	
Algebraic Translation To schematize	Success rate for putting in equation	5 / 24
	Mastery of the translation	Insufficient
	Translation of the mathematical relationships	Abreviative

Figure 6: 9th grade student's cognitive profile

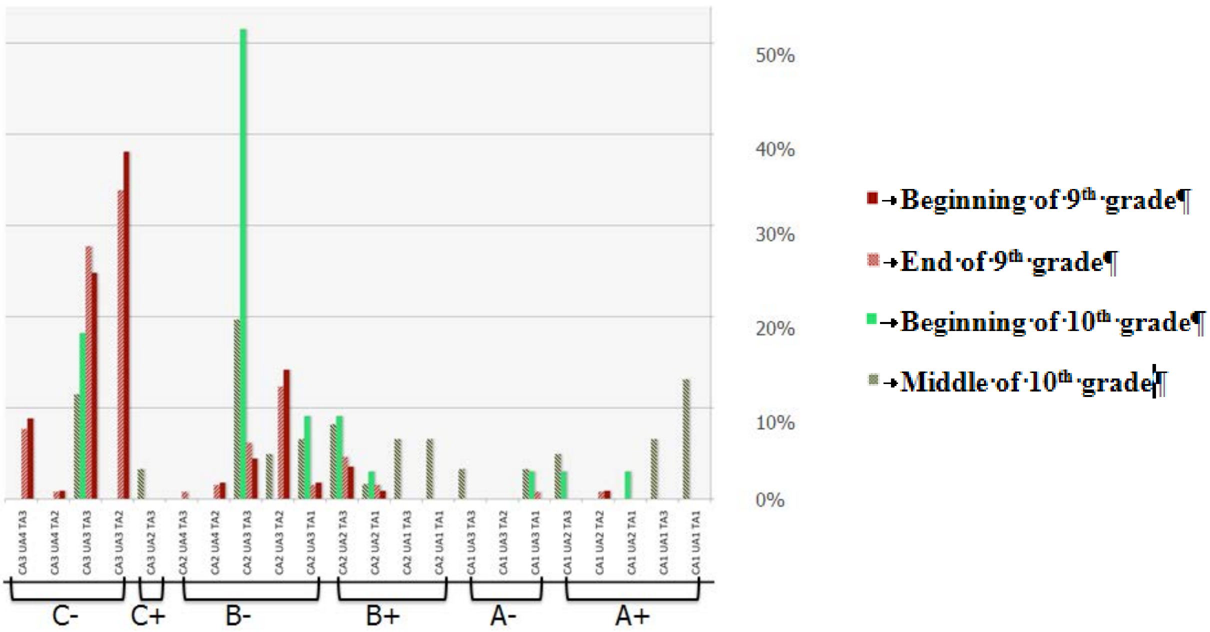


Figure 7: Collective global diagnosis for 9th/10th grade students (191 persons)

uses of numerical calculations (coded CN) and modes on the literacy scale. The CN component is created to take into account that the algebraic skills are built from numerical skills. This is the reason why the computation of the modes on scale on the CA and CN components has a similar algorithm.

Third stage: Collective global diagnosis

Teachers request that the diagnosis allows defining groups of students according to their skills in algebra with the aim of setting up strategies of differentiation. *PépiMep* software automatically calculates three groups of students (groups A, B and C) who have close profiles in algebra (Grugeon-Allys, Pilet, Chenevotot-Quentin, & Delozanne, 2012). For 7th/8th grade, the algorithm to calculate groups takes into account the new CN component.

CONCLUSION

Different studies comparing diagnostic assessment *Pépite* to other forms of assessment show that *Pépite* is reliable and valid, even with open questions (Delozanne et al., 2008, Delozanne et al., 2010). Overall, it provides a tool to save time and to avoid a very tedious work of a human being.

Since 2011, we implemented in *LaboMep* one test for 9th grade students (14–15 years old) and two tests for 9th/10th grade students (15–16 years old). We realized the model of the diagnostic tasks and we designed test and *a priori* analysis for 8th/9th grade students and 7th/8th grade students. We are now programming them in *LaboMep*.

From the theoretical foundations, we define a scale related to the algebraic activity. We can thus follow the evolution of the algebraic skills of the students on different grades. 191 students passed the 9th test and the 9th/10th test and we have already observed that the skills of the students increased. We project to validate this result from the 7th grade to the 10th grade (Figure 7).

What are the research perspectives? We project to extend the tool providing automatically teaching suggestions according to the learning objectives aimed at by the teacher and adapted to students' diagnostic assessment (Pilet et al., 2013), from 7th to 10th grade.

Do the following three calculation programs give the same result?		
Program 1	Program 2	Program 3
Choose a number, Multiply that number by 4, Add 3 to the product obtained.	Choose a number, Multiply that number by 7.	Choose a number, Multiply that number by 4, Add the product obtained triple the number selected.

Figure 8: Learning situation about equivalence of expressions

The first step, transfer of the diagnostic assessment to 7th/8th grade, is already realized. The second step will be to adapt learning situations already defined to study some epistemological algebraic aspects, often not sufficiently explained in curricula: sense of algebraic tool for solving problems of generalization, equivalence of expressions (Grugeon-Allys et al., 2012). The example given in Figure 8 illustrates a situation adapted to the target goal. This research is part of the NéOPRAEVAL project from the French *Agence Nationale de la Recherche*.

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ENDNOTE

1. Using a synthesis of international researches in didactics of algebra, Kieran (2007) proposed the GTG model which differentiates three complementary algebraic activities: (1) Generative activities involve producing various algebraic objects (expressions, formulas, equations and identities), (2) Transformational activities involve the usage of transformational rules (factorization, expansion of products, rules for solving equations and inequalities…), (3) Global/meta-level activities involve the mobilization and use of the algebraic tool to solve different types of problems (modeling, generalization, proof).

Scaling mathematics teachers' professional development in relation to technology – probing the fidelity of implementation through landmark activities

Alison Clark-Wilson, Celia Hoyles and Richard Noss

University College London, UCL Institute of Education, London, UK, a.clark-wilson@ioe.ac.uk

This paper reports research into aspects of 'scaling' classroom access to technology within the context of an English teacher development project, 'Cornerstone Maths'. The aim of this multi-year project is to address issues of underuse of dynamic mathematical technologies by lower secondary students in classrooms through: specially designed web-based software; teacher and student materials; and professional development. The paper proposes the construct of a landmark question as a means to assess the degree of fidelity of the resulting classroom implementations at scale and reports emergent data on this theme.

Keywords: Mathematics, technology, teacher development, scaling, implementation fidelity.

INTRODUCTION

Despite multiple studies over many years that have concluded positive effects of student interaction with transformative digital technologies in mathematics education, teachers and schools find it difficult to integrate such resources within 'normal' mathematics lessons (Clark-Wilson, Robutti, & Sinclair, 2014; Gueudet, Pepin, & Trouche, 2012; Hoyles & Lagrange, 2009). By 'transformative technologies' we mean 'computational tools through which students and teachers (re-)express their mathematical understandings, which are themselves simultaneously externalised and shaped by the interactions with the tools' (Clark-Wilson, Hoyles, Noss, Vahey, & Roschelle, 2015). (For a more substantial elaboration of this, see Hoyles & Noss 2003). The reasons for this lack of engagement include: insufficient time and opportunity for sustained professional development; weak alignment with institutional practices; difficulties installing and maintaining software access; teachers' mathematical knowledge and beliefs, and insufficient access to teaching materials that exploit the affordances of well-designed technologies. The Cornerstone Maths (CM) project has been developed to respond to these concerns.

Cornerstone Maths began in 2009 as a design-based implementation study (DBIR, Kelly, 2004) that seeks to implement a replacement curriculum for hard to teach topics (linear functions, geometric similarity and algebraic expressions), which included professional development, and, research the resulting classroom implementations. This paper concerns the first curriculum unit on linear functions, which evolved from work in the USA. It is organised as a 'curriculum activity system' (Vahey, Knudsen, Rafanan, & Lara-Meloy, 2013) and comprises: web-based interactive software, teacher guide and student workbook [1] and face-to-face/at distance professional development support. Adaptation and pilot studies in England established the efficacy of the materials – a more expansive elaboration of the previous work is reported elsewhere (Clark-Wilson et al., 2015; Hoyles, Noss, Vahey, & Roschelle, 2013).

This paper extends this earlier work by elaborating a theoretical frame and methodological approach for research aiming at assessing the success of the professional development part of a large-scale intervention. Central to the approach is the question of how a teacher's classroom practice comes to align (or not) with the epistemic goals of the CM materials. We will limit this discussion to outcomes relating to classroom implementations of the CM curriculum unit on linear functions.

THEORISING ABOUT SCALING

We have drawn heavily on the work of Hung, Lim and Huang (2010) who, from the context of technology enhanced educational innovations within the Singaporean system, have defined the 'products' and 'processes' of scaling. By products, they mean the mainly quantitative measures such as the number of schools, teachers, and classrooms, the geographical reach and the (school-derived) measures of increased student attainment [2]. The processes describe the means through which such products are achieved, which will differ according to each project. For CM, the processes included the development of a localised PD offer led by a CM project lead who could provide ongoing peer support for teachers to embed CM within local of schemes of work.

However, whilst these products and processes indicate the extent of teachers' and schools' access to (and use of) the CM materials, they mask the more fundamental information about *how* the materials were implemented in classrooms and, crucial to our research interest, whether or not these implementations retain any fidelity to the design principles of the CM innovation. Existing literature on research into the enactment of the mathematics curriculum offers a range of methodologies to this end (Heck, Chval, Weiss, & Ziebarth, 2012 ; Polly & Hannafin, 2011). However, few studies have addressed how to research classroom implementations during large-scale projects involving hundreds of teachers over a timeline of years (Wylie, 2008).

Defining success at scale

Our earlier work (reported in Clark-Wilson et al., 2015, and summarised here) revealed a set of criteria or 'success indicators' at the level of an individual teacher's engagement with CM. These were:

1) Expression of satisfaction with the professional development and teaching materials;

2) Alignment between the professional development and teaching materials and their goals as a teacher;

3) Use of materials and the extent to which they create legitimate adaptations (which align to the design principles of the innovation);

4) Positive outcomes in their classroom;

5) Activity and engagement within the professional community and with the project team.

Ultimately, we were keen to uncover the extent to which teachers redefined powerful learning of their students in the light of the innovation. As our earlier work had concluded that legitimate adaptions [3] enhanced the epistemic value of the tool use (Hoyles et al., 2013), we viewed adaptation as an essential part of teachers' actions as they made sense of (and began to use) the CM materials. However, we were aware that some teachers could adapt the CM materials to produce 'lethal mutations', i.e., implement the materials in ways that are inconsistent or detrimental to the design principles such that they no longer retained their intended epistemic value. For example, an important design principle was for students to have some control of their interactions with the software. Consequently, a teacher who chose to always lead the use of the software from the whole-class display could jeopardise student autonomy in this respect.

As the project scaled to 113 schools (and over 200 teachers) it was evident that observation and interview was no longer a viable methodology. Therefore we chose teachers' self-reports to provide an insight into the teachers' perceptions of the epistemic value of the CM teaching materials and their associated classroom practices.

METHODOLOGICAL DESIGN

The CM unit on linear functions comprised 14 'Investigations', divided into sub-tasks, most of which required direct student interaction with the specially designed web-based software. In this paper, we focus on the first 7 investigations, which address the following mathematical ideas:

Representational:

> Equations, algebraic expressions, graphs and tables are forms of mathematical representation.

> Motion can be represented on a graph of distance versus time.
>
> On a position-time graph, multi-segment graphs can represent characters moving at different speeds.

Relational:

> Linear equations can be derived using differences of position and time in a table or by using the y-intercept and speed/gradient of a graph.
>
> Speed can be determined from different parts of graph and simulation.

Contextual

> For equations of the form $y = mx$, in motion contexts, m is the speed of a moving object.
>
> For equations of the form $y = mx + c$, in motion contexts, c is typically the starting point and m is the speed of a moving object.
>
> Graphs of motion show characters' start position, speed (relative) and places and times where characters meet.

Landmark activities

The notion of a 'landmark' activity originates in the concept of cognitive breakdown, or a 'situation of non-obviousness' (Winograd & Flores, 1986, p. 165), in which established routines are 'replaced by conflict, disagreement or doubt' (Hoyles & Noss, 2002). This resonates with the role of 'contingent moments' within the development of mathematics teachers' knowledge and practice (Turner & Rowland, 2011). In the context of the technology-enhanced mathematics classroom, it is anticipated in the design that the technology would disrupt routine practices in a transformative sense, and that the ensuing breakdowns would provide insights into developing practices.

We define landmark activities as those which indicate a rethinking of the mathematics or an extension of previously held ideas – the 'aha' moments that show surprise – and provide evidence of students' developing appreciation of the underlying concept. Our challenge was to develop a methodological approach that enabled us to research *at scale* teachers' perceptions and use of previously identified landmark activities. There is a blurring as to whether such activities are landmarks for the teacher or for their students. They are derived in fact from the perspective of the teachers although, as will be seen later, teachers' perceptions are invariably influenced and substantiated by their students' responses to tasks, supported by day-to-day formative assessment practices. We recognize the temporal nature of landmark activities in that the teachers' initial selections of landmark activities resulting from their first teaching of the CM unit might evolve and, we conjecture, stabilise over time. At this point, we start from the assumption that, if teachers show awareness of landmark activities that align with the design principles, and foreground [4] them for their students, they have 'got it' with respect to the design principles of the unit.

The process of identification of landmark activities went through several stages. First, the research team (the authors of this paper) made their own selection from the student workbook. Then they discussed their selections and agreed a list of eight activities that were highly aligned to the design principles of the CM curriculum unit under discussion. The tasks included some that were mediated by the software and some that were paper-and-pencil tasks. This process was repeated face-to-face with a focus group of three teachers, selected as they had provided thoughtful reflections to the online surveys, who provided their rationale for their choices. We would have preferred that all teachers gave a wholly open-text response in justification for each landmark question. However, as hundreds of teacher would be responding to the questionnaire over a timeline of years, the project resources did not extend to the resulting qualitative data analysis process. The data gathered from the focus group of teachers, supplemented by responses from the Cornerstone Maths professional development team informed the development of a set of answer prompts that were used within the wider online survey. These prompts are given in Table 1 alongside the teachers' survey responses.

As part of the final questionnaire to teachers, administered after they had completed their first teaching of the CM unit, we asked teachers to report their three most memorable landmark questions and justify their choices by selecting one or more of the answer prompts and/or by selecting 'Other' and providing their own reason.

Rationale for choice of landmark question	**% of total number of selections (n=601) [6]**
Most students were engaged and motivated to complete the activity.	22
It provoked rich mathematical discussion.	46
Most students were able to record their explanations.	20
It revealed important misconceptions.	35
It revealed progress in understanding.	48
Other	8

Table 1: Summary of teachers' reasons for their identification of landmark questions

We conjecture that the teachers would be able to identify particular tasks in the CM materials that could be described as landmark with respect to students' mathematical learning and engagement. The teachers' justifications would be based on their: observations of their students engaging in the various CM activities (with and without the technology); questioning of students about their work (individually, in small groups and during whole-class teaching); and reviews of students' written responses in the workbooks.

FINDINGS AND DISCUSSION

Our initial analyses of 98 of the 111 [5] teachers' responses suggest that there are some trends emerging from the data that offer an insight into the teachers' perceptions and classroom practices with respect to landmark activities. Table 1 summarises the teachers' responses for all of their justifications of their chosen activities as landmark.

Almost half of the responses related to teachers' evaluation of their chosen landmark activities as being important for their role in: provoking rich mathematical discussion; revealing important misconceptions; and revealing progress in understanding. This suggests that the construct of a landmark activity is relatively well-defined.

The 'Other' responses, of which there were 31 comments that related to a particular landmark activity, could be classified as expanding on the particular: mathematical content (i.e., 'stationary= flat line'); mathematical process (i.e., 'it enabled the students to reflect on the connections between the graph, the table and the equation'); and highlighted the particular mathematical difficulties that the students had overcome (i.e., 'students found it difficult to fill in the table from the information given').

We highlight two contrasting findings that lead us to critique the validity of our approach with respect to our underlying aim to develop a methodology that might be appropriate for large-scale studies that seek to research implementation fidelity.

Firstly, we report on the frequency of teachers' alignments with one of our *a priori* landmark activities, supported by qualitative data provided by teachers for their choices. We focus on one landmark activity, as selected by 17 teachers (shown in Figure 1), taken from Investigation 3, which required students to interact with the software to respond to a series of questions that were designed to develop their understanding of the mathematical concepts.

This question was one of a series of sub-questions that required students to generate an appropriate graph by interacting with the software. The teachers' most common justifications for this choice of landmark question were: it provoked rich mathematical discussion (n=7); it revealed important misconceptions

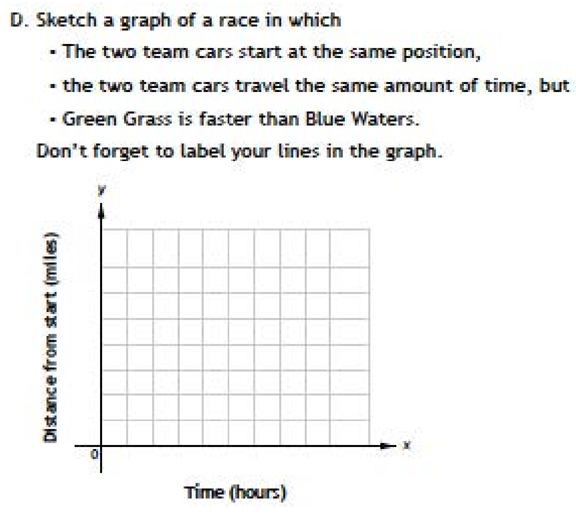

Figure 1: A landmark question from Investigation 3

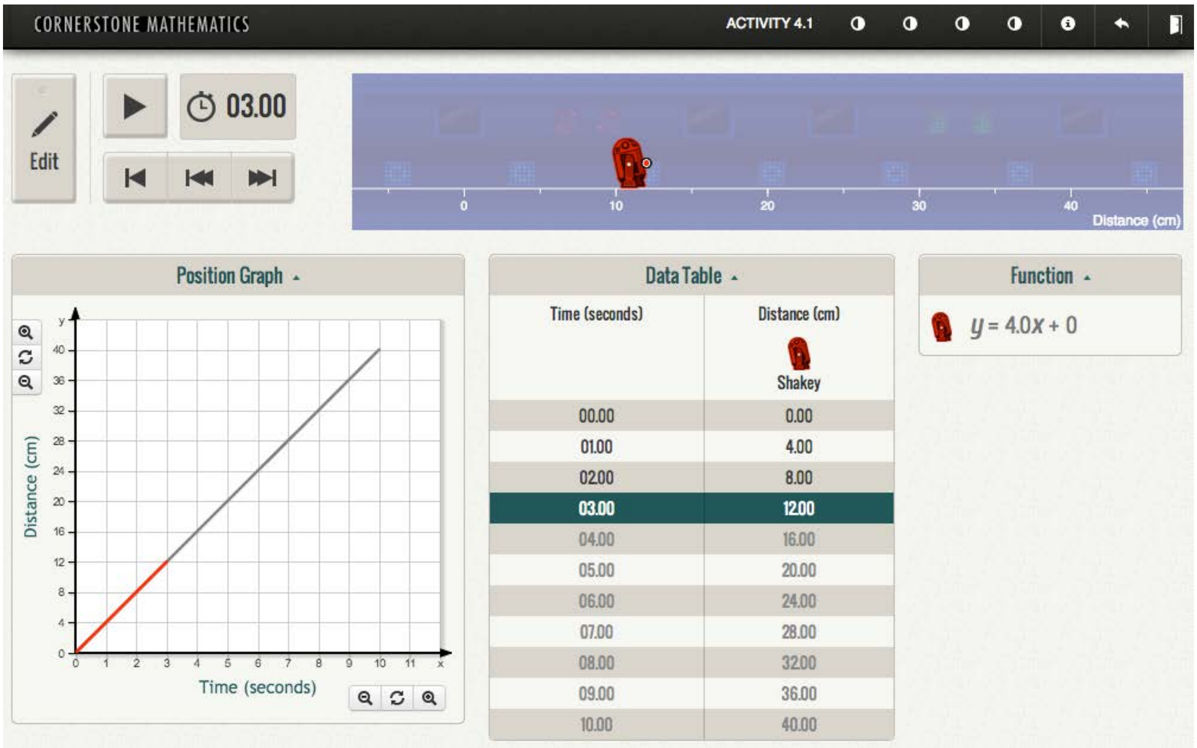

Figure 2: The accompanying software environment

(n=10); and it revealed progress in understanding (n=9). One teacher offered the additional comment,

> When answered together they showed immediately whether a pupil had understood the nature of the graphs, the different axes, the similarities and differences.

Another reported,

> I could easily see which students had a deep understanding of the concept specifically how the position and time were placed on the graph and the meaning of these axes.

By contrast, we were surprised by one landmark question that was identified by 28 teachers, as we had interpreted this activity as simply recording what students were seeing in the dynamic multiple representations and it did not, in itself, highlight any relational concepts. However, our teachers viewed it differently.

This question related to the software screen shown in Figure 2.

In the activity, the students were asked to edit the graph and play the resulting simulation in order to determine how time, position and speed were each represented in the graph, table and equation respectively. They were required to record a written response in their workbooks.

Fifteen of the teachers reported that this question had provoked rich mathematical discussion and fourteen stated that it had revealed important misconceptions on the part of their students.

One teacher justified her choice by saying,

> Because if students understand and can articulate the representation, then everything else follows.

CONCLUSION

Research on the successful scaling of educational innovations, with an emphasis on aspects that impact upon mathematics teacher development, is of primary importance given the funding constraints that many countries are experiencing (see Blömeke, Hoyles, & Rösken-Winter, 2015; Thompson & Wiliam, 2008). Furthermore, innovations that involve mathematical technologies are known to be slow to integrate and scale (Organisation for Economic Co-operation and Development, 2010).

The developing methodology reported in this paper offers a way to ascertain the fidelity of the resulting implementations with a large number of teachers. Although needing further elaboration, the construct of the landmark activity appears to resonate with teachers and their views captured in an online survey methodology with high response rates over time. Follow up interviews would certainly enrich the findings if this could be undertaken with some sample.

We will also explore how the observation of teachers' responses to landmark activities *during* the CM face-to-face professional development might shed further light on teachers' own knowledge development in our ongoing research.

The construct of landmark questions promises to be a productive way to probe teachers' interpretations of the CM unit of work and the mathematical outcomes for their students. Our ongoing work will seek to extend and validate the construct and its application within our studies.

ACKNOWLEDGEMENT

We gratefully acknowledge funding by the Li Ka Shing Foundation and Hutchinson Whampoa. The research has been an intensive collaboration between teams at the London Knowledge Lab, Institute of Education, University of London and at the Center for Technology in Learning, SRI International, Menlo Park, USA.

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ENDNOTES

1. The student workbook is a consumable book that contains the task instructions. Students record their responses to interactions with the technology and other related questions in the workbook.

2. We conjecture that, whilst the efficacy of the Cornerstone Maths materials have been established by its pilot research projects, individual schools and teachers will seek to validate the materials to (re-)establish its efficacy within their institutional settings as an important component of the process of scaling.

3. We adopt the ideas of 'legal' and 'legitimate' mutations of an innovation to describe the extent to which classroom implementations adhere to its original design principles. This is not a bipolar scale.

4. As all teachers have access to the same set of teaching materials, they make individual decisions about which mathematical content and processes to emphasise (or foreground) in their teaching.

5. 111 teachers had completed their teaching of the CM curriculum unit by the end of September 2014.

6. Teachers could make multiple selections. This represents the total number of responses across the teachers' choices of three landmark questions.

Social creativity and meaning generation in a constructionist environment

Dimitris Diamantidis[2], Katerina Economakou[2], Areti Kaitsoti[2], Chronis Kynigos[1] and Foteini Moustaki[1]

1 University of Athens, School of Philosophy, CTI & Press "Diophantus" and Educational Technology Lab, Athens, Greece

2 University of Athens, School of Philosophy, Educational Technology Lab, Athens, Greece, dimitrd@ppp.uoa.gr

The use of digital tools for "doing mathematics" has been studied both from the meaning making perspective and from the point of view of social interactions. In this study, we discuss how the use of digital tools that support collaboration, exchanging ideas and artifacts among students in a dense and intense way fosters the mechanism of meaning making in a group of 9th grade students that interact with a half-baked microworld. We reemploy the UDGS model to describe meaning making, but this time from a social aspect, using the notion of social and sociomathematical norms. In this context of analysis, we search for instances of social creativity, while investigating the connection between creativity and students' joint mathematical thinking.

Keywords: Digital tools, meaning generation, social creativity, theory integration, Logo-turtle geometry.

THEORETICAL FRAMEWORK

The use of digital tools for creative mathematical thinking has mostly been studied from two different perspectives, focusing respectively either on meaning generation or social interaction. Nonetheless, the availability of digital tools that support both math meaning generation and communication among students has recently highlighted the value of drawing from both perspectives. Following this strand, we attempt to investigate meaning generation itself as a social interaction process. In this paper, we study shared meaning making in a context where the focus is on the social interactions among students. The students worked in groups with a digital medium designed to support tinkering with a 3D Turtle Geometry tool using dynamic manipulation and Logo programming. At the same time, this medium allowed students' online collaboration and communication through shared workspaces.

For this study, we have chosen two theoretical tools. Firstly, the notion of social and sociomathematical norms (Yackel & Cobb, 1996; Kynigos & Theodosopoulou, 2001), which we found useful, as it helped us interpret the social interaction of the students during their communication, in terms of "microcultures" and taken-as-shared behaviors of each group. We found especially useful to consider meanings generated through interaction and the taken as shared as work progressed.

Secondly, the UDGS model (Hoyles, 1987), which although it was first used back in the mid eighties, we found it useful for our research, as it is a tool that describes students' meaning making process while they engage in mathematical exploratory activities with digital media. According to this model, there are four phases of the meaning making process: Using, Discriminating, Generalizing and Synthesizing. At first, students use mathematical and non-mathematical concepts, without much attention to their actual meaning. In the next phase they discriminate elements of mathematics in their constructions and the way they use them. Through the observation of patterns in relations or properties of the Logo commands they use, students generalize their ideas. Finally, they make synthesis of these generalized ideas with typical mathematics that these ideas are based on. In this framework a mathematical meaning is the way that a student understands, uses and thinks of a certain mathematical concept.

In this paper, we discuss a classroom study where different group configurations of students experimented with the "Twisted Rectangle" half-baked microworld (Kynigos, 2007). Half-baked microworlds are incomplete by design, challenging students to explore the reason for the buggy behaviour they show, engaging them in the process of mathematical meaning-making.

The Twisted Rectangle's buggy procedure creates an open skewed rectangle, intriguing students to try to fix and express their own mathematical ideas on how to reconstruct it (Figure 1). To conduct their experiments as they tried to find the bug and then work out the mathematics necessary to fix it, the students needed a medium able to support collaboration, joint planning, argumentation and meaning making.

The Metafora Platform (Dragon, Mavrikis, McLaren, Harrer, Kynigos, Wegerif, & Yang, 2013) was built to encourage students to "learn how to learn together". Group members have tools to make plans, to act as designers, and to publish, argue over and discuss their constructions. This act of designing and publishing (Kafai, 2006) is an externalization of an individual's tacit knowledge, or a group's knowledge, in the case of a joint construction of more than one individual. We were interested in studying how meanings were shared and argued over as an integral part of the students' activity. Artifacts were available at all times for inspection and reconstruction, starting from discriminations of the ideas embedded in the procedure by the designers (Kynigos, 2012). It has been a long time now that Papert and Harel (1991) suggested that when artifacts are published intensively and densely in a learning collective, meaning making process happens naturally. We wanted to study this process in detail, to capture the process of shared meaning making and the kinds of socio-mathematical norms generated, as groups of students jointly tried to fix a buggy artifact and use it to build their own.

In this context of social interaction and building on our previous work (Kynigos & Moustaki, 2013), we wanted to give particular focus on creativity in mathematical thinking. Since we talk about groups, we put emphasis on social creativity. We found Fischer's approach as a good tool for us to think with. Arias and Fischer (2000) emphasize that externalization supports social creativity as students move from vague mental conceptualizations of an idea to a more concrete representation of it. It also allows students to interact with, negotiate around and build upon an idea as the diversity of voices and minds increases. Fischer's group approaches creativity as a social process which has four elements: (1) originality: people or, in this case, students have novel ideas or they are capable of applying prior knowledge in new contexts, (2) expression: students should be able to express and externalize these new ideas, (3) social evaluation: students with different perspectives should be able to evaluate these novel contributions, reflect upon them and improve them, and (4) social appreciation: refers to the credits and acknowledgment from the other participants of the group motivating further creative activities (Fischer, Scharff, & Ye, 2004). Fischer, Giaccardi, Eden, Sugimoto, & Ye (2005) have described the characteristics of situations that, in their approach, support this social aspect of creativity: they are *open-ended* and *complex* so that students will be led to unpredictable results and eventually to experiences of *breakdowns*. Breakdowns offer opportunities for reflection and learning, through the procedure of the back-talk of situations (Fischer et al., 2005). Another form of social creativity is *co-creation* which is a situated experience leading to emerging and sharing creative activities with no explicit goal and meanings in a socio-technical environment through synchronization and improvisation as students share emotions, experiences and representations (ibid).

Although there seems to be a connection between aspects of social creativity and the meaning making process when students work in groups, we found research on these two approaches to be rather fragmented with respect to emphasis on one or the other. In our study we tried to see if these separate views can be usefully integrated in a situation where co-constructing students work in groups that communicate with each other exchanging ideas and different versions of the original figure.

TECHNOLOGY

The Metafora platform brings together students from various backgrounds to solve problems of fixing models which we faulty by pedagogical design (Kynigos, 2012). It hosts three types of tools available to the students at all times: a 3D Turtle Geometry environment, an argumentation tool and a shared workspace for students to make shared plans of their actions as a

Figure 1: The half – baked microworld "Twisted Rectangle" in 3D Math

group. The 3D Math tool (Figure 1) affords Logo-based Turtle Geometry with a feature for dynamic manipulation of variable procedures once executed with a set of values (Kynigos & Psycharis, 2003).

THE STUDY

Research design and methodology

In the study we used the methodological tools of "design research" (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003), which is an empirical study of human activity in real settings. In the implementation of the research, students and teachers were engaged, for several sessions, in an activity with the use of digital tools. This research framework is suitable in the classroom, where the learning ecology is described by means of collaboration and effectiveness in addressing tasks that challenge students to take initiative in specific situations (Collins, Joseph, & Bielaczyz, 2004).

Ten 9th grade students, three mathematics teachers of a public Experimental School in Athens and four researchers participated in the research. The implementation took place in the school pc lab, after-class, in the frame of the school Math Club activities, for twelve sessions of two teaching hours each (about two and a half months). Most of the students were not novice users of 2D Turtleworlds. In the time of the study, students had been taught trigonometry, but not stereometry. They were separated in two groups, and each group was divided in two subgroups. The subgroups of the same group were communicating through Metafora communication tools.

Task analysis

The bug in the procedure given to the students was the absence of a relation between a turn and a length of one of the rectangle sides. This resulted in the procedure producing an "open" figure and the students were faced with the challenge of fixing the bug so that it 'closed' no matter what the variable values were. The challenge required students to find simple sin and cos relations between angles and sides in two triangles lying respectively in two different planes joined only by one common side. Once figured out, the students would have to express these relations with functions allowing for the rectangle to be built with one variable for twist around a vertical axis (wmega), one for a horizontal axis (theta) and one for side length. To do that, the students would have to think about the concept of angle in 3D space.

A useful tool for us to recognize the meanings that students generated was the approach of Henderson and Taimina (2005) of students' conceptualization about angle: as a static geometrical figure, as a number that expresses a magnitude, or as a result of turning. These perspectives of an angle correspond to the static or dynamic definition of it (Mitchelmore & White, 1998). Our hypothesis is that ninth grade students mostly conceptualize angle as a static geometrical figure, which corresponds to the static definitions of angle. In Turtle Geometry however, angles are dynamic turns, rather than static direction relations.

Data collection method

Data collection included conversations between teachers and students, or groups of students, their gestures during their discussions, their constructions on the screen or artifacts that they made by hand. For these reasons we used voice recorders and a camera. A screen-capture software (HyperCam2) was used to record students' interactions with the Metafora tools. We also collected students' manuscripts and drawings. The data corpus was completed by the researchers' field notes.

RESULTS

Episode 1: Elements of social creativity in the phase of discrimination used in the process of generalization

The students of the subgroup 1 used dynamically the variation tool in order to find out which variable of the code corresponded to which spatial characteristic of the "Twisted Rectangle". Founding it difficult to come to a conclusion, they had the idea of reconstructing the figure. They decided to use drinking straws, although there were no straws available till then. The sequence of commands "forward(:length) right(90+:theta/2) up(:wmega) forward(:width)" made the turtle go forward for a distance equal to the variable "length", turn right "90+theta/2" degrees, then pitch up "wmega" degrees and go forward for a distance equal to the "width" variable. Turning right "90+theta/2" degrees seems to be complicated, but "theta" was a structural feature of the figure, related with its buggy behavior. Discriminating the role of "theta" was necessary, so that the students could focus on what was missing to fix the bug. Following this sequence of commands, the turtle drew an angle of the figure. Comparing the two representations of this angle, the one on 3D Math

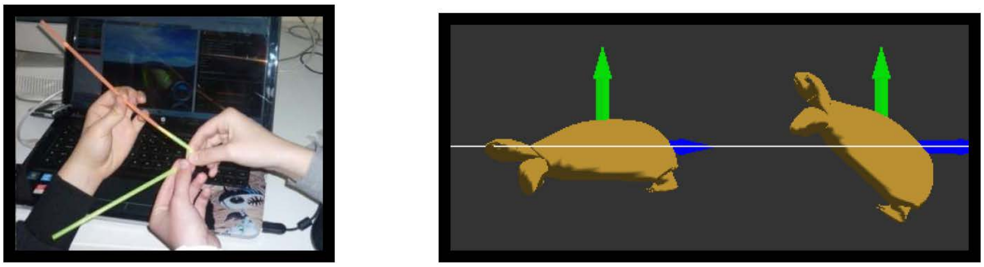

Figure 2: (a) the construction made of drinking straws, (b) the result of command up( )

and the other on the construction of straws, students realized that they are not the same.

Student 1: After fd(:length) it turns right.
Teacher: How much it turns?
Student 1: 90 + theta/2.
Student 2: And after that?
Student 1: up(:wmega). So it goes like this, it has gone up and then goes fd(:width), nice!
Student 2: The figure shouldn't have been like that. Let's do it again...

In this dialogue, the students use interchangeably three representations of the shape: the figure inside 3D Math, the logo code and their construction (Papert & Harel, 1991). Trying to represent the geometrical result of a right turn of "90+theta/2" degrees, they constructed an obtuse angle (Figure 3). This construction of the artifact seemed to be a result of their conceptualization of the angle as a static geometrical figure (Henderson & Taimina, 2005; Mitchelmore & White, 1998). According to the UDGS model, this is the phase of "using", as the students used the mathematical concept of angle without having a complete image of it (Hoyles, 1987).

Furthermore, there seems to be a breakdown in their effort. Thinking that their construction of straws was not totally correct, the students decided to explain the logo code step by step.

Student 1: I have second thoughts about this command...
Teacher: Why?
Student 1: If it turns right 90 plus something, then it goes here (she shows 90-theta/2). But it is not right; the figure ends up being wrong.
Student 2: Thanks God that I do not trust the logo code! (Laughing)
Student 1: Just a moment, give me the straws and the eraser. Ah, it goes here.
Student 2: 90+theta/2 to the right.

In the dialogue above, Students 1 and 2 seemed to evaluate their initial construction and improve it, using an eraser (pointing somewhere) as a representation for the turtle. This modification was revealing of the students' thinking. It seemed that in the first model (without the eraser), the straws were not a signifier for the trace of the turtle. They used the straws as a semantic simply for the sides of the angle. Their idea to use an eraser, in order to add a signifier of the turtle to the model, led to an "improvement" of the use of the straws. The straws instantly became not just the signifier of angle edges, but of the turtle's trace, as well. This "improvement" came up as a more concrete representation of an angle, than their initial conceptualization (the angle as a static geometrical object). The extended model (straws and eraser) was the result of "expressing a new idea" (use an eraser as a turtle), after they interacted with the logo code and the figures. The students reflected upon their model and evaluated it due to the distrust to the code.

The novelty of the students' idea to use the straws, the construction of their model, and the reflection upon it, which led to the evaluation and improvement of the model, according to Fischer's approach, can be interpreted as three of the four elements of social creativity; originality, expression and social evaluation.

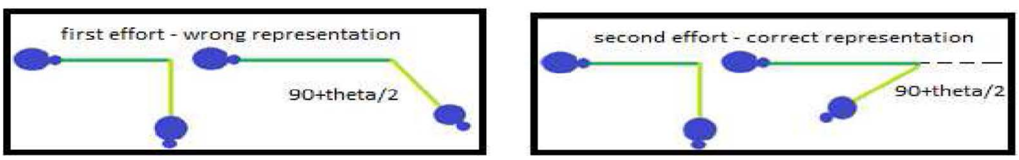

Figure 3: The two representations of the result of the logo command rt(:90+theta/2)

Using their improved model as an instrument (Artigue, 2002), the students started to generate new meanings for the concept of angle, which corresponded to the dynamic definition of it as a turn (Mitchelmore & White, 1998) (Figure 3). In this way, they discriminated the element of turn under their construction and the way they improved it (Hoyles, 1987). Later, as they tried to construct the rest of the figure using straws, they found that the logo commands were following the same pattern.

Student 1: You see? It is fd(:length) lt(90+(:theta)/2) dp(:wmega) fd(:width) instead of fd(:length) rt(90+:theta/2) up(:wmega) fd(:width). Left instead of right and down instead of up.

Student 2: This part of the figure is an angle like the other we have already done.

The students appeared to transfuse a property of the logo code to the figure. They made an abstraction, using a certain pattern of logo commands as a representation of the angle. This observation of patterns of the logo commands is a main characteristic of the "generalization" phase in the UDGS model. When they discovered the same pattern of logo commands elsewhere in the logo code, they recognized it as the symbolic representation of a similar angle, and searched for this angle on the eraser-straw model. In this way, they generated an abstraction, generalizing the concept of an angle and using it. We also suggest that this joint mathematical thinking can be explained through the lens of sociomathematical norms. To be more specific, the argument that a pattern of logo commands defines a certain geometrical figure had been a norm of an accepted mathematical explanation within the group of students.

Episode 2: Creative ideas for synthesizing concepts across context

At the end of the previous session the group managed to address the challenge (Figure 4a). Subgroup 1 and subgroup 2 had different perspectives of the solution (geometrical and algebraic), but they reached to a common solution. The students, reconstructing the code, created a formed and closed "Twisted Rectangle". In this session, the subgroups using the Twisted Rectangle as a building block created their own constructions with 3D Math. Subgroup 1 had constructed a figure that reminded them a logo of a chain store, while subgroup 2 had constructed a "flower". They used the communication tool in order to exchange and combine logo codes and ideas:

Subgroup 2: You could test the code "flower" that we have already tested. We'd like to tell us what you think about it and what you have done... in order to make a combination of our codes!!!

Subgroup 1: We created the code "stem" and combined both codes so we created a new code called "flower with a stem"!

Subgroup 2: Ok. We wanted to make a bigger flower, so we added more variables... We should combine this code now with yours again...

Subgroup 1: Ok! We sent you the new combined code: "The flower with a stem"!

Then, they used the logo of the chain store as a "vase" to put the flower in (Figure 4b).

According to the UDGS model, using the generalized "Twisted Rectangle" as a building block is an instance of meaning making. Observing the students' interactions we notice a dense publishing of their own constructions and an intense exchanging, reflection, combination and improvement of their logo codes (which represent their ideas). Based on the approach

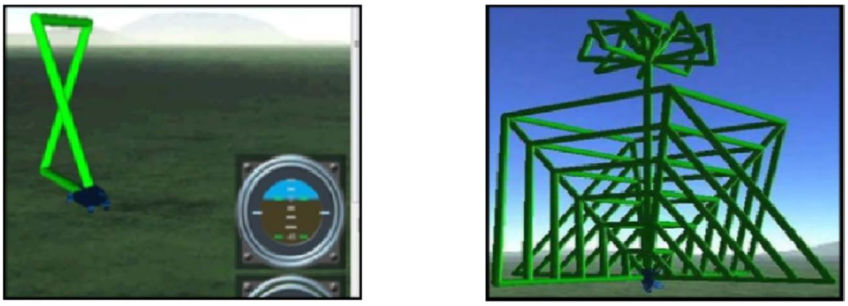

Figure 4: Students' construction: (a) The twisted rectangle, (b) The flower in a vase

of social norms we suggest that this taken-as-shared behaviour was a basis for their communication, which indicated the group "microcultures" (Yackel & Cobb, 1996). This kind of microcultures was characterized by externalization of original ideas and social evaluation of them (through reflection and improvement). According to Fischer and colleague's (2004) approach these are elements of social creativity. Moreover, during the construction of the artifact we noticed the development of taken-as-shared understanding of what was an "appropriate" logo code in order to create a common construction; the two subgroups used variables instead of numbers in their codes.

From our point of view, this shared behaviour of "using variables" is related to the embodiment of the power of generalization that occurs during the meaning making process and indicates a sociomathematical norm. We suggest that this common behaviour was crucial for addressing a task with no explicit goal, like this one. According to Fischer and colleague's (2005) approach and taking under consideration that students used a digital tool which supports communication and sharing of ideas, this situation can be characterized as co-creation which is a form of social creativity.

DISCUSSION

The study discusses two episodes where meaning generation was evident in a context of social creativity in mathematical thinking. In the first episode students of subgroup 1 realized by tinkering with the model of straws that it was not an accurate representation of the Twisted Rectangle because it was static and they improved it using an eraser. Taking a close look at the students' activity, we suggest that they were trying to rebuild their model to be the closer to what the Logo code represented, which was a construction of the Twisted Rectangle, rather than a static result. This novel idea came up early, during the phase of discrimination (of UDGS model) in the meaning generation process. It seems that this idea was shared by the students and used in their attempts to make sense of angle in space which were perceived as joint. The eraser semantic initiated a developing of a socio-mathematical norm about how to think of dynamic angle in space which was then taken as shared in subsequent generalizations of angle and trigonometric relations to twisted rectangle sides.

In the second episode, the students of each subgroup used their shared resolved procedure of a generalized twisted rectangle as a building block to build their own constructions. The two subgroups exchanged their constructions through the Metafora argumentation tool in a dense process focused on negotiating meanings to explain and exchange developing complex models. The ideas of what to build and how it behaves were shared between the two groups which were operating as a new group to show the class what they had done. They were thus conceived and used in a social setting from the beginning and the language developing in the groups seemed to create an atmosphere of social creativity and sharing of these ideas. We take these shared practices and behavior as situated in interaction with the medium of this particular collaboration. From this point of view, we can argue that the Metafora Platform afforded and fostered the emergence of social creativity in the group activities.

In both episodes the learning process seemed to be fostered by social practices, such as the making of shared meaning, common argumentation and beliefs of what is accepted as a solution. Although we were looking at socially emerging meanings, we found the use of UDGS shows helpful to describe meaning generations but this time to try to understand if this meaning making process can be described with these tools as a collective process in emergence of social creativity. More precisely, we found elements of social creativity emerging within the phases of the UDGS model. This study made us reflect that it may be worth readdressing the problem of meaning making in new kinds of collectives now that we have digital media that support communication, collaboration and joint mathematical thinking.

ACKNOWLEDGMENT

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n° 610467 – "M C Squared" project. This publication reflects only the authors' views and Union is not liable for any use that may be made of the information contained therein.

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Instrumental genesis concerning scales and scaling in a dynamic mathematics software environment

Maria Fahlgren

Karlstad University, Department of Mathematics and Computer Science, Karlstad, Sweden, maria.fahlgren@kau.se

It is recognized that the process in which an artefact becomes an instrument for a user, denoted as instrumental genesis, is a complex process. The aim of this paper is to identify elements of the process of instrumental genesis when students are dealing with scales and scaling issues in a dynamic mathematics software environment. This study involves four upper secondary school teachers and their classes. By observing the students' instrumented techniques while working with tasks designed with a hypothetical instrumental genesis in mind, some key elements are identified.

Keywords: Instrumental genesis, scaling of axes, dynamic software environment.

INTRODUCTION

Several decades ago, researchers recognized the affordances provided by graphical technologies, in particularly in the field of functions and graphs (e.g., Goldenberg, 1988; Leinhardt, Zaslavsky, & Stein, 1990). For instance, in comparison to the corresponding work with paper and pencil, they emphasized the ease, and thereby the speed of changing the scales of the axes to obtain several different views of a graph. However, some difficulties have also been identified relating to the issue of scales and scaling of axes (Hennessy, 1999; Mitchelmore & Cavanagh, 2000; Yerushalmy, 1991).

The availability of different kinds of technology in mathematics classrooms is increasing and more and more students are provided with a computer of their own (Valiente, 2010), which entails new possibilities for the integration of technology in mathematics education. However, there is a need for students to learn how to use the technology appropriately so that it becomes an instrument for them. Several researchers use the notion of *instrumental genesis* to describe this process by which an artefact [1] becomes an instrument for a user (e.g., Artigue, 2002; Drijvers & Gravemeijer, 2005; Trouche, 2004). However, there is agreement that the complexity of this process has been underestimated and may have contributed to the recognized difficulty of integrating technology into mathematics teaching and learning (Artigue, 2002).

To address the challenge of integrating technology into the mathematics classroom, Trouche (2005) introduces the notion of *instrumental orchestration*, which also takes account of the social dimension of the instrumental genesis within a classroom. Although many researchers associate instrumental orchestration primarily with the organisation of classroom interaction (e.g., Drijvers, Doorman, Boon, Reed, & Gravemeijer, 2010), the original notion also involves the customization of an artefact in order to create a particular task environment (Ruthven, 2014). This paper concerns the customization of a dynamic mathematics software environment, especially the tools associated with scales and scaling. The tasks were designed with a particular instrumental genesis in mind. The aim of this paper is to pinpoint some elements affecting the process of instrumental genesis in relation to scale and scaling issues within a dynamic mathematics software environment, in this case *GeoGebra*.

INSTRUMENTAL APPROACH

One important aspect of this approach is the process of *instrumental genesis* (Verillon & Rabardel, 1995). Through this process an artefact becomes an instrument for a user. An artefact is an object, material or abstract, available to the user and aimed at performing a certain type of task. For an artefact to become an instrument for a user, there need to exist a meaningful relationship between them (Drijvers & Trouche, 2008). In this way, "...the instrument consists of both the artefact and the accompanying mental schemes..." (p. 367), developed by the user.

It is recognized that the process of instrumental genesis has two directions, one towards the artefact (*instrumentalisation*) and one towards the user (*instrumentation*) (Trouche, 2004). The user shapes the artefact by his/her knowledge and former method of working while the artefact shapes the subject by its constraints and potentialities. However, the fact that the process of instrumental genesis is a rather complex and thereby a time-consuming process has been underestimated (Artigue, 2002). One reason for this, Artigue argues, is the predominant role as a pedagogical tool given to technology.

> Suggesting that instrumentation may be a complex and costly process does not fit visions that consider technology mainly as an easy tool for introducing students to mathematical contents and norms defined independently from it. (2002, p. 253)

Regarding what could be considered as an artefact depends on the situation under consideration. For instance, a symbolic calculator could be considered as a collection of artefacts (Trouche, 2004). Accordingly, this provides students with the possibility to develop several types of instrument while working with this kind of technology. Besides the artefacts, the kinds of instrument being developed depend on the students and the accompanying tasks (Maschietto & Soury-Lavergne, 2013). Hence, it is important for a task designer to be aware of, among other things, the potentialities and constraints of an artefact (Artigue, 2002; Trouche, 2004).

Concerning the mental schemes students develop through the instrumental genesis process, researchers (e.g., Drijvers & Gravemeijer, 2005; Trouche, 2004) distinguish between two categories, *usage schemes* and *instrumented action schemes*. The usage schemes are basic and relate closely to the artefact while instrumented action schemes focus on actions upon objects such as graphs or formulas. In this way "Instrumented action schemes are coherent and meaningful mental schemes, and they are built up from elementary usage schemes by means of instrumental genesis." (Drijvers & Gravemeijer, 2005, p. 167). However, it is not always obvious how to distinguish between these kinds of scheme; it might be a matter of level of capability of the user (Drijvers & Gravemeijer, 2005).

Drijvers and Gravemeijer (2005) argue that instrumented action schemes involve both technical and conceptual aspects. Even if it is the development of conceptual knowledge that is the most interesting, it is the technical activities that are visible and thus the observable part which can be the object of investigation. The technical activities that are developed through instrumental genesis are denoted as *instrumented techniques* (Artigue, 2002) or just *technique* (Lagrange, 1999). In this way, it is the technique that is the "...gateway to the analysis of instrumental genesis." (Drijvers & Gravemeijer, 2005, p. 169).

SCALES AND SCALING WITH TECHNOLOGY

Several decades ago, researchers were already discussing the influence that new technology would have in the field of functions and graphs (e.g. Leinhardt et al., 1990). This section introduces some issues relating to scales and scaling of axes.

In traditional paper-and-pencil work, for example, textbook tasks, graphs often are presented as static diagrams with the coordinate axes scaled in an appropriate way (Zaslavsky, Sela, & Leron, 2002). When working with a graphical technology, on the other hand, the scaling of axes is often left for students, which has proved to cause them some difficulties (Hennessy, 1999; Mitchelmore & Cavanagh, 2000). Further, Leinhardt and colleagues (1990) assert that students' ability to deal with scaling of axes is often taken for granted and argue that "...the construction of axes requires a rather sophisticated set of knowledge and skills." (p. 43). As an example of elements of instrumental genesis observed in a CAS environment, Artigue (2002) discusses "framing schemes":

> When students use function graphs in a computer environment (or a graphic calculator), they are faced with the fact that a function graph is "window-dependent" and they have to develop specific "framing schemes" in order to cope efficiently with this phenomenon. (p. 250)

Mitchelmore and Cavanagh (2000) argue that one reason for students' limited understanding of scaling might be their lack of experience in dealing with graphs where the axes are unequally scaled. In line with this, Goldenberg (1988) discusses students' preference for "symmetric scaling", i.e., *x*- and *y*-axis are equally scaled. He refers to an example where students who had received an appropriate view of a graph, still changed the scales to obtain symmetric scaling. In this way, the students received a visual appearance obscuring important features of the graph. To receive a better visual appearance, they changed the scales of the axes by the same factor, i.e. they used a zoom operation. Goldenberg (1988) argues that one reason for this might be that students' intuition about scale changes is closely connected to real-world experiences: "... our almost automatic approach is to change both scales by the same factor..." (p. 36). However, since usually different units on the axes are required to see the graph in an appropriate way, Goldenberg (1988) stresses the importance for students to deal with unequally scaled axes.

Mitchelmore and Cavanagh (2000) found that students in their study showed limited understanding of the zoom operation. The students used the zoom operation as a magnifying glass but "...were unable to link the operation of zooming with any change in the scale of the graphs displayed in subsequent viewing windows." (Cavanagh & Mitchelmore, 2003, p. 14). As an example, they refer to a case were students who zoomed in on the vertex of a parabola were surprised to see a linear shape of the graph.

Concerning how unequal changes of the scales of the axes (i.e. not zooming) impact on the visual appearance of the graph, Yerushalmy (1991) emphasizes the importance of understanding the difference between the properties of a function and its picture in a graphical view. In her study, students showed difficulties when interpreting the same function graphed in two different scale systems, since the visual appearance differed (Yerushalmy, 1991).

Godwin and Sutherland suggest graphing software as a mean to provide students with experiences of this kind since it provides "The ability to change the scale easily and hence the 'frame of view' of a function..." (p. 134). Further, Hennessy (1999) emphasizes the importance for students not only to realize what will change but also what remains constant as the scales change.

METHOD

This study is embedded within a form of design research project involving two researchers and four upper secondary school teachers with one class each. The focus of the project was on a sequence of three lessons, taught over the course of a school year, in which some use was made of *GeoGebra* to tackle tasks concerning functions and graphs. The researchers and teachers had meetings before and after each lesson. In total, three worksheets, one for each computer lesson, were designed. Although the responsibility for the design was the researchers', the teachers provided valuable information regarding the participating students' capabilities and their current practices.

Each worksheet consists of a sequence of related tasks, denoted as a *task sequence, TS*. The first and the third worksheets concern exponential functions, while the second worksheet is mainly about linear relationships. This paper focuses on issues concerning scale and scaling which are mainly addressed in the task sequences for the first and second lessons. A more comprehensive report of the project will be provided in another paper by the author.

The overarching aim of the task sequences is to enhance both students' conceptual knowledge about functions and graphs and the scope for the dynamic mathematics software to become an instrument for the students (Verillon & Rabardel, 1995). Therefore, on the worksheets the task sequences were intertwined with computer instructions. As noted earlier, the way in which this was done constitutes an instrumental orchestration of the task environment.

The participating students were tenth grade students with no previous experience of working with either dynamic software or graphical calculators. Altogether, 85 students participated in the study. The students were to work in pairs with *one* computer per pair. The purpose of this is that the computer screen should provide a shared object for discussions between students.

The empirical data reported in this paper were mainly collected during the two relevant lessons with the four classes. Each lesson lasted about 60 minutes. In each class one pair of students was video recorded and all teacher-student interactions during the lessons were audio recorded using a microphone attached to the teacher. When necessary in the analysis process, copies of the written responses from the students were used.

In the analysis process, the video recordings were the primary source since these data made it possible to observe students' instrumented techniques which are the observable part of the instrumented action schemes (Drijvers & Gravemeijer, 2005). The audio recordings provided further insight into the frequency of some student responses.

The scope of this paper is restricted to only dealing with aspects concerning scale and scaling in relation to the process of instrumental genesis. Thereby the tools addressed in this paper are those associated with the coordinate system, and particularly the tools needed to obtain appropriate visual graphical appearances.

RESULTS

This section introduces some prototypical observations regarding how students deal with scales and scaling of axes. The first task in the first task sequence (TS 1) was intended to be a routine paper-and-pencil task which introduces the context of TS 1:

> The height of a sunflower is 50 cm when it is measured for the first time (June 1). After that the sunflower grows so that it becomes 30 % higher each week. Calculate the height of the sunflower one week after the first measurement.

The first task is followed by computer instructions and the students are expected to use the software to enter the points known so far. Regarding the first point (0,50), we decided to provide students with both conceptual and technical guidance as follows:

> When the first measurement is performed, $x = 0$ (since 0 weeks have passed) and $y = 50$. The corresponding point in a coordinate system is (0,50).
>
> Insert this point by entering *(0,50)* into the "*Input Bar*": Input: **(0,50)**
>
> *NOTE!* To be able to see the point you must adjust the scale on the *y*-axis. This can be done by "dragging" the *y*-axis. (first mark ✥)

Since we anticipated that several students might be confused when they cannot see the point in the Graphics View [2], we decided to add the note above to draw attention to the scaling of the *y*-axis. However, this note proved to be insufficient since utterances like "Should we not get a point there?" appeared several times both in the audio and video recordings. The video recordings indicate that the reason for this might be students' eagerness to start using the computer and not to spend time reading the instructions. Consequently, the teachers often had to draw students' attention to the *y*-axis making them aware of its scale. Further, since the students tended to use the zoom operation when changing the scales, the teachers frequently had to demonstrate the possibility to only adjust the *y*-axis.

Next, the students were expected to enter the point corresponding to their calculation in Task 1, i.e. the point (1,65). To make students aware of the possibility of adjusting the *x*-axis, i.e. adjusting one axis at time, to obtain an appropriate graphical view, we added the following note:

> If appropriate, adjust the scale on the *x*-axis!

However, this note seemed to be ignored by most of the students. Maybe the note came too early, and by this stage the students could not see the point of it. The empirical data reveal that even students who had adopted the technique of adjusting one axis at a time tend to focus on the *y*-axis and not use this possibility when it comes to the *x*-axis. Both the audio and video recordings reveal that the teachers frequently had to draw students' attention to the grading of the *x*-axis to get a more appropriate visible view of the coordinate system.

The second task sequence, TS 2, introduces a context problem where the students are encouraged to formulate a linear function formula to insert into *GeoGebra*:

> To get money for a class trip, the Class 9b at Sugar School decided to rent a table and sell candy at the market place. The rent for a table is 100 SEK. They purchase candy for 40 SEK per kg. Determine a formula that describes how the total cost depends on the weight in kg of the candy purchased.

As in TS 1, the *y*-axis has to be adjusted to see the object, in this case a graph, in the coordinate system. Therefore, we decided to give the following note:

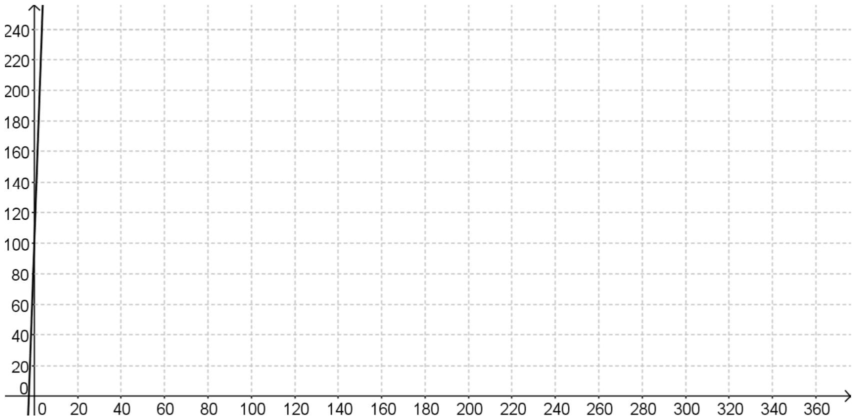

Figure 1: One example of a Graphics View obtained by students

> *NOTE!* To be able to see the graph (the line) there might be a need to adjust the axes.

As in TS 1, both audio and video recording show that several students could not see the object due to the scaling of the y-axis. In their attempts to find out the reason why they cannot see the graph, one pair of students discussed whether they have to type a sign (*) for multiplication between 40 and x. Thus, they did not reflect on the scales. Reminded by the teacher of the scaling of the y-axis, they used the zoom operation, and obtained a graphical view like the one in Figure 1.

The empirical data reveal that graphical views similar to the one in Figure 1 were frequently obtained by students. Actually, all the video recorded pairs obtained a similar view. This, in turn, caused difficulties when they were encouraged to attach a point at the graph and (a) describe how they could use the point to check their calculation in the preceding task, in this case moving the point so that its x-coordinate becomes 10 and read off the corresponding y-value and (b) decide the value of x for a specific value of y. With a graphical view like the one in Figure 1 it turned out to be impossible to use the Graphics View to read off the corresponding value of y when of $x = 10$ without changing the scale of the x-axis. One of the video recorded pairs of students expressed their confusion:

Student 1: It looks like we have done wrong.
Student 2: mm
Student 1: Go back!
Student 2: We check a little more...lock from the start. There is nothing wrong with the equation.

Having convinced themselves that the formula is correct and that they understand what the point presents, they felt that they got stuck and asked the teacher for help. Notably, they did not reflect on the scales of the axes. Reminded of the possibility of changing the x-axis, these students solved the task.

The audio recordings also revealed this kind of confusion on several occasions. Consequently, the teachers often had to remind and instruct the students how to scale the x-axis to receive integer marks and to obtain a better visual appearance of the graph.

However, it was observed how one of the video recorded pairs tackled the problem caused by inappropriate scaling of the x-axes (see Figure 1), by only observing the Algebraic View while solving the task. They moved the point along the graph until the x-coordinate of the point shown in the algebraic view became 10.

Although the participating students when entering the first two task sequences seemed to lack experience of adjusting one axis at a time, to obtain an accessible graphical view, the video recordings indicate an enhanced ability among the students to perform this when tackling the third task sequence.

CONCLUSION AND DISCUSSION

The aim of this paper was to identify elements affecting the process of instrumental genesis concerning appropriate scaling of the coordinate system in a dynamic mathematics software environment. During their work with the tasks, the participating students encountered situations which required rescaling of the axes; to see an object in the coordinate system and/or to obtain an appropriate visible appearance of the object(s). The following closely related elements of instrumented action schemes were recognized:

1) Knowing how to change the scale of the axes, both by using the zoom operation and by adjusting one axis at a time.

2) Realizing when it is necessary to adjust the y-axis to see an object, for example, points or a graph.

3) Realizing when it is appropriate to change the scale of the x-axis to obtain a better visible picture of the objects.

It is possible to distinguish between the technical and conceptual character of the elements (Drijvers & Gravemeijer, 2005). While the first item primarily requires technical capabilities the other two items primarily demand conceptual knowledge. In this study, the mathematical knowledge required to be able to rescale the axes in an appropriate way is about the range and domain of functions representing real world situations. Concerning the instrumented techniques of changing scales of the axes, the result indicates that students are disposed to employ the technique of zoom operation, which aligns with the findings already reported by Goldenberg (1988) more than two decades ago. This sometimes gave rise to obstacles when the visual appearance of an object, for example, a graph, made it hard or even impossible to solve problems graphically.

One reason for students' preference for using the zoom operation technique might be that students already are familiar with this technique from the use of other screen based technologies, for example, smart phones. In comparison, they did not adopt the technique of adjusting one axis at a time so easily. The reason for this might be that students lack of prior experiences of this kind of technique.

Another reason might be the features of this kind of tool in the particular software under consideration, i.e., *GeoGebra*. While the zooming tool, and the associated zoom operation technique, is readily available, the technique of scaling one axis at a time is more demanding, probably because the associated usage schemes involve knowledge of tools with limited accessibility in *GeoGebra*.

A further observation made in this study is that students tend to neglect the written instructions on using the computer, at least when they do not see any immediate need for them. When, later on in the task sequence, they encountered problems due to their inattention to the instructions, they asked the teacher for help. The reason why students tend to disregard such instructions may be that they do not usually read instructions while using digital technology.

To summarize, the findings highlight various obstacles which arose in the course of instrumental genesis under the instrumental orchestration of *GeoGebra* use provided by the teaching sequence and task environment as originally designed. These findings suggested ways in which that orchestration might be redesigned in order to better support students' development of the desired instrumentation schemes. In particular, they highlight the potential importance of taking account of students' previous experiences both regarding the use of technology in mathematics and the use of every-day technology, e.g., smart phones. Furthermore, these findings emphasize how students draw on existing instrumentation schemes, developed in relation to what they perceive as similar tools, when they start to work with a new tool; instrumental orchestration needs to plan for such transitions, taking account both of the continuities and discontinuities between tools.

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ENDNOTES

1. In this paper, the terms of 'tools' and 'artefacts' are used interchangeably.

2. We assume that students have obtained an equally scaled coordinate system showing y-values between 0 and 16 and x-values between 0 and 28.

Scaffolding in e-learning course for gifted children

Antonín Jančařík and Jarmila Novotná

Charles University in Prague, Faculty of Education, Prague, Czech Republic, antonin.jancarik@pedf.cuni.cz

The paper presents partial results of research focusing on teacher-pupil communication in e-learning courses. The authors of the paper study the form of possible teacher's help to a pupil using soft scaffolding methods. The text presents examples of the use of scaffolding in the form of specific instruction in courses for talented pupils run within the frame of the Talnet [1] project. Most of the coursework in this project has the form of off-line discussions.

Keywords: Scaffolding, off-line communication, e-learning, specific instructions.

INTRODUCTION

Information technology and modern technologies have considerable impact on all aspects of our lives, including education. The use of computers in mathematics and in mathematics classroom has been subject to a number of researches and research studies. The goal of the here presented paper is to point at one of the aspects that has not entered the spotlights in research yet, namely the social aspects of the use of modern technology and its impact on communication in mathematics education. Most attention is paid to the teacher-pupil communication and communication in small groups of pupils. The authors want to show the similarities and differences between communication on a computer network and face to face communication in the classroom. Most emphasis is put on what form of support the teacher can offer to their pupils in this communication that result in achieving the desired solution – scaffolding.

Communication offline when compared to face to face oral communication has some specific features. An important difference is the time delay in reactions to other participants' communications. The participants of the communication have the time to search for missing information in other sources (internet, printed documents etc.). No information is lost, as any information can be retraced (on the condition that it was not permanently deleted). This makes group communication easier as all its participants can access the preceding course of the discussion and then find it easier to react.

SCAFFOLDING

It is impossible to imagine *scaffolding* as one specific activity. In literature, we come across a number of similar definitions of scaffolding. For example, Boblett (2012) states that "*Scaffolding* as a metaphor in teaching and learning describes a system of temporary guidance offered to the learner by the teacher, jointly co-constructed, and then removed when the learner no longer needs it." In other words, the concept of scaffolding in education means strategies and methods that help pupils overcome barriers in learning, and techniques that allow pupils to master knowledge and skills. In a broader sense, scaffolding can be understood not as a single activity but as a process that offers clues and strategies to pupils until they are able to cope with the activity on their own (Tejkalová, 2010).

Scaffolding may come in many various forms. Brush and Saye (2002) distinguish between two types of scaffolding: Soft forms of scaffolding (or contingent scaffolding (Li, 2001)) are those activities that come out of situations and have not been planned in advance, they evolved while solving a problem. These activities are, for example, a discussion between teachers and pupils in which the teacher reacts to the pupils' needs and provides support in the extent required by the given situation (Simons & Klein, 2007). In contrast, in hard scaffolding, the teacher tries to foresee the possible problems and plans additional tasks and problems or clues for their pupils.

Tejkalová (2010) presents a detailed list of methods that can be used as scaffolding in a CLIL [2] classroom. Her list includes activation of existing knowledge, decomposition of a more complex task to a single, more easily viable subtasks, showing an example of the expected outcome, a teacher's "thinking out loud", offering hints, use of key words, reformulation, use of models, illustrations etc., use of internet resources, use of mnemonics, use of gestures and pantomime. Of course, the list is not and cannot be exhaustive but offers a better insight into what methods can be included under this umbrella term. It is obvious that some of these methods can be easily transferred to teacher-pupil communication in an e-learning environment (e.g., decomposition of a more complex task to single, more easily viable subtasks, or using of a motivating context). Other methods can be used in work in small groups (e.g., activation of existing knowledge by brainstorming to make a mind map of associations). However, Tejkalová's list also includes methods that cannot be transposed into the e-learning course environment, either because they are not related to mathematics or cannot be applied in an internet course.

Palinscar and Brown (1986) distinguish between four basic types of "scaffolding construction": predictions that are based on existent knowledge, posing questions that arouse interest in the text, summary in which pupils present the text in a somewhat condensed form, and final clarification in which attention is paid to the elements that obstruct comprehension.

Wood and Middleton (1975) differentiate between three categories of scaffolding that can be provided to pupils: general encouragement, specific instructions and direct demonstration.

The goal of the presented study was to analyse communication that took place within the course Talnet in 2010–2014 and to find those phenomena in communication that are characteristic for situations in which a teacher's intervention contributed to solution of the problem and to pupils' progress in the course. When analysing the conversations, elements from Grounded theory (Strauss & Corbin, 1990) were used; the different phenomena were labelled and grouped into broader categories. With respect to literature, it seemed most appropriate to use the classification of scaffolding presented by Wood and Middleton (1975). However, this classification was too rough in the area of specific instructions, which is why more subtle were introduced by the authors of this paper. This classification is still in the process of development, which is why only two subcategories will be presented – moving the limits and revealing the right answer.

OFF-LINE COMMUNICATION – THE COURSE TALNET

The following text demonstrates and develops possible uses of scaffolding in e-learning courses. The descriptions and analyses are based on classification of scaffolding categories of Wood and Middleton (1975). The illustrative examples are from e-learning courses for pupils with talent in mathematics that are run within the frame of the Talnet project and come out of teaching practice of one of the authors (Jancarik, 2013).

The Talnet project offers not only mathematics but also educational, inquiry and communication activities from other disciplines – physics, chemistry, biology, geography and technology. The activities are tailored to the needs of inquiring youth. One of the authors of this paper is involved in designing and implementing the course Mathematics within this project. The course Mathematics is run for a small number of participants (5–10 pupils). Secondary school pupils from the whole country enrol in the course voluntarily, based on their interest.

The course Mathematics introduces the pupils to combinatorial game theory and is structured in such a way to allow pupils discover and prove statements on their own. The goal of this course is not to introduce pupils to a comprehensive theory. Questions and tasks guide pupils to independent discovery. The course is divided into two parts. In the first part the pupils are introduced to different variants of the game NIM [3]; the aim of this activity is to guide the pupils to discovery of the winning strategy of this game (see Bouton, 1901). In the second part, the pupils are introduced to the game Hackenbush [4] and their goal is to assess different positions (see Conway, 1976).

Most of the coursework has the form of off-line discussions. The pupils enter these discussions at their leisure and according to their time possibilities. Thus sometimes the communication is very fast and sometimes there are long time intervals between the individual contributions. This depends on the current possibilities of the pupils and the lecturer. While analysing the conversations, the methods of scaffolding used by the lecturer were identified and then further studied. The following text offers illustrations of some of the methods suitable for e-learning courses. All of them are from the category of soft forms of scaffolding. Examples [5] from practice will be used to demonstrate the use of specific instructions. In analyses of the use of scaffolding in the form of specific instruction, hive categories have been identified and described. This text focuses only on two of these – Moving the limits and Revealing the right answer.

Specific instructions: Moving the limits

Moving the limits is one of the soft forms of scaffolding. The teacher reacts to the conditions that limit the pupil's thinking and tries to motivate the pupil to extend and generalize his/her thoughts. The goal of this form of scaffolding provided by the teacher is to turn the pupil's attention to those aspects of problems the pupil has not noticed or to guide them to relationships they have not realized. This form of scaffolding requires the teacher to carefully follow hints of deeper thoughts in the pupil's statements and to channel the pupil's attention to these rudiments of knowledge.

Let us again demonstrate this by an example from the course. It is a situation from the discussion of the one pile NIM game with extended conditions. The core of the discussion focuses on general characteristics of the winning and losing positions in this game (see Jančařík, 2007).

Example 1 – Moving the limits

Let us start this example by an extract from discussion between the lecturer and pupil A. The goal of this discussion was to clarify the task. It is presented here as introduction to the problem.

Lecturer (06/10/2010, 18:17): Hi, you've asked too many questions in your answer. I'll reply by a counter question. Is it possible that there would be neither winning, nor losing strategy in a game with no element of chance?

Pupil A (07/10/2010, 15:03): Yes, I think that the Tic-Tac-Toe [6], if played according to the official rules, is a perfect example – otherwise there wouldn't be championships in the game, would there?

Lecturer (07/10/2010, 15:22): Are there better moves in Tic-Tac-Toe? If so, how will the game turn out (I'm not requiring here a specific answer but the principle – I can explain this to you later), if you make these moves. Is it determined, or not?

Pupil A (07/10/2010, 16:03): Hmm... I'm a bit ashamed. Determined? I don't quite understand and I don't know the concept. Anyway, what rules are you asking about? Classical or official?

Lecturer (07/10/2010, 16:06): Determined – fixed, stated. Do they matter?

Pupil A (07/10/2010, 16:10): Yes, I think they matter – the official rules are meant to break the advantage of the player who starts. There's no doubt about this advantage.

The goal of the discussion in the following extract is to guide the pupil to the discovery that in games with no element of chance, the result is given if both players play well.

Lecturer (07/10/2010, 19:29): Let me repeat the question: Is it possible that in a game with no element of chance there would be neither winning, nor losing strategy?

Pupil A (07/10/2010, 16:33): Hm... well, simple logic tells me it's not possible (if there's neither winning, nor losing strategy, it's possible to win only by chance, so it can't be without an element of chance), but I can't say how this works in reality. And I think I'm not able to determine this as the number of various combinatorial games is enormous.

Pupil A is aware that it is logical that the result in such games is determined from the beginning; however, he refuses to apply this knowledge generally. As he says, "there are many games and have different rules". The pupil is trying to overcome the obstacle that is preventing him to generalize this knowledge.

Lecturer (07/10/2010, 20:02): Try to think about what you've written. Are you ready to trust logical arguments?

Pupil A accepted the argument, which is evident from his subsequent work in the course. Pupil B, who is solving the same problem, enters the discussion.

Pupil B (27/10/2010, 20:36): I think that if there's no strategy in a game, it's only a matter of chance, or maybe of the skill to take advantage of the opponent's mistake.

Lecturer (27/10/2010, 20:45): Hi, try to reverse the question. Ask when there can be (or must be) a strategy in a game. It's a bit abstract question but sometimes it really helps to start by solving things generally and then apply the general solution to a specific situation.

The teacher reverts the pupil's consideration and hands it back to the pupil in the form of a counter-question. Thus he opens space for new considerations. The teacher makes use of his experience from previous courses and tries to guide the pupil to the point that is the cognitive obstacle, i.e. the fact that in certain kinds of games there must be an optimal strategy, even if we do not know it. This is a high level of abstraction – a non-constructive proof. The course of the following discussion showed that the pupil managed to overcome this difficulty successfully.

Specific instructions: Revealing the right answer

This is again a soft form of scaffolding. Its basic principle is described in its name. Let us illustrate this by two examples.

Example 2 – Hinting at the correct answer, the game TIC TAC TOE

Pupil A (10/10/2010, 19:59): If both players play well, it must be a draw. But only if player two plays as he should. Already in his first move, if he makes the nought/cross incorrectly, then there is a winning strategy for player one.

Pupil A presents the right solution but does not provide any justification or support of this solution. Pupil B, who joins in the discussion later, does not agree and attaches a detailed counterexample.

Pupil B (10/10/2010, 21:08): Hello, I've played this game several times and it seems to me there's a sequence of moves in which the other player has no chance to win. But I may be wrong. Please, look at the attached file:

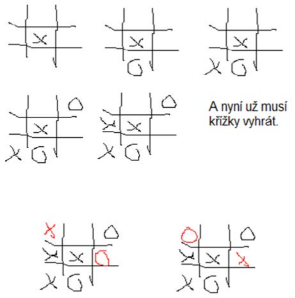

Figure 1: Students' solution to TIC-TAC-TOE game

Player One starts from the most advantageous central field.
Player Two can now make any move, none of them is much more advantageous in his situation. So, he chooses the bottom middle field.
Player Two can now make any move, none of them is much more advantageous in his situation. So, he chooses the bottom middle field.
Player One takes the corner point so that...
Player Two is now forced to prevent the next move which would make three crosses on the diagonal but...
Player One forms scissors, which Player Two cannot beat and so...
Player Two loses.
In my opinion both players play the best moves. However, if we play 'just for fun', Player Two is often the winner...

Lecturer (10/10/2010, 20:49): So, who will tear this argument to pieces?

The teacher does not provide the right solution here but points out that the presented solution is erroneous (which gives a clue to the right answer) and at the same time motivates other pupils to join in the solution.

Pupil B (11/10/2010, 15:09): Well, thinking about it I can see that Player Two is not playing ideally. In fact, his move is quite silly. He does not foresee more moves in advance. If he places his nought into a corner, the described trap disappears. In that case I'd even say that the game will lead to a draw. But I'd better think about it for a while, not jump to a wrong conclusion again.

The clue provided by the teacher was enough to make Filip reconsider his solution, find there a mistake and correct his procedure. Nothing more than a clue was needed.

The primary goal of providing a counterexample is not to draw attention to a pupil's mistake but to give him/her the chance to discover the mistake on his/her own. It is good to find such a counterexample that draws attention to the problematic passages in the pupil's solution. Guiding the pupil to discovery of the mistake accelerates the solving process and also facilitates the communication.

Example 3 – Hinting at the correct answer, Cat and Mouse Game (Tapson, 1977) [7]

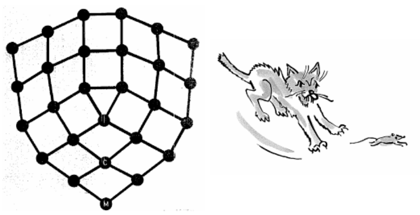

Figure 2: Cat and Mouse Game (taken from Jančařík, 2007)

Pupil A (6/10/2010, 21:27): So far I haven't got into studying situations that could happen if the mouse chose a different direction than the one I'm talking about, or if the same strategy would work if the mouse did not go one field back. So far I perceive as "chance" the fact that the mouse moves in a convenient direction.

Lecturer (6/10/.2010, 21:30): Fine, I'll leave all of you some time to think about the answer once more and to refine it.
Just a question, do you play chess?

Pupil A (06/10/2010, 21:34): To be honest I know the rules and moves of the figures but I'm by no means an outstanding player. I've won a few games with my relatives. But to be good at the game I'd need some training, which is not my case.

Lecturer (6/10/2010, 22:19): This has much to do with the position in chess which is referred to by the beautiful Czech word zugwang. But you need some skill in chess to know. May be there is a better chess player among us. And this hint could help him. :-)

The teacher provides a clue that can be used to solve the situation. This is leaving the move to the opponent and taking the advantage of an unfavourable move. For more information see http://en.wikipedia.org/wiki/Zugzwang.

Pupil B (07/10/2010, 15:00): I'm sorry but I had to google the term Zugwang (Zugzwang) up and now I've grown a bit smarter.: -P But still I can't see any satisfactory solution in the Cat and Mouse – they can be running round and round as mad and the mouse will never be cornered. But how to describe in mathematical terms that it can be caught?

In this case the pupil learnt a new concept but failed to apply it in the situations. An additional hint was needed for the correct solution. The use of hints in this case gave birth to the knowledge that results can be transferred from one game to another, which was very important later in the course.

Example 4 – Hinting at the correct answer, Cat and Mouse Game

Lecturer (14/10/2014 18:10): A very simple idea (I've stepped down a bit): Be the cat, forget about the mouse, go directly to the centre of the board, run around the triangle in the centre and then start chasing the mouse (which has been running somewhere so far), how will this end?

Student (28/12/2014 12:29): I was wondering about this comment and the comment "Has anybody noticed a triangle has three sides and rectangle four sides?" for a long time, so I tried to play in the way that I run around the triangle with the cat and then I got it – if I run around the triangle I'll get to the same place in three turns. There are rectangles everywhere else so that it takes four moves. So I get one turn ahead, which was mentioned by Dominik: "If a mouse is to lose from this position, it must be its turn (if it is the cat's turn, the mouse will simply move to a free field)." In contrast to the starting situation when it was the cat's turn, it will be the mouse which has to move one field and the cat will be able to get to this field in the next move.

The last example borders on specific instruction and direct demonstration. After a lengthy discussion in which the pupils had not been able to get closer to the solution the lecturer almost gave up and described the way that leads to victory. The only thing he did not do was tell the pupils it was the correct solution. It took two and a half months for the above quoted reaction of one of the pupils who then managed to combine all the hints from the discussion and discover the right solution and especially the reason why it worked.

Methods of direct demonstration were scarcely used as soft scaffolding in the course. One of the reasons is the specific nature of the course, whose goal is not the solution of particular problems but development of abilities to experiment and find solutions. Another reason is the pedagogical beliefs of the lecturer that only valuable is only knowledge that is discovered and justified by the pupil on their own.

DISCUSSION

Communication in digital environment resembles common communication in the classroom but at the same time is very different. It is not a face-to-face communication. Neither the teacher nor the pupils have the chance to use non-verbal means of communication – gestures, expressions or intonation. The "dialogue" is not continuous. There are many disruptions and time delays. It is often very concise or abbreviated. Writing of mathematics expressions may be difficult.

The listed aspects clearly show that off-line communication deserves special attention. What is very important is the effort to understand the individual utterances and the teacher's provision of effective support and help.

CONCLUDING REMARKS

The goal of the presented study was an analysis of the scaffolding tools in offline communication. As stated in the introduction, offline communication when compared to face to face communication is specific in that it affect the success of the used form of scaffolding. It became obvious that specific instruction may come in various forms; the paper presents only two types – moving the limits and revealing the right answer. The significant role they play in support leading to successful solution of problems in offline communication is illustrated on selected example situations from the course.

One of the tasks for future research is to study the patterns in off-line communication, to describe basic phenomena that are connected to this communication, and to identify its efficient methods. The examples above are not meant to illustrate the complete range of the different methods of scaffolding that can be used in offline communication. The goal of presenting them was to show the characteristic features of this type of communication and problems its participants (teachers and pupils) must face. It is more a pioneer research indicating the directions for future research in the area.

Another important question to be addressed in our future research is to what extent the studied methods of soft scaffolding can be generalized and modified in such a way to make them implementable into courses in the form of hard scaffolding. Translation of soft scaffolding into the form of hard scaffolding will also enable us to transfer research results also into courses with greater numbers of students.

One of the key questions that is of crucial importance for the educational process and that every teacher must ask is how to state the current pupil's knowledge and the horizon where it can be developed. In this paper, we discussed benefits, based on examples, of the use of scaffolding in an e-learning course for gifted students.

The examples are taken from highly specialized courses for talented pupils. However, the authors of the paper are convinced that the shown methods can be successfully applied not only in e-learning, but also in regular courses of mathematics where there is communication between the teacher and their pupils, as they are helpful at every moment when the pupil faces difficulties trying to overcome obstacles in the cognitive process.

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ENDNOTES

1. See, for example, http://en.wikipedia.org/wiki/Talnet. Talnet is a project for gifted children and teenagers interested in natural sciences and technologies, as well as for their teachers from all country.

2. Content and Language Integrated Learning (CLIL) refers to any teaching of a non-language subject through the medium of a second or foreign language. CLIL suggests equilibrium between content and language learning.

3. See, for example, http://en.wikipedia.org/wiki/Nim.

4. See, for example, http://en.wikipedia.org/wiki/Hackenbush.

5. In case of the above presented forms of scaffolding, it seems that a discussion in the internet environment is in many aspects similar to the usual classroom interaction. The authors deliberately present the examples with dates as they very nicely show how much time elapsed between the different contributions.

6. The game TIC-TAC-TOE is a simplified version of the game Noughts and crosses played on a board with a 3 x 3 square grid. Originally, TIC-TAC-TOE is a paper-and-pencil game for two players, X and O, who take turns marking the spaces in the 3×3 grid. The player who succeeds in placing three respective marks in a horizontal, vertical, or diagonal row wins the game. There are applets on the internet for this game in which we can play the computer. These applets are deliberately not offered within the course as they seduce the pupils to use heuristic strategies. What is expected for the participants is a systematic analysis of all possibilities. And applets do not develop this.

7. The cat and the mouse are represented by tokens that move in the fields connected by a net (the squares represent mouse holes). The cat and the mouse can only move to the nearest field. The goal of the cat is to catch the mouse, the goal of the mouse is to escape. There is a very simple winning strategy for the cat. Still, most pupils are convinced at the beginning that the mouse will always escape and the cat has no chance to catch the mouse if the player plays well.

An examination of secondary mathematics teachers' technological pedagogical content knowledge

Firdevs İclal Karataş[1] and Fatma Aslan Tutak[2]

1 Middle East Technical University, Ankara, Turkey, iclal.karatas@metu.edu.tr

2 Bogazici University, İstanbul, Turkey, fatma.tutak@boun.edu.tr

This paper aims to examine secondary mathematics teachers' technological pedagogical content knowledge (TPACK) and diversities in it due to teachers' gender, age and years of experience. The participants of the study were 138 secondary mathematics teachers in Istanbul, Turkey. An adapted TPACK-M scale with three constructs (TCK, TPK and TPACK) was used to collect data. Results showed that TPACK level of teachers was moderate. According to demographic results, there was no significant difference in TPACK perception of male and female mathematics teachers. Also, small negative correlation was found between age and teachers' TPACK. Furthermore, there was no significant difference in TPACK perception and teaching experience of teachers.

Keywords: Technological pedagogical content knowledge, secondary mathematics teacher, integrating technology in mathematics education.

INTRODUCTION

In accordance with scientific and technologic developments in the world, technological opportunities of schools have increased in Turkey recently. Ministry of National Education (MoNE) has some attempts to integrate technology in schools. The FATIH project (Increasing Opportunities and Improvement of Technology Movement) is among the most significant educational investment of Turkey. The aim of this project is to enable equal opportunities in education and to improve technology in schools for the efficient usage of information and communication technologies (ICT) tools in the learning-teaching processes through providing tablets and LCD interactive boards (MoNE, 2013). However, putting latest technologies into classroom without well trained teachers is not really technology integration (Dockstader, 1999). It can be achieved when technology is used effectively and efficiently in the different content areas to allow students to learn how to apply technology skills in meaningful ways. Although technology has relationship with many domains, it has prominent place in mathematics education due to many reasons. In the last century, technology integration into mathematics education has brought many innovations in the mathematics classroom in terms of development as well as accessibility. According to technology principle of National Council of Teachers Mathematics (NCTM, 2000), "technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning. " (p. 11). Moreover, the effective use of ICT is strongly emphasized in Turkish mathematics education curriculum (MoNE, 2013). From this point of view, mathematics teachers' role in the integration of technology into instruction is crucial. They need to acquire the ability to use technology resources effectively. On the other hand, integrating technology is not just adding technological knowledge in curriculum; it needs a complex mixture of technology, pedagogy and content knowledge. In order to effectively integrate technology in instruction, teachers should have an adequate technological pedagogical content knowledge (TPACK) (Mishra & Koehler, 2006).

LITERATURE REVIEW

The knowledge needed for teachers to use technology strategically in mathematics instruction is a topic that has recently gained much attention (Neiss, Lee, Sadri, & Suharwoto, 2006; Mishra & Koehler, 2006). TPACK, described by Mishra and Koehler, "represents a thoughtful interweaving of all three key sources of knowledge – technology, pedagogy, and content" (2006, p. 14). The TPACK framework describes good teaching with technology by including the components of content, pedagogy, and technology. Shulman's (1986) idea of pedagogical content knowledge (PCK) is the basis for this framework with the inclusion of the domain of educational technology. Technological pedagogical content knowledge (TPACK) provides a useful framework for understanding teacher perceptions and practices of technology integration into curriculum and pedagogy. To integrate technology into their pedagogy and curriculum successfully, teachers must develop confidence in their abilities to integrate technology in the classroom because the integration of technology affects how students learn in the classroom. TPACK for teaching with technology means that teachers should know how particular mathematics concept might be taught with technology so that students understand the concept (Niess, 2005).

After Mishra and Koehler (2006) introduced their TPACK model, it was used in many researches. This framework has been widely adopted for the planning of teacher ICT education (Cox & Graham, 2009) and used as a theoretical underpinning for the development of surveys to measure teachers' TPACK. In terms of measurement, all of the instruments were focused on teachers' self-report, in other words their perception on use of technology. Also, conducting research about development and measuring of TPACK is an important and hard challenge. Since TPACK is a complicated construct and comprises many components, measuring the effectiveness of TPACK depends on the relationships of these components with each other (Koehler, Mishra, Bouck, DeSchryver, Kereluik, Shin, & Wolf, 2011). As a result of the complexity of TPACK, getting reliable and valid results after assessing the TPACK is an important process. Researchers encounter some problems while they are measuring the TPACK of teachers. They face different problems in each different measurement tool of TPACK. Two main problems occurring during the measurement process are understanding the effects of teachers' domain knowledge on their current teaching practices and reliability, validity concerns of TPACK measurement methods (Abbitt, 2011). So, researchers try different methods to measure TPACK because of its dynamic characteristic.

The definition of TPACK concept looks like settled but it continues to be studied. So, most of the studies focused on the definition of TPACK and developing instruments to measure it (Mishra & Koehler, 2006; Koh, Chai, & Tsai, 2010; Schmidt, Baran, Thompson, Mishra, Koehler, & Shin, 2009). Also, many TPACK studies are conducted with pre-service teachers. Similarly to a general trend in TPACK studies in international arena, in Turkey most of the studies focused on adapting surveys or developing new ones to measure TPACK (Öztürk & Horzum, 2011; Yurdakul, Odabasi, Kilicer, Coklar, Birinci, & Kurt, 2012).

Also, the adaptation studies in Turkey did not focus on any specific content. According to the results from a pilot study of the FATIH project, teachers do not feel themselves adequate and confident in their knowledge. Therefore, the purpose of this study were to examine technological pedagogical content knowledge of secondary mathematics teachers at FATIH project schools and their possible gender, age and years of experience diversities related to TPACK by adopting mathematics specific scale (TPACK-M). The specific research problems addressed in this study are the following:

— What are the levels of secondary mathematics teachers' TPACK as measured by TPACK-M scale?

— Is there a significant difference in perception of Turkish secondary mathematics teachers' TPACK in terms of gender?

— Is there a significant relationship between TPACK and age of secondary mathematics teachers?

— Is there a significant difference in perception of Turkish secondary mathematics teachers' TPACK in terms of years of experience?

METHODOLOGY

This study contains the combination of survey and correlational research methodology (Creswell, 2012). In this study, all secondary mathematics teachers at FATIH project schools in Istanbul were identified as the target population. The reason why FATIH project teachers were determined as participants in this study is that they have already been equipped with technological devices such as interactive board and tablets. However, it was not practical to visit all schools to meet the teachers. So, multistage cluster sampling was used in the study. FATIH project schools in Istanbul were considered as clusters. Six districts (three from Anatolian side and three from European side) were chosen from 39 districts. When selecting each district, the ratio of students per teacher and school size in all districts were considered. Firstly, all districts ranged according to the ratio of students per teacher. Then, those districts separated into three groups (low, middle and high). The participants of the study were 138 secondary mathematics teachers working at FATIH project schools in Istanbul. Among the participants, 62 (44.9 %) of them were male and 76 (55.1 %) female, aged between 29 and 62 years (*M=41.1, SD=6.04)*. When teaching experience of teachers was considered, majority of the teachers had teaching experience between 11 and 15 years. As demographic information, it was also asked to participants to indicate whether they had their personal electronic devices (desktop, laptop, tablet and smartphone). The majority of teachers have their own laptop (76.1%), smartphone (73.9%) and tablet (70.3 %), while teachers showed lower percentage of desktop ownership (42.8%).

Technological pedagogical content knowledge of secondary mathematics teachers (TPACK-M) scale was used in the study after an adaptation into Turkish. For the adaptation of the study, the scale was translated into Turkish using a standard protocol. Also, both versions of the test applied twenty five pre-service mathematics teachers who are native in English and fluent in Turkish. The TPACK-M questionnaire was designed by (Handal, Campbell, Cavanagh, Petocz, & Kell, 2013) to identify teachers' TPACK in terms of technological content knowledge (TCK), technological pedagogical knowledge (TPK) and technological pedagogical content knowledge (TPACK). It focuses on secondary mathematics teachers and involves three parts. A 5-point (from 1-strongly disagree to 5- strongly agree) Likert type scale contains a total of 30 items. Each construct has 10 items. Technology knowledge (TK) was not included in the questionnaire because of the research emphasis on discipline related technology. The questionnaire deliberately focused on the concept of ability as a measure of a respondent's capacity to carry out a particular task, rather than focusing on the enactment itself. Hence the examples of the items are, "I am able to use dynamic geometry software (e.g., GeoGebra, Geometer's Sketchpad, Autograph, Cabri)" (for TCK), "I am able to teach a concept using an interactive whiteboard" (for TPK) and "I am able to use technology to demonstrate mathematical models or concepts through learning objects (e.g., animations, simulations, online applications)" (for TPACK). Data was gathered through self-report. Non parametric statistical techniques were used to analyze the data.

RESULTS

Table 1 indicates mean values and standard deviations of participants' TPACK level for three constructs TCK, TPK, TPACK and for whole instrument.

According to Table 1, the whole TPACK mean score is 3.38 *(SD=.83)* in a range of 1 to 5. When three components of scale are examined, the highest mean subscale score belongs to technological content knowledge (*M=3.48, SD=.92*) while the lowest mean subscale belongs to technological pedagogical knowledge *(M=3.28, SD=.77)*. Three dimensions as low, moderate

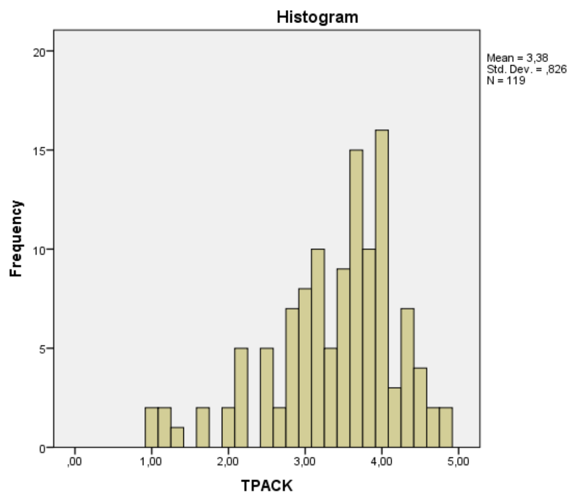

Figure 1: Distributions of TPACK Mean Score

	N	Mean	Std. Deviation
TCK	127	3.48	.92
TPK	131	3.28	.77
TPACK	134	3.39	.90
Whole instrument	119	3.38	.83
Valid N (listwise)	119		

Table 1: Mean TPACK scores

	Sex	N	Mean Rank	Sum of Ranks	U	Z	p
TPACK	male	56	63.71	3567.50	1556.5	-1.105	.269
	female	63	56.71	3572.50			
	Total	119					

Table 2: Mann Whitney U test for gender and TPACK

			Age	TPACK
Spearman's rho	Age	Correlation Coefficient	1.000	-.157
		Sig. (2-tailed)	.	.087
		N	138	119
	TPACK	Correlation Coefficient	-.157	1.000
		Sig. (2-tailed)	.087	.
		N	119	119

Table 3: Correlations between age and TPACK

	TEACHINGEXP	N	Mean Rank	χ^2	df	p
TPACK	Less than 10	7	51.86	6.560	4	.161
	11–15	53	67.71			
	16–20	32	55.69			
	21–25	19	52.53			
	More than 25	7	41.36			
	Total	118				

Table 4: Kruskal- Wallis test for TPACK and teaching experience

and high were determined to interpret the findings. If mean scores of tests are between 1 and 2.33, the level of perception is considered as "low". If mean scores of tests are between 2.34 and 3.67, the level of perception is considered as "moderate". If mean scores of tests are between 3.68 and 5.00, the level of perception is considered as "high" (Yurdakul et al., 2012). Considering these values, it may be inferred that secondary mathematics teachers' perception on their TPACK level is moderate. Also, their perception level for three constructs (TCK, TPK and TPACK) can be considered as moderate. The distributions of the mean TPACK score is illustrated in Figure 1.

In order to explore possible gender differences of secondary mathematics teachers' technological pedagogical content knowledge, non-parametric Mann-Whitney U test was used because of the violation of normality assumption. A Mann-Whitney U test results, shown in Table 2, revealed no significant difference in the TPACK levels of male and female mathematics teachers (*U=1557, z= -1.11, p=. 27*).

Furthermore, the relationship between participants' age and TPACK perceptions was examined using Spearman rho correlation. As shown in Table 3, there were small negative correlation between age and technological pedagogical content knowledge of teachers (= −.16, p > .05)

In order to answer whether there is a difference in TPACK levels across five teaching experience levels, Kruskal-Wallis test was used. Teaching experiences of teachers were categorized as less than 10 years, 11–15 years, 16–20 years, 21–25 years and more than 25 years. Chi-square value, the degrees of freedom (df) and the significance level are shown in Table 4. So, there is not a significant difference in TPACK levels of mathematics teachers across five different teaching experience groups (*6.56, p= .16*).

DISCUSSION AND CONCLUSION

According to the results, the mean score of each construct and the whole instrument showed that secondary mathematics teachers generally rated themselves as moderate. The moderate mean score in TPACK (and three constructs) may be explained by the familiarity with technology in daily life and in-service education on using technology. Overall, the participants reported using various technologies in their personal life to communicate or obtain information. According to study conducted by Menzi, Çalışkan and Çetin (2012), teachers who have personal technological devices see themselves more competent in the field of technology than those who do not have. So, this may explain why participating teachers' perception is moderate in technology related knowledge in this study. However, as shown in other studies (Harris, Mishra, & Koehler, 2009; Lei, 2009), use of technology for communication and information does not necessarily translate into technology integration in the classroom. Teachers used technology to communicate daily but lacked expertise or vision to translate this technology knowledge into use in instruction.

Moreover, in the scope of the FATIH project, teachers took in-service education related to technology use in education. This education is mainly focused on functionality of the hardware and software, but not on content (choice of appropriate media, functionality of the media) or pedagogical integration of the content in strategic ways, including interaction between tablets, interactive whiteboards, teacher and student (ERI, 2014). So, this may be the reason of teachers' moderate perception on technology related knowledge. However, how and in what degree they use technology is a questionable issue. Teachers may still think pedagogical knowledge, content knowledge and technological knowledge separately. This does not mean technology integration.

Considering the specific subscale mean scores, the highest mean value of the teachers' perception corresponds to technological content knowledge (TCK). This means that mathematics teachers feel more competent in content-related technology. According to the report of 2000 National Survey of Science and Mathematics Education, high school mathematics teachers are significantly more likely than middle school teachers to report feeling qualified to teach a number of mathematics topics (Weiss et al., 2001). Therefore, mathematics teachers may prefer to use technology mostly in their good-at construct, content. On the other hand, among the TPACK sub-constructs, the least mean value of the teachers' perception corresponds to technological pedagogical knowledge (TPK). It can be concluded that mathematics teachers do not feel themselves sophisticated use of technology for pedagogical purposes as well as in other constructs. According to the report of 2000 National Survey of Science and Mathematics Education, high school mathematics teachers reported well prepared to use various instructional technologies in their teaching. As yet another lens on teachers' perceptions of pedagogical preparedness, they are least likely to feel prepared in technology-related areas (Weiss, Banilower, McMahon, & Smith, 2001). This result may stem from teachers' lack of general knowledge about technology-related pedagogy.

Based on the existing literature on teacher integration of technology into classroom, gender, age and teaching experience were possible predictors of technology integration. So, demographic diversities of TPACK were discussed in terms of gender, age and teaching experience in this study. Within the sample studied, there were no significant difference between secondary mathematics teachers' TPACK and gender. North and Noyes (2002) suggested that the prevalence of computers in schools could provide both males and females with equal opportunities for computer use, thereby equalizing their perceived differences with respect to computer use. Therefore, if the FATIH project reaches the aim of providing equal technological opportunities to schools, the impact of gender differences on TPACK may become less significant on teachers. Also, teachers in the FATIH project school took in-service education which was constructed on similar content. This may be the reason why male and female teachers perceive themselves similar in terms of TPACK.

Furthermore, weak negative correlation was found between TPACK and age in this study consistent with previous studies (Koh et al., 2010; Öztürk, 2013). When teaching experience of participants was considered, there was no significant difference in TPACK levels of teachers across five teaching experience groups. In this study, the respondents had an average of seventeen years of teaching experience and the categorization was centered on 11–20 years of teaching experience. Therefore, the results of this study may not be generalized because of small numbers of teachers in some teaching experience categories. Previous study showed that age, gender and teaching experience all affect the teachers' response to implementing new ideas in the classroom (Fullan, 2001). Since FATIH project is a new implementation, such demographic information of teachers which can be effect on technology integration was discussed in the present study.

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Boundary crossing in a community of interest while designing an e-book with the aim to foster students' creativity

Chronis Kynigos and Elissavet Kalogeria

University of Athens, Educational Technology Lab and CTI & Press Diophantus, Athens, Greece, kynigos@ppp.uoa.gr, ekalog@ppp.uoa.gr

This paper investigates the process in which teachers of different experiences and subjects collaboratively designed an e-book unit around 'Windmills', aiming to foster students' creativity (call c-book, c for creativity). Our aim was to identify the characteristics of this innovative design, as well as if and how this process can enhance teachers' creativity. We found that the design evolved in four phases incorporating learning mechanisms–indicators for teachers' creativity: Searching for mathematical ideas/concepts, implementation of the raw ideas, pedagogical and /didactical contextualization, transition to the c-book. The c-book was the boundary object around which the participants interacted, reflected and finally transformed their knowledge regarding the way it should be designed to support creativity.

Keywords: Boundary crossing, community of interest, resource design, expressive media.

INTRODUCTION

The present study refers to the process during which designers of different expertise and interests collectively designed digital educational resources in the form of e-books, aiming to challenge and foster its users' creativity in posing and solving mathematical problems through constructionist, exploratory and investigational activity with the use of digital media. Creativity is expected to be at the core of both the process of jointly developing a c-book and the product itself (called c-book, c for creativity). Since this kind of resource constitutes an innovative approach to mediating mathematical content, we were particularly interested in studying both the process of developing as well as the affordances of the product itself, in order to identify their characteristics and their potential for the math teaching and learning.

The designers' online discussions and outcomes that we analyzed here, took place in the context of a European R&D project called 'MC Squared' focusing on technologies affording creativity both in collaborative designs and in using the media to engage in mathematical thinking. The project aims at developing the 'c-book', a technology supporting collaborative authoring, diverse constructionist widgets and data-analytics configurable by its authors for the design of creative educational resources for creative mathematical thinking (CMT) (the c-book units). This technology has been thought to go along with the generation of some specifically generated collectives of designers characterised by their diverse disciplinary backgrounds, expertise, history and membership in different communities of practice, which are defined in literature as Communities of Interest (CoI) (Fisher, 2011). Four such CoIs were formed in the project whose members were as diverse as developers of math digital media, publishers of math educational materials, researchers specialized in creativity or in math education, creative math teachers and students. In this paper, we discuss a group within one of these CoI from Greece.

THEORETICAL FRAMEWORK

We perceive mathematical creativity at the school level "as a process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems and/or as the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle" (Sriraman et al., 2011). Novelty is interpreted as having a local character, i.e.

something that may not be novel to experienced mathematicians but from the perspective of the learners it can be judged as novel and therefore as creative (Askew, 2013). This issue brings to the fore the distinction (Craft, 2000, 2001) between 'high' and 'ordinary' creativity. The former (big C) describes 'great works' by experts or gifted persons which change knowledge and/or our perspective on the world. The latter (small c) recognizes that all pupils can be creative and arises – for example – when a student creates a solution to a novel problem or connects together two seemingly disparate ideas.

Our view on the creativity of educational designers of math resources is in accordance with Ervynck's (1991) who considers as creative math activity every designer's attempt aiming to reform or improve the network of concepts of a math curriculum for pedagogical reasons, even if new mathematics is not generated. According to Bolden and colleagues (2010) some of the creative opportunities that might be offered in the math classroom are the need for mathematical expression and communication (the social aspects of creativity), the construction of meaning and development of personal understandings, the generation of ways for solving problems, hypothesizing about math situations and outcomes, constructing tests of those hypotheses and formulating plans for solving complex problems. In this paper, we sought for creativity in both the process and the products of collaboration among teachers so as to produce material which is expected to offer students some of the above creative opportunities. The kind of creativity which is developed in collectivities through joint enterprises is described as 'middle c' creativity and is needed to create strategies, to find ways to make the differing views of individuals capable of existing together and to produce collective learning outcomes, including an elaborated understanding of the learning topics addressed (Eteläpelto & Lahti, 2008). In MC2, the engagement of teachers in this kind of collaborative design targeted the empowerment of students' creative mathematical thinking (CMT). Although the use of digital media can facilitate the engagement with CMT in unprecedented ways (Healy & Kynigos, 2010), there is a lack of both pedagogical designs targeting CMT and corresponding technologies supporting them. Even in the case of digital tools with great potential for enhancing CMT, such as e-books, the pedagogies that accompany those tools are often outdated, following the traditional teaching and learning models. Moreover, the process of designing them is rather restricted and limited to the authors, instead of being open to collective design that leaves space to the designers for sharing creative ideas. Designing digital educational resources for CMT can be therefore viewed as a 'squared' creativity challenge, since it requires not only fostering students' mathematical creativity but also situating the design process itself within a socio-technical environment that can boost educational designers' creative potential (Kynigos, 2014). In the context of MC2, learners' engagement with CMT was planned to be designed in collectives with the use of digital media, resulting in a new genre of authorable e-book, the *'c-book'*. The technology of a c-book differs from the one of an e-book as it includes dynamic widgets and interoperability, anticipates collective design incorporating an authorable data analytics engine with appropriate interface, drawing on end-users' and resource designers' interactions. This environment is aiming to enhance and stimulate *social creativity* in designing for CMT through the generation of *Communities of Interest* (CoI) (Fischer, 2001). A CoI has a heterogeneous character, as each of the participants of a CoI represents a group of practitioners from different domains or communities while all of them target to resolve collectively a problem and achieve common understandings overcoming their cultural differences. The "symmetry of ignorance" in the process of framing/solving design problems and creating new artifacts/understandings, triggers the emergence of social creativity (ibid). In this context, such CoI is anticipated to operate as a *socio-technical environment,* i.e. a living entity where everyone might be, at the same time, "designer" and/or "consumer", in the process of co-designing dynamic re-useable and re-constructible educational materials for CMT. Interconnectedness, different perspective-taking, knowledge exchange and integration between diverse domains are features of this environment expected to provide more opportunities for creative thinking and learning. The members of a CoI in order to integrate these features in their communication need to cross the boundaries between the different sites and the c-book is anticipated to operate as boundary object by fulfilling a bridging function (Star, 2010). The process of *boundary crossing* entails four learning mechanisms (Akkerman & Bakker, 2011): a) identification of the intersecting practices, b) coordination of both practices through establishing routinized exchanges to facilitate transitions, c) reflection leading to perspective-making and perspective taking and d) transformation that provoke changes in practices or even the creation of a new in-between practice. We perceive these mechanisms as indicators for teachers' creativity, thus in our study we sought for them during the design process.

The present study is a first attempt to describe in detail this kind of innovative design and identify its characteristics as well as if and how this process can enhance teachers' creativity. Particularly, we investigated: (a) designers' ideas in the process of formulating mathematical problems as well as the way they designed and foresaw students' CMT (b) the product of this process in terms of novelty and usefulness as well as its characteristics (c) the potential of the c-book for operating as boundary object between the CoI members.

METHOD

In this paper, we study the development of the c-book unit 'Windmills' which was meant by the researchers to operate as a sparker for social creativity. It was planned to be jointly created by the following 9 members of the Greek CoI: Tom (Computer Science developer), Dimitris (experienced math teacher, designer in math education with the use of digital tools, master degree in the didactics of math), Katerina and Areti (master degrees in math education with technology, but limited teaching experience in math classrooms), Popi (experienced math teacher and teacher educator for the pedagogical use of digital tools in math, with deep knowledge of the available tools), Marios (phd-student and Informatics teacher), Foteini (teacher in engineering education, phd in the pedagogical use of digital tools for the vocational education), Yannis (university teacher) and Elissavet (moderator of this discussion, experienced math teacher, PhD in the domain of math teachers' education with the use of digital tools). Our data were: (a) the 75 contributions in the CoICode, (b) the files attached in the CoICode (c) the pages in the platform of the c-book (the widget instances and the respective narratives). For analysing the content of CoICode, we adopted the data grounded approach (Strauss & Corbin, 1998).

THE C-BOOK ENVIRONMENT

The c-book provides space (CoICode) for organized discussions in two parallel interfaces: a threaded forum discussion view and a mind-map view (Figure 1) and gives user the possibility of switching from one to the other. Users can characterize the nature of their contribution by selecting between different semantics (alternative, contributory, objecting, off task and management) in which they can write text, links, attach files or widget instances (a software such as Geogebra is a widget factory and a microworld of this factory is a widget instance). The platform (Figure 2) is the space for authoring where the students interact with the c-book. It incorporates pages with dynamically manipulated widget instances accompanied by corresponding narratives.

RESULTS

The analysis of the 75 contributions posted in CoICode during the first cycle of MC2 showed that 16 of them were related to technical and organizational issues, 4 were off-topic and the majority (55) concerned the development of the c-book unit. It should be noted that Tom and Marios did not participate in this cycle of the design. From the analysis of our data we identified that the process of developing the c-book unit was evolved in the following four phases incorporated different characteristics:

1) Searching for mathematical ideas and concepts

2) Implementation of the raw ideas

3) Pedagogical and didactical contextualization

4) Transition to the c-book: texts accompanying the constructions

These phases are interconnected and sometimes not clearly limited, as one of them may penetrate or overlap the previous and evolve together overtime. Below, we describe them regarding their characteristics in terms of both processes and products.

Phase 1: Searching for mathematical ideas and concepts

This phase starts with the exchange of files depicting different types of windmills all over Greece (Elissavet and Katerina) or the way they function (Foteini), aiming to operate as starting point and trigger the subsequent expression of the first ideas. The shapes of the windmills and their operation turn the participants' attention on the kind of software that should better integrate those characteristics. Their first ideas are

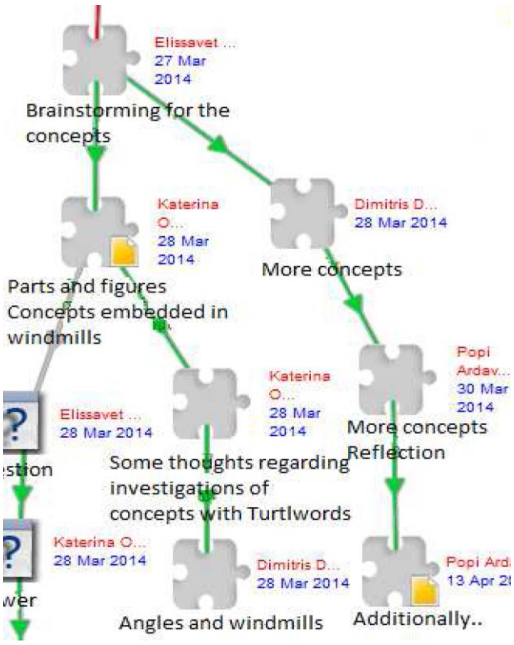

Figure 1: ColCode map

related to Turtlewords (Dimitris, Elissavet and Areti), as they were all fully aware of its functionalities for rotation with the variation tool. Then, Elissavet (Figure 1, 27/3) proposes a brainstorming concerning the concepts embedded in a windmill and mentions some of them: regular polygons, isosceles triangle and rotational symmetry of k grade (depending on the number of sails) for the wheel as well as cone and cylinder for the main building. Dimitris (Figure 1, 28/3) adds some more complicated ideas, for example, when an ant is walking on a wing of a windmill as it turns the ant's orbit might look like a helix. Katerina (Figure 1, 28/3) adds concepts such as parallelograms, rectangles, angles, turns and curves in 3d space. She proposes ways of connecting them and the use of Turtlwords to investigate the properties of parallelograms and of different kinds of triangles as sails of the windmill. Popi (Figure 1, 30/3) raises some questions about the mechanics and architecture of windmills, the locations which favor their operation, the time required for a complete rotation of two similar windmills with similarity ratio ½, the way the sprockets work etc.

In this part of the discussion we observe that most of the concepts come from geometry and have to do with the perceptions of windmill and its parts as geometrical figures. Popi, who is experienced in both mathematics teaching and technology introduces the time parameter and searches for a context where a windmill would be integrated. At the same time, the participants start thinking which of the available software should better be related to those concepts. The discussions of this phase seem to be focused rather on ideas than on actions.

Phase 2: Implementation of the raw ideas

At this point the CoI members start implementing the ideas expressed during the 1st phase. The constructions are individual and we note a lack of intervention in the other members' constructions. Simultaneously new ideas are invented concerning mathematical concepts, technological tools and problems. Elissavet gives a logo-code in Turtleworlds that initially constructs an equilateral triangle and then rotates it in order to take the shape of a six sails wheel. Taking this perspective, Dimitris and Katerina give half-baked logo-codes constructing an equilateral triangle using trigonometry and a parallelogram for representing the sails respectively. Areti also attaches a file with a half-baked code in Turtlwords (variables a, b, c) that initially constructs an open jagged line of two equal sides of length a, turns depended on c and the third side of length b. The dynamic manipulation of variables with the variation tool, constructs an isosceles triangle. Subsequently, the triangle turns and it is rotated n times with the variation tool, so as to give the impression of the sails of the wheel. At the same time, Dimitris and Popi start expressing their ideas with the use of Geogebra. Dimitris constructs a dynamic figure of a sail and Popi a simple model of the wheel in which she has incorporated the time parameter manipulated by a corresponding slider. The dynamic manipulation of time, brought to the fore algebraic concepts and relationships. The investigation of the model led to direct proportional amounts, linear and multiple branch functions. Dimitris makes a further refinement of the concepts and classifies them into: (1) structural relationships and construction of a windmill as a 2d shape and (2) movement of the windmill with the slider. Elissavet adds to this distinction a third group: the view of the wheel from different perspectives as a solid 3d shape, i.e. its transformation from a plane to a solid shape. She also emphasizes on the complementarity of the activities with Turtlewolds and Geogebra: the former needs a constructionist activity from simple mathematical concepts as structural units to more complicated ones, while the latter requires a de-construction of the model, resulting in the embedded concepts.

The discussions of this phase include both the expression of ideas and their implementation individually. The mathematical concepts are classified according to the way they are used, while they are also related to the functionalities of specific software tools. As the CoI members exchange their constructions we observe their smooth transition from one practice to another, effortlessly, just crossing the boundaries, without reconstructing them.

Phase 3: Pedagogical and didactical contextualization

In this phase the majority of constructions in the form of widget instances has already been completed and the next step was to put them on the platform as different pages. The CoI members discussed the optimal sequence of the activities encompassing their constructions, the targeted students' ages, the relation to the official curriculum and the kind of activities regarding their openness.

Foteini believes that this c-book unit should be addressed to specific school grades, while Dimitris' opinion is to put 'neither floor nor ceiling' in the targeted ages. From this disagreement the following dilemmas emerged: "*How can we expect our students to do creative mathematics with the use of the c-book, without having previously defined their ages?*" or "*Did defining in advance the ages of students limit our own creativity?*" Finally, they decided to include in the c-book a wide range of activities, starting from the simpler and gradually address them to students 12–17 years old, with the structure of the concepts not aligned to the official curriculum. Then, they had to choose which of the constructions should be incorporated in the c-book as well as their sequence on the different pages. They all agreed to start from the constructive activities with Turtlewords, giving students the opportunity to explore separately the figures – and the concepts of isosceles, equilateral triangle, parallelogram – which represent the sails of the windmill and then to continue towards more complicated concepts (for example rotational symmetry), using the initially investigated ones as structural units. The investigations of the model with Geogebra were decided to be posted after the 6th page, with the aim to offer students the experience of understanding functional relationships through their use (for example multiple branch functions, representing the operation of the windmill in different time intervals). We note that the multiple branch functions are taught in the 4th grade of secondary education; however, the investigation of the model could lead students of lower grades to understand this notion through its use (a typical example of mathematics-in-use). At the end of this phase Dimitris expressed the following alternative opinion: "*It does not really matter which activities we'll choose from Turtlewords or Geogebra. The widget will change when we start writing the texts. I believe that text and widget are a unit and not two different pieces*". The interrelation of widgets and texts was an issue of less importance till then and Dimitris' contribution revealed his conception of the c-book as a whole, integrating both instances and texts. Another issue was the form of the activities around these concepts, i.e. their openness and the

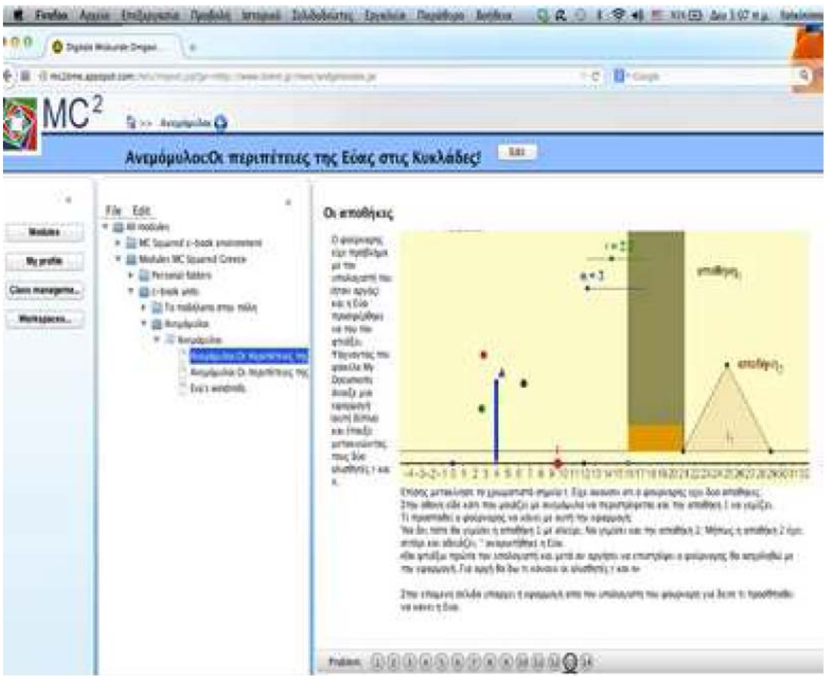

Figure 2: Screenshot of the c-book

level of instruction. Since most of the constructions involved activities on given microworlds the participants expressed the need for thinking up ideas that would give students the opportunity to create their own constructions around windmills. Thus, for the last page of the c-book unit they chose to ask students to construct their own windmill.

Summarizing the content of this phase, we observe that the conversations are rather oriented to pedagogical and didactical ideas than in action, which was prominent in the previous phase. The CoI members are facing dilemmas regarding the pedagogy of their constructions (widget instances) and have to think of ways to stimulate students' creativity without decreasing their own creativity in design. The coherence of the activities had also provoked a number of contributions, since till this phase the constructions were products of individual work and, at this point, the need to be synthesized emerged. These constructions expressed different pedagogical and didactical approaches of the embedded mathematical concepts and the participants had to deepen on these concepts, their interrelations with the available tools and the way they should be communicated through corresponding activities. The CoI members made clear their perspectives (perspective making) and extended or synthesized the ideas and perspectives of the others (perspective taking). Now, the artifacts are accompanied with pedagogy and the participants start thinking about the form of the c-book, expressing ideas alternative to their traditional way of designing.

Phase 4: Transition to the c-book

This phase begins when the CoI members' constructions have just been put on the platform as widget instances, with the sequence which had already been discussed in phase 4. Now the participants are discussing the number of available tools and the kind of functionalities they should provide to students. They are also trying to connect them through narratives and real life situations, so as to make more attractive the students' involvement with the c-book. 'Eva's adventures at Cyclades!' was the title of the c-book unit, in which Eva, her father and a local windmill owner were some of the heroes of the story. The difficulty in connecting widgets and texts is evident in the following discussion:

Areti: Dimitris, how would you find a hypothetical scenario, such as 'a boy trying to construct with a logo-code the windmill's wheel doesn't finally manage it, can the students help him?'

Dimitris: I believe that we don't need to ask something. If you have a story and a widget connected to it, the text needs to be challenging, otherwise the student would not use it, even in case he is asked to. I don't know if it's better to ask clearly students what we want them to do or not. If not, do we lose the story's coherence, as we continuously interrupt it with questions?

Areti: I'm confused. Do you disagree on the way me and Katerina changed the texts?

Dimitris: Texts resulting in a question are ok and constitute a safe teaching method. My objection is that I imagine the c-book unit as a book with texts, which is also a tool, something like the interactive books in Harry Potter. This kind of books does not need any questions in order to attract you... Trying to correct the texts of Popi's model in page 6, I inspired a new activity and put it on the twelfth page, in the form 'text-widget' (Figure 2).

Foteini: It seems that the texts have the potential to re-formulate the widgets.

In this part of the discussions, the CoI members seem to reconceptualise the use of the c-book unit: Areti expresses the need for framing the widget with a real problem to stimulate students' interest. This approach differentiates the c-book from a microworld accompanied by a formal mathematical problem. The development of the c-book unit was an unknown and unfamiliar territory for all and the perspective of the experienced Dimitris favoured the development of new understandings and practices around its use. In this process, the c-book was the boundary object around which the participants interacted, reflected and finally transformed their knowledge. At the same time, new ideas incorporating both a story and a widget instance are generated. For example, Yiannis proposes to give students a photo of a ruined windmill or without sails or roof and ask them to repair or complete it or to put inside its conic roof a rectangular parallelepiped reservoir of maximum volume, for water storage. Furthermore new widget factories are introduced: Katerina exploits available tools of the platform and creates a widget instance with the respective story in which students starting from a 2d plan are challenged to create a 3d building with a windmill. These ideas reflect a new holistic perception of the c-book and the need for making their constructions more attractive using 'non-mathematical' material (photos, plans) in challenging situations.

RESULTS

The results of our research reveal that co-designing the c-book unit was a process evolved in four phases that incorporated different characteristics and underlying learning mechanisms for the CoI members. The ideas about the product and the product itself were modified over time. Phases 1 and 3 include discussions rather concentrated on ideas, while phases 2 and 4 are oriented to practice. In phases 1 and 2 the CoI members coordinate their work, collecting individual ideas and widget instances respectively, which become objects of pedagogical and didactical reflection in phase 3, in order to be synthesized as a coherent whole. Before the end of phase 3 and at the beginning of phase 4, the form of the c-book is outlined and the participants gradually acquire a more precise picture of what the c-book unit might be. Holistic perception of the c-book and the importance of the narratives are the outstanding characteristics of phase 4. At the end of this phase the c-book hybrid form is progressively crystallized and the transformation of the CoI members' knowledge on how to design a c-book aiming to stimulate students' CMT is evident, due to the contributions of the more experienced members. In conclusion, our findings showed that the c-book was the boundary object around which the participants coordinated their activities, reflected on their pedagogy and transformed their knowledge about how to design for creativity.

ACKNOWLEDGMENT

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n° 610467 – project "M C Squared", http://mc2-project.eu. This publication reflects only the author's views and Union is not liable for any use that may be made of the information contained therein. The c-book technology is based on the widely used Freudenthal Institute's DME portal and is being developed by a consortium of nine partner organisations, led by CTI&Press 'Diophantus'.

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What can be learned from online public-generated mathematical content? A case-study of the comments on a viral mathematical video

Alik Palatnik

Technion – Israel Institute of Technology, Haifa, Israel, umapalatnik@gmail.com

The study explores the phenomenon of "viral mathematics", providing insights about the availability and the structure of mathematical content in public-generated media (in particular in comments on videos). The study offers the research methodology, relying on original mix of qualitative, quantitative and data mining paradigms. The findings show that communication using scientific argumentation can occur in comments on a recreational mathematical video; the growth of public interest in mathematical themes can be triggered by the viral video; and the arguments used by the comment-makers contain mathematical derivations, terminology and views on science. Public-generated content can be harnessed as a unique open educational resource and a new source for engaging mathematical problems.

Keywords: Public-generated content, viral video, web cloud analysis, argumentation, divergent series.

INTRODUCTION

My first account with the video in question occurred in February 2014. My 10th grade students were intrigued and asked for my opinion about the "*one plus two plus three plus four video*". The challenge was accepted and I designed the subsequent lesson to discuss the mathematical ideas in the video. For me it was an example of blurring the boundaries between the classroom and out-of school learning. In turn, I was intrigued by the thought that a You Tube video succeeded to engage my students in university level mathematics. At that time, the video "Astounding: 1 + 2 + 3 + 4+… = -1/12" had about two and a half million views and about four and a half thousands comments, which defines it as a viral mathematical video. In general, can public interest in such a video facilitate learning of mathematics? What can students learn from the comments on a video? To answer the above questions we have to understand the nature of communication through the comments, the structure of the comment and to anticipate its application to education. This research is an exploratory study that used a mixed methodology approach in a relatively new field of learning.

THEORETICAL BACKGROUND

The advent of personal computers, electronic mobile devices and the Internet has increased the blurring between formal (institutional) and informal education in the 21st century. Blogs, wikis, multimedia sharing sites, podcasting, and social networks facilitate individual production and user generated content, harnessing the power of the crowd through an architecture of participation (Andersen, 2007). Hossain and Quin (2012) argue that the interactive features of Web 2.0 technologies enable mathematics teachers and students to create collaborative learning environments inside and outside of the classroom, providing new type of resources and ways of instruction.

Educational resources, their design, sharing and implementation receive a renewal attention from the mathematics education community (see, e.g., Ruthven, 2013 for a comprehensive review). One of the growing tendencies, closely associated with online sharing is a development of open educational resources (OER). Theoretical lens of instrumental approach was used by Trgalova, Soury-Lavergne and Jahn (2011) and Trgalova and Jahn (2013) to study the particular OER (dynamic geometry content repositories) and members of their communities. Kynigos (2014) urges to study new types of OER, which are able to "generate more powerful and relevant sociotechnical communities for the learning of mathematics" (p. 252).

One of the social phenomena related to Web 2.0 is the 'viral video': a video clip content which gains widespread popularity through the process of Internet sharing ("Viral video", n.d.). This particular type of media can be harnessed as a unique OER. As shown in 2011 by "The periodic table of videos" channel, educational videos explaining basic notions in chemistry can become viral – garnering millions of views. The impact of the channel awarded its creators with a *Science* Prize for Online Resources in Education (Haran, & Poliakoff, 2011). This explores the potential of a general public interest, manifested in public-generated content, as an educational resource in a mathematical classroom.

Research questions

This study aims to explore the phenomenon of "viral mathematics", providing insights about the availability and the structure of mathematical content in public-generated media (in particular in comments on videos with a mathematical topic). It raises following research problems. First, little is known about the nature of comments on mathematical video. Whether they consist mainly of expressions such as: "lol" or contain a valuable mathematical component? Second, within the interest on the viral video itself, the public interest in the mathematical concepts and ideas mentioned in the video has to be isolated. Third, the characteristics of comments on the mathematical videos are unclear and have to be typified. To do so, a case study of the comments on the "Astounding: 1+2+3+4… = -1/12" video (Haran, 2014) was undertaken.

The research questions, following from the research problems are:

1) What mathematical content can be found in public-generated online resources, such as comments on a video?

2) Can viral video elevate public interest in the underlined mathematical content?

3) What is the structure of comments on a mathematical viral video?

RESEARCH METHODOLOGY AND APPROACH

Viral videos and comments are an unused resource in mathematics education and to date have not been the subject of mathematics educational research. Therefore, one of the objectives of this study is to propose suitable research methodology, which relies on original mix of qualitative, quantitative and data mining paradigms.

The video "Astounding: 1 + 2 + 3 + 4 + 5 + … = -1/12" was published on YouTube on 9/01/2014 and is one of 232 (while writing this article) videos of Numberphile channel. Nearly all of those videos have hundreds of thousands of views. The "Astounding…" video introduced the paradoxical result: $1 + 2 + 3 + 4 + \ldots = -1/12$ through a manipulation with series (divergent and convergent) and infinity. The creators of the video present the following "proof". First, the following sums were denoted: $S_1 = 1 - 1 + 1 - 1 + \ldots$; $S_2 = 1 - 2 + 3 - 4 + \ldots$; and $S = 1 + 2 + 3 + 4 + \ldots$ Then the presenter claims and explains that $S_1 = 1 - 1 + 1 - 1 + \ldots = 1/2$. Based on this result he proves that $S_2 = 1 - 2 + 3 - 4 + \ldots = 1/4$. Through further manipulations the presenter derives that $S - S_2 = 4S$ and hence $S = -1/12$. In short, the paradoxical result is obtained due to inconsistencies of various natures: different meanings of "=" sign in mathematics; attribution a numerical value to a divergent series; rearrangement of infinite series using the commutative property of addition, etc. Quoting Niels Henrik Abel:

> The divergent series are the invention of the devil, and it is a shame to base on them any demonstration whatsoever. By using them, one may draw any conclusion he pleases and that is why these series have produced so many fallacies and so many paradoxes. [1]

To answer the research questions different methodologies were applied:

> To answer the first question and to identify the main themes of the discourse in comments on videos, *word cloud* analytics were applied. A word cloud is a special visualization of text where the more frequently used words are highlighted by their proportional size (Feiberg, 2013). Use of word cloud as an analytical tool is still rare in educational research, however it appeared to be a fast and visually rich way to overview the data (McNaught & Lam, 2010). Those features make a *word cloud* especially suitable for preliminary analysis and for validation of previous findings (ibid). Wordle and Tagxego programs were applied to obtain word clouds, containing 50 most frequently used words [2] out of 38452 words

1a 1b

Figure 1: Word clouds containing 50 most frequently used words in all the comments on the "Astounding..." video (2a- out of 38452 words; 2b- out of 5332 words of the sample)

in all the comments to the video. Comparison revealed no significant differences between the word clouds. The further analysis was made using Wordle software (Figure 1).

The data was prepared for analysis in the following way [3]: To open all of the pages of comments I selected "Show more" at the end of each page; to open all replies to each of the comments I selected the "View all N replies" notifications (58); to show all the lines in lengthy comments I selected "Read more" notifications (340). The obtained HTML document was copied to the MS Word using "Keep text only procedure" resulting in a 209 pages document. The utility phrases (e.g., Hide replies; Show less) were deleted from the text file before the data mining stage.

To answer the second question, namely to estimate the impact of the video on public interest to the featured scientific content, the changes in information seeking behavior as expressed by *Google Trends* were used. The methodology, using Google Trends tool was proposed and applied by Segev and Baram-Tsabari (2012). The mathematical concepts Grandi series and Cesaro sums were chosen as the subjects of investigation for the several reason:

— Grandi series and Cesaro sums are the key concepts of the video. The Cesàro sums method using the limit of the average of the first n partial sums of the series, as n goes to infinity, enables "summation" of the divergent series, thus Grandi series $1 - 1 + 1 - 1 + \ldots$ has Cesàro sums of 1/2.

— Terms Grandi series and Cesaro sums were not mentioned explicitly in the video and were mentioned in the comments

— Grandi series and Cesaro sums are usually unknown to the lay public and for the non-mathematical academic audience unlike other key concepts of the video (e.g., those present in the university curriculum or related to Riemann or Euler).

The hyperlinks to mathematical sites and blogs in the comments are additional indicators of the video's impact. The standard Find function of Word with "http" argument was used to find the external links. Each link was labelled according to its content (e.g., Blog, Wiki, scientific paper).

Descriptive statistics of the data sample and qualitative analysis of argumentation in the comments was applied to classify the comments and arguments, to answer the third question. Out of 209 pages of comments 25 pages (≈12 % of pages) were randomly selected. The sample contains 85 comments. An additional instrument for establishing the representativeness of the sample is a word cloud, produced from 5332 words of the sample (≈14 % of text) (cf. Figure 1a with Figure 1b). The main themes and topics (e.g., sum, infinity, series) are intact. All the comments in the sample was numbered, and coded according to whether they contain the following indicators: presence of mathematical derivations/ scientific terminology; characteristics of scientific enquiry/ epistemology; attitudes toward science or scientists; emotions in support or against the video. In addition, each comment was analyzed in order to locate the essential parts of an argument: a claim, a data and a warrant (Toulmin, 1958/ 2003). For example, comment #80 on the "Astounding..." video: "*Assumption: the sum of an oscillating sequence is its arithmetic mean. Conclusion from assumption: the sum of a set of positive integers*

Mathematical derivations/ scientific content	Characteristics of scientific enquiry/ epistemology	Attitudes toward science and scientists	Emotions	Argument
1	1	0	0	1

Table 1

is negative. You've just proved by contradiction that the sum of an oscillating sequence cannot be its arithmetic mean." was labelled as shown in Table 1.

The claim: the sum of an oscillating sequence cannot be its arithmetic mean, thus the main statement of the video is false; the data: a fragment of the video, where the statement $1-1+1-1\ldots=1/2$ is proven; the warrant is a proof by reduction ad absurdum. Consequently, the abovementioned comment is an argument, which uses mathematical calculations and terminology and free of emotion. First stage of the research will provide a list of key words, which are the core of scientific communication established in community of commenters on the video. The data collection was completed by simple filtering technique using the key words. The comment-makers use pseudonyms and usually do not share any personal details such as gender, age, occupation and level of expertise in mathematics. However, in some cases it was possible to make grounded suggestions about the comment-maker's background. For example search phrases: "teacher", "class", "students" revealed math teachers/ students among the comment-makers. I read the mathematical derivations line by line and checked the mathematical terminology used by comment-makers to find the experts in mathematics among them.

RESULTS

What mathematical content can be found in public-generated online resources, such as comments on a video?

The word clouds (Figure 1) show that the comment-makers were involved in scientific discussion, which main topics and themes were connected to a sum of infinite series. The word count confirms this finding. 30 words out of top 50 in the word cloud belong to mathematical/ scientific terminology (general or specific to the content of the video). See Table 2 for the details.

#	Word	Frequency in the comments		#	Word	Frequency in the comments
1	sum	233		16	average	72
2	series	220		*17	shift (technique)	72
3	infinity	213		18	positive	67
*4	S	191		19	negative	59
5	number	184		20	adding	55
*6	S2	161		21	theory	55
*7	1/2	142		22	mathematics	54
*8	1/12	134		23	sums	51
9	numbers	127		24	prove	51
10	infinite	113		25	physics	50
11	value	100		26	equation	50
12	answer	92		27	divergent	48
13	proof	92		28	sequence	47
*14	S1	80		29	summation	43
15	math	76		30	converge	41

Table 1: Word cloud frequency (* indicates the words specific to the video)

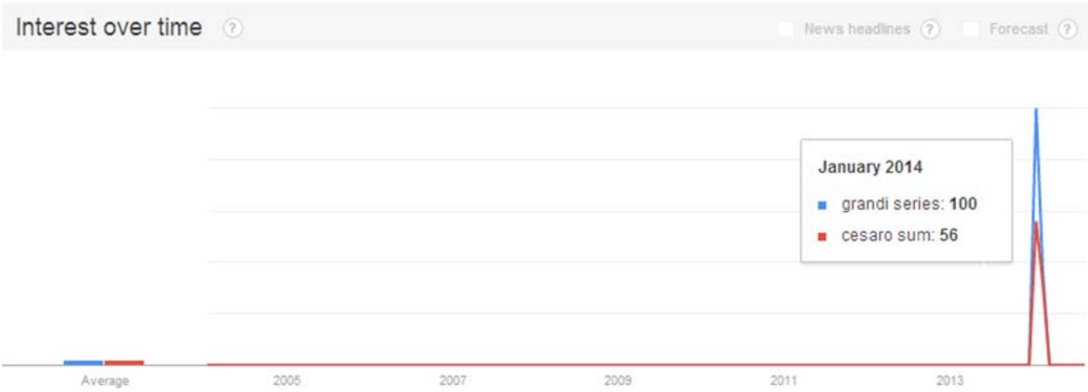

Figure 2: Google trends graph showing a percent of searches on Grandi series and to Cesaro sum

Can viral video elevate public interest in the underlined mathematical content?

Figure 2 indicates the growth of public interest in Grandi series and Cesaro sum (abstract and unknown to the lay public mathematical concepts which are explicitly mentioned in the comments and not in the video). The peak of interest to the concepts in question was registered in January 2014, corresponding to the release date of the video–9/01/2014. It is of note, that there are 37 different hyperlinks pointing to 32 different mathematical sites in comments (Figure 3). Most of the links led to various Web 2.0 platforms: Wikipedia articles about mathematical concepts related to the video (e.g., Divergent series, Riemann zeta function), mathematical blogs, social network, question and answer site or YouTube. Publication of the video has created a teachable moment and new resources for learning were generated/ brought into account.

What is the structure of comments on a mathematical virus video?

The analysis of the sample shows that the comment-makers use mathematical derivations and scientific terminology to the broad extent (28% and 36% of comments resp.) The data analysis process revealed that 73% of the comments contained arguments (63 out of 85). These arguments were categorized according to the possible roles that they play within the discourse [4]. In short the most-used categories of argument were: common sense; mathematical laws; mathematical intuition (based on non-infinite mathematics); counterintuitive thinking, proof; and authority. More than half of the comments (52%) in the sample contained indicators of positive or negative emotions toward video and its creators. For example: the fragment of comment #28: "*You know what is more amazing? WHY anyone would ask for the result of 1+2+3+4... ? It is obvious it is infinity!! You must be a genius to think that maybe not.*" Table 3 shows the distribution of the arguments/ emotions in the comments.

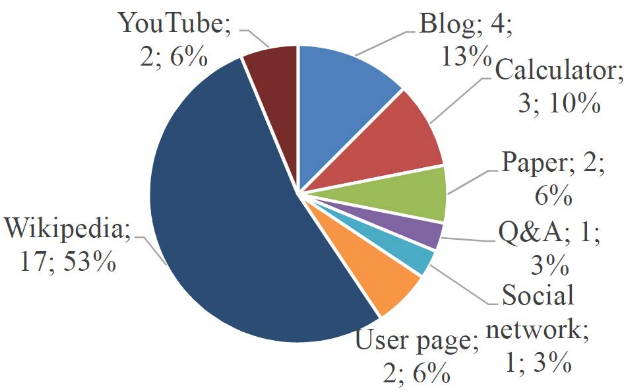

Figure 3: Distribution of hyperlinks in the comments

The next step was to apply the filtering technique to the text file of all the comments. It revealed data, which the two previous techniques could not disclose. There were evidences of various stakeholders to the "viral math" phenomenon. The math teachers/ lecturers were among the comment-makers. Most of them were concerned with misconceptions which students may develop as a result of watching the video. Comment-maker X wrote: "*My complaint is, this video is leading my students to believe that 1-1+1-1+... converges to 1/2, and is encouraging them to work with limits of sequences without knowing if they converge, which is a huge problem...*" Nevertheless, most of the teachers appreciate the "*arousing interest in discussing the problem*". The reading of the mathematical derivations line by line reveals mathematicians/ physicists among the comment-makers. Some of the comments were mathematically perfect and could be a part of a textbook, as previously cited in the Methodology section comment #80.

The comment contains:	Emotional expressions supporting / refuting the result	No emotional expressions supporting / refuting the result
Argumentation supporting / refuting the result: 1+2+3+...=-1/12	33	30
No argumentation supporting / refuting the result: 1+2+3+...=-1/12	11	11

Table 3: Argumentative/ emotional component in the sample of the comments on "Astounding…" video

There were some emotionally loaded comments toward the video creators and comments expressing attitudes towards science. The following excerpt of the comments should raise the awareness of mathematics educators, willing to use this resource to the issue of public understanding of the science and its role in society:

— *Mathematicians are frauds (in Russian)*

— *this equation means infinity is different from what we believe, or perhaps our whole concept of mathematics is wrong*

— *Why are you bashing on this channel? Blame the mathematicians who derived this and use it*

— *Unbelievable that scientists make "science" based on fallacies.*

DISCUSSION

This study explores a viral mathematical video and a corpus of corresponding comments as an open resource which was adapted for the needs of (mathematics) education. Trgalova and Jahn (2013) studied more conventional OER (web-based repository specifically designed for the educational needs) and registered a relatively small number of reviews and modifications of the content by users. The authors suggest that lack of tools for communication and collaboration in the repository may decrease users' motivation to contribute. In the list of key ideas of Web 2.0 application to education Andersen (2007) named individual production and user-generated content in first place. The current study, through analysis of the comments, makes a step toward better understanding of the learning in dynamic web environment with active user participation. The findings show that comment-makers on a recreational mathematical video can create, share, and communicate mathematical content using scientific argumentation. The active public participation through embedded tools of communication and engagement provided an educational component to an open resource.

The arguments used by the comment-makers contain mathematical derivations, terminology and views on science and scientists. The high presence of arguments in the comments advocates the further study of arguments as a promising direction in the research of mathematical communication based on Web 2.0 platforms as file sharing sites, social networks and blogs. This study treats learning from comments as a special case of learning through argumentation. Learning from argumentation has been shown by Asterhan and Schwarz (2007) to lead to students' conceptual gains where the introduction of the dialog of disagreeing discussants proposing reasons for their views about a problem resulted in the students' discussion becoming more argumentative. Lachmy and Koichu (2014) use their analysis of students' arguments to understand the interplay of empirical and deductive reasoning when proving in a computer-enhanced environment. To this end the categories that originated in the comments can facilitate the critical thinking of the students when applying to the public-generated content. For example, the most "innocent" step in the underlying proof is the implicit claim that the sums 1-1+1-1+...; 1-2+3-4+...; 1+2+3+4+... can be denoted and the numerical values can be attributed to S_1, S_2 and S. The argument belongs to a common sense category and must be revisited.

The growth of public interest in mathematical themes triggered by the viral video was shown in this study and is a phenomenon that should be studied intensively as a part of wider vein of research on a public interest in science (e.g., Baram-Tsabari & Segev, 2013).

Despite the study's limitations (it is a case study of the comments on one video), the claim is that this particular type of media can be harnessed as a unique open educational resource. First, it embodies the cyber-social learning opportunities typical of the Web 2.0 environment. Second –it combines the benefits of multimedia learning with mathematically rich content in public-generated comments on the video.

ACKNOWLEDGMENT

The author would like to thank Dor Abrahamson, Ayelet Baram-Tsabari and Boris Koichu for the valuable help. This research was partially supported by the Technion Graduate School, the Mandel Leadership Institute and the Israeli Science Foundation (grant 1596/13).

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ENDNOTES

1. From letter (Jan 1828) to his former teacher Berndt Holmböe. In (Morris Kline. Mathematics: The Loss of Certainty (1982), p. 170).

2. Except common words of English vocabulary.

3. As far as I know, there is no commercially available solution to the YouTube/ blog commentary's page import to CSV format. One of the possible contributions of the study is to define parameters and variables to be used in the algorithm.

4. The results of the categorization are omitted due to space limitations.

Representations of creative mathematical thinking in collaborative designs of c-book units

Ioannis Papadopoulos[1], Berta Barquero[2], Andrea Richter[2], Maria Daskolia[3], Mario Barajas[2] and Chronis Kynigos[3]

1 Aristotle University of Thessaloniki and CTI & Press Diophantus, Thessaloniki, Greece, ypapadop@eled.auth.gr

2 University of Barcelona, Barcelona, Spain, bertabf@gmail.com, a.richterboix@gmail.com, ma.barajas@gmail.com

3 University of Athens and CTI & Press Diophantus, Athens, Greece, mdaskol@ppp.uoa.gr, kynigos@ppp.uoa.gr

The study presented here focuses on the representations of creative mathematical thinking (CMT) held by two Communities of Interest (CoI) before and after designing educational resources with an innovative tool affording diverse expressive media for creativity in classrooms. Our analysis revealed similarities but also distinct differences in the way these communities operationalized their CMT representations in their products. Both CoI (Greek and Spanish) appreciated the novel affordance of diverse expressive media to foster CMT in students through open, real life, interdisciplinary problems amenable to multiple solutions. However, the CMT affordances appreciated by the CoI members were distinctly different with respect to contextual factors involving personal and schooling issues as well as influences by the research culture of the two corresponding teams.

Keywords: Creative mathematical thinking, technology.

INTRODUCTION: CMT AND TECHNOLOGY

Various definitions regarding mathematical creativity have been proposed during the last decades. Poincaré (1948) viewed mathematical creativity as the interplay between a period of conscious and unconscious work, in which any choices in terms of mathematical reasoning and proof are made in the pursuit of elegance and usefulness. Recently, Sternberg and Lubart's (2000) identified mathematical creativity as the ability to predict some 'non-predictable' conclusions that are at the same time useful and applicable. More recently discussion on math creativity has been led towards sorting out whether to approach it in terms of a *product* or a *process* (Shriki, 2010), as a *general ability* (associated with applying analogous problem-solving processes from one field to the other) or as some *domain-specific* skill (exemplified within a particular disciplinary or other activity field and taking into account the logico-deductive procedures already developed in it) (Lev-Zamir & Leikin, 2011). More particularly, the three most prominent strands in the research tradition of math creativity are as follows: the first and most popular is situated in the so-called the 'genius' approach of creativity (Sriraman, 2005); the second, represented by Silver (1997), is based on the claim that mathematical creativity is susceptible to instructional and experiential influences and gives equal emphasis on problem posing as on problem solving, with preponderance on the use of ill-structured and open-ended problems; finally, a more recent but rather under-researched approach is that of 'techno-mathematical literacies', the focus of which is on identifying and addressing mathematical creativity in out-of-school contexts and in out of mathematics (non-disciplinary) situations, such as looking for and applying mathematical concepts and relations in technology-based workplace practices (Noss & Hoyles, 2013).

Only the latter of these approaches tightly addresses the use and role of digital media for CMT. Although often implied, the development of CMT with the use of exploratory and expressive digital media has not often been centrally addressed in providing users with an access to and a potential for creative engagements with mathematical concepts (Healy & Kynigos, 2010). During the last decades several digital tools, applets and authoring systems have appeared designed to foster CMT affording dynamic manipulation, interconnected representations (inc. mathematical formalism), simulations of phenomena and situations embedding mathematical rules, visualisations of data representations and handling of probability. Some of these technological advances have been developed within the frame of approaches with a constructionist, argumentational, inquiry-based, problem-based, experimentation origin, all connected but not centrally matched with creativity. Nevertheless, even though research points to the potential of expressive digital media to foster CMT, the uses of these same media in educational practice at large are conversely frequently instrumented towards contexts of traditional lecturing and demonstration of exercise solutions (Ruthven, 2008). This looks like a paradox situation where digital media are being used very differently with respect to their potential for providing users with access to mathematical ideas hitherto obscure and inaccessible, and at the same time allowing them to creatively engage with experiential constructionist, dialogical and social generation of mathematical meanings, understandings and social norms has long been elaborated (Hoyles & Noss, 2003).

Our paper therefore addresses a real wide-scale problem and it is within that context we attempt to contribute to our understanding of teachers designing of affordances for CMT. No doubt, the clarification of shared understandings of the essence of CMT and the spreading of respective activity in education are slow and non-linearly developing processes. New media affording *collaborative* designs for CMT may provide us with the means to generate socio-technical environments (Fischer, 2011) more conducive to addressing these issues and designing for learning cultures cultivating CMT. This paper studies designers' representations of CMT (in the context of a European project called 'M C Squared') as they became operationalized in the collaborative production of materials with an innovative e-book author aiming to afford CMT (we call it the 'c-book', 'c' for creativity) to the end users. The designer communities have been orchestrated by four research teams so as to ensure the diversity among the members in relation to the disciplinary backgrounds, expertise, history and membership in different communities of practice which are defined in literature as Communities of Interest (CoI) (Fischer, 2011). Four such CoI (English, French, Greek and Spanish) were thus formed in MC2 project. The CoI were put together to include a variety of expertise involving designers and developers of digital resources for math education, publishers of mathematics educational materials, researchers in (and outside) math education or in creativity, teacher educators, school math teachers and students.

Given the diverse approaches to CMT outlined above, and the relative lack of connection to math activity with expressive digital media, the concept remains fuzzy in the literature. So, the aim of research is to illuminate how a shared understanding of CMT can be generated in CoI using technology specially built to afford collaborative design for CMT. Such research may contribute in crafting ways to orchestrate and moderate teacher communities of practice potentially playing a role in spreading the use of expressive media for CMT related activities in classrooms. Our research focus was thus twofold: (a) to illuminate the designers initial representations of CMT in two (of the four) CoI, and (b) to illuminate the process of operationalization of these representations during collaborative designs of such technologies for students which would be done by examining how the CoI members evaluated their product in terms of CMT affordances. In the sections to follow we briefly outline our research method, and then discuss the CMT representations of the two CoI so far revealed by our analyses (the Greek and Spanish respectively) in two ways. Firstly as they were expressed in their responses to two semi-structured questionnaires before they were engaged in the design, and subsequently, as they were manifested in their evaluation of the CMT affordances of their collaboratively produced c-book units. Finally, through a synthetic analysis we aimed to identify common patterns and differences in the ways the two CoI approached the CMT criteria with regards to their task at hand with the c-book technology.

DESCRIPTION OF THE STUDY AND METHODOLOGY

The decision to work with four (here two) CoI was a methodological one for two reasons: The first one was that we aimed to obtain diversity in relation to the expertise of the CoI members within each CoI. The second was that each CoI would correspond to a particular educational context. This decision would contribute to deepening our awareness of the role of the context diversity both by the exercise of de-contextualising our findings and by using our synthetic knowledge to later try to develop methods to re-contextualize generic findings (Lagrange & Kynigos, 2014) by distinguishing between commonalities and differences in the operationalized representations of the members of the two CoI.

In order to identify the CoI members' initial representations of CMT, an exploratory approach was chosen independently by the two research teams, mainly based on the use of questionnaire as the main instrument for data collection. Each research team developed its own questionnaire and followed a different but nevertheless comparable procedure. In the Greek case the questionnaire was administered to the 17 participants in the Greek CoI. It consisted of: (a) open-ended questions aiming to explore the CoI members' personal definition of CMT, and (b) a list of 17 statements concerning the nature and characteristics of CMT, based on a 6-point Likert type scale (ranging from 1=completely disagree to 6=completely agree). The Spanish research team developed a different questionnaire, which was administered to the 17 members of the Spanish CoI. This questionnaire consisted of: (a) a list of 26 statements (which respondents were asked to respond to by using a 5-point Likert scale, ranging from 1=completely disagree to 5=completely agree) concerning the nature of creativity and CMT, the characteristics of creativity in their professional background and the profile of a creative student and a creative math teacher; (b) three open-ended questions to collect features, criteria and examples of tasks and activities that could foster CMT. The qualitative data from the open-ended questions were subjected to thematic analysis (Braun & Clarke, 2006). This is a method for identifying, analysing, and reporting patterns (themes) within data and goes through the following six phases: (i) familiarizing yourself with your data, (ii) generating initial codes, (iii) searching for themes, (iv) reviewing themes, (v) defining and naming themes, and (v) producing the report. Quantitative data from close questions were statistically analysed, calculating, for each item, the median and the Inter-Quartile Range (Mogey, 1999). Then, a configuration of members from each CoI produced one c-book unit each, the '*Windmills*' [1] c-book unit (jointly created by 7 members of the Greek CoI), and the '*Viral behaviour of the Social Networks*' [2] (designed by 8 members of the Spanish CoI). After completing these c-book units, the two CoI were asked to evaluate the CMT affordances of their c-book units on the basis of specific criteria previously defined by them. Responses from the CoI were subjected to thematic analysis and the criteria identified were organized according to whether they (a) were aligned with established criteria already suggested in the literature, (b) were specific to the CoI, (c) were shared between the two CoI, and (d) were in accordance to the two CoI initial CMT representations. Finally, comparative analysis was employed to trace similarities between the criteria proposed by the two CoI.

INITIAL REPRESENTATIONS OF CMT IN THE TWO COI

The analysis of the data gathered from the Greek CoI responses to the open-ended question asking them to provide a definition of CMT revealed a set of themes ascertaining that the participants conceptualised CMT both in terms of *process* and *product*.

The first theme matched CMT with the 'construction' of either math ideas or some concrete mathematical 'objects'. This is in line with a 'constructionist' approach to creativity giving emphasis to the learners' creative expression and learning through the active exploration, modification and creation of digital artefacts (Daskolia & Kynigos, 2012).The next two themes that emerged are in accordance to what literature identifies as the criteria of novelty/originality (Liljedahl & Sriraman, 2006) and usability/applicability (Stenberg & Lubart, 2000). The former relates CMT to 'mathematical productions' that are new/unusual, and/or new or unexpected ways of applying mathematical knowledge in posing and solving mathematical problems. The latter was perceived by the Greek CoI as the purposeful combination of math knowledge from different math domains or with knowledge from other scientific areas. Some other themes identified in the Greek CoI representations of CMT were those pertaining to fluency, flexibility and imagination. The literature views all three of them as characteristics of a creative mathematical process (Silver, 1997; Leikin, 2009), interconnected but not necessarily all present at the same time (Baer, 1993).

The analysis of the close questions showed that almost all Greek CoI members seemed to agree that mathematical creativity entailed that is an ability which can be fostered through interaction with other people in a collaborative context (*Mdn=6, IQR=1*) and within a milieu which is rich in many alternative –and even contradictory- ideas (*Mdn=6, IQR=1*), which allows openness to other disciplinary fields (*Mdn=6, IQR=1*), and includes problems inspired from real life (*Mdn=6, IQR=0.25*). Most Greek CoI members agreed that CMT has to be based on a deep and well-rooted mathematical background, and that it can emerge from the people's ability (a) to use many, different and unusual ways to solve a problem (*Mdn=5, IQR=0.25*), (b) involving many and diverse mathematical representations for approaching a math problem (*Mdn=5, IQR=1.5*).

In the case of the Spanish CoI, thematic analysis was also employed for the analysis of the data from the open-ended questions. The analysis revealed the key traits which could trigger CMT in students: a) allowing for multiplicity in the approaches and techniques to resolve questions (16 respondents out of 17); b) promoting multiple representations of a particular mathematical concept (16 / 17); c) engaging students in math problem situations close to their reality (13 / 17); d) situating math problems in interdisciplinary contexts, by combining math knowledge with other disciplines and showing its functionality in other domains (13 / 17); and e) including communication tasks within the activities to share students proposals and opinions (10 / 17).

Concerning the nature and characteristics of CMT, most Spanish CoI members consider creativity being a quality that can be developed through instruction (*Mdn=4, IQR=1*) and that interaction with people with different perspectives can enrich the creative process (*Mdn=5, IQR=1*). Most of them consider themselves as creative professionals to a great extent (*Mdn=4, IQR=0*), and acknowledged that mathematics help promoting creativity in other disciplines (*Mdn=4, IQR=1*). In describing a creative student, they emphasised as requisites the ability to: (a) formulate questions and initiate investigations (*Mdn=4.5, IQR=1*); and (b) find different ways to solve problems combining different tools and representations (*Mdn=4.5, IQR=1*). The Spanish CoI members claimed that, in order to foster CMT, the teacher has an essential role, by holding a deep mathematical background (*Mdn=5, IQR=1*), and by encouraging the advent of diverse students' responses when approaching the mathematical questions (*Mdn=4, IQR=1*).

Comparing the two cases it seems that both CoI shared several criteria, which the literature identifies as closely related to CMT. To mention some: the belief that CMT can be fostered in the classroom, through appropriate education (Silver, 1997), the recognition that the interaction between the learning actors is important (Leikin, 2009) and the connection of creativity with mathematics (Silver, 1997). The promotion of CMT is understood through tasks of an exploratory nature (Mann, 2006), through open problems amenable to multiple solutions (Lev-Zamir & Leikin, 2011), through situations that demand the combination of various tools and/or representations, or through interdisciplinary tasks (Perry-Smith & Shalley, 2003).

EXTRACTING THE COI CRITERIA OF CMT

The CoIs' initial representations of CMT allowed researchers to build a shared understanding and 'vocabulary' on how to address CMT within their CoI, which helped them reflect on and develop their own criteria for assessing the c-book units' CMT potential impact to prospective students. The two CoI have designed and produced a series of c-book units until now. However, from a research point of view we were interested in identifying whether and how these CMT representations were employed by the two CoI immediately after they had completed their first c-book unit. In the following paragraph, we offer a short description of these two first c-book units, namely 'Windmills' and 'Viral Behaviour of Social Networks'.

The main idea of 'Windmills' was to challenge students to explore, identify and use the mathematical concepts underlying the construction and operation of Greek windmills, an integral part of the Greek islands' scenery, quite familiar to the students. The CoI intention was to design a c-book unit that would invite students to foster 'unexpected' mathematical ideas, surprising even to the initial designers of the activities. The construction of the c-book unit was initiated with the CoI members developing and exchanging through CoICode (the c-book author collaboration tool), Turtleworlds (a Logo-based Software), Geogebra (Dynamic Geometry Software) and DME Draw3D widget [3] instances (3D grid using cubes as building blocks). The rationale was to start addressing simpler and gradually move to more complex mathematical concepts. This 'low threshold – high ceiling' rationale adhered to the idea of making the c-book unit appropriate for students within a large range of school grades (lower to higher secondary). The first four pages included activities with Turtleworlds targeting the construction of geometrical figures – parts of a windmill (triangles, parallelograms, regular polygons, solid shapes) and use them to create a windmill. The following pages used GeoGebra to engage students in exploring the operation of ready-made models of windmills and identify the underlying mathematical concepts (linear functions, direct proportional amounts, multi-branch functions, periodic functions, co-variation of geometric magnitudes). Finally, students would have to repair a deserted windmill to explore geometric figures.

The Spanish c-book unit focused on fitting mathematical tools and models to describe and understand the 'Viral Behaviour of Social Networks'. Mathematical modelling was thought central to allow students to: a) formulate assumptions on different phenomena related to social network users, b) use mathematical tools to analyse relations and patterns on real data, c) look for, fit and test mathematical models to forecast social networks behaviour, d) use math (and its models) to obtain responses about reality, and e) be able, if necessary, to reformulate assumptions and models. The CoI wished to put forth an interdisciplinary approach where math (mainly arithmetic and pre-functional modelling) would provide tools for analysing social phenomena. The c-book unit consists of 3 parts: The *first part* focuses on introducing students to a sequence of questions about how the number of friends evolves depending on the degree of friendship, in order to explore exponential properties acting under social networks. The *second part* presents the 'theory of the six degree of separation', by asking students to estimate the number of friends a teenager can have and, then, forecast how this evolves depending on the degree of friendship. Finally, the *third part* allows students to work on their own data by asking them to search their friends and connections in Facebook and estimate up to what degree of friendship they are connected with any other person in the world. The c-book unit was developed in 14 pages and includes widgets from 3 different widget factories: GeoGebra, DME (probability tree, answers box, tables, etc.) and a special Cinderella widget to visualize and compare graphs of points using linear and algorithmic scale.

After completing the two c-book units, the CoI members were invited to evaluate the potential of the units as educational resources fostering users' CMT. Each research team conducted a separate analysis of the data gathered from their CoI evaluation. This analysis examined whether the criteria used by each CoI were in alignment with their initial CMT representations, with the criteria identified in the literature review, and with the educational traditions of the specific CoI. A comparative analysis between the two cases brought evidence about commonalities and differences between the two CoI. The findings of this analysis are as follow:

a) *A focus on exploring 'real' problems and questions*: Both CoI emphasised the use of real-life situations (close to students' reality) in their c-book units. This is recognized as a characteristic of mathematical creativity and many mathematicians describe it as an invaluable aspect of their craft (Sriraman, 2005). The same criterion was traced in the initial CMT representations of both CoI (a, c, d [4]).

b) *Promote openness, exploration, and diversity in the approach taken:* According to their initial representations the two CoI appreciated the fact that the activities included in the c-book units were based on open questions allowing the students to suggest their own answers. Silver (1997) suggests that such open questions encourage the development of students' creative fluency, associating fluency with exploration as a way of thinking. On the first hand, there is an explorative nature in the questions included in both c-book units. The students are prompted to explore a problematic situation using different mathematical tools and also including simulation of different mathematical models. On the second hand, both c-book units include specific widgets to collect the diversity of students' proposals, in a more open format as windows to write down their answers to be collected or as interactive widgets. For instance, the Turtleworlds widget allows students to build and work with their windmill construction with different blade's forms; the Cinderella widget facilitates that each user can work with their own Facebook data, combining the point table with the graphical representation in different scales (a, c, d).

c) *Enable multiplicity of mathematical approaches/ representations/techniques/ solutions:* According to both CoI, one of the strongest didactical qualities of the c-book units was the one of multiplicity. This refers to the case of embedded activities that promote multiple solutions for a problem or multiple paths toward the solution of a problem (Leikin, 2009). The technology involved in the two c-book units facilitates the students to work with, and combine, different representations of mathematical concepts. One example might be the variety of numerical and graphical representations (trees, point graphs, etc,) of geometrical sequences (a, c, d).

d) *Promote inter/intra-disciplinarity:* According to Perry-Smith and Shalley (2003), a person's exposition to diverse contexts and functional areas may lead to the production of different and unusual ideas. Such manifestation is an indication of novelty/originality, which is considered an inherent characteristic of CMT. The Greek c-book unit deals with mathematics through some pure engineering and environmental issues, such as the preservation of old windmills, the use of alternative energy sources, etc. The Spanish c-book unit connects mathematics with social and technology issues such as the use of Facebook. Both c-book units combine a variety of mathematical topics belonging to different mathematical domains. 'Windmills' constructions invite students to apply knowledge from trigonometry (angles, construction of triangles, etc), functions, coordinates, and graphs, whereas the 'Social Networks' promotes the construction of models based on exponential and logarithmic properties, geometric sequences, and combines diversity of representations: probability trees, point tables and graphs with ordinary and logarithmic scale (a, c, d).

e) *Promote a progressive modelling process:* Mathematical modelling penetrates the whole activity in the Spanish c-book unit. The c-book unit starts with an initial generating question that was broken into some derived questions placed along the unit (and complemented with the most appropriate widget instances). Each of its three phases leads to consider different mathematical models (more complex at each step) that appears from the analysis of real data about social networks, at the same time models evolve thanks to its evaluation and contrast with reality (a, b, d).

f) *Involve a constructionist aspect:* In the case of the Greek c-book unit the students are invited to create their own windmill or to complete half-constructed ones. This constructionist aspect is very much based on the previous experience and educational tradition of the Greek CoI members. All of them were involved in two wide-scale initiatives of the Ministry of Education in Greece. In the first, the aim was to expose mathematics teachers (amongst others) to constructionist epistemology, technologies and activity designs. The second project concerned the digital enhancement of the traditional textbooks which was also penetrated by the flavour of constructionism (a, b, d).

CONCLUSIONS

The study was carried out in the context of a growing realisation that large scale use of digital media in educational practice is widely dis-aligned from CMT affordances. Understanding and promoting CMT representations in designer communities has had a dual purpose of suggesting a strategy to spread CMT cultures and illuminating our own understanding of the fuzzy notion of how CMT is represented amongst designers. We thus tried to identify some initial CMT representations amongst diverse designers and then to illuminate how these representations were operationalized in the process of collaborative design. Both CoI appreciated the potentiality for the c-book units to foster CMT in students appreciating the potential for integrating real-life problems that are open and amenable to exploration, multiple solutions and the combination of various disciplines and/or various mathematical topics. But equally important were the operationalization of representations of CMT linked to a specific context. The Greek CoI associated the chance to foster students' CMT with the ability of the c-book technology to support constructionist activities. The Spanish CoI made a corresponding association of this technology with its ability to support a progressive modelling process.

ACKNOWLEDGMENT

The research leading to these results has received funding from the European Union Seventh Framework Programme (FP7/2007–2013) under grant agreement n° 610467 – project "M C Squared", http://mc2-project.eu. This publication reflects only the authors' views and Union is not liable for any use that may be made of the information contained therein. The c-book technology is based on the widely used Freudenthal Institute's DME portal and is being developed by a consortium of nine partner organisations, led by CTI & Press 'Diophantus'.

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ENDNOTES

1. http://mc2-project.eu/index.php/technology-and-production/c-books/100-windmills

2. http://mc2-project.eu/index.php/technology-and-production/c-books/101-viral-behaviour-of-social-networks

3. *Widgets* – c-book-widgets (widgets for short) are small pieces of software which can be included into c-books via the c-book environment in order to allow interactive content. *Widget instance* – a widget that has actually been inserted into a c-book page is called a widget instance. They can still be configured by the c-book author in order to fulfill the specific needs. For example, for visualizing the graph of a function, the c-book author may specify the ranges to be used, etc.

4. Letters a, b, c, and d refer to the four ways for organizing the evaluation criteria used by the CoI members as they are described in p. 4.

Formalising functional dependencies: The potential of technology

Giorgos Psycharis

University of Athens, Department of Mathematics, Athens, Greece, gpsych@math.uoa.gr

This paper proposes "formalising functional dependencies" as an approach to address critical aspects of the potential of digital technologies for the teaching and learning of functions. This approach focuses on the role of the available tools in supporting students' transition from experiencing dependencies in terms of non-algebraic digital representations to expressing these dependencies formally. To illustrate the approach, data from two studies based on the use of two distinct computational systems are analysed. Key aspects of their potential include: work with dependencies at the level of magnitudes, specially designed functionalities and dynamic interplay between symbolic and non-symbolic representations of functions.

Keywords: Functions, dependency, formalism, digital technologies.

INTRODUCTION

The notion of function occupies a central position in school mathematics curricula but it constitutes a rather difficult topic for many students. Evidence identified by research concerns issues such as students' difficulties in understanding function as covariation and dealing with algebraic symbolism (Kieran, 2007). The development of new modes of representation within specially designed technological tools that allow considering functional dependencies through the use of non-standard representations (including non-algebraic ones) has generated further interest as regards their potential to deal with the above mentioned difficulties. One distinct feature of these tools is that they are designed to make the symbolic aspect of function more accessible and meaningful to students, especially through multiple linked representations including some sort of combination of visual or geometric representations (e.g., dynamic geometry) and algebraic multirepresentation, possibly including Computer Algebra Systems (CAS) or other symbolic forms (e.g., algebraic-like formalism) (Mackrell, 2011). The interactive and dynamic character of the corresponding digital representations have brought to the fore the need to acknowledge both the transformative potential of the corresponding technologies and the opportunities provided for meaning generation. This need was reinforced by the fact that such systems encourage different levels of interplay between symbolic and non-symbolic representations (e.g., direct manipulation of mathematical objects) and different kinds of algebraic expression that can be aligned or not with standard mathematical notation. In this study, I am particularly sensible to the possibility offered by particular computational environments to students to make sense of function through modelling dependencies in a non-algebraic/symbolic (e.g., geometrical, iconic) setting before passing to formal expressions of these dependencies and to mathematical functions. In this process, there is always a transition from experiencing dependencies through the use of non-algebraic/symbolic digital representations to recognising which of these dependencies constitute functional ones and expressing them formally using the available symbolic representational forms. This transition is far from trivial for students. Apart from the novelty of the used representations, well-known problems that students face with functions and algebra are also brought to the fore (Kynigos et al., 2010). Thus, the potential of the corresponding technologies needs to be addressed. This is the general goal of this paper. Based on the integrated framework developed by Lagrange and Psycharis (2014), I adopt a similar approach – I call it *formalising functional dependencies* – to address such potential.

FORMALISING FUNCTIONAL DEPENDENCIES

The approach formalising functional dependencies is developed around the need to address the following issues as regards the students' transition from non-symbolic/algebraic to symbolic/algebraic representations of function and the coordination between them in technology enhanced mathematics: (1) the role of the available tools in supporting different levels of students' work with dependencies (e.g., modelling, exploration), (2) the students' activity to 'translate' the modelled functional relations in symbolic language and to conceptualise the connections between different representations of function, and (3) critical aspects of the students' difficulties with functions (i.e., covariation, symbolism). Below, I present briefly the three parts of the theoretical work that underlie these issues and constitute the approach.

Levels of students' work with dependencies. Lagrange and Artigue (2009) developed a conceptual framework for functions and algebra in order to address students' work with dependencies. Taking an epistemological point of view, they situated students' activities for approaching functions at three levels: (1) activity in a physical system (e.g., dynamic geometry, a simulation): students can experience dependencies sensually in a physical system through observation of mutual variations of objects; (2) activity on magnitudes: the idea of function is linked to dependencies between magnitudes which is expected to support students' consideration of functions as models of physical dependencies; (3) activity on mathematical functions: students work with mathematical functions of one real variable, with formulas, graphs, tables and other possible algebraic representations.

Situated abstraction. Noss and Hoyles (1996) introduced the notion of situated abstraction to address abstraction within computational media as a meaning generation process in which mathematical meanings are expressed as invariant relationships, but yet remain tied up within the conceptual web of resources provided by the available computational tool. In this perspective, a 'situated abstraction' approach to students' conceptualisation of function within a particular computer-based setting involves meaning generation evident in the concretion of generalized relationships by students through the use of the available tools and structures.

Function as co-variation and the role of symbolism. The essence of a co-variation view of function is related to the understanding of the manner in which dependent and independent variables change as well as the coordination between these changes (Thompson, 1994). However, this dynamic conception of simultaneously variation between magnitudes is rather difficult for the students especially when mathematical symbolism is involved. Research has been showing rather conclusively that the idea of independent variable, the algebraic expression of functions and its connection to other representations constitute obstacles for many students even for those beginning to study more advanced mathematics (Kieran, 2007).

In order to concretize the approach, I consider here two computational systems, both dealing with functions through innovative representations and functionalities, but different in many other aspects. One is *eXpresser*, a microworld designed to support 11–14 year-old students in their reasoning and problem-solving of generalisation tasks (Noss et al., 2009). It provides an 'algebraic' language, which involves the use of numbers and variables, with the aim to support students to construct relationships between patterns. The other system is *Casyopée* (Lagrange, 2010). It offers a dynamic geometry window incorporating representations of measures and of their covariation connected to a symbolic environment designed to support students' work on mathematical functions. Both systems offer opportunities for students to understand key actions in the process of modelling a dependency into a functional relation. However, eXpresser uses non-standard 'symbolic' representations while Casyopée's symbolic forms are consistent with current notations at secondary level. Next, I adopt the approach formalising functional dependencies to analyse data of eXpresser's and Casyopée's use in two respective studies so as to make sense of their potentialities for functional meaning making.

FUNCTIONAL RELATIONS IN FIGURAL PATTERN TASKS

eXpresser. The microworld affords the creation of coloured patterns in the construction area (Figure 1) by repeating a building block of several tiles ('unit of repeat'). The students can select tiles of different colours to construct the unit of repeat and then to define a pattern by specifying the number of repetitions and the appropriate number of coloured tiles in its property window (e.g., Figure 4). The number of coloured tiles can be represented iconically through expressions involving numbers that appear tied in a grey frame and 'unlocked numbers' – i.e. variables – and appear tied in a pink frame. A variable can be defined through a pop-up menu by 'unlocking' a particular number corresponding to an attribute of a construction (e.g., the number of red tiles, the number of repetitions) and provides a representation of an independent variable and its current value. Variables can be copied, deleted or used in operations (e.g., addition). Thus, through the use of variables, students can create relationships between two patterns of different colours based on dependencies (e.g., between the numbers of tiles of different colours). A pattern is shown dynamically (i.e., animated) by pressing the button Play (Figure 1). Then, the microworld picks random values for every variable and the model is shown dynamically in the construction area. Thus, students have the opportunity to see how their construction would look if the values of the independent variables of their current construction had different values.

It is important to mention that a model is always coloured only if the same independent variable has been used to build appropriate general expressions representing the total number of tiles for each one of the patterns used for the construction of the model. In different cases, the unfolded model/pattern appears to be distorted ('messed-up') and it is not coloured (e.g., see Figure 4b, c). Another feature of eXpresser is that of 'General Model' window (Figure 1): when the students animate a pattern this window shows different instances of the construction for different values of the various parameters in relation to the values assigned by the system in the representation appearing in the construction area. In order to colour their pattern in the 'General Model', the students have to build a general expression (i.e., the *Model Rule*, Figure 1) that always gives the total number of all tiles in the model (i.e., not just any pattern).

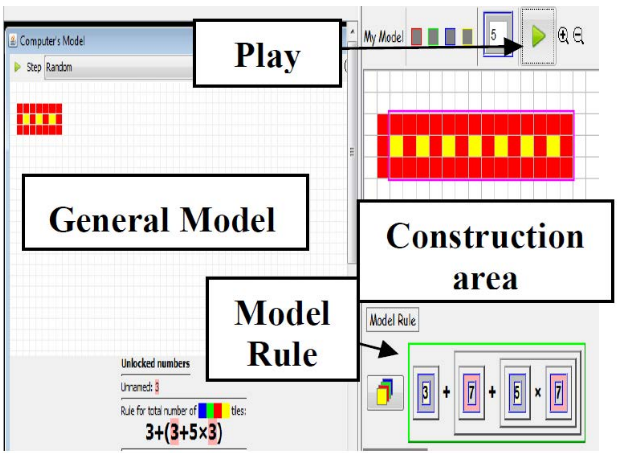

Figure 1: A pattern in eXpresser

The experiment. In the study with eXpresser (Zoupa, 2013), three case study groups of 13-year-old students (6 sessions for each group) were asked to construct and validate patterns through general expressions that underpin them. Since the students had not had any experience with patterns in their school lessons, the aim was to investigate if and how the microworld could help them to understand dependencies and express them using the system's structures and symbolic language. The data consisted of screen capture software files, files of students' work and video recordings. The data was analysed under a data grounded approach. Through the analysis of students' interaction with the available tools, episodes were selected to highlight the evolution of meaning generation for function.

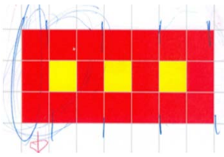

Figure 2

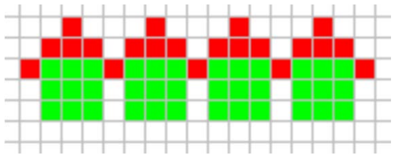

Figure 3

After an initial familiarization with eXpresser, the students were engaged in constructing the patterns shown in Figure 2 and Figure 3 consecutively. They had to allocate the correct number of tiles of each colour that were needed for the construction and then to create appropriate general relationships by using the same independent variable in the task. Next, I describe one group of students' work in four phases and corresponding steps that took place in the second and the third session.

Phase 1: Exploring the role of numbers and variables

Task: Construction of the pattern in Figure 2. (a) Constructing two building blocks (patterns): the first one constituted by the first column (3 red tiles) and the second one constituted by the second and the third column (6 tiles: 5 red and 1 yellow). (b) Considering the first pattern as specific. (c) Constructing the second pattern specifically for three repetitions (Fig 4a). (d) Unlocking the number of repetitions in the second pattern but keeping constant the numbers of red and yellow tiles. Feedback showing the construction was not coloured for different numbers of repetition in the construction area (Figure 4b). (e) Unlocking the number of red and yellow tiles without linking these (new) variables to the variable defined in the previous

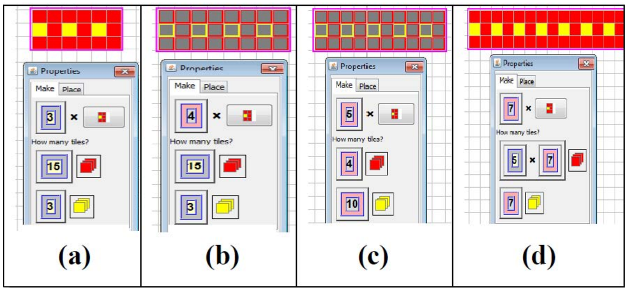

Figure 4: The properties window and the visual outcome (second pattern)

step. Feedback showing that the construction was not coloured in the construction area (Figure 4c).

Phase 2: Building functional expressions within a pattern

Task: Construction of the pattern in Figure 2. (a) Recognising the number of repetitions of the second pattern as an independent variable in the task. (b) Using this variable to express the number or red and yellow tiles (i.e., the number of red tiles is the same as the number of repetitions of the second pattern, while the number of yellow tiles is five times the number of repetitions of the second pattern). Inserting these expressions in the properties window through the choice 'replace' (Figure 4d). (c) Animating the model dynamically in the construction area. Feedback confirming the correct animation of it (Figure 4d).

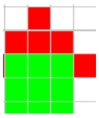

Figure 5a

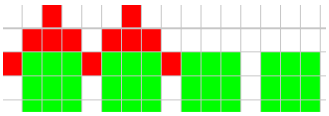

Figure 5b

Phase 3: Building functional expressions between patterns

Task: Construction of the pattern in Figure 3. (a) Constructing three building blocks (patterns): the first one constituted by the one red tile (i.e. the red tile on the left part of the first house), the second one for the roof with 5 red tiles and the third one for the green square with 9 green tiles (Figure 5a). (b) Considering the first pattern as specific and constructing the second and the third ones as general by unlocking the numbers of repetitions and thus creating one independent variable in each one of them. (c) Running the models dynamically. Feedback showing that the construction was 'messed-up' due to fact that the two variables representing the number of repetitions in each pattern changed according to different (randomly chosen) values (Figure 5b). Recognising that the two variables had to take the same value. (d) Linking the two patterns by replacing the one independent variable with the other through dragging and the choice 'replace'. (e) Building appropriate functional expressions for the numbers of green and red tiles in the two patterns.

Phase 4: Expressing the general rule of the total number of tiles in the model

Task: Construction of the pattern in Figure 3. (a) Constructing a general expression giving the total number of all tiles (i.e. not just any pattern) in the Model Rule window through the use of the only independent variable (Figure 6 shows an immediate instantiation of this expression for three repetitions of the model). (b) Expressing the general rule through traditional algebraic notation with paper-and-pencil (i.e., 5x+9x+1).

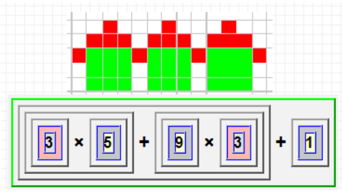

Figure 6

As regards the levels of dependencies, the physical system in eXpresser involves the dynamic reproduction of patterns. At the level of magnitudes, numbers of tiles and numbers of repetitions are involved. These magnitudes are concretized in the system as numbers in grey frames and variables in pink frames showing current instances of their values. I note that the dynamic change of the values (assigned automatically at random) to one variable is shown inside the pink frame of a variable when the corresponding pattern unfolds dynamically in the construction area. Thus, measures are 'encapsulated' within the corresponding magnitudes. As regards the notion of function, the independent variable is the number of repetitions of a building block created by the students while the dependent variable is the total number of tiles (of each colour). In this linear function, the input is the number of repetitions and the output the animated model. In all phases, while exploring the role of numbers and variables and experimenting with building different symbolic forms of general relations, the students considered together the physical system and the dependency between magnitudes. Thus, they worked with dependency and co-variation together at the level of magnitudes and at the level of magnitudes represented through variables.

From a situated abstraction point of view, the domains of students' meaning generation here involve: (a) making sense of the structure of the requested models in terms of specific and general patterns, (b) conceptualising the construction of appropriate building blocks, (c) identifying the independent and the dependent variables, and (d) conceptualising the formulation of functional relations between these variables. Building appropriate functional relations indicated the students' transition to the world of functions as it is embedded in the structures of the system.

As regards students' conceptualisation of covariation and symbolism, the role of feedback was critical. eXpresser provided a dynamic representation of covariation: animating the pattern had the effect of the construction dynamically changing as the values of the respective parameters changed automatically. In this process, 'messing-up' and 'correct colouring' challenged students to create and undertake changes in the symbolic form of the corresponding relations and at the same time to progress in their conceptualisation of the involved covariations as functions.

LINKING GEOMETRICAL DEPENDENCIES AND FUNCTIONS

Casyopée deals with various representations of functions consistent with school mathematics and curriculum. It provides a symbolic window with three registers: numeric, graphic and symbolic (Figure 8). Casyopée also includes a dynamic geometry window linked to the symbolic window. The geometric window allows defining independent magnitudes (related to free points) and also dependent ones (i.e., through the use of the "geometric calculation" functionality, see Figure 7 on the right) involving distances (e.g., lengths), *x*-coordinates or *y*-coordinates. The two windows are interconnected: objects defined in one window can be used in the other. Couples of magnitudes that are in functional dependency can be exported to the symbolic window and the system automatically can define a function. This function can be further treated by the students with all the available tools. This functionality –called "automatic modelling"– is expected to help students in modelling dependencies.

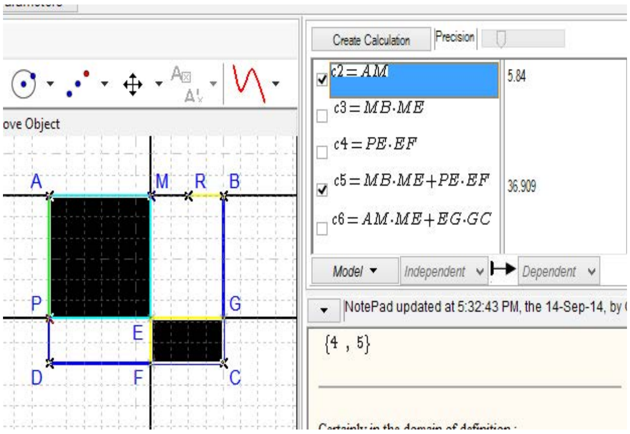

Figure 7

The experiment. In the study with Casyopée (Kafetzopoulos, 2014), three case study groups of 17 year-old students (six sessions for each group) were engaged in solving optimization problems through modelling geometrical dependencies. The aim of the study was to investigate how a computational medium linking CAS and dynamic geometry could help students make sense of function as covariation through: (a) conceptualising dependencies in geometrical situations and (b) modelling them as functions in order to solve the given tasks. Here I will report on students' work in the following task: "The owner of a rectangular estate *ABCD* (*AB*=10m, *AD*=8m) wants to design two gardens and two buildings inside it. In the given geometrical figure (Figure 7), we consider *A*(0;8), *B*(10;8), *C*(10;0), *D*(0;0). *M* is a point on *AR* (*AR*=8m) and *P* a point on *AD* so as *AMEP* to be a square. The figure *EGCF* is a rectangle. The gardens will cover the shaded part of the figure and the two buildings the rest of it. By moving *M* between *A* and *R*, different shapes of the gardens are designed. (1) Is there a position of *M* in *AR* for which the two gardens have the same area? (2) Is there a position of *M* in which the two gardens will have the same area as the two buildings? Justify by: (a) exploring the dynamic figure, (b) using the software to create functions modeling these questions, and (c) using the available tools to find the solutions". Since the main focus of the task was on students' conceptualization of function as covariation, the figure was already prepared for them. The students had to: (a) make sense of *M* as the only free point in the figure, (b) recognize *AM* as an independent variable and use it to define functions of areas, (c) work with different representations of functions. Next, I describe the work of one group of students in four phases and corresponding steps that took place in the last two sessions. The data and method of analysis were similar to the ones described in the experiment with eXpresser.

Phase 1: Exploring dependencies in the geometrical model. (a) Experimenting with the dynamic aspects of the figure by dragging points. (b) Recognising *M* as the only free point in the model. (b) Defining measures for different magnitudes (e.g., the areas of *AMEP* and *EGCF* and their sum) as geometrical calculations (Figure 7, right). (c) Observing covariation at a perceptual level (i.e. how dragging *M* changes the shapes of the shaded parts) and numerically (i.e. through the changes in the values of the corresponding magnitudes). *Phase 2: Identifying independent and dependent variables.* (a) Choosing an independent variable after recognizing dependencies between co-varying magnitudes (e.g., *AM* for the area of *AMEP*). (b) Using the same independent variable for the area of *EGCF* after checking that dragging *M* changes the area *EGCF*. (c) Choosing *AM* as an independent variable for the sum of areas (i.e. *AMEP+EGCF*, *MBGE+PEFD*). *Phase 3: Working with algebraic functions through automatic modelling.* (a) Exporting functions to obtain formulas for four functions (two for question 1 and two for question 2, Figure 8, left) through automatic modeling. (b) Working further on the algebraic functions to solve the problems, i.e. making equal two functions and solve. Identifying the position of M in question 1 (AM=40/9) and question 2 (AM=4 or 5). *Phase 4: Linking different representations of algebraic functions.* (a) Interpreting the answers to questions 1 and 2 given through the equality of functions by coordinating different representations, e.g., linking table, geometry and graphics, focusing on the common point of the two graphs, changing the step in the corresponding tables for more precise values (Figure 9). (b) Conceptualizing the addition of two already defined functions as a new function (e.g., *AM→AM•ME+EG•GC*).

As regards the levels of dependencies, the physical system in Casyopée is the dynamic geometry window which provides the context for modelling the problem in the geometrical setting and opportunities for creating and animating geometrical objects. At the level of magnitudes, the students can use the geometric calculation functionality to construct magnitudes in the form of symbolic objects that can be dependent to the geometrical situation (e.g., expressions of areas). These magnitudes have a dual status in the system since they are concretized symbolically as parameters (*c0*, *c1*, etc.) and numerically as measures whose values can change dynamically (e.g., through dynamic manipulation of the dependent geometrical objects). Thus, the students can be engaged in exploring covariation of pairs of magnitudes, modeling functional dependencies algebraically (through automatic modeling) and working further with mathematical functions. Thus, the combined use of geometric calculation and automatic modeling supports students' transition from the world of measures to the world of mathematical functions through work with magnitudes. This transition is evident in students' activities described above: work in the physical system (phase 1a) was followed by work with magnitudes (e.g., definition of geometric calculations for areas) and observation of covariations (phases 1b, 1c), and then further extended to include identification of independent variable (phase 2), definition of functions through automatic modeling (phase 3) and problem solving by linking different representations of functions (phase 4).

From a situated abstraction perspective, the layers of meanings for function here involve: (a) making

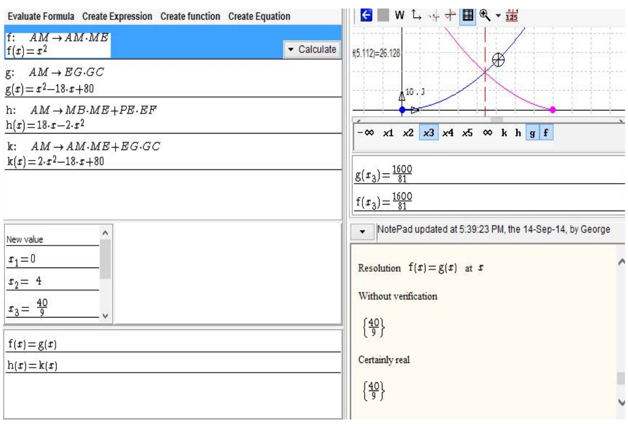
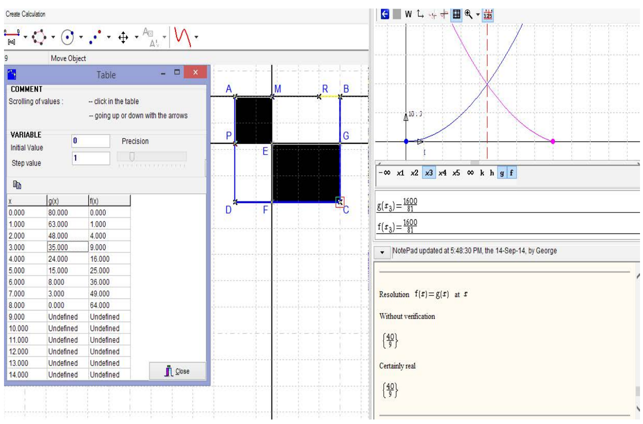

Figure 9

sense of the dependency between *M* and the areas influenced by its move, (b) conceptualizing the creation of relevant geometric calculations, (c) conceptualizing the idea of independent variable representing a geometrical object and using it to express functional dependencies through automatic modelling, (d) abstracting these dependencies at the algebraic level and use the corresponding functions to solve the requested problems.

As regards the symbolic aspect of function, the formulas taken in Casyopée seemed to have legitimized the use of all the available representations of functions by the students. Although the students were able to explain the provided formulas of functions, they chose to work with these formulas at an operational level for solving the given tasks (i.e. through the equality of functions). In doing so, they were engaged in linking different representations of functions.

CONCLUSION

As regards the levels of dependencies, the analysis revealed that eXpresser and Casyopée favour students' work with dependencies at the level of magnitudes as a critical part of their passage to formalisation. In particular, the analysis indicated the importance of working with magnitudes as a bridge between sensual experience of dependencies and symbolic expression of functional relations. In eXpresser, dependencies between magnitudes and visual representation of their covariation (i.e., through dynamic reproduction of patterns for random values) seemed to support the students' articulation of general relations and favour a structural understanding of patterns. Dependencies of this kind in Casyopée seemed to facilitate the students' transition from dynamic geometry and the world of measures to the world of mathematical functions. The analysis of the students' activity according to situated abstraction helped us to capture the progression of their conceptualization of functional dependencies taking into account the role of particular functionalities (e.g., automatic modelling) and feedback (e.g., messing-up). Experiencing covariation through dynamic interplay between symbolic and non-symbolic representations of functions helped the students to make sense of the role of independent variable and symbolism in expressing functional relations and connecting the formalism embedded in the tools to the formalism of school mathematics. This is an indication that the two systems can be used to address well-known and researched difficulties of students with algebra and functions (e.g., recognising independent variable, articulating functional relations and expressing them symbolically) and promote a meaningful transition to algebraic thinking. Thus, the potential of the two systems can be recognised in the direction of enriching representations of functions with new non-symbolic and symbolic ones and of enlarging students' possibilities to construct functional meaning by making connections between these representations.

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Developing future mathematics teachers' ability to identify specific skills needed for work in GeoGebra

Jarmila Robová[1] and Naďa Vondrová[2]

1 Charles University in Prague, Faculty of Mathematics and Physics, Prague, Czech Republic, jarmila.robova@mff.cuni.cz

2 Charles University in Prague, Faculty of Education, Prague, Czech Republic, nada.vondrova@pedf.cuni.cz

Our research aims at some aspects of the teachers' TPACK, namely their ability to identify specific skills needed for work in GeoGebra and to develop these skills in their pupils. Future mathematics teachers were acquainted with specific skills via selected tasks to be solved in GeoGebra. The data consist of the pre-service teachers' written solutions and proposals for teaching with GeoGebra and transcripts of discussions. The data were analysed in a qualitative way. Explicit work with the skills is needed for pre-service teachers to realise their importance for pupils' work in GeoGebra. While their technology knowledge was good, their proposals for teaching were mostly static and provided pupils with step by step directions. Implications for educating future teachers to teach effectively with GeoGebra are given.

Keywords: Dynamic geometry, TPACK, pre-service teachers, GeoGebra skills.

INTRODUCTION

In our work as educators of pre-service teachers, we strive to find ways of developing their knowledge and skills so that they use ICT tools in their future practice productively. The programme for future teachers naturally includes mathematical courses (developing their content knowledge), a technology course (developing their technology knowledge or TK) and a general didactics course (developing their pedagogical knowledge). All these strands should meet in a mathematics education course. In this article, we will report on a part of a design experiment (Cobb et al., 2003) aimed at designing a part of a regular mathematics education course focused on ICT. To meet the time constraint within the course, we decided to explore the merits and limitations of one ICT tool only. We chose GeoGebra (or GG) as it mutually connects different representations of mathematical concepts (geometric, algebraic, tabular, graphic) and can assist pre-service teachers during all their mathematical learning. Two areas of mathematics were selected for investigation; geometry (synthetic and analytic) and, thanks to GG's multi-representational nature, functions.

THEORETICAL FRAMEWORK

To work with ICT tools in mathematics, differences between computer and theoretical mathematics must be observed, i.e., issues caused by the representation of mathematical concepts, operations, objects, etc. on the computer screen. For dynamic geometry software, Laborde (1998) points out a necessity to understand differences between theoretical-geometric and computer-graphic worlds (e.g., the necessity to construct such figures whose properties are preserved when the objects are moved – i.e., robust constructions). Robová (2013) presents what she calls *Specific Skills* for work in GG. For the purpose of this article, we chose only those pertinent to functions: I. making functions visible (on the screen), II. changing visual appearance of graphs, III. interpreting points on objects, IV. interpreting numerical results, V. using dynamic features of GG, VI. using graphic styles. Skills I. to IV. are part of technology content knowledge (TCK) while V. is between TCK and technology pedagogical knowledge (TPK) and VI. is part of TPK .

We suggest that the teachers' awareness of the Specific Skills and their ability to design teaching which takes them into account belong to their technological pedagogical content knowledge (or TPACK).

> [TPACK is] the basis of good teaching with technology and requires an understanding of the representation of concepts using technologies; pedagogical techniques that use technologies in constructive ways to teach content; knowledge of what makes concepts difficult or easy to learn and how technology can help redress some of the problems that students face; knowledge of students' prior knowledge and theories of epistemology; and knowledge of how technologies can be used to build on existing knowledge and to develop new epistemologies or strengthen old ones. (Mishra & Koehler, 2006, p. 1029)

There is a growing body of literature on developing TPACK. For example, Balgalmis, Shafer and Cakiroglu (2013) claim that "focusing on the mathematical concept more than technology and using technology when it is really necessary were the basic criteria for effective technology based lesson" (p. 2534). Bowers and Stephens (2011) suggest that rather than elaborate specific types of knowledge for TPACK, the best we can do is to "engage prospective teachers in technology-enhanced mathematical explorations with the explicit goal of discussing the ways in which technology enabled them to describe relationships among objects on the screen that could not have been developed without the tools employed" (p. 291). They describe a course on the Geometer's Sketchpad; they let pre-service teachers explore the software by solving problems and then they discussed the model of TPACK with them. Sfard's (2008) commognitive perspective of learning is of particular importance here: "*Thinking* is an individualized version of (interpersonal) communicating." (p. 81). Thus, "teaching can be seen as the practice of orchestrating mathematical discourses" and "learning can be seen as the ways in which students engage in these discourses" (Bowers & Stephens, 2011, p. 287).

Abbitt (2011) summarises performance-based TPACK measures which are based on the idea that "by examining the design and planning process, it is possible to assess the knowledge of a preservice teacher in the TPACK domains" (p. 292). To achieve this goal, Harris, Grandgenett and Hofer (2010) developed Technology Integration Assessment Rubric for analysing lesson plans. The rubric is to be used with pre-service teachers during a teacher preparation programme and thus it is specifically useful for us. According to the rubric, the plan is assigned 1 to 4 points in four measures: 'Curriculum Goals & Technologies' (curriculum-based technology use), 'Instructional Strategies & Technologies' (using technology in teaching/learning), 'Technology Selection(s)' (compatibility with curriculum goals and instructional strategies), 'Fit' (content, pedagogy and technology together).

Bowers and Stephens (2011) describe a rubric for assessing the quality of pre-service teachers' teaching plans made after 18 lessons of work with the Geometer's Sketchpad. They determined for each plan:

> the degree to which the student demonstrated TK, TCK, TPK, or TPACK with the assumption that these types of knowledge are additive. [...] If a student demonstrated a good use of the technology to examine a particular content area but did not include any particular presentation affordances, such as use of colour or scripting tools, then he or she was characterized as having knowledge at the level of TCK, but not TPACK. (p. 293)

Both rubrics, slightly modified, were used in our research. Finally, we made use of a Tool Competence Model for symbolic calculators suggested by Weigand (2011) which specifically deals with functions: Static Mode (producing static representations), Dynamic Mode (creating dynamic representations), Multiple Mode (using the ICT tool as a multiple representational tool).

METHODOLOGY

The research questions were the following: (1) Are pre-service teachers with good technology knowledge aware of the Specific Skills? (2) How do pre-service teachers connect their content knowledge, pedagogical content knowledge and technology knowledge to prepare proposals for teaching with the help of GG?

By good technology knowledge, we mean that pre-service teachers were able to work in GG, they could orientate themselves in its workspace, they knew where to find required tools and how to use them. In view with literature, rather than presenting them with ideas of effective use of GG, we prepared *worksheets with tasks* in which they had to use Specific Skills which emerge from the differences between computer and theoretical mathematics. It was hoped that by confronting pre-service teachers with such tasks, the

Specific Skills will be brought to their attention and they will realise their importance for their future pupils, too. Two worksheets concerned geometry and two concerned functions. The first set of worksheets on each topic comprised less demanding tasks, while the second set put the pre-service teachers' in pupils' role (the tasks were mathematically more demanding). There was a space after each task in which any skills needed for its solution in GG should be written.

The *first (pilot) stage* of the design experiment took place in spring 2013 (see Robová & Vondrová, 2014) with 19 future lower and upper secondary mathematics teachers, within their second mathematics education course. One of the results was that pre-service teachers should meet more tasks in which technology fails as they are "forced" to reason mathematically then. If the task leads them towards using technology (the solution via software seems to be obvious), they might forget to check the appropriateness of the solution by mathematical means. Thus, we modified worksheets accordingly. It also transpired that we underestimated the importance of discussing the emerging issues. Thus, two discussion periods were to be organised in the second stage in order to make the Specific Skills more visible for pre-service teachers with the hope that they will use them in their teaching proposals. Finally, the quality of the pre-service teachers' proposals for teaching was not very high. Bowers and Stephens' study (2011) suggests that such proposals should be subject to class discussion and the basis for bringing out pre-service teachers' own metacognitive processes (e.g., by asking them to speak about the development of the proposals). Thus, a written peer review was to be included in the next stage.

Main study

Twenty three future lower and upper secondary mathematics teachers participated in the study. It took place during a mathematics education course taught by the second author (Table 1). In the two discussions, the course teacher asked general questions first "what knowledge was needed in order to solve the tasks" and then chose questions on the basis of the pre-service teachers' reactions. The audiotaped discussions focused on specific skills and on merits and drawbacks of GG. The pre-service teachers' written work was collected. Seven pre-service teachers' work on the screen was captured by Camtasia and saved as a videorecording.

After completing 4 worksheets, the students were to prepare a *Project*, that is, a proposal for teaching with the support of GG. The project was to include tasks which would lead to pupils' autonomous investigation of a topic in GG, solutions to the tasks, GG figures, the goal(s) of activities, pupils' prior mathematics and technology knowledge, their expected problems and a suggestion of their remedy, etc. The pre-service teachers submitted their Projects via a Moodle module Workshop. Afterwards, they were randomly assigned their peers' Projects to evaluate.

This paper only focuses on the part of the study dealing with functions. Worksheet 1 on functions comprised tasks on determining the domain and range of functions, their monotony, zero points and x and y-intercepts. They were quite easy to solve as the aim was to make the pre-service teachers aware of the Specific Skills without having to concentrate on the mathematical aspect too much. Worksheet 2 tasks also referred to the Specific Skills but they were mathematically more demanding. Examples of tasks are given in (Robová & Vondrová, 2014).

Analysis of data

In the analysis of the solutions to worksheet tasks, it was followed (research question 1): the pre-service teachers' TK, content knowledge, the Specific Skills. When in doubt, Camtasia recordings were used to get more information. For the analysis of the pre-service teachers' Projects (research question 2), the modified Harris, Grandgenett and Hofer's (2010) and Bowers and Stephens' (2011) rubrics were used as well as

Session 1 **4 lessons**	**Session 2** **4 lessons**	**Session 3** **4 lessons**	**Home study**	**Home study**
Individual work: Geometry worksheet 1, 2	*Individual work:* Geometry worksh. 2 *Discussion 1* *Individual work:* Function worksheet 1	*Individual work:* Function worksh. 2 *Discussion 2* *Assigning project proposals*	*Individual work:* Project proposals either on geometry, or function	*Individual work:* Peer review of project proposals

Table 1: Organisation of the study

Weigand's Tool Competence Model. As the tasks were to lead to pupils' autonomous work, we also used the measure of hypothetical pupils' role in gaining new knowledge (from A1 – pupils are given step by step instructions, to A4 – it is up to pupils to decide how they will solve the tasks). The two authors coded the projects independently and discussed their coding until 100% agreement was reached.

Two of Harris, Grandgenett and Hofer's (2010) measures were not used in our coding: 'Curriculum Goals & Technologies' (as all the Projects were aligned with the curriculum) and 'Technology Selection(s)' (GG was prescribed). Instead, we used 'Types of Goals' measure which concern explicitly stated goals: Content Goals (related to specific mathematical topics), Skill Goals (related to the development of some skill), General Competence Goals (outside of mathematics). The two remaining measures were used: 'Instructional strategies and technologies' was coded from I1 (technology use *does not support* instructional strategies) to I4 (*optimally supports*); 'Fit' was coded from F1 (content, instructional strategies and technology *do not fit together* with the instructional plan) to F4 (*fit together strongly*).

While Bowers and Stephens' (2011) pre-service teachers' projects consisted of one sketch, our pre-service teachers proposed several tasks for more than one lesson. Thus, we had to assess what feature prevailed in the Projects. Moreover, we did not fully embrace the authors' assertion that the elements of TPACK are additive (they coded pre-service teachers' projects either TK, or TCK, or TPK, or TPACK), thus introducing a kind of hierarchy which, in our opinion, is not in the original model of TPACK. Unfortunately, the authors do not explain what kinds of projects were coded as TPK and we had to use our own modification. Instead of TPK and TPACK codes, we used TPACK1 and TPACK2. The former means that the pre-service teacher demonstrates an understanding of using technology for more informative, quick and effective teaching of mathematics. Tasks proposed make understanding easier but do not lead to argumentation. The latter means that the pre-service teacher realises the potential of the software for developing mathematical reasoning up to the level of argumentation and proof; it goes beyond what can be observed on the screen. This is, we believe, Bowers and Stephens' code TPACK.

Each Project was peer reviewed in writing by 1 to 3 pre-service teachers. In their analysis, we followed to what extent they commented on the phenomena which we identified as important for TPACK (and used for the analysis of Projects): E1 – a shallow evaluation, a pre-service teacher does not comment on obvious deficiencies or merits of the Project, his/her comments are general; E2 – he/she comments on some aspect only, such as the content; E3 – he/she comments on both positive and negative features, uses most measures (Tables 2 and 3); E4 – expert evaluation.

RESULTS

Research question 1 (pre-service teachers' solutions to worksheet tasks)

All pre-service teachers demonstrated good TK. Some, though, did not connect technology and content knowledge sufficiently. For example, when they did not get a result via predefined tools in GG or when no direct tool is available, they did not use other tools of GG to solve the task but made paper calculations. On the other hand, in some cases TCK was evident; e.g., as for f: $y = \sqrt{3x - x^3}$, GG does not find x-intercepts and some pre-service teachers used a function with the same zero points instead to solve the task: g: $y = 3x - x^3$. In terms of pupils' required skills, the pre-service teachers mostly recorded content knowledge, and some TCK (such as "to input an equation of a function", "to determine intercepts", etc.). They least *commented* on the fact that GG rounds off numbers and thus its numerical results must be critically evaluated (for irrational or periodic numbers).

In terms of the Specific Skills, nearly all participants demonstrated the use of skills I., III. and V. Ten of them failed in skill II. *Input a bounded domain.* The most problematic was skill IV. *Interpret numerical results;* seven pre-service teachers did not record 1,7320 as $\sqrt{3}$, thirteen left 0,33333 instead of 1/3 and sixteen did not recognise $\sqrt{2}/2$ in its decimal expansion (or did not use their mathematical knowledge to solve the task without the help of GG to see the result in the exact form).

The group discussion confirmed the above results. The pre-service teachers were able to solve the tasks and commented on many of the differences between computer and theoretical mathematics as they manifested themselves in the tasks (e.g., the point of discontinuity for f: $y = \frac{x^2 + x - 2}{x - 1}$ is not depicted in the graph). The necessity to recognise decimal expansions of common real numbers was not mentioned and when it was brought out by the teacher in the discussion, it transpired that some pre-service teachers did not use this skill for evaluating results GG produced for them, leaving them in decimal expansions. This corroborates our above analysis.

Research question 2 (pre-service teachers' Projects)

Table 2 shows what Specific Skills the pre-service teachers mentioned (or not) in their Projects. Unsurprisingly, the skill to input the equation of the function was the most frequent one. However, the opposite relation, i.e., the fact that GG can display an equation of the function for a given graph (and when the graph is moved, the equation changes), was only given once and once used but not mentioned. Most pre-service teachers used dynamic features and noted a necessity to work with a slider.

Consistently with the above results, the skill to interpret numerical results was only mentioned once and most alarmingly, in five pre-service teachers' Projects the skill was needed but not demonstrated. Almost half of the pre-service teachers used such functions in their Project where this skill was not needed at all and we cannot say whether it was a chance or intention – they might have realised a possible problem and wanted their pupils to avoid it rather than confront it.

Table 3 shows that the quality of Projects varied. It was not our intention to rank them but to assess as many features as possible. But still, differences between pre-service teachers can be seen. On the one hand, student F3 reached very good evaluations in all measures, her Project was of a high quality. On the other hand, student M4's project was quite poor, he only demonstrated TK and did not make any use of the potential of GG. Let us now look at individual measures.

If a pre-service teacher provided some goals for their Projects, they were only about content, that is, what mathematical knowledge they want to develop. No competence outside of mathematics was mentioned nor a skill such as "to experiment", "to make hypotheses based on the observation of what is happening on the screen", etc. It is quite important as skills like that are stressed in Czech curricular documents. The instructional strategy as proposed in Projects was mostly supported by technology. It concerned pupils' individual work with the teacher helping them. The marks for the measure to which 'content, instructional strategies and GG fit together within the instructional plans' were mostly average. The same can be seen from the sixth column: only a minority of Projects included dynamic features of GG (mostly using sliders for functions – "observe what happens on the screen – how the graph changes – when the slider

Specific Skills in the Projects (I. to VI., see above) Fi – females, Mi –males	Explicitly stated	Not stated, but used	Not used but required by the task
I. Input the equation of the function	F2, M1, F3, M2, F4, F5, M4, F6	M3, M5	
I. Display the equation of the function for a graph	F3	M1	
II. Input a suitable range of coordinates	F1	F4	
II. Input a bounded domain	F2, F5, F6		
II. Input multiples of pi on the axes	F3	F2	
III. Determine x and y intercepts	F2, F4	M4	
III. Determine intercept of objects via tools of GeoGebra	M1, F4		
IV. Interpret numerical results produced by GeoGebra	F3		F2, M1, M2, F4, F6
V. Work with a parameter – slider	F1, F2, F3, M2, F4, F5, F6	M3, M5	
VI. Work with graphic styles of objects (including hiding objects)	M1, F3, F6	F1, F2, M3, F4, F6	
VI. Use texts and symbolic records in GeoGebra		F4	

Table 2: Specific Skills for tasks in Projects on functions (n = 11)

is moved"). In other words, the proposed tasks could be solved without the software and GG only made their solutions more illustrative and quicker.

Student M4's Project was coded as TK only, he does not seem to grasp the ways technology can support the development of mathematical knowledge. Most of the Projects were rated as TCK which means that the pre-service teachers could use GG productively for developing their own mathematical knowledge but did not concentrate on the ways tools of GeoGebra can support their pupils' learning. No Project included tasks which would lead pupils towards the level of argumentation and proof. Just to compare: in Bowers and Stephens' study (2011) 3 out of 21 pre-service teachers were coded as TPACK and 9 pre-service teachers as TK only.

Even though the pre-service teachers were asked to devise teaching in which pupils will work autonomously, the Projects of about half of them were too instructive. They gave pupils tasks which lead them step by step towards the desired goal and provided them with explicit instructions what they should notice and how to continue work.

The pre-service teachers' peer reviews of the Projects were mostly quite superficial and did not comment on even very obvious drawbacks. In the vast majority, the pre-service teachers evaluated the mathematical content only. Their comments were often general ("the project will contribute to pupils' understanding", "the tasks are well chosen", "the use of GeoGebra is effective"). The Projects of students F3 and M2 were evaluated by us as the best ones in most measures. This was confirmed by the analysis of their reviews of their peers' Projects. They wrote detailed and specific evaluations and commented on most of the measures as given in Tables 2 and 3.

DISCUSSION AND CONCLUSIONS

As in the pilot study stage, we conclude that it is far from straightforward that the pre-service teachers will connect their technology, content and pedagogical knowledge without deliberate support from the educator. The presented study remedied some of the deficiencies of the pilot study. We organised more discussion about the TCK and TPACK aspects and included a peer evaluation process. Still, the quality of the pre-service teachers' Projects did not meet our expectation. A good example is the skill to interpret numerical results produced by GeoGebra. The pre-service teachers solved many problems which required this skill, the skill was emphasised in the whole group discussions and still, some of them did not use this skill when it was required and many did not seem to realise its importance for their future pupils. Some of them might not have good understanding of real numbers and do not see that a decimal number with an infinite expansion can sometimes be written in a precise way as a fraction or square root. More content knowledge is needed.

Fi – female Mi – male	**Types of goals**	**Instr. strate-gies and techn.**	**Fit**	**TK, TCK, TPACK1**	**Static Mode, Dynamic Mode**	**Pupils' auto-no-my**	**Peer eval.**
F1 Absolute value	Content	I2	F2	TCK	SM	A1	E3, E2
F2 Goniometric functions	Content	I3	F3	TCK	SM	A1	E1, E1
M1 Power functions	Content	I2	F2	TCK	SM	A2	E1, E1
F3 Sine and cosine	Content	I4	F4	TPACK1	DM	A3	E1
M2 Fractional linear function	Not given	I3	F3	TPACK1	DM	A3	E1, E2
M3 Quadratic function	Not given	I3	F3	TCK	DM	A3	E1, E1
F4 Quadratic function	Content	I2	F2	TCK	SM	A2	E1, E2
F5 Logarithmic function	Not given	I2	F1	TCK	SM	A3	E1, E1
M4 Transformations of graphs	Content	I2	F2	TK	SM	A1	E3, E1, E1
M5 Power functions	Content	I3	F3	TCK	DM	A3	E1, E1
F6 Quadratic functions	Content	I2	F2	TCK	SM	A2	E1, E1

Table 3: Overview of measures used for analysing Projects (n = 11)

One of the implications of our study is that a good choice of tasks must be accompanied by a more explicit discussion about the features of GG use in teaching (represented, e.g., by the measures in Tables 2 and 3). It can "guide students' own metacognitive processes as they reflect on their learning and development efforts" (Bowers & Stephens, 2011, p. 301). This discussion, however, should be in person. In Bowers and Stephens' (2011) study the worst results were reached by pre-service teachers who chose the completely online version of the course: "This could suggest that these students did learn how to use the program, but did not engage in the discursive practices of using technology to probe either their own, or their future students' thinking in the conceptual ways that were discussed during in-person sessions." (p. 294) Students in the blended course had better results and the best were reached by students in the "in person" course. Thus, the discussion should be organised during and after the work on the worksheets and after writing the Projects, commenting on the peers' Projects and seeing the feedback from the peers. The instructions for the peer evaluation must be specific in order to provide pre-service teachers with more guidance by drawing their attention to measures from Tables 2 and 3. We see the peer review as an important part of the development of TPACK.

Due to the number of participants, the results cannot be generalised. The geometry part of the study awaits the analysis and its results together with the presented results will inform the next round of the design experiment in spring 2015.

ACKNOWLEDGEMENT

The paper was supported by the project PRVOUK P15 The school and teaching profession in the context of growing educational demands and GAČR P407/11/1740.

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Knowledge for teaching mathematics with technology and the search for a suitable viewing window to represent functions

Helena Rocha

Universidade Nova de Lisboa, Faculdade de Ciências e Tecnologia, Lisbon, Portugal, hcr@fct.unl.pt

The usual difficulties of students regarding the choice of an appropriate window when using the graphing calculator in the study of functions and the importance of the teachers' knowledge to overcoming them, led to this study. The main goal was to characterize the way teachers address the viewing window in the classroom, trying to infer aspects of the Knowledge for Teaching Mathematics with Technology that can justify that practice. The conclusions reached point to the importance of a set of specific knowledge where I highlight the knowledge of the students' difficulties, the knowledge of mathematical content necessary to understand the impact of the viewing window on the graphic, and the knowledge of teaching strategies that address both the students' difficulties and the relevant mathematical knowledge.

Keywords: Teacher knowledge, technology, functions, viewing window.

INTRODUCTION

Nowadays it is widely recognized the potential of technology for the teaching and learning of mathematics (Goos & Bennison, 2008). However, some studies point to difficulties on the integration of the technology, suggesting that teacher's knowledge is central in classroom use (Doerr & Zangor, 2000; Hoyles & Lagrange, 2010). One of these difficulties is related to the choice of a suitable viewing window to represent the graphic of a function (Cavanagh, 2006).

This study focuses on the teacher's use of graphing calculators during the study of functions at 10th grade and intends to characterize the teacher's practice in what concerns to the use of the viewing window and to infer from that characterization aspects of the teacher's knowledge that can justify that practice.

KNOWLEDGE FOR TEACHING MATHEMATICS WITH TECHNOLOGY

Shulman's work (1986) constituted a starting point for a whole set of investigations that have sought to characterize the teacher's professional knowledge. In recent years many authors dedicated themselves to this theme, trying to clarify concepts, to propose new characterizations and, in some cases, to give a special attention to technology. A look at those characterizations allows the identification of a set of domains that, with greater or lesser emphasis, appear to be consensual: Mathematics, Teaching and Learning, Technology, and Curriculum. These are the base domains of Knowledge for Teaching Mathematics with Technology (KTMT). Curriculum is however a domain that differs from the others in the sense that it is conceptualized in a transversal way, and therefore influential over all the others.

In addition to knowledge base domains, this model particularly values knowledge developed at the confluence of more than one domain. This importance given to a knowledge that goes beyond a particular domain is somehow recognized by Hill and Ball (2009), which consider in their model, for instance, knowledge such as Knowledge of Content and Students. Mishra and Koehler (2006) adopt a similar view in their TPACK model, considering also knowledge domains which consist of the mutual influence among other domains. But the first author who actually considers more than some influence among domains of knowledge, understanding this "influence" as a new domain of knowledge that is added to the initial ones, is Shulman (1986). Here I adopt a similar perspective. I consider two sets of knowledge, that I call inter-domain knowledge: the Mathematics and Technology Knowledge (MTK), and the Teaching and

Learning and Technology Knowledge (TLTK). In both cases the curriculum is considered to be a transversal influence, always present. MTK focuses on how technology influences mathematics, enhancing or constraining certain aspects, and TLTK focuses on how technology affects the teaching and learning process, enhancing or constraining certain approaches. One of the differences between KTMT and TPACK is that the inter-domains knowledge require some knowledge on the base domains but do not develop straightforward from that knowledge. However, the main difference between these two models is the way how KTMT intends to integrate in a single model the research developed on teachers' knowledge and on the integration of technology in teachers' practice. This is why MTK includes knowledge of technology's mathematics fidelity, knowledge of new emphasis that technology puts on mathematical content, knowledge of new sequences of content, and representational fluency; and TLTK includes knowledge of new issues that technology requires students to deal, knowledge of mathematical concordance of the proposed tasks, and knowledge of the potential of technology to the teaching and learning of mathematics. Finally, the KTMT includes Integrated Knowledge (IK). This is a knowledge held by the teacher that articulates simultaneously the knowledge of each of the base domains and the two sets of inter-domain knowledge. It is a knowledge that develops from the interactions between all domains and is characterized by its global and comprehensive nature but, at the same time, by its particularity, in the sense that it is that knowledge that maximizes the specific potential of technology to provide better mathematics learning. It is this knowledge that is the true essence of KTMT (Rocha, 2013).

DEFINING A SUITABLE VIEWING WINDOW

In the literature there are many references to issues related to the choice of the graphing calculator viewing window. Cavanagh (2006) mentions the students' difficulty in understanding the impact on the graph of adopting different values for each axis. Hodges and Kissane (1994) refer to the lack of awareness of students regarding the impact that any change in the viewing window has on the observed graph. Rocha (2002) emphasizes the role of the teacher to minimize the students difficulties related to the definition of a suitable viewing window. This author, as well as Cavanagh and Mitchelmore (2003), highlights the need for the teacher to be aware of these difficulties. And she also stresses the importance of allowing the students the opportunity to face these difficulties in a way that allows them to develop an understanding of the impact of the window over the graph displayed on the calculator's screen. However, Cavanagh (2006) is cautious about the best moment to confront students with these difficulties, stating that it should not occur to soon.

According to Doerr and Zangor (2000) and Kastberg and Leatham (2005), teachers' knowledge is determinant in how technology is integrated in their practice, and in how the teachers highlight the link between knowledge from different sources (such as algebraic and graphic) and the development of critical thinking. And at this level, Cavanagh and Mitchelmore (2003) believe that teachers must do more than simply inform students about the limitations of the calculator and about how this machine can present information in a misleading manner. It is necessary that the teachers strengthen the connections between different representations, continually drawing attention to discrepancies between, for example, the expected graph and the one that is displayed by the graphing calculator. It is also essential, with regard to the aspects concerning the choice of the viewing window, that the teachers appeal to the meaning of the values represented, so that they make sense for students (Rocha, 2002).

However, much of what has been done in research in this area is focused in telling the teachers what the integration of the graphing calculator should involve, and not on analyzing and understanding their practice (Kissane, 2003). And a better understanding of teachers' practice is fundamental if we want to promote teachers' professional development in a relevant way.

METHODOLOGY

Given the nature of the problem under study and in line with the ideas advocated by Yin (2003), the study adopts a qualitative and interpretative methodological approach, undertaking one teacher case study (in the part of the study presented here). Data were collected by semi-structured interviews, class observation, and documental data gathering. The teacher's classes reported here were followed for one school year while she taught functions using the graphing calculator. It was performed an interview before each class, with the purpose of knowing the intentions of the teacher and the underlying reasons. This interview was also used to decide which classes would be observed. During the study, fourteen classes of 90 minutes at 10th grade (age 16) were observed. After each of the observed classes, it was performed an interview with the intention of knowing the analysis that the teacher did of the events. These interviews were based on data from the classes and discuss some episodes selected by the researcher. Data analysis was mainly descriptive and interpretive in nature, considering the problem under study. The process started with the identification of the episodes where the viewing window has been addressed, and then the KTMT model and the respective domains of knowledge (which encompasses the research results around the issue of the viewing window) were used to structure the analysis of the episodes. All interviews and observed classes were audio-taped and later transcribed.

THE TASKS AND THEIR IMPLEMENTATION

Throughout the study of functions, the teacher offers students a wide range of tasks. In this section, I briefly present some of these tasks and their implementation, paying a special attention to the emphasis given to the viewing window.

In parabola's axis

In this task, students should start by marking two points at opposite sides of the axis of symmetry of $y = x^2$ and find the slope of the straight line that goes through them (Figure 1). After trying some examples, they should formulate a conjecture for the slope of the line that passes through any two points in the same conditions. And afterwards they must demonstrate the veracity of their conjecture.

The viewing window is one of the aspects considered by the teacher when she planned the lesson. Given that the experience of the students with the graphing calculator is still not much and that there are already several other aspects that they will have to face on the calculator (mark points on the graph, draw the line, calculate its slope, etc.), she decides to include information about a suitable viewing window on the task:

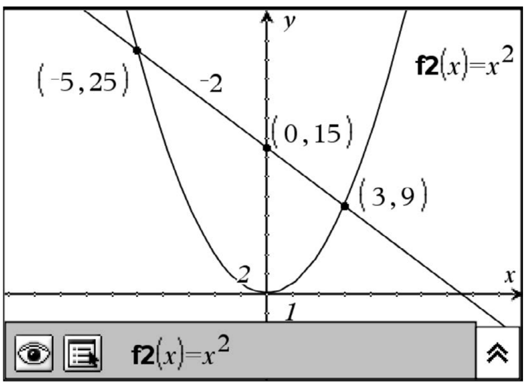

Figure 1: In parabola's

Teacher: You cannot do everything at once. We are just starting to work with the calculator. The students aren't used to it and on this task they already have to be able to do several things with the machine. We have to focus. And in this task the focus is not on the window. So I think it is better to give the window information. This way you avoid wasting time. (pre-lesson 2)

Still, in class the teacher values the importance of the viewing window. She takes the opportunity to introduce the expression 'viewing window' and to point to students two different ways to represent it:

Teacher: To make your job easier, I give you the viewing window. (...) But what is the viewing window?... When I draw a graph, it has no reading if I don't include the scale. I have always insisted on this. A value on the x-axis and a value on the y-axis... for us to realize the scale. (...) When I'm looking at a graph on a calculator (...), I see it in a particular window. We know that the graph will continue. In this case we know the shape of the graph outside this window. In some other cases we don't know. Now, this window has x ranging from -10 to 10 and y ranging from -6.65 to 6.67. So, rather than include the scale on the graph, or together with that, I can write it in two ways: $[-10, 10] \times [-6.65, 6.67]$ or $x \in [-10, 10]$ and $y \in [-6.65, 6.67]$. (class 2)

The box

The goal of this task is to find the length of the side of the square that should be cut out of a cardboard of 1.2 m × 80 cm, to build a box with maximum volume (Figure 2). Working in groups, the students should analyze the problem and come to the conclusion that the volume of the box will be given by $V(x) = 4x^3 - 4x^2 + 0.96x$. Afterwards they are expected to use the graphing calculator to draw the graph of the function and find the value of x that maximizes V.

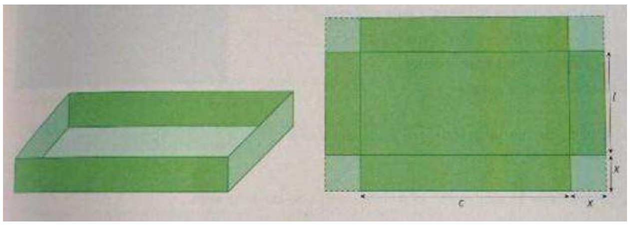

Figure 2: The box

This is a task where finding an acceptable viewing window requires some work. And the teacher seeks to support her students, trying to help them making sense of the values to consider:

Teacher: So let's change the window to get a better view. (...) Let's think a little bit further. What values can x take?
Student: (...) Between 0 and 1.
Teacher: No. What's the size of the cardboard? This is a cardboard.
Student: 80.
Teacher: 80. How far can you cut? You cut at one side and at the other side. How much can you cut?
Student: Ah! 40.
Teacher: Ok, it can change between 0 and 40. So the x can go from 0 up to... instead of 1 I can choose?...
Student: 0.4.
Teacher: 0.4. (...) Let's have a look. Notice I already have the function in the part that interests me. (...) Here it is so low, so low. So what can I do? I can put here a negative value and cut a little bit here (points to the top of the screen) (...) So, let's go, window... You're starting at 0, I'll put -0.5, right? (...) yMin -0.5. And what about yMax? You have 1. 1 is up there, so it has to be a smaller value. What value do you want to try? (...) Try it! (class 7)

But the teacher has also in mind that the calculator provides several ways to change the viewing window. Thus, throughout the lesson she points to her students different possibilities. And, by the end of the lesson, when she addresses the problem with the whole class, she begins by discussing suitable values for x and then she uses successive zoom box until she gets a suitable viewing window:

Teacher: Well, most people did this and got a graphic like this. (...) And now we can't forget that we are in the context of a problem. (...) This means that I could adjust the calculator window according to the values involved on the problem. And one of the possibilities (...) was to go to the window settings and put 0 for xMin, or a little bit less, and for xMax... how much?
Student: 0.4.
Teacher: 0.4. Ok. If I did this, what happened? Then I see almost nothing, but I can notice that the curve is here, but it's so close to the x-axis that is not possible to see. Now I have two choices. I can try to estimate the values assumed by y or I can use (...) the zoom box. Because what I want is to get a better view in this area here, around here. (class 7)

But it is not only the diversity of forms available to change the viewing window and its explanation to the students that are valued by the teacher. More than knowing the commands of the calculator, the teacher wants the students to understand what was done and why it was done, so that in future they are able to deal with a similar situation by themselves. In this sense, the teacher addresses the search for a good viewing window as a process. The viewing window is successively changed taking into account the problem and using the features provided by the calculator. And this is a conscious option of the teacher:

Teacher: I think that if I just do it right, at the next time they are going to face difficulties again. Don´t you think?
Researcher: So, it's a way of showing them the process. Is that so?
Teacher: It is a way. (...) There may be situations where I realize what the optimum window is, but if I don't, I need to have knowledge of some ways to look for it. And that was what I tried to do. (...) Because if not, if I go there and just do it right, they don't even realize that there is some difficulty. They fail to do it, but that's the right window and that's it. (post-class 7)

Folding the corner of the sheet

In this task the students should fold a sheet of paper so that the upper left corner touch the bottom side

(Figure 3) and then find the triangle of largest area that appears in the lower left corner of the sheet of paper. Students work in pairs and begin by doing several folds and taking some measurements. This way they get a set of data that allows them to find a function that fits the data and then find a solution to the problem.

During the task the students are confronted with a set of data that suggests a quadratic function, when the

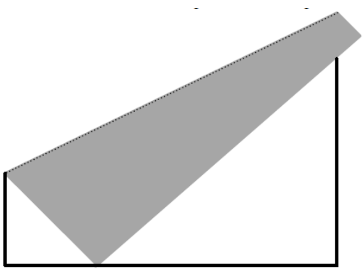

Figure 3: Folding the corner of the sheet

corresponding function is actually a cubic. Students realize that the quadratic function does not fit perfectly to the data but, when they think of alternative functions, they are guided by data visualization and think on a quartic function:

Student: Here we tried a quadratic (...) because the points are like... in parable form, but this is not quite...

Teacher: Yes, it doesn't go through all the points.

Student: It can be like that?

Teacher: (...) We are working with experimental data, so either you weren't very accurate in measurements (and there is always an error in the measurements we make), or that's not the function that best fits your data. Did you try another function?

Student: Yes, we tried this one, the quartic.

Teacher: The quartic. And what happened?

Student: It's better here at the beginning, but then it also doesn't go through the points.

Teacher: Right. And did you try other functions?

Student: No, we just tried these (...) the points are like this... (the student points to the U-shaped distribution of points), so these functions will be the best ones. Unless is grade 6... is it? (class 12)

Most students sought for the function that best fitted the data between polynomial functions of the 2nd and 4th degree. Only the students who tried all kinds of functions provided by the calculator (using regression), considered a polynomial function of 3rd degree:

Teacher: What about you? What conclusion did you reach?

Student: We think it's this one.

Teacher: A cubic. And why did you choose it?

Student: Well, we started by the quadratic because we thought it was. But it doesn't go through all the points. And then we decided to try all that appear here in the calculator and this is the best one... (class 12)

The discussion of the task, performed in the following lesson, is based on the students work. The functions chosen by some students and their graphical representation in conjunction with the data collected are presented. The discussion starts with presentations of 2nd and 4th degree polynomial functions, since these are the dominant responses in the class. The teacher has to insist, asking for something different, in order to get someone presenting a 3rd degree function. Given the obviously better way how this function fits the data, the teacher takes the opportunity to highlight some aspects of the viewing window and the cases where it does not show a global picture of the graph:

Teacher: Most of you looked at these points, and thought it was a quadratic function, but you can't conclude like that. You have to keep in mind that you are only seeing the graphic in a window and what you see doesn't tell you anything about what you don't see. (...) When I introduce a polynomial function of the 2nd degree in the calculator and ask for the graph, I know it will be a parable because I know these functions, because my knowledge

> about Mathematics allows me to know it. (...) But when I see a graphic looking like a U, without more information I can't conclude that this function is a quadratic... because the graph can reverse the concavity outside my window... I don't know... (class 13)

Thus, the teacher analyzes the importance of the viewing window and how a partial view of the graph can easily lead to wrong conclusions.

DATA ANALYSIS

The practice of this teacher seems to be characterized by three different stages in what concerns to the viewing window. At an initial stage, the teacher avoids situations where the students need to look for a suitable viewing window. However, she does not give up of implementing what she believes to be a valuable task, just because the required window is not trivial. Whenever the standard window is not suitable for a full exploration of the task, she informs students about the values they should use on the viewing window. At an intermediate stage, the teacher already proposes to students tasks where they actually need to find a suitable viewing window. At this point the teacher discusses with the students how to find a suitable window, trying to make them develop an understanding over the searching process. At an advance stage, the teacher confront students with misleading situations, where the students need to understand the impact of the viewing window in order to avoid an uncritical acceptance of the graphic displayed on the calculator's screen. All over these stages, the teacher highlights the importance of some kind of written record that gives information about the viewing window being used. This option turns the window into a noticeable and important aspect of drawing graphs on the graphing calculator since the first moment.

The teacher's practice is certainly a consequence of her professional knowledge. So it makes sense to analyze the teacher's practice trying to infer which domain of the KTMT model seems to be involved.

The teacher participant in this study recognizes students' difficulties in finding an appropriate viewing window. As such, initially she avoids complex situations, providing students with explicit information about a suitable viewing window (TLTK). She is also aware of the difference between plotting a graph with and without technology. Namely, she recognizes the importance of recording the scale/viewing window along with the graph obtained on the calculator (MTK). As so, the teacher highlights to the students the relevance of the chosen viewing window. With this purpose in mind, she selects tasks that allow her to emphasize the importance of mathematical knowledge for proper reading and interpretation of graphs, focusing on the usual difficulties of students in a phased manner (IK). Besides that, the teacher is concerned about explaining the different options available to find a suitable viewing window (includes TK), presenting the demand for a suitable viewing window as a process (IK). Thus, the analysis of the teacher's practice emphasizes the relevance of knowledge in the domains of KTMT involving technology: KT, TLTK, MTK and IK.

CONCLUSION

The conclusions of this study point to an approach to the viewing window which takes into account the degree of difficulty involved. Starting from a practice that ensures that the students are aware of the viewing window, but where any change is based on available information, it evolves to a practice where the students are expected to actually find a suitable window, and afterwards it evolves again to include misleading situations. This practice of the teacher is aligned with the perspective of authors such as Cavanagh (2006), who emphasize the importance of a phased contact of the students with situations where finding a suitable viewing window is more complex, recognizing the difficulties usually faced by the students.

The analysis of the teacher's practice also suggests that the way the demand for a suitable viewing window is addressed in the classroom is influenced by the teacher knowledge regarding KT, TLTK, MTK and IK. Namely, it seems to be highly relevant the teacher's knowledge in what concerns to:

— the students' difficulties in finding a suitable viewing window;

— the new emphases that technology puts on certain mathematical knowledge during the search for a viewing window;

— the teaching strategies that allow the teacher to address mathematical knowledge relevant to guide the search for a suitable viewing window, while focusing on the usual difficulties of students in a phased manner;

— and the different options available to find a suitable viewing window, as well as the importance of explaining this options to the students.

In the future, it would be particularly interesting to investigate the impact, on the teachers' approach to the viewing window, of an education program (pre-service or in-service) that emphasizes this set of knowledge. After all, more than being a characterization of the teacher's knowledge, the KTMT model intends to help identify aspects of the professional knowledge that, when included in training programs, can contribute to the professional development of those enrolled.

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Designing a didactical situation with mobile and web technologies

Håkan Sollervall[1,2] and Didac Gil de la Iglesia[2]

1 Malmö University, Malmö, Sweden, hakan.sollervall@mah.se

2 Linnaeus University, Kalmar, Sweden

We have designed, implemented and evaluated a didactical situation where 27 students in grade 4 created large triangular constructions in an outdoor environment by involving a customised GPS-based mobile application. The students' strategies for construction were reflected upon during a teacher-led discussion involving web technologies and aiming at the formulation of principles for construction. The effective communication of underpinning theories and possible learning objectives, in combination with user-friendly mobile and web technologies, served to scaffold the teacher's successful orchestration of a logos-oriented mathematical discourse.

Keywords: Didactical situation, mobile technologies, web technologies, design.

INTRODUCTION

Many tasks in mathematics textbooks relate to contexts in the real world that are beyond the student's reach and sight in the classroom. Such tasks are commonly treated in micro-space (Brousseau, 1986) on the student's desk. Students seldom get the opportunity to experience mathematics in large, full-sized space, which is crucial for considering spatial ideas and not only visual ideas as represented in models and in drawings on paper (Bishop, 1980). However, doing mathematics in meso-space, outside the classroom, calls for the teacher to provide and orchestrate meaningful teaching activities. Further didactical challenges include how to connect such activities with mathematically meaningful follow-up activities in the classroom.

Recent developments in the field of technology-enhanced learning show promising attempts to design outdoor teaching activities involving the use of mobile and web technologies (e.g., Sollervall & Milrad, 2012). While mobile technologies can support the design and orchestration of outdoor teaching activities, they pose technological challenges regarding stability (Gil, Andersson, Milrad, & Sollervall, 2012) and pedagogical challenges regarding usability and instrumental genesis (Verillon & Rabardel, 1995).

Beyond these pedagogical and technological challenges, we have to consider how to connect the activities with the regular mathematics curriculum and how to implement complete didactical situations (Brousseau, 1997) in specific schools with specific groups of children, preferably with limited resources for the purpose of scaling up.

For these reasons, we have adopted a co-design methodology (Penuel, Roschelle, & Shechtman, 2007) where researchers in mathematics education and media technology work together with schoolteachers to design and implement didactical situations with innovative technologies such as augmented reality (Nilsson, Sollervall, & Spikol, 2010) and customised mobile applications (Sollervall & Milrad, 2012).

The inherent complexity of the designed activities has on several occasions led to the researchers controlling the implementation of the outdoor activities and neglecting the follow-up indoor activities. Although our ambition is to design didactical situations where the teacher has full agency of the implementation, our conclusion – based on the outcomes of several similar projects – is that a first design cycle should prioritize the functionality of the technologies and the didactical flow of the situation. During the first cycle, we are satisfied if the technologies perform acceptably and the students experience a sequence of meaningful and enjoyable mathematical activities. To support the achievement of such outcomes, the researchers have

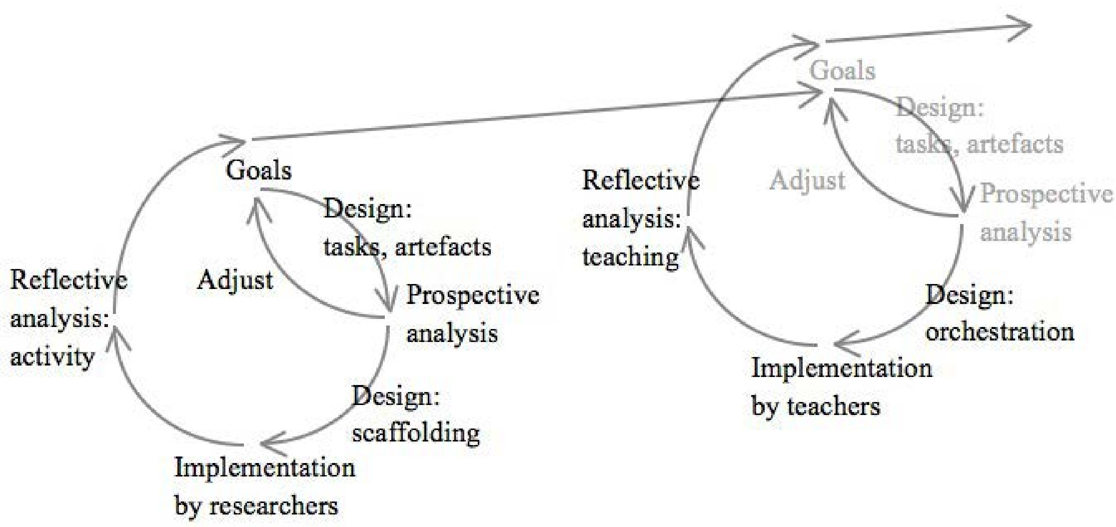


Figure 1: First and second design cycle, as implemented in the current project

been actively involved in implementing the activities (Figure 1).

During the second design cycle, the researchers provide support for new teachers to orchestrate and adapt the didactical situation by enhancing the customised technologies, communicating learning objectives, and indicating how these objectives may be achieved by unfolding didactical affordances of the situation. The teacher is fully responsible for pedagogical design and implementation (Figure 1), but is not involved in the initial phase where the activity, including its tasks and artefacts, is modified by the researchers who also prepare guidelines for orchestration. These guidelines are communicated to the teacher during a short preparatory session, directed at enabling the teacher to interpret and unfold the learning opportunities that are embedded in the didactical situation.

In this paper, we report on the second cycle implementation of a complete didactical situation and characterize the possible learning opportunities in terms of dimensions within a mathematical praxeology (Rodríguez, Bosch, & Gascón, 2008).

Our research question addresses the evaluation of an implemented didactical situation, specifically designed to promote a logos-oriented mathematical discourse:

> How do the scaffolds provided support logos-orientation in a teacher's orchestration of a didactical situation that is specifically designed with mobile and web technologies to provide opportunities for a logos-oriented discourse?

THEORETICAL AND METHODOLOGICAL CONSIDERATIONS

A teacher's design cycle includes the analysis of a teaching activity both before and after it is implemented with students. Such prospective and reflective analyses are central features of design-based research (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). In our prospective analyses, we put focus on students' hypothetical action trajectories, analysed from a socio-cultural perspective. In the reflective analyses, and particularly when advanced technologies are involved, we shift our focus of attention between the cycles. In the first cycle, we simply evaluate the activity flow. In the second cycle we evaluate the teaching outcomes, and in the third cycle we analyse the learning effects. The current paper reports on a second cycle and will put focus on evaluating teaching outcomes, for the purpose of informing future implementations of the activity by modifying and improving the guidelines for orchestration. This approach is underpinned by an ambition to find a reasonable distribution of ownership between researchers and teachers.

In the process of designing teaching activities with advanced technologies, we have noticed that fundamental didactical principles may become neglected when the research efforts favour enhancing the performance of the technologies themselves. For example, involving technologies that provide extensive feedback may remove didactical challenges that are actually needed to promote students' learning processes. For this reason, we have adopted the theory of didactical situations (Brousseau, 1997) as a design model. The fundamental structure of a didactical situ-

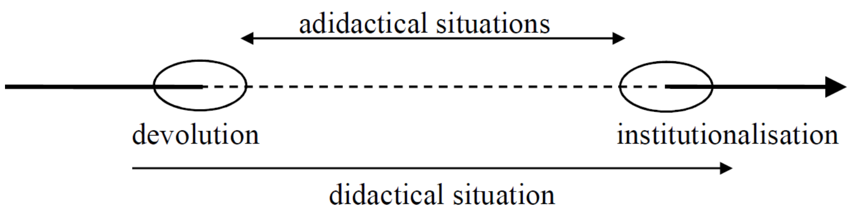


Figure 2: Structure of a didactical situation

ation – devolution, adidactical situations, institutionalisation (Figure 2; adapted from Balacheff, 2013) – fits particularly well when designing with mobile technologies across physical contexts.

The three phases within a didactical situation become naturally separated if adidacticity is promoted by giving the students full responsibility for the technology-supported exploration of mathematical tasks by retroacting only with the milieu and not the teacher, as indicated in Figure 3 (left pane, adapted from Bessot, 2003, p. 7).

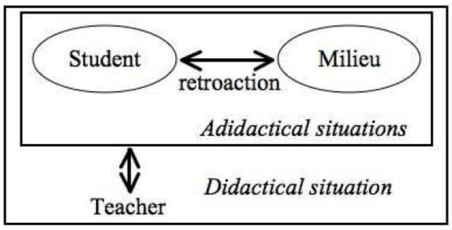
While the theory of didactical situations provides a structure for a teaching activity, with focus on achieving mathematical learning objectives, we utilize the notion of praxeologies to capture qualitative differences of the learning opportunities that are offered to the students during a didactical situation. In our analysis, we will charac-terise such teaching outcomes in terms of praxeological elements (Figure 3, right pane).

While praxis, that is, tasks and techniques, naturally dominates within the adidactical situations, the logos dimension may emerge in the phase of institutionalisation where the students are invited to reflect on their experiences by engaging in technological and theoretical discussions about how and why the techniques work. A technology-oriented discourse may include describing techniques, explaining how they work and when they work, while a theory-oriented discourse aims at justifying the techniques and the technological claims (Rodríguez et al., 2008).

In previous projects (e.g., Perez, 2014) we have observed situations dominated by praxis-oriented activities where the teacher has not unfolded the logos-oriented affordances that were embedded in the situations. In the current study, we chose to make explicit the notion of praxeologies for the teacher and discussed a variety of affordances for a logos-oriented discourse during the institutionalisation phase, based on the students' experiences from exploring tasks in the outdoor environment.

DESIGN CONSIDERATIONS

In the first cycle, a teaching activity in an outdoor environment was designed for the purpose of investigating spatial orientation ability (Peng & Sollervall, 2014). The teaching activity involved ten similar tasks that each called for the coordination of two given distances with respect to two given reference points. Each such task can be interpreted as the construction of a triangle with three given sides, a construction that is treated in Euclid's Elements (Heath, 1908, p. 292; Figure 4).



Praxis	Logos
Tasks	Technology
Techniques	Theory

Figure 3: A didactical situation (left pane) and elements of a praxeology (right pane)

Book I Proposition 22

Out of three straight lines, which are equal to three given straight lines, to construct a triangle: thus it is necessary that two of the straight lines taken together in any manner should be greater than the remaining one.

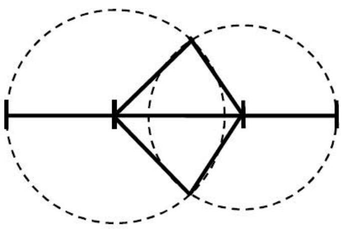


Figure 4: Euclidean construction of a triangle with three given sides

During the second cycle, the purpose was to stimulate a logos-oriented discourse during the phase of institutionalisation, involving technology as well as theory:

— technology: identifying and comparing strategies for construction, distinguishing between possible and impossible constructions;

— theory: justifying the strategies particularly the circle strategy, formulating criteria and arguing why some constructions are possible and others are not.

Based on their experiences from the first design cycle, the researchers designed a complete didactical situation encompassing devolution, an outdoor activity, and institutionalisation. The outdoor activity involved ten tasks, that each called for the coordination of two given distances with respect to two given reference points. Seven of the tasks involved possible constructions while three were impossible, such as "10 15" when the distance between the reference points was 35 meters. The impossible constructions were included for the purpose of stimulating a technology-oriented discourse during the institutionalisation phase.

The researchers' ambition was that the students should work with their ten tasks in small groups, simultaneously and independently, in the schoolyard. To achieve variation of tasks between the groups, it was decided to place six different reference points in the schoolyard (Figure 5) and to vary the order of the tasks. For example, Group 7 had their first task "20 30" against the house and the tree (Figure 6, right pane) while Group 5 had "20 30" as their eight task against the bicycle and the car.

The didactical situation was designed for up to 14 groups and targeting students in grades 4–6. The 14 sets of tasks were deployed on mobile phones (androids) supporting a customised technological application (Figure 6, left pane).

Although Euclidean constructions are not in the grade 4 mathematics curriculum in Sweden, the presented didactical situation connects well with mathematical content such as distances, measurements, and circles. Moreover, the institutionalisation phase involved mathematical communication and reasoning, as competencies that are strongly emphasised in the steering documents for mathematics education in Sweden.

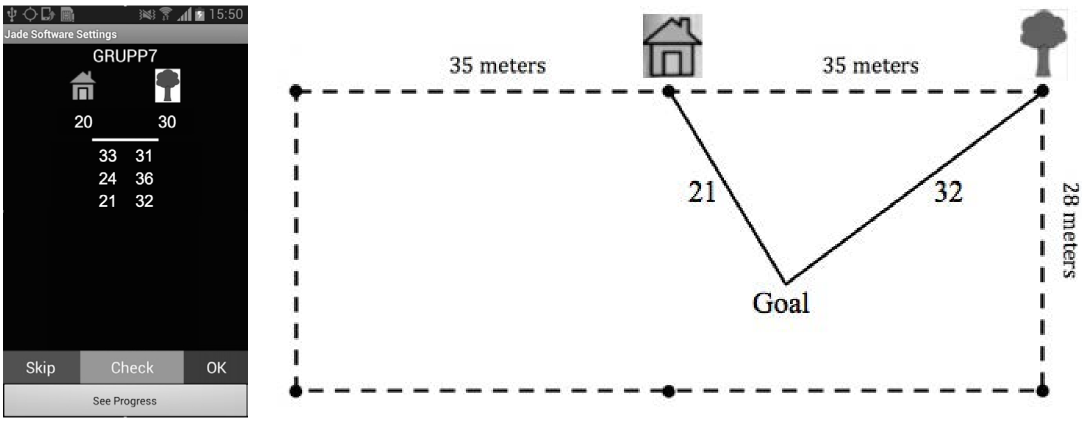


Figure 5: The presented Google Map (left pane) and the actual field (right pane)

Figure 6: The display (left pane) and an illustration of the solved task (right pane)

SCENARIO FOR THE DIDACTICAL SITUATION

Two days before the activity was implemented, with 27 students in grade 4, the researchers met the teacher at her school. After a short outdoor session where the mobile technologies were tested hands-on, she was informed about the researchers' desire to promote a logos-oriented discourse. The schematic structures of a didactical situation and a praxeology (Figure 2 and 3), the Euclidean circle strategy (Figure 4), together with possible logos-oriented learning objectives, were presented on a single sheet of paper and was briefly discussed. A sample set of 10 tasks was also presented and discussed. Furthermore, the teacher was informed about the possibility to show the students' results on a Google Map. It was made clear that these ideas should serve only to inspire her and that she was completely free to orchestrate the activity according to what she believed would be best for her students, not for the researchers.

The entire activity including devolution, the outdoor activity, and institutionalisation, was video-taped and lasted 1 hour 45 minutes (8.00–9.45 on a Friday morning).

Scenario for devolution

The 27 students arrived to the regular classroom at 8 o'clock in the morning. They had been told in advance that they would engage in an outdoor activity and use mobile phones. Before the students arrived, the teacher had divided them into 12 groups that were displayed with on the whiteboard. The three visiting researchers introduced themselves and the teacher informed the students that they were going to work with mobile phones to solve mathematical tasks outdoors on a field where the researchers had placed six coloured markers (cat, house, tree, bicycle, car, horse).

The field and the markers were displayed on a Google Map (Figure 5, left pane). When the teacher asked the students if they recognised the field in the picture they immediately answered yes. The teacher informed the students that they were going to look for "magic points" that were located specific distances from two of the markers, and that they were going to use the mobile phones to check the distances. The teacher asked the students what they would do if they were not satisfied with the measurements and they readily answered that they would try again. Each group received a phone from the teacher and got them started after a few instructions.

When all students had opened their first tasks that were all different (an example is shown in Figure 6, left pane) the teacher told them to go to the field (Figure 5, right pane) and try to solve the tasks. The time was now 8.15. The researchers noticed that the teacher had not informed the students about the inherent inaccuracy in the GPS values that may cause a measurement error of a few meters.

Scenario for the outdoor activity

As mentioned earlier, the researchers had prepared 14 sets of 10 tasks. The 10 tasks were identical with respect to distances but were presented to the groups in different order and with respect to different markers. Each task referred to distances to two of the six markers. For example, for Group 7 the distances "20 30" were shown on the display of their mobile phone directly under pictures of "house" and "tree" (Figure 6, left pane). The objective was to find the point on the field that was located 20 meters and 30 meters from the markers, respectively (Figure 6, right pane). The response "21 32" could be considered as acceptable. On a few occasions, the teacher negotiated this issue of non-exact measurements with the students directly on the field.

The students showed no signs of confusion either regarding how to handle the mobile phones, how to interpret the tasks, and even accepted the somewhat inaccurate measurements. They were enthusiastic and engaged fully in the tasks, although some phones did not give correct measurements due to thick clouds that caused large errors in some of the GPS-values. After half an hour some students complained that it was cold outside and the teacher decided at 8.50 to ask them to go back to the classroom.

Scenario for institutionalisation

At 8.56, everyone was back in the classroom. After a short discussion about some incorrect values and asking if the students liked the activity (which they did) the teacher asked the groups to present their strategies for finding the "magic points". Most of the groups were eager to present and the teacher promised that they would all get to do it. The first group gave their mobile phone to one of the researchers who downloaded its log file to a computer that was connected to the classroom projector. Their tasks and their attempts

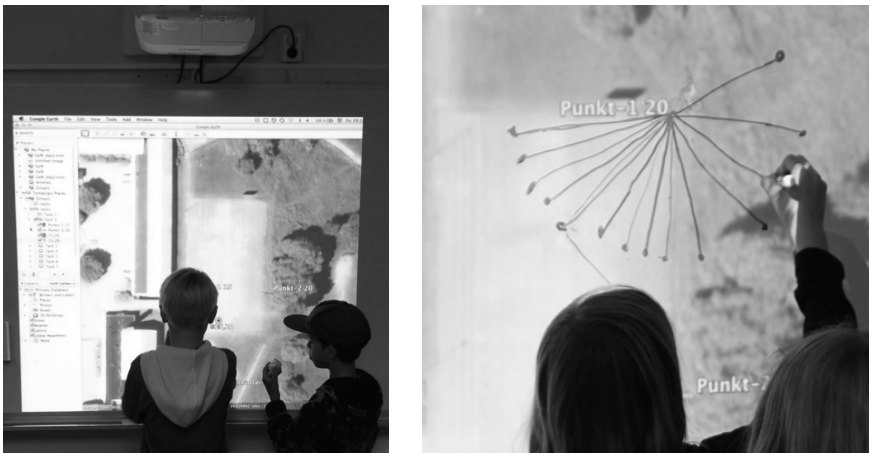


Figure 7: One group presenting (left pane) and the whole class contributing (right pane)

became visible (numerically) on the board (a regular whiteboard, not interactive) to the left of the Google Earth (Figure 7, left pane). They chose the task they wanted to present and what attempts they wanted to be shown with "pins" on the Google map visualisation.

For several of the groups, the teacher had to tell the students to describe the task before they started talking about how they worked with it. During the presentations, she repeatedly asked technology-oriented questions like "How did you think when you did that?", "Why did you do walk like that?", and "How did you get those values?". Several of our previously identified strategies were confirmed (Peng and Sollervall, 2014) but the targeted circle strategy did not appear in the presentations.

However, when all the groups had presented, the teacher continued the discussion, focusing on the last group's presentation. They had marked a point located 20 meters away from the bicycle marker (upper right corner in Figure 7, right pane) and had drawn a line segment from the point to the marker. The teacher asked the class if somebody could mark another point that was also located 20 meters away from the bicycle. Several students tried, but failed. They seemed confused about what to do but were eager to contribute. The teacher commented on their attempts, for example "Oh that is more than 20 meters", "That point is too close", "That is too far away". She tried to guide the students by asking questions: "If you stand there and it is 20 meters, how can you walk to keep 20 meters?", "Where else can you find 20 meters?". Finally, one student managed to mark a point that seemed to be the same distance from the marker. The teacher confirmed the attempt by saying: "Yes! You found it!" and then "How did you know how to do it?". The student responded: "I just thought it out". The teacher continued with "Now I want each of you to mark a new point, that is also located 20 meters away from the bicycle", and "Don't worry, there are infinitely many such points and each of you will get a chance to mark one". Most of the students caught on to the idea about keeping the distance 20 meters but changing directions, and occasional mistakes were quickly corrected. When about ten points had been marked, all located on the field, the teacher commented: "Oh, nobody is being brave today". One student understood what she referred to and readily marked a point in the bushy area behind the field (Figure 7, right pane). A few more points were marked outside the field. A crucial scaffolding question was asked.

Teacher: Do you begin to see a pattern? You can walk in any direction.
Student 1: Oh it is a circle!
Student 2: A spider web!
Teacher: Yes! A circle! Can you all see that?

The teacher drew a circle through the points.

Teacher: Now I have 20 here and how can I find 30 down there?

The students were invited to mark points that were initially not connected with the first circle. These points were corrected after comments from the teacher, who wrapped up the discussion at 9.45 by saying "If you can find the point where the two circles meet then you have found the magic point". Although enforced by the teacher, the concluding theory-oriented comment completed a didactical situation addressing all the four dimensions of an emerging point praxeology.

CONCLUDING DISCUSSION

The institutionalising discourse may be characterised as teacher-driven but student-centred. The teacher was informed about the researchers' desire to promote logos-oriented discussions and was prepared for orchestrating the session towards issues relating to technology and theory. Knowing about possible strategies for construction guided her to ask logos-oriented questions aiming particularly at the circle strategy. She patiently awaited the students to catch on to the mathematical ideas that were embedded in the didactical situation. She amplified the students' presentations by adding interpretations that led them to unfold ideas that were shared among all the students by involving them in making new constructions.

The teacher's orchestration was influenced by the preparatory session two days before the trial, particularly regarding the logos-orientation and the circle strategy. The customised mobile applications inspired the students to engage in the outdoor activity, while the web technologies served to underpin the presentations and connect their obtained measurements with their field experiences. All students could readily relate to what the presenting students were referring to on the Google map application. These technical design features scaffold effective communication in the classroom and enabled the teacher to put focus on asking logos-oriented questions.

A few incidents occurred during the implemented outdoor activity. The inaccurate and sometimes failing GPS-values (due to cloudy weather) caused confusion among some of the students. However, the teacher swiftly handled all such incidents.

It may be noted that the teacher did not address the issue of possible and impossible constructions. This may be considered as a natural decision due to the fact that the students were quite young (grade 4) and possibly not yet ready to engage in conditional reasoning. Instead, the teacher engaged the students in interpreting their triangular constructions in terms of circles, as one of several suggested discourses.

The mobile-assisted outdoor activity offered opportunities for the participating 27 students to simultaneously engage in similar coordination tasks, involving the same pairs of distances but with respect to different markers. Being informed about possible logos-oriented discourses and having observed the students acting in the outdoor environment, the teacher cleverly managed to institutionalise their common experience with respect to the circle strategy. The customised mobile and web technologies inspired the students to engage in the activities and supported transitions between outdoor and indoor contexts. These supporting technologies enabled the teacher to put focus on pursuing mathematically meaningful institutionalising activities, thus successfully finalizing a complete and complex didactical situation.

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Theory of semiotic mediation in teaching and learning linear algebra: In search of a viewpoint in the use of ICT

Melih Turgut

Eskisehir Osmangazi University, Eskişehir, Turkey, mturgut@ogu.edu.tr

Utrecht University, Freudenthal Institute, Utrecht, The Netherlands, M.Turgut@uu.nl

This paper presents a research project on the description of a linear algebra course within the perspective of the Theory of Semiotic Mediation (TSM), in order to construct mathematical meanings prioritizing artefact–sign (i.e., ICT tools) and social context relationships. I will first describe the project and, following a theoretical framework of the TSM, discuss a priori epistemological analysis of GeoGebra's potentiality to be used as an artefact to bring out key linear algebra concepts in future didactic interventions. I also present the results of a pilot study elaborating the potentiality of GeoGebra for students' construction of the mathematical meaning of free variables. Future steps of the project are also outlined.

Keywords: Theory of semiotic mediation, teaching and learning linear algebra, ICT.

LEARNING LINEAR ALGEBRA: EPISTEMOLOGICAL ISSUES

Because of the strength of generalization, linear algebra is a powerful theory used to frame problems belonging to quite different contexts, but at the same time, it may be difficult to construct in itself. From a didactic point of view, merely taking a theoretical approach, i.e., introducing *vector space* concept to students, that is, only giving axioms, means asking students to enter a meaningless game of symbols, because the historical genesis of linear algebra indicates numerous lengthy steps to form an evolution of the idea of vector spaces. The notion of a *vector* begins with Aristotle who represented 'force' in geometric terms (Chong, 1985). However, after approximately 2000 years, mathematical representations of vectors were used as points and directed line segments by Gauss and Hamilton in terms of Cartesian geometry. Researchers used *matrices* to represent these geometrical ideas, as well as linear equations, and, by practical discernment, they obtained certain extensions to elimination techniques, n-tuples, determinants and transformations. However, all these developments were operational, sometimes having different theoretical elements, and so used different mathematical languages. There was, however, a missing *unified*-general approach covering and connecting all of them. In 1888, Giuseppe Peano defined the vector space concept with an axiomatic system, as *a set* satisfying certain axioms (Dorier, 2000). This was a formal definition, and opened a door to non-geometric vector spaces, such as polynomials and square matrices. Since we, as linear algebra lecturers, use all of these concepts together, one can conclude that the use of different notations, depictions and axiomatic language is essential for the teaching of linear algebra. From a didactical point of linear algebra education, the axiomatic-formal system of the course reveals a learning obstacle in teachers' hands; the *obstacle of formalism* (Dorier, Robert, Robinet, & Rogalski, 2000). This obstacle is associated with the specific *terminology* of linear algebra and appears when students are faced with the mathematical *symbol language* triangle, composed of equations, matrices and vectors. Students waver 'under an avalanche of new words, symbols, definitions and theorems' and therefore, for many students, 'linear algebra is no more than a catalogue of very abstract notions' (ibid. p. 95). In summary, the formalism obstacle can be considered as students' failure to grasp linear algebra's symbols and their associated-corresponding *mathematical meanings*. From the viewpoint of *semiotic registers* (Duval, 2006) of such mathematical meanings, learning linear algebra needs *conversion* between different registers; 'graphical', 'tabular' and 'symbolic languages' of linear algebra or, in other words, students should have 'cognitive flexibility' to overcome the obstacle of formalism (Dorier & Artigue, 2001, p. 270).

The construction of mathematical meanings cannot be easily achieved through a direct use of ICT, needing a careful *didactic design* of tasks to exploit the use of *artefacts* (Mariotti, 2012). I will try to overcome the obstacle of formalism using such didactic designs enriched with ICT tools, in particular GeoGebra (5.0 version) (its potentiality will be presented in detail), for students' construction of mathematical meanings of linear algebra concepts. Taking a semiotic approach, I will focus on the *Theory of Semiotic Mediation* (TSM) (Bartolini Bussi & Mariotti, 2008); both the design of the tasks and analyses of the processes.

RESEARCH QUESTIONS AND THEORETICAL FRAMEWORK

This study is a one-year, post-doctoral research project, planned to start in February 2015, focusing on the following research questions:

— *How should a linear algebra course be designed within a didactic-semiotic perspective?*

— *Does this approach overcome the formalism obstacle of linear algebra students?*

Taking into account the TSM, I intend to address both research questions. This is because the TSM is a Vygotskian-rooted approach in math education proposed by Bartolini Bussi and Mariotti (2008). It relates to the *semiosis* feature of math, which bases the teacher's actions in a *social context* and on the hypothesis that the *production of signs* can be elaborated on when the teacher intentionally uses an artefact to accomplish a math task within a communication-oriented process. By use of specific artefacts in the mediation process, the TSM aims to construct math meanings in students' mental schemes; in other words, to transform *personal meanings* into *math meanings,* while they solve the proposed tasks as *mediator*. In this process, the main focus is on the *emergence of signs* that foster students' possible math learning. Within this aim, the TSM is constructed on two key elements: *the notion of the semiotic potential of an artefact* and *the notion of a didactic cycle*. The semiotic potential of an artefact is associated with its '... *evocative power*, stressing the distinction between meanings emerging from the activity with the artefact and the math meanings evoked by such activity' (Mariotti, 2013, p. 442). In other words, it is related to the potential for math meanings to emerge whilst students solve a mathematical task. The notion of a *didactic cycle* is about the design of the teaching-learning process, especially describing semiotic processes: (i) *activities with artefact* (students work in pairs or in small groups), (ii) *individual production of signs* and (iii) *collective production of signs* (Bartolini Bussi & Mariotti, 2008, pp. 754–755). Iteration of such didactic-semiotic (environment) processes aims to foster the evolution of personal signs-meanings to (desired) math signs-meanings. This is because the (desired) evolution of the signs is 'artefact signs', 'pivot (hinge) signs' and culturally accepted 'mathematical signs' (ibid, pp. 756–757). Therefore, as an important component of this polysemy of the artefact, the teacher has a central role; she should orchestrate mediation with specific social activities to exploit the semiotic potential of the artefact (Mariotti, 2013), i.e., her role should be surrounded by an interacting triangle of, *use of artefact, personal meanings*, and *math meanings* in a socially-communicative environment.

Why I select TSM in this project to overcome obstacle of formalism

Harel (2000) proposes three teaching principles for learning linear algebra (p. 180):

— the *concreteness principle* associated with results stemming from the use of the *axiomatic language* of linear algebra and students' pedagogical needs, in particular, the transition from geometric to abstract features. For instance, the concept of a polynomial as a vector is not concrete for students if they cannot comprehend the mathematical notion of linear independency.

— the *necessity principle* that refers to instructional activities which should form *a problematic environment* for the construction of mathematical concepts, with students seeing an 'intellectual need' (ibid, p. 185).

— the *generalization principle* that reflects students' generalization of driven concepts after the problem-solving process; in particular, with the help of the *intellectual need* character of the proposed learning environment.

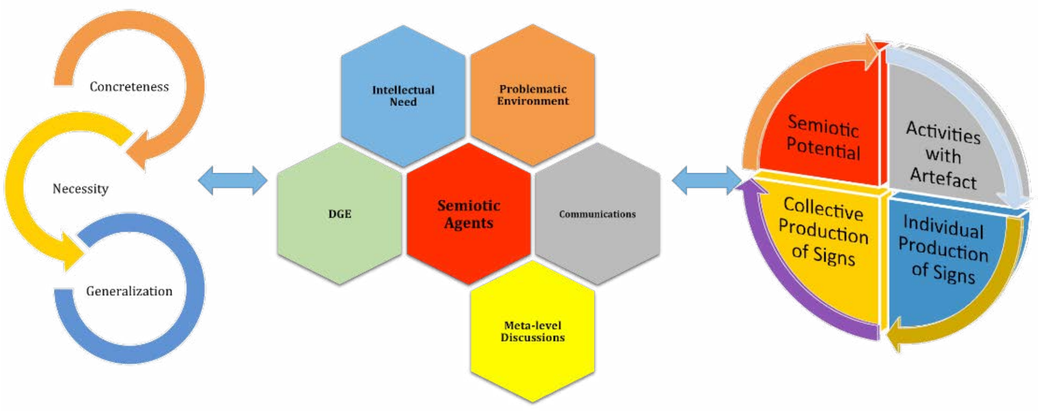

Figure 1: Key epistemological concepts in relation to the TSM

In summary, one can conclude that construction of math meanings of the geometric features of linear algebra through ICT tools can provide dialectic relationships among abstract features. The use of ICT didactic designs should provide a problematic environment (necessity) (Turgut, 2013); preparatory insights (concreteness) for non-geometric linear algebra (Harel, 2000), and math discussions for comprehending abstract concepts. In other words, as Dorier and Artigue (2001) emphasize, the lecturer should create an environment for students, which provides an opportunity to reflect on '*meta-level discussions*' of problems, referring to the generalization principle. This can also be an opportunity to *unify* linear algebraic concepts (p. 271).

Several attempts have been done within the semiotic perspective to analyse and construct math meanings of linear algebra with ICT tools. Sierpinska, Trgalová, Hillel and Dreyfus (1999) designed a research program with Cabri–geometry II of a geometric model of vector space in order to overcome the obstacle of formalism. Lengthy teaching experiments (within different conceptual perspectives) reveal that students are able to grasp math meanings, in particular specific concepts, such as linear combination (ibid, p. 129). Hillel and Dreyfus (2005) investigate how the conditions of *communication* influence the construction of math meanings, whilst students are attempting to solve tasks regarding the projection of vectors and an approximation with Maple CAS. Episodes were constructed on different semiotic systems, in particular *agents* for communication; the students themselves, an observer, the computer and Maple, classroom teacher, classroom notes and text. At the end of the sessions, the agents (communicative semiotic environment) were able to contribute to students' construction of new meanings of linear algebra concepts. The researchers also emphasize the Maple role as a 'silent mediator' that stems from students' wait for-know-decide-use process of the commands. In summary, this was a glimpse of the use of dynamic geometry environments (DGE) as an artefact in the teaching/ learning linear algebra.

To sum up, all these together imply a puzzle including several keywords. I postulate and summarize the following key concepts to overcome the obstacle of formalism that, all together, fit with the TSM (Figure 1).

The keywords in Figure 1 have dialectic relationships with the TSM's core elements. For instance, intellectual need and problematic environment are related, but the dialectic relationship between them also implies the notion of the semiotic potential of the artefact and task designs. Consequently, the TSM underpins this process, because the TSM is a powerful framework for both the design of didactic cycles and analysis of the signs. Another fact is that the communication-meta level discussions process corresponds to Individual and Collective Productions of Signs and so on.

Within the framework of Figure 1, I hypothesize that GeoGebra may be a powerful artefact to use in task designs because one main potentiality of the use of GeoGebra is its involvement of a 3D interface, and hypothetically, this might help students to *unify* the different representations of linear algebra. However, it needs a grain analysis to describe the key points that we want to emerge from its use.

Semiotic potential of some tools of GeoGebra: A priori analysis

The following *priori* epistemological analysis of GeoGebra will describe our goals and answer the following questions; what key linear algebra concept can

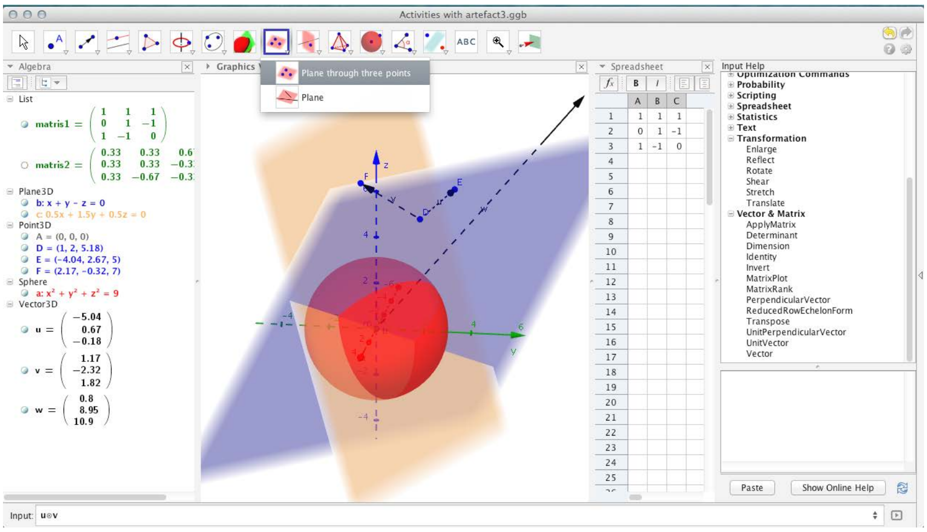

Figure 2: GeoGebra interface with different windows

emerge from the artefact, and what can we do with GeoGebra? As a consequence, this process leads us to discuss the *semiotic potential of the artefact* that is the core element of the TSM. The analyses of semiotic potential has a *dialectic* relationship between explication of goals and attempting to see what happens when we present the designed tasks to students, and then see which of their *utilization schemes* are useful to transform into mathematical meanings. To sum up, the process of outlining the semiotic potential of an artefact needs a deep analysis encompassing an epistemological and didactic-cognitive perspective that will give us a framework to assist in the design of the second phase of the TSM. Let us first describe this artefact's tools in relation to linear algebra.

In the GeoGebra interface (Figure 2), all the windows can be viewed together; the CAS, Graphics, 3D Graphics, Spreadsheet and Algebra. Using a spreadsheet table, such as Excel, one can form matrices with any rows or columns. They immediately appear in the Algebra window. With the help of the Input Help column (right side), other forms can also be accessed. The key element, a vector, can be composed by the Input line or in the Spreadsheet window in the form of co-ordinates and vectors (also with manipulations, cross product or forming a line) that appear simultaneously in the Graphics window. The Graphics window enables the plotting of a 3D view of the lines, planes and surfaces. At the same time, the tools of this window provide certain 3D applications, but the important one is the plane through three points. In addition, the Transformation part in the Input Help window provides a matrix transformations application in the plane, but not in 3D. As a limitation, one can see several tools in the windows, but the tools may be limited for the purpose of the tasks to prevent possible cognitive loads on the students. The viewing of these windows all together can create an environment for the possible *conversions* of different semiotic registers by students that may help them unify linear algebraic concepts. Therefore, I decided to consider the following four different semiotic registers (in sense of Duval, 2006): *algebraic register* (AR), 2D *graphics register* (2DGR), 3D *graphics register* (3DGR), and *spreadsheet register* (SR). Conversion of these registers might help students to shift different representations of linear algebraic concepts. Besides this powerful feature of the artefact, 2DGR has a *slider* component that forms an environment as a *dynamic variation*. This process is also applicable in other registers by *moving*, or the controlled movement of the mouse. These may be important in enabling students to evolve meanings of particular notions, because *variation* in 2DGR and 3DGR, provided by the slider and AR, may be key elements in the construction of associated meanings in the design of cycles for our future didactic interventions. I limit myself to the emergence of 'free variables in R^3' in the system of linear equations and associated geometric invariants with the following pilot study, which aforementioned registers can evoke students' learning.

PILOT STUDY AS A TEACHING EXPERIMENT

In this part, I attempt to elaborate the semiotic potential of GeoGebra, in particular, the use of AR, 2DGR (slider tool) and 3DGR, in the construction of a link between the system of linear equations, augmented matrices and intersection of planes. In other words, I focus on the question, 'will the use of those semiotic registers evoke construction of mathematical meaning of free variables?' For this purpose, I designed a task involving manipulation on different registers (inspired by the problem in Anton, 1981, p. 54). The educational goals of the experiment are:

— conversions among 2DGR (slider), 3DGR and AR,

— evolution of personal meanings to mathematical meanings of a free variable,

— fostering construction of the mathematical link among the concepts.

Using three registers, I prepared a task, as described in Figure 3. The participants of the experiment were two sophomore level undergraduate mathematics education students, and the experiment was implemented at the beginning of a linear algebra course, following the topic of solving the system of linear equations. A number of students volunteered to participate, but only two were selected according to their mathematical background and communication skills. They had performed moderately on former courses, and had taken only three mathematical courses; general mathematics, abstract mathematics and geometry. The fact is that they knew the equations of a plane, and the augmented matrices corresponding to the system of linear equations. The students were unable to use GeoGebra since, prior to the experiment, I had introduced the main tools of the software by removing any unnecessary tools, i.e., the only tools in the 3DGR were *move* and *rotate*. Thereafter, they practised dragging and shifting in the windows.

The expected situations in this experiment were; (i) the students' analyses using two sliders for a and b (2DGR), (ii) at the same time analysing the variations of the equations in AR, (iii) and the positions and manipulations of the plane in the 3DGR. Overall, the task was prepared to analyse the system of linear equations in terms of the variations of a and b, (for instance when a=0, b=2, the planes coincide). I estimated that, after they analysed such variations, they would focus on the system in AR, and thereafter, they would build an augmented matrix in order to attempt to solve the system. As a next step, they would compare their results within dynamic variations in 2DGR and 3DGR, also associated with the *intellectual needs* of the task. In the end, they would first construct a mathematical meaning of *two free variables*, as well as its corresponding meanings on the planes' movements and intersections (c, d, e in AR, with yellow, red and purple planes in 3DGR).

Procedure and data analysis

The experiment was implemented as a teacher-researcher, involving Buse and Deniz (pseudonyms), working as a pair on a computer screen. The data consisted of video-recorded interviews, screen recorder software and students' productions, with the teaching episode ending after 40 minutes. Within a semiotic lens, the data was analysed with respect to the TSM's frame of categories of signs (Bartolini Bussi & Mariotti, p. 756); emergence of *artefact signs*, *pivot-hinge signs* and *math signs*. Artefact signs are a production driven as a result of an immediate use of the artefact characterizing the proposed task; math signs refer to math meanings, such as definitions, a proposition or a math proof. Pivot signs refer to hy-

Task: Move the sliders a and b, and explore and explain what is happening in the GeoGebra interface systematically.

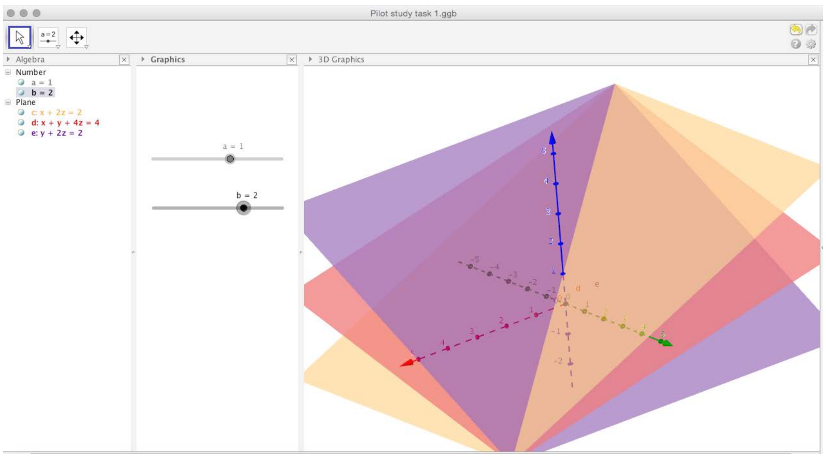

Figure 3: GeoGebra interface of the proposed task

brid terms in natural language, such as 'object' or 'thing', associated with math terminology (ibid.).

Summary of the results

After the teacher had introduced the task, the students tried to make certain interpretations by moving the sliders and analyzing the different registers together, and, thereafter, they focused on the equations in the AR. In this way, they comprehended what was differentiating in the plane equations whilst they dragged the sliders, since they realized that a system of linear equations existed with the key values being a=0 and b=2. They also formed the planes' equations with respect to a and b, and Buse pointed out that the solution of the system must be related to these values. In this process, the teacher was orienting the students to focus on the relationship between the a and b values as well as the solution of the system of linear equations. The following excerpt is drawn from the discussion from which the signs evolved: from artefact type to mathematical signs.

45 Teacher: You said at the moment, values of a=0 and b=2 must be in relation to a solution of this [indicating the system on the paper sheet] system, how you are sure about that?

46 Buse: Because changing the values here [indicating the 2DGR], shows us different type intersections here [indicating 3DGR], in fact, sliders are affecting the equations [meaning the AR]. These intersections must be similar to the position of the lines that we discussed.

47 Deniz: Exactly, look [moving the sliders] if a=0 and b=2, all the planes coincide. Oh yes, it is already obvious here [referring AR], in all the equations z=1, as in the case of coinciding lines, therefore, I think, the solution must be infinite here.

Thereafter, the students also explain other cases using their knowledge, stemming from an analysis of the system of linear equations in R^2: when $a \neq 0$, $b \neq 2$, the intersection of the planes is 'single point', 'exact solution' and 'consistent system'. Similarly, if a = 0, $b \neq 2$, they mention 'parallel planes', 'no solution', and 'inconsistent system'. Thereafter, the teacher asks how to find the relationship between the augmented matrix and the interpretations expressed by the students. They use the Gauss-Jordan elimination method. The following excerpt is drawn from this discussion.

71 Buse: Look [showing the solution her pair], I put a=0 and b=2 to see what will happen, I could only calculate z=1. I can not find x or y.

72 Deniz: We can not find, look, the second and third rows are completely zero.

73 Teacher: What does it mean? How can you relate this fact with your initial interpretation?

74 Deniz: Any real number can satisfy this system if we put it instead of x or y. They are independent from the plane equations, since, this is consistent with the picture here [indicating 3DGR], where there are two variables that we cannot find, but the solution, I mean, the intersection is a plane: a two-dimensional thing. But I am not sure whether this hypothesis is valid for other cases.

75 Teacher: Let's analyse other cases then.

76 Deniz: [She is moving the sliders, checking her hypothesis] If we take a=1 and b=2, there is again an infinite solution, [looking at the matrix form] and we can not find y either x, but x=y, since we have one free variable, the intersection is a line; a one dimensional thing.

As a consequence, Buse points out the exact solution, 'no free variable', and therefore, 'single point and 0-dimensional thing'. To sum up, I observed a semiotic chain (Bartolini Bussi & Mariotti, 2008) in the use of different semiotic registers, from artefact signs to math signs of *free variables* such as: 'chancing value' (item 46); 'intersection of the planes' (item 46); 'solution types' (item 47); 'independence from equation' (item 74); 'variable' (item 74); 'dimension' (items 74–75); and 'free variable' (item 76).

CONCLUSIONS

Even if linear algebra does not consist of only geometrical features, as Harel (2000) states, it can be a powerful 'corridor to the more abstract algebraic concepts' (p. 185). In this project, I aim to construct math meanings of geometrical features in linear algebra concepts within the semiotic potential of an artefact; with GeoGebra, and particularly in this work, I focus on the notion of a free variable. Through this experiment, I conclude that the emergence of the notion of a free variable has a strong link with student recognition of geometry and to relating geometric objects' invariants with their infinitive characteristics, such as variables. The use of different registers helped students to articulate the different cases. In the experiment, I realized that the main feature was the students' use of the software. Because they did not know how to use it as an instrument, they were not able to master the use of the sliders when they changed windows. The project will continue with underpinning teaching experiments and case studies to analyze the semiotic potential of the mentioned software in terms of, 'what key linear algebra concepts can emerge' through its use, and I will also point out students' utilization schemes in sense of Rabardel (1995). The description of such utilization schemes may also be a basis for describing possible meanings that may emerge during the designed activities.

ACKNOWLEDGEMENT

I would like to thank Maria Alessandra Mariotti and Paul Drijvers for their kind and endless contributions during the preparation of this paper. This project is supported by *The Scientific and Technological Research Council of Turkey* (TÜBİTAK BİDEB) under 2219–International Postdoctoral Research Fellowship Programme.

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TWG15

Posters

Using spreadsheets in learning equations

Osman Bagdat

Anadolu University in Eskisehir, Faculty of Education, Eskişehir, Turkey, osmanbagdat@anadolu.edu.tr

Spreadsheets as a very functional tool has been used in many studies to facilitate transition process from arithmetic to algebra. In this study, we describe middle-grade students' experiences about using spreadsheets in learning equations. For this purpose, ten 7th grade students who had been taken an elective mathematics course participated in a spreadsheet activity about exploring equations with two unknowns. As a result, working on the problem arithmetically with spreadsheets helped students to conceive the spreadsheet cells as variables and it facilitate the students' formulization of arithmetic generalisations.

Keywords: Spreadsheets, arithmetic, algebra.

INTRODUCTION

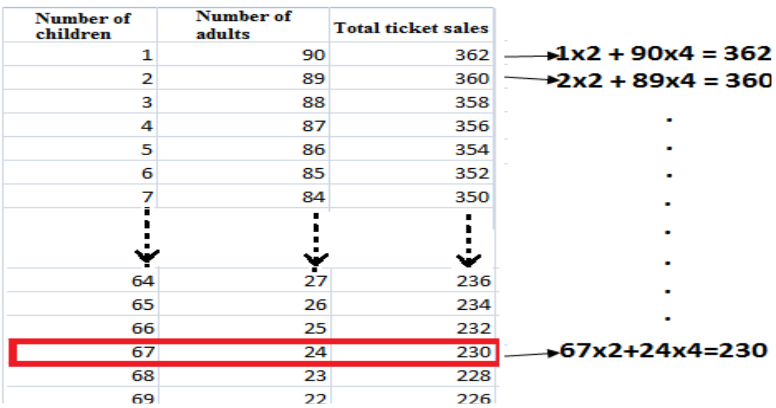
Transition from arithmetic to algebra is considered as one of the crucial point of mathematics. Many studies indicate that majority of students have difficulties in this process. In recent years, using technological tools have been seen as an alternative approach to overcome these difficulties. Since spreadsheet activities provide instant feedback in an interactive environment, it appeared as an important learning tool for students. Lots of studies have been done in math education about spreadsheets, many of which in algebra. Spreadsheets, in particular, provide students with a learning environment in which students can explore equations (Lagrange & Erdogan, 2008), and make pattern generalizations which are one of the crucial components of algebraic reasoning. Students can provide variety of solutions and have chance to see relationships among the variables as numbers change (Edwards & Bitter, 1989).

Although the importance of using spreadsheets in teaching algebra, we do not encounter many examples in actual Turkish classroom settings. This study contributes the efforts of utilizing technology in learning and teaching mathematics through a real example from a middle school classroom setting.

METHODOLOGY

This study was conducted with ten 7th grade students in a public middle school in Eskisehir, Turkey. Data were collected through a word problem (see below) which can be solved both arithmetically and algebraically. Students worked in pairs throughout the classroom period. The whole problem solving process was recorded on all computers through using the software, Camtasia Studio 7.

> *The problem*: "On Tuesday a total of 91 people visited the local zoo. Entrance fees: 4 TL (Turkish Liras) for adults and 2 TL for children. At the end



Number of children	Number of adults	Total ticket sales
1	90	362
2	89	360
3	88	358
4	87	356
5	86	354
6	85	352
7	84	350
64	27	236
65	26	234
66	25	232
67	24	230
68	23	228
69	22	226

Figure 1: Sample student solution

of the day the total ticket sales were 230 TL. So how many children visited the zoo?"

FINDINGS AND DISCUSSION

Initially, students used the "trial and error" strategy to find the total number of children visiting the zoo. The teacher provided a table drawn on a paper for each group for draft solutions and asked students to think about the possible numbers of children and adults visiting the zoo. In this moment, students used only calculators. After a few trials, the teacher asked students to continue working on the problem using pre-designed spreadsheets. While some groups randomly entered numbers for children and adults and tried to find the total ticket cost (i.e., 230 TL), others used a more systematic way (see Figure 1).

Later the teacher asked students to solve the problem algebraically. Specifically, the teacher asked students to create algebraic representations for the total number of people visited the zoo, and the total ticket sale. Following algebraic equations were provided from a teacher-led group discussion:

Teacher: What does x+y=91 represent?
Student A: It is the total number of people visited the zoo. While "x" is the number of students, "y" is the number of adults. The total number of people who visited the zoo is 91.
Teacher: Okay. How about 2x+4y=230
Student B: Since children pay 2 TL for each ticket, I wrote "2x" for the total amount of money they pay for entrance to zoo. Then I did the same thing for adults (4y). The total ticket sales should be 230 TL. So I wrote "2x+4y=230."

Working on the problem arithmetically with spreadsheets helped students to conceive the spreadsheet cells as variables. Thus, it facilitated making sense of variable and helped students make a smooth transition from arithmetic to algebra.

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Influence of dominant cognitive structure on the way of students' thinking during problem solving

Edyta Juskowiak

Adam Mickiewicz University, Poznań, Poland, edyta@amu.edu.pl

In this contribution, I present a part of a research project, whose goal is to study and describe the ways in which 14-year-old students and students of the Department of Mathematics and Computer Science at Adam Mickiewicz University in Poznań, Poland, use respectively the graphic calculator and Geogebra for solving particular tasks. During this study, I attempted to answer the following question: what is the influence of dominant cognitive structure, predicative or functional, on the way of thinking of students during specific problem solving?

Keywords: Dominant cognitive structure, problem solving with technology, mathematical thinking, GeoGebra, graphic calculator.

INTRODUCTION

The appearance of graphing calculators and GeoGebra computer software in Polish schools has raised the interest of the research community in the students' learning process with the use of those tools, aiming at recognizing both their advantages and disadvantages as well as their role and place in the process of learning and teaching mathematics. In this context, it is important to know the differences concerning the ways in which people work on the same task with or without the use of technology (Demana & Waits, 2000).

THEORETICAL FRAMEWORK

For this study, I have used Schwank's theory of functional and predicative thinking (Schwank, 1995, 2001). Schwank's study shows that in every human being we can observe a relatively stable tendency to exhibit a way of thinking characteristic of one of the two cognitive structures: either the predicative or the functional one.

QUESTIONS

How do pupils and students of mathematics (future teachers) solve mathematics problems using the graphing calculator and GeoGebra in relation to their dominant cognitive structure, be it functional or predicative? Does the analysis of solution of the problem allow determining the dominant cognitive structure of each student? Is this structure the same during the student's work on a task with or without technology?

METHOD

This research study involved first grade junior high school students and students of Math and Computer Science Department of Poznań Adam Mickiewicz University, specializing in teaching. The work of the high school students was observed in the context of an examination requiring solving non-standard tasks with a graphic calculator; the work of the university students was observed in the context of a research project focusing on the ways of working on non-standard tasks with or without the use of new technologies. The data gathered consisted of the records of the students' work on the task with technology, the records and notes of a discussion with the students after the activities with technology, and the students' worksheet (Juskowiak, 2014).

RESEARCH RESULTS AND IMPLICATIONS

The attempt to define the way of thinking of the examined junior high school students and university students, based on their solutions, has turned out not to be an easy task. It required a specific methodology, which entailed a precise monitoring of the students' work on the task through the use of tools for recording the students' activity with technology. Such methodology allowed observing the problem solving process in a very precise and detailed way, so helping to define the students' way of thinking. Though this is a preliminary small-scale study, it seems to show that on the basis of tasks solution one can draw conclusions about the students' dominant thinking structure. The same dominant cognitive structure manifested both in working with technology and without its use. Only in one case, this did not occur: a student with a dominant predicative cognitive structure solved the math problem with technology showing a functional cognitive structure. This confirms the fact, which had already been observed by Schwank, that the tendency to a specific way of thinking does not exclude thinking in other ways (Schwank, 1995, 2001).

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Integrating technology into primary and secondary school teaching to enhance mathematics education in Hungary

Zsolt Lavicza[1], Istvan Juhos[2], Balazs Koren[3], Kristof Fenyvesi[4], Csaba Csapodi[2], Marta Kis[2] and Jose-Diego Mantecón[5]

1 University of Cambridge, Cambridge, UK, zl221@cam.ac.uk
2 University of Applied Sciences Budapest, Budapest, Hungary
3 Eötvös Loránd University, Budapest, Hungary
4 University of Jyväskylä, Jyväskylän, Finland
5 University of Cantabria, Santander, Spain

Keywords: Technology, teaching traditions, design experiments.

During the past decades, technology has been becoming an integral part of everyday life and slowly shaping mathematics and science teaching and learning (e.g., Heid & Blume). Although there has been enormous investment on educational technologies in many countries, technology has yet to make a sizable impact on education (e.g., Drijvers et al., 2010). On the one hand, students are becoming increasingly proficient users of technology while, on the other hand, opportunities offered by technologies have still little been utilized. Nevertheless, technologies are becoming more integrated into education providing new opportunities for pedagogical approaches and classroom organisation. For example, mathematicians stated that they use technology because in this way they can more easily treat students as mathematicians and nurture their knowledge through discovery and experimentation (Lavicza, 2010). To utilise the opportunities technology offers we developed a large-scale project, GEOMATECH (http://geomatech.hu), in Hungary integrating teaching traditions of the country as well as good practices from around the world.

Many Hungarian mathematicians, scientist and mathematics educators have a world-wide respect. In addition, mathematics education theorists and practitioners, George Pólya, Zoltán Dienes, Imre Lakatos, Tamás Varga among others, are often quoted as great innovators and founders of modern theories and practices in mathematics education. In the GEOMATECH project, we are developing new approaches for technology integration into Hungarian schools utilising Hungarian teaching traditions, successful international examples, and experiences of Hungarian teachers. The research team of the project is closely working with 45 teachers (25 mathematics, 20 science) and their students (1200 mathematics, 800 science) for 8 months utilising Design Experiment and the Community of Practice frameworks (Cobb et al., 2003; Jaworski, 2006) to develop teaching materials and employing theories of inquiry-based education (Artigue & Blomhoj, 2013). The research team offers continuous feedback to the material development, teacher training and software development teams to extend this work to other Hungarian schools. The GEOMATECH project (owing to the generous 8million Euro EU Funding, TÁMOP-3.1.12) will develop high-quality teaching and learning materials for all grades in primary and secondary schools in Hungary. These materials (1200+ Mathematics, 600+ Science) will be embedded into an on-line communication and collaboration environment that can be used as an electronic textbook, a homework system, and a virtual classroom environment. In addition to material development, we will offer 60-hour professional development courses for more than 2400 teachers in 800 schools in Hungary. Furthermore, we will organize a wide-range of teacher and student activities including competitions, mathematics and science fairs, and develop a network of schools for the long-term sustainability of the GEOMATECH project. The technology background of the project is offered by GeoGebra (http://geogebra.org), which an open-source, dynamic mathematics software widely used around the world.

All activities in GEOMATECH will be evaluated by the research team and questionnaires to measure students' various beliefs and conceptions will utilise the modified instruments of Andrews and Mantecón (2015). Currently, we are in the initial phase of the project starting the Design Experimental phase and preparing the project team (almost 200 people) for further tasks in the project. In our poster, we will outlined the theoretical frameworks, the different stages of the project and highlighted initial results from the Design Experimental phase.

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On the usage of media when solving exercises – the activity theory viewpoint

Hannes Stoppel

Westfälische Wilhelms-University Münster, Institute of Education of Mathematics and Computer Science, Münster, Germany, hannes.stoppel@uni-muenster.de

A lot of digital media are used in mathematics lessons. Students take recourse in them when difficulties arrive while solving exercises, but other difficulties often appear when choosing an inappropriate method to solve an exercise under usage of digital media. Students need to know whether, when, and in which way (wwww) to use digital media for procedures. Wwww differs for the same procedures in dependence on the digital media used. Activity Theory (AT) will be helpful to understand procedures used in solving exercises as described with examples of Computer Algebra Systems (CAS).

Keywords: Activity, method, object, digital media, CAS.

INTRODUCTION

In accordance with earlier findings (Stoppel, 2012), solving exercises include the development or the choice of *methods* like algorithms and their application to *objects* like equations or geometric figures. The analysis of students' solutions from two universities and several grammar schools showed that solving exercises depends on the ability of students to choose or develop an appropriate method to design or select any useful object for the solution of the exercise afterwards.

Based on these results, some exercises from calculus and probability theory were solved with different CAS. The analysis of solutions showed that different syntax of CAS leads to differences in the choice or the development of methods and activities for solving the exercises.

THEORETICAL BACKGROUND

Several possibilities for usage of CAS in different phases of a solving process are shown in Zehavi and Mann (2005). Examples of the applicability of CAS are described in Pointon and Sangwin (2001). We used Activity Theory to observe connections between the syntax of operations of media and the solution processes. Studies concerning activities when using digital media already exist (e.g., Kaptelinin & Nardi, 2006). In contrast hereto we draw on the concept of activity system by Leontiev (1978) to study the methods used inside of solutions, and representations of the activity system of Engeström (1987) to investigate artefacts regarding the usage of media and their interconnections to methods.

Wwww lead to two questions: *1. In what situations are students unable to find methods or connections of methods to objects? 2. What might be helpful for students to find legitimacy methods and connections?* An analysis of several correct and wrong solutions in the abovementioned study shows that AT is helpful to understand the phases of the solving processes.

RESULTS

On the way to the application of a method, the learners need to decide whether to choose an already familiar method or to construct a method by themselves. Independent of being useful for solving a given exercise, the hierarchy of the approaches to solutions, the types of solutions and the way through solutions play a big role for students. When using different digital media, students might need to solve parts of exercises employing different command syntax. The command syntax could lead to methods that differ from usual methods of solutions without digital media, so that, with digital media, students might be distracted or even unable to recognize the solution at the end.

To find answers to question 2, we need to take a look at the usage of media by students in mathematics lessons.

While solving exercises, activities are important for the development of methods. Their complexities are visible in Engeström's (1987) activity system while the inclusion of methods into solutions is shown by Leontiev's (1978) concept.

With help of the systems the importance of various aspects and the interrelationships between them for a successful solution of an exercise becomes evident. Teachers need to consider them to support their students and to decide on wwww to use digital media.

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TWG16

Students learning mathematics with resources and technology

Introduction to the papers of TWG16: Students learning mathematics with resources and technology

Matija Lokar[1], Ornella Robutti[2], Nathalie Sinclair[3] and Hans-Georg Weigand[4]

1 University of Ljubljana, Ljubljana, Slovenia, Matija.Lokar@fmf.uni-lj.si
2 University of Torino, Torino, Italy, ornella.robutti@unito.it
3 Simon Fraser University, Burnaby, Canada, nathsinc@sfu.ca
4 University of Würzburg, Würzburg, Germany, weigand@dmuw.de

SCOPE AND FOCUS OF THE WORKING GROUP

Recent discussions within the CERME technology group have confirmed the relevance of introducing information and communication technology (ICT) in mathematics education with a range of various resources, such as software, hand-held devices and online classroom activities. The scope of the working group at CERME9 was to address the opportunities and possibilities, as well as the challenges and limitations that resources and technology present for student learning. Due to the large number of participants in the technology group at the last CERME conferences, the group was divided into two strands. TWG15 concentrated on "teaching with resources and technology", while the focus of TWG16 was on "students learning with the help of resources and technologies".

In TWG16, we were interested in empirical results obtained in the course of the last few years and decades concerning, for example, the introduction of new learning methods and concepts, the improvement of learning within the frame of old concepts or new forms of exercises in the whole range of mathematics education. Moreover, we wanted to set the scope towards the future, to think about general conclusions for mathematics education, and the research required as well as the implications for the classroom.

A range of different issues was covered, including:

— The design and/or use of the current software and technologies concerning students' formal and informal learning, at school and out of school;

— The design of teaching experiments with software and technologies concerning student learning;

— The influence of computers, notebooks, tablets or handheld-technology on the learning of mathematical concepts and the construction of learning environments;

— The results of empirical studies and investigations, especially those concerning long-term learning with ICT, massive courses, national programmes relating to the teachers' professional development;

— The possibilities of present and up-coming mobile communication and representation tools like tablets, smartphones;

— The possibilities of different methods of assessment using ICT;

— The influence of web-resources and online courses on the learning of mathematics;

— The examples of "best practice" in the classroom;

— The examples of the use of technologies designed for the support of students with disabilities.

RESULTS OF TWG16

In TWG16 we received 23 papers and 3 posters from 15 different countries. In the following sections, the topics are classified as *old questions* and *up-coming* *topics* strongly represented in TWG16, which are discussed first, and finally the topics referred to as *missing topics* are discussed.

"Old question" strongly represented in TWG16

Starting with the Dynamic Geometrical Systems *Cabri-Geomètre* and *The Geometer's Sketchpad* in the late 1980s, dynamic mathematics has always been a strong argument for the use of new or digital technologies. Nowadays, there are numerous simulations in mathematical and environmental settings, which use dynamism in connection with *touch screen technology*. An example is the "place value chart" (Figure 1), which visualizes the changes while moving the tokens from one column to the other (see the article of Behrens). See also the papers by Brunström and Fahlgren and Bray, Oldham and Tangney.

Multi-linked representations are another often mentioned argument for using digital technologies in the learning process. Figure 2 shows the symbolical, numerical and graphical representations of a quadratic function, which enable, for example, identifying structural analogies in the different representations (see the articles of Pinkernell and Swidan).

There are many more topics that are important and will stay important in the following years:

- ICT and problem solving and/or guided inquiry learning (see for example the paper by Schumacher and Roth)
- The relationship between hands-on materials and ICT
- The transition from activities to mathematical thinking
- The transition between ICT-tools and paper-and-pencil activities
- The benefits of ICT for the learning of content, e. g., calculus
- Programming and its connection to mathematics (see the paper by Misfeldt and Ejsing-Duun)

Up-coming topics

a) Touch Screens and Human-Computer Interaction

Touch screen working with tablets is already a frequently discussed topic in present classrooms. Gestures will become more important – for example, while zooming into a graph or diminishing an interval – to help visualizing and, hopefully, understanding mathematical concepts. The complexity of simultaneously changing more than one variable (by moving the fingers) should by no means be underestimated. But the empirical basis concerning these new features is still quite small. See the paper by Bairral and Arzarello.

b) Internet resources – Social media or networks

Although Internet resources such as Facebook, etc. are not constructed for doing mathematics, social media or networks open new possibilities, for example, "peer learning". There might be automatic help systems available, but there will also be the possibility to get personal help from experts, tutors, students or other Internet users. See the article of Puga and Aguilar.

c) Automatic Testing – Automatic Feedback

Intelligent tutoring systems have been developed and tested in mathematics classrooms for many years (see the article of Borba, Azevedo and Barreto). Nowadays, there seems to be a new approach possible in forma-

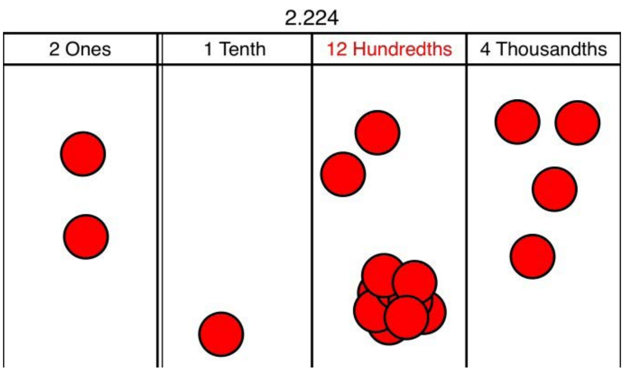


Figure 1: Changing the number's representation within the place value chart by moving tokens

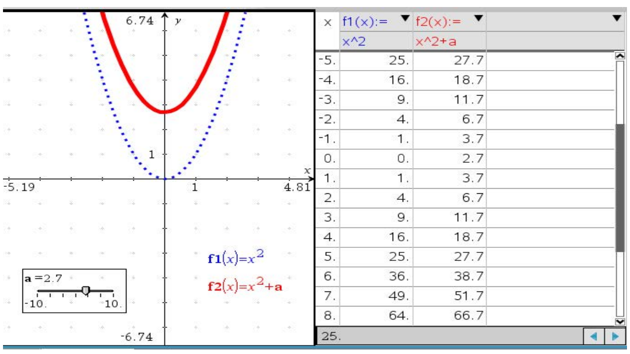
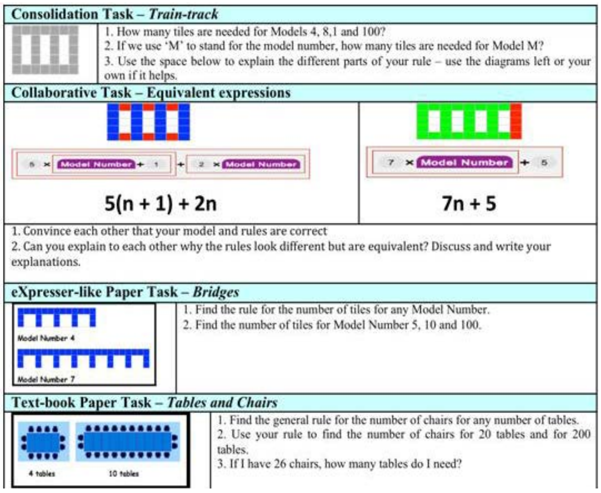


Figure 2: Dynamic multiple representation environment for exploring the effects of a on the representations of $f(x)=x^2+a$


Consolidation Task – *Train-track*

1. How many tiles are needed for Models 4, 8,1 and 100?
2. If we use 'M' to stand for the model number, how many tiles are needed for Model M?
3. Use the space below to explain the different parts of your rule – use the diagrams left or your own if it helps.

Collaborative Task – Equivalent expressions

5 × Model Number + 1 + 2 × Model Number

5(n + 1) + 2n

7 × Model Number + 5

7n + 5

1. Convince each other that your model and rules are correct
2. Can you explain to each other why the rules look different but are equivalent? Discuss and write your explanations.

eXpresser-like Paper Task – *Bridges*

Model Number 4

Model Number 7

1. Find the rule for the number of tiles for any Model Number.
2. Find the number of tiles for Model Number 5, 10 and 100.

Text-book Paper Task – *Tables and Chairs*

4 tables

10 tables

1. Find the general rule for the number of chairs for any number of tables.
2. Use your rule to find the number of chairs for 20 tables and for 200 tables.
3. If I have 26 chairs, how many tables do I need?

Figure 3: Transition from arithmetic to (formal) algebra (Geraniou & Mavrikis)

tive assessment within the frame of automatic testing and automatic feedback, especially in arithmetic and algebra.

The system described in the paper by Geraniou and Mavrikis supports the transition from arithmetic to algebra by visualizing the variables in different representations. The article of Mackrell shows how real-world problems will be solved in a CABRI environment through the provision of effective feedback. There are a lot of didactical problems in the construction of systems like these: the problem of the right feedback in problem solving activities, the possibility of different kinds of feedback and feedback as step-wise help, feedback on task level or/and process level, and so on. Haddif and Yerushalmy focus their research on e-assessment of challenging calculus construction e-tasks designed to function as a dynamic interactive environment of multiple linked representations (MLR) that provide feedback to the learner (Figure 4).

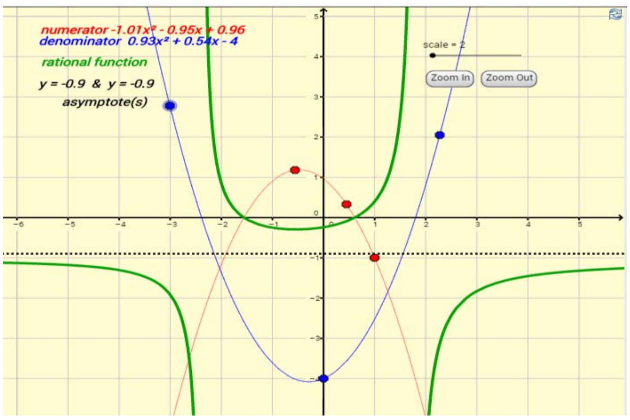


Figure 4: Construction of rational functions by given "dots" of the graph of the function

d) The significance of "games" for mathematics learning

Young students often play games on their tablets and smartphones, which raises the possibility of integrating mathematical games into the environment that is already present. Soldano, Arzarello and Robutti want the users to improve their proving abilities in a geometric environment, while Avraamidou wants the users to build houses, while adjusting the budget in relation to some variables such as the size of the building, the cost of the furniture and the mood meter.

e) ICT-support and special needs students

We only received one contribution referring to the learning style of a special needs student. Particularly students with physical disabilities might benefit from the possibilities of digital technologies. However, empirical research is still quite rare and the results are quite individual anyway, and generalizations almost impossible because of the great variety of special needs.

Missing topics

In TWG16 we have "only" received 26 papers and posters and, of course, these contributions represent a relatively small array of all possible topics. This just might be an indication of the waning importance of some topics, which were quite intensively discussed only a few years ago.

— The new possibilities of ICT concerning creating videos, doing screen recording, gathering and analysing big data … were not discussed in TWG16.

— Digital technologies allow more individuality, for example, the creation of portfolios and personalized e-textbooks.

— Communication is the norm and the goal in the NCTM standards (2000) and nowadays in many curricula all over the world. DT might support cooperation between communities and enable the "opening" of the classrooms.

— Completely new and emerging technologies have not been discussed in our group: sensor devices, augmented reality, mobile technology …

— Symbolic calculators are not discussed in any paper. They seem to be disappearing because they cannot compete with laptops, tablets and smartphones. But there are still strict regulations (in some countries, for example in Germany), and these tools are not allowed during examinations.

There are also some topics that might be called traditional and will be an ongoing problem in mathematics education, not only in relation to the use of digital technologies:

— We need more long-standing quantitative empirical investigations;

— We – as researchers – have to use proper research methods, which are informed by contemporary theories, and base our work on the existing and growing research in mathematics education;

— We have to get some more information about the actual use of digital technologies in real classrooms.

TWG16

Research papers

Children's perception of the affordances of the mathematical tools

Yasmine Abtahi

University of Ottawa, Ottawa, Canada

Vygotsky's theory proposes a sign/tool-mediated view of learning. Tools and signs are inseparable parts of teaching and learning of Mathematics. Vygotsky's theory provokes questions: how are tools perceived, how are signs tied to the use of tools and consequently how are tools being used in the mathematics classroom. In this paper, I look at Vygotsky's perspectives on the perception of the tools, through the lens of Gibson's view of affordances. I analyse three children's interactions with the mathematical tools, as they gradually begin to tie signs to them, while working on addition of fractions problems.

Keywords: Perception, affordances, tools, fractions, meaning.

INTRODUCTION

Fractions are one of the most challenging concepts to teach and learn in elementary-school mathematics (Steffe & Olive, 2010). Lamon (2007) noted that fractions are one of the topics in elementary-school mathematics that are among 'the most difficult to teach, the most mathematically complex, the most cognitively challenging, and the most essential to success in higher mathematics and science' (p. 23). One way to assist children in learning fractions is to employ different mathematical tools in the classroom (Cramer & Henry, 2002; Misquitta, 2011; Mendiburo, 2011; DeCastro, 2008; Mills, 2011; Cramer, Post, & del Mas, 2002). Extending Swan and Marshall's (2010) definition, mathematical tools are any *tool-like* objects that can be handled by an individual during which mathematical thinking is fostered. Objects – any foci of attention (Engestrom, 2009) – are inseparable parts of any mathematics classroom. They include tools such as an abacus, symbols such as x^2, and graphs. The use of *tool-like* objects refers to Marx's view of the use of *working tools*; where man uses the physical and mechanical properties of objects to reach his goals. Hence, an abacus is a tool-like object and x^2 is not, because children are able to use the physical and mechanical properties of an abacus to achieve a mathematical goal.

Although the use of mathematical tools is conceptualised as being useful in the learning of fractions, children encounter difficulties in grasping the relationship between mathematical tools and the mathematical meanings that they are intended to represent (Norman, 1993; McNeil & Uttal, 2009; Rabardel & Samurçay, 2001). The process within which the child grasps the interrelationship between mathematical tools and the meaning of a mathematical concept is a highly complex one.

McNeil and Uttal (1997) explained that any mathematical tool "can be thought of in two different ways: (a) as an object in its own right and (b) as a representation of something else" (p. 43). For example, if the relationship between the sizes of the pieces in fraction circles and the concept of the additions of fractions is not clear to a child, then he/she needs to learn not only the mathematical concept, but also the functionality of the fraction circles as a system and its relationship with the mathematical concept; in other words, he/she needs to learn two separate systems and the relationship between them.

In this paper, I look at the interrelationship between mathematical tools and the mathematical meanings represented by the tools, in the context of fractions learning. Within the mathematics education research, the relationship between the mathematical tools and the mathematical concepts has been studied under different theoretical framework, in particular Cultural Historical Activity Theory (CHAT) and Actor-Network Theory (ANT). CHAT providing a detailed theoretical lens to analyse the mediated actions of a subject, investigates the subjects activity in relation to a particular goal (Engestrom, 1999). ANT, on the other hand examines the association of human and non-human entities as nodes of a network (Fenwick & Edwards, 2010) nodes of the network. Both these theories, even though seemingly relevant, have limitations in the particular context of my study. My focus in this study is on how children, with their own perceptions, utilise the physical properties of tools to construct meaning for the mathematical concept, through solving a mathematical task. Neither CHAT nor ANT provides me with a lens to thoroughly examine the role of the physical properties of the tools in children's understanding of fractions. Therefore, to look at the physical properties of mathematical tools – as objects in their own right – I used Gibson's notion of *affordances*. And to look into children's mathematical perception as they interact with the tools (i.e., the interrelationship between the tools and mathematical meanings). I introduce the concept of *perception*, as viewed by Vygotsky in the *object/meaning* ratio.

GIBSON'S VIEW OF AFFORDANCES

Concerned with how the environment supports cognitive activity, Gibson (1977) contended, "in any interaction involving an agent with some other system, conditions that enable that interaction include some properties of the agent along with some properties of the other system" (Gibson, 1977, p. 72). Gibson's notion of *affordance* focuses on the contribution of the physical system to the cognitive activity. The term affordance refers to whatever it is about the environment that contributes to the kind of interaction that occurs. In relation to the learning of mathematics, the term affordances refer to whatever it is about a mathematical tool that contributes to the interaction of the child with these tools in the process of solving a mathematical task. I consequently refer to the term *perception* as whatever it is about the child's thinking that contributes to the interaction with the tools.

In a child's interaction with the tools, it is crucial to highlight where to locate the reference of the term affordance. For example, is the affordance that fraction circles provide for making half a unit or 1/2, a property of the fraction circles, a property of the child interacting with it, or properties of both? Fraction circles are a set of nine circles of different colours. Each circle is broken into different equal fractional parts, which use the same size as a whole. Gibson argued "affordance is a property of whatever the person interacts with, but to be in the category of properties, we call affordances, it has to be a property that interacts with a property of an agent in such a way that an activity can be supported" (p. 341). Hence, for the properties of a mathematical tool to be called its affordances, they need to be perceived by the child in such a way that a mathematical activity can be supported. For example, the physical properties of fraction circles that might assist a child to grasp a fractional concept are called their affordances *only* if the child perceives the interrelationship between the physical components of the fraction circles and the mathematical task, e.g., to solve 1/3 + 1/2. The physical properties of fraction circles might also be perceived as useful to build a bridge; these properties are called affordances if the task at hand is bridge making.

The above argument implies that interacting with an environment that provides an affordance for some activity does not entail that the activity will happen; the occurrence of the activity is intertwined with: the activity of the agent in that situation – that is his/her perception – and the task at hand. Assuming the task at hand is a particular mathematical task to be solved by the child, it is then the perception of the child that needs in-depth analysis; that is how the child perceives the tool and its affordances, in accordance with the mathematical task. Vygotsky's notion of Object/Meaning ratio offers a systematic approach to look at the gradual yet complex process of change in the child's perception as she/he interacts with the mathematical tools to make meanings for the tool as well as the mathematical concept.

VYGOTSKY'S VIEW OF PERCEPTION

A special feature of human perception is the perception of real objects (Vygotsky, 1976). The perception of the real objects involves the perception of not only colours and shapes but also of meaning; we do not see a round object with two hands, we see a 'clock'. The attachment of meaning to an object is a process that develops through the use of signs in interactions with tools (Vygotsky, 1978). Signs, such as language, drawings and the various systems of counting, are 'means of internal activity aimed at mastering oneself' (Vygotsky, 1978, p. 55). To better explain the process of attaching signs to the use of tool-like objects, I use Vygotsky's object/meaning ratios.

Vygotsky argued that at first the perception of a human being could be expressed figuratively as a ratio in which the numerator is the object and the meaning is denominator – object/meaning. This means that for a young child the object is dominant and the meaning of the object is subordinate. At this stage, the physical properties of things play an important role in a child's interaction with them. For instance, a stick can be a horse in child's play but a box of matches cannot be a horse. It is only later, when the child can make use of signs and symbols in her/his interaction with the objects, that the meaning becomes the central point and objects are moved from being dominant to being subordinate, thus giving rise to the meaning/object ratio. At this stage Vygotsky noted that, for example, to show a location of a horse on a map a child could put a box of matches down and say, 'This is a horse'. The perception of the child can now be expressed as a meaning/object. This figure of perception in which the meaning dominates is the result of tying signs to the tools; the box of matches is a symbol (sign) to represent the horse.

Vygotsky's object/meaning view has a two-fold theoretical implication for the study of children's perceptions as they interact with the mathematical tools to solve a task. On the one hand, it provides a base for analysing the gradual changes of the child's perception of the affordances of the tools as the child interact the tools to work on a mathematical activity. On the other hand, it provides a base for analysing the gradual changes of the child's perception of the interrelationship between the tools and the mathematical meaning they are intending to represent.

Provided that the child is interacting with a mathematical tool to solve a mathematical task, at the initial stage of the child's encounter with the mathematical tool, the child's perception can be presented by the object/meaning. This figure of perception applies to how the child perceives the affordances of the tool (i.e., the meaning of the tools as an object in relation to the task at hand) as well as how the child perceives the mathematical concept represented by the tool (i.e., the mathematical meaning). At this stage, the tool is dominant and its meaning(s) – as an object or its mathematical meaning – is subordinate. Hence, this is the stage that the physical properties of the mathematical tool play an important role in the child's interaction with them, both to perceive the affordances provided by the tool and to perceive the mathematical concept presented by the tool. For example, in fraction circles, the relationship between sizes and colours of the pieces plays an important role in how the child perceives the affordances of the fraction circles and the fractional concepts they are presenting.

In order to invert this ratio, that is, in order for a tool to be used as a symbol (a sign) for a mathematical concept, the child needs to increasingly tie signs to their use of the tool. Children do this by talking about what they do, talking about the tasks, drawing, and using mathematical symbols. It is in the gradual process of inverting the object/meaning ratio to a meaning/object ratio that children grasp the interrelationship between the affordances of mathematical tools and the meaning of the mathematical concepts that they are intending to represent.

In the following sections, I illustrate these ideas with two examples of children's interactions with mathematical tools as they attempted to solve addition of fractions problems. In the first example, N and J used their perception of the addition of fractions to make meaning of the affordances of a newly designed mathematical tool (i.e. the fraction board) in relation to the task. During the interviews, N and J used different mathematical tools, such as fraction strips and Cuisenaire Rods to solve different addition of fractions problems. My rationale to report their interaction with the fraction board is that this tool was designed by me, hence children had no previous encounters with the tools. Children's first interaction with the tools gave me an opportunity to examine how they used the mathematical perception of the addition of fractions to perceive the affordances of the fraction board. In the second example, Teresa used the affordances provided by a mathematical tool to make meaning of the addition of fractions. In both cases, however, the children's perception, both of the mathematical concept and of the tool, changed gradually and through tying signs to their interactions with the tools as they worked on the specific mathematical tasks. The reason for selecting these two pieces of data is to implicitly illustrate how the children's perceptual change of the affordance of the mathematical tools goes through similar gradual and complex process as children's gradual changes in perception of the mathematical meanings presented by the tool.

THE CASE OF J AND N: WORKING WITH THE FRACTION BOARD

J and N, in grade 5, participated in a small-scale research study in which they were asked to use a mathematical tool called the fraction board to solve 1/6 + 2/5. The fraction board is designed to help students with the addition of fractions. It contains fraction strips of half, third, fourth, fifth and sixth, a board which frames a fraction chart from one to 1/30, and a wooden roller which holds the fraction strips and moved up and down the board (Figure 1).

Both N and J has previously demonstrated an understanding of the addition of fractions through other tasks, but neither of them had had previous encounters with the fraction board. Hence, they initiated the task by attempting to grasp the mathematical affordances of the fraction board, as an object on its own right. With the initial help of the researcher (Y), N and J started to perceive the general physical properties of the tool, not in any particular relation to the task at hand. For example when Y began to introduce the physical components of the board both N and J quickly perceived the affordances of the strips by assigning a fractional amount to them:

Y. Okay I just very quickly tell you that this is a fraction board. These are called fraction strips and we have different kinds
J. This is 1/2 and this is 1/4 [*pointing to different fraction strips*]

At this stage, the physical properties of the tool played an important role in N and J's perception of the tool. For example, the ways in which different fraction strips are partitioned into different equal sized parts, assisted N and J to perceive their affordances.

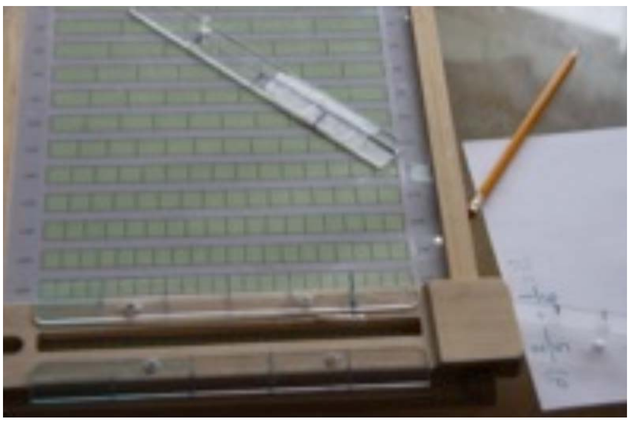


Figure 1

Through the gradual increase of the use of signs in their interaction with the tool, N and J's perception of the affordances of the strips gradually changed. This time in relation to the particular task at hand (i.e., to solve 1/6 + 2/5), they picked the strips of 6ths and 5ths as stated:

J. that is the 1/6 so I need one of that
N. yes and for that you need two (*pointing to the 1/5 fraction strips*).
[J colours the strips, one part on the 6ths and 2 parts on 5ths]

It was only later, that N and J were able to tie signs related to the mathematical meaning of the addition of fractions to their use of fraction board, perceive other affordances of the tool, and solve the mathematical task at hand. After selecting the two useful fraction strips of (6ths and 5ths). N and J started to perceive the "adding" affordances of the tool: "N. and I ...I...I guess they would go in here (pointing to the roller)". After loading the strips on the roller, N and J did not immediately perceived the affordance that the tool provided for finding the common denominator. So they used their perception of the mathematical concept of the need of a common denominator to perceive the affordances of the tool. They knew that to add 1/6 + 2/5, they needed to "turn the 5 and 6 into something". So they started to randomly moving the roller down the fraction board to see what number on the chart fit in both 1/6ths and 1/5ths. They unsuccessfully tried 12ths, 9ths and 18ths:

J. we can turn this (6ths) into two twelve's. *[they stopped at the 12ths line, noting 5ths did not fit]*
N. Lets try the nines
J. nop...
N. No definitely not
L. try 18 ... 18 going to 18?

After a few trial and errors they again used their perception of the concept of addition of fraction to conclude that the number that fits in both 5ths and 6ths is the thirtieths.

J. Oh... I get it, I can do it.
[...]
J. 1, 2, 3, 4, 5
N. five... five thirtieth
J. so we can trade this into five thirtieths

N. and then counting on the sixths... (*counting from the top of the roller...*), so 1, 2, 3, 4,5, 6,7 ,8, 9, 10, 11, 12... twelve thirtieth

J. so we can trade this (*pointing to the one sixth strip on the fraction board*) with 12/30. So we can change it into 17/30.

N and J's interaction with the fraction board is an example of how perceiving the mathematical affordances of a mathematical tool is a gradual process, which is intertwined with the mathematical task. Moreover, this example shows how N and J's object/meaning ratio (where they interacted with physical properties the fraction strips and the board to perceive the affordances of the fraction board) was inverted to a meaning/object ratio (where they used the affordances of the fraction board to solve the task).

THE CASE OF TERESA: WORKING WITH FRACTION KITS

This example is borrowed from Pirie and Kieren's (1989) study, in which Teresa used fraction kits to solve an addition of fraction problem. The reason for including this episode is that Pirie and Kieren's (1989) study was conducted over a period of time, which made it possible to look at Teresa's gradual changes in perception over a longer time span. Fraction Kits were designed by Tom Kieren; they contain rectangles, based on a common standard sheet as a unit, representing halves, thirds, fourths, sixths, eighths, twelfths, and twenty-fourths.

Teresa began the task of adding two fractions, not knowing what to do. She said 'I don't know' and 'I think you just add the tops and the bottoms' (Pirie & Kieren, 1989, p. 163). She was then given the fractions kits and a series of tasks. By perceiving the affordances provided by the fraction kit, 'she noticed that one fourth, three eights, and two sixteenths together exactly cover three fourths' (Pirie & Kieren, 1989, p. 163). Later, Teresa could 'add' 1/3 + 1/6 + 6/12 using the affordances of the kit. This is the stage in which the physical properties of the fraction kit, for example, the relationship between the sizes and colours of the pieces, played an important role in Teresa's interaction with the kit.

After a while, with a gradual change in her perception of the mathematical concept of the addition of fractions, she was able to tie signs to her interaction with the tool: 'You can do 2/3 + 5/6 because twelfths fit on both' (Pirie & Kieren, 1989, p. 167). Later, when asked 'What is 1/2 + 3/4 + 2/5 + 7/10?', Teresa, without using the kit, she said:

> Twentieths will fit on all of them. Two times ten makes twenty, so one times ten or ten twentieths. Four times five makes twenty so three times five is fifteen twentieths... (p. 169).

Teresa's gradual perceptual development for the addition of fractions, through the use of the fraction kit, made it possible for her to make statements like:

> Addition is easy. You can make up the right kind of fractions just by multiplying the denominators and then just get the right numerators by multiplying by the right amounts (p. 169).

This example shows how in the process of Teresa's interaction with the mathematical tool her object/meaning perception (where she used the fraction kit to solve the task) was inverted to become a meaning/object ratio (where she used signs and symbols to solve the task). In this process, Teresa increasingly used signs, while interacting with the fraction kits.

DISCUSSION

In this paper, I have employed Gibson's concept of affordances and Vygotsky's notion of object/meaning ratio to analyse children's interactions with two different mathematical tools. Teresa's case demonstrated a gradual perceptual change in the mathematical concept of the addition of fractions as she used a fraction kit to solve the addition of fractions tasks. J and N's case, by contrast, showed a gradual perceptual change in the affordances of the mathematical tool (i.e., the fraction board). These two examples show how children's perception of a mathematical tool and their perception of the mathematical meanings presented by the tool go through similar gradual and complex processes. Moreover, in both cases the children perceptions, of the tools and of the mathematical meanings, are highly intertwined with the children's attempt to solve the mathematical task. For example, in J and N's case, they would have perceived different affordances provided by the fraction board, had the task been to find the equivalent fractions of a particular fraction.

Perceiving the affordances of a mathematical tool is highly intertwined with perceiving the mathematical concepts there are intended to represent and with the task at hand. However, the main reason for contrasting the case of Teresa (implicit focus on changes in perception of mathematical meaning) with the case of N and J (implicit focus on changes in perception on the affordance of the tool) in this paper, is to illustrate significance of combining the Gibson's view of affordances and Vygotsky's view of perceptual changes.

Based on Gibson's view, the affordance of an object only become apparent in the ways in which the object is being used in a particular task. Based on this perspective the meaning of the object is interrelated with not only what the child is doing with the object, in relation to the task at hand, but also the mathematical meanings that may or may not be apparent to the child using the tool. Consequently, the Gibson's view of an object makes Vygotsky's notion perception in the object/meaning ratio more dynamic.

Moreover, the combination of Gibson's view of affordances with Vygotsky's perspective of object/meaning ratio may shed some light on how the children use the physical properties of the mathematical tools to perceive their affordance and their relationships to the mathematical task.

Further research is required to examine how the physical properties of mathematical tools play a role in children's interaction with them.

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Gaming the system: An opportunity to analyse difficulties in arithmetical problem solving

David Arnau[1], José Antonio González-Calero[2] and Miguel Arevalillo-Herráez[3]

1 University of Valencia, Department of Didactics of Mathematics, Spain, david.arnau@uv.es

2 University of Castilla-La Mancha, Department of Mathematics, Spain

3 University of Valencia, Department of Computer Science, Spain

Gaming the system is a behaviour that must be avoided when interactive learning environments are designed. However, from the point of view of the research in mathematics education, the observation of this behaviour may bring to light the students' knowledge. In this paper, we provide results of a study in which primary education students (10–11 years old), grouped in pairs, solved problems in an arithmetical way using an intelligent tutoring system. We analyse cases in which the students were able to refine a fuzzy idea of how to calculate a quantity based on the belief that they necessarily had to use certain quantities, operations or conceptual schemes. We also provide examples of how sometimes such students' certainties can become obstacles.

Keywords: Word problems, intelligent tutoring systems, gaming, primary education, arithmetic.

INTRODUCTION AND AIMS

One-to-one tutoring (one teacher for one student) produces better results than instruction in ordinary groups (Bloom, 1984). However, the high cost of this kind of teaching prevents a widespread use. One way to circumvent these difficulties is to use intelligent tutoring systems (ITSs) that reproduce some of the actions that are carried out by a human tutor. The learning of arithmetical word problem solving has been a field where the design of new ITSs has abounded. Some examples would be AnimalWatch (Beal, Arroyo, Cohen, & Woolf, 2010); HERON (Reusser, 1993); or MathCal (Chang, Sung, & Lin, 2006).

Gaming the system is an unwanted behaviour when ITSs are used in teaching situations. We use the definition of gaming provided by Baker and colleagues (2009): "attempting to succeed in an interactive learning environment by exploiting properties of the system rather than by learning the material (by systematically guessing or abusing hints)" (p. 475). To avoid this behaviour, some control elements should be introduced during the ITSs design. However, from the point of view of the research in mathematics education, the observation of systematic guessing in a computer environment can provide us with information about students' prior knowledge when they make a mistake and the degree of certainty associated with this knowledge (see examples about the importance of analysing these performances in Mavrikis, 2010; Shih, Koedinger, & Scheines, 2008).

In this report we briefly describe the operation of the ITS called Hypergraph Based Problem Solver (HBPS) when it is used to solve word problems in an arithmetical way. We put forward an analysis of the performances of primary education students (10–11 years old) when they solve problems arithmetically using HBPS. It is an exploratory study that is the first step of an ongoing research about the effect of gaming when teaching to solve word problems with an ITS. In particular, in this paper, our analysis focuses on the moments in which the students resort to gaming the system to advance into the resolution process when a verbal cue acts as a source of confusion. The aim of this report is to show that the analysis of this behaviour provides relevant information about students' knowledge and misconceptions.

THEORETICAL FRAMEWORK

The arithmetical solving of word problems implies the repeated execution of analytical-synthetic processes (Bogolyubov, 1972). During the analytical process, solvers try to link the unknown quantities with the known ones through conceptual schemes that typically arise from the described situations in the statement (Riley, Greeno, & Heller, 1983). For example, a problem in which the time spent by a person going from one point to another is mentioned will usually evoke typical conceptual schemes of distance-rate-time (Mayer, 1981) such as the distance covered by a person is obtained through multiplying the speed of this person by the time spent. In an arithmetical solution, conceptual schemes let solvers link one unknown quantity to other known ones, and therefore make possible directly calculating the unknown quantity. Obviously, when students solve word problems, they may find difficulties in matching up the conceptual scheme to the right operation. These difficulties are reflected in the construction of incorrect operations.

On the other hand, when solvers read a problem statement some key words, which act as verbal cues, can lead them to identify not only the conceptual scheme, but also the arithmetical operation necessary for finding the solution: e.g., altogether, lost, gained, etc. However, as Nesher and Teubal (1975) pointed out, these key words also might act as a source of confusion leading the solver to an incorrect mathematical operation. In this way, Lewis and Mayer (1987) called statements such as "Joe runs 6 miles a week. He runs 3 times as many miles a week as Ken does. How many miles does Ken run in 4 weeks?" (p. 366) inconsistent compare problems, because the solution requires division of two numbers but the structure *times as many as* is involved. Several studies have shown that students tend to generate more solution errors in word problems that contain key words that suggest the opposite operation to that which is required in the solution process (Hegarty, Mayer, & Green, 1992; Lewis & Mayer, 1987; Nesher & Teubal, 1975).

MATERIAL AND METHODS

Material

In this study we used the program HBPS (Arnau, Arevalillo-Herráez, Puig, & González-Calero, 2013; Arnau, Arevalillo-Herráez, & González-Calero, 2014; González-Calero, Arnau, Puig, & Arevalillo-Herráez, 2014), which is able to supervise both the arithmetical and algebraic ways of solving word problems [1]. In fact, HBPS was designed around the main actions that an expert human tutor would do when supervising a student at solving an arithmetical problem. In particular, HBPS can infer the line of reasoning that a student is following and provide him/her adapted feedback messages using natural language. In this sense, results from Arnau, Arevalillo-Herráez, and González-Calero (2014) back the idea that HBPS is able to provide conceptual support as human experts do. These capabilities enable the system to offer a human-like personalized tutoring. This feature may be especially relevant taken into account that one of the potentials of ITSs is to make individualized tutoring widely and inexpensively available (Woolf, 2009). For this study we employed a version of HBPS, in which *help on demand* were deactivated and error messages were reduced to a simple notification. To illustrate the operation of HBPS we use the problem *The plans*, whose statement and a brief analysis is offered below.

The plans: Ann and Mike earned 36,000 € for designing a bridge. Since they did not work the same amount of time it should be distributed so that Ann will receive five parts of what they earned and Mike, seven parts. How much money did Mike earn?

The problem may be reduced to the known quantities (total of money, M (36000); number of parts that corresponds to Ann, Pa (5); and number of parts that corresponds to Mike, Pm (7)) and to the unknown ones (total of number of parts, P; money that corresponds to one part, Mp; and money that Mike receives, Mm). These quantities can be linked by the relations: $P = Pm + Pa$, $Mp = M/P$ and $Mm = Mp \cdot Pm$ [2]. If a solver follows this solution path, the last operation, which is used to calculate the part of money that corresponds to Mike, would be multiplication. When students are introduced to the solving of proportional sharing problems, this may cause difficulties due to the fact that in equal sharing problems the determination of these quantities usually implies dividing. In addition, in these problems it is common to find the key word *part*, which is usually associated with the mathematical operation division.

In the version of HBPS used in this study, the students can see the statement of the problem and find the name of the quantities that must be employed in a drop-down menu. When they select a name of a known quantity, they must assign the appropriate numerical value to it (see screenshot on the left in Figure 1). When they select a name of an unknown quantity they must assign an arithmetic operation that HBPS calculates automatically. To assign this operation, the expressions are introduced by using a calculator-like component. It is composed of a button

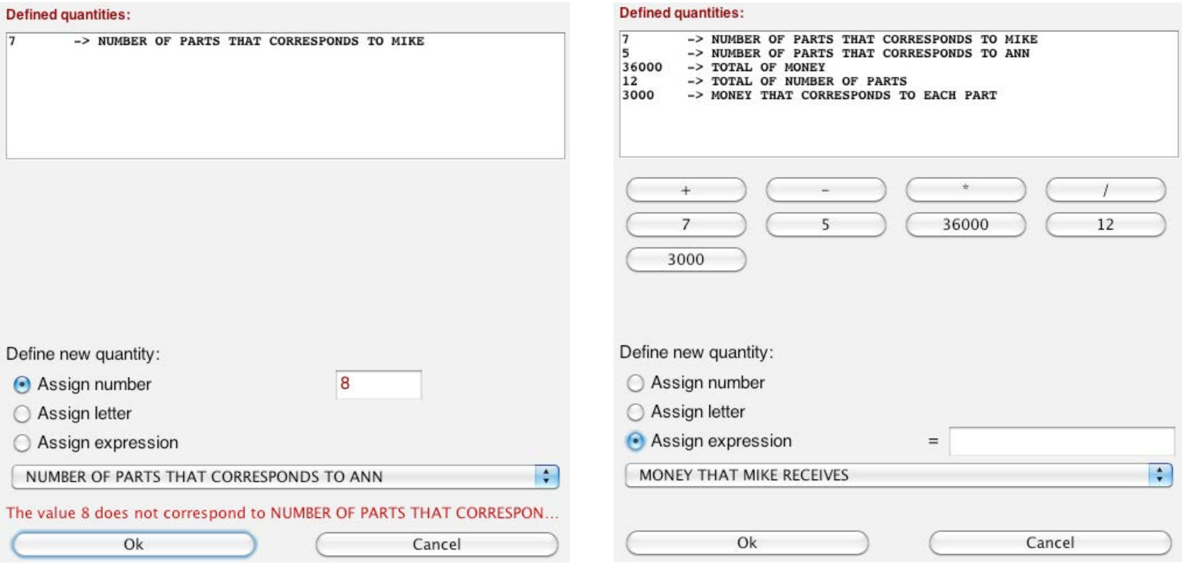

Figure 1: Error message (left) and construction of an arithmetical expression (right)

for each arithmetic operation, and another button for each different quantity that has been introduced into the system by the student (see screenshot on the right in Figure 1). In case of incorrect assignment of value or operation, this version of HBPS generates a simple error message by which the user is reported that last action is not correct but no additional information is offered (see screenshot on the left in Figure 1).

Empirical study design

The empirical study was carried out in a natural group of primary education composed by 20 students between 10 and 11 years old. As part of a wider study aimed at analysing how HBPS influences the students' proficiency in arithmetic word problem solving and that is not described in this report, the students had used HBPS to solve word problems in the classroom. Then, six students were chosen and they were grouped in pairs. The criterion adopted to select the students pursued to pair students who had shown difficulties in the same problems during the former phase of the study. Each pair was offered the problems that both members had previously failed to solve when they had been working individually in paper and pencil at the end of the instructional phase with HBPS. One of the problems the three pairs had to solve was *The plans*. They were asked to try to solve it using HBPS and they were encouraged to verbalise their actions. The sessions were video-recorded and afterwards the performances were transcribed to written language. As our investigation aimed to observe the process of problem solving, the researcher had a very low degree of intervention (Schoenfeld, 1985). The researcher just provided the sequence of problems and helped with technical problems that could appear. The analysis of these transcriptions makes possible to identify at each moment what relationship among quantities was established by the pairs and which mathematical operation was employed to link quantities in order to evaluate how students translate information from the statement into the mathematical language over the resolution process.

RESULTS

Next, we describe episodes of systematic guessing produced when the three pairs were solving *The plans* problem. All cases start from a situation in which the unknown quantities "total of number of parts" (12) and "money that corresponds to each part" (3000) have been already determined and the only remaining quantity to be determined is "money that Mike receives". At this point the determination of this quantity requires the use of the conceptual scheme "total money = money per part × number of parts". First we offer the sequence of operations that each pair introduced from that moment onwards in order to calculate the only unknown quantity.

Lola-Juan: (1) 36000 – 3000; (2) 36000/3000; (3) 36000/12; (4) 3000/12; (5) 3000·12; (6) 3000/7; (7) 3000 – 7; (8) 3000 – 12; (9) 3000·7.

Julia-Roberto: (1) 36000/7; (2) 3000/12; (3) 3000/7; (4) 3000/12; (5) 3000/7; (6) 3000/7; (7) 36000/12; (8) 3000/12.

Pablo-Helena: (1) 36000/3000; (2) 36000/7; (3) 3000/7; (4) 36000 – 3000; (5) 3000/12; (6) 3000·7.

The pair Lola-Juan

The pair Lola-Juan starts the sequence performing three operations in which the quantity "total of money" is involved. In the dialog that follows, there are evidences that they are initially convinced that that quantity must be employed to calculate "money that Mike receives". After rereading the statement, Juan says that the quantity that must be used is "money that corresponds to each part" (3000) (item 7, Lola-Juan). By contrast, Lola is still convinced of having to use "total of money" (36000) (item 6, Lola-Juan) and, in fact, she introduces 36000/12 (item 10, Lola-Juan).

1 Lola: Then...
2 Juan: 36000 divided by 3...? ... divided by 3000 (he introduces 36000 − 3000 and an error message is shown).
3 Juan: I think it is divided by.
4 Lola: So divide it. (Juan introduces 36000/3000 and an error message is shown.)
5 (It is heard how they reread the statement quietly.)
6 Lola: It certainly has to be 36000 (she introduces 36000).
7 Juan: No, it has to be 3000.
8 Lola: But then...
9 Juan: But then, divided by 12?
10 Lola: 36000 divided by 12 (Lola introduces 36000/12).
11 Juan: You have already done it and you got 3000! (He says so before the error message appears.)

From the fourth attempt on (item 14, Lola-Juan) the quantity "money that corresponds to each part" (3000) appears in all the operations they perform. However, as it is shown in the following dialogue, the erratic sequence of operations seems to point out that they are not so sure about which quantity will be the other one or which operation they will have to use.

12 Lola: Let's see... 3000 divided by 12.
13 Juan: Or 3000 times 12 or minus 12.
14 (Lola introduces 3000/12 and an error message is shown.)

During the rest of the resolution Lola seems to have no doubts about the necessity of using the quantity "number of parts that corresponds to Mike" (7). In actual fact, it seems like she were trying to make use of the scheme money = money by part × number of parts (item 15, Lola-Juan). However, she divides by 7 instead of multiplying by it. This could be a consequence of associating sharing with division. After introducing 3000/7, Lola deduces that a multiplication must be done, whereas Juan carries on trying operations with the only criterion that "money that corresponds to each part" (3000) must appear.

15 Lola: Wait. If he receives 7 parts: 3000 divided by 7.
16 (Juan introduces 3000/7 and an error message is shown.)
17 Lola: Or multiplying, it will be multiplying.
18 (Juan introduces 3000 − 7 and an error message is shown.)
19 (Juan introduces 3000 − 12 and an error message is shown.)
20 Lola: 3000 times 7 (she says so before Juan introduces the last expression).
21 Juan: That's true, minus 7 or minus 12 (Juan introduces 3000 − 12 and an error message is shown).
22 Lola: 3000 multiplied by 7.
23 Juan: Yes (he introduces 3000·7 and it is shown the message of problem solved correctly).

The pair Julia-Roberto

This pair opted to use division in all the attempts. This fact suggests that the pair erroneously considers a certainty, beyond any doubt, that the money an individual receives must be calculated using the mathematical operation division. Maybe the pair thinks that, given the fact that "the money Mike receives" is a part of "the total of money" and that there is a quantity "number of parts that corresponds to Mike", the problem could be interpreted as a partition division one. As it shown in the following excerpt, the solving process is conditioned by this belief, which causes the pair not to manage to solve the problem. The dialogue starts when they introduce the first expression of their sequence.

1 (Julia introduces 36000/7 and an error message is shown.)
2 Roberto: So then, that number (he points out 3000 on the screen) must be there for some reason.
3 Julia: Yes. This one divided by 12 (while she is saying this, she writes 3000/12).

4 Roberto: 3000 divided by 12?
5 Julia: I don't know (she introduces 3000/12 and an error message is shown).
6 Roberto: Divided by 7, I think so.
7 Julia: Yes.
8 Roberto: And if it isn't like that, nothing.
9 (Julia introduces 3000/7 and an error message is shown.)
10 (They seem to read the name of the quantities inaudibly.)
11 Julia: It is divided by 7.
[...]
12 Roberto: It would be money divided by the parts that Mike receives and it doesn't come to anything (while he is speaking, he introduces 3000/7 and an error message is shown).

The pair Pablo-Helena

After performing, at the request of Helena, the operation 3000/7 (third in their sequence) and obtaining an error message, they decide to read, in the window *Defined quantities* (see screenshot on the right in Figure 1), the name of the quantities and their values. Then, the following dialogue started in which it is evidenced how the Pablo's suggestion of performing a multiplication was the trigger for solving the problem (item 9, Pablo-Helena). Indeed, Helena immediately reacts to this idea by identifying the correct operation (item 10, Pablo-Helena). It should be underlined that when Pablo mentions the operation multiplication to Helena she straightaway turns her attention to the quantity "number of parts that corresponds to Mike".

1 Pablo: Ah, I think it is (he introduces 36000−3000 and an error message is shown)...!
2 Helena: There is nothing else, only this one (she says this while Pablo is opening the drop-down menu where the name of the only non-defined quantity appears).
3 Pablo: So we have to find it out, because we have already done a lot.
4 Helena: Let's see.
5 Pablo: And can't it be 3000 divided by 5?
6 Helena: And 3000 divided by 12?
7 Pablo: It may be.
8 (Pablo introduces 3000/12 and an error message is shown.)
9 Pablo: It may be multiplying.
10 Helena: By 7!!
11 Pablo: I'll try it then (he introduces 3000·7 and a message of problem solved correctly is shown).

DISCUSSION

All the three cases can be classified as gaming the system performances. However, this does not mean that all the students' actions correspond to trial and error efforts without any trace of reflection. In fact, what is observed is a repeated analytical-synthetic process by which students try to establish the relation that the quantity "money that Mike receives" has with the rest of quantities already determined. The process makes visible the students' conceptions about what quantities should be used to solve the problem. Additionally the protocols reveal how the pairs' certainties evolve. In particular, the three pairs turn their attention to the quantity "money that corresponds to each part" (3000) after having been working with the quantity "total of money" (36000) (for example, at first Lola stated that this quantity had to be used).

The pairs that solved the problem correctly posed a multiplication just after having proposed a division. However, the trigger for writing the correct operation wasn't the same. In the case of the pair Lola-Juan, Lola proposes multiplying after the system had reported an error when introducing the operation 3000/7, which had been proposed by her. This may be attributed to Lola's capacity of synthesising the correct operation once she had identified the conceptual structure that linked the quantities. In the case of the pair Pablo-Helena, Pablo merely suggested using a multiplication and it was Helena who proposed to multiply by "number of parts that corresponds to Mike" (7). Perhaps, Pablo's suggestion acted as a hint for Helena, since, up to that moment, she had seemed to be sure about the need of performing a division.

The pair that did not solve the problem correctly insisted that it was necessary to divide by "number of parts that corresponds to Mike" (7). In fact, they posed the operation 3000/7 up to three times. This suggests that they had identified part of the conceptual scheme that linked the quantities, but they were not able to make a synthesis correctly. Possibly, the fact that a share of the total money had to be done could induce them to consider that a division had to be performed.

In this case, the pair's certainty that a division was mandatory could become an insurmountable obstacle.

This report illustrates how students' conceptions play an important role in gaming the system behaviours. The analysed cases are far from being irrational trial and error sequences in which the pairs abuse of the system's capabilities without engaging with the problem. Although, even in the case of the pairs that finally solve the problem, more evidences apart from the excerpts are needed to state that learning took place, these pairs seems to have gain insight into the conceptual structure of the problem either by refining their previous certainties or by completing them. It should be recognised that the fact that some pairs managed to solve the problem can be favoured by the setting of the study in which the students worked collaboratively in pairs. However, according to the purpose of our work, we opted to use this setting because two-person protocols are appropriate for analysing decision-making behaviour instead of using one-person protocols that could give rise to the purest students' cognitions (Schoenfeld, 1985). Besides these considerations, the reported cases demonstrate that gaming behaviours can be an important source of information about students' knowledge and the difficulties associated with some problems. Often, this kind of performances is classified as mere trial and error attempts and little attention is paid to it. However, intelligent tutoring systems are now capable of logging and processing all the user's actions when he/she is solving a problem. In this context, future studies may address how this massive information could be employed in intelligent learning environments in order to scaffold students' learning.

ACKNOWLEDGEMENT

This work has been partly supported by the Spanish Ministry of Economy and Competitiveness through projects EDU2012-35638 and TIN2011-29221-C03-02.

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ENDNOTES

1. We consider that a solving process will be arithmetical when the solver calculates all unknown quantities going from the known to the unknown quantities. On the other hand, a solution will be algebraic when the solver uses one or more letters to represent unknown quantities, which will result in one or more equations.

2. Although relations have been represented by using algebraic language, we are not asserting that the relations are represented in this way on the mental level. We have just offered a schematic representation of the net of quantities and relations that can be taken from the statement. To do so we have represented the relations by algebraic language although this does not mean anything about the algebraic or arithmetical character of the solving process.

Instances of mathematical thinking through collaborative gameplay

Antri Avraamidou

University of Leeds, School of Education, Leeds, UK, antri.av@gmail.com

This paper examines the computer gameplay of two 12 year old children. In the analysis of gameplay in building a house without budget constraints and then a second house with budget constraints within "The Sims 3" video game in out-of-school settings, four variables appeared: Budget; Size of the House; Cost of furniture/devices; and Sims mood meter. This paper argues that those four variables influenced the mathematical relationships in building the two houses and the ways players' used the game's artefacts in line with their everyday and mathematical prior understandings affected their gameplay and therefore the exploration of mathematical relationships that form instances of players' mathematical thinking.

Keywords: Gameplay, mathematical thinking, out-of-school settings.

INTRODUCTION

The aim of this paper is to present instances of mathematical thinking as it occurred during the gameplay of a group of two 12-year old children who were, collaboratively, building two houses in "The Sims 3" video game, in out-of-school settings. The building of the first house had no budget constraints and thus the gameplay was less constrained than it was in the building of the second house with budget constraints. Much of this paper argues that four variables that occurred during gameplay, influenced the mathematical relationships in building those 2 houses. These variables were *Budget*, *Size of house*, *Cost of furniture/devices* and *Sims mood meter*. The paper is structured as follows: an integrated review of literature and presentation of the theoretical framework; the setting and methodology of the study; results pertinent to the focus of this paper; a discussion of issues arising.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Everyday experience and daily interaction with the world are perceived as starting points of mathematics (Bonotto, 2005). European Commission recommends mathematical competence as one of the eight key competences a citizen should develop for lifelong learning by being able to "develop and apply mathematical thinking in order to solve a range of problems in everyday situations" (European Commission, 2007, p. 6). Mathematical thinking is often described as a process and Stylianides (2009, p. 258), for example, refers to mathematical thinking as a process that includes: "exploring mathematical relationships to identify and arrange significant facts into meaningful patterns" that can be later used in order to make conjectures. Hoyles (2003, p. 4) posits that these mathematical relationships may be captured within the tools of an environment "but these relationships lie dormant until they are mobilised, and it is in their mobilisation that meanings are created". Several researchers have explored the mathematics used in everyday situations: workplace settings (Magajna & Monaghan, 2003); in the streets (Nunes, Schliemann, & Carraher, 1993; Saxe, 1991); and in supermarket shopping (Lave, 1988). They argue that the mathematics used in these settings is not formal mathematics but, rather, 'street mathematics', 'supermarket mathematics', and so on. Given that a quite common activity that children do in their everyday life is playing video games (ESA, 2012), the question arises: What sort of mathematics (if any) might emerge, is developed and is applied during gameplay?

Devlin (2011, p. ix) argues that "well-designed video games are going to play a major role in school mathematics education in the future". However, Bragg (2006, p. 233) argues that "assumptions that students will see the usefulness of mathematics games in classroom are problematic" and Bourgonjon, Valcke, Soetaert and Schellens (2010) argued that not all students appreciate the use of games, for educational purposes, in the classroom. That might be because, in most 'computer off the shelf (COTS) games' context and gameplay are inextricably linked, whereas in most educational games, the educational content and context often demote the gameplay (Reiber, 1996). During the past decade, several researchers have turned their interest to the way COTS games might enhance players' learning. For example, Squire (2006) illustrated the way the game *Civilization* helped students learn history in afternoon activities at school. Additionally, Gee and Hayes (2010), after exploring the Sims series video game, argued that building in such virtual worlds requires geometry knowledge and skills. The educational potentials of COTS video games exist, but how does mathematics enter gameplay in out-of-school settings?

Players' actions are central to the construction of their meanings and computers can provide a window making players' meanings visible (Noss & Hoyles, 1996). Within such actions, the construction of new knowledge is linked to the negotiation and consolidation of previous knowledge as in constructionism (Papert, 1993). In this research, actions are considered to be directed by players' goals as they are playing the game (Leont'ev, 1978). 'Goals' are mainly interpreted in two ways in literature related to mathematics' research; as the goal that directs an action of an individual which is part of an object-oriented activity (i.e., Leont'ev, 1978) and as 'emergent goals', essential 'deeds' that need to be done and are emerged during the activity (i.e., Saxe, 1991). During an activity, an individual consciously performs goal-directed actions (Leont'ev, 1978). However, in practice-related situations, there are instances where plans need to be changed and other goals emerge. These emergent goals are very important for understanding the activity because they emerge as a result of the interaction of individuals with their material world, through practice (Saxe, 1991).

In his research regarding candy sellers' real-life activities in Brazil, Saxe (1991) developed a research framework, consisting of three components in order to investigate the strategies and techniques that children used in the practice of selling candies in the streets. Within such a framework, one is able to analyze the interplay of culture and individuals' goals (and cognitive process) as it occurs within a practice (Saxe, 1991). Saxe's (1991) emergent goals component consists of four parameters: *social interaction, activity structures, prior understandings* and *conventions/artefacts*. This model suggests that candy sellers' goals that emerge during their activities in their situated practice, are shaped and affected by their *social interaction* with each other or with other individuals involved in the practice (i.e., buyers), by their *prior understandings* that they bring in the practice (i.e., existing knowledge), by *conventions and artefacts* related to the activity (such as money and currency conventions) and lastly, by the whole *activity structure*. A previous study that examined the gameplay of an 11-year old boy, who was building houses in "The Sims 2" video game (the precursor of "The Sims 3") using Saxe's model to analyze the goals that emerged during gameplay, argues that the creation and use of an artefact-strategy that the player had developed during gameplay was a mathematical abstraction (Avraamidou, Monaghan, & Walker, 2012).

WHAT IS "THE SIMS 3"?

"The Sims" series (http://www.thesims3.com/) are popular real life simulation games that allow players to control the lives and relationships of game characters and create houses for them, in an open-ended basis. Unlike many popular games, "The Sims" series do not have an explicit goal/objective that the players need to accomplish and players do not 'typically' compete with each other (or with a computer) in order to win. In fact, players can decide on the way they want to play the game. There are three game modes when playing The Sims (initial game): the Live Mode; the Buy Mode; and the Build Mode. While playing the Live Mode of the Sims, players can control the Sims characters during their virtual living in the Sims houses and observe their lives, maintaining (or not!) a balance between each Sims character's mood meter by fulfilling their desires or making their fears come true, while the Sims virtual time passes by. Players can pause the virtual time of the game and edit and/or delete constructions of the Sims virtual houses and neighbourhoods and also, build from scratch virtual houses for the Sims in the Build Mode. Lastly, players can decorate and furnish their Sims' houses while in the Buy Mode of the game, by selecting items that appear in the menu. All items in the Build and Buy mode have prices in the game's currency 'Simolins' indicated with the letter 's'. An important aspect of the Sims series is the budget. The Sims families have specific budgets and when players need to build, edit, delete and/or buy items for a family, they need to do so within the budget of that specific family. When the budget is exceeded, players need to 'sell' or delete items so as to increase the amount. When playing the Live Mode of the game, players can increase the family's budget by finding jobs for unemployed Sims or making sure that employed Sims go to work and earn money.

RESEARCH DESIGN

This paper reports on an exploratory case study (Yin, 2003) of the gameplay of two 12-year olds, Marios and Christina (M & C), who were selected on the following criteria: a. they were friends that usually met to play together; b. they had not played "The Sims" series before; c. both wanted to play the specific game; and d. both participants and their parents gave their consent to the research. The researcher approached the parents of one participant, informed them and when both parents and participant gave their consents, the participant was asked to name one of his/her friend that they wanted to play the game with. Then participant's friend and friend's parents were also informed and gave their consent. Participants were informed that they were being recorded for research's purposes but were not informed that the research was related to mathematics. M & C played the game in Christina's bedroom, using the researcher's laptop. They collaboratively built three houses: a. built a house without budget constraints, b. built a house for a Sims family with a specific budget and c. reshaped an existing house of the game following a scenario given by the researcher. Participants' talk and on-screen activity were recorded using screen recording software (Ambrosia) with audio recording, in order to ensure a rich data collection of approximately 5 hours of recorded gameplay. The researcher was present during their gameplay and acted as observer participant but her role was restricted to only assist them with any technical difficulties. Overall, three meetings took place, where M & C built and edited the three houses but only the first two houses will be used for the purposes of this paper.

Data were transcribed in Greek (both talk and descriptions of their actions) data analysis was conducted in three stages. The first stage produced open codes *a la* Strauss and Corbin (1998), describing participants' actions during gameplay that were later grouped into categories. During the second stage of analysis, players' actions and goals that directed those actions were identified. Whilst conducting the second stage of analysis, a pattern was observed: there were goals that emerged whilst actions were performed in order to achieve a previous goal. Thus, a third stage of analysis was conducted, where emergent goals were identified and players' actions were grouped into episodes consisting of related goals and emergent goals.

RESULTS

The selected results that follow are structured in a way to support the discussion section and the focus of this paper regarding the way the four variables (Budget, Size of the house, Cost of furniture/devices and Sims mood meter) influenced the mathematical relationships in building the two houses.

The four variables were initially identified through the first stage of analysis that produced 13 categories that are illustrated in Table 1 below. The four variables arose from the analysis as overlapping sets whose elements are the categories.

Overall, 206 goals were identified in the second and third stages of analysis (144 in House 1 and 62 in House

Variable	Related categories
V1: Budget	tensions/disagreements, feelings when referring to budget issues, ways to spend less / save money, interaction with the game's features (errors), experiencing the reality of the game
V2: Size of the house	tensions/disagreements, referring to previous house, experiencing the reality of the game, appearance considerations, ways to spend less / save money
V3: Cost of furniture	tensions/disagreements, feelings when referring to budget issues, ways to spend less / save money, referring to previous house, appearance considerations, ways to spend less / save money
V4: Sims Mood meter	tensions/disagreements, experiencing the reality of the game

Table 1: Related categories of first stage of analysis and the four variables

House 1 ground floor – kitchen, living room, bathroom

House 2 – kitchen, living room, bedroom, bathroom

Figure 1: House 1 and House 2 (almost) final versions

2) of which a total of 60 were emergent goals. Four of those emergent goals are illustrated next in the four variables-related results.

During the construction of House 1 there were in fact only two variables enabled; Cost and Size. Budget and Sims Mood meter variables were not affecting players' actions because there was no family and therefore no budget restrictions. As soon as players selected a family to build House 2, the Budget and Sims Mood meter variables were enabled. This affected the other two variables, Size and Cost. The relationship of the four variables will be discussed in the Discussion section. First, a description of the variables is needed.

Variable 1: The budget

The 1st house that M & C built (House 1), was a house without any budget constraints. They created a two-storey house that was valued by the game as 224.023s including furniture and devices. When M & C selected the Williams family (husband, wife and a baby) to build a house for (House 2) the family's available budget of 17.491s appeared on the gameplay screen and as soon as they added or deleted something the available budget increased or decreased respectively. When the available budget was almost used, the game excluded the unavailable options of the Buy mode menu that the family could not afford with a red colour. When M & C wanted to furnish the bathroom (Goal: *To furnish the bathroom*), they saw that they only had money (127s) to buy a baby's toilet seat because it was the only available option in the menu.

Christina: It's red! Everything is red! (non-affordable)
Marios: Where are they going to go when they will want to go to the toilet?

A new, emergent goal then occurred: *To increase the available budget.* They needed to achieve this emergent goal in order to be able to achieve the initial goal of furnishing the bathroom. They decided to move the bedroom furniture in the living room and delete the parents' and baby's bedrooms. The family's available budget increased to 5.009s and they created a smaller room which they decided to be a room for the parents and the baby. They then furnished the bathroom and the rest of the house.

Variable 2: The size of the house

House 1 was a relatively large house that cost 86.388s unfurnished (including foundations, walls, tiles, windows, doors and lights). When they started building House 2 with a budget that was ≈1/10th of the first house's total value, they stated that they would do and maintain a smaller house by saying *"We will do one floor, because it is a small family, that's OK"*. Indeed, they did eventually create a smaller-sized house, having only a ground floor instead of two which cost 15.508s, unfurnished (see Figure 1).

M & C reshaped House 2 several times by deleting rooms in order to make the house smaller and increase the budget as described earlier. Category 'space-size-arrangement issues' showed that they had issues with the size of House 1 as well. While furnishing the kitchen of House 1, Christina stated *"What are we going to do with all this space?"*. The initial table they added was the largest table in the menu but they said that it was too small for their kitchen. Then another goal emerged: *To reduce 'all that' space of the kitchen.* They talked about reducing the size of the kitchen but preferred to add two more tables and ten chairs around them to make the table look bigger instead.

Variable 3: The cost of the house including furniture/devices

The total cost of House 1 was 224.023s in which 137.635s was the cost of furniture and devices, whereas in House 2 it was 17.491s including a cost of 1.983s for the furniture and devices. Indeed, House 1 was filled with a, sometimes unreasonable, number of luxurious items that were mostly chosen according to appearance criteria (see Figure 1). In fact, the category 'appearance considerations' most frequently appeared in House 1. In contrary, House 2 was equipped with essential furniture and devices bearing in mind the family's budget. The Cost variable was noted by the players when they started adding furniture in House 2 and more specifically when the budget was decreased significantly. Table 2 below illustrates selected quotes of players' talk during the building process of House 1 and House 2. Each row maps quotes of talk while performing similar actions in each House.

M & C's talk changed when adding the same items in House 2, as it included expressions showing that they compared the cost of the items and chose the ones that were not expensive bearing in mind the budget.

Variable 4: The Sims mood meter

M & C stated that they were done with House 2 and they were happy leaving the family with a budget of 861s. They then switched to the Live mode of the game to play with the family. While exploring the mood meter of their Sims they noticed that their Sims did not have a device to cook (emergent goal: *To provide food for the family*) and they added an oven/hob in the kitchen. They then noticed that they were not happy with an "unfinished" room looking at the mood meter. Thus a new goal was emerged: *To make the Sims happier*. They added the cheapest available wallpaper that they saw (they did not notice that there were free wallpapers in the menu) and added the missing tiles in the kitchen in order to get their Sims mood meter higher and thus, their Sims happier, reducing the budget to only 9s.

DISCUSSION

In this section, I look at the relationships of the four variables Size, Cost, Budget and Sims mood meter and the way they influenced mathematical relationships in the two houses, with particular regard to literature on mathematical thinking and Saxe's emergent goals model.

Going back to Stylianides' (2009) description of mathematical thinking as a process of exploring relationships, M & C recognized that there were relationships between the four variables. They figured out that increasing either the Size or the Cost of the House resulted in decreasing the available budget. Furthermore, they recognized that in order to increase the available Budget, they needed to decrease either the Size or the Cost of the House. For example, when M & C needed to increase the budget of the family in House 2 in order to continue building the house, they either decreased the Size of the house or selected furniture with low Cost or did both. It is not claimed here that M & C were fully aware of the exact interrelation of those four variables. However, stating that they would do *"a smaller house because it is a small family"*, reducing the size of the house to increase the budget and selecting the cheapest available options indicate that they recognized the way each variable affected the other.

The mathematical relationships that were influenced by the four variables were 'invisible' as they were *"lying dormant"* (Hoyles, 2003, p. 4) within the artefacts of The Sims 3 game. However, those relationships became 'visible' during the gameplay, through the way M & C used the game's artefacts. For example, the Sims

House 1	House 2
"Add more windows here to make the room brighter"	"No, we will not add any more windows, look (pointing at the budget).
"This wallpaper looks nice, add this one (worth 10s per column)"	"No, that's 10 (wallpaper worth 10s per column), that's expensive. That's 4, that's good".
"No, choose this one or this one" (the two most expensive available TVs)	"Not that one (the more expensive TV), that one (the cheapest), it doesn't need to be nice-looking"
"It was 100.000s before and now we added so many expensive things and is now only 200.000s? That's fine" (third meeting)	"No no, that's expensive. We can't"

Table 2: Players' talk while building House 1 and House 2

mood meter and the Budget are two important artefacts of the game. When M & C chose to play with the family in the Live mode, the meter was enabled. When M & C needed to increase the mother's Mood meter, by adding devices in House 2, they realised that in order to do so, they had to spend more money from the family's available Budget and therefore increase the Cost of House 2 so as to increase the level of the meter by providing the family with a device to eat. Even though their initial plan was to leave the family with an available Budget of 861s, they chose to spend that budget, in order to make their Sims happier.

The interpretation of the Budget, the Sims Mood meter and the Sims' needs in House 2 was in fact shaped by M & C's everyday knowledge of a house's structure and content and also of real life needs. For example, Marios' statement: *Where are they going to go when they will want to go to the toilet?* was enough for them to start deleting parts of House 2 and rearrange the furniture and rooms. It is important in real life to have a bathroom/toilet in a house and within the Sims game, they recognized that deleting items resulted in getting a refund and therefore increasing the Budget. Going back to Saxe's (1991) emergent goal model, the goal of *'increasing the budget'* emerged and was achieved as a result of M & C's prior everyday and mathematical understandings during their interaction with the game's artefacts. Examining the other three emergent goals that were presented earlier (i. to reduce 'all that' space of the kitchen, ii. to provide food for the family and iii. to make the Sims happier), the way M & C used the artefacts of the game's features, including the available Budget, the Sims Mood meter and artefacts, in line with their prior understandings in order to achieve those emergent goals, influenced their gameplay. In addition, the four variables that occurred during gameplay, influenced the mathematical relationships that were hidden within those artefacts. In fact, this constrained gameplay of M & C, was an "activity with relationships" (Noss & Hoyles, 1996, p. 124) that M & C explored during their gameplay.

This paper argued that there were instances of players' mathematical thinking during their collaborative gameplay of The Sims 3 video game in out-of-school settings. Those instances are not school-alike formal mathematical processes. Rather, the mathematical thinking occurred informally but meaningfully for the players in order to achieve their goals during gameplay. These instances were captured by players' exploration of the mathematical relationships, hidden within the artefacts of the game. Those relationships were influenced by the four variables that emerged during the gameplay. The use of those artefacts, which was highly affected and shaped by players' everyday and mathematical prior understandings, was crucial for the gameplay.

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The use of hands and manipulation touchscreen in high school geometry classes

Marcelo Bairral[1] and Ferdinando Arzarello[2]

1 UFRRJ, Seropédica Brazil, mbairral@ufrrj.br
2 Università di Torino, Torino, Italy, ferdinando.arzarello@unito.it

Touchscreen dynamic environments user interfaces employ a specialized interaction model on screen. In this paper, we analyse students' manipulation to explore and justify their geometrical reasoning on a free online touch device: the Geometric Constructer (GC) software. We discuss data from a teaching experiment with Italian High School students. The experiment was videotaped. Based on this we observe two domains (constructive and relational) regarding the development of geometrical thinking on GC. Students' manipulation on constructive domain is basically done to make construction and it contributes to exploration and to arise conjecturing. Indeed, manipulation in relational domain can suitably support and improve students' justifying and proving performances.

Keywords: Touchscreen device, GC software, constructive domain, relational domain, dragging to approach.

INTRODUCTION

The emergence of multi-touch devices - such as iPods, iPhones and iPads - will promote new impact and challenges in learning and instruction in general, and in mathematics in particular. Although in Mathematics Education some touch devices have been developed (for instance, Geometer Sketchpad Explorer, Geometric Constructer, GeoGebra app, Sketchometry and Math Tappers apps) research is still scarce concerning mathematical learning through touchscreen manipulation.

In our current research project[1] we are interested in the way of manipulation of tablet resources as iPad. Particularly, how ways of touchscreen manipulation can improve students' geometrical thinking. In this paper we are addressing issues regarding the question: during the process of solving geometric problems using the software GC which domain (constructive or relational) of manipulation touchscreen could be fruitful to improve student's strategies for justifying and proving? We assume (i) that manipulation on tablet is different from a mouse click and (ii) that mathematics used by students to solve a geometrical task in a paper-and-pencil environment is different from what they use in a touchscreen device.

Gesture and touchscreen manipulation

The role of gesture, particularly the touchscreen, in supporting mathematical reasoning in technological context is an emerging field of research in mathematics education (Arzarello et al., 2013; Nicholas, 2013). Regarding their usage, environment mobile touchscreen user interfaces employ a specialized interaction model.

Interaction through current mobile touchscreens basically occurs with the computer recognizing and tracking the location of the user's input within the display area. In other words, interactivity occurs in response to two dimensions of the input action (Yook, 2009; Park, 2011). This enables six basic finger actions for input: tap, double tap, long tap (hold), drag, flick, and multi-touch (rotate). According to Sinclair and Pimm (2014), these types of manipulations "describe specific configurations and actions of the finger(s) on the screen and they are different from those discussed in the mathematics education literature in two ways: they involve contact with a screen and they perform an action" (p. 210).

Even though we are not looking only for ways of touch that represent mathematical concepts (for instance, rotation) we agree with Boncoddo and colleagues (2013) that a particular way of manipulation may

1 In Brazil, the research project is granted by Capes (Ministry of Education).

serve as an important function of grounding mathematical ideas in bodily form and they may also communicate spatial and relational concepts. Specifically for geometrical thinking, inspired in (Hostetter & Alibali, 2008), we consider important to stress that, in touchscreen devices, manipulations are based on visuospatial images, linguistic factors influence gestures and ways of touchscreen are communicatively intended.

Adopting an embodied cognition perspective in our research we highlight reciprocal connections between ways of touchscreen and cognition. Contrary to what happens in clicking, manipulating touchscreen interface implies a continuity of action, the spatiality of the screen, the movement simultaneousness and movement combination and, depending on the resource device, the feedback speed. On the following Figure, we observe one student trying to explain one of the properties of the isosceles trapezoid. He uses hands to represent the sides that are not parallel.

Touchscreen manipulation in dynamic geometric devices

Inspired in (Tang et al., 2010) we assume that touchscreen manipulation is not the same as mouse clicks (Arzarello et al., 2014; Bairral et al., 2015). There are differences if we use a usual PC, where dragging is produced with the help of a mouse, or we use a touchscreen of a tablet, where we can use our fingers to move the figures, and there are differences if we can use more than one finger (as in multi-touch environments) or only one finger.

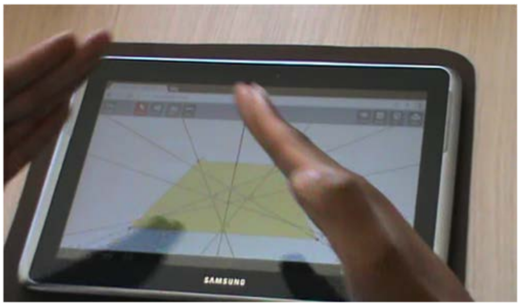


Figure 1: Student construction on GC

As we have had a first shift and improving passing from paper and pencil environments to DGS with drag and drop activities (e.g., Cabri Géomètre, Sketchpad, etc.), now we have a further shift and improvement with the transition to multi-touch environments (e.g., Geometric Constructor, SketchPad Explorer, Sketchometry) and to the variety of simultaneous fingers' actions they allow.

When we manipulate the screens in our device with touchscreen interface, we perform a set of movements. Some manipulations we perform have specific mathematics cognition, like when we want to enlarge (or reduce the size of) a picture in some image editor (Paintbrush), or when we perform this through a touchscreen manipulation.

On such occasions we "pull" the image diagonally, upwards or downwards, or we "click" on one of its vertices, so both dimensions (width and height) are reduced or enlarged proportionately. If we do not perform this type of movement, i.e., if we manipulate only one dimension, the image will come out deformed.

Nevertheless, although all these manipulations are based on one mathematical concept (method of the diagonal as a way to generate similar figures), they are not necessarily the same in cognitive terms (the action of enlarging without deforming), epistemological (the simultaneous changing in different parts of the shape) and spatiality (work and manipulating area on the screen). By the way, we still have to go further on these singularities.

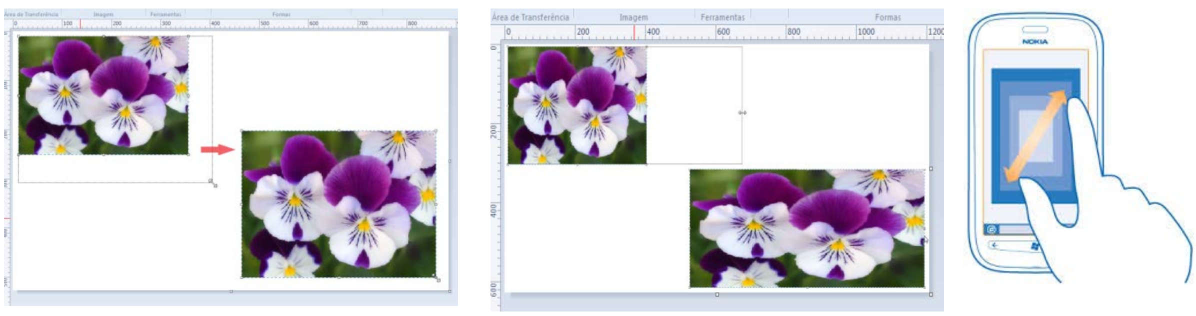


Figure 2: (a) Illustration of an enlargement in a drawing program; (b) distortion in a drawing program; (c) enlargement through sliding on the screen (Bairral et al., 2015)

Domains of manipulation and geometric learning in dynamic touchscreen devices

Touchscreen dynamic environments user interfaces employ a specialized interaction model on screen. In agreement with Arzarello and colleagues (2002), within this type of environment the interaction also concerns deeply perceptual aspects, which involve not only the objects (e.g., drawings) but also the physical perceptions of students, their motions, gestures, languages, etc. and the artefacts that they use as mediating instruments. Perceptual aspects which must be analysed concern many components, i.e. visual phenomena, motion, kinaesthesia, inner time(s); on the other hand, the most typical theoretical features are the structured mathematical objects, their invariant properties, conjectures, theorems, proofs.

Even though in the relational domain students also construct geometric objects we observed (Arzarello et al., 2014) that it is in this particular domain where they show more interacting and reflecting about the construction.

According to Arzarello and colleagues (2014), a cognitive process within a GC device could be seen in two interrelated domains of manipulation: the construction domain, where students basically refer either to tap and hold, which are the basic actions, or to isolated ways of constructing geometric objects (point, line, circle, shape, etc.) with a touch interface. What we call relational domain is a combination of this constructional and the performed touchscreen actions, which include drag, flick, free or rotate.

While in a construction domain student act as discrete observation (focused on some specific construction or constructed object or even doing some touch on the screen) in the relational domain their manipulation seemed more focused on their questioning, conceptual understanding and other emergent demands concerning their manipulation as a whole construction.

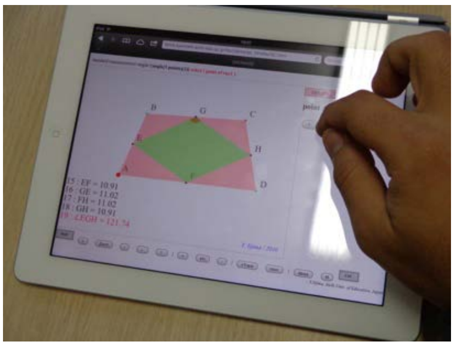


Figure 3: Student construction on GC

METHODOLOGICAL ASPECTS OF THE STUDY

We are conducting teaching experiments (TE) with High School (Brazilian and Italian) students and Brazilian prospective mathematics teachers. In this paper, we discuss data from one TE: five High School students (16–17 years old) at Liceo Volta (Turin, Italy) working on software Geometric Constructer (GC). All of them had previous experience with dynamic geometric environments (DGE). Each session took about two hours long and it was videotaped. In each session the students worked out on proposed tasks.

Geometric constructer features

The choice of GC software is because, as far as we know, it is the only software which incorporates all the potentialities of usual DGE in a fully touch-screen device. By 'potentialities' we mean two main facts (Arzarello et al., 2014): (i) the possibilities of using more than one digit (multi-touch) on the screen to interact with the software and (ii) the possibility of making constructions and not only explorations. As far as we know, at the moment there are very few types of mathematical software that satisfy both these features.

Some of the haptic devices on the market (for instance, GeoGebra app and FreeGeo) satisfy (ii) but not (i): in fact, they allow users to move only one point each time, which makes them very similar to environments where dragging is done with the mouse. A very few, for example Sketch-explorer, satisfy (i) but not (ii). GC satisfies both[2]. Using GC we may construct basic geometrical objects (points, segments, lines, circles), measure them, drag and make traces of geometrical objects and so on. The Student using different colors to edit the construction and measuring internal angles from the quadrilateral EGHF for the Varignon theorem task[3].

2 It has been designed by Professor Iijima Yasuyuki (Aichi University of Education, Japan2) and we used its version in English.

3 The Varignon Theorem proposed task: In quadrilateral *ABCD*, the middle points (*E*, *F*, *G* and *H*) on each side have been drawn, forming quadrilateral *EFGH*. What characteristics does *EFGH* have? What happens if *ABCD* is a rectangle? What if it is a square? What if it is any quadrilateral? Demonstrate.

The proposed and analyzed task: Constructing square[4]

Build a quadrilateral ABCD. On every one of its sides build a square external to the quadrilateral with one side coinciding with the side of the quadrilateral. Consider the centers of the squares that have been built: R, S, T, U. Consider the quadrilateral RSTU: what can you observe? What commands do you use in order to verify your conjecture?

Data analysis

Due to continuity of motion and spatiality on the screen we consider that with touchscreen devices analysis should be about paths of interaction rather than points of interaction. Further, it would be mathematically inappropriate (in most cases) to reduce data of a trace to a single point, as we observe in device without touch action. The analytical process was done in two main steps: (1) identification of each type of manipulation (Arzarello et al., 2014; Park et al., 2011; Yook, 2009) and (2) construction of timeline to gain information of the global cognitive movement throughout interaction on GC software. Based on videotaping the timeline illustrates the ways of touchscreen and shows geometric aspects from students' interaction on the GC software (Arzarello et al., 2014, p. 47). For the first step we adopted Yooks' (2009) framework as summarized in the following Chart 1.

RESULTS

In the following two Charts (2a and 2b), we show part of a timeline elaborated by students' solving the task with the software GC performing four types of basic actions[5] (tap single, scale, hold single and hold multi).

Although in order to make a construction (point, line, angle, circle etc.) the user has to use the software icons, we observed all the manipulation on the screen. We didn't consider touch on the icon as an example, for instance, of the tap or hold touchscreen. Rather, in some interval of time we could observe more than

Action		Type	Motion
Basic	Refers to tap and hold which are the basic ways of interacting with a touch interface.	Tap (single) Tap (double) Hold (single) Hold (multi)	Closed
Active[1]	It is a combination of the basic action and the performed finger action, which includes drag, flick, free, or rotate.	Drag Flick Free Rotate	Open

Chart 1: Yook framework quoted by Park (2011, p. 23)

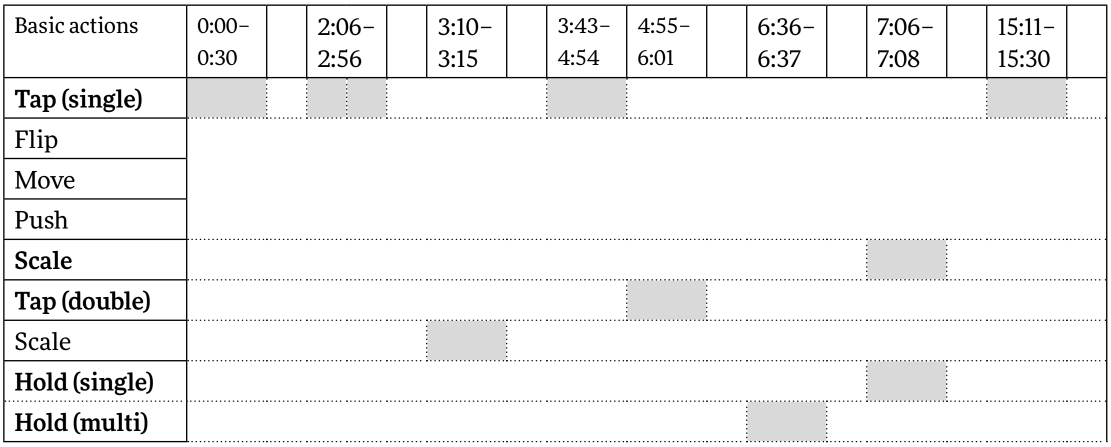


Chart 2a: Part of the timeline illustrating basic actions

4 This activity was thought as a task to introduce curiosity among students for the next task (Napoleon Theorem).

5 To fix the timeline on the CERME template, we cut down some time interval.

Active actions	0:00–0:30	0:30–0:50	1:28	1:46–1:54		3:15–3:20		6:05–6:09		8:31 / ... / 15:02		15:35–16:55
Drag free												
Drag approach												
Flick												

Chart 2b: Part of the timeline illustrating active actions

one way of touch, but we selected some in which the exemplified type has predominance.

Due to the nature of the task (with open construction and exploration) we identified the predominance of touchscreen types on the relational domain and basically touch such drag (free or approach) and flick. The rotate didn't occur in this task. As we can see on the Chart 2b the usage of drag to approach was dominant.

As we observed in a previous analysis (Arzarello et al. 2014) the dragging to approach works as a refreshing, a quite stabilizing and reflecting area for deep understanding of the geometric properties that emerge from the manipulation on drag free or other way of touchscreen. This type of manipulation seems to be an appropriated moment to improve justification and proving.

According to Arzarello and colleagues (2014), manipulation in the constructive domain seemed to be focused on only predetermined motion, whereas motion through relational manipulations is open in the sense that it can generate more unpredictable processes. We still have to research further on the issue of open motion.

Manipulation on construction domain seems focused on only predetermined motion although motion through relational manipulations provides motion open in a sense that they can generate more unpredictable processes. By the way, we still have to go further on the issue of open motion and on the issue of the two domains of manipulation on GC software.

To summarize the reflection above, we illustrate on Chart 3 how we are relating the two domains of touchscreen with geometrical thinking and the motion through touchscreen. Although students dealt naturally with the device, their manipulation apparently was related with the software constraints (or advantages) or with the proposal task.

FINAL REMARKS

As simultaneous touchscreen manipulation of spots on the screen brings about implications of an epistemological order, it also adds complexity to our cognitive structures. This particular feature was observed by one of the students in our research. According to him, "*in a very complex figure, moving several elements at the same time can become a bit difficult*". Besides this cognitive implication, the use of touchscreen devices in the teaching of mathematics brings about transformations in didactic and epistemological realms, and educational research is still lacking.

Another relevant issue to consider is the way using a multi-touch-screen allows changing the task design in a substantial way. More precisely, multi-touch screen devices allow designing geometrical problems in a different way from the usual one, which would be very difficult within non-multi-touch screens environments. For example, within multi touch screens it is possible to ask two students, who use the same screen, to play mathematical games, where each of them pursues antagonist aims: exploiting the strategy they use to win they can so enter into the mathematical property upon which the game has been built (Arzarello et al., to appear).

We identified the touch "to approach" as a predominant way in this type of environment. This sort of touchscreen should be seen as a cognitive tool to empower learners conjecturing and exploring for argumentation during the process of solving the task. This allows us to ascertain that the drag-approach allowed by the multi-touch environment can suitably support and improve students' justifying (exploring) and proving (conjecturing) performances.

We think that manipulation that promotes open motion (relational ways of touching) can be appropriate to provide new epistemological challenges regarding geometric knowledge and different ways of proving. Since the drag to approach is a relational action, it

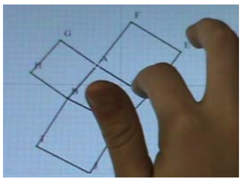
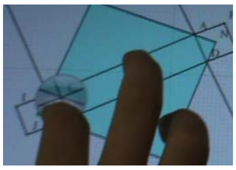
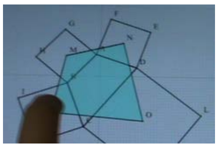
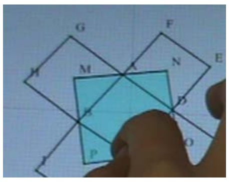
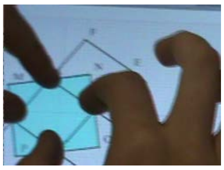
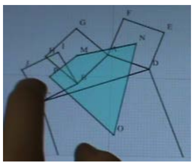
Video time	Screen example	Students geometrical thinking on GC			
		Type of touch-screen manipulation	Geometric strategy by touchscreen	Nature of motion	Domain of manipulation
2:30–2:40		Tap (single or double)	Student constructing square tapping with one finger and making the construction point by point	Closed motion, predetermined (specific goal, basically construction)	Discrete construction and isolated observation (perception). Usually students make constructions for exploration.
6:30		Hold (single)	Student making a zoom at one point		
0:03		Drag free	After having constructed the last square on each side of the quadrilateral ABCD, a student drags freely point P to see what happen with the shapes	Open motion, but focused on emergent conceptual demand of the task	Related construction and global observation for analyze conjectures and geometric properties and shapes. In this domain manipulation on screen is predominant.
5:35–5:45		Drag-approach	Student approaching MNOP to a rectangle to analyze how the shapes become constructed on the side		
			Student using 4 fingers (2 from each hand) for dragging 4 points simultaneously and shaping them as a square		
5:18–5:28		Free	Student moving freely point C		

Chart 3: Students' geometrical thinking on GC device

seems to be an appropriated moment to improve justification and proving within mathematics classrooms using touchscreen devices. But we would say that, depending on the aim of the teacher, the nature of the task is important and the teacher may let students work freely on the task, using naturally their own way of touch.

A new organization of lessons and of the nature of proposed mathematical tasks (didactic), a view on the touchscreen manipulation that is different from mouse dragging (cognitive), and attention to the changes in mathematics when simultaneously moving different points in a figure (epistemological) are examples of changes and will be an object for reflection on our results in CERME9 (TWG16).

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ENDNOTE

1. According to Yooks’ (2009) framework, the four active actions can be associated to multi hold manipulation.

How a digital place value chart could foster substantial understanding of the decimal place value system

Daniela Behrens

University of Bremen, Bremen, Germany, daniela.behrens@uni-bremen.de

When using digital technology in the mathematics classroom, teachers should always consider whether the tool supports the students' learning process in a way "traditional tools" do not allow. In this sense, the aim of this paper is to discuss theoretically how an interactive place value chart could foster substantial understanding of the decimal place value system in the context of decimal fractions, by analysing the artefact's potential for learning. Here, special attention is dedicated to the mathematical, structural aspects of the place value system included in the software. This theoretical discussion builds the foundation for future empirical investigation that will be outlined at the end of the paper.

Keywords: Decimal fractions, place value system, interactive technology, semiotic resources.

INTRODUCTION

Different studies emphasize that students build several misconceptions concerning decimal notation (e.g., Heckmann, 2006; Steinle & Stacey, 2004). The majority of these misconceptions indicate a lack of understanding of the decimal place value system which is fundamental to understand our decimal notation of numbers and which therefore plays an important role in learning mathematics.

Besides practical learning approaches based on activities with concrete material (e.g., Montessori) also research on human cognition and especially on mathematical thinking emphasizes that knowledge formation does not evolve only in human mind, but rather through activities concerning the body, the interaction with an artefact and linguistic and symbolic resources (Radford, 2005, p. 111).

In this sense a digital, interactive place value chart (designed by Kortenkamp) offers special opportunities to actively explore the place value system for a substantial understanding of our decimal number notation in a way traditional tools (e.g., paper-pencil place value charts, base-ten-blocks) do not allow.

To discuss this theory, I will first show how the mathematical structure of the place value system is build up and how this is included in the digital tool, so that a first analysis of the mathematical content within the tool is given in this paper. Based on this I will furthermore reason why especially this tool could support students' understanding of the place value system in comparison to other materials. As a last part I will present my research approach to investigate empirically how this tool might support students' understanding of the place value system regarding decimals. This project, called DeciPlace, is a cooperative project between Ulrich Kortenkamp (University of Potsdam) and Angelika Bikner-Ahsbahs (University of Bremen).

AN INTERACTIVE PLACE VALUE CHART TO FOSTER THE CONCEPT OF PLACE VALUE CONCERNING DECIMAL FRACTIONS

The expansion of the number system from natural to rational numbers often gives rise to difficulties in the learning process in relation to fractions but also concerning decimals.

Different research projects emphasize that students build several misconceptions on decimal notation (e.g., Heckmann, 2006; Steinle & Stacey, 2004). These misconceptions often lead to inadequate strategies, for example to order decimals by size like "longer-is-larger" or "shorter-is-larger" (Steinle & Stacey, 2004). Heckmann (2006) considers transmissions from natural numbers and fractions (especially if fractions are taught before decimals) as the main causes for misconceptions (ibid, pp. 77–89). This suggests the assumption that some students have a non-viable understanding of our decimal place value system or do not use it, so that they are not able to arrange decimals according to size correctly (ibid, p. 51).

That is why it is recommended to focus explicitly on the place value system while learning the concept of decimals (Heckmann, 2006, pp. 52, 562ff.; Padberg, 2009, p. 166; Steinle & Stacey, 2004, p. 548).

Place value

The decimal place value system is based on four properties that are important to understand (Ross, 1989, p. 47):

1) Positional property. The quantities represented by the individual digits are determined by the position they hold in the whole numeral.

2) Base-ten property. The values of the positions increase in powers of ten from right to left.

3) Multiplicative property. The value of an individual digit is found by multiplying the face value of the digit by the value assigned to its position.

4) Additive property. The quantity represented by the whole numeral is the sum of the values represented by the individual digits.

Within the mathematical structure of the place value system defined by these properties, the decimal fraction is a particular structure that is determined by a fixed value represented by a number in decimal notation. In order to preserve the decimal fraction and its structure, only transformations like bundling and de-bundling are allowed in the place value system.

Although learners already know the decimal place value system and its properties, at least unconsciously, while ordering natural numbers and calculating with them, they normally do not pursue this concept in relation to decimals. Furthermore the extension of the place value system from natural numbers to decimal numbers is not self-explanatory (Padberg, 2009, p. 167), since there are not only analogies but serious differences between both (ibid, p. 170). Therefore the emerging extensions and general properties of the place value system should be addressed adequately when teaching decimals to avoid inappropriate transfers from natural numbers and to foster a suitable understanding of our decimal place value system related to decimals.

The interactive place value chart by Kortenkamp and its benefits

One of the main reasons for difficulties in mathematics learning is the fact that mathematical objects and structures are not directly accessible through our sensory organs. Nevertheless we can represent mathematical objects and structures through symbols as well as through visuals and we can gather experience engaging in mathematical activities on concrete material to get access to mathematical concepts (Gersten et al., 2009; Schipper, 2003).

That is why it is recommended to use visuals and concrete activities to develop an adequate understanding of place value (e.g., Heckmann, 2006, p. 577).

A digital, interactive place value chart (designed by Kortenkamp, see Ladel & Kortenkamp, 2013) allows the combination of concrete acting and iconic and symbolic representation of numbers. The following a priori analysis of the artefact examines the possible functions of the interactive place value chart and their corresponding mathematical meanings within the place value system. The main functionalities of the interactive place value chart are the representation of a number by tokens in a place value chart and the variation of the number's representation by changing the quantity within place values moving tokens, while keeping the number invariant.

At the upper row of the place value chart the symbolic representation is described through the particular place values (Tens, Ones, Tenths, Hundredths, etc.), structured in columns. Additionally, the represented number can be displayed in standard notation. The bottom row as the main part shows tokens representing the quantity of each place value within the whole number (see Figure 1).

A number can be created by adding tokens in the given place value columns by tapping the screen. Mathematically, this signifies an increase of the quantity in a particular place value by the quantity of added tokens. Here, the relation between particular place values and the number in standard notation can be explored addressing indirectly the *positional*

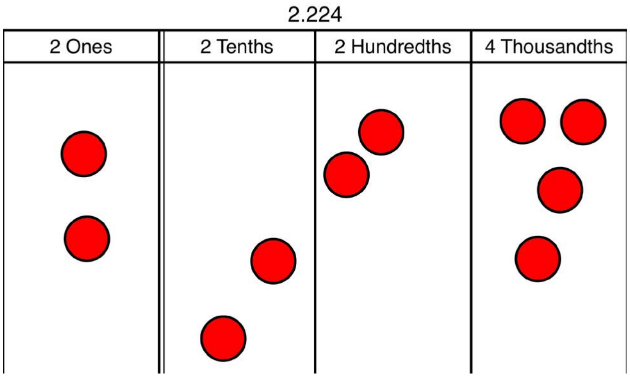

Figure 1: The digital place value chart

and the additive property of the place value system. Additionally, the function of the decimal zero is illustrated by a blank column of tokens.

The fundamental characteristic of this interactive place value chart concerns the invariance of the number, while varying its representation within the particular place values. Specifically, the user can slide tokens directly on the touchscreen, either within the same column or to other columns. Here, the former action can be done to structure the token's arrangement, e.g. to facilitate counting the tokens within a column. For the latter activity of moving a token to another column there are three different possibilities of the artefact's reaction, which are guided by the mathematical structure of the decimal place value system:

a) When the user slides one token to one column rightwards, the token multiplies into ten tokens within the new column. If the user moves for example one token of tenths to the hundredths, in the hundredths-column there will emerge 10 tokens (see Figure 2).

   Here, the artefact represents the transformation of the representation through de-bundling according to the *base-ten-property* of the decimal place value system. The artefact furthermore follows the *multiplicative property* of the place value system, which implies that the quantity in each place value has to be multiplied with the according place value that decreases in powers of ten from left to right, so that the quantity in place values has to increase in powers of ten while de-bundling from left to right.

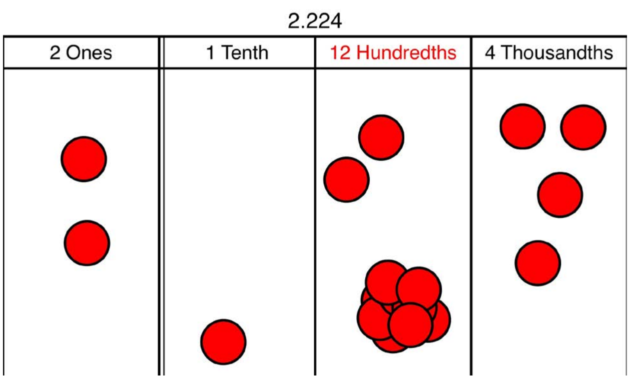

Figure 2: Changing the number's representation within the place value chart by moving tokens

b) When the user slides one token to one column leftwards, there are two cases:

   i) If there are at least ten tokens in the initial column, nine tokens go along with the slid token and the ten tokens merge to one token in the new column. Here, the artefact automatically bundles according to the *multiplicative* structure and the *base-ten-property* of decimal place value.

   ii) If there are less than ten tokens in the initial column, the token slides back to its initial position, because the condition for bundling only groups of ten is not given.

To sum up, the analysis of the relation between artefact signs and the underlying mathematical meaning shows that the interactive place value chart follows the particular structure and corresponding properties of the decimal place value system by performing and illustrating bundling and de-bundling activities.

As an additional functionality, the number of place values as well as the basis of place values can be changed, so that the chart can address further number systems (basis 2–16) and can be used on different levels of learning.

How this interactive place value chart might support substantial learning activities related to decimal place value in comparison to other learning materials for place value systems will be the focus of the following passage.

The interactive place value chart in comparison to other materials

To explore the structure of decimal fractions and the allowed rules within the decimal place value system, students need a tool that follows the rules keeping the structure invariant. In contrast to "traditional" place value charts (paper-pencil or with real tokens), the interactive place value chart allows and even performs bundling and de-bundling activities (see above). Ladel and Kortenkamp (2013) emphasize that this allows to "operate with the tokens *while keeping the represented number unchanged*" (p. 191, italic in the original), whereas in traditional place value charts the manipulation of tokens means a change of the number and therefore a change of the underlying structure. In this sense the "modern approach emphasizes the human activity and is ruled by the object (i.e. the numbers), not the artefact" (ibid). That means that the use of traditional place value charts requires to reflect and understand the artefact, so that the artefact and its function become additional issues to understanding. In contrast, the interactive place value chart follows and even illustrates the structure and the rules of the place value system, so that it could be helpful especially for low-performing learners.

Compared to the base-ten-blocks as well as linear-arithmetic-blocks materials, the interactive place value chart does not illustrate particular place values by different sizes and aspects, but by its position within the chart. Therefore the positional property of the place value system is emphasized. This can be seen as a preparation for the symbolic notation through digits, because the digits of a number equally do not differ in aspect, instead their position within the number is essential (positional property). This preparation is important for disconnecting from concrete activities on the tool to understand and symbolise decimal numbers (Scherer & Moser-Opitz, 2010, p. 85; Schipper, 2003, p. 223). Hasemann (1995, p. 13) points out that exactly this transition from the concrete situation to formalising and symbolising is the main difficulty of understanding mathematics for many students.

Another chart that is used to illustrate the structure of our decimal number system is the Gattegno Chart (see Figure 3), where "each column has a *digit name* associated with it (*one, two, three,..., nine)* and each row has a *value name* associated with it (*thousand, hundred, -ty, [nothing])*. Thus, a number name can be thought of as made up from *digit names* and *value names*" (Hewitt, 2005, p. 45, italic in the original). With respect to this structure the Gattegno Chart aims to "[help] students to learn to say and write number names" (ibid, p. 44) as well as it is meant to "[provide] a basis for work on place value, decimals, addition and subtraction, multiplication in powers of 10 and standard form" (ibid). Various tasks are suggested for exploring the chart's structure to invent calculating methods (see, e.g., Hewitt, 2005). However, these calculating methods are rather based on the chart's structure than on conceptual understanding of the decimal place value system. The properties of our decimal place value system could be addressed by specific tasks within this chart, but in contrast to the interactive place value chart they are not illustrated directly within the table. Moreover the Gattegno Chart is focused on the symbolic representation of numbers in contrast to the interactive place value chart, where the iconic representation includes in particular the illustration of the multiplicative and base-ten property.

OUTLOOK INTO THE EMPRICIAL RESEARCH TO BE CONDUCTED

With this analysis of the mathematical structure of the place value system included in the interactive place value chart and the theoretical considerations on its benefits, the basis for an empirical investigation of the artefact's impact on students' learning is provided. On this basis my research approach will be to investigate empirically how students interact with the digital place value chart in the process of conceptualising the place value system as a main factor for constructing a fruitful concept of decimals.

In learning processes and their profound analysis, the use of artefacts seems to play a crucial role, especially while investigating the interactive place value chart, because the culturally construed structure of the place value system is already included in this artefact. Recognising this structure "may occurs [sic] for the

0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900

Figure 3: An example of the Gattegno Chart

expert automatically and unconsciously" (Mariotti, 2012, p. 28), but students firstly have to explore and get to know this structure within the place value chart through the use of the tool (ibid.). That is why Radford emphasizes that artefacts in the mathematics classroom "are not merely devices that provide stimuli for cognitive development" (Radford, 2014, p. 360), but "they are part of cognition, which we see simultaneously as ideational and material" (ibid).

With the neuropsychological model "gestures as simulated action" from Hostetter and Alibali (2008) we can moreover understand why also the dynamics and relations between different semiotic resources (gestures, inscriptions, language, etc.) might be important in the analysis of learning processes. Hostetter and Alibali (2008) state that "gestures emerge from the perceptual and motor simulations that underlie embodied language and mental imagery" (p. 502). That is why this approach leads to suppose that for example not only gestures and language, but also gestures and activities on artefacts in a learning situation are intertwined with each other and so both have to be considered in the analysis also concerning their relationship.

Recent studies about the role of gestures in mathematics learning revealed that especially high-performing learners use gestures while "generalising" a mathematical concept, so that it seems as if they bridge the transfer from concrete acting to mathematical thinking through gestures (e.g., Goldin-Meadow & Beilock, 2010).

Regarding the interactive place value chart we can note that the sliding-actions needed to operate with the tokens originate in concrete sliding-actions of real tokens, but as well they are gesture-based because of the chart's touchpad-surface. Furthermore the place value chart emphasizes the spatial property of the place value system within the chart, which also can be experienced by the sliding-actions on the touchpad. The virtuality of the digital place value chart, its gestural basis of sliding-actions and the spatial property seem to be suitable to transfer the chart and the sliding-actions into the gesture space as a virtual and dynamic representation, that can similarly be used to reason (verbally) about the place value system and its properties in relation to decimals.

Equally it would be possible that students transfer the virtual place value chart into an inscription, where the sliding-action and therefore the bundling- and de-bundling-activities of place value can only be represented statically and fixed in contrast to gestures.

In summary, all semiotic resources, the mathematical concept of the place value system and its impact on the artefact should be part of the analysis of learning processes. Thus the focus of investigation will be on the role of bodily and artefact-mediated action and perception together with linguistic and symbolic activity, so that the following research questions are traced:

— How do students construct decimals with the interactive place value chart? What conditions foster or hinder this process?

— How do students use semiotic resources interacting with and disconnecting from the interactive place value chart, and what functions do these resources accomplish within the process of learning?

To examine the mentioned research questions video-recorded interviews with student-pairs in grade 5 will be conducted before these students are formally introduced to decimal fractions. The students will be on different levels of their mathematical development and come from an inclusive setting within a comprehensive school (Oberschule) in Bremen. Within these interviews the students receive tasks dealing with the properties of the place value system in relation to decimal fractions, where they can use the interactive place value chart.

The final goal of my research project will be to develop further the tool of the interactive place value chart and to design a learning environment that fosters substantial understanding of the place value system regarding decimal fractions taking into account different semiotic resources. Therefore, the tasks as well as the design of the tool will be developed further during the process of data collection based on the emerging results of analysis. That is why the realisation and the analysis of the single interviews will be organised in an alternating way.

Concerning the design of the tasks it is important to allow a non-predetermined use of different semiotic resources besides the tool (gestures, inscriptions, language, etc.), so that it will be possible to investigate the impact and function of different, spontaneously used resources on the students' process of understanding. Here, the semiotic bundle (Arzarello, 2006, p. 267) as a refined notion of semiotic system provides a suitable framework for the analysis of semiotic resources. This framework of semiotics combines the enlarged notion of semiotic system stated by Ernest (2006) and the Vygotskian approach of psychological processes (Arzarello, 2006, pp. 278f.). This means that besides the classic semiotic resources like words and inscriptions also gestures and artefacts are included in this modern notion of semiotic system (Radford, 2002, footnote 7). Thereby the Vygotskian approach allows "a deeper understanding of its [the semiotic system's] dynamics" (Arzarello, 2006, p. 279). That this enlarged semiotic approach of semiotic bundle could be fruitful in the analysis of learning processes has already been shown in particular studies. Different research projects on the role of gestures in mathematical learning processes (e.g., Behrens, Krause, & Bikner-Ahsbahs, 2014; Krause, 2015 in preparation; Sabena, 2007) revealed for example that within the whole semiotic activity, gestures play an important role in the students' process of generalisation. In this sense Radford describes the role of gestures by supplying "the unperceivable general with something concrete" (Radford, 2005, p. 115).

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Peer learning in mathematics forum on Facebook: A case study

Yaniv Biton[1,2], Sara Hershkovitz[1,2] and Maureen Hoch[2]

1 Shaanan College, Haifa, Izrael, yanivb@cet.ac.il, Sarah@cet.ac.il

2 Center for Educational Technology, Tel Aviv, Israel, MaureenH@cet.ac.il

To help students prepare for the resit exam of the mathematics Bagrut (Israeli matriculation) of 2013, the Center for Educational Technology established a virtual review session using Facebook, for four days before the exam. 614 students and 16 teachers participated. We examined three central questions, each about using Facebook to prepare for the mathematics Bagrut: What opportunities for learning were created? What are the students' opinions? What are the teachers' opinions? Analysis of the posts on Facebook revealed five types of situations with potential for learning. Answers to on-line questionnaires show that both students and teachers hold positive opinions towards the solution for learning provided by Facebook. We recommend researching opportunities for learning afforded by the social networks.

Keywords: Facebook, learning opportunities, student interactions.

INTRODUCTION

Online social networking sites such as Facebook have developed in recent years and have become the most popular meeting places for youth and adults (Boyd, 2010). Many studies have investigated the potential of using these networks to promote learning (e.g., Forkosh-Baruch & Hershkovitz, 2013; Neman, Lev, & Amit, 2013). In some of these studies of teacher-student interactions the student has the status of the teacher's friend (Madge et al., 2009) and hierarchies are formed as a result of this friendship status (Steinfield, Ellison, & Lampe, 2008). Asterhan and colleagues (2013) discuss whether and how teachers may use Facebook for innovative, collaborative forms of online learning that extend beyond the traditional classroom, and whether this is at all recommendable or feasible.

Recently many researches have studied the Facebook option of creating a group where teacher and students belong but do not need to be "friends". Students perceived learning in this environment as very intensive and collaborative in nature (Meishar-Tal, Kurtz, & Pieterse, 2012). In the learning of mathematics, social network sites have been found to invite student collaboration and encourage learning (Baya'a & Daher, 2013).

In this present research the learners were members of a group on Facebook opened specially for preparation for the mathematics Bagrut (Israeli matriculation) exam. We investigated what opportunities for learning were created as a result of the interactions that formed within the group and examined the viewpoints of students and teachers who took part in the study group. We present the results of a pilot study, in preparation for a wider research on this subject. In this pilot we chose the medium of Facebook as being familiar and convenient. In a larger study we recommend investigating other possible platforms and comparing the influence of different platforms.

Our research questions were:

1) What opportunities for learning could be identified as a result of interactions on the Facebook forum?

2) What were students' and teachers' attitudes towards these interactions on the Facebook forum?

OUTLINE OF THE RESEARCH

Eight groups – four in Hebrew and four in Arabic – were opened on Facebook for four days, twelve hours a day, before the resit of the mathematics Bagrut exam. Teachers were on call to respond to students (three shifts of four hours). The Hebrew speakers' group comprised 513 students, and the Arabic group 101 students. The groups were divided according to the questionnaires in the Bagrut exams at intermediate and advanced levels. The teachers were trained in on-line teaching, in the principles of a forum, and in the Facebook tools, and were given technical instructions on how to provide responses in the forum.

During the activity the students raised questions in whatever subject they wished. The questions were uploaded to the forum as photographs or as details of book, page, and exercise number (the teachers were provided with all the relevant textbooks). On receipt of a question the teacher sent a reply, "I will upload an answer soon" and after several minutes (on average 10 minutes) he uploaded a response to the forum as a photo of the page on which he wrote the solution, or hints on how to reach it. At the end of the study session online questionnaires were sent to the students and teachers who took part in the forums. 105 students and 15 teachers completed the questionnaires.

RESEARCH METHODS AND TOOLS

We used a mixed methods research model (Johnston & Onwuegbuzie, 2004) which combines qualitative and quantitative data analysis. The research tools were two online questionnaires, one for students and one for teachers, comprising open and closed questions. The open questions for the student included those on his background, which exam paper he was taking, how he heard about the study group, and his suggestions for what he would like to preserve in the study group and what he would like to improve. The open questions for the teacher included those on his seniority, his online teaching experience and the classes he usually teaches. Teachers were also asked to write down their feelings about teaching through Facebook, to describe interactions they remember favourably, etc. The closed questions in both questionnaires comprised statements on a Likert scale from 1 (disagree) to 4 (strongly agree). These statements included issues such as the use of technology, peer learning, motivation to continue learning/teaching in a similar manner in the future, interactions with students, etc.

The participants answered the questionnaire at the end of the Facebook review session. The answers to the open questions were analysed by three mathematics education experts to improve validity reliability by triangulation (Denzin & Lincoln, 2000). The analysis was carried out in four stages: first the answers were collected; in the second stage all the answers were divided into short sentences; subsequently each sentence was classified according to general subject matter; and finally the sentences in the same subject matter group were collected together and arranged according to categories. After much discussion 100% agreement was achieved between the judges about the categorisation of the data.

In order to learn what opportunities for learning were created as a result of revising for the Bagrut exam in mathematics through the medium of Facebook an analysis was made of the content appearing in Facebook throughout the review session. First we divided up all the content into analysis units according to the participants in the interactions: teacher-student and student-student. In the second stage we analysed all the interactions and learning opportunities that arose. The analysis consisted of identifying and characterizing the students' questions and the correctness of their explanations. Each analysis unit was examined with the aim of pinpointing the development of mathematical knowledge during the interaction. From this analysis we identified learning opportunities which can be seen in Figure 2. In order to preserve the privacy of the participants care was taken to ensure anonymity.

FINDINGS

Figure 1 shows the number of students in each group according to the levels of the exam papers (804 and 805 – intermediate, 806 and 807 – advanced) and the number of questions or discussions raised (the posts).

Figure 2 presents a map of the opportunities for learning observed throughout the review session.

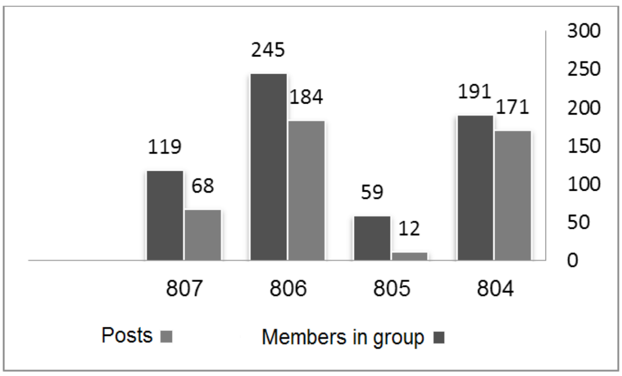

Figure 1: Number of participants and number of posts in each study group

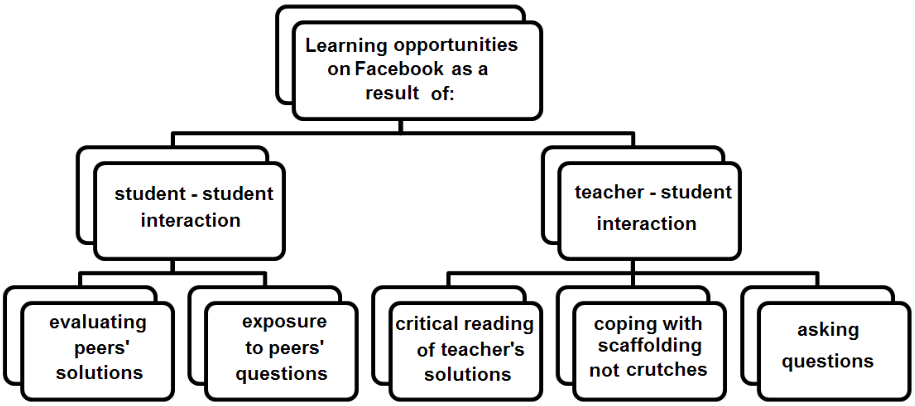

Figure 2: Learning opportunities on Facebook resulting from teacher-student interactions and student-student interactions

We now provide a short description of each opportunity and some episodes from the forum.

Evaluating peers' solutions

During the review session students asked for help in pinpointing the mistakes they had apparently made in their solution, intending that the teacher would evaluate their work and find the mistake. We observed that during the time that passed between a student uploading his solution and receiving a reply from the teacher (perceived as the source of authority in the forum) other students responded and tried by themselves to pinpoint the source of their peer's error. The students' attempts created a cognitive appeal to the correctness or incorrectness of the evaluation and thus started a chain of responses until a final response was given by the teacher. Similarly we noticed that throughout the review session students had considerable success in taking the teacher's role by attempting to provide explanations through the forum. This finding is strengthened by the students' answers to the questionnaire at the end of the review session. 72% (N=104) stated that they learned from responses given by other students.

We now provide an episode from the forum. Here we give an example of a student-student interaction in which a student uploads his solution to a problem and requests help to find his mistake. Sagi refers to the following problem (of which only the relevant sections are replicated here).

> A sequence of 2n+1 numbers satisfies the following conditions: $a_1 = 10, a_{n+1} = 5a_n$.
>
> a. Express in terms of n the sum of the terms in the even-numbered places.
>
> b. The last term in the sequence is 3,906,250. Calculate:
>
> 1) the sum of the terms in the even-numbered places
>
> 2) the sum of the terms in the odd-numbered places
>
> 3) the term in the middle of the sequence.

Sagi posts the following:

> I need help with general understanding of this problem. It's clear to me that the sequence is geometric and that the q is 5.... The general term, according to the given information, should be $a_n = 10 \cdot 5^{2n+1-1}$. And so a_2 is 6,250. It's clear that's incorrect because it should be 50 but I would be happy for an explanation of what I am not understanding here, and for the solution. Thanks.

Sagi attempts to phrase his difficulty in his own words. We can see that he recognizes that the sequence is geometric and knows the formula for the general term in a geometric sequence $a_n = a_1 \cdot q^{n-1}$. However he substitutes 2n+1 for n in the formula thus finding the last term instead of the general term. Avichay replies to Sagi and the following dialogue takes place:

Avichay: The general term in a geometric sequence is $a_n = a_1 \cdot q_{n-1}$ and so $a_2 = 10 \cdot 5^1 = 50$.

Sagi: Right, but here the last term is in place 2n+1.

Avichay: That doesn't change anything buddy.

Sagi: I'm pretty sure it does because in section b you have to use the general term $a_n = 10 \cdot 5^{2n}$.

Avichay: That formula is for every geometric sequence. The number of terms doesn't matter for this formula, only for the sum formula.

Sagi: I'm pretty sure you're wrong. Let's wait for one of the tutors to answer.

Avichay: Okay, but there you're looking at the sum of terms. You asked what's a_2?

Sagi: Let's wait for an answer. I want to be sure. But really, thanks pal.

Avichay: I think I understand what your problem is...... The n-1 in the power of q isn't according to the number of terms in the sequence but the place of the term in the sequence.

Avichay: And if you want to find the value of a_2 you need to do $q^{(2-1)}$.

Avichay is determined to help Sagi and tries to fine-tune the mathematical language he uses in his explanation. Sagi appears doubtful but is grateful for Avichay's efforts. The improvement in Avichay's language throughout the episode is evident. At the end of the discussion between Avichay and Sagi, Achinoam joins in as follows:

Achinoam: In my opinion if you substitute 2 in the first formula given $a_{n+1} = 5a_n$ you get $a_2 = a_1 \cdot 5$, that is $10 \cdot 5 = 50$.

Sagi: Right, that's clear, but I'm trying to understand why the formula for the general term, and it's correct, doesn't give me $a_2 =50$.

Achinoam: Why not? It works out terrific...If I understood you correctly you meant the formula for the general term $a_n = a_1 \cdot q^{n-1}$ where you substitute 2n + 1 for n?

Achinoam: If so, I can help.

Sagi: Yes.

Sagi: That's what I meant.

Achinoam: Terrific so: $a_{2n+1} = a_1 \cdot q^{2n+1-1}$ and that's in fact equal to $a_{2n+1} = a_1 \cdot q^{2n}$ and then if you want a_2 your n will have to be 1/2 and then $a_2= a_1 \cdot q^{2*1/2}$ and that works out $a_2= a_1 \cdot q^1$ and that's exactly right if you substitute the givens and you do get $a_2=$ 50.

Achinoam: I hope you understand me.

Sagi: Wow thanks a lot!! Simply all the time I read my general term to be a_n instead of a_{2n+1}. Really thank-you. I was getting stressed out.

Achinoam: Hey – my pleasure. Just glad I was able to help.

Achinoam has shown Sagi how he could have got the correct answer using his method. Sagi says he now understands and clarifies his mistake in his own words. Ironically Achinoam's convincing argument is mathematically flawed. Here the intervention of the teacher is required to make sure that Sagi and Achinoam will understand that n cannot take the value 1/2.

From an analysis of the present dialogue we can learn about learning opportunities that occurred among the students: exposure to different solution methods, improvement in mathematical language through giving explanations and asking questions, perseverance and motivation on the part of Sagi to find his mistake on the one hand and the desire of his peers (Avichay and Achinoam) to help him on the other.

Exposure to peers' questions

Throughout the review session students were exposed to questions raised by other students and tried to answer these questions themselves. This finding is based on the number of observers of each post in the forum, on the students' reports in the questionnaire, and on the responses of the students in the forum itself. Exposure to peers' questions expanded the available pool of exercises and presented the additional challenge of dealing with questions that were difficult for their peers to solve. This finding is supported by the students' questionnaires where 78% reported that they learned from questions raised by other students.

Critical reading of teachers' solutions

The most significant learning opportunities that occurred during the review session were the chance to read, to analyse, and to understand the teachers' solutions on the forum. On some of the posts, after reading the teachers' solution the student returned to his own solution to compare the two methods. In this excerpt we can see the comparison one student made after receiving the teacher's answer to his question. At the end of this post an error was found in the book, thanks to the student's "stubbornness".

> Thanks. But somehow in the answers they put 3/4 instead of 3 root 3 divided by 2. And according to the volume of the prism that you found I got the correct t but the maximum volume is different. Maybe they made a mistake? I'd like you to solve the rest because I didn't get the same answer ...

Coping with scaffolding not crutches

In not a few cases the teacher's response was advice for continuing the solution and the student had to deal with the problem on his own. 87% of the students claimed that the teachers' tips helped them learn. In this excerpt we see a hint given by the teacher and the student's satisfied response that it helped him to solve the problem.

> Teacher: I recommend you to try to finish this by yourself. If not, let me know and I'll post the solution. Tip: the lateral area is the sum of the areas of the rectangular faces without the bases.
>
> Student: Thank you very much for the help. I got it right! ☺

Asking questions

Throughout the review session, in addition to the problems the students posted as photos or text, they asked concrete questions on particular parts of a solution, and expressed doubts that arose during a solution. In contrast to questions asked face to face, here asking questions requires another skill – the ability to formulate the question in writing, with suitable emphasis for the teacher who is supposed to answer.

The following excerpt shows a student's questions after a solution has been posted by the teacher. It includes a search for explanation/proof, indicating critical reading of the solution.

> It's not clear to me why you can deduce from the sketch of the graph alone that there are no maximum or minimum points? Who says there isn't one before the asymptote? And how can you tell without a table if the function is increasing or decreasing from the asymptote? Thanks!!

Instructional interactions on the social network

The answers to the questionnaires were analyzed as described in the section on research methods. In Table 1, we show examples of students' and teachers' remarks in each of the categories: motivation for continued learning, peer learning, technology utilisation, and supportive learning climate.

As can be seen in the table, students' and teachers' responses were mainly positive, and in general the participants' responses indicate great satisfaction with the use of Facebook in preparing for the exam. 75% of the students (N=105) stated that it was easy for them to ask questions and receive replies through Facebook, 79% stated that they would like to use Facebook in this way also for learning other subjects, and 87% stated that they would like to continue learning in a similar

Categories	Student questionnaire	Teacher questionnaire
Motivation for continued learning	I'm glad I got the chance of the Facebook forum. It gave me the option with exercises that I couldn't solve, not to give up like I usually do, but to get the solutions from a teacher – that really helped me.	I really liked the fact that the students asked relevant questions, related to the answers, and didn't give up until they understood.
Peer learning	The forum was a very good idea. We could learn from other students' questions and answers.	A student posted a question after a lesson, and I noticed that students started to help each other in the forum, and succeeded in solving some parts of it.
Technology utilisation	I would recommend improving the method of posting pictures on Facebook.	The idea of photographing the problem or the solution and posting is brilliant and effective in making best use of the time and for presenting the solution.
Supportive learning climate	I would be very happy to get this kind of help throughout the year. It is all over and above what a student can expect for success. Thank you so much for all the help.	The students' appreciation was heart-warming.

Table 1: Students' and teachers' remarks about the integration of Facebook in preparing for the Bagrut exam in mathematics

manner throughout the year. 93% of the teachers (N = 15) stated that the environment encourages meaningful learning and that the project justifies the investment of resources. There was 100% agreement among the teachers on willingness to continue in a similar manner next year. 93% stated that they would be interested in opening similar learning environments for their own students during the year.

A little criticism on the use of technology was heard from both students and teachers, relating to the uploading of pictures that were sometimes not clear, thus making it difficult to understand and respond to the problem. In addition, teachers in charge of forums where there was a lot of activity indicated the need for extra staff to help manage the responses where necessary.

DISCUSSION AND CONCLUSION

In this research we asked what opportunities for learning could be identified as a result of interactions on the Facebook forum. The Facebook forum encouraged different interactions between teachers and students and among the students themselves. These interactions provided the learners with learning opportunities which included: asking questions, peer learning, different methods of problem solving, and critical reading of solutions, and were exposed to questions from different textbooks and to solution methods of different teachers. These learning opportunities carry extra value and are important in the learning process leading up to the Bagrut exam and in general. Individual study without interactions with peers or with a teacher is unlikely to afford any of these opportunities.

We also asked what were students' and teachers' attitudes towards these interactions on the Facebook forum. The findings in this research indicate students' great satisfaction with the opportunity given them to study for the mathematics Bagrut exam through the medium of Facebook. They were motivated to deal with questions their peers found difficult. The findings relating to the students' positive opinions of learning in a Facebook environment strengthen findings of earlier studies about learning on social networks (Meishar-Tal, Kurtz, & Pieterse, 2012). The teachers also expressed great satisfaction with the Facebook environment for learning and declared their intention to adopt a similar environment preparing for Bagrut exams in the following years and for teaching during the school year.

Advantages of this technological platform included the neutralizing of the time factor – the learning could, and did, take place at all hours of the day and night. Weaker students were able to take advantage of the greater abilities of their peers and to learn from them. Students who lack confidence to ask questions in a face to face situation felt freer to express themselves. The activity was open to students from different schools with different backgrounds. These are only some features that are not present in a traditional classroom.

This research was an initial testing of teacher-student and student-student interactions on Facebook in a four-day review session in preparation for the Bagrut mathematics exam. The results encourage continuation and further research into these and other related aspects, on wider groups of teachers and students, and for longer time periods. A wide based research in the subject would be likely to lead to peer learning also among the teachers themselves – on how to characterize students' questions leading up to the exam, and in general. In addition, we recommend that continued research on these issues could provide educational policy makers with an understanding of the value of investing in similar projects in the future.

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Using tree diagrams to develop combinatorial reasoning of children and adults in early schooling

Rute Borba[1], Juliana Azevedo[2] and Fernanda Barreto[3]

1 Universidade Federal de Pernambuco (UFPE), Programa de Pós-Graduação em Educação Matemática e Técnológica (Edumatec), Recife, Brasil, resrborba@gmail.com

2 UFPE, Edumatec, Recife, Brasil, azevedo.juliana1987@gmail.com

3 Prefeitura da Cidade do Recife, Recife, Brasil, fernandasabarreto@gmail.com

Combinatorial reasoning is very important in mathematical development, providing students with contact with essential problem solving. One form of symbolic representation of combinatorial situations is a tree diagram. Two experimental studies were designed and implemented, the first with adults in initial schooling and the second with Elementary School children. Comparisons were performed in order to observe the impact that the use of tree diagrams – virtual or built with pencil and paper – had on combinatorial reasoning. From initial poor performance, both children and adults in initial schooling benefitted from instruction by use of tree diagrams that enabled them to perceive how elements can be combined in a systematic manner and helped them develop their combinatorial reasoning.

Keywords: Combinatorial reasoning, tree diagrams, children and adults; initial schooling.

THE ROLE OF SYMBOLIC REPRESENTATIONS ON COMBINATORIAL REASONING

Combinatorial reasoning is a way of thinking very useful in general mathematical learning. According to Batanero, Godino and Pelayo (1996), Combinatorics is a key element of discrete mathematics, being essential for the construction of formal thought. The nature of combinatorial situations – counting techniques of possible groupings of a given set of elements that meet certain conditions, without necessarily having to count them one by one – provides students contact with essential problem solving and may help their development in Mathematics and other subjects.

Diverse forms of symbolic representation may be used in solving combinatorial problems, such as: drawings, lists, tree diagrams, tables, formulas and other forms. These distinct symbolic representations may provide means of systematization and help students understand how to obtain the total number of combinations.

The role of symbolic representations in mathematical development is pointed out by Vergnaud (1997) as a key element in conceptualisation. Considering Combinatorics, Fischbein (1975) emphasizes that the use of tree diagrams can enable advances in the development of combinatorial reasoning because this representation helps systematization by pointing out the necessary steps in choosing elements to compose combinations.

In a longitudinal study, Maher and Yankelewitz (2010) investigated the initial understanding of eight and nine year olds in a problem of *Cartesian product*. The authors defend that it is necessary to invite children to use various representations to express their ideas and ways of thinking, because representations give meaning to the problems and communicate ideas. Thus, children can find patterns, be systematic and generalize results.

Sandoval, Trigueiros and Lozano (2007) proposed the learning of Combinatorics by use of the software *Árbol*. The study was conducted with 25 Mexican children, aged 11 to 13, and the authors observed improvements in student performance, especially regarding the choice of strategies for efficient resolution. Thus, it is emphasized that this software, through tree dia-

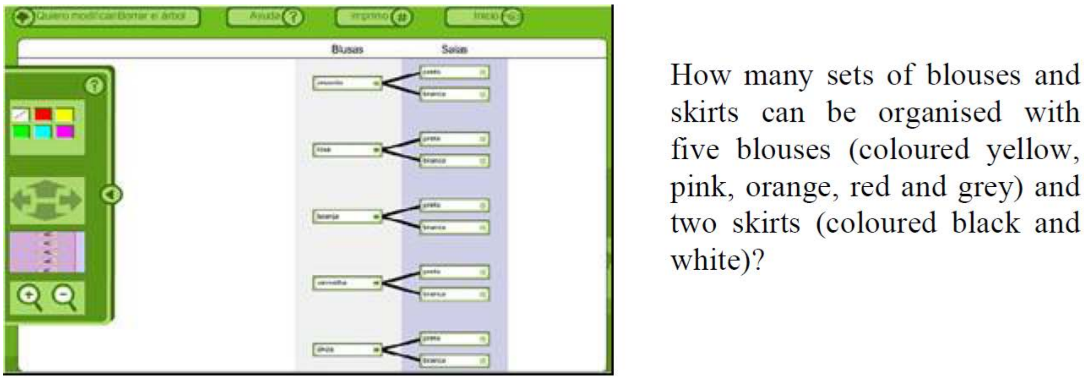


Figure 1: Árbol screen of a Cartesian product problem

grams, favours the use by children at initial level of schooling, because it provides possible combinations in all types of combinatorial problems (*Cartesian products*, *combinations*, *arrangements* and *permutations*). Figure 1 shows a screen of the software *Árbol* of a *Cartesian product* problem in which in it possible to visualise the total number of combinations in a tree diagram.

COMBINATORIAL SITUATIONS

According to Vergnaud (1997), to study and understand how mathematical concepts develop in students' minds through their experience in school and outside school, one needs to consider a concept as a three-uple of three sets: the set of situations (S) that make the concept useful and meaningful; the set of operational invariants (I) that can be used to deal with these situations; and the set of symbolic representations (R) that can be used to represent invariants, situations and procedures. Thus, the immense role played by symbols cannot be ignored in mathematical teaching and learning, as means to articulate invariants – conceptual properties and relations – situations and strategies used in problem solving.

Vergnaud (1997) also points out that combinatorial problems are part of what he calls *multiplicative structures*. In the same direction, Pessoa and Borba (2010) defend that as connected concepts, different types of combinatorial problems should be taught in the classroom and present examples of these distinct situations:

Cartesian product: At the square dance three boys and four girls want to dance. If all the boys dance with all the girls, how many pairs will be formed?

Simple permutation (without repetition): Calculate the number of anagrams that can be formed with the letters of the word LOVE.

Simple arrangement: The semi-finals of the World Cup will be played by: Brazil, France, Germany and Argentina. In how many distinct ways can the three first places be formed?

Simple combination: A school has nine teachers and five of them will represent the school in a congress. How many groups of five teachers can be formed?

TWO STUDIES ON COMBINATORIAL REASONING DEVELOPMENT

With the aim of investigating the role of symbolic representations – in particular the use of tree diagrams – on the development of combinatorial reasoning, we designed and implemented two experimental studies. The first one involved adults in initial schooling and in the second study took part Elementary School students (5th grade, 10 year olds).

Method of the 1st Study: Adults in early schooling using tree diagrams and lists

The adults taking part in the study were 24 students of classes corresponding to the 4th and 5th years of regular Elementary School with no previous systematic instruction on Combinatorics. They were separated into three groups, each group consisting of eight students. After solving an eight item pre-test (two problems of each type), they were taught in groups that varied in terms of symbolic representations used: G1 – lists and tree diagrams; G2 – tree diagrams; and G3 – lists. After the learning session they solved an eight problem post-test.

Method of the 2nd Study: Children using virtual or written tree diagrams

This study was conducted with 40 students from the 5th grade of Elementary School, divided into four groups, that took part in a pre-test (eight Combinatorics problems), followed by different forms of intervention and two post-tests (also with two of the four types of problems), which assessed the progress achieved a few days after the teaching session (immediate post-test) and nine weeks after the intervention (delayed post-test). Just as the adults, the children had no previous systematic instruction on Combinatorics. During the teaching session, the students worked in pairs. The first experimental group (EG1) worked with the software *Árbol* (Aguirre, 2005) in which diagram trees are constructed; the second experimental group (EG2) constructed tree diagrams with pencil and paper; the third group, a control group (CG1), worked, through drawings, multiplicative problems, according to the classification proposed by Nunes and Bryant (1996) (excluding Combinatorics); and the fourth group was an unassisted control group (CG2), that took part only in the pre and the post-tests.

HOW DO TREE DIAGRAMS HELP COMBINATORIAL REASONING?

Results of the 1st Study: Adults progress on combinatorial reasoning

Initially (at pre-test) the adults presented only incorrect answers or partially correct answers with very few correct combinations presented. They preferred to use lists in their answers but, usually, were not systematic in their usages and could not obtain the total correct number of combinations.

After the teaching session, all three groups progressed in their combinatorial reasoning. An Analysis of Variance (ANOVA) showed no significant differences between groups ($F (2, 21) = .78$; $p > .05$). Thus, both forms of symbolic representations helped the adults to understand the combinatorial relations involved in the problems, but tree diagrams more clearly helped them to be systematic in their answers.

Table 1 shows that at post-test the adults presented more correct answers or much more answers very close to the correct ones.

Progress in understanding was, however, sometimes limited. Figure 2 shows two examples of partially correct answers at post-test.

The first example (a *Cartesian product*), asked to indicate possible couples by choosing a man, out of a group of four, and a woman, out of a group of six. The adult that answered in this manner presented only four couples, considering there were only four men available. The second example (a *combination*) asked to form pairs, out of a group of five people. The adult in this case incorrectly considered, for example, Luíza

Problem type	Incorrect Answer		Partially correct answer		Correct answer	
	Pre-test	Post-test	Pre-test	Post-test	Pre-test	Post-test
A	75	14,6	25	81.2	0	4,2
C	47,9	6,2	52,1	83.3	0	10.5
P	91,7	50	8,3	50	0	0
CP	91,7	12.5	8,3	75	0	12.5

A – Arrangements; C – Combinations; P – Permutations; CP: Cartesian products

Table 1: Percentage of types of answers in each problem type, at pre and at post-test

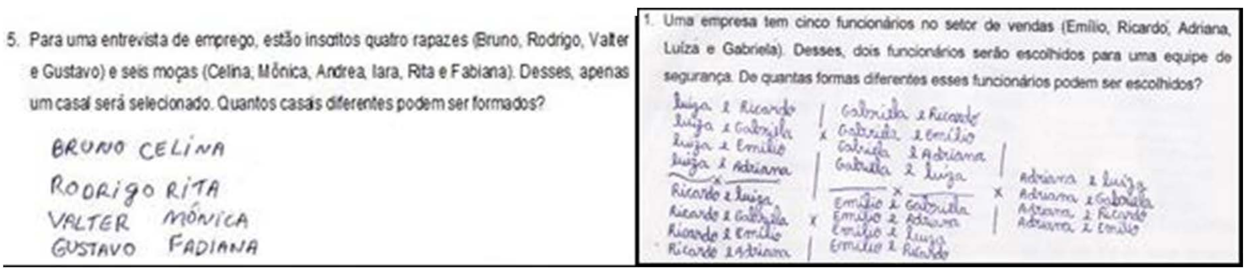


Figure 2: Partially correct answers at post-test

and Ricardo as a different pair from Ricardo and Luíza. In this way, 20 pairs were listed instead of only 10.

In three of the problem types, correct answers were presented at post-test, but difficulties with *permutations* still remained. When using lists, the adults listed some of the *permutations* but not all of them. The lists were sometimes limiting in the understanding that in *permutations* all elements are used in all possible orders.

Some preferred, after the teaching session, to use tree diagrams and those that used this symbolic representation tended to do so in a more systematic manner that led to correct answers. Figure 3 is an example of a correct answer presented at post-test of the problem of *combination* of two elements out of five. In this type of problem, and others, the adults benefitted from use of tree diagrams because this form of representation highlighted the need to be systematic in the combination of elements in distinct combinatorial situations. The adult in this case noticed that if Emília and Ricardo have been already marked as a pair (top left hand side), than the branches started with Ricardo must not include Ricardo and Emília as a distinct pair.

Results of the 2nd Study: Children's advances in combinatorics

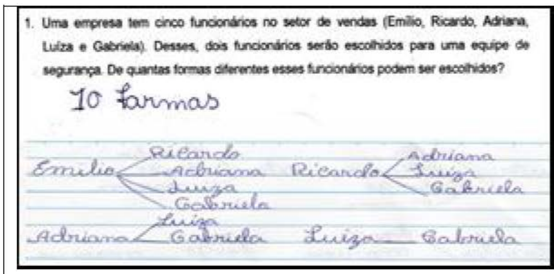
In the study with 5th grade children, each of the eight problems was scored zero to four, depending on how correct the answer was. If no combinatorial relation was observed the answer was scored zero; if only one combination case was presented, the score was one; if a limited number was presented, the score was two; if a larger number was presented, but not the total number of cases, the answer was scored three; and, finally, the score four was given to answers that were completely correct – with the total number of cases asked for. Thus, the total possible score was 32.

Table 2 shows children's performance at pre-test, immediate post-test and delayed post-test. The initial means were very similar because the children were distributed in the groups by pairing their scores and all four groups had very low initial scores.

By use of paired t-tests, significant differences were observed when performance at pre-test and immediate post-test were compared, for both experimental groups (EG1: t (8) = -2.920; p = .0019; EG2: t (8) = -3,447; p = .0009). Thus, the teaching session that used tree diagrams (either virtual or in pencil and paper) was effective in developing children's combinatorial reasoning.

No significant differences were observed between performance at immediate post-test and delayed post-test for the two experimental groups (EG1: t (8) = -0.472; p = .649; EG2: t (8) = -1.541; p =.162). This indicates that learning was retained by children of both experimental groups because, after nine weeks, the children still were able to recognize the distinct combinatorial relations involved in the problems and also were still able to successfully present correct combinations.

The children in the control groups presented no significant differences in performance, neither when


1. Uma empresa tem cinco funcionários no setor de vendas (Emílio, Ricardo, Adriana, Luíza e Gabriela). Desses, dois funcionários serão escolhidos para uma equipe de segurança. De quantas formas diferentes esses funcionários podem ser escolhidos?

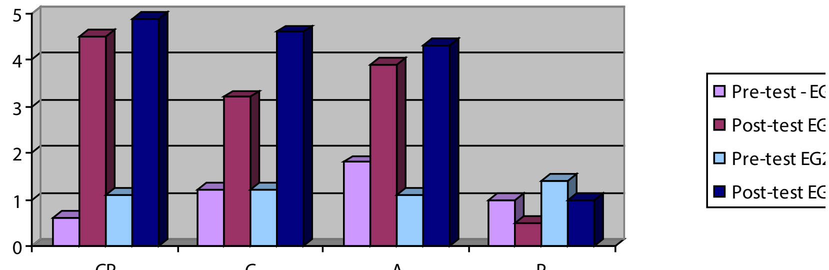
10 formas

Since Ricardo and Adriana have already been marked (top right hand side), than Adriana's branches do not include Ricardo and Luiza's branch only includes Gabriela, since Luiza has already been paired with all the previous persons.

Figure 3: Correct answer at post-test with tree diagram

Groups	Pre-test	Immediate post-test	Delayed post-test
EG1 – Software *Árbol*	4,6	12,1	13,22
EG2 – Pencil and paper	4,8	14,8	16,44
CG1 – Multiplicative problems	4,7	4,1	4,0
CG2 – Unassisted	4,9	2,8	4,2

Table 2: Means of groups at pre-test and two post-tests



CP: Cartesian product; C: Combination; A: Arrangement; P: Permutation

Graph 1: Means of experimental groups at pre-test and immediate post-test according to problem type

pre and immediate post-test were compared (CG1: t (9) = 0.751; p = .472; CG2: t (9) = 0.391; p = .705), nor when immediate and delayed post-test were compared (CG1: t (9) = 0.0750; p = .946; CG2: t (9) = -0.782; p = .454). Thus, only solving other multiplicative problems (not including combinatorial ones) or just taking part in other regular school activities was not sufficient to improve combinatorial reasoning.

Table 2 also indicates a slightly better improvement in EG2 when compared to EG1. This performance somewhat higher may be related to the fact that students in this second group solved the situations using the same representation (writing with pencil and paper) adopted in the pre-test and post-tests, while students in the first experimental group solved the problems aided by the software and at post-tests had to use a pencil and paper representation. In addition, EG2 may have been benefited by having to think about the combinatorial relations concurrently with the construction of the tree diagrams, while EG1 had the tree diagrams built by the software, and it was necessary to think of these relations only when selecting the valid cases constructed. A positive result, observed mainly by the students of the group that learned with the software, was that at post-test the children used different types of strategies – tree diagrams, lists and diagrams – showing that students did not merely learn a procedure but understood the relations involved in distinct types of combinatorial problems.

Taking in consideration problem type, Graph 1 indicates that there were huge improvements at post-test in *Cartesian products*, *combinations* and *arrangements*.

Means in these types of problems were higher than four (considering that there were two problems of each type and that for each problem the maximum score was four, and the total maximum was eight). This indicates that children in the experimental groups tended to obtain correct answers in at least one problem of these types. The graph shows no improvement in *permutations* and, just as what was observed with adults, the children possibly needed more time to better understand the tree diagram construction of this problem type, in which all elements are used in distinct orders.

Figure 4 shows a child, from the first experimental group, solving similar *combination* problems at pre and at immediate post-test. At pre-test the problem involved selecting two pets out of three animals (a dog, a bird and a turtle) and at immediate post-test the problem involved the choice of two teachers out of four (Ricardo, Tânia, Luíza and Sérgio). Initially the child incorrectly answered that there was only one way of choosing two pets out of three animals. At post-test the child used a tree diagram and correctly answered that there were six different ways.

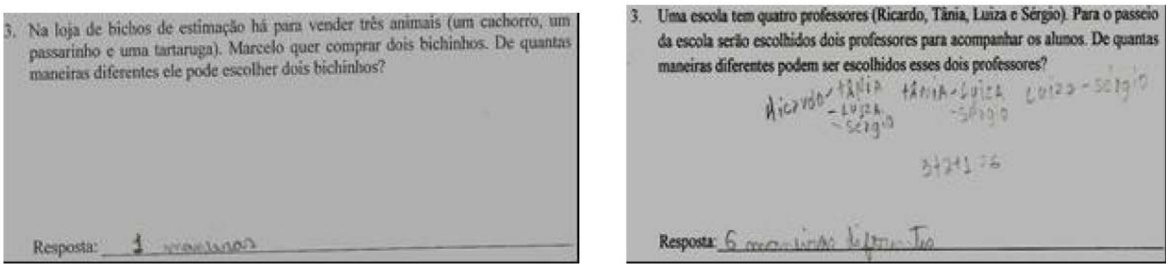
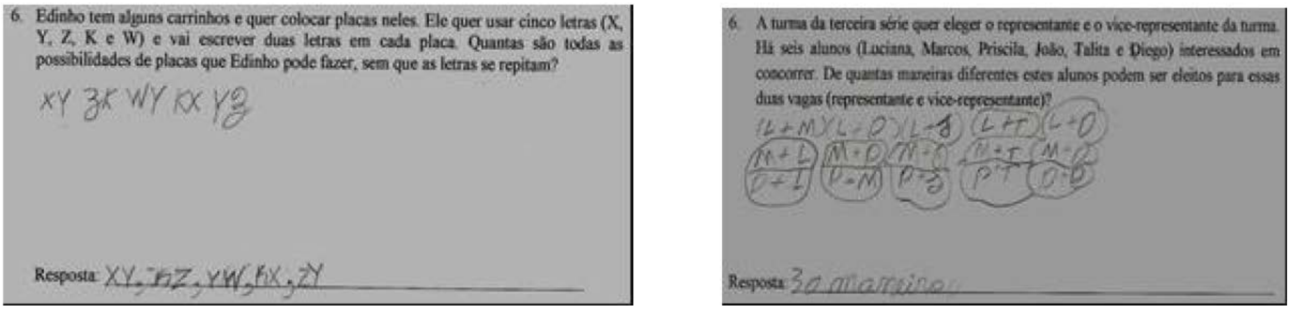


Figure 4: Incorrect answer at pre-test and correct answer at immediate post-test of a child from the first experimental group


6. Edinho tem alguns carrinhos e quer colocar placas neles. Ele quer usar cinco letras (X, Y, Z, K e W) e vai escrever duas letras em cada placa. Quantas são todas as possibilidades de placas que Edinho pode fazer, sem que as letras se repitam?

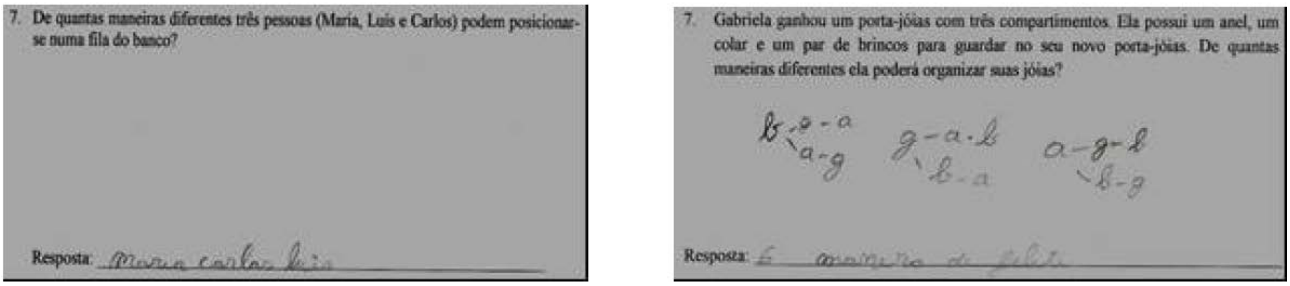
XY ZK WY KX YZ

Resposta: XY, KZ, YW, KX, ZY

6. A turma da terceira série quer eleger o representante e o vice-representante da turma. Há seis alunos (Luciana, Marcos, Priscila, João, Talita e Diego) interessados em concorrer. De quantas maneiras diferentes estes alunos podem ser eleitos para essas duas vagas (representante e vice-representante)?

Resposta: 30 maneiras

Figure 5: Incorrect answer at pre-test and correct answer at immediate post-test of a child from the second experimental group


7. De quantas maneiras diferentes três pessoas (Maria, Luís e Carlos) podem posicionar-se numa fila do banco?

Resposta: Maria carlos luis

7. Gabriela ganhou um porta-jóias com três compartimentos. Ela possui um anel, um colar e um par de brincos para guardar no seu novo porta-jóias. De quantas maneiras diferentes ela poderá organizar suas jóias?

Resposta: 6 maneira de colocar

Figure 6: Incorrect answer at pre-test and correct answer at immediate post-test of a child from the second experimental group in a *permutation* problem

Figure 5 shows a child, from the second experimental group, solving similar *arrangements* problems at pre and at immediate post-test.

The pre-test problem involved choosing two out of five letters (X, Y, Z, K and W) for license plates. The child listed only five options out of the 20 possible *arrangements*. The list was not systematic and the child did not consider that the order of choice implied in different options (the licence plate XY is different from the YX one). The immediate post-test problem involved the choice of a representative and vice-representative of a class out of six students (Luciana, Marcos, Priscila, João, Talita and Diego). The child listed 15 *arrangements* and answered that there were 30 in all. The child was very systematic in the listing produced. First all the choices with Luciana (represented by the initial L) as representative were listed, followed by Marcos (represented by the initial M) as representative and Diego as representative (represented by the initial D). The child at this point generalized that for each student as representative there were five options, so with all six children there would be 30 distinct *arrangements*.

Progress was not generally observed in *permutation* problems but Figure 6 shows how a child (from the second experimental group) solved this type of problem at pre and at post-test. The pre-test problem involved ordering three people (Maria, Luís and Carlos) in a line. The child presented only one option.

At post-test, the child used a tree diagram and was successful in doing so. The problem involved putting a ring (anel), a necklace (colar) and a pair of earrings (brincos) on three trays of a jewel box. The child correctly considered all possible branches of the tree and concluded there were six different ways of putting the jewellery in the box.

One interesting aspect is that many children from the experimental groups preferred to use listings at post-tests, despite having had the experience – by use of software or pencil and paper – of using tree diagrams. What was observed was that learning with tree diagrams enabled systematic listing, not present at pre-test. Figure 7 is an example of a child, of the second experimental group, that at post-test correctly listed the 24 possible couples, chosen from a group of six boys (Gabriel, Thiago, Matheus, Rebato, Otávio and Felipe) and four girls (Taciana, Eduarda, Letícia and Rayssa).

This was also observed amongst adults – the preference of use of listing was maintained but the child or adult used systematic lists, after instruction. This seems to be strong evidence that the child or adult did not simply learn a procedure but understood what relations were involved in distinct combinatorial problems.

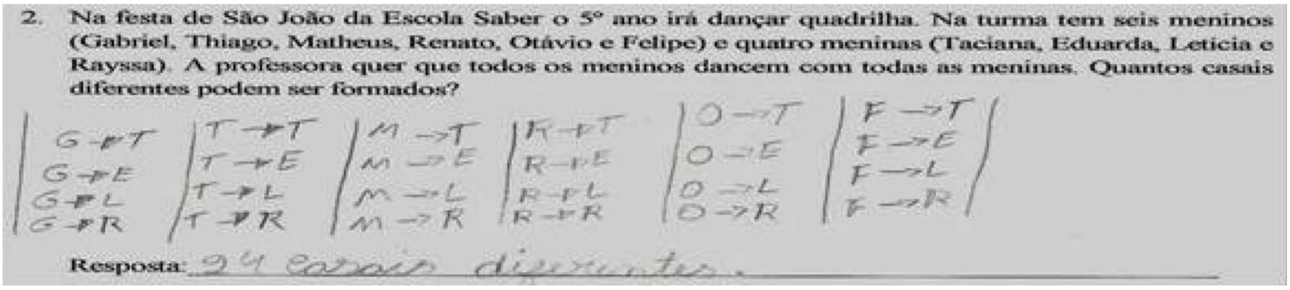


Figure 7: Correct answer at delayed post-test using listing of a child from the first experimental group

USING TREE DIAGRAMS IN INITIAL SCHOOLING

The two studies presented show that despite initially not understanding combinatorial problems, children and adults in early schooling can develop their combinatorial reasoning by use of robust symbolic representations (Vergnaud, 1997), such as tree diagrams, that aid the systematic enumeration of combinations.

Tree diagrams – built by software use or in writing – may help students' understanding of combinatorial situations, because this form of representation may aid systematic choice of elements to compose combinations, as pointed out by Fischbein (1975), and also attested by Maher and Yankelewitz (2010); and Sandoval, Trigueiros and Lozano (2007). The use of tree diagrams also enabled improvement in listings that at post-test were used in a systematic manner by adults and children. The better use of listings shows that the participants had not merely developed procedural knowledge (how to use tree diagrams), but had developed a better understanding of combinatorial situations.

One aspect that must be pointed out is that the use of the software *Árbol* helped children develop their combinatorial reasoning but it required an extra effort when the students answered the problems by use of pencil and paper. In this case, what had been learnt using the software, had to be *transferred* to pencil and paper solutions. This aspect must be considered in teaching situations and future studies may look into how the use of technology may enable the use of varied forms of symbolic representation in solving combinatorial problems.

Care is required in using tree diagrams in *combinations* and *permutations*. In *combinations* care is needed in not considering twice equivalent cases and in *permutations* the tree may have many steps of choice that must all be considered.

Combinatorial reasoning is a very relevant aspect in mathematical development and schooling is an important factor in this progress. How Combinatorics is taught can aid combinatorial reasoning development and tree diagrams may be used as tools that effectively represent combinatorial situations and the relations involved. This symbolic representation is especially useful at initial schooling by its visual aspect that enables both children and adults to perceive how elements can be combined in a systematic manner and help them develop their combinatorial reasoning.

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Technology-mediated realistic mathematics education and the bridge21 model: A teaching experiment

Aibhín Bray, Elizabeth Oldham and Brendan Tangney

Trinity College Dublin, Dublin, Ireland, braya@tcd.ie

Many recent curriculum reforms aim to address shortfalls with regard to student engagement with mathematics by harnessing the affordances of technology, social constructivist pedagogies, contextual scenarios, and/or approaches aligned with Realistic Mathematics Education (RME). However, these may not sit well within a conventional classroom setting; a 21st Century (21C) learning model may be more appropriate. This paper describes a teaching experiment in Ireland, supporting an ongoing curriculum reform; it used technology-mediated activities consonant with social constructivism, RME, and 21C learning. The study involved twenty students (aged 15–17) over a two-day period. Results suggest that the approach has the potential to increase student engagement with and confidence in mathematics.

Keywords: RME, contextualised learning, twenty-first century learning, technology-mediated, post-primary education.

INTRODUCTION

Debate regarding the quality of mathematics education at post-primary level is ongoing in many countries. Recent curriculum reforms have typically focused on developing students' conceptual understanding, problem-solving ability and productive disposition (National Research Council, 2001), with the intention that students would be able to apply their mathematics confidently in real-life and other contexts. However, on leaving school, many students' views of the subject are still fragmented and de-contextualised (Gross, Hudson, & Price, 2009), resulting in low levels of mathematical confidence and engagement. Research indicates that factors contributing to these attitudes include a formal, abstract and assessment-driven approach that reinforces behaviourist and didactic tendencies in teaching and learning, with content and procedure prized over literacy and understanding (Conway & Sloane, 2005; Ozdamli, Karabey, & Nizamoglu, 2013). Mathematical creativity is generally not encouraged, leading to a perception of mathematics as involving memorisation and execution of set procedures that lead to unique, correct answers (Dede, 2010; Ernest, 1997), and a belief that mathematics is "hard, right or wrong, routinized and boring" (Noss & Hoyles, 1996, p. 223).

It has been suggested that, within an appropriate pedagogical framework, the use of technology in the classroom can make mathematics more meaningful, practical, and engaging (Drijvers, Mariotti, Olive, & Sacristán, 2010; Olive et al., 2010). Social constructivist educational theories have been shown to align particularly well with the affordances of technology (Bray & Tangney, 2013; Patten, Arnedillo Sánchez, & Tangney, 2006). Another approach seen as addressing limitations in traditional mathematics education is that of Realistic Mathematics Education (RME) (Gravemeijer, 1994; van den Heuvel-Panhuizen, 2002), which also sits well with social constructivist pedagogy. However, activities combining a technology-mediated, social constructivist and RME approach to mathematics learning do not fit easily into the conventional classroom with its didactic teaching and short class periods (Wijers, Jonker, & Kerstens, 2008). So-called 21st Century (21C) learning models – emphasising a student-centred, active approach and key skills such as collaboration, communication, creativity and problem-solving, as well as content – may be more appropriate (Dede, 2010; Voogt & Roblin, 2012).

In Ireland, a reformed post-primary mathematics curriculum is being introduced (Cosgrove, Perkins, Shiel, Fish, & McGuinness, 2012). The reform initiative, known as ‘Project Maths’, aims to increase students’ understanding, problem-solving ability and engagement, particularly with regard to problems set in context; it recommends a focus on constructivist learning and an emphasis on the meaningful use of technology. Research is being undertaken, not only to evaluate the effectiveness of the project on a national scale (Jeffes et al., 2013), but also to examine specific teaching experiments. In particular, Jeffes and colleagues (2013) refer to the problem that “teachers are currently emphasising the content of the revised syllabuses rather than the processes promoted within it”, and that “students need to be regularly given high quality tasks that require them to engage with the processes promoted by the revised syllabuses” (p. 5). Within this context, we aim to investigate whether the combination of a technology-mediated approach, RME and a particular model – Bridge21 (Lawlor, Marshall, & Tangney, 2015) – of 21C learning facilitates the development of mathematics learning activities that increase student engagement and confidence. To provide a framework, the key features of RME and of the Bridge21 model are described and different levels of technology usage are discussed. The combination of the three elements is then illustrated through the description of a two-day experiment in a school setting. Preliminary results are discussed and tentative conclusions drawn.

FRAMEWORK

In this section, the three elements of the framework are outlined briefly.

Realistic Mathematics Education (RME)

RME is an approach to mathematics education that involves students developing their understanding by exploring and solving problems set in contexts that engage their interest, with teachers scaffolding their reinvention of the mathematics that they encounter (Freudenthal, 1991). Five characteristics of RME are identified: (i) the importance of problems set in contexts that are real to the students; (ii) the attention paid to the development of models; (iii) the contributions of the students by means of their own productions and constructions; (iv) the interactive character of the learning process; and (v) the intertwinement of learning strands. It should be noted that the contexts do not have to be drawn from the real world; the important aspect is that the students find them meaningful (van den Heuvel-Panhuizen, 2002).

The five characteristics guide a process called ‘progressive mathematisation’ (Gravemeijer, 1994). This involves: starting from a problem set in a context; identifying the relevant mathematical concepts involved; gradually refining the problem so that it becomes a mathematical one representing the original situation; solving that problem; and interpreting the solution in terms of the original situation. Mathematisation has two components, designated as ‘horizontal’ and ‘vertical’. They are described by Dickinson, Hough, Searle, and Barmby in terms of modelling: “The process of using a model to solve a particular problem is known as ‘horizontal mathematisation’, while that of using the model to make generalisations, formalisations etc. is known as ‘vertical mathematisation’” (2011, p. 48). As the students engage in progressive mathematisation, they encounter the concepts first informally, then ‘pre-formally’, and only eventually at a formal level. The mathematisation and formalisation processes are illustrated in the teaching experiment described below.

The Bridge21 Model of 21C Learning

Bridge21 is a particular model of 21C learning developed in the authors’ institution (Lawlor, Conneely, & Tangney, 2010). It was originally used in an out-of-school outreach programme, and in recent years has been adapted for use in Irish post-primary schools. Currently it is being trialled in a number of schools as part of a systemic reform process in Irish education (Johnston, Conneely, Murchan, & Tangney, 2014). In this team-based pedagogical model, adults act as guides and mentors, scaffolding and orchestrating the learning experience. The model is innovative in that it offers a structured approach to the implementation of a 21C Learning activity, providing a set of steps to facilitate a successful intervention. The steps typically include: team formation; a divergent-thinking, ‘warm-up’ activity; investigation of the problem/challenge; planning; an iterative phase of task execution/problem solving/artefact creation; presentation; and reflection. Strict deadlines are enforced to encourage planning and ensure the teams stay on-task. The physical learning space is configured to support a collaborative, project-based, cross-curricular and technology-mediated approach, with an emphasis on individual and group reflection.

Technology usage (enhancement and transformation)

Use of digital technologies in mathematics education has the capacity to open up diverse pathways for students to construct and engage with mathematical knowledge, embedding the subject in authentic contexts and returning the agency to create meaning to the students. It can facilitate an emphasis on practical applications of mathematics, through modelling, visualisation, manipulation and more complex scenarios (Olive et al., 2010). However, Olive and colleagues (2010) also note that "it is not the technology itself that facilitates new knowledge and practice, but technology's affordances for development of tasks and processes that forge new pathways" (p. 154). The SAMR Hierarchy (Puentedura, 2006) offers a useful tool for describing different levels of technology integration in activities (Figure 1). The Bridge21 approach focuses on the creation of activities that fall within the Transformation space on the hierarchy. However, within the field of mathematics education, the use of technology to augment traditional approaches – outsourcing the calculation, increasing speed and accuracy, and thus permitting more focus on underlying concepts – is also seen as important. In the activities developed in this project, technology is incorporated in such a way as to create and support tasks that are meaningful and realistic for the students; it is not used merely to re-instantiate aspects of traditional mathematics teaching.

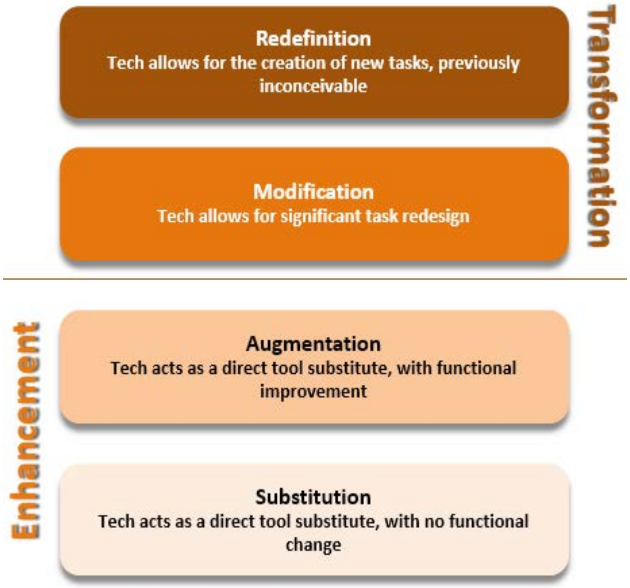

Figure 1: SAMR Hierarchy (Puentadura, 2006)

RESEARCH METHODS

The experiment discussed in this paper makes up one embedded unit within an overarching explanatory case study (Yin, 2014). To date, three such experiments have taken place within school settings and initial results are currently being analysed. A mixed methods approach to data collection and analysis has been taken, with considerable emphasis placed on qualitative data (Creswell, 2003; Yin, 2014). Qualitative analysis uses a directed content approach, and a pre-experimental design is used to analyse the quantitative data.

The Mathematics and Technologies Attitudes Scale (MTAS) (Pierce, Stacey, & Barkatsas, 2007) was utilised to gather quantitative data. MTAS is a 20-item questionnaire with a Likert-type scoring system. It has five subscales:

1) Behavioural Engagement: how students behave when learning mathematics

2) Affective Engagement: how students feel about the subject

3) Mathematical Confidence: students' conceptions of their ability to do well in the subject and to handle difficulties

4) Confidence with Technology: students' confidence in their ability to master technological procedures required of them and resolve difficulties

5) Attitude to using Technology in Mathematics: the degree to which students feel that technology provides relevance, aids their learning, and contributes to their achievement in mathematics.

The instrument was administered to students before and after the interventions, and paired t-tests were used to analyse the data (Creswell, 2003). While it is ambitious to expect meaningful data about such large and important issues from a 20-item questionnaire, the descriptors of the MTAS subcategories have been very useful to guide the qualitative analysis, permitting a more in-depth investigation of the themes.

Qualitative data came from focus-group interviews conducted 2 to 4 weeks after each intervention. The MTAS subscales were used as a-priori codes to direct content analysis of the interviews using NVivo10. Use was also made of codes drawn from a set of design principles for mathematics learning activities that fit within the technology-mediated, Bridge21/RME paradigm; their development is described by Bray, Oldham, and Tangney (2013). Some of the elements used as codes include: task design that is realistic, practical, and open-ended; teamwork; and transformative and computational use of technology. Matrix coding was used to identify associations between elements of the design principles and subscales of MTAS.

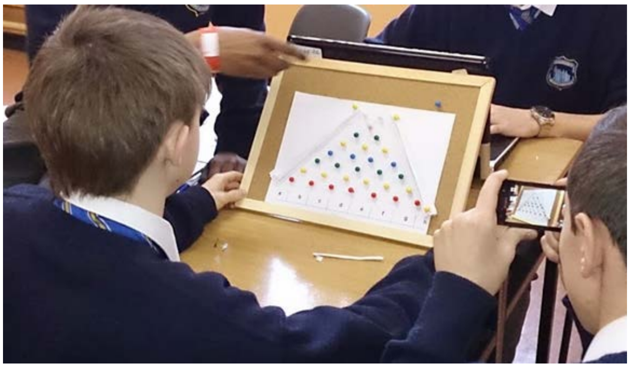

Figure 2: Plinko and probability

THE TEACHING EXPERIMENT

The students involved were from year 10 (age 15/16), which in the Irish system is known as Transition Year. This is a one-year school programme in which the focus is on personal, social, vocational and educational development, providing opportunities for students to experience diverse educational inputs in a year that is free from formal examinations (Department of Education and Science, 2004). Timetabling is more flexible than is the case for other school years, facilitating teaching experiments that are not constrained by short class periods. The first author had access to students for two days, from 10 am to 4 pm. During this period, she acted as the main teacher, or facilitator, with one classroom assistant. The class consisted of 20 male students of mixed ability, assigned by the class teacher to 5 groups of 4 students each, in such a way as to balance abilities. The environment was a large room with movable desks; each team was allocated a workstation, where they could work together. Laptops (two per team, to enhance collaboration), smartphones and other resources were provided.

Each of the two days followed the same general structure, based on the Bridge21 learning model of warm-up, investigation, planning and implementation. Throughout the day the facilitators interacted with the students, scaffolding their exploration of the mathematics and the technology. Based on the activity, the final section of the day was dedicated to a 'sales pitch' on day 1 and a competition on day 2; in each case this was followed by group presentations, in which the students discussed what they had accomplished and the mathematics they had understood. A whole-group discussion concluded the sessions.

The first day's activity was 'Plinko and Probability', which encourages students to develop a deep conceptual understanding of patterns, Pascal's triangle, probability and bias. Plinko is a game of chance based on a Galton board: a board with evenly spaced pegs arranged in staggered order, to form a triangle (Figure 2). Balls should be funnelled onto the board from directly above the top peg. If the pegs are symmetrically placed, the marbles have equal probability of bouncing left or right. A number of evenly placed slots form the base of the board, into which the marbles fall.

The students were informed that they were going to be developing a game for a casino and would have to devise the rules and scoring system in such a way that the game would be appealing to players, but that the casino owners would win overall. They were provided with a Plinko board template, a cork-board and some pins and marbles, smartphones, and laptops with open-source spreadsheet software and the free video analysis tool, Kinovea[1] (Figure 2). They were also given a sheet of exploratory questions relating to the possible paths on a Galton/Plinko board.

The aim of the activity was to encourage the students to make sense of what appears to be random behaviour. In particular, they were encouraged to identify that the number of routes to the pegs in the grid (starting from the top) forms Pascal's Triangle, and also to understand the probability of a marble landing in a particular bin if the board were perfect. In addition, they analysed their own boards, using the spreadsheet to tabulate and visualise 100 rolls. They were thus able to see how well their game conformed to a digitally generated one,[2] introducing the notions of bias and fairness. They used video tracking to see if any of the marbles they rolled followed the same path to any one bin, developing a practical understanding of the concept of probability.

1 www.kinovea.org

2 http://phet.colorado.edu/en/simulation/plinko-probability

Tasks that involve odds and chance are familiar to Irish students. In order to engage the students further, and add to the realistic aspect of the activity, they were required to decorate their boards, and develop the rules and scoring for their game, with the purpose of making a sales pitch to the facilitators ('casino owners') and to the year 12 (aged 17/18) students of the school ('players'). The successful team would be the one that was able to persuade both groups of the validity and attractiveness of their model. This aspect of the activity led to much heated discussion regarding the best way to organise the game, for example: "The stats are here!", "We get more money if we do it my way!", and "Just think about it, we make more money!"

In terms of the process of mathematisation, the first part of the activity, in which the students developed a physical board, and from that a mathematical model of the probabilities of their board, exemplifies horizontal mathematisation. Vertical mathematisation is evident in their generalisation of these probabilities into the set of rules for their games; in particular, some teams advanced beyond the basics and began to use the AND/OR rules of probability – beyond the curriculum for this year group – to set up more general models. Progress from informal through pre-formal and on to formal conceptualisation was facilitated by the tools available to the students, with the formal language introduced as the concepts were developed. In particular, the development and exploration of the boards encouraged the identification of the pattern of Pascal's Triangle, and the use of video-analysis and spreadsheet technology supported the development of formal mathematical models.

The work on day two involved collection, representation and analysis of data, line of best fit, correlation, causality, and extrapolation. It used the Barbie Bungee activity described by Bray and Tangney (2014) and Tangney, Bray, and Oldham (2015).

RESULTS OF THE TWO-DAY EXPERIMENT

Quantitative data

The sampling distributions were normally distributed according to the Shapiro-Wilk test. Paired *t*-tests identified gains in all MTAS subscale scores; the differences were significant ($p<0.5$) for all subscales except Mathematical Confidence.

Qualitative data

Coding matrices generated by NVivo facilitated comparison between MTAS and the design principles, in order to generate conjectures as to the primary factors that caused the gains in student engagement and confidence indicated by the MTAS scores.

The coding process is in its early stages. However, initial results suggest that the aspects of the design principles most associated with Affective Engagement were the realistic (in the RME sense), cross-curricular and guided discovery aspects of the task design; the Bridge21 activity structure; and the transformative use of technology, which facilitated the realistic nature of the tasks. Behavioural Engagement was also positively associated with the realistic, practical and guided discovery aspects of the task design, the activity structure and the transformative use of technology, but the impact of working in a team also appeared to have a positive effect. Mathematical Confidence was positively associated with the real, guided, and practical tasks, with the use of technology also appearing influential. The use of technology, both transformative and computational, was most significantly related to Confidence using Technology, with the variety of technology noted as adding to flexibility and adaptability. The transformative and computational use of technology, in conjunction with the task design, appeared to have the most influence on students' Attitude to using Technology in Mathematics.

DISCUSSION AND CONCLUSIONS

There is evidence from the results that technology-mediated interventions using the Bridge21 model and embodying an RME-style task design can have a positive impact on student experience in the classroom. The qualitative results in particular, indicate a encouraging increase in student engagement with and confidence in mathematics. The quantitative results also showed gains on all the MTAS subscales, although it should be noted that the gain for Mathematical Confidence did not reach significance.

It is worth noting, however, that the students taking part in the experiment described here appeared particularly favourably disposed to the approach – one student went so far as to say "... it changed the way I look at maths... it was a life-changing experience!" (Groups in other schools were positive but not quite so ebulliently so.) Also worth noting is that, while the impact of the collaborative, team-based approach was primarily positive, this is the one area in which some misgivings were expressed; one negative association was recorded between it and both Behavioural and Affective Engagement: "The groups [...] in our class, we all like, know each other, and people just like got pushed aside and lost motivation to do anything and were just a bit bored."

Ongoing data analysis is using an inductive approach, looking for emergent themes, not directly related to MTAS or the design principles. One of the most interesting themes to emerge thus far is the students' positive sense of ownership of their learning, which they are associating with mathematical confidence, reasoning that "because you can have your own idea, even if the teacher is explaining it wrong, or ... in a different way, it's like you have your own idea about it and you can add to what they are telling you to do".

This study set out to identify whether activities designed within a technology-mediated, socially constructivist, RME setting could increase student engagement with and confidence in mathematics, in line with some of the aims of the Project Maths syllabus. While initial results are promising, the relative novelty of the approach may be a contributing factor, and although the experiment took place in a school, it did so in a year in the Irish school system that allows for flexible approaches to curriculum and timetabling. If however, the findings can be replicated – both for repeated use with similar students, and for classes following syllabi leading to state examinations – it would augur well for addressing some of the shortcomings identified in the implementation of Project Maths to date.

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Crossing the bridge: From a constructionist learning environment to formal algebra

Eirini Geraniou and Manolis Mavrikis

Institute of Education, University College London, UK, e.geraniou@ioe.ac.uk, m.mavrikis@ioe.ac.uk

In the digital era, it is crucial to explore how digital technologies can be successfully integrated in the mathematics classroom and what their potential impact on learning is. This paper presents some reflections based on data gathered as part of the MiGen project (www.migen.org[1]) from studies aimed to investigate ways to support the transition to formal Algebra, through the use of a constructionist learning environment and carefully designed 'bridging' activities that consolidate, support and sustain students' algebraic ways of thinking. Our claim is that explicit links need to be made to Algebra through those specially designed activities so that such a digital tool can support students' learning of formal Algebra in order to be successfully integrated in the mathematics classroom.

Keywords: Generalisation, microworlds, transition, algebra.

INTRODUCTION

In the last few decades, the number of appearances of digital technologies designed for mathematics learning keeps growing. Relevant research (e.g., EACEA Eurydice Report, 2011), though, has shown that these technologies are not always used to their full or intended potential and also, students rarely use ideas, concepts or strategies they seem to have acquired through their interactions with such technologies. For example, Gurtner (1992), referring to the Logo environment, demonstrated that the tool's features which are designed to support students when faced with complex mathematical problems may impede them from making connections between their work in Logo and any mathematical or geometrical ideas they are already familiar with and use when problems seem less complex. Also, the lack of information on why and how to build bridges to formal mathematics, which were not often made in standard Logo situations, led to the lack of connections to formal mathematics (Gurtner, 1992). In this paper, we discuss our approach to support students' transition of moving back and forth from paper-and-pencil to interacting with digital tools and therefore consider ways of facilitating the integration of digital technologies in mathematics classrooms. In particular, our focus is on the transition to formal Algebra and how students 'transfer' their knowledge from their interactions with a digital tool, namely eXpresser, specially designed to support and address students' difficulties with learning algebra, to paper-and-pencil (PaP) activities.

There is a lot of research on the issue of 'transfer' (e.g., DiSessa & Wagner, 2005). Our interpretation is closely aligned with Beach (2003) who has argued that the metaphor should be viewed as transition instead of transfer. Crossing boundaries from one location to another is in fact a process of transition and therefore people are the ones who move and not knowledge or learning. In the case of Logo, Gurtner (1992) considered "the type of connections generally expected, and very seldom observed, between Logo practice and mathematics" (p. 247) as *transfer* and suggested that there is a need for a long period of practicing in Logo, especially one which is rich in reflection, so that some transfer to mathematics can happen.

Going back to our focus on Algebra, the transition to formal Algebra has been investigated by various authors (e.g., Radford, 2014) and the literature is replete with examples of student difficulties (e.g., Stacey & MacGregor, 2002). Students struggle to understand the idea behind using letters to represent *any* value (Duke & Graham, 2007) and are inexperienced with

1 The MiGen project was funded by the ESRC/EPSRC Technology Enhanced Learning Programme (RES-139-25-0381 2007–2011). Part of the research reported here was in the context of an ESRC 'Follow-on' project (ES/J02077X/1) and the M C Squared project, which was co-funded by the EU, under FP7 Strategic Objective ICT-2013.8.1 "Technologies and scientific foundations in the field of creativity" (Project No. 610467).

mathematical vocabulary. Even students capable of expressing a general rule through the use of words, like 'always' or 'every', struggle to use letters and symbols and form algebraic expressions.

Similarly to Radford (2014), who claimed that there is a need for specially designed classroom activity to support students' developmental path to formal Algebra, and to Gurtner (1992), who suggested presenting structured tasks, using appropriate microworlds and making explicit interventions during students' interactions, we claim that a digital tool specially designed to support the development of algebraic ways of thinking (AWOT) together with carefully designed bridging activities should 'smoothen' the transition to formal algebra without rendering it impossible for students to reach the mathematical 'bank of algebra'. Besides 'learning' the tool and developing expertise in using it, students should make the connections to mathematics. The issue is to find out ways for supporting students to make such connections.

EXPRESSER AND THE TRANSITION TO ALGEBRA

The MiGen system is a pedagogical and technical environment that improves 11–14 year-old students' learning of algebraic generalisation. Its core component consists of a microworld, eXpresser, which has been specially designed to help students develop AWOT through a series of generalisation tasks (Noss et al., 2012). In eXpresser, students construct figural patterns by expressing their structure through repeated building blocks of square tiles, and articulating the rules that underpin the calculation of the number of tiles in the patterns. A typical activity in eXpresser asks students to reproduce a dynamic model (or part of it) presented in a window that appears on the side of the activity screen.

Figure 1 shows a model where a row of red tiles is surrounded by grey tiles. Students are asked to construct a model that works for *any number* of red tiles, and find a rule for the total number of tiles surrounding the red tiles. They can test generality by *animating the model:* that is, by letting the computer change the number of red tiles at random. The design of eXpresser capitalises on animated feedback and on the simultaneous representation of a *specific* and *general* model ('My Model' and 'Computer's Model' in Figure 2), built by combining patterns and on the close alignment of the symbolic expression, the *Model Rule* and the structure of the model. All numbers in eXpresser are *constants* by default, referred to as 'locked' numbers. When the user 'unlocks a number', it is possible to change its value; it becomes a *variable*. In the *Computer's Model*, a value of the variable ('Num of Red Tiles' in this example) is chosen automatically at random (it is '10' in Figure 2) which will generally be different from that in the specific model ('6' in Figure 2). So the *Computer's Model* indicates to students whether their constructions are *structurally* correct for the different values of the variable(s). Students also construct a *model rule* for the total number of tiles, and validation of its correctness is made evident by colouring: tilings are *only* coloured if the rule for the number required is correct.

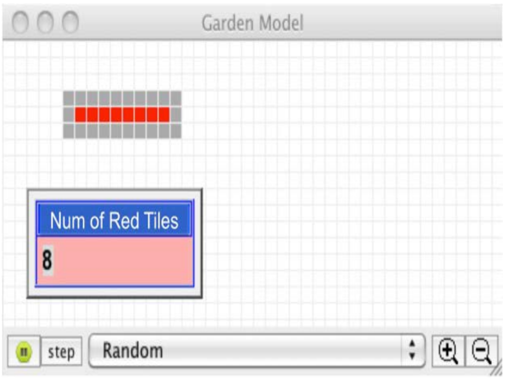


Figure 1: A model for 8 red tiles surrounded by grey tiles. Students must construct a general model and find the general rule

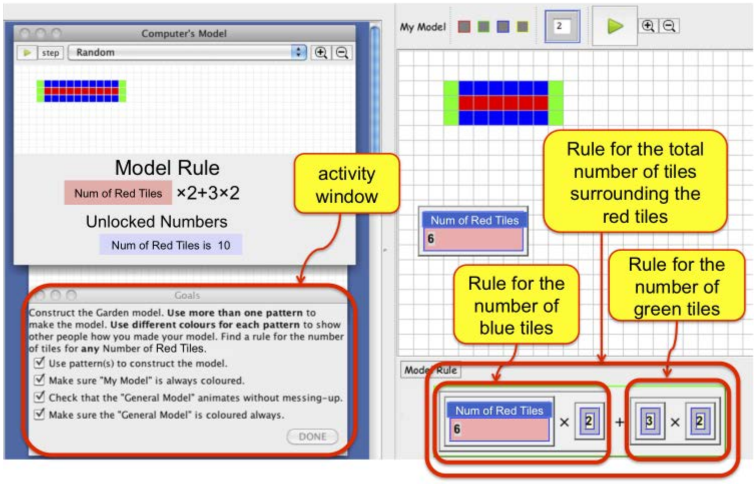


Figure 2: The eXpresser screen showing the general and specific models (Computer's Model on the left, and My Model on the right), and a correct rule for the total number of surrounding tiles. The task goals are shown in the "Activity window" (lower left-hand corner)

To make connections to formal Algebra feasible and support the transition from interactions with the eXpresser tool to PaP Algebra, we had to consider what characterizes formal Algebra and more specifically AWOT. Algebra involves a number of mathematical concepts, from numbers, to variables, from numerical expressions to expressions that involve the use of 'unknown' numbers and functions. Various authors have characterised algebra as 'generalised arithmetic' (e.g., Kieran & Chalouh, 1993). For example, Sfard and Linchevksi (1994) distinguished between the operational phase, where "the focus is on numerical processes and there is no hint of abstract objects rather than numbers" (p. 197) and the structural phase, which involves processes of manipulations of symbols. They argued, therefore, that there are "two crucial transitions: from the purely operational algebra to the structural algebra 'of a fixed value' (of an unknown) and then from here to the functional algebra (of a variable)" (p. 191). Leading on from these distinctions, Radford (2014) considered three conditions that characterise algebraic thinking: (i) *indeterminacy*, which is about recognising the use of 'unknown' values in the form of variables, parameters, etc.; (ii) *denotation*, involving the symbolisation of the undetermined values of the problem in question that can include the use of natural language, gestures, signs, as well as a mixture of these or symbols and (iii) *analyticity*, involving the skill of manipulating the indeterminate quantities like known values.

In the case of the eXpresser tool in our previous work (Mavrikis et al., 2013), we have identified two AWOT. The first one is:(i) *Perceiving structure and exploiting its power*, which is about noticing what stays the same and what is repeated in a figural sequence so as to understand how the sequence is 'structured', supporting therefore "the development of structural reasoning" and the habits of "breaking things into parts" by identifying "the building blocks of a structure" (Cuoco, Goldenberg, & Mark, 1996, p. 69). This AWOT, especially as it is operationalized in eXpresser that encourages students to construct what they perceive and manipulate the various properties of their constructions, could relate to Radford's (2014) indeterminacy and analyticity conditions, but also to the initial transition from the operational to the structural algebra of a fixed value of an unknown as described by Sfard and Linchevski (1994) above. The second AWOT is: (ii) *Recognising and articulating generalisations, including expressing them symbolically*, which is the process of translating the observed structure in an algebraic expression, using formal algebraic notation to write general rules for numerical sequences. This AWOT can be linked to Radford's (2014) denotation condition as well as the second transition from the structural algebra to the more functional algebra of a variable (Sfard & Linchevski, 1994), as its focus is on the production of formal algebraic expressions.

BRIDGING ACTIVITIES

We designed a sequence of activities both to help students become familiar with the tool but also to facilitate the transition to algebra. The sequence starts with introductory and practice tasks that ask students to construct figural models. It continues with individual activities, such as the one described above (see Figure 1). Students were asked to construct the task model in eXpresser using different patterns and combinations of patterns, depending on their perceptions of the task model's structure and derive a general rule for the number of square tiles needed for any Model Number. In our initial studies, students were presented with off-computer tasks, immediately after the final eXpresser task in an effort to reveal their strategies on solving similar tasks on paper and whether eXpresser had an impact on those strategies or not. In later studies, though, and after close collaboration with teachers, we recognised the need of activities, which promote students' reflections upon mathematical concepts and problem-solving strategies they used *throughout* their interactions with eXpresser and not just at the end. These we referred to as consolidation tasks. So, throughout their interactions with eXpresser and immediately afterwards, students were presented with four types of bridging activities (examples are given in Figure 3), which are designed to support their transition to paper-and-pencil tasks: (i) *Consolidation tasks*, which are usually short tasks that are used to intervene and encourage students to reflect on their interactions with eXpresser throughout a sequence of eXpresser tasks, (ii) *Collaborative tasks*, which are presented at the end of an eXpresser task and focus on students' justification strategies regarding rule equivalence, (iii) *eXpresser-like paper tasks*, which are figural pattern generalisation tasks on paper, and (iv) *text-book or exam like tasks*, which are the traditional generalisation tasks given to students on paper.

STUDENT DATA

Over the past 7 years, we have carried out studies in 6 different schools in London, worked with 11 mathe-

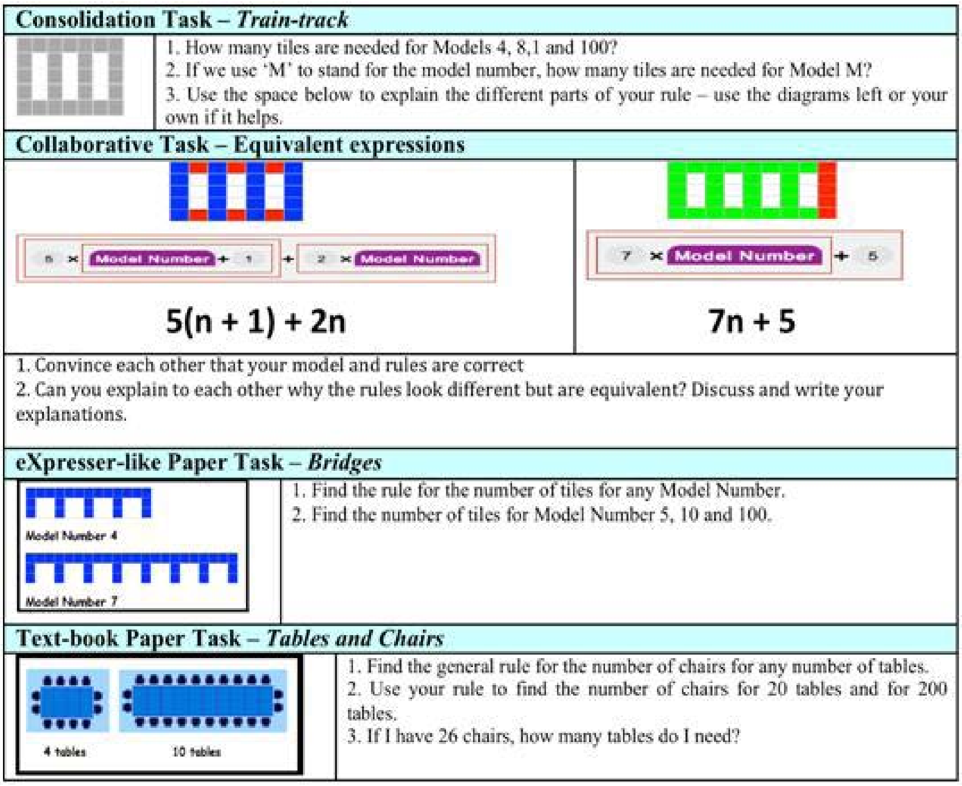


Figure 3: Examples of Bridging Activities

matics teachers and collected data from 553 students aged 11–14 years old. Each study was carried out over the course of four consecutive lessons, during which students became familiarised with the tool through some simple tasks, worked on one or two main activities and then were given bridging activities. A sample of students was interviewed at the end of their interactions with eXpresser. All students had been introduced to Algebra at school before their interactions with eXpresser, but of course their experience varied based on their age. Our data comprise one-to-one and small groups of students' and teachers' interviews and transcripts, video and audio files from interviews, one-to-one, small groups and classroom observations, detailed logs from students' interactions in the form of a database and bridging activities. Results from our studies are presented in a number of papers (e.g., Mavrikis et al., 2013; Noss et al., 2012; Geraniou et al., 2011). In this paper, we focus on the data collected from the bridging activities students worked on independently (or in pairs/groups of 3 for the collaborative tasks) during, but mostly after their final interaction with eXpresser. Using the two AWOT described in Mavrikis and colleagues (2013), as an analytical framework for interpreting students' strategies, we present our initial results under those two headings.

(i) Perceiving structure and exploiting its power. For the consolidation tasks, which were used with 175 students as their necessity was identified later in our studies, most of the 175 students demonstrated on the model figures presented on paper how they visualised the structure of the given model. In Figures 4, 5, 6 and 7, we present some examples of students' answers on the four bridging activities presented in Figure 3. Students clearly marked the different parts that would remain the same in any instance of the pattern and the parts, which, repeated every time, create the different instances of the pattern. Especially for the collaborative task, students verbally identified their building blocks in their models and rules and compared them to conclude about their equivalence. An example of two students' collaboration and its outcome is presented in Figure 5. Students demonstrated a variety of ways to visualise the task patterns and it was evident how influenced they were by the eXpresser's features as they were using the eXpresser terminology, e.g., number of building blocks or models. For example, in Figure 6 [F], [G] and [H], students drew the 2 building blocks that they could use if they were solving this task in eXpresser, that of a column of 3 square tiles and that of an 'L'- shaped one of 5 tiles. For example, Janet named her independent variable as "number of red BBs" (BBs stands for Building Blocks), and even though

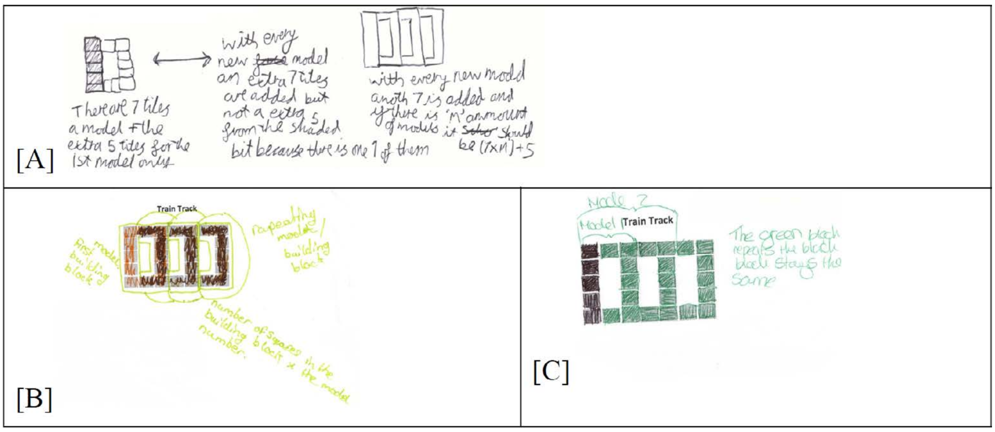


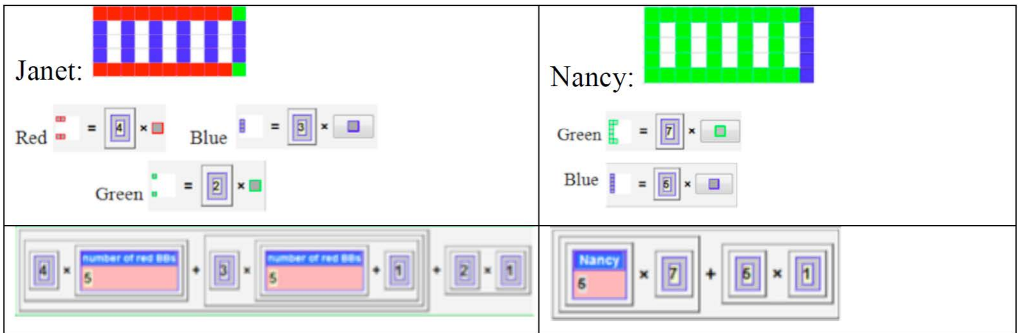
Figure 4: 13 year-old students' answers on the Train-track consolidation activity

Nancy, named hers as 'Nancy', she used eXpresser's terminology in her discussions with Janet.

(ii) *Recognising and articulating generalisations, expressing them symbolically*. Students seemed to rely on the structure of the given task model in order to articulate a general rule. Most of them provided clear explanations to justify their derived rules.

Their work revealed some fluency in using the formal algebraic language. They identified what stayed the same and translated that into a constant in their rule. For example, in Figure 6 [H], the student annotated their rule (5xM)+3 and showed that the coefficient 5 is the number of repeated building blocks in their second building block. The constant 3 is the number of tiles in their first building block, which is not repeated. Similarly, the student in Figure 6 [G] successfully identified 2 building blocks, that produce the task model, and indicated which building block stays the same and which is repeated.

Students' answers revealed their ability to articulate general statements, such as "with every new model, another 7 is added and if there's 'M' amount of models, it should be (7xM)+5" (Figure 4[A]) or "there is always 2 chairs to the ends of the single tables, then 2 chairs on the end of all tables put together" (Figure 7[I]). But the crucial step was their ability to translate that generalisation in parallel to their visualised structures into general rules and argue about similarities (or differences) between their models and derived general rules, when discussing rule equivalence (e.g., Figure 5). Most students used the eXpresser language and terms such as 'model number' to represent the variable in their rule (e.g., "5xwhatever model number n is+3", Figure 6[D]), as an intermediate step before expressing their derived rules in a formal algebraic expression (e.g., "(5xM)+3", Figure 6[H]). During collaboration, most students seemed to reach similar conclusions. Janet and Nancy for example recognized that the simplified general rule for their models is 7n+5



4n + 3(n+1) + 2x1	7n + 5x1

Nancy:	Yeah it's one red building block plus one blue building block so that would actually kind of make the…
Janet:	yeah, it would make the same shape...
Nancy:	because one red building block added to one blue building block…
Janet:	and that's the same as one of my green building blocks.

Figure 5: 12 year-old students' discussion on the Collaborative bridging activity

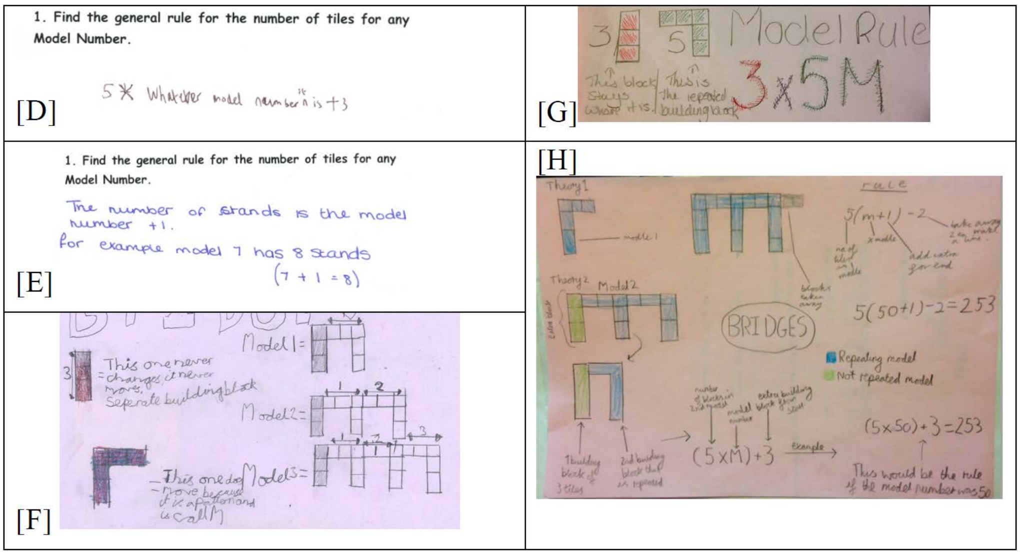


Figure 6: 12 year-old students' work on the eXpresser-like Bridges bridging activity

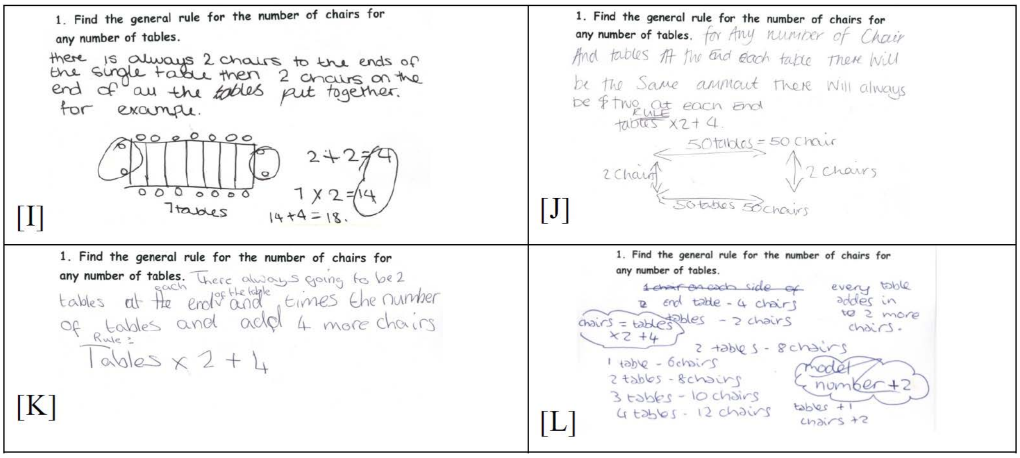


Figure 7: 12 year-old students' work on Tables and Chairs textbook-like activity

and that 'n' represents any model number. eXpresser seems to have played a crucial role in this outcome, as it encourages students to name their variables ('unlocked' numbers) based on what their values represent and therefore allows students to give meaning to that variable, thus easing students' transition to formal algebraic language.

Even though the bridging activities have been carefully designed to prevent students from looking for the term-to-term rule in a sequence, there were some students, especially in the text-book like bridging activities, who reverted to their past experiences and worked out the answers for each consecutive term in a sequence. For example, in Figure 7 [L], the student calculates the number of chairs when having 1 table, 2 tables, 3 tables, etc. Despite, their focus on the term-to-term rule, they spotted the correct general rule and wrote "Chairs=tablesx2+4". Such an outcome though may be ephemeral and more work is needed to support the sustainability and longevity of any AWOT formed soon after interacting with eXpresser.

CONCLUSION

When solving problems, mathematicians do not need to stop and think, but instead get into a "mechanical mode" (Sfard & Linchevski, 1994). Similarly, students who become experts in a digital tool may learn how to interact with it procedurally and provide right answers, but not necessarily reflect on and consolidate their knowledge during their interactions. Consequently, they may fail in developing a robust understanding of the mathematical concepts (and procedures) the tool was designed to help them with and may not be able to offer mathematically valid justifications for their actions. Such an outcome can discourage teachers from using digital tools in their mathematics lessons, as they are not convinced of their value.

In the case of eXpresser, the examples presented above reveal how students seem to successfully cross the 'bridge' from eXpresser algebra to formal algebra. Students demonstrated a conceptual understanding behind the development of general rules and generalised and adopted AWOT when solving PaP generalisation tasks. eXpresser, through the use of its language, supported students in their transition from numbers to 'unknown' numbers and variables and made the transition to symbolic thinking successful. In our experience, for such transitions to be successful, there is a need for bridging activities making the connections to algebra explicit. Their need and value have been mentioned by Gurtner (1992) too, who argued that 'the do-math-without-noticing-it' philosophy of Logo can be abandoned in favour of techniques that *explicitly* present looking for connections" (p. 253). We also recognised, similarly to Gurtner's (1992) research that "In contrast to the more classical transfer model [...] useful bridges can be built from the beginning, as soon as work has started in both domains" (p. 265). This was addressed by the consolidation tasks. There also seems to be the need for a long period of practice with eXpresser, rich in reflection and consolidation, before transfer to mathematics can be possible.

We have investigated the initial transition from a constructionist learning environment to the PaP algebraic generalisation tasks, and we have only started looking at the further transition to tasks that focus on abstract algebra, as described by Sfard and Linchevski (1994). The main concern is to identify and make more explicit the residual knowledge that gets noticed particularly by the interaction with constructionist learning environments. A successful integration in our view involves the successful transition from interacting with a digital tool to the awareness of the knowledge that can potentially be transferred to PaP activities and identifying ways to encourage the sustainability of such knowledge. Our aim remains to investigate further the issues of 'Transfer' and 'Bridging' and support the implementation of digital tools in the classroom through carefully designed and innovative bridging activities that consolidate and sustain students' mathematical ways of thinking.

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Digital interactive assessment in mathematics: The case of construction e-tasks

Galit Nagari Haddif[1] and Michal Yerushalmy[2]

University of Haifa, Faculty of Education, Haifa, Israel, gnagarih@campus.haifa.ac.il, myerushalmy@univ.haifa.ac.il

Our research focuses on the e-assessment of challenging calculus construction e-tasks designed to function as a dynamic interactive environment of multiple linked representations (MLR) that provide feedback to the learner. A construction e-task requires students to use technological affordances to construct examples that satisfy specific conditions. The e-task is checked automatically and intermediate actions and submitted answers are reported. We present here an example of a construction e-task and report on a pilot experiment designed to elucidate the role of the dynamic MLR environment in solving and assessing construction e-tasks. Specifically, we examine the student's submitted solutions and analyse whether it helps reflect the reasoning behind the answer.

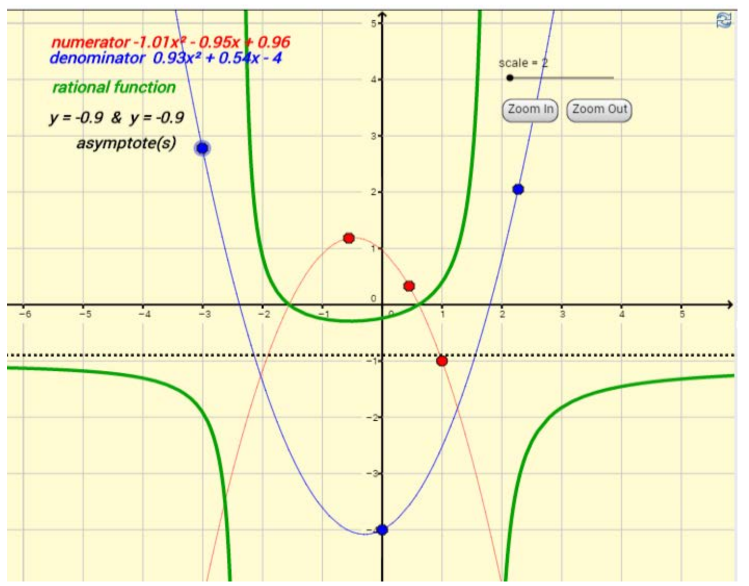
Keywords: Calculus, assessment, representations, technology, examples.

THRORETICAL BACKGROUND

The Joint Information Systems Committee (JISC)[1] defines e-assessment as the end-to-end electronic assessment process that uses information and communications technology (ICT) to present the assessment activity and to record the responses. By the term e-assessment we refer to forms of assessment that are created to be delivered, answered, managed, and marked mostly automatically, using ICT. We are especially interested in e-assessment of conceptual understanding of the content of calculus by high school students. In the present study we define an e-task as a technology-aided mathematical activity that engages students in seeing and doing mathematics. Scalise and Gifford (2006) introduced a categorization of innovative item types that may be useful in e-assessment. These types are based on categories of ordering involving successively decreasing response constraints, from fully selected responses (as in conventional multiple-choice questions) to fully constructed one (as in the traditional essay). The latter can be a challenge for computers to analyze meaningfully, even using sophisticated tools. According to the authors' review, technology makes a limited contribution to tasks that attempt to assess higher-order mathematical skills. Our research focuses on designing and studying challenging e-tasks in a dynamic MLR environment that provides reflecting feedback through e-assessment. We are particularly interested in high-school level e-tasks, which until recently have been limited mostly to closed multiple-choice questions.

The dynamic linkage of MLRs has been used as a design strategy that attracts attention to the relation between different representations. Yerushalmy (1997) and Yerushalmy and Schwartz (1999) have incorporated these strategies in the design of e-tasks in the areas of functions and calculus. The ability of digital tools to translate instantaneously across representations enable students to exhibit and evaluate actions in more than one representation system. Cognitive and pedagogical research suggests that appreciating the manner in which multiple representations are related is not automatic (e.g., Tall, 1991). Using multiple representations supports and requires tasks that involve decision-making and other problem-solving skills, such as estimation, selecting a representation, and mapping the changes across representations. E-tasks involving MLRs provide feedback to students that reflects the process of inquiry during the examination (reflecting feedback).

In computer-based instruction, feedback is any message or display that follows the learner's action or response. Vasilyeva and colleagues (2007) provided an overview of feedback studies and classified feedback into several types. For more than two decades, Yerushalmy and colleagues have explored the main differences between reflecting and judgmental feedback. Reflecting feedback provides immediate feedback to students about



(a) Construct a rational function: define the numerator and the denominator functions by dragging the dots marked on the functions.
(b) Fit the appropriate asymptote(s) to the rational function you have determined.
(c) Alternatively, choose a set of asymptotes and try to find a rational function with those asymptotes.
You can use this environment to explore the behavior of rational functions formed by dividing a constant, linear, or quadratic function by a linear or quadratic function.

Figure 1: Dynamic MLR learning object: rational functions and asymptotic behavior, http://tube.geogebra.org/student/m97472

their actions in multiple linked representations, and offers students the opportunity to judge and reflect upon the action taken. By contrast, judgmental feedback provides a short right/wrong answer, without other representations or explanations. Yerushalmy (1991) compared three groups of students performing symbol manipulations in algebra: one group received a reflecting graph feedback, the second judgmental feedback without a graph, and the third group used a symbolic manipulation aid. Yerushalmy found that judgmental feedback had a positive effect on the process, but that the students still lacked the motivation or ability to complete the tasks correctly. The effect of feedback was more significant on the first and third groups than on the second one (the manipulator aid motivated the students to obtain a correct product, and the graph feedback motivated them to make algebraic investigations).

Examples may be used for assessment in several ways. One obvious use is in refuting conjectures, either by citing standard counter-examples or by constructing new ones. Higher-level skills are needed when constructing instances of mathematical objects that satisfy certain properties, because typically there are many correct solutions but no general method by which such a solution can be constructed (Sangwin, 2003). When students are asked to create their own examples, they experience the discovery, construction, or assembly of objects and of their relationships (Liz, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). Occasionally, students create an example based on a ready-made example, which helps them reach the correct answer. A major affordance of technology is that it allows the easy creation of many different examples.

Figure 1 shows the GeoGebra applet, a challenging dynamic MLR learning object (http://tube.geogebra.org/student/m97472). To support inquiry learning of rational functions and of their asymptotic behavior, this object provides graphic and symbolic representations of the numerator and denominator (linear and quadratic functions of a single variable) and of their quotient. Users create instances of the rational function (Figures 1 and 2), explore and discover which functions have specific types of asymptotes, how many, and what are the reasons for it. During

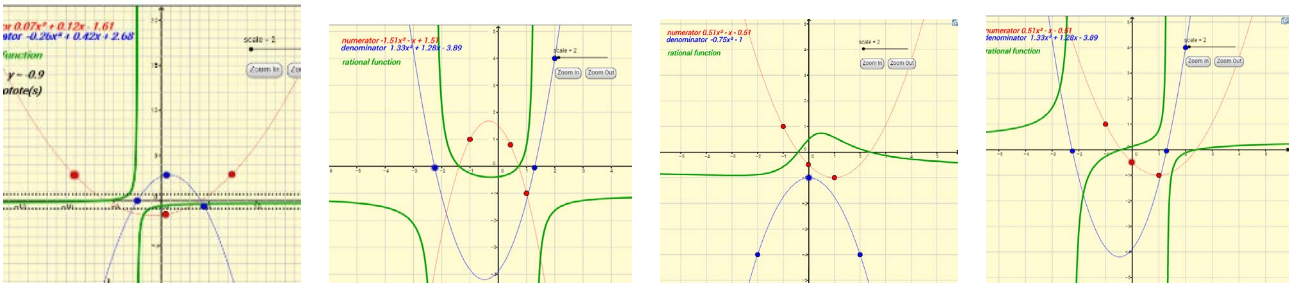
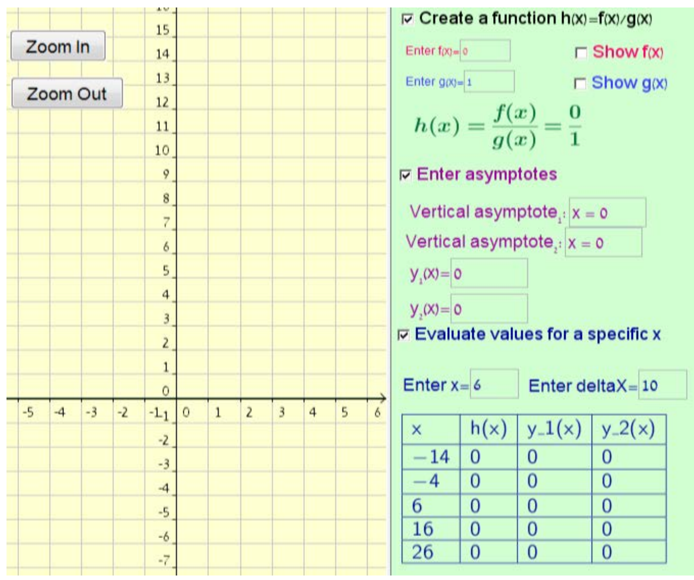


Figure 2: Examples/instances of rational functions of two quadratic functions

the exploration, the GeoGebra applet provides many instances of the same object (Figure 2), and students can evaluate their actions, reflect upon the feedback they receive, and change their conjectures accordingly (Santos-Trigo & Camacho-Machín, 2013).

GOALS AND FRAMEWORK

The current research is part of the Digital Interactive Assessment project at the University of Haifa (http://assess.gigaclass.com). A main challenge for the developers is to design e-tasks that on one hand invite opportunities for active personal learning and on the other set limits that pertain either to pedagogy or content (Watson & Mason, 2006). Our research arises from the challenge to design e-tasks that faithfully assess future learning and teaching. We are also seeking to create e-tasks that check automatically not only the correctness of the answer but also its justification, without the need for human check of written explanations. Because examples play an important role in justifying answers (e.g., Buchbinder & Zaslavsky, 2013) and can be checked automatically, we decided to include them in the present study. In this paper, we focus on construction tasks requiring students to construct examples that satisfy specific conditions. These e-tasks can be used to generate examples and provide tools for exploration. The tools are interactive MLR artefacts designed for mathematical experimentation, and may be part of the GeoGebra applet (the words tools and artefacts are used without any theoretical connotation; the tools in the e-task shown in Figure 3 are the value table, the coordinate system, and the symbolic input line). Each designed e-task appears as multimodal text. The designed tools are suitable for experimentation, and the answers submitted are checked automatically, almost without human intervention. The answers appear as live screenshots of the relevant representations within the tools, for example symbolic expressions, tables of values, and graphs. Therefore, they can be checked automatically. Figures 3 and 4 contain an example of a construction e-task (http://tube.geogebra.org/student/m440111) created by the authors. The e-task, which was inspired by the applet shown in Figure 1, also deals with vertical and horizontal asymptotes and has three representations: numeric (the value table), graphic (coordinate system), and symbolic (input line for the algebraic expression of the function). Students are asked to construct the requested function by typing an appropriate symbolic expression. Before the exercise, students receive instruction about using the numeric table (by typing x and deltaX as shown in Figures 3, 4), dragging points to create different instances of functions, using sliders (Figures 1, 2) using the zoom in and zoom out buttons (Figures 1, 3, 4), and about other technical issues. At this initial stage of the study, we intend to follow the students' reasoning mainly by asking them to submit their answers and to highlight the relevant component that justifies the answer. To make this possible, the task must provide multiple tools or operations for sending the answer. Moreover, students must be able to choose an appropriate numeric interval or a point in the table, or indicate on the graph the relevant segment. The value table in the e-task shown in Figure 3 is initially empty. Students



In each of the above parts you are required to submit a screenshot that supports your answer. You may write further explanations, if necessary. The diagram allows construction of the function in the form: $h(x) = \frac{f(x)}{g(x)}$. Define the numerator function f(x), the denominator function g(x), and the asymptotes by entering their expressions. You may use the value table and change x, deltaX.

Construct a function $h(x) = \frac{f(x)}{g(x)}$:

1) With one vertical asymptote $x = 4$ and one horizontal asymptote $y = 2$
2) With two vertical asymptotes $x = 4$, $x = 2$
3) With two vertical asymptotes $x = 4$, $x = 2$ and one horizontal asymptote $y = 2$
4) With two horizontal asymptotes $y = 2$, $y = -2$.

Figure 3: Example of construction e-tasks

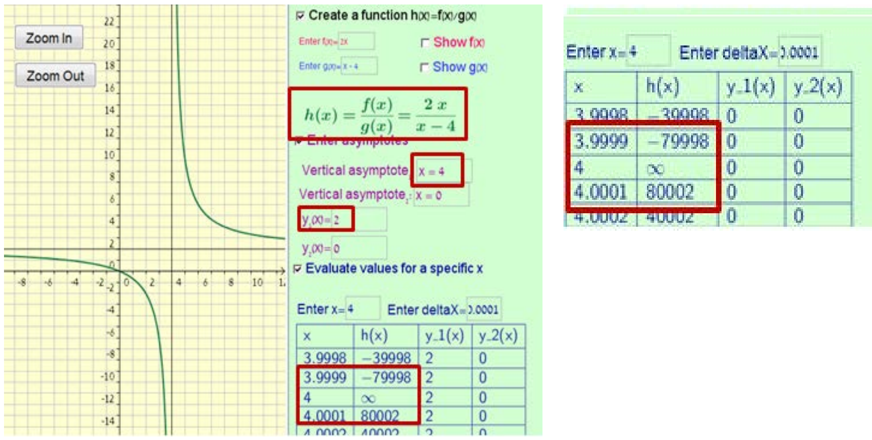
The task:	Possible correct solution: $h(x) = \frac{2x}{x-4}$
Construct a function in the form: $h(x) = \frac{f(x)}{g(x)}$ with one vertical asymptote, $x = 4$, and one horizontal asymptote, $y = 2$.	

Figure 4: A possible correct submission for a construction e-task

may enter an infinite number of values of x and its surroundings (deltaX). To support their construction, students must choose the appropriate values of x and its surroundings. Figure 4 shows a possible correct solution (the relevant components are surrounded by rectangles): the student constructs the function $h(x) = \frac{2x}{x-4}$ and builds the value table around $x = 4$, with $\Delta x = 10000$ to support the argument that $\lim_{x\to\pm\infty} h(x) = 2$. The second value table around $x = 4$, with $\Delta x = 0.0001$ supports the arguments that $\lim_{x\to 4^+} h(x) = \infty$, $\lim_{x\to 4^-} h(x) = -\infty$. Both tables support the student's construction and demonstrate that $y = 2$ is a horizontal asymptote and $x = 4$ is a vertical asymptote of the function $h(x) = \frac{2x}{x-4}$.

The automated checking system is under construction, therefore we can report only on a pilot experiment. The submission and checking of the answers were conducted manually rather than automatically. Our research question is: What is the role of the dynamic MLR environment in completing construction e-tasks. Specifically, we explore how students use the environment to solve the task and what is the added value of marking certain parts on the screenshot they submit as the solution in reflecting reasoning behind the answer. To obtain initial answers to the above questions, we conducted a pilot experiment.

PILOT STUDY

We report a few excerpts from the experiment we conducted with two pairs of 11th grade high school students: Iddo and Ilay, and Shira and Ayala. Each pair studied with the same teacher but in different schools. They studied the standard curriculum of functions and calculus in a regular classroom, without special emphasis on technology, and they successfully passed the Israeli matriculation exams.3 Each pair worked on eight calculus e-tasks created with the GeoGebra software (each e-task had up to four parts). Most of the e-tasks in the pilot experiment were construction e-tasks related to calculus, not necessarily asymptotes. In each e-task students were asked to mark certain parts on the submitted screenshot that reflect additional reasoning on their part regarding the answer. One of these e-tasks is shown in Figures 3 and 4. At the beginning of the experiment, the first author demonstrated all the functions of the applet. She was present during the experiment and answered technical questions (for example, how to enter the square root function into the input bar). The experiment was videotaped to capture the complete sound track and everything that happened on the computer screen. All subject matters included in the e-tasks are from the standard curriculum, and the participants were already tested on these topics at their matriculation exams. The students were asked to say out loud whatever they were looking at, thinking about, doing, and feeling as they went through their task. This enabled us to see the process of task completion as it was taking place, rather than only its final product, and to listen in on the problem-solving process. Figures 5 and 6 present different trials of the two pairs, including their conversation, as they were working on the e-task that appears in Figure 3. Both pairs were asked to construct a function in the form $h(x) = \frac{f(x)}{g(x)}$ with two horizontal asymptotes, $y = 2$, $y = -2$.

First pair: Ilay and Iddo

Ilay: It is difficult... It (the result) has to be square of something.

(Writing in their papers for several minutes.)

Iddo: Aaah! When you have $\sqrt{x^2}$, you have plus and minus. Try to do $4x^2$ divided by... no... it'll be reduced.

One minute later:

Iddo: Let's try $\frac{2x}{\sqrt{x^2}}$, then think that you divide the numerator in x and it equals 2, and then you divide the denominator in x and you put it into the $\sqrt{x^2}$ so that it equals 1. But there is another option of -1.

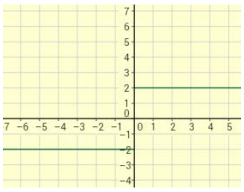
(Iddo refers to: $h(x) = \frac{2x}{\sqrt{x^2}} = \frac{2x}{\sqrt{x^2}} : \frac{x}{x} = \frac{2}{\frac{\sqrt{x^2}}{x}} \Rightarrow \frac{2}{\pm\sqrt{\frac{x^2}{x}}} = \frac{2}{\pm 1} = \pm 2$)

(First try on the screen.)

Iddo: Yes, right,

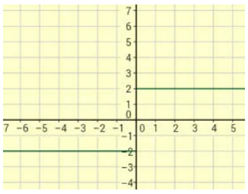
Ilay: It's all from our yellow book...

Iddo: Let's see what's wrong here.



On the screen:
$h(x) = \frac{2x}{\sqrt{x^2}}$
$y = 2, y = -2$

Iddo: These are not the asymptotes.

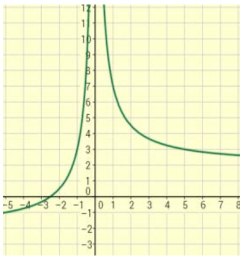


On the screen: $h(x) = \frac{2x}{\sqrt{x^2}}$
They remove the horizontal lines $y = 2, y = -2$

Two minutes later:

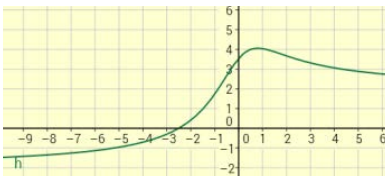
Iddo: Add 5 to the numerator. No, no... actually yes, it has to work, I think.

Iddo: Oh, add 2 to the denominator, because then if you have a negative value for x, then $x^2 + 2$ is positive.



On the screen:
$h(x) = \frac{2x+5}{\sqrt{x^2}}$

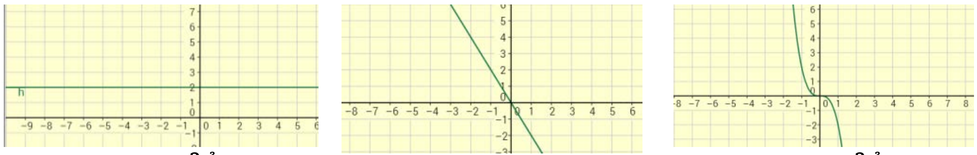
Ilay: Great!



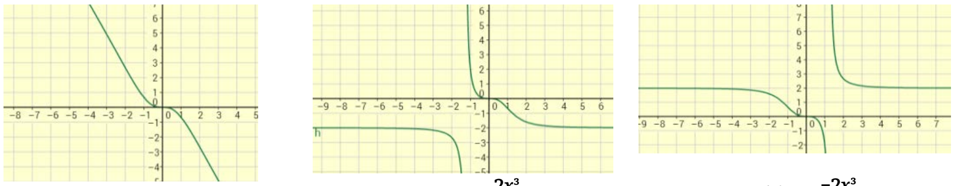
$h(x) = \frac{2x+5}{\sqrt{x^2+2}}$

Figure 5: First pair's conversation and attempts at construction

Second pair: Ayala and Shira

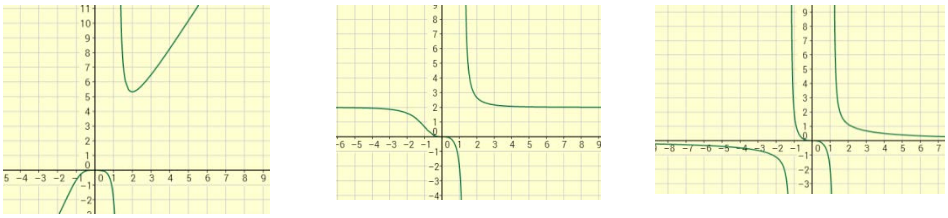


1. $h(x) = \frac{2x^3}{-x^3}$ 2. $h(x) = \frac{2x^3}{-x^2}$ 3. $h(x) = \frac{2x^3}{1}$



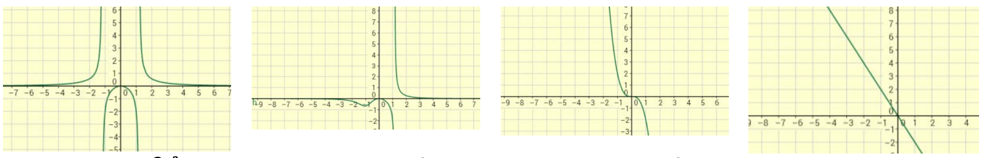
4. $h(x) = \frac{2x^3}{-x^3+2}$ 5. $h(x) = \frac{2x^3}{x^3+2}$ 6. $h(x) = \frac{-2x^3}{-x^3+2}$

Ayala: Do you remember that Felix (the teacher) read us from the book about horizontal asymptotes? The asymptote is the relation between the highest position, here and here (numerator and denominator)?
Shira: You know what? We can cheat a little.

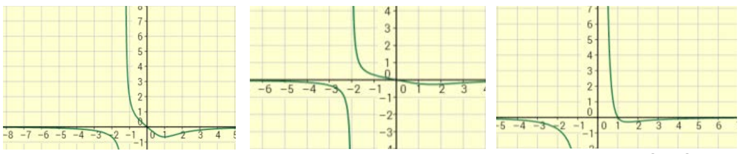


7. $h(x) = \frac{2x^4}{-x^3+2}$ 8. $h(x) = \frac{-2x^3}{-x^3+2}$ 9. $h(x) = \frac{-2x^3}{-x^4+2}$

Ayala: Now there is a horizontal asymptote y=0. Think about it mathematically, not graphically. If you divide this and this in x^4, it is cancelled.



10. $h(x) = \frac{2x^2}{-x^4+2}$ 11. $h(x) = \frac{2x^2}{-x^5+2}$ 12. $h(x) = \frac{-2x^3}{1}$ 13. $h(x) = \frac{-2x}{1}$



14. $h(x) = \frac{-2x}{x^3+2}$ 15. $h(x) = \frac{-2x}{x^3+8}$ 16. $h(x) = \frac{-2x+2}{x^3}$

Shira: I'm sure that if we had more time we would have found the answer.

Figure 6: Second pair's conversation and attempts at construction

DISSCUSSION AND CONCLUSIONS

Construction e-tasks have a great potential for e-assessment of higher-order mathematical skills. The e-tasks may have an infinite number of possible solutions, which all can be checked automatically. The solution stages (Figure 5) attest to the fact that submitting the correct answer is not accidental. Below we discuss the functionality of the dynamic MLR environment in solving and assessing construction e-tasks.

The first pair had no difficulties constructing the required examples (Figure 5). In each of the three trials they obtained a function with the required horizontal asymptotes, $y = 2$, $y = -2$. They did not submit the first trial because they were not familiar with the Heaviside step function, and did not submit the second trial because it had a vertical asymptote as well. Finally, they decided to submit the third trial. They appear to have started working with a firm conjecture in place regarding the functions with two horizontal asymptotes, and therefore constructed successfully the appropriate functions after a small number of educated trials. They used the dynamic MLR environment merely as a control, to provide feedback about their results. The conversation transcript (Figure 5) shows that their trials were accompanied by correct mathematical explanations. By contrast, the second pair used the MLR as a tool for empty trial and error experimentation. They made 16 trials (Figure 6), all of them involving polynomial or rational functions, which cannot lead to a function with two different horizontal asymptotes. Their conversation transcript (Figure 6) implies that they had no ideas or direction, and guessed without method. Shira suggested cheating the computer, and Ayala tried to recall the rules of her teacher for finding asymptotes. They did not have a ready-made example, which may have helped them reach a correct answer, and the tools were of no help in constructing such an example. They could not meet the challenge and therefore did not submit a function. The dynamic MLR environment provided the second pair with feedback regarding their incorrect answers. We may carefully state that the MLR tools in construction e-tasks can help those who are close to the correct answer, but when the correct answer is too far removed, the tool may encourage trial and error behavior. When the students constructed the correct function, they submitted the screenshot with the appropriate value table, as shown in Figure 4. In this case, other justifications, beyond the construction itself, are redundant. Although we report here only on the asymptote task, the evidence is consistent with other construction e-tasks included in the pilot experiment, but not reported here because of lack of space.

In the pilot experiment we saw repeatedly that the correct construction was accompanied by appropriate value tables, mathematical explanations, and educated trials. We therefore cautiously suggest that a correct final answer, if produced in a compound environment such as the MLR, eliminates the need for manually checking all the solution stages, enabling automatic checking of the solution. Tracking the solution process may be important to assess a partial solution, however, when the submitted construction is partially correct. The experimental setting would allow triangulation of human checks of scanned paper submissions, tracks of intermediate stages and simultaneous computerized video records.

ACKNOWLEDGEMENT

This research was supported by the Israel Science Foundation (grant 522/13).

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ENDNOTES

1. JISC 2009- http://www.jisc.ac.uk/assessment.html, accessed: December 2009.

2. Clearly, a value of a function cannot be infinite (±∞). But in the GeoGebra software it is the way to indicate that the value goes to infinity. This may explain the "∞" shown in the value table.

3. http://cms.education.gov.il/EducationCMS/Units/Mazkirut_Pedagogit/Matematika/VaadatMkzoa/BaaretzBaolam/.

A framework for describing techno-mathematical fluency in beyond-school problem solving

Hélia Jacinto[1] and Susana Carreira[1,2]

1 University of Lisbon, Research Unit of the Institute of Education, Lisbon, Portugal, helia_jacinto@hotmail.com

2 University of Algarve, Faro, Portugal

This study seeks to characterize the mathematical problem solving activity with digital tools that emerges from students' participation in an online mathematics competition. Using a qualitative approach, we elected the case of a 13-year-old participant aiming to understand the ways in which she interweaves her mathematical competence and her technological fluency for solving problems, using GeoGebra. Main results expose the role of the digital tool that permeates every stage of the problem solving process, since the tool is used for solving and for expressing the solution. We further propose a framework for describing the processes that may capture the interplay between mathematical knowledge and technological fluency in solving problems, termed techno-mathematical fluency.

Keywords: Beyond school mathematics, digital literacy, mathematical competitions, problem solving, techno-mathematical fluency.

INTRODUCTION

The constant immersion in a technologically pervaded world is changing the "kind of mathematical abilities that are needed for success beyond schools" (Lesh, 2000, p. 177), especially since the new and powerful tools made available are introducing "new kinds of problem-solving situations in which mathematics is useful, (...) and they radically expand the kinds of mathematical understanding and abilities that contribute to success in these situations" (p. 178). Whilst the kinds of mathematical thinking needed beyond school are shifting, the types of problem solving situations that demand some form of mathematical thinking are also changing. Furthermore, little is still known about the problem solving that occurs beyond the classroom and additional research is needed specially to understand the role of digital technologies in such activity (English & Sriraman, 2010; Santos-Trigo & Barrera-Mora, 2007).

A glimpse on the context: The mathematical competition SUB14

SUB14® is a web-based mathematical problem solving competition organised by the University of Algarve. Addressing 12-13-year-old students, it is supported by a website where the problems are published. The Qualifying consists of ten problems, each one published every two weeks. Participants may solve the problems using their favourite methods or tools, but they are explicitly required to report on their solving process offering a complete and detailed explanation of their reasoning. At this stage, the rules allow and encourage help seeking from relevant others. Participants who answer correctly to eight or more problems may attend the Final stage, which consists of a one-day tournament at the University of Algarve (see Carreira, 2012).

Our goal is to investigate mathematical problem solving with digital tools in this beyond-school competition, where participants may use their favourite digital tools but, at the same time, are required to use a mathematical stance. We report our progress on analysing how they combine their mathematical knowledge and their technological fluency in solving the competition's problems.

THEORETICAL BACKGROUND

This study is supported by a conception of inseparability between the subject and the digital tool in the development of mathematical thinking. Thus, we consider *humans-with-media* (Borba & Villarreal, 2005) as a central unit in understanding problem solving activity with technology. This metaphor brings forth the idea that mathematical thinking is an outcome of this symbiosis between people and the digital tools they use.

Mathematical problem solving – the mainstream view

The competition poses non-routine problems, whose context is fully and clearly expressed in the statement, which allow different techniques, procedures or tools. As these problems are not aligned with the mathematics curriculum and a diversity of approaches and tools is encouraged, this problem solving activity can be seen as the development of a productive way of thinking about a challenging situation (Lesh & Zawojewski, 2007). Current frameworks consider a mathematical literate person as someone who is an *active problem solver*, in other words, who has the "capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena" (OECD, 2013, p. 5).

Looking for a way of explaining student's and expert's problem solving performance, Schoenfeld (1985) proposed a model comprised of four dimensions: basic resources, heuristics, control, and belief systems. The processes followed by the solvers were organized into five stages: *read* – time spent "ingesting the problems conditions" (p. 297); *analysis* – attempt to fully understand the problem "sticking rather closely to the conditions or goals" (p. 298) that may include a selection of ways of approaching the solution; *exploration* – a "search for relevant information" (p. 298) that moves away from the context of the problem; *planning and implementation* – defining a sequence of actions and carrying them out orderly; *verification* – the solver reviews and assesses the solution. While paper and pencil were the predominant tools used in Schoenfeld's studies, today's wide dissemination of powerful technological tools is raising new queries, namely if and to what extent these frameworks still account for the mathematical problem solving proficiency in the presence of digital tools (Santos-Trigo, 2007, Santos-Trigo & Camacho-Machín, 2013).

Bringing together mathematical and technological literacies

In reporting studies that sought to identify the mathematical competencies needed in several workplaces, Hoyles, Noss, Kent and Bakker (2010) highlight an interrelationship of the technology use and the mathematical skills of the workers, proposing the notion of *Techno-mathematical Literacies* as encapsulating both the technological and the mathematical skills needed within those specific work contexts.

Debates concerning the digital skills needed in our daily activities are undergoing. The European project

Process	Problem or Digital Task
Statement	State clearly the problem to be solved or task to be achieved and the actions required.
Identification	Identify the digital resources required to solve a problem or complete a task.
Accession	Locate and obtain the required digital resources.
Evaluation	Assess the objectivity, accuracy, reliability and relevance of digital resources.
Interpretation	Understand the meaning conveyed by a digital resource.
Organisation	Organise and set out digital resources in a way that will enable the solution of the problem or achievement of the task.
Integration	Bring digital resources together in combinations relevant to the problem or task.
Analysis	Examine digital resources using concepts and models which will enable solution of the problem or achievement of the task.
Synthesis	Recombine digital resources in new ways which will enable the solution of the problem or achievement of the task.
Creation	Create new knowledge objects, units of information, media products or other digital outputs which will contribute to the solution of the problem or achievement of the task.
Communication	Interact with relevant others whilst dealing with the problem or task.
Dissemination	Present the solutions or outputs to relevant others.
Reflection	Consider the success of the problem-solving or task-achievement process, and reflect upon one's own development as a digitally literate person.

Table 1: Processes of digital literacy (Martin & Grudziecki, 2006)

DigEuLit developed a theoretical framework addressing the meaning and operationalization of "digital literacy" by describing the activity of a digital literate person when dealing with a digital task (Martin & Grudziecki, 2006). Those processes (Table 1) can be summarized as actions required before solving the problem (stating, identifying, accessing, evaluating, interpreting, organizing), while producing the solution (integrating, analysing, synthesising, creating, communicating), and actions that occur afterwards (disseminating and reflecting).

To some extent, this list of processes resembles the problem solving stages proposed by Schoenfeld (1985). Due to the rules of the competition, a mathematical problem may be considered as a digital task that requires a number of technological skills, so we conjecture that these two frameworks can provide the necessary level of detail for describing problem-solving-with-technologies. We, therefore, ponder an association of the *stages* and processes: *read* – statement (i.e., appropriation of the situation and the conditions in the problem); *analyse* – identification, accession, evaluation, interpretation (i.e., initial attempt to comprehend what is at stake, namely the mathematics that may be relevant and the tools that may be necessary); *explore* – organization, integration, analysis (i.e., the quest for a combination of mathematical and technological tools within a plausible strategy); *plan and implement* – synthesis, creation, communication (i.e., carrying out the outlined strategy); and *verify* could be complemented by dissemination and reflection since, at this point, there isn't an overlap between the two frameworks. Thus, the notion of *techno-mathematical fluency* stresses the need to be fluent in a language that entails mathematical and technological knowledge, promoting the skilful use of digital tools, and the efficient interpretation and communication of the mathematical solution produced.

Dynamic Geometry Software in problem solving

Constructing figures and investigating their properties are two of the most commonly known affordances of Dynamic Geometry Software (DGS). In particular, one of the features of DGS is that it "visually make[s] explicit the implicit dynamism of thinking about mathematical geometrical concepts" (Leung, 2008, p. 135) embedded in a challenging task. However, such dynamic environments are not only helpful in visualizing geometric concepts and understanding rules, but also in producing conjectures and generalizations, and finding connections amongst concepts (Baccaglini-Frank & Mariotti, 2010; Jones, 2000). When tackling model eliciting problems, students often perceive and make use of the affordances of the software in order to develop a conceptual model of the situation. They undertake a construction, revealing how they are interpreting the problem and depicting the mathematics concealed; they explore and investigate properties (Iranzo & Fortuny, 2011).

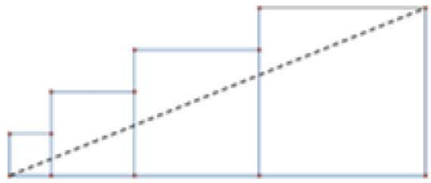
Carreira, Jones, Amado, Jacinto and Nobre (to appear) have identified six general affordances of GeoGebra for solving geometrical problems within the competition SUB14: immediate constructions, measurement, referential constructions, setting properties, constructions using parameters or variables, and drag and explore. Claiming that the perception of the affordances of GeoGebra is crucial to this problem solving activity, the researchers observed how several students tackling the same problem with GeoGebra produce different digital solutions that comprise qualitatively different conceptual models. We now argue that such differences may be explained by the solvers' latent techno-mathematical skills, regardless of where the mathematical and technological knowledge were learnt or developed.

RESEARCH METHODS

Our main goal is to develop a deep understanding of the interplay among mathematical knowledge and technological fluency during the development of the solving process within SUB14. Thus, we developed an interpretative study where the research methods were steered by qualitative techniques for gathering, organizing and analysing empirical data (Quivy & Campenhoudt, 2008).

We report the case of Jessica (fictitious name), a participant whose productions stood out due to the sophisticated use of technology, namely GeoGebra, for solving these problems (Jacinto & Carreira, 2013). Data include the solutions sent by Jessica in two editions of SUB14; an in-depth interview with Jessica, audio and video recorded, focusing her problem solving activity in the classroom and at SUB14, and asking her to remember and retrace solutions submitted to the competition; and two specific solutions, where she used GeoGebra, were also selected for a deeper analysis. Whilst the data from the interview provide a view of Jessica as a

Consider a sequence of squares of sides 1, 2, 3, 4,… cm, arranged so that they are connected to each other, as illustrated on the right. Once together, cut up all the squares by a line drawn from the bottom left corner of the smaller square to the upper right corner of the larger square. What is the area above the cut line, if the sequence has 8 squares?



Don't forget to explain your resolution process!

Figure 1: The problem "United and Cropped"

student, a problem solver and a technology user, the GeoGebra's construction protocols and the written explanations shed light upon the interplay between mathematical knowledge and technological fluency. As the rules of the competition explicitly require to document the solving processes, we assume that the final versions of the constructions performed with GeoGebra account for the interplay between Jessica's technological and mathematical knowledge to solve the problems and express her reasoning.

The several types of data were organized using NVivo®, where audio and video data were transcribed. The analysis followed an interpretative perspective providing a holistic description of the case, by combining Jessica's perception of her own problem solving activity (interview), with the analysis of the participant's productions (GeoGebra file), enlightened by the theoretical ideas discussed above.

THE CASE OF JESSICA

Jessica is a 13 years-old girl who engaged in SUB14 during her 7th and 8th grades. Her answers to the problems are always on time, she describes and properly justifies her processes in a clear language. She developed a particular interest on GeoGebra that stemmed out from her school experience, since her teacher used it often as a way to present geometrical contents. Despite being teacher-centred, it has driven Jessica to download, install and explore GeoGebra at home, independently.

Jessica: As I said, we use technology a lot. We have a board… a white board, and we also have an interactive board. We used GeoGebra very often when we were studying geometry and geometric transformations.

Researcher: When you say "we used", you mean the teacher?

Jessica: Precisely. And we watched it.

When asked to recall and retrace her solution to the problem "United and Cropped" (Figure 1), she claimed to enjoy solving geometry problems because of the possibility of improving the solutions' graphical display, afforded by GeoGebra.

Jessica: I think I went straight to GeoGebra. I knew it had something to do with geometry. I realise it was a triangle (…) and that by rearranging it in a simpler manner all I had to do was calculate the whole area and then subtracting the area of this triangle, which is easy: base times height divided by two. And then I thought… Oh, great! Geometry! I'm getting it neat!

Jessica usually resorts to a notepad, coloured pens, a calculator and the computer. Initially, she thinks that GeoGebra only affords "dressing up" the solution that she finds by using paper-and-pencil but, later, she acknowledges that manipulating the construction also led her to a powerful understanding of the problem.

Jessica: Hum… usually I look for the notepad and a pen, then [go to] Word and then I always… well I always use GeoGebra or some other software to add something to the text, for presenting a more complete work.

Researcher: So… you use it [GeoGebra] only after you solved the problem?

Jessica: Yes, but… it depends. If GeoGebra or some other tools would help me understand the problem, then I'd use it firstly and later I'd move to Word.

The image shows a large square divided into 14 smaller squares, coloured in yellow, of different but integer dimensions, and 1 white rectangle, also of integer dimensions. The white rectangle has an area of 30 464 cm^2.
Which is the area of the larger square?

Don't forget to explain your resolution process!

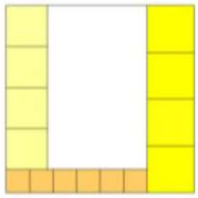


Figure 2: Statement of the problem "A divided square"

Researcher: Ok, so you also use them while you're still looking for the solution...

Jessica: Yes, for instance, in this case [the problem United and Cropped] I started by going to GeoGebra to understand it properly, and then I discovered 'Oh, that is a triangle right there, therefore I have to subtract the area of that triangle'. In that case, I started with GeoGebra for a better understanding.

Her solving activity starts outside the screen but she easily recognizes that digital tools afford powerful approaches to the SUB14's problems. The following section reports Jessica's work with GeoGebra while solving another geometry problem.

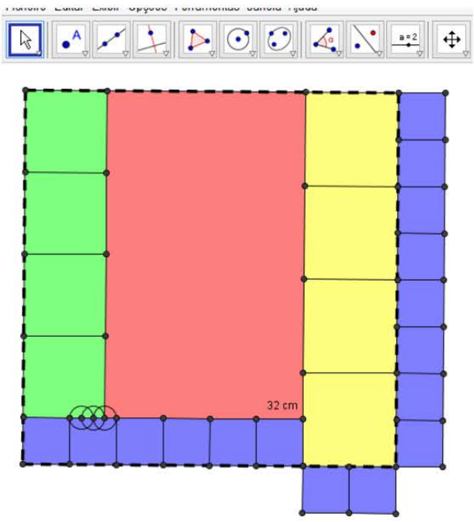
The problem "A divided square"

Jessica's solution (Figure 3) combines the construction of the figure presented in the statement and a written explanation where she presents a 'label' that helps in interpreting the image and her problem solving processes, including finding the area. A closer analysis of the 'construction protocol', which allows tracking steps of the construction, reveals that GeoGebra's role goes beyond 'embellishment'.

Resorting to immediate and referential constructions she represents the larger square that supports the construction: draws two perpendicular lines and a circle centred at their intersection and a radius of length CD. Then, she constructs four squares on the right by finding midpoints, using parallel and perpendicular lines and their intersections. Finally, she builds four squares and colours them in yellow (Figure 4).

As for the lower squares (Figure 5), Jessica marks R as the midpoint of FQ, then uses a central reflection of the point F over R and obtains F'$_1$ which coincides with Q. She marks I' as the central reflection of I over F'$_1$ and, using another central reflection of F over F'$_1$ marks F' and constructs the segment I'F'$_1$. She then marks S as the midpoint of I'F' and proceeds by using circles with given centre and radius, finds intersections and midpoints, and drawing parallel lines to complete the representation of the lower squares. Similarly, she constructs the remaining squares on the left side (Figure 6).

Finally, by setting properties, she colours polygons, adds several squares along the exterior of the initial square and some circumferences whose centres divide the side of a smaller square in four parts (Figure 3). These items emphasise a visual perception of the



The boundaries of the larger square are dashed. I have changed the colours of the smaller squares so I can differentiate them easily. Same colour squares have the same area.
yellow square side = 1/4 larger square side
blue square side = 1/8 larger square side
blue square side = 1/2 yellow square side
green square side = 1,75 the blue square side
Let us assign x to the side of the blue square.
Length of the red rectangle = $4{,}25\,x$
Height of the red rectangle = $7\,x$
Area of the red rectangle = $4{,}25\,x \times 7x = 30\,464 cm^2$
$2975\,x^2$=30464
$x^2 = 30\,464 \div 2975 = 1024$
$x = \sqrt{1024} = 32 =$ length of the side of the blue square
Length of the red rectangle = $32 \times 7 = 224$
Side of the larger square = $32 + 224 = 256cm$
Area of the larger square = $256cm \times 256cm = 65\,536cm^2$
Answer: The area of the smaller square is 65 536 square centimetres.

Figure 3: Solution of the problem "A divided square"

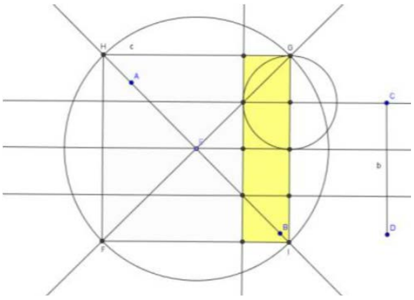


Figure 4: Constructing the initial squares

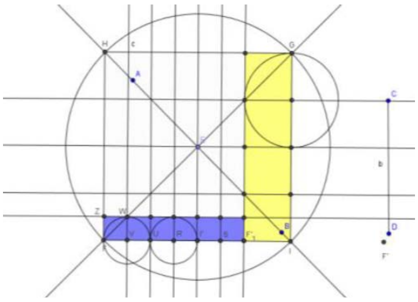


Figure 5: Constructing the lower squares

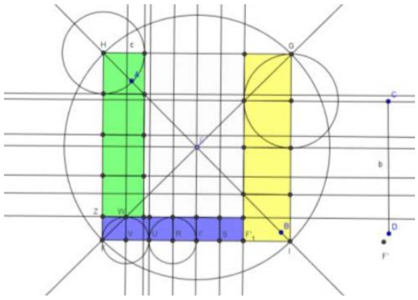


Figure 6: Constructing the left squares

existing relations between several lengths of the geometrical figures. On the right, a label helps interpreting the construction and establishing numerical relations between lengths of the sides of the squares. The unknown is defined as the length of the blue square and, using those relations, she formulates an equation that will provide the measurement that is missing: . With this value she determines the length of the side of the larger square and, then, its area.

This case illustrates how GeoGebra is crucial at several stages of the problem solving activity: it is not only affording the construction but, most importantly, the construction activity is allowing to uncover the relations between the geometrical objects, it is transforming what is invisible and concealed inside the proposed figure, into visible and usable ideas for the development of a way of solving this problem and expressing the solution. In fact, the constructions become part of the reasoning, of the process and the solution itself. This can be interpreted as an instance of the problem solving activity of a student-with-GeoGebra (Borba & Villarreal, 2005).

REFRAMING TECHNO-MATHEMATICAL FLUENCY

Taking the problem solving stages proposed by Schoenfeld (1985) and the processes of the digital literacy framework developed by Martin and Grudziecki (2006) as leverage, we have set an analytical tool to account for the ways a participant deals with digital tools whilst solving a mathematical problem and expressing its solution.

Jessica starts by skimming the mathematical topic enclosed in the problem, such as geometrical notions, rules, procedures, and recognizing GeoGebra as a key digital resource (*Read/statement*). The following stage (*Analyse*) is patent through the *identification* of a mathematical repertoire and a technological repertoire (geometry and GeoGebra's affordances) that are due to her previous knowledge about them and her *access* to them. Moreover, Jessica's choice relies on her *assessment* of the techno-mathematical resources and her *interpretation* of the techno-mathematical outcomes.

She then *Explores* ways of organizing different resources such as notepad, coloured pens, calculator, GeoGebra, text and image editor, e-mail, and several mathematical resources such as properties of parallel and perpendicular lines, circumferences and their representations, areas, algebraic expressions, and combines them in a relevant way to the development of her strategy (*organisation*, *integration*, and *analysis*).

Based on the constructions and their manipulation, she *Plans* and *Implements* her strategy recombining the techno-mathematical resources (*synthesis*) in order to produce new knowledge objects: strategies, representations, conceptual models (*creation*). During these processes, she may ask for the assistance of relevant others (like her teacher) to proceed in finding the solution (*communication*). It is important to note that the activity reported by Jessica and the analysis of the construction protocol suggest that the understanding of the problem and the decision on the actions necessary to solve it are not limited to the initial stage but it develops throughout the analysis and exploration stages and it is deepened during the construction and manipulation of the geometrical figures.

The last stage consists of reviewing the process and the solution (*Verify*) but it also includes presenting the solution to others, in this case, the GeoGebra construction and a detailed explanation of the procedure (*dissemination*). As for the personal *reflexion* on the success accomplished during the problem solving activity, there are no other concrete evidences to support it than the fact that Jessica has decided to present this solution to the judges of the competition.

CONCLUDING REMARKS

The two frameworks selected were meant to characterize the problem solving stages and the processes of digital literacy and, as such, their combination seems to offer powerful tools to approach a description of the latent processes underlying the notion of techno-mathematical fluency. However, some improvements must be pondered. Firstly, the processes of digital literacy are a set of actions that occur in a relatively ordered sequence, unlike the stages of problem solving that, as Schoenfeld (1985) disclosed, are flexible enough to describe failed attempts or new appropriations of the problem. So, the descriptors of the techno-mathematical fluency involved in problem solving within SUB14 must comply with this flexibility. There may also be an overstatement of the digital literacy processes, particularly in the level of detail included in the original framework. Our comprehensive knowledge about the competition and the participants allow us to assume that: i) they often choose the tools they are most familiar with, namely everyday digital tools available in their home environment, hence accession, interpretation and evaluation could result in some benefit if they were agglutinated in a broader process, bringing together knowledge and decisions about the digital resources; ii) the communication process, which relates to possible help seeking, permeates other stages of the problem solving activity, namely in understanding the problem or in devising a path; iii) the verification of the solution is not clearly addressed in the digital literacy processes, but it is a very important metacognitive process for assuring the completeness of the solution; iv) the dissemination process, not considered in Schoenfeld's model, is extremely important given the competitive nature of this activity and the unavoidable fact of having to submit a solution to those who are responsible for their acceptance and from whom a return is expected; v) solving and expressing are inter-related activities that are often inseparable (Jacinto, Nobre, Carreira, & Amado, 2014).

In light of the data and the theory, the notion of techno-mathematical fluency that emerges from the 'problem solving with technologies' activity is a useful way of accounting for the intertwining of mathematical knowledge and technological fluency. Future developments will concentrate on the refinement of the framework descriptors based on further analysis of other participants' problem solving activities, observed within the same informal learning context.

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Feedback and formative assessment with Cabri

Kate Mackrell

University College London, Institute of Education, London, UK, katemackrell@me.com

The University of Chicago Number Stories project aims to enhance student engagement in solving real-world problems in a Cabri environment through the provision of effective feedback. The relevant literature concerning feedback and formative assessment in technology situations is hence reviewed in light of the affordances of Cabri, and issues arising in the project, such as providing feedback in open-ended situations, are discussed.

Keywords: Cabri, feedback, formative assessment, real-world problems, technology.

INTRODUCTION

The *Number Stories* (NS) project (University of Chicago, 2014) involves the development of an online database of number stories, which are real-world questions based on real-world contexts supported by factual sources, targeted to individual users (or solvers) including school students at any level, teachers, teacher-educators, home-schoolers, district supervisors, and the general public. Unlike a traditional curriculum project where real-world problems may be used as a means to achieve specific mathematics learning goals, the main aim of the NS project is to promote understanding about how mathematics is used in daily life and to enable solvers to gain in their confidence and ability to apply mathematics in real situations.

Each number story consists of a collection of Cabri files in which the context is established, a question or problem is posed, and a solution is given. There is a wide range in the mathematics required. For example, questions from the "Chain Letters" context, in which one person sends a letter to n people who then send letters to a further n people, may be solved by techniques ranging from dragging representations of letters into mailboxes to using geometric series. Questions may be specific or more open-ended, such as "Who got the better end of the deal when Manhattan was purchased by the Dutch from the Indians in 1626?", which allows a number of approaches and does not have a well-defined solution. Solvers may post their own solutions to the database and explore the solutions posted by others.

A major concern of the project is to make the questions as engaging as possible to solvers through their intrinsic interest, the use of relevant digital manipulatives, and the provision of feedback that will enable solvers to be successful. This paper accordingly reviews research evidence concerning feedback and discusses some of the issues and the ways in which this evidence is being implemented.

The relevant mathematics education literature on feedback comes from classroom contexts where the aim of feedback is to enable the student to meet specific learning goals, not the aim of the NS project. More general learning goals are common, however, as is successful task completion, the difference being that in a classroom context this is taken as evidence that the learning goals incorporated into the task have been met.

DEFINITIONS AND DIFFICULTIES

Feedback may be defined as any information provided to the learner about a response (Wiliam, 2013). Based on Shute (2008), we define formative feedback as feedback given to the learner with the purpose of enabling the learner to modify their thinking or behavior in order to meet the goals of the activity in which they are engaged. Much of the relevant literature on feedback is concerned with formative assessment, which is broader than formative feedback in that learner responses are used more generally to adapt teaching and learning to meet student needs (Sangwin, Cazes, Lee, & Wong, 2010); this might, for instance, involve directing the student to an entirely different activity rather than giving formative feedback on the existing activity.

Digital learning environments such as Cabri provide direct manipulation feedback in which the technology responds to user actions, ranging from displaying text to changing the appearance of a graph or other screen object as its parameters are changed. Such feedback is neutral; any evaluation comes from the student themselves, as they discover which actions enable them to meet their goals.

Formative assessment, in contrast, involves the provision of a more deliberate goal-directed response, ranging from evaluating a student action as correct or incorrect through giving formative feedback such as hints, or changing the task completely by presenting a simpler version or directing the student to external learning resources.

Despite growing awareness of its effectiveness, (academic achievement gains ranging from half to a full course grade have consistently been found (Black & Wiliam, 1998) formative assessment is not widely implemented. Hattie and Timperley (2007) state that there is little feedback at all in classrooms, and that much of this is ineffective. Bellman, Foshay, and Gremillion (2014) identify the implicit, unrealistic expectation that most teachers can immediately adopt the formative assessment model and differentiate instruction. Lee, Feldman, and Beatty (2012) note that teachers find it difficult to gather and respond appropriately to student ideas during whole class discussion. Hence, although technology may offer new opportunities for effective formative assessment (as is currently being explored by projects such as FaSMED (2015)) we cannot assume that implementing effective feedback within Number Stories will be unproblematic.

We will next consider a particular framework for feedback and then explore the affordances and constraints for such feedback in a digital environment.

FRAMEWORK FOR FEEDBACK

Hattie and Timperley (2007) categorize feedback according to its purpose and have developed the following framework for feedback:

a) Feedback at the *task* level (FT) provides information about how well a task is being accomplished in relation to a goal.

b) Feedback at the *process* level (FP) provides information about the processes being used to accomplish a task.

c) Feedback at the *self-regulation* level (FR) provides information to help learners monitor and regulate their own actions towards a goal.

d) Feedback at the *self* level (FS) provides an evaluation of the student as a person.

This framework aligns well with the aims of the NS project, highlighting task completion, but in the context of developing solvers' problem-solving abilities in an environment where solvers are self-directed in their choice of problems to solve.

Hattie and Timperley's review shows that the effects of feedback vary according to level, though there are many mediating factors that still need to be explored. In particular, they argue that FS is least effective, FP and FR are the most effective for deep learning, and FT can be effective when the information in the feedback is "useful for improving strategy processing and enhancing self-regulation" (p. 91).

Feedback at the task level (FT) is the most common type of feedback, but too much feedback at this level may encourage students to focus on the immediate goal and not the strategies to attain the goal. It is more powerful when it results from faulty interpretations rather than a lack of understanding (where it may be better to have elaborations provided through instruction), and enables student to examine erroneous hypotheses and ideas, particularly when the learner has expected the response to be correct), and is most effective when it leads to the development of better strategies for processing and understanding the material. Simple tasks benefit most from FT, and simple FT is the most effective.

Process feedback (FP) is aimed at the process used to create a product or complete a task. FP is more effective than FT for enhancing deeper learning, and is most beneficial when it enables students reject erroneous hypotheses and gives cues to directions for searching and strategizing. Ideally, it moves from the task to the processes necessary to learn the task to regulation about continuing beyond the task to more challenging tasks and goals.

Self-regulation involves the way students monitor, direct and regulate actions towards the learning goal. It includes the capability to self-assess and to seek out feedback. Self-regulation feedback (FR) can lead to further engagement with the task, and enhanced self-efficacy. Bellman, Foshay, and Gremillion (2014) and Sangwin, Cazes, Lee, and Wong (2010) also identify the importance of FR and the way that formative assessment can develop self-regulation capacities.

The most problematic type of feedback is FS. Praise directed at this level bears little relationship to student achievement, and negative FS can undermine self-efficacy.

Some of the implications for NS are straightforward, such as replacing "smiling" and "frowning" faces with more neutral correct/incorrect symbols to avoid FS feedback, and keeping all feedback as simple and goal-focused as possible. Another implication is that feedback may not be enough and that sources of instruction may be needed. Other implications are more problematic: it is clear that, particularly for more open-ended tasks, FP and FR are crucial to sustain engagement, to experience successful task completion and to enhance problem-solving ability.

We next consider the extent to which feedback can be given in a digital environment.

FEEDBACK IN A DIGITAL ENVIRONMENT

The MiGen project (Mavrikis, Noss, Hoyles, & Geraniou, 2013) is a digital environment that has been deliberately designed to incorporate formative feedback and is aimed at supporting children's learning of mathematical generalization. The system includes an open environment in which students can construct and explore the structure within patterns, and an intelligent support system that gives students hints and clues about what to do next. Two major issues have been identified in the development of this intelligent support system which need to be dealt with in any digital environment incorporating feedback.

Interaction bandwidth

The interaction bandwidth of a situation refers to the available modalities and speed of communication that the situation affords (Mavrikis & Gutierrez-Santos, 2010). There are crucial differences between human communication modalities and those that are available to a computer-based system.

> The computer is limited in the amount of information it can obtain from the student, and is also limited in the amount and types of feedback it can provide. These limitations are both technical (c.f., state of the art in natural language processing and generation) and pragmatic (e.g., humans behave differently with computers, i.e. they listen and read with different interest or attention) (Mavrikis & Gutierrez-Santos, 2010, p. 642).

Examples of these limitations include the awkwardness of keyboard input of mathematical expressions and the tendency of students to do written calculations on scrap paper in paper-based mode but to try to work mentally in computer-based assessment (Stacey & Wiliam, 2013).

In contrast, in face-to-face communication in a technology environment, a human facilitator can, for example, speak, point to screen objects, take control of actions, and draw inferences based on facial expressions and gaze direction, far beyond what can be achieved by most computer-based systems. This has an impact on the quality of assessment possible. Sangwin, Cazes, Lee, and Wong (2010, p. 231) note that "The process with which a human teacher engages when assessing work at even this micro-level is both complex and subtle. Both for formative and summative purposes it involves them making many judgments rapidly."

Designing a feedback system involves finding ways to compensate for the reduced interaction bandwidth of a technology environment. This is of particular importance in the NS project; learning may be enhanced by teacher feedback if the materials are used in a classroom, but materials need to be designed to be accessed and used by individuals without teacher support. We next describe some of the possibilities for action, interpretation, and response in a technology environment.

Actions possible

A digital environment may be impoverished compared to a face-to-face environment, but is significantly richer than a paper-and pencil environment. Students may be given dynamic objects to manipulate, such as a 3D object to rotate, or linked representations of equations and graphs (Stacey & Wiliam, 2013), or spreadsheets (Sangwin, Cazes, Lee, & Wong, 2010). Students may also be given tools to construct or operate upon mathematical objects; tools for calculating, graphing, and manipulating algebraic expressions enable student attention to focus on problem-solving strategies, concepts, and structures, rather than mechanical processes. Students may also choose operations which are then performed by the computer (Sangwin et al., 2010).

Non-verbal ways of responding are also possible. Stacey and Wiliam (2013) argue that if a student needs to estimate the height of a tree after reduction by 30%, then dragging a slider to the estimated position provides a better test of estimation than selecting an answer from a list presented on paper. Student choices may also be constrained in ways that are pedagogically useful; given only the choice of "multiply" or "add", solving the equation $3x = 14$ requires the awareness that division by 3 is equivalent to multiplication by 1/3 (Sangwin, Cazes, Lee, & Wong, 2010).

Cabri, incorporating dynamic geometry together with graphs, calculation and the ability to create and evaluate expressions, gives the possibility for a wide range of dynamic objects to manipulate and tools to use, as well as containing more traditional features such as input boxes and multiple choice questions. Importantly, the task designer has control over the availability to students of both objects and tools, enabling a focus that is unusual in a DGS.

Interpretation of actions

The fundamental principle of mathematical assessment with technology is that various mathematical properties of a student's work on a specific mathematical question are established (Sangwin et al., 2010). Technology can also identify other potentially task-related aspects such as the sequence of actions taken, the number of attempts and the response time (Stacey & Wiliam, 2013; Pellegrino, 2010).

Input boxes and multiple-choice questions are easily checked for accuracy. However, feedback based solely on this may not be of particular value to the student, as it may consist of FT alone. These also constrain the type of task that can be given. In particular, geometric constructions and more open-ended questions cannot easily be assessed using these means.

The challenge is to find means to gain more sophisticated evidence on the basis of which FP and FR feedback may be given. This is dependent on both the way the task is modeled (discussed below) and the affordances of the technology. Sangwin, Cazes, Lee, and Wong (2010) note that some DGS (e.g., Cinderella and C.a.R.) allow teachers to set up assignments or exercises that can automatically check student constructions. All DGS can capture information about parameters such as coordinates, lengths, and angles which enables feedback to be given on tasks such as rotating a segment by 90 degrees, or dragging a point to make two segments parallel.

Cabri will soon be able to check constructions, can detect any of the above parameters and also the number of objects inside certain types of other objects. Such objects may also be assigned values that can be the basis of calculations. Cabri also has booleans, which enable sophisticated evaluation of numbers, such as how far the response given is from the correct response, or whether the response is congruent with an earlier estimate. It is also possible to measure response time and number of attempts, and student can make and save a recording of the steps taken.

Response to evidence

There are a number of ways in which the system can engage the student in evaluating their response, and hence enhance the possibility of FR feedback. One is to represent the student response in a different register, or otherwise work out its implications. For example, Sangwin, Cazes, Lee, and Wong (2010) ask students to give an example of a function with certain properties. The function given is then graphed, and the student can evaluate whether or not it has the required properties. Another example is where students are asked to integrate a function. The system finds the derivative of the function provided and asks the student to compare this with the original function. This is designed to encourage the students to differentiate in order to check the result for themselves. Geiger and Redmond (2013) ask students to create a function to model particular data. The system then graphs this function against the data in order to give immediate feedback about the suitability of the model. It is also possible to provide feedback about the immediate correctness of a step, leaving it to the student to evaluate their overall success in achieving the goal. An example here is the Digital Mathematics Environment facility to solve equations step by step, with the program providing feedback on accuracy at each step – but not indicating whether or not the particular steps taken were useful in reaching the goal (Drijvers, Boon, Doorman, Bokhove, & Tacoma, 2013).

In MiGen, responses come from the system, based on the properties of the student evidence (to be discussed below), and may range from variations on "correct" and "incorrect" to feedback that progressively provides more specific help through 'nudge' questions, comments, suggestions, and interventions (Gutierrez-Santos, Mavrikis, & Magoulas, 2010). There is, for example, a suggestion button which students are free to ignore that only lights up if the system observes an action that implies that help is warranted, and drop-down menus that allow students to choose a sentence and ask for help (Mavrikis, Noss, Hoyles, & Geraniou 2013). The advantage of technology here is the immediacy of the feedback, not possible from a teacher (Sangwin, Cazes, Lee, & Wong, 2010).

Apart from the calculus example (due to insufficient CAS) all the specific possibilities above may be implemented using Cabri.

It would hence appear that Cabri enables a rich enough bandwidth to open a number of possibilities for feedback.

Modeling the task to design feedback

The second major issue identified in the MiGEN project is that of modeling the task to design feedback. The importance of such a model is recognized by Pellegrino (2010), who, in describing systems which explicitly make formative interventions and give detailed diagnoses of student understanding, states that such systems are based on an underlying analysis or model of learning and performance in the content domain. Pellegrino recognizes the difficulty in achieving such a system, however, stating that systems with extensive diagnostic assessment capability, while desirable, require considerable research and development efforts. Even for an apparently straightforward task such as the Chain Letter problem in which the solver might be asked to find the number of letters sent after a certain number of iterations all possible solution processes and possible misconceptions would need to be identified.

This is particularly difficult in an open-ended environment. Mavrikis and Gutierrez-Santos (2010) identify a number of difficulties in the MiGen project including eliciting precise, concise, and operationalised knowledge from 'experts' such as teachers and the nature of the microworld:

> the freedom provided to the students to explore such environments makes it difficult to model the task, the learner and the relationship between the target domain and the knowledge which the system affords. (Mavrikis & Gutierrez-Santos, 2010, p. 641)

They found it crucial to involve students at all stages of development, as difficulties were problematic to predict; an in-depth understanding of user behaviour required observing and analyzing situations in their actual context. This modeling was necessary in order to generate the detailed set of messages used as feedback in the intelligent support system.

An implication for NS is that, with its large number of contexts and open-ended questions, the resources in terms of time to create detailed task models and software capability to analyze and appropriately respond (Cabri does not incorporate a programming language) are not currently sufficient to create an intelligent support system of the sophistication of that developed in the MiGen project.

NUMBER STORIES

Starting in July 2014, when feedback consisted simply of "correct" or "incorrect", we have incorporated an increasing range of types of feedback, starting with giving further characteristics of the solver response (such as its distance to the correct answer), but increasingly using techniques such as either on request or after a certain number of incorrect responses representing aspects of the question with further visual manipulatives, or asking solvers to make decisions as to the type of help that they would like.

One particular strategy that we are incorporating is to build in a level of direct manipulation feedback in which the solver can choose to see the consequences of their response, often translating an answer into a visual representation such as a graph or other visual model. For example, distances on the Earth may be shown with a model of the Earth cut with a plane at an appropriate distance from a point. One of the aims of such strategies is to increase the ability of the solver to self-assess.

We are finding that, at least for more straightforward questions, our feedback systems may be elaborated in the process of creating the activities as further possibilities occur, meaning that we do not necessarily need to develop our feedback support system in advance, and will be able to adapt and elaborate it further when the activities are field-tested.

We are looking at ways in which solvers can identify the mathematics that they need to solve a particular problem, particularly in problems that can be solved in a variety of ways. The chain letter context introduced above is an example, where a solution might involve dragging icons, multiplying, or using geometric series. The challenge is to direct the solver to the appropriate solution method while at the same time minimizing the amount of structure imposed. A possibility is to create separate problems, such as a situation involving smaller numbers for less advanced learners rather than attempt to address all possible solution methods in one question.

Some of the challenges we face arise from the scope of the project itself. The lack of specific learning goals, for instance, means that we have the temptation to focus feedback on the FT level, which may lead to merely instrumental problem-solving.

A second issue is that instead of giving any explanation of the mathematics required for learners who lack the necessary understanding to solve a particular NS problem we anticipate directing solvers to other sources. A tension is that such sources will use different contexts and hence may not be seen as immediately relevant. One possibility is to create guided discovery environments using the question contexts.

Another issue is that problems are designed to be independent, rather than follow any particular sequence. This mans that all our feedback needs to be problem-specific, and relatively exhaustive: we cannot refer to other problems which may help the solver with the current problem. It also means that there are a large number of content domains to analyze.

Our main challenge is that many of the most interesting and engaging real-world questions involving mathematics are both open-ended in approach and do not have well-defined answers. An example is the NS about the purchase of Manhattan. It is not possible to create detailed intelligent support systems for each of these questions, in which all solver actions could be responded to appropriately by the digital environment, and we are committed to the use of the materials not being dependent on teacher feedback. One possibility is to structure these questions to give specific choices of direction, or specific sub-questions to solve. However, this is likely to defeat the broader aim of enabling solvers to increase their problem-solving skills, a major part of which needs to be the ability to make appropriate choices. We are considering scaffolding the question around the mathematical modeling process, asking solvers to identify what they need to do to formulate a mathematical representation of the real-world situation, use mathematics to derive results, interpret the results in terms of the given situation, and if necessary revise the model (Geiger & Redmond, 2013, p. 121). Another possibility is to ask solvers to self-evaluate and evaluate others through exploring solutions posted to the database by other solvers. Strategies that enhance FR are hence not just desirable, but necessary, as the most informative feedback possible must come from the solvers themselves.

CONCLUSION

The Hattie and Timperley (2007) feedback framework is a useful fit with the aims of the NS project, involving task completion at the FT level, and, at the FP and FR levels, enhanced ability to engage confidently in the processes of mathematical problem-solving and greater awareness of the use of mathematics in the world. Their findings, together with those of others, give us ideas to incorporate, but also remind us that the best feedback to enable our most obvious goal, successful task completion, will also involve enabling the learner to achieve higher-order learning goals.

Cabri's comparatively rich variety of possibilities for student action, interpretation and response means that we have many possibilities for feedback to explore for structured tasks. However, it is not currently possible to develop a full intelligent support system in which all the processes in which solvers engage can be identified and appropriately responded to by the digital environment. We have hence identified FR feedback as critical and are seeking to develop the ability of solvers to self-assess through strategies such as enhanced direct manipulation feedback and comparison of their work with the work of others.

Field testing will commence in May, 2015, and will enable us to evaluate the success of the strategies that we are currently planning.

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Learning mathematics through programming: An instrumental approach to potentials and pitfalls

Morten Misfeldt and Stine Ejsing-Duun

Aalborg University, Research Lab: ICT and Design for Learning, Copenhagen, Denmark, misfeldt@learning.aau.dk

In this paper, we explore the potentials for learning mathematics through programming by a combination of theoretically derived potentials and cases of practical pedagogical work. We propose a model with three interdependent learning potentials as programming which can: (1) help reframe the students as producers of knowledge and artifacts, (2) support abstraction and encapsulation, and (3) promote thinking in algorithms. Programming is a topic that has recently gained interest in primary and lower secondary education levels in various countries, and hence a specific analysis of the potentials in relation to mathematics is paramount. Analyzing two cases, we suggest a number of ways in which didactical attention to epistemic mediation can support learning mathematics.

Keywords: Programming, Constructionism, APOS theory, Algorithmic thinking, instrumental approach.

INTRODUCTION

Programming and mathematics are often thought of as strongly connected activities. Partly because of their shared genes – the first computers were conceptualized and build by mathematicians – but also because programmers attend to logic, procedures, and functions in order to obtain their goals. Over the years a number of projects in mathematics education aimed at utilizing programming to obtain mathematical learning goals with the students. The earliest of these projects tended to collapse in mainstream implementation due a complex combination of lacking technological readiness of the school system, teacher competences, and more principal didactical difficulties with connecting programming activities to accepted mathematical curricular goals.

Recently, several countries have included basic programming in the national curriculum. In some of these countries (such as Estonia and France) programming is placed in direct curricular connection to mathematics, whereas in others (England, and Sweden) programming is related more to a design and engineering agenda. However, in all cases the focus is not on developing general "humanistic" skills with technology, rather it is on thinking in algorithms, writing programs, and developing technology. In other countries such curricular changes are being discussed and tested on a small scale. Hence, it makes sense to take a closer look at the arguments that have previously been proposed for utilising programming in mathematics education. In this paper, we will modestly attempt to describe these arguments, however in order to compare and combine previous thoughts on this topic we will employ the instrumental approach to the use of Information and Communication Technology (ICT) in mathematics education. The instrumental approach was developed in a French didactical tradition to meet the challenge that computer algebra systems posed to mathematics education and it has in the last decade become a European mainstream framework for addressing ICT in mathematics education.

In this paper, we investigate the mathematical learning potentials in programming activities by a combination of literature and empirical observations in classroom settings. Furthermore we explore if the instrumental approach can be activated in order to study these potentials empirically. We will describe some of the main intellectual projects and frameworks in mathematics education that used programming as a means to obtain mathematical learning goals. We suggest classifying these projects in three clusters; (1) *viewing students as producers*, (2) *supporting abstract thinking,* and (3) *developing algorithmic thinking*. Using the instrumental approach as theoretical framework we describe two educational situations utilizing pupils' programming activities in order to learn mathematics.

WAYS OF THINKING ABOUT TEACHING MATHEMATICS WITH PROGRAMMING

Investigation the intersection between mathematics and programming has many aspects. Mathematics and logic gave birth to programming with the pioneering work of Turing and others. Furthermore computing is influencing mathematical work in many areas of society. However the interaction between programming and mathematics that we focus on here, relates to curricular activities in primary and lower secondary school and we aim at understanding the potential synergies between learning mathematics and learning programming.

The tools that we choose to bring to mathematics students do influence the learning of mathematics that becomes likely or possible (Guin et al., 2005, Ainley, Pratt, & Hansen, 2006). And in that sense bringing programming into mathematics teaching does support certain types of learning. Bringing programming into the classroom with the purpose of learning mathematics easily leads to a version of the *planning paradox*; the more detailed the teacher articulates the mathematical learning goal, the more difficult it can be for pupils to appropriate programming as a personal instrument (Ainley et al., 2006).

Students as producers: Constructionism and a different mathematics

Serious attempts to use programming in teaching mathematics in primary and lower secondary school started with Seymour Papert. Papert's idea was simple–to create an interactive universe (microworld) that children access through mathematics, which prompts them to think mathematically by embedding nuggets of mathematical knowledge into the microworld that the pupils playfully stumble upon while developing projects.

As a means to obtain this goal, Seymour Papert developed the programming language LOGO, where the child steers a small turtle around the screen with commands such as "forward 10" and "right 90". The turtle can leave a trace allowing the child to create various geometrical figures. Papert's pedagogical strategy, constructionism, suggests that children learn in a particularly efficient way when they are engaged in developing constructs such as beautiful patterns, interactive art, computer games, etc., and in his bestseller, Mindstorms (1980), he describes LOGO as a 'mathematical microworld' that allows children to engage in such projects. The teacher's role in such work is to connect the children's work and intentions to "powerful ideas" from our mathematical heritage (Papert, 2000).

During the 1980s there was great enthusiasm and confidence that LOGO and similar programming languages would radically reform mathematics teaching in primary schools, and the first ICMI study on technology in mathematics education was focussed on how technology influenced mathematics as a topic (Churchhouse & International Commission on Mathematical Instruction, 1986). However, the results in mainstream implementation did not entirely live up to the expectations. There are a number of reasons for the disappointing results; for instance, students easily overlook the nuggets of mathematical knowledge (Noss & Hoyles, 1992, Ainley et al., 2006), making their work in the microworld non-mathematical.

Abstraction and concept formation: APOS theory

The idea that programming could be helpful in mathematics education in the late 1980s also developed in the context of teaching mathematics in high school and college. Here the geometric and artistically framed LOGO program was less popular. On the contrary, teachers often utilized common programming languages such as BASIC, COMAL and PASCAL to support learning. One of the outspoken hopes was to create a process-oriented approach to abstract mathematics, basing abstract constructions in concrete numerical computations. The arguments for this approach were often based in constructivism and radical constructivism, which claims that all abstract learning has a concrete starting point, as well as in the and in the discussions of process-object duality (Sfard, 1991). Ed Dubinsky's work is probably the clearest description of the learning potential of programming (see Breidenbach et al., 1992). His theory is often referred to as APOS theory and it is situated in a radical constructivist framework (Glasersfeld, 1995). APOS is an acronym for action, process, object, and scheme. The theory describes mathematical concept formation as beginning with performing actions on well-understood mathematical objects; these actions can be organized in processes and encapsulated into objects. These objects can be related to one another in schemas. The encapsulation stage is, as famously described by Sfard (1991), crucial and hard. And the schematic aspects of concept formation is similar to Skemp's relational understanding (1971). This rather general learning theory of mathematical concept formation relates to the use of computers because they can significantly empower and enrich the concrete numerical calculations that are – in this conception – the necessary foundation for concept formation.

Process approach to mathematics: Algorithmic thinking

The ability to think in algorithms and procedures is promoted as an important learning goal in mathematics. Algorithmic thinking describes students' ability to work with algorithms understood as systematic descriptions of problem-solving and construction strategies, cause-effect relationships, and events. A recipe is a good example of an algorithm: (1) Add all dry ingredients together. (2) Stir. (3) Add 2/3 cup of the water and stir. (4) If the dough is steady, then stir for 2 minutes. Otherwise, go to step (3) and add more water. Algorithmic thinking is about being able to develop, execute, and make machines to perform such algorithms. Donald Knuth (1985) views algorithms as a crucial phenomenon constituting the intersection between computer science and mathematics. He traces the study of algorithms to the mathematical masterpiece Al Kwarizm from the 9th century (Katz, 1993). Knuth defines algorithms as follows (Knuth, 1985, p. 170):

> I tend to think of algorithms as encompassing the whole range of concepts dealing with well-defined processes, including the structure of data that is being acted upon as well as the structure of the sequence of operations being performed; some other people think of algorithms merely as miscellaneous methods for the solution of particular problems, analogous to individual theorems in mathematics.

Hence algorithms, according to Knuth, consist of both a recipe and the actual objects dealt with by the recipe. Knuth analyzes the difference between mathematical thinking and algorithmic thinking. He finds that a first approximation algorithmic thinking relates to (1) representation, (2) reduction, (3) abstract reasoning, (4) information structures, and (5) algorithms. Mathematical thinking can, according to Knuth, relate to all of these, however other aspects are also present such as (a) formula manipulation, (b) behavior of functions, (c) dealing with infinity, and (d) generalization. Hence algorithmic thinking is strongly related to mathematical thinking but emphasizes specific and slightly different aspects than other types of mathematical thinking.

Before we introduce classroom examples exemplifying these learning potentials, we will introduce the instrumental approach that we use as a general theoretical framework for the use of ICT for mathematics teaching. This framework will be used to analyze the cases and create a connected description of the different learning potentials.

THE INSTRUMENTAL APPROACH

The instrumental approach (Guin, Ruthven, & Trouche, 2005) addresses students' use of technology when learning mathematics from the perspective of appropriating digital tools for solving mathematical tasks. It builds on (Verillon & Rabardel, 1995), and views computational artifacts as mediating between user and goal (Rabardel & Bourmaud, 2003). The approach presupposes a continuation and dialectic between design and use, in the sense that a pupil's goal-directed activity is shaped by his use of a tool (this process is often referred to as *instrumentation*), and simultaneously the goal-directed activity of the pupil reshapes the tool (this process is often referred to as *instrumentalization*) (Rabardel & Bourmaud, 2003, p. 673). In students' work with technology the distinction between *epistemic mediations* and *pragmatic mediations* (Guin et al., 2005; Rabardel & Bourmaud, 2003) operationalize the difference between learning with technology and just using technology to solve tasks. Epistemic mediations relate to goals internal to the user – affecting his or her conception of, overview of, or knowledge about something Rabardel & Bourmaud (2003) use the example of a microscope, and Lagrange (in Guin et al., 2005, ch. 5) refers to experimental uses of computers) and pragmatic mediations related to goals outside of the user – making a change in the world (Rabardel & Bourmaud use the example of a hammer, Lagrange (in Guin et al., 2005, ch. 5) refers to the mathematical technique of "pushing buttons"). Finally, Rabardel & Bourmaud (p. 669) introduce sensitivity to a broader conception of the *orientation* of the mediation. Instrumented mediations can be directed towards (a combination of) the objects of an activity (the solution of a task), other subjects (classmates, the teacher), and oneself (as a reflective or heuristic process). Hence the theoretical framework consists of the concepts: *instrumental genesis*, as consisting of *instrumentation and instrumentalization*, the concepts *epistemic* and *pragmatic mediations*, as well as a sensitivity towards the *orientation* of an instrumented mediation. The orientation of the mediation can be towards oneself, external objects, and other subjects.

EXAMPLES OF LEARNING MATHEMATICS WITH PROGRAMMING

These classroom observations are taken from the project *Children as Learning Designers in a Digital School*. The project is the realization of a research call from the Danish Ministry of Education. The research project is directed toward the area students own production and student involvement (Levinsen et al., 2014) and it explores:

1) How students' digital production impact on learning processes and the qualification of learning results regarding subjects and trans-disciplines; and

2) How ICT involves designs for learning that allow students to act as learning designers of their own learning practice in terms of form, framing, and content on their learning, engagement, and motivation.

The project comprises of a number of interventions in different schools. The examples in this paper come from a mathematics class where children in 5th grade (approximately 11 years old) program games for peer pupils to play and discuss using iPads and the software program Hopscotch. We present two activities that we suggest are related to the three different learning potentials described earlier.

Creating a good game: "It has to be fun"

The first example relates directly to the students' potential as artifacts producers. Oliver is trying to solve a problem–he wants to move his figure using tilt (i.e., by tilting the iPad). He asks the others for help. Instead of suggesting a solution, Ally asks him, "Why aren't you just tapping it?" Oliver answers, "Because it's a game, Ally. It should be fun."

The motivation for Oliver is obviously that the game he creates should be fun. Programming is merely a means for obtaining that goal. Throughout the course of the intervention Oliver gets really far in the process of making games. He is very independent and on his own he examines other games in order to, e.g., make points.

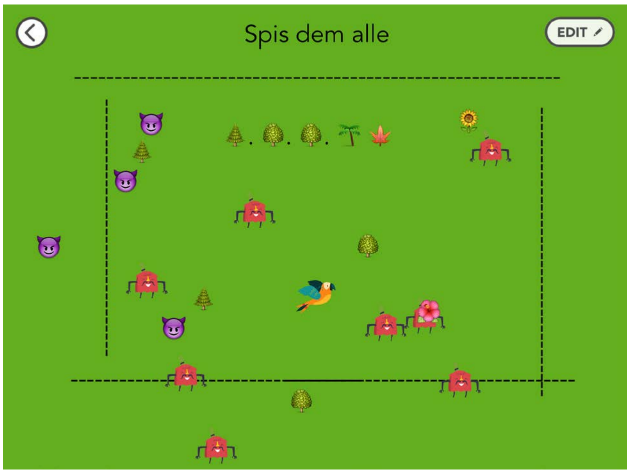

Figure 1: Oliver's first game, "Eat them all". The player controls the parrot by tilting the iPad. The goal is to eat the toasts and avoid the purple devils

For the same reason – wanting to develop good games – more pupils want to make countdowns, scoring systems, control with arrows, etc. They know the game genre well and what is needed to make a good game. These elements can only be done using variables. Despite the lack of algebra knowledge (algebra is considered "above their level" in the school), half the class voluntarily and with a high level of focus attends as the teacher demonstrates how to use algebraic concepts (variable and coordinate systems) to make an arrow control. In order to move one object (e.g., the avatar) by touching another object (e.g., an arrow) the pupils need to make a move-variable. Despite the emergence of this rapid algebra course, the pupils are not working with a task defined by the teacher. The teacher has merely defined a frame, "make a game", and the pupils themselves start defining tasks within it.

From an instrumental perspective the pupils' interest in understanding the mathematical concepts that the teacher are oriented towards can be viewed as a pragmatic end of creating a good game. Such an end is indirectly oriented towards peer students as players of their games. Obtaining the pragmatic goal of creating a game does require students to obtain epistemic goals – in this case about variables and the coordinate system – as sub-goals along the way. But understanding and acknowledging that there are mathematical sub-goals might not be so easy. In this case, mathematical sub-goals are strongly supported by the teacher's choice to create a "what kind of math do I need in order to make my game" crash course, deliberately focusing the pupils' attention on the mathematical aspects of creating arrow control and scoring systems.

Thinking in algorithms

The second example we initially see as relating to algorithmic thinking, but also to using programming as a way to support later abstraction and reification. The activity is introductory (just after the teacher has introduced the course structure and learning objectives). In plenary, the pupils and the teacher program a small cardboard figure that the teacher has set up on the whiteboard. They decide to call him 'Puff'. The teacher challenges the pupils by asking how to make Puff do various things and the following dialogue happened (translated from field notes):

Teacher:	Puff can only speak mathematics. How can I make him go right?
Zack:	Go right.
Teacher:	He does not know how far he should go.
Marc:	Go 2 centimeters to the right.
Teacher:	Yes, but unfortunately he does not know centimeter on the screen.
Austin:	Displace two units to the right.
Teacher:	Yes, units he understands. But he does not know what right is.
Ann:	You must move to a coordinate.
Oliver:	If he should go to the left, then: Go -3.
Ann:	Could you get him to go to a coordinate?
Teacher:	Yes, he would go there–but he would then fly around.

The teacher demonstrates her point by moving the cardboard figure from one point to another instead of sliding between points. She then shows how moving with a positive number makes Puff go right and negative number, as suggested by Oliver, will make him move left. After some discussion the teacher raises another issue:

Teacher:	They [the sprites in Hopscotch] are ego–they see the world from their own noses. How do you think he can go downwards?
Girl:	I am just guessing...can he rotate degrees?
Teacher:	Yes he can.
Zack:	Rotate 90° clockwise.
Teacher:	Now, you have to try programming each other. Give each other a rule and a signal. Use rotation and units. Make a square by controlling the other orally.

Pupils work together in pairs. They try to control each other. Zack and his teammate come over to the teacher and are frustrated. Zack says that he does not know which way to turn when she just says "turn 90° degrees". His teammate complains that he "just lies down on the floor" instead of moving around. The teacher talks with them about being precise and setting an x- and y-axis on the floor.

This example shows how pupils struggle with translating programming the figure on the whiteboard, which has two dimensions, to programming each other in three dimensions. Zack understood that the language should be precise, but he also teases on purpose. Several pupils have this kind of negotiation with mathematical concepts (turning, coordinate system, etc.), but some are also getting away with just saying "turn 90 degrees right/clockwise" without their partners correcting them. Those who are being controlled sometimes find that if they follow the instructions they end up walking into things, especially as the units are not precisely specified. Most use one step as a unit, some are using one foot as a unit.

By having pupils translate the programming activity to a classroom situation the teacher promotes reflection on the relationship between spatiality and algorithms. Another consequence of having pupils mediate programs by playing roles is that they get an understanding of what precision means. After this introduction the pupils described programming as a mathematical language that you 'speak'. From an instrumental perspective, the pupils here aim at affecting other subjects directly through programming and it has the effect that they negotiate what a good algorithm is and what it means to be precise in such instructions. It is a classical point that learning to program can benefit from attempting both to act as the creator of algorithms and as the performer (this is described as "playing turtle" by Papert, 1980). But it is interesting that the pupils' negotiation of the instructions is resolved by the teacher through introducing mathematical concepts (the 2-dimensional coordinate system). The teacher consistently introduces the solutions to pupils' problems in mathematical terms; this seems like a strong didactical strategy that supports the pupils talking and thinking about mathematics when they work.

THE CHALLENGE OF MAINTAINING AN EPISTEMIC FOCUS

In this section, we will discuss whether the mathematics learning potentials that have previously been suggested in the literature about programming and mathematics education can be viewed as genuine mathematics learning potentials in the sense that they involve epistemic mediations towards mathematical concepts. It is obvious from the cases that the pupils need help with mathematical concepts when they try to appropriate a programming language to develop their games. The two examples show how such situations can facilitate the engagement with classical mathematical concepts such as numbers, the coordinate system, and orientation/angles. In both cases the pupils interact with the teacher, each other, and games developed by others in order to handle the challenges. But one can discuss whether the overall pragmatic purpose of improving their skills with Hopscotch and potentially making a better game support or hinder the pupils' epistemic focus on mathematical concepts. The analysis shows that it is – in these specific cases – not reasonable to disregard this as only a pragmatic mediation with little educational value. But this could very well have been the case if the teacher had not been so careful in attracting the pupils' attention to explicit and relevant mathematical ideas. However, the pupils also bring in mathematical ideas (for instance, about angles) without being prompted by the teacher. Hence it would be meaningful to investigate further how the classroom norms and shared ideas about mathematics (the sociomathematical norms discussed by Cobb, Stephan, McClain, & Gravemeijer, 2001) affect the mathematical value of introducing programming.

By using the instrumental approach it is apparent that pupils' epistemic relation to mathematics is necessary for programming to be successful in mathematics education. We see several ways that this epistemic relation can be strengthened or hindered. The teacher did acknowledge and talk about the different goal-levels of an activity; this allowed her to talk directly about mathematical goals, even though these where sub-goals of the larger goal of creating a good game. Constantly focusing attention on mathematical concepts as problem solvers and conflict settlers were also actively applied by the teacher, especially when the pupils programmed each other. When the pupils are either negotiating or in cognitive conflict, this teacher turns to mathematical concepts and principles as part of the way forward for the pupils. In that sense our analysis suggest that the potentials for learning mathematics through programming, as previously described in the literature, depends largely on the teacher's approach and didactical principles.

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Reasoning with dynamically linked multiple representations of functions

Guido Pinkernell

University of Education Heidelberg, Heidelberg, Germany

In video-taped interviews, teacher-students were asked to describe and explain effects of a parameter a on the standard representations of $f(x)=x^2+a$ in a computer based dynamic learning environment. An analysis of one particular interesting yet typical misconception leads to differentiating between surface perception and structural insight. Theoretical considerations based on Duval and Davydov lead to postulating that for a full understanding of the relation between a and $f(x)=x^2+a$, a learner needs to identify to structural analogies between the representations of f. A qualitative analysis of further interviews results in a category model of student responses that can be used for diagnostic purposes.

Keywords: Multiple representations, functions, abstraction, ICT, qualitative analysis.

INTRODUCTION

A typical task connected with the use of mathematical software in classroom is to explore the relations between the standard representations of functions. For example, pupils could be asked to explore the effects of a parameter a in $f(x) = x^2 + a$ on the shape and position of the graph of f by means of a dynamically linked multiple representation learning environment, where the value of a is controlled by a slider (Figure 1).

Usually, it seems sufficient to observe that changing the value of a causes the parabola to move upwards while a gives the distance. But is it really as simple as that? Just describe what seems, literally, obvious?

Duval (2002) argues that for showing a full understanding of the concept of function, a learner needs to be able to change within and between various representations of a function, for example, equation, table and graph. This means that properties of one representation are explained by properties of another. If a learner is not able to perform such a change then, following Duval's rationale, he does not understand to the full extent, even if his observations within one representation appear to be perfectly valid.

This article begins with a case study that illustrates Duval's concept of understanding functions. A student describes the effects of the parameter a on the graph of $f(x) = x^2 + a$ (Figure 1). She then realises that one of her observations contradicts what she has learned about how graphs and equations connect, but she is not able to resolve this contradiction. She knows a lot, yet she does not quite understand.

So it appears that learning about functions based on visual perception only is not sufficient. For developing a sound understanding a learner needs to turn his attention from the visual properties of the different representations to structural analogies between them. These analogies do not equate to perceivable similarities between the representations. For identifying structural analogies one needs to "see through" the specific appearance of each representation. In this sense, the identification of structural analogies is a form of abstraction. Hence this article then turns to the concept of learning by scientific abstraction by

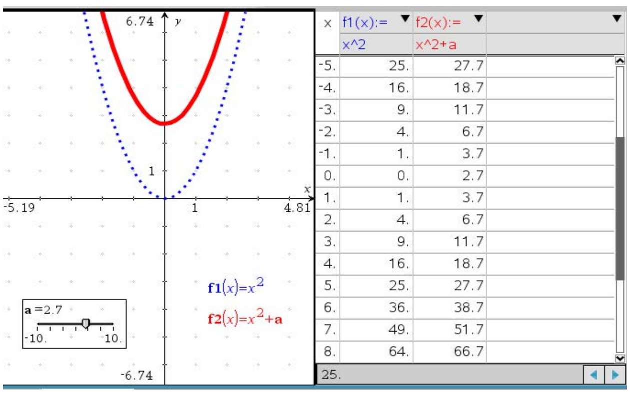

Figure 1: Dynamic multiple representation environment for exploring the effects of a on the representations of $f(x) = x^2 + a$

Davydov (1972). It serves as a suitable theoretical basis for justifying the postulation that learners need to refer to a structural level.

When theoretical considerations lead to expect that learners behave in a specific way then we must be certain that they are able to do so. The last part of this article reports on a qualitative analysis of further interviews with teacher-students where three categories of understanding could be identified, among them references to structural analogies between representations as required by theory.

A CASE STUDY

In videotaped interviews, teacher-students of the University of Education Heidelberg were given the task as shown in Figure 1. They first were asked to describe the changes within graph, table and term when the slider is operated. Then they were asked to explain why they thought their descriptions were correct. One student described the effects of the slider as both a translation and a change of shape:

1 Student: [moves slider to the right] well the parabola moves upwards along the y-axis ... the um how is it called the width [moves both palms repeatedly towards each other as if clapping hands]
2 Interviewer: yesyes, ok
3 Student: changes ... anyway when one moves it to the right ... towards the positive ... [moves slider to the far right such that the parabola nearly vanishes from the screen] ... and when one moves it downwards to the negative [moves slider to the left such that the parabola's vertex nearly touches the lower screen edge] and here the parabola becomes wider but still opens upwards
4 Interviewer: can you explain why the parabola moves up or down when one changes the slider value [points at the slider with a pen]
5 Student: [looks a the slider, murmurs] hm what is a ... what ... is a [leans back, talks louder] a is ... a was something ... a is ... the y-direction
6 Interviewer: ah
7 Student: when I ... upwards ... well then it is not a normal parabola any more then it is, like, somehow narrower

When asking the student to explain her observations (line 4) the interviewer, who is the author of this text, ignores that she describes the effects of a as a translation as well as a change of shape of the curve (lines 1 and 3). The student answers by referring to knowledge about the effects of a she probably had acquired at school (line 5). But, again, she immediately points out that the curve looks shrunken (line 7). This seems to be a problem for her as it conflicts with what she has learned about parameters and graphs (meanwhile the value of a has been set to 1):

8 Student: this is simply a normal parabola that has been moved upwards by one, but the width changes too.
9 Interviewer: why do you say 'but'?
10 Student: the parabola does not move upward only but it becomes narrower too, and then there should be something in front of the [x] squared
11 Interviewer: ah, why?
12 Student: because that's how I learned it

That the parabola has been reduced in size is, with respect to the algebraic structure of x^2+a, wrong. The student is aware hereof since she expects a factor in front of x^2 (line 10) which could explain that the curve's shape seems changed. Yet, considering her descriptions of the effects on the graph only, she proves to be a careful observer. The parabola does appear to be shrunken (Figure 2).

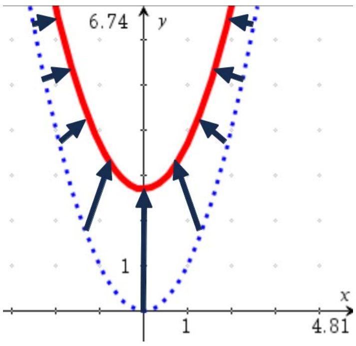

Figure 2: Moved upwards only or shrunken too?

The student realises that her description contradicts what she knows about the effects of a parameter change on the function graph. To resolve this contradiction she needs to explain the effects of a on the graph by means of the specific algebraic properties of the equation, i. e. she needs to perform a coherent representation change. This is achieved by considering the additive structure of the expression x^2+a:

Every single value of $f(x)=x^2$ is increased by a, so each single point $(x|x^2+a)$ of the parabola has been moved upwards from its original position $(x|x^2)$ by a, which affirms that the effect of a can only be interpreted as a vertical translation. So a valid interpretation needs to be based on an analysis of the term structure. Hence algebra plays a decisive role for describing and explaining relations between parameters and functional representations. When a learner describes the effects of a parameter change without giving a sufficient explanation on the base of the algebraic representation it is not clear whether his descriptions are based on his visual impressions only or whether they are based on a structural insight into the situation. This is particularly problematic when these descriptions fulfil the expectations of the teacher: If a learner only mentions a vertical translation by a, does he just reproduce what he sees, or does his description reflect an understanding of the `mechanism´ of how a affects each representation of x^2+a? One risks confusing a correct verbal description with an understanding of the situation, pupils as well as teachers.

Referring to possible restraints of the tool and its use – e.g., lack of visual aids or failing to adjust the window – misses the point this article is trying to make. The fact that misinterpretations can appear indicate a fundamental difference between a concept and its representations. A learning environment with a "perfect" design that avoids misinterpretations does not guarantee that a learner understands how the different representations interact. While giving all the correct answers, he might only reproduce what he sees on the screen, thus confusing the concept with its representations. Each learning environment needs to activate the learner's reflection (Yerushalmy 2005), which means – as it is argued here – that he turns from describing perceptually accessible properties of each representation towards analysing analogies between the three representational forms on a structural level.

UNDERSTANDING MULTIPLE REPRESENTATIONS BY ABSTRACTION

Does the student from the case study understand the effects of a on the representations of x^2+a? She gives a careful description of what she sees on the screen. And she knows, too, how the effects of a parameter on a function graph can be read from a given equation. But she cannot resolve the conflict between her observations and her knowledge. In this sense, she shows a lack of understanding.

So what does she – or a learner in general – need to achieve so that he or she shows a full understanding of the relation between parameters and the representations of function? Giving a description of his observations only is not sufficient as we have argued, even if the description is correct. He needs to give reasons for why his observations are valid, i.e., why they are consistent with the whole of the multiple representation environment. Reasoning by referring to well-known rules might be acceptable, but in the case of the student more was needed to clarify the conflict between what she saw and what she knew. It would be helpful to explain the mechanism of how a and representation of x^2+a connect. This means to refrain from reproducing visual information but to analyse analogies between the representations on a structural level. Focussing on structures instead of surface leads to a cognitive activity that is central for the learning of mathematical concepts: abstracting.

In the context of learning with a dynamic multiple representation environment abstracting means extracting the essential information from representations by conceiving structural analogies between representational forms while eliminating irrelevant surface properties. By referring to the concept of scientific abstraction by Davydov (1972) and others we will show that conceiving structural analogies is achieved by identifying invariants in the dynamic multiple representation environment while the structure of the algebraic representation is decisive.

Mathematical concept formation as the result of abstraction

Mitchelmore and White (2007) identify two different approaches to abstraction among theories of mathematical learning, which they call empirical and theoretical.

Empirical abstraction refers to a cognitive representation of knowledge that results from identifying common properties in a set of examples. "Abstracting is an activity by which we become aware of similarities [...] among our experiences. [...] An abstraction is some kind of lasting change, the result of abstracting, which enables us to recognise new experiences as having the similarities of an already formed class" (Skemp 1986). However, empirical abstraction that is limited to a perceptual analysis of real or cognitive objects can hardly explain the formation of such concepts that meet the scientific requirements of generality and precision. Hence, for the formation of a scientific concept, a theoretical basis is needed which supports argumentation that is independent from perceptual evidence.

Theoretical abstraction: To be valid beyond experience, knowledge needs to be developed within a theoretical system of its own, which comes with specific symbolic representations and rules of argumentation. Following Vygotsky (1934/1986), this symbolic representation form does not need to resemble any physical features of the knowledge that it represents. In fact, perceptually or otherwise empirically accessible properties are unsuitable for forming an abstract concept. "A theoretical idea or concept should bring together things that are dissimilar, different, multifaceted, and not coincident, and should indicate their proportion in the whole [...] Such a concept, in contrast to an empirical one, does not find something identical in every particular object in a class, but traces the interconnection of particular objects within the whole, within the system in its formation" (Davydov 1972/1990, 255).

For continuing our case analysis the concept of theoretical abstraction – or "scientific abstraction" as Davydov puts it (1972/1990) – appears to be suitable: The effects of a on x^2+a can be described as a function $a \rightarrow x^2+a$, which is a function different from f. Its properties can only be derived from changes within the representations of f. The change within the graphical representation of f appears to be a translation and a dilation, here the student is perfectly right. To decide whether this interpretation conforms with the rest of the multiple representation environment, the algebraic expression of $a \rightarrow x^2+a$ needs to be taken into consideration. Its additive structure decides which of the two interpretations of the effect of a on x^2+a is valid. So it is knowledge about symbolic algebra that forms the necessary theoretical basis for understanding relations between the multiple representations of a dynamic learning environment. However, the student is not able to apply her knowledge about algebra to explaining how, or whether at all, her descriptions are valid for the whole of changes within the multiple representations of $f(x) = x^2+a$.

Abstracting as conceiving structural analogies

In a teaching concept of "ascending from the abstract to the concrete" based on Davydov, Giest (2011) states that, with each new learning process, "initial abstractions" are gained from examining a learning material that allows change and variation. From varying the material, invariants become apparent that initiate the necessary reasoning for identifying a constant structure within change. With a dynamic multiple representation environment (e.g., Figure 1) the necessary variation here is twofold. First by operating the slider, thus changing the visible appearance of each representation and second, by switching between the three representation forms. These changes correspond to Duval's (2002) forms of representational changes that characterise a full understanding of the concept of function.

Obviously, the student from the interview meets the first of Duval's requirement at least partially. Within the graphical representation, she gives two pertinent interpretations of the effects of the parameter change on the actual function graph, and from the algebraic representation she can read correct information about the effects of parameters on function graphs in gene. But she does not fulfil the second criterion of Duval. She is not able to make a coherent change between the algebraic and the geometric form of representation here. Or to say it with Giest: She is not able to identify the necessary invariants within the multiple representation environment.

Conceiving structural analogies as identifying invariants

In Figure 1, the invariant in question is not visible, it becomes apparent as changes between and within representational forms. Considering the additive structure of the term of x^2+a, the invariant is characterised by the common operator $+a$. In the equation, the invariant is the summand $+a$ that redefines $f1(x)=x^2$ to $f2(x)=x^2+a$ (Figure 3). In the table, the invariant is the constant difference between the values of $f1$ and $f2$ in all table lines. In the graphical representation, the invariant is the constant vertical distance between the two graphs of $f1$ and $f2$, which is the same at all points. Thus, the invariant has a specific meaning in each representation form, yet, in each form, it can be visualised by an arrow with constant direction and length. Especially the arrows from the geometric representation form show that the effect of a on the graph of x^2+a must be interpreted as a translation only.

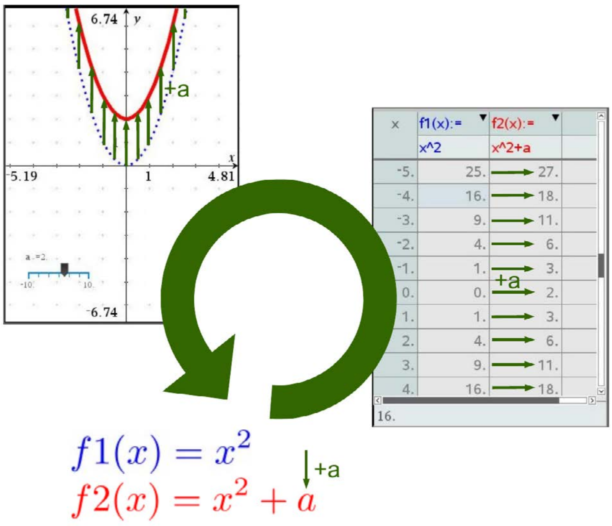

Figure 3: The invariant + *a*, identified in all three representation forms, shows that the effect of *a* on $x^2 + a$ is indeed a translation

We can sum up now: To show understanding of the relation between a parameter *a* and the representations of *f(x)+a* a learner needs to identify the operator *+a* as an invariant within each representation and between all representation forms of *f*.

This seems expecting much from pupils. However, next we report on a study where references to structural analogies in further interviews could in fact be identified.

CATEGORIES OF REASONING: AN INTERVIEW STUDY

Aim and methods

Together with the interview from our case study further interviews were analysed with the aim of categorising students' answers regarding to what extent they showed structural insight into the relations between graph, table and term of *f*. The interviewees were teacher-students of the University of Education Heidelberg from their first to their third year of study, all having selected mathematics as one of their compulsory subjects. The interviews contained questions about various tasks about for exploring the relation between parameters and quadratic functions. The task from Figure 1 was the first. All were accompanied by dynamic multiple representation learning environments, prepared in advance by means of the TI-Inspire CAS software on a laptop. Apart from two questions – the initial one that asked for a description of how the given parameter affected the appearance of graph, table and equation and one that asked the students to explain why they thought their observations were correct – no other questions were fixed in advance.

From the case study and subsequent theoretical considerations above, two a-priori categories were formulated, "reason with reference to a rule" and "reason with structural reference." Roughly, the first comprises all statements where students refer to what they have learned and believe to be generally true, while the second covers all statements that refer to invariants as described above. A refined definition will be presented in the results section. The research questions were as follows:

(A) To what extent can the descriptions and explanations from the interviews be assigned to the two a-priori categories?

(B) How did interviewees exemplify reasons with structural reference?

For the analysis, six interviews were selected which, on first view, promised a sufficiently large range of students' observations and explanations. The categories were developed by means of a qualitative content analysis (QIA, Mayring, 2010). The QIA is a systematic method for text analysis guided by pre-set coding rules. In the variant of the deductive category application these coding rules are derived from the relevant theoretical framework, which then are applied repeatedly by trained coders with the aim of refining the rules by enhancing inter-subjective comprehension. The coders were three teacher-students who did not take part in the interviews selected for coding. As mathematics students they were able to understand the specific terminology of coding rules while, as students, they still relied on a very precise formulation of the rules to agree on common coding.

Results and discussion

Apart from the two a-priori categories two more emerged from the coding process. First a category "proposition" that separated observations prompted by the interviewer's request to describe from explanations prompted by the request to explain, which often were very similar. For this, the well-known Toulmin model of argumentation was introduced to the coders; the model itself was not object of analysis here. Second a category "reason with reference to an example" which became necessary to allow for reasons that explained rules by referring to single values of *x* or *a*, mostly *x=0* or the actual value of *a*. Eventually, the third and most successful coding round in terms of intercoder reliability was based on the following rules:

- Category A "Proposition": All statements or gestures that can be seen as answers to the interviewer's initial prompt "Describe what you see when you operate the slider" and that show characteristics of a general rule. They refer to the effects of changes of the slider on the representations of the given function. If the interviewee modifies his propositions later during interview, these modifications are coded within this category too.

- Category B.1 "Reason with reference to a rule": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer to a rule or sound like one. Such rules often refer to connections between the parameter and the curve's shape or position. The statement does not refer to single parameter values but shows that the interviewee implies a global validity. Such rules do not need to be stated explicitly but the interviewee can also indicate that he knows of such a rule (e.g. "that's how I have learned it"). Even when the interviewee shows understanding at a structural level in some other part of the interview, reasons with reference to rules or knowledge are to be assigned to this category.

- Category B.2 "Reasons with reference to an example": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that refer a single value of x or to the actual value of the parameter or the actual state of the visible graphical configuration. These statements often refer to connections between the actual value of the parameter or a single value of x and the curve's present shape or position. Even when the interviewee refers to rules or shows understanding at a structural level elsewhere, reasons with reference to rules or knowledge are to be assigned to this category.

- Category B.3 "Reasons with reference to structure": All statements or gestures that can be considered as reasons for the observations assigned to category A or their modifications and that apparently refer to an invariant between different representation forms or within one representation. The invariant here is the value of *a* which takes on a specific appearance in each representation: In the algebraic representation it is the summand *+a*, in the numerical representation it is the difference between the old and new

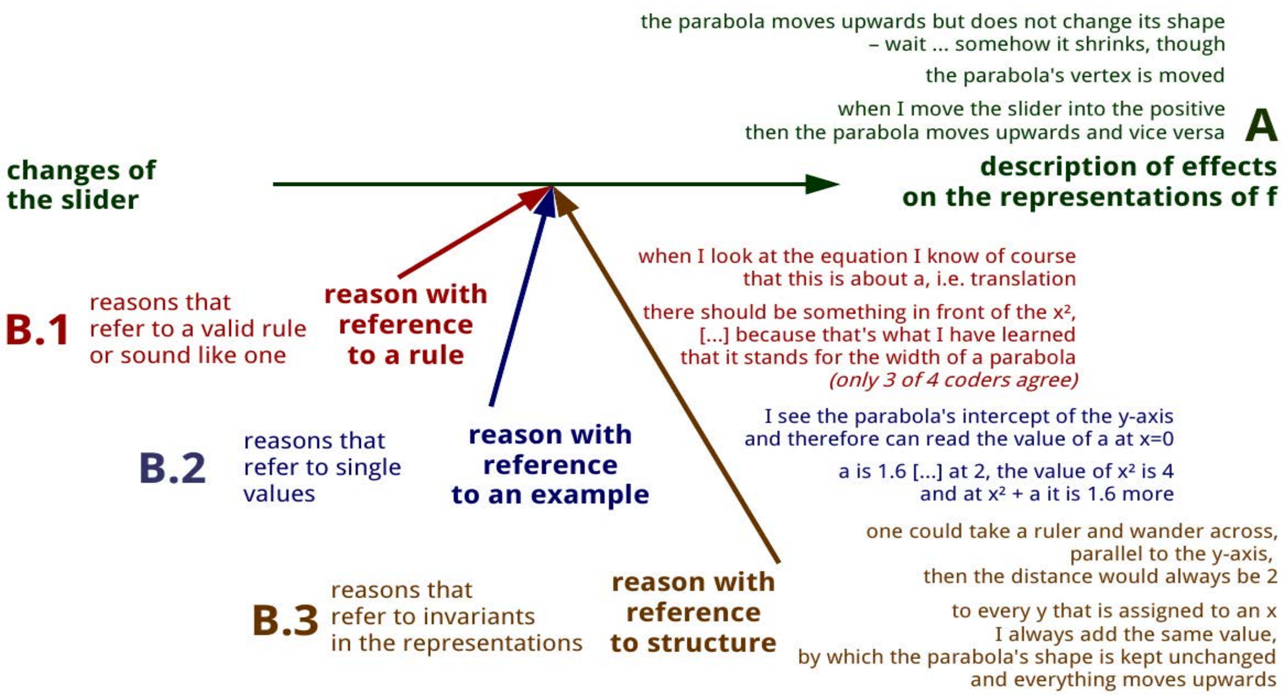

Figure 4: Categories of reasoning while exploring the effects of a parameter *a* on the epresentations of $x^2 + a$ in a computer based learning environment

function value in each table line. In the geometric representation it is the vertical distance between the graphs of both functions at each point. It differs from category B.2 insofar as statements here do not refer to single values of a but to any value of a in general. It differs from category B.1 insofar as a has been globally (i.e., for all values of x) identified within a representation.

The layout of Figure 4 places the four categories at appropriate places within the Toulmin model of argumentation. The proposition (category A) corresponds to Toulmin's "claim" which are descriptions of the effects of slider changes. The three categories of reasoning (B.1, B.2 and B.3) are placed as warrants into the diagram. All categories are illustrated by statements taken from interviews after little linguistic polishing. Generally a statement was considered exemplary when all four coders including the author agreed. One statement for category B.1 is an exception which, to the author, still appears to be a significant example for this category.

For answering research question (A), both a-priori categories were suitable for categorizing students' answers when accompanied by the two more categories as defined by the coding rules above. As for question (B), the two statements cited here indicate a range of possible structural references from evoking dynamic images – here a ruler that moves vertically across the coordinate plane while measuring a constant distance between the two graphs – and a more static view on how the different representational forms connect – here pointing out that, for all x, a is added to the corresponding $f(x)$, which explains the congruence preserving effect of a. The fact that structural references were in fact observable shows that such references can be expected from students. However, these statements also show that references to structural analogies between representations do not need to be as formal as indicated above. The "moving ruler" argument is a convincing example.

The category system from Figure 4 covers responses from the case $a \to x^2 + a$ only, where the invariant is easily identified as the constant vertical distance between the two curves or the constant difference between the two function values, each illustrated by an arrow with constant length (Figure 3). With parameters in other places of the algebraic expression this is different: For example, with $b \to b \cdot x^2$ or $c \to (x + c)^2$ other (mis) interpretations of the effects of the parameter can be expected. However, while coding rules need to be adjusted to these cases, the need for a structural reference still holds on theoretical grounds.

Prospects and consequences

Presently, standardised interviews are being developed for diagnostic purposes based on these results. Apart from diagnostic use, these results may be significant for classroom teaching too. Nearly all students and, in recent interviews, pupils reported that they had little, if any, experience with computers in school. Many were not able to explain the movements on screen which, to them, were totally new. These results plead for an extended use of dynamic software in mathematics teaching. In a dynamic environment, insufficient conceptions about function representations become apparent and can be dealt with openly. Last, these results show that, for an exploratory learning with computer based dynamic multiple representations too, a sound basic knowledge in algebra is necessary. Knowledge about term structure turns out to be essential as it plays a decisive role when validating the explorations.

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Looking for help on the Internet: An exploratory study of mathematical help-seeking practices among Mexican engineering students

Danelly Esparza Puga[1] and Mario Sánchez Aguilar[2]

1 Autonomous University of Ciudad Juárez, Juárez, Mexico, danelly.esparza@uacj.mx

2 National Polytechnic Institute, Ciudad de Mexico, Mexico

We report an exploratory study focused on identifying some of the Internet resources used by Mexican engineering students when they need help for their mathematics studies. The study consisted of an initial phase in which a questionnaire was designed and piloted, and a second phase in which such questionnaire was used to conduct individual and group interviews. The results show that looking for mathematical help on the Internet is a widespread practice among students and it is even preferred over traditional sources of help such as the library. Among the most widely used resources are YouTube and Facebook. These and other sites are used to find different ways to solve a problem, to clarify doubts and reinforce knowledge, to get ready-made results, to compare results, and when they skip class and need to catch up.

Keywords: Mathematical help-seeking, Internet resources, social networks.

INTRODUCTION

The phenomenon of help-seeking has captured the interest of sociologists and social psychologists for decades. Generally speaking, these studies have focused on identifying and explaining what demographic, sociocultural, and psychological factors encourage people to seek the help of others. These studies have been conducted mainly in medical, work-related, welfare and social security contexts (see, for example, Cornally & McCarthy, 2011). Early conceptualizations of the concept of help-seeking favoured a perception of it as an undesirable activity, that is, some researchers began to interpret help-seeking as an indicator of lack of independence, which can have high costs for individuals' self-esteem and sense of competitiveness (e.g., Shapiro, 1978). The reconceptualization of the concept of help-seeking provided by Nelson-Le Gall (1985) contributed to assign a positive meaning to the concept in the field of educational research; this in the sense that seeking help can be seen as a useful skill or competence for students to address problems that otherwise would be difficult to address. Help-seeking can be seen as a useful skill for students' self-learning.

Looking for help (or help-seeking) refers to the process of trying to find support from the people and resources around us, when we have questions or need help related to our mathematics lessons. Looking for help is a basic process when studying mathematics, however, the studies addressing help-seeking practices in mathematics instruction are still scarce. Some of these studies have focused on identifying affective elements that motivate students to seek help or avoid it (Ryan & Pintrich, 1999), while other studies have characterized seeking behaviours displayed by mathematics students (Kempler & Linnenbrink, 2006), there are also studies investigating mathematics teachers' knowledge of students' help seeking behaviors when they solve mathematical word problems (Marais, Van der Westhuizen, & Tillema, 2013). A common feature of these studies is that the only considered sources of help are people —teachers, classmates— but technological resources are not considered as potential sources of mathematical help.

Contemporary students' help-seeking practices, however, are changing due to the pervasiveness of the Internet and mobile electronic devices; for instance, there are studies showing that mathematics students from different regions of the world use open online forums to find mathematical help (Puustinen, Volckaert-Legrier, Coquin, & Bernicot, 2009; van de Sande, 2011).

Both studies, by analysing students' exchanges and posts, focus on characterizing the help seeking behaviours that mathematics students manifest when they look for help in open online discussion forums. There is a need for more studies showing what other technological tools mathematics students use nowadays as sources of help and how they use them.

In this paper, we report an exploratory study focused on identifying the Internet sites and web-based tools that some Mexican engineering students use as a source of help for their mathematics lessons. In particular we explore three aspects of their Internet-based help-seeking practices:

1) What sites or tools do they consult when they need help in mathematics and how often do they use them?

2) What are the reasons that lead them to use those sites?

3) What are the reasons why students trust in the information provided by such websites?

The motivation for this study comes from trying to understand how Internet resources are affecting the study processes of contemporary mathematics students: how and why today's students use the Internet to support their mathematical studies? Our study is relevant to the area of research focused on students' learning of mathematics with resources and technology as it helps to expand our understanding about how students independently use the Internet (web-based tools, social networks) as a source of help for their mathematical studies.

CONCEPTUAL FRAMEWORK

As we mentioned before, help-seeking in mathematics education is usually conceptualized as the ability of a student to rely on people —teachers, classmates, relatives— to get help that could be useful to overcome the difficulties that arise when learning mathematics. Karabenick and Puustinen (2013) confirm this when they point out how educational research on help-seeking has focused primarily on interactions in the classroom where students ask for help to the teacher or their classmates. These authors raise the need to expand this area of research in order to understand how new technologies are affecting students' help-seeking practices.

Following the line of thought of Karabenick and Puustinen (2013), in this work we adopted a wider definition of help-seeking in mathematics that includes non-human elements as possible sources of help; thus, in this study, mathematical help-seeking is defined as the ability to use the people and resources around us (including technological resources such as the Internet and mobile devices) as sources of help to overcome the difficulties and doubts that arise during our mathematics learning process.

CONTEXT

The study was conducted at the Institute of Engineering and Technology of the Autonomous University of Ciudad Juarez (UACJ) of Mexico. The Institute is a public institution of higher education that is located in the city of Ciudad Juarez, on Mexico's northern border, south of the city of El Paso, Texas, in the U.S. The Institute has an approximated population of 4,500 students from this area and with socioeconomic backgrounds ranging from medium to low.

One reason for selecting this institution for developing our study was our interest in exploring the Internet-based help-seeking practices among tertiary students. An additional reason for selecting the UACJ was that the first author of this paper works as a teacher at the institution, so this gave us access to both the students and the university facilities to conduct the study.

STUDY POPULATION

Undergraduate students who met two conditions were selected for the study: first, they were taking at least one course in mathematics at the time of the study; and second, they wanted to participate in the study voluntarily. The study involved a total of 21 engineering students of both sexes, distributed into two phases: an initial-exploratory phase and the second phase of the study. More details on the structure of these phases are found in the section "procedure".

All the participants in the study were students from different engineering specialties at the UACJ. We selected students from different engineering specialties because all of them were taking several mathematics courses (calculus, differential equations, linear algebra) as part of their engineering education. At the time of the study their ages varied between 19 and 38 years.

In the initial-exploratory phase four students from three different careers participated: industrial engineering, digital systems engineering, and electrical engineering. At the time they participated in the study, two of the students were enrolled in their first semester, one of them was in the third semester, and the last one was studying the forth semester (engineering studies at the UACJ last eight semesters).

The second phase of the study involved the participation of 17 students from five different engineering specialties: civil, electrical, biomedical, industrial and mechatronics. For this stage we only selected students from intermediate and advanced semesters (from fourth to eighth semester).

PROCEDURE

As mentioned above, the study was divided into two phases. The study began with an initial-exploratory phase in which a questionnaire of seven open-ended questions was used to guide semi-structured interviews; these interviews were conducted between November the 19th and November the 26th, 2012. The aim of this phase was twofold: on the one hand we sought to confirm the hypothesis that some university students use the Internet when they need help or have doubts related to their mathematics lessons; on the other hand we wanted to assess how well the designed questionnaire worked to generate empirical data, this is, to assess whether the wording of the questions was understood by the students or whether the questionnaire produced the type of information needed to answer the questions raised. Examples of the questions included in the initial version of the questionnaire used in this phase were: have you ever used a website to look for help for your mathematics lessons?, if yes, how often do you use these sites?, in your opinion, what Internet sites or tools are trustworthy to look for help in mathematics and what makes them reliable?

This initial phase allowed us to refine the guiding questionnaire to conduct interviews during the second phase of the study. The questions that constituted the final version of the questionnaire are:

1) Have you ever used a website to find help for your mathematics lessons?

2) If yes, what are the websites that you have used for your mathematics lessons?

3) How often do you use these sites?

4) How do you use these sites for your mathematics studies?

5) Mention the benefits of using Internet sites as a source of help for your mathematical studies at the university

6) Mention the drawbacks of using Internet sites as a source of help for your mathematical studies at the university

7) In your opinion, what Internet site or tool is more reliable and what makes it reliable?

This final version of the questionnaire was used during the second phase of the study to interview 17 engineering students from the UACJ. Five of them were interviewed individually, while the remaining 12 were interviewed through two focus groups of six members each. The second phase of the study was developed on three different days: February the 13th, March the 12th, and June the 26th, 2013. The answers given by these 17 students are the main empirical data on which the results of our study are based. All the interviews, both from the initial-exploratory phase and the second phase, were audio recorded for later analysis and transcription.

To analyse the empirical data consisting of individual and group interviews, it was necessary to repeatedly listen to the audio recordings in order to locate the answers to each of the survey questions, and thus try to make a categorization of the answers. During this process it was evident that some students' responses had common elements, which effectively allowed us to produce a categorization of the data. These results are shown in the following section.

RESULTS

This part of the report refers only to the interviews conducted with the 17 students who participated in the second phase of the study, and is divided into four sections: in the first section, the frequency with which students use Internet sites as a source of mathematical help is presented. In the second section we show the most popular sites among the engineering students who participated in the study. Then the students' perceptions of the benefits and limitations of using websites as a source of mathematical help will be described, and finally the reasons given by the students to trust (or mistrust) an Internet site or tool as a source of help are presented.

Frequency of use of websites
All the 17 respondents reported using at least one website as a source of help for their mathematics lessons. Most students are frequent users since 12 out of 17 respondents reported using these sites on a daily basis, while the remaining five students declared that they use these resources during the weekend.

Popular Internet sites and their functionality
Figure 1 shows the websites and tools most frequently mentioned as a source of mathematical help by the engineering students who participated in the study.

The website most mentioned by the students was YouTube. 12 out of the 17 respondents agree that this is the site they turn to when they need to revise and fully understand the topics covered in class.

Facebook is the second most mentioned site, 11 out of 17 students declared to have used it for their mathematics lessons. They claim that Facebook is a tool through which they can contact their classmates or the teacher to clarify questions about the homework or even logistic concerns (like finding out in which room a lesson will be taught). For example, the student number 2 from the focus group number 1 mentioned the following (all the presented excerpts were translated from Spanish to English):

> *Student 2, focus group 1*:
> [...] I use Facebook with my classmates when I have doubts about what was explained in class, because sometimes when I come home I do not remember what I saw.

The next most mentioned site is Google with six mentions. This is the place where some students start their searches of the topics they want to investigate. Most users do specialised searches in Google Books and Google Scholar to delimit the sites or documents where they want to focus their search.

Other websites such as Wikipedia, buenastareas.com, Yahoo! Answers and Slide Share, and Internet tools such as the email, and the online calculator Wolfram Mathematica, were also mentioned because of the different features they offer, like ready-made answers to their mathematical problems:

> *Student 2, focus group 2*:
> [...] I use Wolfram Mathematica to get the results, but sometimes it is not very reliable because omits a few steps [...]

> *Student 4, individual interview*:
> [...] Sometimes you are given a problem and the answers are already uploaded ... I would use Yahoo! Answers when I am struggling and I feel like oh no [...].

In the category "Others" shown in Figure 1 are included tools such as Skype, and websites like Profesores Universia (http://profesores.universia.es), Vitutor (http://www.vitutor.com) among others. These sites

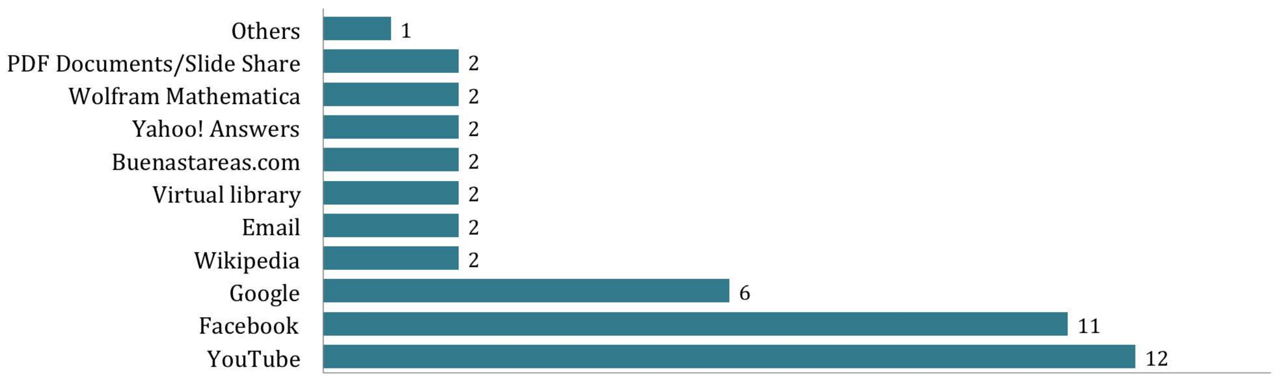

Table 1: Internet sites and tools most mentioned as a source of mathematical help-seeking among the students who participated in this study

are used to find alternate solving methods, comparing results, and finding step-by-step solutions:

> *Student 1, individual interview*:
> [...] There is one called Web Profesores (Teachers Web) which is a site where teachers upload documents or teaching techniques [...] they are written documents [...] once I put a formula that needed on Google and so I got to this page, and there is another one that it is the same... Profesores Universia.

> *Student 5, focus group 2*:
> [...] There is a page called vitutor.com, there are many exercises of many subjects... solved mathematics problems and others to be solved [...] they give you everything step-by-step.

In sum we can say that most of the sites and tools that students turn to are used to: (1) find different ways to solve a mathematical problem, (2) clarify doubts and reinforce knowledge, (3) get ready-made results or mathematical problems solved, (4) compare results, and (5) when they skip a class and need to catch up.

Benefits and limitations as perceived by the students

Students' perceived benefits in using the Internet as a source of mathematical help are diverse. Some students highlight the geographical and temporal unlimited access, making them prefer the Internet over more traditional sources of help:

> *Student 5, individual interview*:
> I think I save... for example, if I go to the library I have to drive... [by using the Internet] I save time and money.

Some students perceive the Internet as a source of help to clarify doubts that is infallible, as stated by this student:

> *Student 3, focus group 2*:
> The benefit that I find is that anything that I look for, I will find it; I get thousands of pages and I will always find something, I will never be in doubt.

Furthermore, we identified four limitations that the interviewed students expressed, these are: (1) restricted access to the sources due to copyright and/or language, (2) lack of reliability of the sources, (3) the Internet can function as a distractor, and (4) there is too much information on the web. To illustrate these limitations, next we present a few excerpts from the interviews:

> *Student 6, focus group 2*:
> The problem is that unless you know what you are looking for, you get very vague answers, there is ambiguous information and copyrights are regulated on the Internet, if you want to read a book you can only see part of it [...] the useful material is restricted.

> *Student 5, focus group 2*:
> It happened to me once while I was studying for an exam, there was a topic that I knew nothing about [...] I found a method that had nothing to do, it was not what I needed [...] at times you put something in and you get like thousand things that are not what you are looking for.

Reasons to trust the information contained on a website

Several students who participated in this study based the reliability of the Internet sources that they consult on the authority provided by the academic degree of the author or the prestige of the institution that produces the resource:

> *Student 3, focus group 1*:
> YouTube seems reliable to me because university teachers upload the videos.

> *Student 5, individual interview*:
> SlideShare... I think is more reliable because there the doctors send [slides presentations].

> *Student 1, individual interview*:
> [...] Profesores Universia because it is supported by several Latin American universities.

> *Student 3, focus group 1*:
> For example in YouTube, well there the videos are from university teachers and themselves make the videos to clarify doubts, for me it is very reliable because they are university teachers and I think they even receive an economic benefit by making the videos [...]

DISCUSSION

In this exploratory study we investigated the Internet-based help-seeking practices among some Mexican engineering students. As other studies indicate (e.g. van de Sande, 2011), we found that the search for mathematical help on the Internet is widespread (12 out of 17 respondents reported using these sites for this purposes on a daily basis). According to the participants in our study, the most commonly used sites are: (1) YouTube, where students can find video recorded lessons to review and deepen the mathematical content covered in class; and (2) Facebook, where they can contact their peers or their teacher to clarify conceptual and logistic doubts related to their mathematical lessons. However, students also report using sites like Yahoo! Answers where they can find ready-made answers to their mathematical tasks.

The students who participated in the study look for mathematical help on the Internet to find different ways to solve a mathematical problem, to clarify doubts and reinforce knowledge, to get ready-made results or mathematical problems solved step-by-step, to compare the results that they obtain, and when they skip class and need to catch up. However, when it comes to assessing the reliability of the information they get from these sites, students seem not to pay attention to the intrinsic mathematical properties of the information obtained (Lithner, 2003), but rather base their assessment on features not related to mathematics, such as the academic prestige of the person or institution that publishes information.

The exploratory nature of this study allows us to see only the surface of a widespread practice that is affecting the way that contemporary students are studying mathematics. Next we outline future avenues of research that could be deepened in order to better understand this phenomenon.

Future avenues of research

One limitation of our study is that it is based on self-report measures of technology use, which may be less accurate than the observations of actual behaviours (Junco, 2014). A future avenue of research could focus on using more direct methods of observation as proposed by Junco (2014) in which recording software is used to document and characterize help-seeking behaviours manifested by mathematics students in different kinds of devices such as computers, tablets and mobile phones.

Another limitation of our study is that it does not deepen into how each site or web tool are used as a source of mathematical help. It is necessary to produce detailed characterizations of how students use tools like Facebook and YouTube to find mathematical help. Also, one could delve into the selection and exclusion criteria applied by students when selecting a particular piece of mathematical information from the sea of information offered by the Internet.

Another relevant line of research would be to include the perspective of mathematics teachers about these practices of mathematical help-seeking: what do they think about these student's practices? Do they consider it a desirable practice? Do they integrate these sources of help into the mathematical instruction that they provide to their students?

We believe that mathematical help-seeking on the Internet is an emergent area of research that is not only fertile but also relevant to study because it is a widespread practice that affects the way in which contemporary students approach school mathematics.

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Guided Inquiry learning of fractions – a representational approach

Stefan Schumacher and Jürgen Roth

University of Koblenz-Landau, Institute for Mathematics, Mainz, Germany, Schumacher@uni-landau.de, Roth@uni-landau.de

We present a theoretical model of a representational approach to inquiry based learning (IBL) in this paper. In IBL-environments, students investigate a mathematical domain by using multiple representations such as dynamic simulations and hands-on material guided by specially designed textbooks. In the empirical part, we describe a study focussing on self-generated representations by students with the aim of representing procedures and results.

Keywords: Self-generated representations, protocols, guided inquiry learning, fractions.

CONCEPTUAL FRAMEWORK

In order to introduce the representational approach to inquiry based learning (IBL) in mathematics we define the concept of representational competence as two aspects that are two sides of the same coin. One side is the ability to manipulate and interpret prescribed external representations (representational input) and the other side is the ability to generate own external representations (representational output) (e.g., Izsák, 2011; Schnotz, Baadte, Müller, & Rasch, 2010; Cox, 1999). To build the link between IBL and representational competence, we need to define IBL:

> Inquiry or scientific discovery learning environments are environments in which a domain is not directly offered to learners but in which learners have to induce the domain by experiments or examples. (de Jong, 2005, p. 215)

To specify this definition of IBL to inquiry in mathematics, we need to clarify what makes the work of a mathematician.

> Mathematics proper might be regarded as the science of significant structure. Thus mathematics studies the representation of one structure by another, and much of the actual work of mathematics is to determine exactly what structure is preserved in that representation. (Kaput, 1987, p. 23)

Combining the essence of these two quotes it becomes clear that in IBL-environments in mathematics the goal for students should be to investigate a domain by analysing structures of given representations through examples or experiments with, e.g., hands-on usable material or dynamic representations. This represents the first side of representational competence, the processing of representational input.

In IBL-environments the processing of the representational input in the cognitive system of each participating students is mediated by (1) social interaction within the group the students are working in, (2) interaction of the group with the learning environment or (3) personal interaction of an individual with the learning environment. After the processing in the individual cognitive system of each student, the students are supposed to generate representational output, which is also mediated by social or personal interaction. Our approach is in line with Tytler, Prain, Hubber and Waldrip (2013, p. 3) who see the need for the development of and research on IBL-environments in science learning *"with a strong explicit emphasis on student-generated representational work"*. One major goal of generating representations during IBL is to represent results and solution steps externally. When it comes to student-generated representational work with the aim of presenting results and solution steps, we have to introduce the term "protocol". A protocol can be defined as a record, notation or description of essential stages phases and products of a learning process (e.g., an IBL-process) by using external representations such as texts, other symbols or diagrams (Dörfler, 2000, p. 111f).

The aim of those protocols is to help the students to reflect on the IBL-process. By presenting results in form of protocols, these protocols become part of the learning environment and can therefore be part of the (social or personal) interaction with the learning environment in later stages of the inquiry-process. Furthermore, students can revise their protocols repeatedly during the inquiry process. What we described so far is our theoretical model of the representational approach to IBL (see Figure 1). We have derived this model from classic input-output oriented information-processing models. In our model, the information-processing system is the cognitive system of the participating student. For the presented research, we consider the cognitive system (see Figure 1) and therefore mental processes as a "black box". To get deeper knowledge on IBL from a representational point of view, we might have to open this "black box" in further research. For example the "integrated model of text and picture comprehension" of Schnotz (e.g., Schnotz et al., 2010) or other theories of cognitive processing in multimedia-learning could be used to open this black box.

In this study, we focus on the individual student's ability to generate protocols. We want to investigate

- how the individual's ability to generate protocols (in the case of fractions) develops over time,
- in which way this development can be supported and
- how it is related to the individual's content knowledge (in this case knowledge on fractions).

See the chapter "Research Design and Questions" for a detailed list of the investigated research questions. To investigate the research-questions we conducted a quasi-experimental intervention study with two IBL-conditions and one control condition. The focus of this article is to present the quantitative data from three measurement occasions (pre-, post-, follow-up test). We do not focus on the qualitative analysis of the interaction in the IBL-conditions, even though we videotaped some of the groups during IBL. Nevertheless we want to introduce the IBL-environment we used in the study to make the research more clear.

THE IBL-ENVIRONMENT

In the IBL-environment of the presented study, the students discovered fractions by analysing artworks of Max Bill who is one of the most famous representatives of the so-called "concrete art". Max Bills "progression in five squares" (see the left side of Figure 2) is one of the artworks we used. In this artwork, Max Bill arranged five equal squares in a column and split them progressively into smaller, but in each square equal sized rectangles. Because of the described struc-

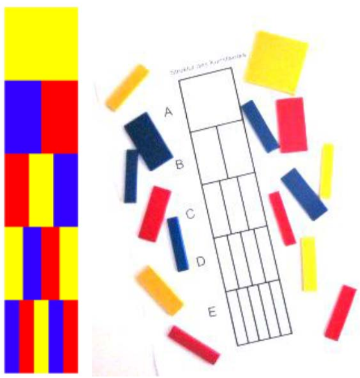

Figure 2: Max Bill "progressi-on in five squares" (left) and an equivalent fraction puzzle (right)

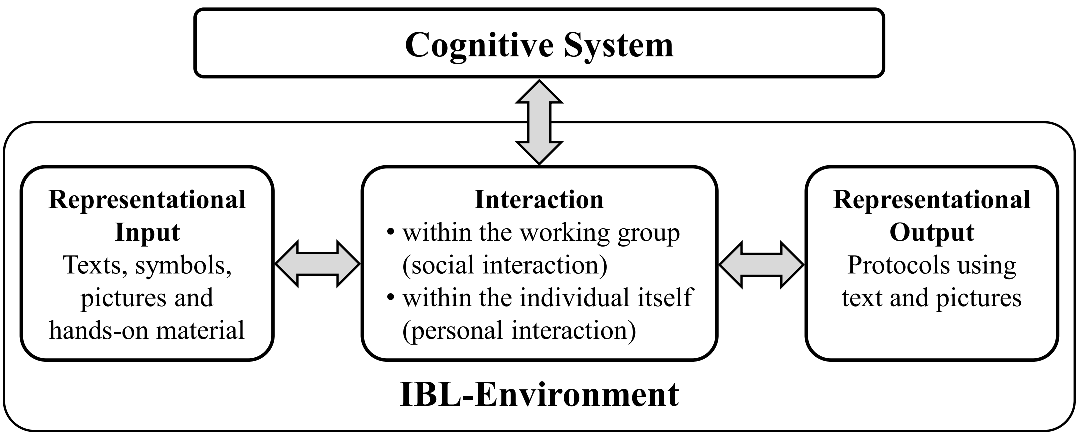

Figure 1: Theoretical model of the representational approach to IBL

Figure 3: Students during an IBL-process

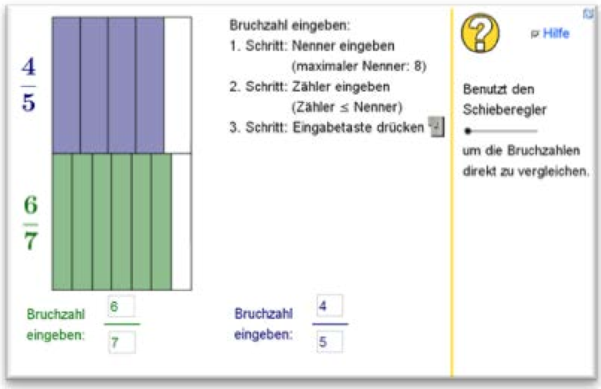

Figure 4: Screenshot of a dynamic visualization to compare fractions

ture, the artwork is suitable for students to explore the underlying structures of the "part-whole-concept of fractions".

For every artwork presented in the IBL-environment hands-on material based on this artwork (see the right side of Figure 2) was available for the students. It consists of the artworks outline structure on a laminated template and puzzle pieces of each coloured sub-area of the artwork. One can interpret each puzzle piece as a fraction of the whole artwork (or in the case of Figure 2 as a fraction of one of the five squares). Therefore, we named those hands-on materials "fraction puzzles". In Figure 3, you can see a student using the fraction-puzzles to reason on an argument on the comparison of unit fractions, the working group discovered in their IBL-process. The students' argument was that one third has to be "bigger" than one fourth, since in the case of the third the whole (the square) gets split into three parts, while in the case of the fourth the whole gets split into four parts. After writing down this argument in form of a protocol, the student in the upper right took a one half puzzle piece and two one fourth pieces (see Figure 3) and said that now the whole also is split into three parts as well. A discussion started after that and some students decided to revise their so far produced protocol. They added that the parts the whole is split into have to be of equal seize.

In addition to the fraction puzzles, students also had the opportunity to use dynamic visualizations (constructed using GeoGebra), again with a structure based on the artworks. Students can use these dynamic visualizations to test hypothesis they put up while dealing with hands-on material by further examples. Using the dynamic representation presented in Figure 4 students can for example test hypothesis on different comparison strategies for fractions.

Since guided IBL can be considered more successful than unguided IBL, we chose to implement a "triple support scheme" (De Jong, 2005; Reid Zhan & Chen, 2003) in the textbooks of our IBL-environment. We implemented *interpretative support* to help the students to interpret the prescribed representations. Whenever we present a task in the textbook that we anticipated as probably hard to solve for at least some of the students, a question mark icon indicates that there is some help provided in a special textbook. In this textbook, we present additional questions, hints or at the most solution steps but never the solution itself. We carried out *experimental support* to guide students while setting up experiments. This for example could be hints how to use the fraction-puzzles or the dynamic visualizations. In the presented study, we focus on *reflective support* with the aim of supporting students to generate protocols. We consider generating protocols as reflection, since students need to reflect on their learning process to be able to generate protocols. Prompts seem to be a promising approach to support the generation of protocols (Rau, Aleven, & Rummel, 2009; Berthold, Eysink, & Renkl, 2009; Berthold, Nückles, & Renkl, 2004). They are defined as "requests that require the learners to process the to-be-learned contents in a specific way" (Berthold, Eysink, & Renkl, 2009). An example for a prompt we used is the following: "Represent the result of the task you just solved and reason why your result is correct using a sketch and a text." In our learning environment, we provide requests like this next to a framed space in the textbook in which the students can represent their results bearing in mind the prompt. When the framed space next to the prompt is empty, we consider

First Unit	– (Unit) fractions in the meaning of the part-whole concept – Comparing Fractions using meaningful semantic strategies on the basis of pictorial representations
Second Unit	– Repetition: Unit fractions in the meaning of the part-whole concept – Equivalence of fractions using graphical representations – Problematization of adding fractions on a semantic level
Third Unit	– Adding fractions on a semantic level through pictorial representations – Fractions with a value greater than one using the part-whole concept – Application of the reached results on a realistic problem situation

Table 1: Content of the learning unit

this as *low instructional level*. In contrast to that, we consider to some extent pre-filled framed spaces as *higher instructional level*. In pre-filled framed spaces, we provide for example the beginning of a sentence or a first rough outline of a sketch, which the students have to accomplish in order to represent their results.

RESEARCH DESIGN AND QUESTIONS

To investigate the students' ability to generate protocols in more detail we conducted a quasi-experimental study with two experimental IBL-conditions comparing two different instruction levels of reflective support through prompts to a control-group. The control-group was taught in a teacher-centred setting. The content of these teacher-centred lessons was the same as in the IBL-environment and in all three conditions students learned in three 90-minute lessons (see Table 1).

We kept learning-time and content consistent over all three conditions. Experimental-Group 1 (EG1) learned with textbooks using prompts on a higher instructional level. Experimental-Group 2 (EG2) learned with textbooks providing prompts on a low instructional level as described earlier.

In this research, eight sixth grade classes from two different German grammar schools took part. We randomly choose one class from each school as the control-group (CG). The other three classes of each school learned in the IBL-environment. The students in the IBL-environment learned together in groups of three to four students. We randomly assigned the students of each class to these working-groups and then each working-group randomly to one of the experimental conditions, by either providing them with textbooks containing prompts on higher (EG1) or low instructional level (EG2). By this distribution a total of $N = 81$ students were assigned to EG1, a total of $N = 68$ students were assigned to EG2 and a total of $N = 50$ students were assigned to CG, which means a total of $N = 199$ students took part in the study (including later dropouts).

We carried out the study in a pre-, post-, follow-up-test design and collected data on two variables at each of the three measurement-occasions. First data on the students' knowledge on fractions was collected and in a second step the students' ability to generate protocols was measured. For the measurement on fraction-knowledge a paper and pencil test was developed. This test has a special focus on the part-whole concept and operations amongst fractions (based on this concept) that were part of the intervention. The items in the test all focus on some kind of switch between representations. A typical task is to find the right pictorial representation for a given fraction in a multiple-choice-item.

For measuring the ability to generate protocols, we developed a new instrument based on so-called "video items". The underlying idea behind video items is to present a short video to the students during the test situation. This video shows a complete problem-solving process simulating IBL. In the case of the presented study, we used videos demonstrating a problem solving process on fractions, using hands-on mate-

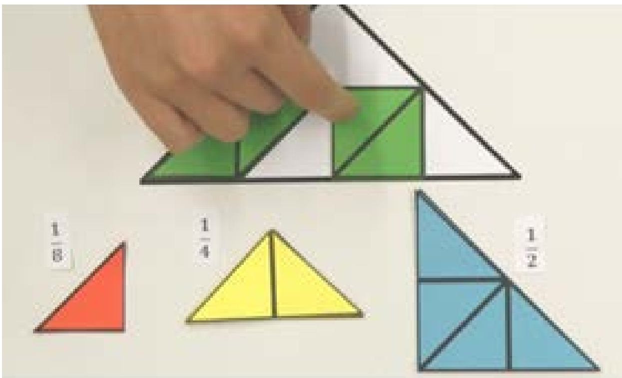

Figure 5: Screenshot of a video item

rials like the fraction puzzles described earlier. For a screenshot of one the videos, see Figure 5. By using videos, we want to simulate an IBL-process, which is in line with our theoretical model of the representational approach to IBL.

The task for students in the test situation is to generate a protocol of the video-content using pen and paper after watching the video.

Raters can evaluate these protocols using categories like the correctness and completeness of the *represented contents* and of the *represented relations between the contents*. In Engl and colleagues (2014), we describe the concept of using video items to measure the ability to generate protocols in detail. See Figure 6 for an example of a protocol based on a video-item.

To close this chapter we list the detailed research questions that contribute to the overall research-question we want to answer with this design:

Research question 1:
Do students in an IBL-setting achieve at least the same learning success as students in a teacher-centred setting?

Research question 2:
In which way does the ability to generate protocols develop over time under the three different conditions and is it possible to identify differences between the groups?

Research question 3:
Is there a correlation between the ability to generate protocols and the knowledge on fractions?

In the following, we present and discuss the results in detail. As said we put a special focus on the ability to generate protocols.

RESULTS

To investigate research question 1 we conducted a repeated measures ANOVA on the fractions-test scores, comparing the three conditions. The main-effect shows a highly significant increase in fraction knowledge over time for the three groups ($F(2,336) = 443{,}793$; $p < .01$). However no significant difference on the interaction between time and group could be detected ($F(4,336) = .986$; $p = .415$). This result shows that not only the teacher-centered setting can be considered successful. It indicates that the used support strategies for the IBL-environment lead to satisfactory learning-outcomes. With a focus on the treatment condition (reflective support through prompts), we can conclude that according to knowledge acquisition it makes no difference whether students are prompted on a high or low instructional-level in the case of the presented IBL-environment. This leads to the conclusion, that other design-principles (experimental and interpretative support) have more impact on knowledge acquisition than the different treatment conditions. What it does not mean is that students should not be prompted to generate protocols at all. We can say that independent of the instructional level

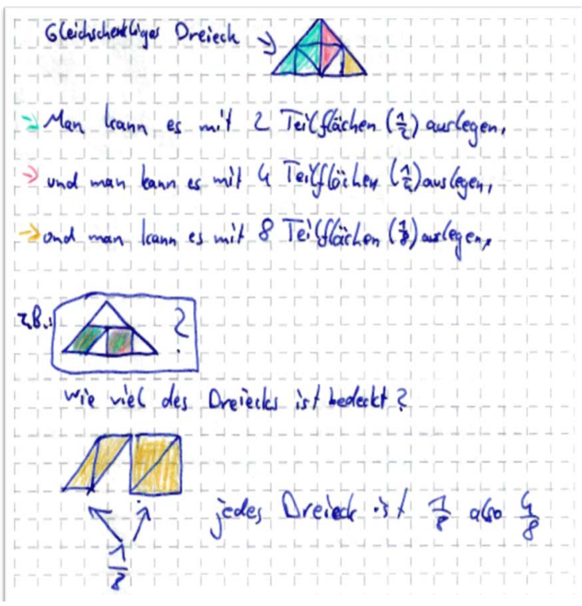

Figure 6: Student generated protocol on the basis of a video-item

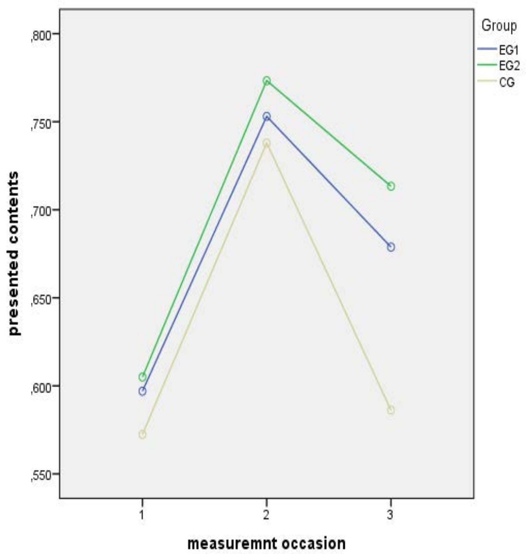

Figure 7: Relative values of presented contents in the protocols

of prompts, students learn successfully in the IBL-environment with the implemented support.

Let us turn to the results relating the ability to generate protocols now. According to the main-effect, the repeated measures ANOVA reveals a highly significant increase in the numbers of correctly presented contents over time ($F(2,306) = 37{,}282; p < .01$). Looking at the interaction between group and time the repeated measures ANOVA does not reveal significant differences between the groups. ($F(4,304) = 1{,}142; p = .337$). However, a Tuckey-HSD Post-Hoc-Test detects a significant difference between EG2 and CG ($p < 0.05$).[1] This result becomes clearer when looking at the difference between EG2 and CG at the third measurement occasion in Figure 7. The interpretation of this result is that EG2 shows a more sustainable ability to generate protocols than the CG. This is interesting because we provided EG2 with prompts on a low instructional level. EG1 also seems to show a more sustainable ability to generate protocols in comparison to CG, even though we cannot detect a significant difference.

What we can conclude from the significant main effect of the ANOVA on the generated protocols is that students develop the ability to represent results by learning on the topic, whether it is in an IBL-environment or a teacher-centered setting. However, students learning in the IBL-environment with prompts on a lower instructional level show a more sustainable ability to represent results.

Regarding research question 3, we detected low correlations between content knowledge and the ability to generate protocols at each measurement occasion (see Table 2). There are high achieving students regarding to fraction knowledge who fail to generate protocols and the other way around. This leads to the conclusion that content knowledge and the ability to generate protocols are two different constructs. We will discuss this interesting result amongst the other results in the closing section of this article.

Pre-Test	$r(180) = .217$	$p < .01$
Post-Test	$r(187) = .210$	$p < .01$
Follow-Up-Test	$r(167) = .224$	$p < .01$

Table 2: Pearson Correlations

1 Since the Tuckey-HSD Post-Hoc-Test uses pairwise testing, we can apply it without the ANOVA showing significant results (Hsu, 1996, pp. 175f.).

CONCLUSIONS AND PRACTICAL IMPLICATIONS

As shown in the results section students improve their ability to generate protocols significantly over time. However, the ability to generate protocols is more sustainable for students who learned in the IBL condition. As for the students of EG2, who were guided to generate protocols by prompts on low instructional level (request to represent results next to empty framed spaces), the described effect was significant. Therefore, when it comes to reflective support, we recommend a low instructional level of prompts. Here we provided the prompts next to empty framed spaces. This means students are open to generate their own creative protocols, considering the hints how to generate the protocol given in the prompts. The benefit of the lower instructional level is that right or wrong, the students generate these protocols truly on their own and therefore represent their way of thinking. Therefore, the teacher can use them to get more detailed information on the students´ thinking process. This would not be possible with pre-filled framed spaces, like in the first research-condition. When interpreting protocols we have to take into account that the quality of a generated protocol is not to be mixed up with high content knowledge. The low correlations between these two constructs clearly indicate this. To find out more about why the correlations are low, it might be interesting to open the black box and try to gain insight into students' mental models and the way they use them to generate protocols.

Another open research question is, whether students we consider "good representers" due to the results of the video items, really use their skills in IBL-settings and if not, how we have to design prompts to make them use their skills. Concerning this, it might also be interesting which factors influence the use of such skills. Motivation might have a huge impact, since our experience from watching students in IBL-environments indicates that the motivation to generate protocols is generally very low.

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Game approach with the use of technology: A possible way to enhance mathematical thinking

Carlotta Soldano, Ferdinando Arzarello and Ornella Robutti

University of Torino, Torino, Italy, carlotta.soldano@gmail.com, ferdinando.arzarello@unito.it, ornella.robutti@unito.it

The purpose of this paper is to draw attention to the strategic thinking that students develop facing mathematical games. We hypothesise that prompting a strategic way of thinking within a didactical intervention, called the 'game-approach', could improve students' proving processes and support them in the production of proofs. More precisely, we are designing tasks based on two players' games in a Multi-touch Dynamic Geometric Environment, in which the discovery of the winning strategy coincides with the discovery of a geometric property. We aim to contribute to the debate on the possibility of cognitive and epistemic unity and to deepen the studies based on dragging practises and their cognitive counterpart.

Keywords: Games-approach, logic of inquiry, tablet.

INTRODUCTION AND THEORETICAL FRAMEWORK

Proving has always been one of the most difficult tasks in teaching and learning mathematics: its formal and rigorous features collide with the fallibility and guessing aspects of the processes that produce it. There are not many teachers in Italian schools who think it is necessary that students experience an exploration and an argumentation phase in order to grasp what a proof is. Teachers' convictions are encouraged by traditional textbooks, where students find the usual definition-statement-proof model. The conditions under which students can make conjectures and validate them are left in the shadow. Furthermore, teachers forget that the formalistic aspects are not the main concern in proving processes (see the discussion of "formal" in Arzarello, 2007).

Some scholars have pointed out this aspect of proofs both from an epistemological (Thurstone, 1994; Tymochcko, 1998) and a didactical (Hanna & de Villers, 2012) standpoint. For example, in the book "Proof and Refutations" (1984) the philosopher Imre Lakatos expressed a different view of mathematical statements. As it is well known, based on a detailed discussion of Euler's errors in his search for a topological classification of polyhedra, Lakatos pointed out a dialectic process in this search. He showed that definitions are not carved in stone, but often have to be patched up in the light of later insights, in particular flawed proofs. This gives mathematics a somewhat experimental flavour. Balacheff agrees with Lakatos' approach, and in the introduction to the French editions of the book, he writes:

> Les mathématiques sont aussi prises en compte, non en tant que text de savoir, mais en tant que savoir construit socialment et donc l'acquisition par l'individu doit etre contrôlée comme sens et pas seulement comme langage. (Lakatos, 1984, p. XVIII)[1]

Following Balacheff's point of view, we are studying a fresh way of introducing pupils into mathematics' rules of thinking in order to produce proofs in elementary geometry. The paper aims to present some reflection of a PhD work in progress research. The hypothesis we are checking consists of investigating if and how the introduction of some multi-touch Dynamic Geometric Environment for Tablets offers important facilities to work in this direction. Here we are referring in particular to applications or software that allow users to work on the same screen with more than one finger at the same time. This peculiarity gives the possibility of designing teaching/learning situations, where students are asked to build up and investigate geometric objects in a new and shared way. The aim of such multi-touch activities is to put students into a geometric game situation, and making them ask why it is... / it is not... /may be... / cannot be... / so; and try to answer the question. We hypothesise that questioning develops a specific type of rational behaviour, which can help students understand and produce proofs.

In the literature our idea finds a foundation in the epistemological research of Jaakko Hintikka (1998, 1999). His aim is to replace the classical and static logic by a dynamic and dialectical model: the *Logic of Inquiry*, based on the analysis of the way people develop their strategic thinking in games. According to him, two types of rules characterize any goal directed activity: *definitory rules* and *strategic rules*. For instance, in chess the definitory rules tell you which moves are possible, while the strategic rules tell you which moves is advisable to make in a given situation.

We can apply this idea to the teaching of deductive logic. It is clear that the rules of inference are definitory rules, not strategic ones. At each stage of a deductive argument, there are normally several propositions that can be used as premises of valid deductive inferences. The so-called rules of inference will tell you which of these alternative applications of the rules of inference are admissible. They do not say anything as to which of these rule applications one ought to make or which ones are better than others. For that purpose you need strategic rules.

Students should learn and keep in mind both these rules during the construction of arguments. However, teachers generally explain to students the rules of inference and the way to use them correctly (definitory rules), not which rules are more advisable to use or how to select new arguments (strategic rules). Therefore, students learn how to avoid making mistakes, but not how to discover proofs or to find out new truths by means of deductive inferences.

> In concentrating their teaching on the so-called rules of inference, logic instructors are merely training their students in how to maintain their logical virtue, not how to reason well. (Hintikka, 1999, p. 3).

Clearly, it is not easy to teach strategic rules. If you are engaged in a game, like chess, it is more natural and taken for granted the use of strategic thinking, than when you are doing mathematics. For this reason, we decided to design games (of mathematical kind), whose solution is the discovering of a geometric property. We hypothesize that the result of this choice could lead students to use strategic thinking within the mathematical one.

In literature the idea of using games for educational purposes has come back in fashion thank to the widespread diffusion of mobile devices and virtual games. "A game can provide a structure for the learning that takes place in the environment" (Devlin, 2011, p. 32). We believe that, in order to bring these innovations into the classroom, *epistemic games* (Shaffer & Gee, 2005) need to be designed. The authors define these games as follows:

> Epistemic games are about knowledge, but they are about knowledge in action- about making knowledge, applying knowledge, and sharing knowledge. (Shaffer & Gee, 2005, p. 16).

Games based on geometric property have to be played in an environment that allows students to come back to what has been done or seen, produce interpretations and possibly explanations, anticipate facts and situations, produce forecasts and hypothetical discourses. In other terms, an environment that allow students to answer such questions, as "How is it?", "What is best for me to do?", "How will it be?", "How could it be?", namely they apply strategic rules. DGEs are a powerful tool to support students in the formulation of such questions and, therefore, in the application of strategic ways of thinking within mathematics. Indeed they allow the design of dynamic game situations, where, in order to win, students have to discover/use suitable mathematical properties, which allow them to develop suitable strategic moves. Discussing their strategies and why they were suitable for winning, the teacher can coach them towards the formulation and the proof of the mathematical properties that were behind the game, according to the Logic of Inquiry approach.

A further necessary condition for developing the Logic of Inquiry in the classroom, is to let students be used to questioning by themselves during mathematics activities. Hintikka (1999) formulated the structure of the interrogative model in the form of a game between an idealized *inquirer* and a source of answers called *nature* or *oracle*. The inquirer starts from a given theoretical premise T and his/her aim is to establish a certain given conclusion C. At each stage of the game, instead of making a deductive move, the inquirer may address a question to the answerer (oracle, nature, or whatever the source of new information may be). If nature responds, the answer becomes an additional premise. Hintikka calls such a move an *interrogative move*. After that, the process starts again until all the information added to the premise T lead the inquirer to the conclusion C.

The following example (Hintikka, 1999, p. 31) is illuminating about the interrogative model and its importance in reasoning. It shows that we are able to rewrite the solution of any Sherlock Holmes' story in an interrogative form. The episode we analyse is "the curious incident of the dog in the night-time", extracted from the story called "Silver Blaze". The background is this: the famous racing-horse Silver Blaze has been stolen from the stables in the middle of the night, and in the morning its trainer, the stable master, finds it dead out in the heath. All sorts of suspects crop up, but everybody is very much in the dark as to what really happened during the night.

Watson: "Is there any point to which you would wish to draw my attention?"

Sherlock Holmes: "To the curious incident of the dog in the night-time."

Inspector: "The dog did nothing in the night-time."

Sherlock Holmes: "That was the curious incident."

Even Watson can see that Holmes is in effect asking three questions: "Was there a watchdog in the stables when the horse disappeared?", "Did the dog bark when the horse was stolen?", "Who is it that a trained watchdog does not bark at in the middle of the night?". The following deductive argument is the exact transposition of the three questions of Holmes' inquiry: "There was a watchdog in the stables." "The dog did not bark when the horse was stolen." "A trained watchdog does not bark only at its owner." "Hence, the thief was the owner."

Each question is the source of a new *abduction* [2] and it is also an abduction that marks the transition from an inquiring to a deductive approach. Hintikka's analysis shows that this way of thinking does feature the epistemological basis of mathematics altogether and not only of game theory. In particular, it shows an *epistemic unity* between the argumentation phase, represented by the process of questioning, and the proof, represented by the reorganization of the answers in a deductive chain. In the literature, it has already been studied the cognitive unity between argumentation and proof (Boero, Douek, Morselli, & Pedemonte, 2010). Therefore, the result of epistemic unity can deepen these previous studies. In the interrogative model, we find both natural representations of non-logical reasoning (argumentations), and representations of formal logic. Hintikka observes a similarity between the two types of reasoning, founded on the role of presuppositions in the interrogative inquiry. In fact, before the inquirer is in a position to ask a convening question, i.e. "Who did it?" he or she must establish its presupposition "Someone did it". From the point of view of the transition from one proposition to another, an interrogative step looks rather similar to a deductive step: the latter takes the inquirer from one or more premises to a conclusion, while the former takes the inquirer from the presupposition of a question to its answer.

THE GAME-APPROACH

We divide what we call the game-approach into two phases: the game-task design and the so called Devil's Advocate reflection.

The game-task design consists of the transformation of the geometric properties in a *non-cooperative game,* in which each student has a different aim to reach, that contrasts with that of the other player. The task contains the rules of the game, the players' aims and some questions to answer. During the game, there is a silent inquiring activity in the students' mind. Thank to a Tablet and the schoolmate's feedback the inquiring process develops throughout the game, producing interrogative and deductive moves deeply intertwined: deductions are needed for establishing presuppositions for interrogative moves and interrogative moves are needed to add possible new hypothesis to the process of inquiry. In order to create a significant and relevant mathematical experience, we support students in the construction and development of strategies with questions like "Can you write someone else a way for winning?" which indirectly guides their attention to switch from the particular to the general. John Mason summarized these two-way processes as follows:

> …'to see the general through the particular and the particular in the general' and 'to be aware of

> what is invariant in the midst of change' is how human beings cope with the sense-impressions which form their experience, often implicitly. The aim of scientific thought is to do this explicitly. (Mason, 2005, p. 8)

Finally, we ask students: "How do you know that the method always works?". We hypothesize that, with this question, students can gradually discover the geometric property on which the game is designed, so avoiding a possible gap between it and the mathematics' theory. Students play the game on a shared Tablet and answer the questions working in pairs. Generally, this activity requires a one-hour lesson. After that, we withdraw students' worksheets and prepare a PowerPoint presentation in which we introduce the character called "The Devil's Advocate". This is the second phase, the Devil's Advocate reflection. In this phase, the Devil's Advocate (the teacher or the researcher) makes the Logic of Inquiry more explicit to students. In fact, he questions students to make them think theoretically on what they have found and she insinuates doubts on their deductions and sentences.

THE ANALYSES OF AN EXAMPLE

The episode described below is part of a teaching experiment developed in a tenth grade science class at a private high school. During the activity, the classroom was composed by eight students, working in pairs: they have to read the task, play the game on the tablet and answer some questions on a worksheet. They use GC/htlm5 [3] a newest version of Geometric Constructor (one of the free dynamic geometry software used in Japan since 1989) compatible both with iPad and Android tablets. During this first phase, the role of the teacher is to observe students and to help them if they are in trouble, whereas the role of the researcher is to videotape a single group. The work in pairs activity is followed by the teacher's systematisation of the mathematical content at the blackboard (at that moment the Devil's Advocate reflection has not been designed yet). In the second phase, the teacher asks students what they have discovered during the game in order to engage them as much as possible in the process. The systematisation generally takes place at the beginning of the subsequent lesson for matter of time (each class lasted 50 minutes).

The teaching experiment deals with some themes related to a classical topic included in the National Curriculum 2012 (Indicazioni Nazionali): the circle. The teacher commits almost twelve lessons to the project, developing six themes: the reciprocal position between two circles, the reciprocal position between line and circle, the chords theorem, the angles at the centre and at the circumference, the circumcentre and incentre of triangles, the inscribed and circumscribed quadrilaterals.

The example we will show, describes a non-cooperative game situation involving two players Z and Y. The final aim of the activity is discovering the geometric property that describes the reciprocal positions between two circles. Here is the task given to students: "Play the chase with your schoolmate. Z's aim is changing the length of segment AB by dragging its endpoint in order to make the two circle intersect; Y's aim is changing the length of segment CD by dragging its endpoint in order to avoid the intersection. When does Z win? When does Y win? Move the centre of the circles to examine the possible cases."

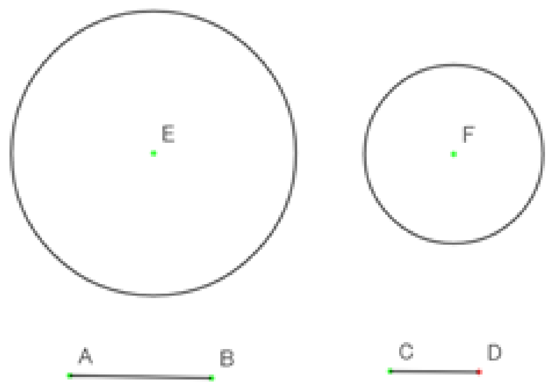

Figure 1: The picture shows what students see on their Tablet

1 Student Y: Have I to run away? *(Reducing radius of circle E more and more)*
2 Student Z: I think you could also move... *(Enlarging more and more the radius of circle F)*
3 Student Y: Yes, but if you are enlarging it, what could I do?
4 Student Z: I think you could also move... you can try to move this one *(pointing the centre E)*
5 Student Y: Wait, wait, wait!
6 Student Z: I think you can try to move this one
7 Student Y: Yes but if I run away...

2a 2b 2c

Figure 2: 2a) First strategy "Running away"; 2b) Second strategy "A circle inside the other"; 2c) Third strategy "One centre"

8 Student Z: I will make it bigger [the circumference F]!
I stay here so I always catch you!

9 Student Y: If I put myself [circumference F] inside *(pointing the area of circumference E)*, you can make yourself [circumference E] bigger and catch me, but I can... if I tighten...

10 Student Z: No! In this way it should be... it should be... Points should coincide *(pointing the two centre E and F)* because I catch you however!

During the game, students switch between two modalities:

- the *played game* in which students play against each other or collaborate and their only aim is to win;

- the *reflected game* in which students take distance from the game to analyse and make judgments about what has happened. Their aim is collaborate to discover the strategy to win.

In the example, line 5 marks the switch from the played game to the reflected game. When students enter into the second modality, they start analysing the different types of situations that could arise during the game. First, they examine the case in which player Y tries to escape by moving the centre of the circle (line 6–8). In this case, player Z always catches him by making his radius larger. Then they experiment with a second strategy: putting the centre of circle Y inside the circle Z. This second solution leads player Z to win over player Y as well. Finally, student Z suggests to student Y a different situation in which the centres of the circles coincide (line 6).

It is important to notice that during the reflected game, students exchange their roles (line 5), since they not only think about their movement, but at the opponent's movement as well, and they identify themselves with the geometric object: each student is the circle that he/she moves on the screen. While students try to discover the strategy to win, they explore unconsciously the reciprocal position between two circles and they implicitly discover the link between it (the reciprocal position between the circles) and the position of centres or the length of the radiuses. Even if the mathematical theory remains implicit in students' actions, words and visualisation during the whole game, students build strong concept images (Tall & Vinner, 1981), which help them in the construction of mathematical concepts.

As in the case of mouse dragging practises (Arzarello, Olivero, Paola, & Robutti, 2002), we aim to analyse the modality of dragging in order to notice if there is a correspondence with the cognitive level. In particular, in mouse dragging practises, there are two main cognitive typologies (Saada-Robert, 1989; Arzarello, 2007):

- the *ascending processes*, from drawings to theory, in order to explore freely a situation, looking for regularities, invariants, etc.

- the *descending processes* from theory to drawings, in order to validate or refute conjectures, to check properties, etc.

In designing the tasks, we started from the two main cognitive typologies which characterize mouse dragging practises and tried to readapt them in order to describe that of games practices. Insofar, we began transferring the results on mono-touch to multi-touch dynamic geometry software: the aim is to observe what remains invariant and what changes in the students' approaches and processes. In particular, in the played game we distinguish between *ascending processes* when students enter into the game, explore the situation freely, look for strategies and *descending processes* when students play with a strategy in act. In the reflected game, instead, we recognize *ascending* *processes* when students explicitly use the strategy and *descending processes* when they try to check it. We are interested in observing whether or not students share the ascending and descending processes.

In line 1 students are in the played game descending processes, they are playing with a strategy in act: Y tries to escape reducing the radius more and more and Z makes his radius larger following Y's movement. The students realise that with this implicit strategy Z always win. Y is going to give up, when Z suggests her the first explicit strategy (line 6): "moving the centre of the circle to escape". Students are now in the reflected game ascending processes, they play with the explicit strategy and they immediately understand that student Y always loses. Line 8 shows the moment in which students are in the reflected game descending processes, because they check the strategy and decide to abandon it. Students come back to the reflected game ascending process and explicit another strategy: "make a circle inside the other". They immediately pass on the descending process and understand that Z continue to win. They abandon the strategy and make explicit a third one: "make the centres coincide". They are one more time in the ascending process.

Under the first two strategies, there is the implicit mathematical property: "if the distance between centres is less than the sum of their radius and major of their difference, circles intersect at two different points". Since for every Y's movement, there are infinite Z's movements such that the distance between centres is minor then the sum of the radiuses and major then their difference, students are lead to the conclusion that Z always wins. Students do not know the mathematical property that leads them to this conclusion, they only experiment the property in an empirical way, through the game.

CONCLUSION AND POSSIBILITY FOR FURTHER RESEARCH

The analysis of the protocols reveals that the game approach makes students explicit their strategic rules of thinking, but it is not enough to give them insights on the impact of the use of strategic rules of thinking on the mathematical reasoning. In particular, we observed that students do not mention the mathematical property on which the game is based. For this reason we have designed two didactical interventions, which are fundamental in order to both make the mathematics rise from the game and overwhelm the possible cognitive discontinuity between the inquiry phase and the deductive phase:

1) The introduction of specific questions, such as "Can you write someone else a way for winning?" and "How do you know that the method always works?"

2) The introduction of the Devil's Advocate reflection.

The first one helps students thinking the reasons why one wins and detect the geometric property, while the second one helps students in the deductive transpositions of their arguments. Both these interventions make the Logic of Inquiry more explicit to students.

Another issue we are addressing now aims at deepening the technological possibilities offered by DGEs in order to make the game more challenging and engaging. For instance, we are designing more complex games, where players must overcome some intermediate steps in order to win. Such steps correspond to parallel steps in a possible proof of the mathematical properties upon which the game is built. For example we are introducing the opportunity for a player to choose from time to time between two alternative possible constructions in the environment. Only one of them will facilitate her/him: exploiting which is the right one to choose corresponds to a mathematical property, which can facilitate the successive proving phase.

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ENDNOTES

1. "Mathematics is also taken into account, not as a text to know, but as knowledge socially constructed and therefore the acquisition by the individual must be controlled as sense and not as just as language."

2. The following example (Peirce, 1960, p. 372) clarify what an abduction is. Suppose I know that a certain bag is plenty of white beans. Consider the sentences: a) these beans are white; b) the beans of that bag are white; c) these beans are from that bag. A deduction is a concatenation of the form: b and c, hence a; an abduction is: a and b, hence c. An induction is: a and c, hence b. For more details, see Magnani and colleagues (2001) and Arzarello and colleagues (2000).

3. The inventor of GC/htlm5 is Yasuyuki Ijjma from Aichi University of Education (Japan). To visit the software go to http://iijima.auemath.aichi-edu.ac.jp/ftp/yiijima/gc_html5e/gc.htm.

Online platforms for practising mathematics in German and English speaking countries – a systematic comparison

Martin Stein

Westfälische Wilhelms-Universität, Münster, Germany

During the last years, a multitude of web-based computer assisted systems for learning mathematics were developed. Eva-CBTM (Evaluation of Computer Based Platforms for Training in Mathematics) makes a complete survey of all such platforms (German or English as language). To do so, an evaluation scheme has been developed which assesses the help system, system structure and other features. A significant number of learning platforms will be compared using this assessment system.

Keywords: Software evaluation, learning program, fractions, Khan academy, IXL, Tenmarks.

THE EVA-CBTM-PROJECT

Eva-CBTM is a project for the Evaluation of Computer Based programs for learning and Teaching Mathematics. The aim of Eva-CBTM is to develop a complete system for such an evaluation which has its focus on mathematics didactics, and to apply this system to all relevant international platforms (German and English as language). The evaluation is carried out under the perspective of a formal analysis of the fulfilment of didactical criteria, and the perspective of pupils working with these systems will not be studied in this paper.

Due to the limited space in this paper, the *main focus* will lie here on the description of the *evaluation system*, whereas the results cannot be given in full. For the interested reader, (Stein, 2012) contains detailed results and descriptions of all platforms which were evaluated.

ASPECTS OF EVALUATING SOFTWARE IN EVA-CBTM

The main components of the evaluation system

An analysis of existing criteria catalogues for the evaluation of (mathematical) learning software (for instance, Grosser, 2000 (in German); Handal, Handal, & Herrington, 2006 (in English)) unveils that these are often written from a media-pedagogical perspective or test the user-friendliness of such systems. Such catalogues can be used to evaluate individual programs (and then possibly to compare them). Since they cover a broad range of different types of websites – "drills, tutorials, games, simulations, hypermedia based materials and tools and open-ended learning environments" (Handal & Herrington, 2003, p. 278), they are not specialised enough for analysing and comparing) programs which focus on practise in mathematics only.

School children working with such programs need *feedback* about their work (Handal, Handal, & Herrington, 2006, p. 8), and they normally also need *help,* resp., *assistance* (ibid, p. 13). Every practising software therefore needs an *assessment system* and an *assistance/help system.* Both of these components operate closely together in many cases, such as when the evaluation system notices a mistake and triggers the help section, but it still makes sense to describe the two systems separately.

Additionally, the systems require a *pool of tasks* that must be *structured* in some way. The follow-up-task should be provided by the system, according to whether or not the user was successful in the previous task. With this, we have a *system structure* and a *follow-up system for task selection* as further characteristics of a CBTM-System that are in need of evaluation (ibid, p.

11). Systems can also differ significantly in the question of how many decisions they take autonomously, and how many they leave to the user. This is measured by the *degree of freedom*.

Finally, the *completeness* by which the covered topics are handled has to be evaluated. This aspect is measured in *thematic completeness*. It is given as a percentage that is based on how completely the subject under consideration is covered. Using this percentage value on the point score already given to a system yields the overall evaluation score of the system.

A process model for the interplay between user activities and system activities

Doing mathematics, solving problems, calculating formulas... always follows certain steps: it is a *process*. This work process runs along a time axis that begins with the presentation of the task and ends when the solution is submitted. In *teaching*, we can describe three stages: pre-active, interactive, and post-active (Kysilka et al., 2002, p. 60). Looking at *learning platforms*, we prefer the wording pre-active, formative, and summative, as used in literature about evaluation (see, for instance, Schulmeister, 2002) since the learning software evaluates / assesses the outcomes of the pupils' work.

Explanations

Assistance without request.

Communication between the assessment system and the assistance system.

Assistance provided by the system at the user's request.

User activity, the user's actions are communicated to the assessment system.

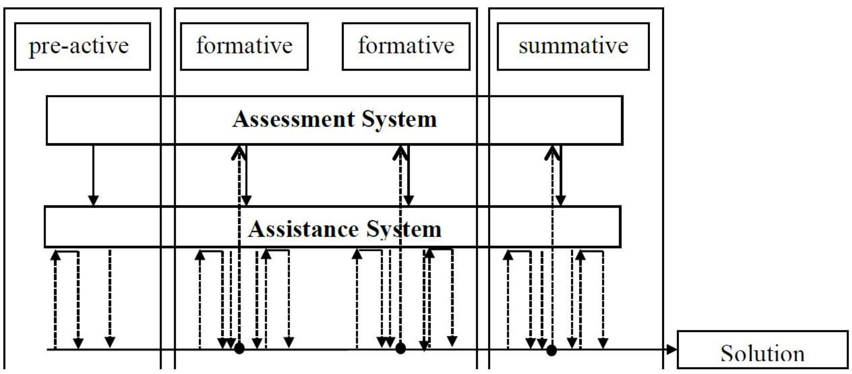

Figure 1

Pre-active: Before the start of the work process – it is clear that user input cannot be evaluated before the user starts to solve the task. However, assistance can be available at this point.

Formative: During the work process – the evaluation of user activities takes place based on the user's inputs. This temporal process is a basic part of any attempt to solve a task. Whether or not the system „notices" this work, is another matter. For that to happen, the system needs to enable the user to enter the consecutive steps of his/her work. Only then we can talk of formative assessment / assistance.

Summative: At the end of the work process – this is the standard kind of evaluation that every teaching and practice system has.

This model is applied at two levels:

- *Micro level*: Each task requires several solution steps. If the system has been programmed to work in this way, it can pick up on the individual steps that the user takes and react appropriately.
- *Macro level*: CBTM systems do not provide tasks in isolation from each other. Instead, they provide sets resp. sequences of tasks. The sequence of work on these tasks can be described and analysed with the same model.

THE EVA-CBTM EVALUATION SYSTEM

Assessment

According to the process model, assessment can take place pre-actively, formatively or summative. Apart from that we have to differentiate between assessments that have a procedural component that touches on questions of heuristic strategy and assessments that solely focus on the current content or input. Three levels are defined for both assessment strategies of which six types of assessment follow, as shown in the Table 1.

Assistance

Literature on e-learning acknowledges demand for feedback and assistance, but lacks any detailed information which could cover the richness of types of assistance by an appropriate evaluation system: Schwier and Misanchuk (1993) demand a gradual increase on the "types, amount, and layers of stimuli and feedback presented" (p. 20). Bescherer and colleagues (2012, p. 152) say, "When students work on their exercises at home, however, lecturers and tutors generally are not available to help out. In such cases, computer-based approaches, where hints and feedback are provided on demand by using software, can be a real asset to learning. However, such feedback needs to be more differentiated than merely pointing out 'right', 'wrong', or 'reread chapter X.' "

One perspective on the topic of assistance in work processes and solution-finding processes offered by literature on mathematics education (see, e.g., Zech, 2002) leads to a catalogue of terms, listed and briefly explained here:

Strategic Assistance: Solution strategies are described, a specific strategy is recommended; *Content-based assistance*: Users are provided with assistance by the task currently in front of them; *Motivating assistance:* The help has an encouraging character („Keep it up", „That was again very good", etc.); *Responsive assistance*: Typical feedback. This contains the basic information as to whether the solution step was correct or incorrect or if the solution method will take you to your goal; *Diagnostic assistance*: The assistance contains information about the cause of the mistake.

Solution-generating assistance: The assistance leads to the solution.

These types can be used

- *Situation specific*: The assistance is focused on the task currently in front of the user, using the concrete numbers of this specific task. For example, there would be two separate, pre-calculated solution methods to the equations $2x+3=7$ and $3x+4=9$.
- *Generalised*: The assistance is general in nature. So, for the solution to the equations $2x+3=7$ and $3x+4=9$, one piece of assistance is given for each task, perhaps for the equation $5x+8=17$.
- *Global topical*: The assistance includes several types of task. For example, the assistance for expansion, finding the common denominator and addition of fractions is all contained within a single explanation.

This system of 18 possible combinations is so nuanced in the case of motivation that a classification of learning systems would not be reliable. For example, it can be argued that the mention of addition

You have added on the right side. Think about it!

Strategic	Content based
simple normative Checks whether the user has found the correct solution method.	**simple normative** Checks if the solution method corresponds to a predefined target. If not, it is regarded as incorrect. If $\frac{1}{2}$ *is expected,* $\frac{2}{4}$ is considered as wrong.
complex normative Choosing an alternative correct solution method is accepted or is reacted to flexibly. E.g., „This method is also possible, but we will continue our calculations as follows..."	**complex normative** Equivalent solution methods are recognised, and are both designated correct or incorrect. $\frac{1}{2}$ and $\frac{2}{4}$ are both recognized as wrong or both as right.
diagnostic The reason behind an incorrect solution method is recognized. „You have carried out the steps in a wrong order."	**diagnostic** The cause of a mistake is communicated. In case of $\frac{2}{3}+\frac{1}{2}=\frac{3}{5}$ the system says, for instance: „You have made the mistake ‚Numerator plus numerator and denominator plus denominator'."

Table 1

… is *strategic assistance* (choose the correct *operation* as a solution strategy), or if it should be seen as *content-based assistance* (it is about the *addition*).

Such a discussion is in any case superfluous – the fact that even in the theoretical analysis uncertainty is creeping in shows that these four categories are much too finely structured to be the basis of an evaluation schema. Using them would lead to unreliable coding.

As such, some items of the classification system can be dropped and in the end the following items remain in the list:

> Motivating Assistance | Feedback on the assessment | Strategic, solution-generating, situation-specific | Strategic, solution-generating, generalised | Strategic, solution-generating, global topical | Content-based, solution-generating, situation-specific | Content-based, solution-generating, generalised | Content-based, solution-generating, global topic

Special *types* of help – for instance, the use of multiple representations or interactive features – are not measured individually in this system. They are scored as content-based, solution generating and can be situation specific or generalized.

The use of *tools* like computer algebra systems or dynamic geometry software can help students learning mathematics and can play an important role in the assistance section of a CBTM-system. However, this was not made an issue of this study: Eva-CBTM analyzed the *fraction arithmetic part of those systems*, and here those tools do not play an important role. If the same study shall be executed for the algebra or geometry section of such systems, the assistance part of the Eva-CBTM evaluation system has to be extended to cover such features.

System structure and choice of exercise

Before we can discuss system structure and exercise choice in a system, some remarks about the framework of exercises are necessary.

Looking at the set of exercises, it becomes immediately clear that the minimal structure is that of a *sequence*. Every system of exercises gives exercises in a certain order – the *exercise series*. Exercise series of this type can be assembled „*ad hoc*", or they can be put together *according to internal evaluation systems*. This is often done according to *difficulty*, although there are other conceivable systems.

Every exercise is provided with certain labels and pointers. A *label* is a piece of information assigned to the task. A *pointer* directs the user to another exercise or sequence. There can be multiple pointers, such as if different follow-up exercises should be chosen based on information about the mistakes made by the user. A *backwards pointer* defines a relationship with a previous exercise or exercise series, which can be defined by labels that determine which previous exercises should be chosen based on certain events.

The interaction between the information about the exercises contained in the labels, the follow-up structure defined by the pointers and the information about previous exercises defined by the backwards pointers leads potentially to an extraordinarily complicated structure of exercises through which the system has to navigate. For practical use, the following *selection structures* can be designated.

Architecture of exercise sequences and the overall system

Ad-hoc arrangement: The program moves from one exercise to the next in such a way that no system is noticeable.

Internal arrangement through evaluation of exercises: Every exercise in a sequence receives a label through which it is evaluated. This is often „difficulty" or „complexity".

External arrangement through evaluation: For every sequence of exercises, preceding and following sequences are defined by an evaluation – e.g., by the level of difficulty. Preceding sequences for the addition of two three-figure numbers are for example the addition of two two-figure numbers or one two-figure number and one three-figure number. *External arrangement by content-based analysis*: Sequences of exercises are connected to others by follow-up relationships. This can be based on different types of exercise analysis: e.g., it is possible to define sequences by the basic skills necessary to solve them. In the case of solving the equation $3x + 4 = 17$ those are addition, division, subtraction, etc. It is also possible to determine which follow-up sequences make the best sense.

The system of choosing exercises

The system of choosing exercises touches on the architecture of the system. More complex systems make decisions dependent on the successes or failures of the learner. They navigate within the defined architecture.

Depending on the starting point in the process model, there are different evaluation possibilities on the Pre-activity, the Formative and the Summative stage. Due to the space restriction we cannot describe this here and have to refer to (Stein, 2012).

Degree of freedom

The above classification only captures the method and the complexity with which a system selects exercises or exercise sequences. CBTM-systems, however, can also be very different in the degree of freedom that they allow the users in their work. The fine grain evaluation system of Eva-CBTM cannot be discussed here; it is described fully in (Stein, 2012).

Thematic completeness

The question of how completely a given system covers its chosen area plays an important role in its evaluation. The evaluation begins with a pedagogical analysis of the topic under observation, based on existing pedagogical and methodological knowledge. Every topic recorded in the evaluation is given an appropriate marker. If in the course of the evaluation a particular system introduces a new aspect, the table is expanded to include this aspect. At the end, the percentage of possible practical topics that each system covers is calculated.

ALLOCATION OF POINT SCORES IN THE EVA-CBTM-SYSTEM

The combination of the different aspects of the evaluation (assistance, assessment, and so on) with the three stages of the process model, each being divided in different categories, leads to a multidimensional matrix. Each cell of this matrix describes one possible feature of the training system. In most cases for each cell a score of 1 point is allocated, if applicable (the few exceptions cannot be discussed here), and 120 allocations have to be placed.

The allocation of one point for each feature is a personal view of the author, other allocations are possible. If, e.g., *assistance* (besides a feedback right / wrong) seems not a desirable feature of a system, for this part of the matrix 0 points would be given. And if the formative part of the matrix is not seen as useful, the same will be done with all cells of the matrix belonging to the formative stage. In the result section, we give a brief summary of the outcome under the heading *Alternative scoring*.

Because the available contents can differ depending on the region for which they were principally designed, this evaluation is carried out in a thematic area that all systems contain, i.e. *calculations with fractions*.

Each of the observed systems is evaluated by three independent evaluators – the evaluation was part of their master thesis (Frankewitsch, Menke, & Wietholt 2012 / available from this author) written under supervision of this author. After a training phase in which examples from different systems were scored by the evaluators and this author, the evaluation was carried out independently by the three evaluators, the scores were compared in a joint meeting of the three evaluators. When the evaluators failed to come to a unanimous evaluation of an item, the reasons for the different ratings of the point under discussion were then discussed by them further, and the final decision was protocolled, together with the reasons for it (see Frankewitsch et al., 2012, pp. 112 ff.). The independent evaluation lead to nearly unanimous results, with a minimum of 2 (4 platforms), a maximum of 6 disagreements (1 Platform), the median is 4 disagreements (out of 120 scores) only.

The scores for assessment, assistance, and so on, are multiplied with 10, which leads to the following values:

> Assessment Strength: AS-value | Assistance (Help) Strength: HS-value | System Structure: SyS-value | Choice of Exercise System: CES-value | Degree of freedom: DF-value | Thematic Completeness: TC-value

The thematic completeness is evaluated as has been described before. The percentage is divided by 100 to yield a decimal number between 0 and 1 which is called TC. The result of the multiplication of TC with the AS-value is the weighted AS-value W-AS, and so on.

CONSIDERATIONS FOR CHOOSING WHICH ONLINE PLATFORMS FOR LEARNING AND PRACTISING MATHEMATICS TO EVALUATE

In the face of the over 60 mathematics-platforms to be found in the internet, often with very different objectives, a definition of the platforms to be evaluated is necessary. The following conditions must be met to be included on the list:

The software can be used completely without a tutor, it serves principally to practise mathematical skills, the mathematical topics of grades 5 to 10 are covered as completely as possible, and the software can be used by interested school children or their parents without involving an external institution (such as the school).

The following list gives an overview of the finally selected platforms.

German language platforms

Abfrager: abfrager.de
Bettermarks: Bettermarks.de
Mathegym: *mathegym.de*
Realmath: realmath.de
Scolaris: scolaris.de
Skilltime: Lernen.skilltime.de

English language platforms

AAA Math: aaamath.com
ALEKS: aleks.com
Aplia: aplia.com
Bettermarks: bettermarks.de
BrainPop: brainpop.com
Ixl: ixl.com
Khan Academy: khanacademy.org
Mangahigh: mangahigh.com
Mathletics: mathletics.eu
MathsOnline: mathsonline.com.au
TenMarks: tenmarks.com

RESULTS

10 points per item

Due to limitation of space, we give the results for those platforms only which yield a score over Q3(=120), and add the result for the weakest platform, to show the spread. (Table 2)

Alternative scoring

If we do not consider the formative stage, and find assistance beyond saying “right” or “wrong” not desirable, the results are as follows in Table 3.

DISCUSSION

The evaluation system developed in this paper leads to a ranking order in which bettermarks is leader far ahead all other systems. An important reason for this is the construction of bettermarks as a system with a strong emphasis on the formative stage. This decision leads to a system in which not only the end result of a computation is assessed, but the intermediate steps as well. In consequence, the evaluation of the assessment system AS-score) and the assistance system (HS-score) of bettermarks leads to far higher scores than that of the other systems – but even if the formative stage is not rated as in the alternative scoring, the leading position of bettermarks remains untouched.

	TC	W-AS	W-HS	W-SyS	W-CES	W-DF	Total
Scolaris	0,22	2	9	2	0	1	14
Mangahigh	0,58	15	58	17	29	1	120
Khan Academy	0,44	15	52	13	17	26	124
IXL	0,69	21	55	21	28	3	128
Tenmarks	0,67	13	87	13	13	1	128
Mathegym	0,62	31	62	12	12	25	142
Bettermarks	0,65	65	164	26	65	46	367

Table 2

Scolaris	Mangahigh	Khan	IXL	Tenmarks	Mathegym	Bettermarks
14	99	89	93	61	102	223

Table 3

Other important aspects under which many of the systems are very weak, and which must be improved to yield higher scores, are *deficits in the assistance system* (IXL: no pre-active help, no formative help, HS-value = 80 (of 410), W-HS=55), and the *assessment system* (Mangahigh: simple normative only, AS-value=25, W-AS=15).

ACKNOWLEDGEMENT

EVA-CBTM was carried out with support from *Bettermarks*, Berlin. The author wants to thank the bettermarks team for many fruitful discussion, and Th. Frankewitsch, V. Menke and M. Wietholt for doing the evaluations with great accuracy.

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A learning trajectory of the accumulation function in multiple-linked representational environment

Osama Swidan

University of Haifa, Faculty of Education, Haifa, Israel, osamasw@gmail.com

Our study was designed to identify a learning trajectory for the accumulation function as it learned with multiple representational technology based artifact. Thirteen pairs of 17-year-old students were asked to explain the connections between multiple-linked representations and to conjecture about the mathematical relationship embedded in the artifact. The study was guided by the semiotic mediation theory and the theory of knowledge objectification. We include two rounds of analysis: one to detect mathematical elements involved in learning the accumulation function, another to identify the progression process of learning the accumulation function. The data analysis identified four phases in the processes of objectification along which we suggest a trajectory for learning the accumulation function concept.

Keywords: Accumulation function, multiple-linked representation, semiotic mediation, objectification, learning path.

INTRODUCTION

Constructing a learning trajectory for students is one of the most daunting and urgent issues facing mathematics education today (Steffe, 2004). The main issue concerns the promotion of students' development of new mathematical concepts, especially concepts the development of which is challenging. The accumulation concept, which central to the idea of integration, is one of such concept. Learning the integral concept based on the accumulation function may create opportunity for the student to make the connection between the derivative and the integral which is essential for meaningful learning of calculus (Thompson et al., 2013). Simon (1995) offered the Learning trajectory as a way to explicate an important aspect of pedagogical thinking involved in teaching mathematical understanding. According to Simon, learning trajectory consists of three components: (a) the students' learning goal, (b) a hypothesis about the process of the students' learning of mathematical concept, and (c) tasks to be used to promote the students' learning (Simon, 1995).

Studies regards the learning of the accumulation function concept (Thompson, Byerley, & Hatfield, 2013), have examined the role of graphic and numeric artefacts in learning the accumulation function. In these studies, technology was used mostly in attempts to relate a function with its accumulation function in a simulated-physical situation such that of time graphs involving velocity and position (e.g., Noble, Nemirovsky, Wright, & Tierney, 2001). The present study examines the learning of the integral concepts analytically by high school students as they learn it with dynamic, interactive, and multiple-representation artifacts.

Multiple-representational artifacts enable the development of learning environments that present in separate linked windows the accumulation and its rate of change functions. In an attempt to identify a learning trajectory which help students conceptualizing the accumulation function, our study was designed to explore the objectification processes of the accumulation function when it learn by multiple-linked representations and interactive artifact. To do so, we explore the ways in which high school students learn the accumulation function with multiple-linked representation artifact.

THEORTICAL FRAMEWORK

The study was carried out using the theoretical framework of semiotic mediation (Bartolini Bussi

& Mariotti, 2008) and the theory of knowledge objectification (Radford, 2003). The semiotic mediation theory considers learning as an alignment between the personal meanings arising from the use of a certain artifact for the accomplishment of a task and the mathematical meanings that are deposited in the artifact. Personal meanings are to refer "to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not)," and a mathematical meanings are to refer, "to the extent that its usage is congruent with its usage by the mathematical community" (Berger, 2005, p. 83). In the context of using artifacts, the semiotic mediation theory describes the relations between personal meanings and mathematical meanings as a double semiotic relationship. On one hand, concentrating on the use of the artifact for accomplishing a task, recognizing the construction of knowledge within the solution of the task. On the other hand, analysing the use of the artifact, distinguishing between the personal meanings arising from the use of the artifact in accomplishing the task and meanings that an expert recognizes as mathematical when observing the students' use of the artifact in order to complete the task. Radford (2003) suggests a semiotic tool to analyse the dynamically evolving relationship between personal and mathematical meanings. The basic components of the semiotic tool are the students' progressive attention and awareness of the mathematical object. Varieties of semiotic means of objectification that have a representational function attract the students' attention to mathematical objects. Furthermore, the properties of the artifact can help students attend to the mathematical objects related to the activity under consideration. Paying attention to the necessary aspects of the mathematical phenomenon and using various semiotic means of objectification, students become aware of the attributes of mathematical objects within that phenomenon. Being aware, students attain objectification of the mathematical objects, which then become apparent to them through various devices and signs.

Given the cultural knowledge deposited in any designed artifact, we consider the linked graphs, the table of values, and the control tools that are the focus of the present study as culturally determined signs that convey knowledge regard the accumulation function. The accumulation function is represented symbolically as $\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$. The mathematical structure of the accumulation function involve variety of mathematical elements. Such as Δx, product $f(x)\Delta x$, and sum of products $\sum f(x)\Delta x$. These elements are deposited in the artifact used in this study.

The students' learning goal is to come up with conjectures about the mathematical relationship of the accumulation function. We hypothesize the students should become aware of the mathematical elements of the accumulation function as they interact with the artifact. Our aim is to explore the ways in which students become aware of the idea of accumulation as it is represented in the artifact. We propose to answer the following research question:

How and what are the mathematical elements used by the students in the process of becoming aware of the mathematical meanings of the accumulation function when it is learned with multiple-linked representational artifact?

SEMIOTIC POTENTIAL OF THE ARTIFACT

The interactive integral and accumulation artifacts at the heart of our study are part of the calculus unlimited (CUL) artifact. As a multi-representational artifact, CUL contains different types of tools that we grouped into four categories.

I) *Graphing tools*: Two vertically aligned Cartesian systems, coordinated vertically. The trajectory in the upper Cartesian system signifies a function. The function is defined symbolically by the free input of a single variable expression. The trajectory in the bottom system presents the values of Riemann sums $\sum_{i=1}^{n} f(x_i) \cdot \Delta x_i$.

II) *Numeric tools:* The associated table of values contains three columns. The left column presents the upper boundary values, the middle column presents the delta x values and the right column shows the accumulated values.

3) *Accumulation tools*: Because our study focused on the computation of rectangles, the rectangles that appear on request in the upper Cartesian system (the function system) represent the product of $f(x_i + \Delta x_i)\Delta x_i$. Rectangles are color-coded to reflect the product sign (positive or negative) (Figure 1).

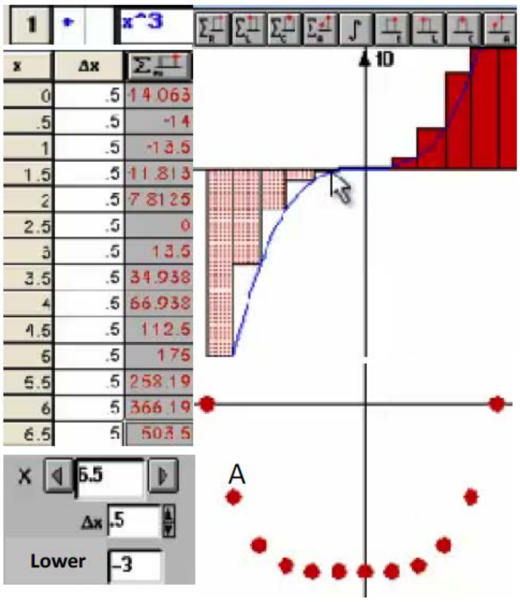

Figure 1: CUL interface

IV) *Boundaries tool:* After the function *f* and the method of accumulation are specified, three parameters determine the value of the accumulation at a given point: the lower and upper boundaries of a bounded region and the width of each interval into which the region should be divided. The design of CUL attempts to direct the attention of the user to these parameters and emphasizes especially the control over the upper boundary value, with its immediate visual feedback of a value of accumulation points. Students control the bounded area by using arrows to move a marker in intervals of Δx to the left or to the right of the lower boundary (Δx is an absolute measure of the interval, thus always achieving a positive value both right and left of the lower boundary). The upper boundary value is represented in the accumulation function graph by a marked coloured point and in the table of values.

STUDY DESIGN

The present study explores approximately 12 hours of learning by 13 pairs of 17-year-old students from two different schools in Israel. The students volunteered to participate in five after-school meetings. At the time the meetings took place, the students had already learned the concepts of function and derivative. The Author introduced them briefly to the interface and illustrated how to use it. He explained, for example, how to input the symbolic expression, how to change the controlling parameters. In particular, the students were told about the technical functionality of the artifact.

To promote the processes of objectification of the accumulation function, we asked the students to explain and explore the possible connection between two given function graphs and the table of values. They were given the instructions in Figure 2.

The learning took place in the computer lab at school. Students were video-recorded and the corresponding computer screens were captured.

Data Analysis. The data analysed through two levels: (a) the macro level: identifies the mathematical elements involved in objectifying the accumulation function. Mathematical elements were defined as segments of discourse in which the students sought to discover the mathematical relationship inherent the accumulation function, for example, delta x, product, sum of product, and the positions of the accumulation function graph. (b) at the meso level, we analyzed the data in two stages: (a) we distinguished personal meanings that are accepted mathematically from those that are not. Statements that in the context of the accumulation function are mathematically wrong were defined as personal meanings that are not accepted mathematically. By contrast, statements that are mathematically correct in the context of the accumulation function were defined as personal meanings that are accepted mathematically. For example, statements like "the y-value of the initial point in the accumulation function is equal to the height of the rectangle," which refers to the objectification product

The task is to explore and explain possible connections between graphs in the upper and the lower Cartesian systems and the linked table of values. You should be able to input function expressions and obtain a pair of graphs such as f(x) = x^2, x^2-9, x^3, (x+3)(x-1)(x+4). While working on the task you may use the CUL artifact to generate graphs by means of symbolic expression, and change the value of X by pressing the ◄ ► arrows in the upper part of the value box, as needed. As default values you can ask to specify 0.5 in Δx box, -3 in the lower box, and -3 in the X box.

Figure 2

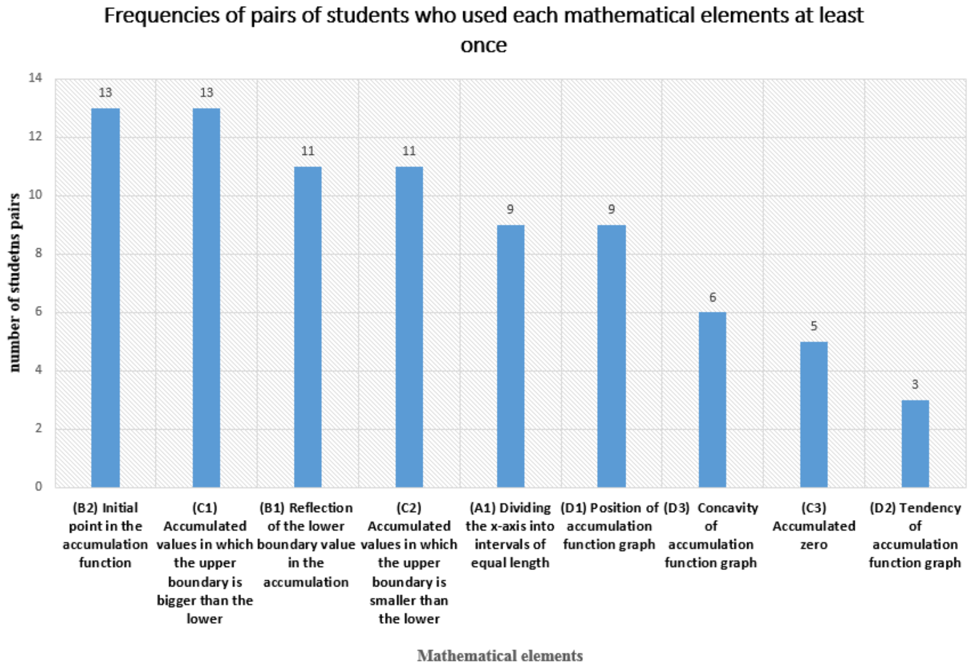

Figure 3: Frequencies of pairs of students who used each elements at least once

were defined as personal meanings that are not accepted mathematically. For each value of delta *x* other than 1, the initial point in the accumulation function is equal to the product of the rectangle dimensions, but not equal to the height of the rectangle. For this reason, we defined this statement as a personal meaning that is not accepted mathematically. Statements like "the y-value of the initial point in the accumulation function is the same as the first rectangle area" were defined as personal meanings that are accepted mathematically. (b) we applied the strategy of evolution over time to each pair of students, to analyze the evolution of personal meanings into mathematical ones over time.

FINDINGS

The detected elements were grouped into four categories: (a) objectifying delta x as splitting the x-axis into equal-length segments, (b) objectifying the product, (c) objectifying the sum of product, and (d) objectifying the properties of the accumulation function graph. Categories (b), (c), and (d) were divided into subcategories based on the mathematical element that is under consideration. In the graph below, we summarized the frequencies of pairs of students who used each element at least once in the learning process.

We found that all the students tried to objectify the product $f(x)\Delta x$, which are represented by the initial value point in the accumulation function graph (point A, Figure 1). Furthermore, all of them tried to objectify the sum of product in a domain in which the upper boundary values are bigger than the lower boundary values (B2, C1). The majority of students (11 pairs) considered the reflection of the lower boundary value in the accumulation function graph (B1). Eleven pairs of students also tried to identify the accumulation function in a domain in which the upper boundary values are smaller than the lower boundary values (C2). Nine pairs tried implicitly to identify delta x as dividing the x-axis into segments of equal length (A1). Nine pairs also considered the position of the accumulation function graph relative to the x-axis (D1). Six pairs tried to objectify the concavity of the accumulation function graph (D3). Five pairs considered the accumulated zero (C3), and three pairs considered the tendency of the accumulation function graph (D2)

Paths followed to learn the accumulation function

In Table 1 we present a distinction between personal meanings that are accepted mathematically and those that are not. The number of episodes associated with each category is also shown in Table 1. For example, in row B2 the number 41 indicates that 41 of all the episodes are associated with the mathematical element 'product' represented by the "Initial value point in the accumulation function." Eighteen of these were considered to be accepted mathematically and 23 were not.

We counted a total of 211 episodes containing mathematical elements. Five percent (10/211) of the episodes were devoted to identifying delta x as dividing the x-axis into intervals of equal length. Thirteen percent (27/211) were devoted to identifying the reflection of the lower boundary value in the accumulation function graph. All of these but one were coded as meanings accepted mathematically. These findings may indicate that the objectification of these two mathematical elements was not challenging for the students. It is possibly that the design of the artifact helped students objectify these mathematical elements. Twenty one percent of the episodes (41/211) were devoted to the objectification of the initial value point. Twenty-three of these were coded as personal meanings that are not accepted mathematically. This finding may indicate that objectification of the product was challenging for the students.

Twenty seven percent of the episodes (57/211) were devoted to objectification of the accumulated values when the upper boundary value is bigger than the lower one. Forty-four episodes of these were coded as personal meanings that are accepted mathematically. In contrast, among the 31 episodes that were devoted to objectification of the accumulated values when the upper boundary value is smaller than the lower one, only four were coded as personal meanings that are accepted mathematically. This finding indicates that the students were better able to objectify the sum of product when the upper boundary is bigger than the lower boundary than the other way around.

Seventeen percent of the episodes (35/211) were devoted to objectification of the properties of the accumulation function. Only three episodes of these were coded as personal meanings that are not accepted mathematically. Twenty-two episodes were devoted to objectifying the position of the accumulation function graph. Two of these were coded as personal meanings that are not accepted mathematically. Four episodes were devoted to the tendency of the accumulation function, and all of them were coded as accepted mathematically. Nine episodes were devoted to the concavity of the accumulation function graph and coded as accepted mathematically. This finding indicates that the students who noticed the last two elements also became aware of their mathematical meaning.

Below we present evolutionary path in the learning process, which we identified through the data analysis. A learning path refers to the organization of learning activities in a proper order composed of knowledge elements, designed task, and artifact such that students can effectively study a subject area.

The chart in Figure 4 show the order in which students were engaged with various mathematical elements, showing also the time elapsed. The data are color-coded. The mathematical elements appear in the left column. The numbers at the bottom represent the time in minutes. We divided each minute into four squares, each one representing 15 seconds. The colours distinguish between personal and mathematical meanings. The grey colour represents personal

Elements	*AcceptedΔ*	Not accepted	*Total*
A: Delta x			
A1: Dividing the x-axis into intervals of equal length	10	0	10
B: Objectifying the product $f(x) \cdot \Delta x$			
B1: Reflection of the lower boundary value in the accumulation function	26	1	27
B2: Initial value point in the accumulation function graph	18	23	41
C: Objectifying the accumulation function as sum of products			
C1: Accumulated value in which the upper boundary is bigger than the lower	44	13	57
C2: Accumulated value in which the upper boundary is smaller than the lower	4	27	31
C3: Accumulated zero	10	0	10
D: Objectifying properties of the accumulation function graph			
D1: Position of the accumulation function graph	20	2	22
D2: Tendency of the accumulation function graph	4	0	4
D3: Concavity of the accumulation function graph	8	1	9

Table 1: Frequency of mathematical elements distinguishing personal meanings that are and are not accepted mathematically

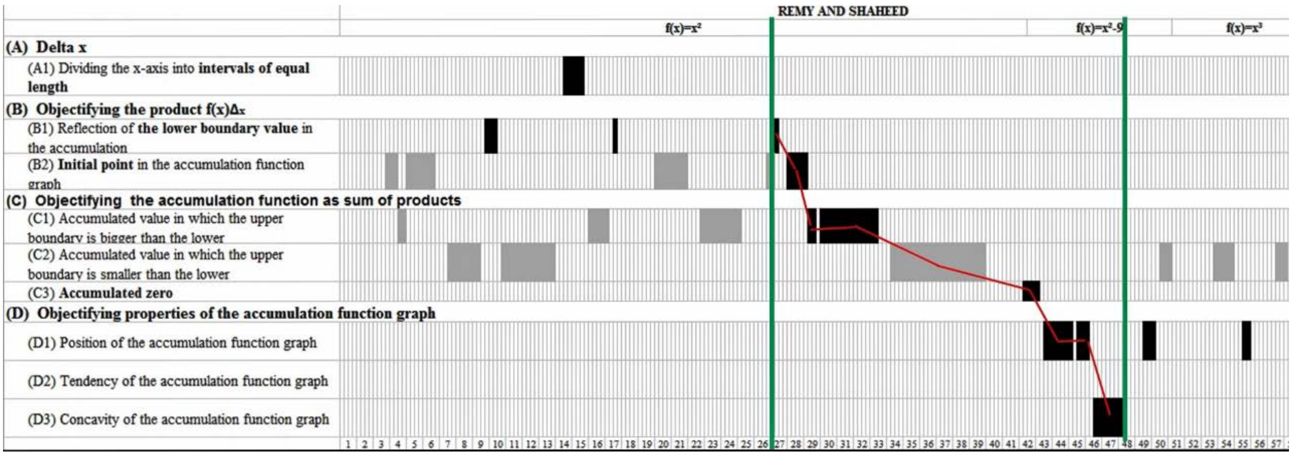

Figure 4: Learning path for Remy and Shaheed

meanings that are not accepted mathematically; the black colour represents personal meanings that are accepted mathematically.

Analysis of the charts suggests a possible path for learning the accumulation function graphically. The chart (Figure 4) illustrate the suggested path, which consists of four phases: (a) objectifying delta x, (b) objectifying the product, (c) objectifying the sum of product, and (d) objectifying the properties of the accumulation function graph. Despite the differences between the charts, the similarity between them is apparent in the trend line.

The students initially identified delta x and ascribed mathematical meaning to it as a divider of the x-axis into intervals of equal length. It seems that objectifying delta x and becoming aware of its role was an important aspect of objectifying the product $f(x)\Delta x$. Students who initially tried to objectify mathematical elements other than delta x returned to objectify delta x after several unsuccessful attempts at identifying the other elements (Figure 3). Objectifying the reflection of the lower boundary value in the accumulation function occurred before the objectification of the initial accumulated point in the accumulation function graph. This finding may point to the principal role played by the lower boundary in students becoming aware of the accumulation function. Although our analysis shows that becoming aware of the product was a complex process (row B2, Table 1), eventually most students were able to objectify the product and ascribe mathematical meaning to it. However, analysis of the chart shows that these students undertook several attempts before becoming aware of the mathematical meaning of the product (Figure 4). This finding indicates that students were able to overcome the complexity of objectifying the product.

Objectifying the sum of the products usually occurred after the students have objectified the product $f(x)\Delta x$. Students who tried to objectify the sum of the products before becoming aware of the product $f(x)\Delta x$ were not able to do so (grey segments in row C1 in Figure 3). Analysis of the chart reveals that only three episodes out of 13 that were coded as not accepted mathematically in category C1 (accumulated value in which the upper boundary is bigger than the lower, Table 1) occurred after the students objectified the product $f(x)\Delta x$. This finding indicates that objectifying the product $f(x)\Delta x$ made possible the objectification of the sum of the products $\sum f(x)\Delta x$.

DISCUSSION

The learning trajectory suggested in this study is comprised of four phases of objectification: (a) delta x, (b) product, (c) sum of product, and (d) accumulation function graph properties. Since delta x is central element of the mathematical structure of the accumulation function, becoming aware to it meanings is an essential part in understanding the accumulation function. The linked dynamic representation play a central role in objectifying delta x. Our decision to determine that delta x is a fix-equal value was inspired by pedagogical rather than mathematical considerations. This decision found useful in drew the students attention which afford them to become aware of the mathematical meanings of delta x.

Objectifying the product was challenging for the students. This finding is consistent with those of Sealey's (2014) study, claiming that difficulties in understanding the product are not necessarily related to performing calculations but rather to how the product is formed. Our findings indicate that the complexity of becoming aware of the product was not related to performing the calculation but to the semiotic structure of the multi-representations. Although, the complexity that accompany the objectification of the product, eventually the students were able to overcome this complexity. The properties of the function graph and controlling the upper boundary afford the students to objectify the product. Varying the upper boundary parameter in a discrete manner and its reflection on the multi-representations drew the students' attention to the connection between the emerging rectangles and the accumulation function graph. The semiotic potential of the artifact encourages the emergence of personal meanings and that the control tools promote evolution toward the accepted mathematical meaning. Concerning objectifying the sum of product, the students have partially become aware of the sum of product. On one hand, objectifying the sum of product in which the upper boundary is bigger than the lower was less challenging for them (Table 1). On the other hand, despite our attempt to provide students with a wide variety of graphs to help them objectify the meaning of the accumulation function for each value of x, they were not able to objectify the accumulation graph in which the upper boundary is smaller than the lower boundary value. This finding is not completely consistent with Sealey's findings that "None of the students spent much time explicitly discussing the concepts represented in the summation layer" (p. 240).

In spite of, the properties of the accumulation function graph is not mentioned implicitly in the mathematical structure. However, becoming aware to the properties of the accumulation function graph is epistemologically important since the transition from local to global observation means an evolution of the meanings because of theoretical reasons. Hence, we believe that this phase should be included in the learning path for objectifying the accumulation function.

ACKNOWLEDGEMENT

This study is a part of Ph.D. dissertation under supervision of prof. Michal Yerushalmy.

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Student perceptions on learning with online resources in a flipped mathematics classroom

Evangelia Triantafyllou and Olga Timcenko

Aalborg University Copenhagen, Department of Media Technology, Copenhagen, Denmark, evt@create.aau.dk

This article discusses student perceptions of if and how online resources contribute to mathematics learning and motivation. It includes results from an online survey we conducted at the Media Technology department of Aalborg University, Copenhagen, Denmark. For this study, students were given links to various online resources (screencasts, online readings and quizzes, and lecture notes) for out-of-class preparation in a flipped classroom in mathematics. The survey results show support for student perceptions that online resources enhance learning, by providing visual and in depth explanations, and they can motivate students. However, students stated that they miss just-in-time explanations when learning with online resources and they questioned the quality and validity of some of them.

Keywords: Student perceptions, screencasts, Khan Academy, flipped classroom, mathematics education.

INTRODUCTION AND CONTEXT

One of the recent developments in teaching is the flipped (or inversed) classroom approach (Bergmann & Sams, 2012). In a flipped classroom the traditional lecture and homework sessions are inverted. Students are provided with online material in order to gain necessary knowledge before class, while class time is devoted to clarifications and application of this knowledge. The course content, which is provided for self-study, may be delivered in the form of screencasts and/or pre-class reading and exercises, while class time is mainly used for group work activities. The hypothesis is that there could be deep and creative discussions when the teacher and students physically meet. This teaching and learning approach endeavours to make students owners of their learning trajectories, and relies heavily on current technology.

Various researchers and instructional designers have sought to investigate the advances in flipped learning environments (Bishop & Verleger, 2013). According to such studies, students were very positive about their experience and instructional video components in flipped classrooms (Love, Hodge, Grandgenett, & Swift, 2014) and suggested that flipped classroom approach (1) provided them with an engaging learning experience, (2) was effective in helping them learn the content, and (3) increased self-efficacy in their ability to learn independently (Enfield, 2013).

While the aforementioned approaches report on benefits of the flipped classroom, there are also critics to this approach (Kellinger, 2012; Nielsen, 2012). Concerns include among others: criticism about the accessibility to online instructional resources, the growing move towards no homework, lack of accountability for students to complete the out-of-class instruction, poor quality video production, and inability to monitor comprehension and provide just-in-time information when needed.

Taking into consideration the reported strengths and weaknesses, we introduced this instructional model to a statistics course for Media Technology students (Triantafyllou & Timcenko, 2014). The results of this study revealed appealing qualities but also drawbacks of flipped mathematics classrooms. One of our biggest concerns was the resources, which students would get as a preparation for the class. In that study, the students got links to online mathematics courses, lecture slides from a mathematics course they had attended, and screencasts of problem solutions. The screencasts were produced by us using a smart pen. With regard to the screencasts, the results showed that students found them helpful, they appreciated the fact that they could skip or re-watch parts of the solution, and they felt they were supported when studying challenging concepts. Nevertheless, less than half of them watched most of them and they mentioned as a weakness the fact that they cannot ask questions for clarification during a recorded lesson. Moreover, our personal experience was that creating quality screen casts is time consuming and hard.

Therefore, we decided to conduct a survey study in order to further investigate student perceptions and preferences on online resources and especially screencasts as part of a flipped mathematics classroom. The study was carried out during a mathematics workshop, which was implemented using the flipped classroom model. The workshop was offered to fifth semester Media Technology students as an introduction to their computers graphics rendering and computers graphics programming courses. In the following sections, we discuss previous studies on student perceptions on online resources and describe our methodological approach. Afterwards, we present and analyse the results of this survey study. We conclude this paper with a discussion and an outline of future work.

BACKGROUND

Various researchers have sought to investigate student perceptions and learning with online resources, both in traditional and flipped classrooms (McGarr, 2009). In the case of traditional classrooms, online resources are given to students for revision or preparation for assessments. Regarding student use of such resources, Biehler et al. introduced interactive modules containing domain knowledge, exercises, diagnostic tests and illustrations within blended bridging courses in mathematics and found that slightly more than 50% of the students used the diagnostic tests (Biehler, Fischer, Hochmuth, & Wassong, 2012). Kay and Kletskin introduced problem-based screencasts covering key areas in mathematics. The screencasts were created as self-study tools, and used by higher education students to acquire pre-calculus skills (Kay & Kletskin, 2012). The results indicated that a majority of students used the screencasts frequently.

As far as student perceptions of online resources are concerned, Biehler et al. found that students considered them helpful, while Kay and Kletskin reported that students viewed online resources as easy to use, effective learning tools, rated them as useful or very useful, and reported significant knowledge gains in pre-calculus concepts.

Factors that determine student perceptions of online resources have been also investigated. In the study by Biehler et al., the results showed that the use of such resources was highly depended on the learning type of the student. Yoon and Sneddon conducted a study on student perceptions of the effective use of lecture recordings (screencasts) in undergraduate mathematics courses and they identified a set of factors that determine student perceptions of live and recorded lectures as competing or complementary. Personal learning styles, study habits, esteem for the lecturer and the possibility of interaction in the lecture can namely make students prefer live lectures rather than lecture recordings (Yoon, Oates, & Sneddon, 2014).

Nevertheless, there are researchers, who challenge the learning that unfolds in online environments (Parslow, 2012; Schwartz, 2013). Such critics claim, for instance, that some online resources have no pedagogical underpinnings, don't allow learners to build knowledge hierarchically, and don't offer meaningful or personalized feedback.

Our past research revealed that Media Technology students encounter challenges in their mathematics learning, because they lack motivation and basic skills in mathematics (Triantafyllou & Timcenko, 2013). In our previous study these students perceived screencasts to be helpful and supporting for out-of-classroom learning, but we wanted to investigate further if and why these students choose to learn using such resources.

METHODOLOGY

In order to explore student perceptions and preferences on online resources, we conducted an online survey study at the Media Technology Department of Aalborg University Copenhagen. We surveyed fifth semester students, who had just finished a mathematics workshop, which served as an introduction to two semester courses: computer graphics rendering and computer graphics programming. This workshop aimed at recapitulating prerequisite mathematics knowledge for these courses (i.e. linear algebra, geometry and trigonometry).

The mathematics workshop followed a flipped classroom model of instruction. To facilitate this, we created a list of various online resources to provide students with instruction outside of the classroom (before the lectures). We gave students a detailed reading guide that provided information on the topics covered by each resource and a studying sequence. The online resources included: (1) Selected Khan Academy screencasts and related practice problems (www.khanacademy.org/), (2) Selected sections of the www.mathisfun.com webpage, that contains both explanations, visualizations and quizzes, (3) Selected readings from the www.betterexplained.com webpage, which aims at presenting mathematics in an intuitive way, using text and visualizations, (4) scanned lecture notes from their past mathematics course covering the relevant subjects. We chose different sources, in order to provide support in solving exercises, brief and simple introduction to the related concepts and real-life examples and intuitive explanations. Students had to take an online quiz out of classroom before attending the workshop. The quiz contained questions similar to past mathematics exams. Therefore, the students were aware with this kind of questions. We used the quiz in order to observe student understanding, recurrent misconceptions and common mistakes, since the vast majority of students had passed these exams in the past. The information exchange between the teacher and the students (i.e., resources for out of classroom learning, assignments, news forum) and the quizzes were facilitated by the Moodle VLE.

The online survey used a Likert scale in order to collect student responses on use of the assigned or other online resources, and on perceptions of learning when using them. Items in the survey were measured using 5-point rating scales, with the range of answers from "strongly disagree" to "strongly agree." Moreover, there were items, which gave students the opportunity to provide information in an open-ended manner.

The survey was sent to the 100 students subscribed to the mathematics workshop. Since the survey was optional, it was not possible to ensure all students completed it. Forty six students responded to the survey, yielding a response rate of 46%. The response rate is relatively high, but it should be noted that it was not a simple random sample. For example, there could be bias towards more diligent students (who may be potentially more likely to give positive feedback), or only the students who actually used the online resources. The results of our past study and the large sample size help to some extent to mitigate the influence of this bias.

RESULTS AND ANALYSIS

From the analysis of the survey results, we excluded two responses, because they were incomplete. Therefore we had a sample size of forty four responses (N=44, 61% male and 39% female). Nearly all survey respondents (91%) said that they had used some online resources in mathematics either for the mathematics workshop or in the past (Figure 1). The distribution of usage for the different resources is shown in Figure 2 (students were allowed to select more than one option). One third of the respondents having used some online

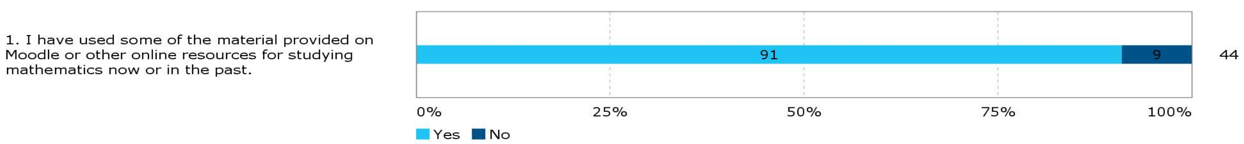

Figure 1: Proportion of students surveyed who have used online resources for mathematics

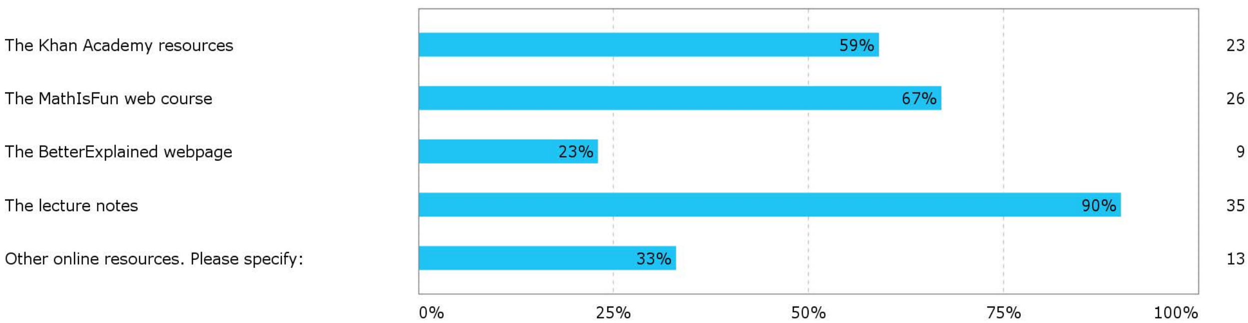

Figure 2: Distribution of answers to the question "Which material have you used?"

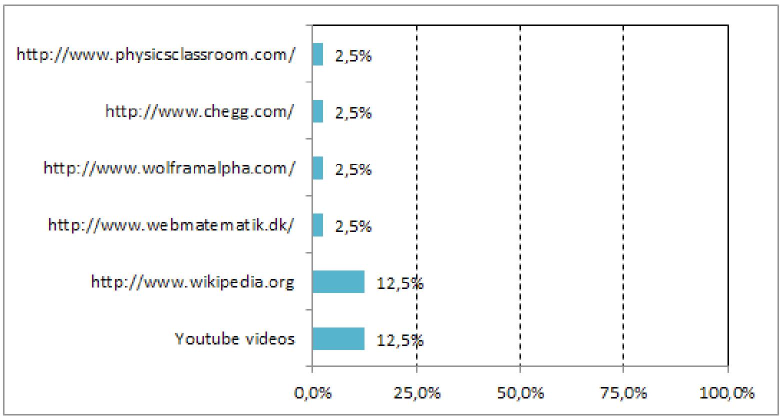

Figure 3: Distribution of other online resources among survey responses

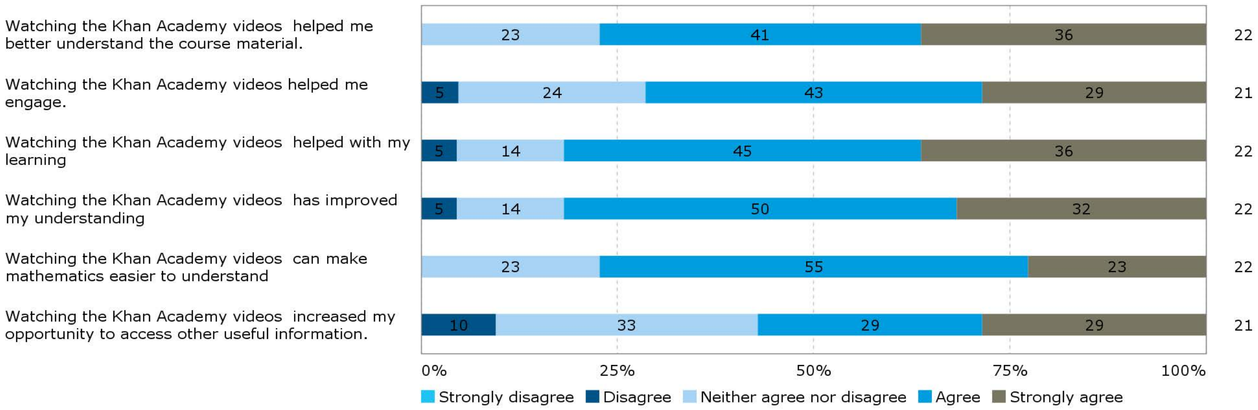

Figure 4: Percentage agreement and disagreement of students with statements on watching Khan Academy videos (screencasts)

resources, made also use of resources we did not suggest. These resources are shown in Figure 3.

Students who used online resources were invited to respond to a series of statements on watching Khan Academy screencasts and reading online resources. We made this separation, because watching a problem solution triggers different thinking from reading a text or even a written solution to a problem. In the following analysis, we combine "Strongly agree" and "Agree" answers to summarize those people who agreed with the statements and "Strongly disagree" and "Disagree" answers to summarize people who disagreed with the statements.

Figure 4 summarizes the results for the statements on Khan Academy screencasts. A significant majority of students who used these screencasts found that watching them has helped with their learning and has improved their understanding (82% and 81% respectively). The statements on screencast contribution on understanding the course material and making mathematics easier received also high agreement scores (78% and 77% respectively). Less strong agreement (72%) was seen with the second statement, which relates to their use for increasing engagement and even less (58%) with the statement about screencasts increasing opportunities to access other useful information.

Figure 5 summarizes the results for the statements on other online resources. The vast majority of students who used some kind of online resources found that reading online resources has helped with understanding the course material and has improved their understanding (90% and 86% respectively). The statements on screencast contribution on improved learning and on increasing opportunities to access other useful information received also high agreement scores (82% and 79% respectively). Less strong agreement (76%) was seen with the fifth statement, which relates reading online resources to making mathematics easier to understand and even less (58%) with the statement on online resources helping students engage.

In addition to agreeing or not with these statements, we asked students to define the strong and weak points of these resources. We included these two open-ended

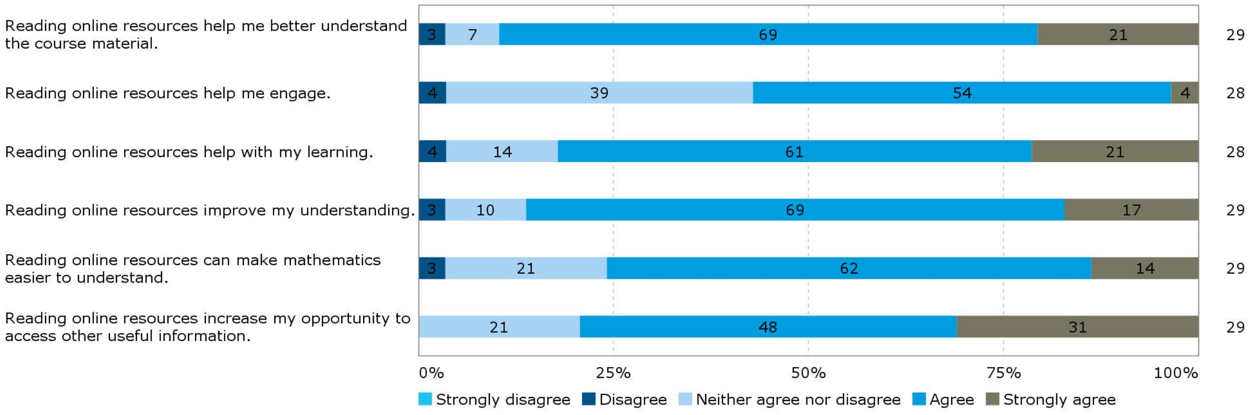

Figure 5: Percentage agreement and disagreement of students with statements on reading online resources

Strong points	Frequency	Weak points	Frequency
The narrator is good at explaining	27.3%	None	13.6%
You can pause/rewind	18.2%	They are long	13.6%
Visual explanations	18.2%	I don't know	13.6%
They come along with quizzes	18.2%	No one to ask questions	9.1%
They helped me focus	13.6%	Complicated language	9.1%
Clear structure	9.1%	They are boring	4.6%
Good tempo/pace of explanation	9.1%	Too abstract	4.6%
Easy to understand	4.6%	Sometimes confusing	4.6%
Step by step explanations	4.6%	Knowledge does not "stick" in memory if you just watch and don't practice yourself	4.6%
They are fun	4.6%		
Easily accessible	4.6%	You don't know which video to watch, in case you experience knowledge gaps	4.6%

Table 1: Student perceptions on strong and weak points of Khan Academy screencasts

Strong points	Frequency	Weak points	Frequency
Different explanations for the same topic	31.0%	No one to ask questions	20.7%
Easily accessible	17.2%	Debatable quality / validity	20.7%
Visual explanations, animations	17.2%	Possibly time consuming to find what needed	13.8%
Can be fun	6.9%	Overwhelming amount of information	10.34%
Explanations on specific topics	6.9%	Sometimes confusing	6.9%
A fast way to find information	6.9%	Notation and categorization may differ	6.9%
Easy to understand	6.9%	Possibly inaccessible due to technical problems	3.5%
Reproducibility	3.4%	No one to check your understanding	3.5%
You can get help from others	3.4%	Our brain gets used to search for knowledge, not to memorize it	3.5%
They are fun	4.6%		
Some come along with quizzes	4.6%		

Table 2: Student perceptions on strong and weak points of reading online resources

questions, because we wanted to give students the opportunity to provide further information on their perceptions in an open-ended manner. The strong and weak points of Khan Academy screencasts as pointed out by the students are shown in Table 1, while the strong and weak points of reading online resources are shown in Table 2. For building these tables, we have grouped answers with the same meaning but different wording.

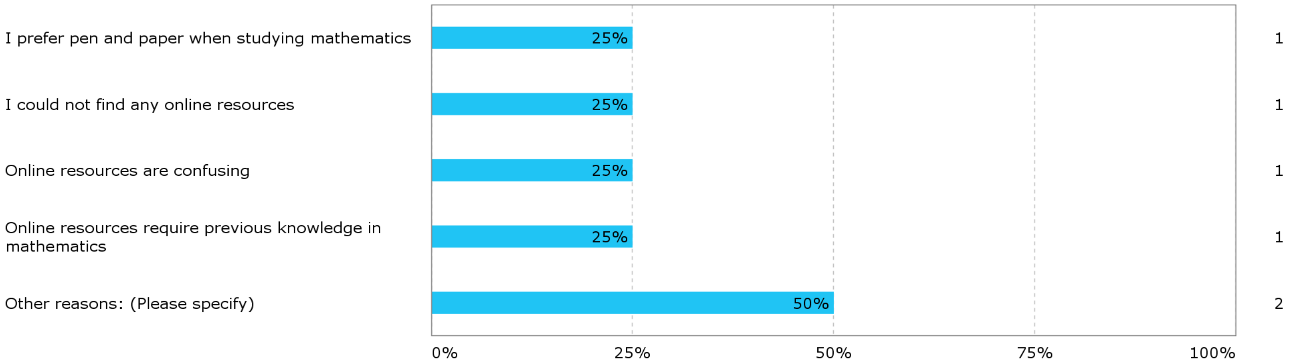

Figure 6: Distribution of answers to the question "Why have you never used online resources?"

Respondents who said that they did not use any online resource for mathematics were asked the reasons for it. Fifty percent of the students mentioned other reasons, namely "I was never introduced to anything good" and "(they are) too complex, no explanations, (I) can't even solve the very first exercise". The other statements were equally favored by students (Figure 6). Students were also asked which tools they use for studying mathematics. Books and notes were mentioned by 50% of the students, pen and paper by 25%, Matlab by 25% and study groups by 25%.

DISCUSSION AND CONCLUSION

In this online study, we surveyed Media Technology students on their perceptions on online resources in mathematics. The survey consisted of both close- and open-ended questions in order to better sketch student perceptions. We observed a general consistency between close- and open-ended questions. Regarding Khan Academy screencasts, students found them to provide good and detailed explanations, although some mentioned that they look messy or confusing sometimes. Students perceived them also as being engaging and helping creating focus. The least supported statement on screencasts is the one on screencasts creating links to other useful information. This can be explained by the mission of the Khan Academy website to be a closed environment, where students should find all the information and support they would need (screencasts, practice problems, feedback from teachers/peers, etc.). What students missed in screencasts was mostly the ability for just-in-time explanations, being shorter and using less complicated language. However, the last one can be attributed to the fact, that Media Technology students lack basic skills and thus also terminology in mathematics. Another fact worth mentioning is that 13.6% of the students stated that screencasts have no weak points. We would like here to acknowledge the fact that questionnaire data can be biased since survey respondents tend to answer questions in a manner that will be viewed favorably by others (King & Bruner, 2000). Therefore, we plan on incorporating more reliable methods for collecting data on resource usage, such as analyzing log data from Moodle.

Reading online resources were perceived by students to help answering specific questions and understanding the course material. They are perceived also as a means to find links to other useful information and to find different explanations on the same topic. Animations and visualizations are other aspects that students valued with regard to online resources. Nevertheless, 20.7% of the students mentioned the lack of just-in-time explanations and the matter of the quality and validity of such resources. There were also students (13.8%) stating that it could be time consuming to browse the internet for specific information.

As to the reasons for students not to choose online resources, the survey results show some indication that there are students who still prefer books and pen and paper for studying mathematics. However, the number of students who completed the survey and did not use online resources is very small (N=4), therefore we cannot draw any statistically significant conclusions based on these answers.

The survey results indicate that online resources were seen by students completing the online survey as valuable and useful as an aid to learning. We conducted this study in the context of a flipped classroom, where students were asked to prepare themselves before the lectures, by using this aid. Since working with mathematics by themselves is perceived by students the most important learning (Sikko & Pepin, 2013), we believe that the decision of which tools should support this individual learning is a crucial one. Our survey has indicated that watching screencasts is perceived more engaging than reading online resources, while reading online can help to find explanations that make sense to the individual. However, students still perceive face-to-face instruction as paramount, since the problems of the inability to follow comprehension and the lack of just-in-time explanations are still to be solved.

Although the results revealed that students perceive online resources as contributing to their learning and understanding, it is difficult to draw firm conclusions in terms of improvements to student learning as at this stage it has not been possible to measure this quantitatively. In the literature, there are few studies on the flipped classroom that examined student performance throughout a semester. In such cases, mainly pre- and posttest methods have been employed for student assessment. While the results from such studies are encouraging, there is not sufficient evidence for generalization beyond specific contexts (Bishop & Verleger, 2013). Thus, a further quantitative study will be designed for student assessment throughout a whole semester.

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Discrete or continuous? – A model for a technology-supported discrete approach to calculus

Hans-Georg Weigand

University of Würzburg, Würzburg, Germany, weigand@dmuw.de

In recent decades, the approach to calculus in mathematics classrooms has changed: a quite formal approach—closely linked to the teaching of calculus at university and based on the sequence concept—has been transformed to an intuitive access to the concepts of limit and derivative. The importance of sequences has so far decreased that they are sometimes no longer even taught in the calculus course. In recent years, this concept has been criticized for not developing adequate perceptions of the basic concepts of calculus and not sufficiently preparing the students for scientific courses at university. In this paper, we will present an alternative discrete step-by-step approach to the basic concepts of calculus. It shows the theoretical basis of the on-going project "ABC – A discrete Approach to the Basics of Calculus" (see Weigand, 2014).

Keywords: Calculus, discrete mathematics, limit, derivative, digital technologies.

During recent decades, teaching the concept of derivative in mathematics classrooms has changed. In the seventies and eighties of the last century, especially in continental European countries such as France, Italy, Germany, and the Eastern European countries, the limit concept was based—in close relationship to college or university mathematics—on extensive work with sequences. A formal definition of the limit of a sequence and the proving of certain theorems concerning the convergence of sequences were the basis for the definition of the derivative of real-valued functions. In the late eighties and nineties—based on the calculus courses of Emile Artin (1957) and Serge Lang (1964/1973)—the model of an *intuitive limit concept* was introduced in mathematics education, which was adopted as a concept for high schools (Blum 1975) and has since then been widely accepted in schools and is the dominant concept today (Törner et al., 2014).

Within the frame of the intuitive limit concept, no formal definition of a limit occurs. The access to the derivative starts with the discussion of real-valued continuous functions, using an intuitive limit concept such as "... coming closer and closer ..." and calculating the derivative of polynomial functions by algebraic transformations without any formal definition of the limit. Using sequences is not necessary in this concept, at least as long as one does not consider more complex functions, for example trigonometric and exponential functions. The idea of the original concept of a simplified limit concept in the 1980s has been to consider sequences "later," but in reality, for example in many new curricula such as in Germany, sequences are no longer part of calculus in mathematics lessons.

It becomes clear that this has been a turning point regarding the concept of calculus in mathematics lessons. The changes in the access to the derivative concept changed the contents and the structure of the entire calculus curriculum. A concept-oriented approach to calculus was substituted by an application-oriented approach. There is a danger that learners stay on an intuitive and technical level and that basic ideas or conceptions for a content-oriented or integrated understanding of the mathematical concepts are not given.

In the following, we ask for the *understanding* of the basic concepts of calculus, concerning:

— the present situation and (empirical) results if we look at the knowledge of students and freshmen at the university;

- the theoretical basis or the basic knowledge to be able to understand the mathematical concepts of limit and derivative;
- a constructive strategy to develop these basic concepts in the classroom.

This article gives a theoretical framework for a new or an alternative access to the basic concepts of calculus. It uses digital technologies as a calculation and drawing tool to present sequences or discrete functions and to compare the properties of different functions or sequences. It has to be seen as a basis for a follow-up empirical investigation.

CONCERNING THE UNDERSTANDING OF THE BASIC CONCEPTS OF CALCULUS

The understanding of the *concept of limit* has quite often been the subject of theoretical reflections and empirical investigations (Keene et al., 2014). It is well known that many students have problems with the formal definitions of the concepts of limit and derivative. They are either not able to use the definition properly in a given context, or they are able to solve problems on a formal level, but lack an advanced understanding of the concepts (e.g., Tall & Vinner, 1981). The main results of the investigations concerning the learning, teaching, and understanding of the limit concept in the last decades are:

a) Conceptual *understanding* of the formal limit concept is challenging for high school students as well as for some college and university students and requires explanations and visualizations using different representations (beyond the symbolic representation).

b) The understanding of the *process of the construction or calculation of limits* in the sense of *step-by-step processes on numerical and graphical levels* is essential for the understanding of the limit concept beyond a formal definition. This can be supported by computer visualizations.

To understand the *concept of derivative*, it is necessary—besides understanding limit processes—to have adequate conceptions of the *rate of change* and to understand—in relation to limit processes—the transformation from the *average* rate of change to the *local* rate of change (see Rasmussen et al., 2014).

There are numerous propositions concerning the use of digital technologies and their dynamic possibilities of visualizing the approximation processes on a numerical and graphical level (e.g., Kidron & Zehavi, 2002; Martinovic & Karadag, 2012). All those suggestions have in common that they work with real-valued continuous functions and visualize—with programs such as Geogebra[1]—the limit processes dynamically with a sequence of secants converging to the tangent in a point of the graph of the function and/or the numerical process of convergence in the frame of a table (in a spreadsheet). The necessary transition from the continuous perspective to the discrete stepwise process—the discretization process—concerning the limit process, which includes selecting either a sequence of points on the graph or a sequence of numerical values converging to a selected value of the function or to a point on the graph, has to be made by the learners on their own.

The detailed (re-)construction of the limit process and the possibility of step-by-step thinking in the frame of this process has always been the strongest argument for working with sequences and their limits before starting to work with the limit of real-valued functions and their limit processes, for example the first derivative.

SEQUENCES AND DIGITAL TECHNOLOGIES

As a consequence of the increasing role of digital technologies in mathematics and mathematics education, discrete mathematics, and hence sequences, have gained importance. This was emphasized by the NCTM *Standards for School Mathematics* (1989), which included discrete mathematics as a separate standard for grades 9 to 12: “Sequences and series ... should receive more attention, with a greater emphasis on their descriptions in terms of recurrence relations.”[2] Sequences are prototypes of discrete objects in mathematics.

In the *Principles and Standards for School Mathematics* (NCTM 2000), however, discrete mathematics is no longer a separate standard but is now distributed across the standards and spans the years from kindergarten through twelfth grade. *Iteration and Recursion*

1 www.geogebra.org

2 http://standards.nctm.org/Previous/CurrEvStds/9-12s12.htm

are explicitly emphasized as one of the three important areas of discrete mathematics.

Even though sequences are not explicitly defined or introduced as a separate concept in the mathematics curriculum, they are used quite often implicitly or in an intuitive way: Many real-life problems allow mathematical representations with sequences, for example growth processes or problems involving goods and their cost, or approximation algorithms such as the Heron-method for calculating irrational numbers or the Newton-method for calculating zeroes of functions are based on iteration sequences.

Nowadays, computers or digital technologies make it possible to generate sequences, to create symbolic, numerical, and graphical representations, and to switch between different representations—by just pressing of a button. In the following, digital technologies are a tool allowing a discrete access to the concept of limit and derivative as a preliminary stage working with these concepts on a continuous level.

A STEP-BY-STEP CONCEPT FOR A DISCRETE APPROACH TO CALCULUS

We will now present a concept of a discrete access to calculus, which develops the concept of the average rate of change based on a discussion of various sequences by looking at discrete functions. The advantage of this concept is not presented in the beginning, the concept of rate of change is instead developed by using discrete examples. By gradually changing the step size of the discrete actions at hand, limit processes are prepared by comprehensible step-by-step actions and are thus easier to understand. Here, the computer is used both as a tool for the representation and visualization of sequences and functions, and as well as a tool for creating recursively defined sequences in particular, which allows the user to switch between symbolic, numerical, and graphical representations (see Weigand, 2014).

Level 1: Sequences and growth processes

Sequences can be explained or defined on a formal level via an explicit mapping $a_n: \mathbb{N} \to \mathbb{R}$, or they can be defined recursively. This is widely used for the representation of growth processes, for example *linear* growth by , *exponential* growth by $a_{n+1} = A \cdot a_n$, and *limited* growth by $a_{n+1} = a_n + P \cdot (B - a_n)$, $n \in \mathbb{N}$, while all other variables are being real numbers. These sequences can easily be visualized using a spreadsheet or a computer algebra system such as Geogebra. The main goal of this first level is to become acquainted with the recursive kind of definition of sequences, to see the relationship between local aspects, between successive elements, and global aspects of the whole sequence, and to see the dependence of elements of the sequence on the initial value and the parameters. In (Thies & Weigand, 2003) and (Weigand, 2004) it is shown that high school students (grade 11) can solve problems in the frame of growth processes while working experimentally with digital representations of recursively defined sequences.

Level 2: Difference sequences

The aim of this second level is the introduction of the concept of difference sequences $(\Delta a_n)_{\mathbb{N}}$, $\Delta a_n := a_{n+1} - a_n$, of a given sequence $(a_n)_{\mathbb{N}}$. The concept may be introduced in connection with real-life problems, for example the average air temperature in one year during the last 100 years, which may be presented in a table and a graph. The given relations in examples like this are not based on algebraic formulas. This would encourage students to not immediately working on a formal level and foster the understanding of the relationship between the sequence and the difference sequence by operating s step-by-step.

Level 3: The concept Z-functions and their difference functions

3.1 Quadratic Z-functions

Starting with sequences or functions defined on the domain $\mathbb{N}$, we gradually extend the concept of sequence to functions defined on $\mathbb{Z}$, $f: \mathbb{Z} \to \mathbb{R}$, and advance to more subdivided discrete domains. We call functions $f: \mathbb{Z} \to \mathbb{R}$ "Z-functions." These functions f with $y = f(z)$ are "extended sequences," defined on integers $z \in \mathbb{Z}$,

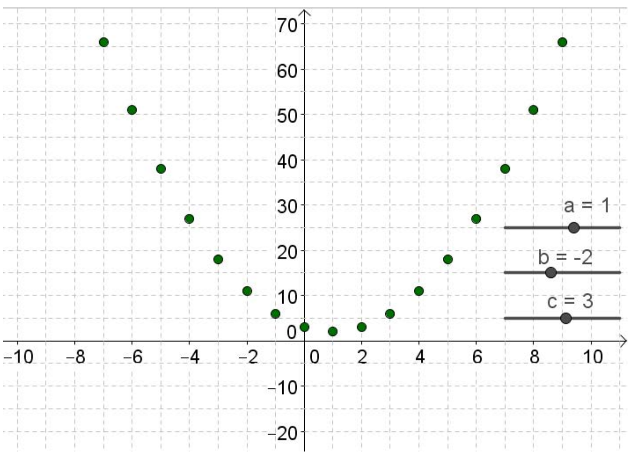

Figure 1: $f(z) = z^2 - 2z + 3$

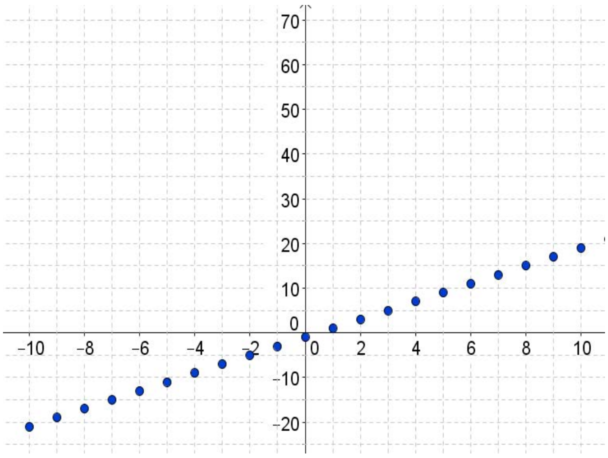

Figure 2: $D_f(z) = f(z + 1) - f(z)$

for example $f(z) = z^2 - 2z + 3$. We will now look at these Z-functions in relation to their difference-Z-functions D_f: $D_f(z) = f(z + 1) - f(z)$.

The values of D_f can be interpreted as the slope of a right-angled triangle with two legs of lengths $|f(z + 1) - f(z)|$ and $|\Delta z| = 1$. $D_f(z)$ is the *rate of change* of the graph between the points $(z, f(z))$ and $(z + 1, f(z + 1))$. Digital technologies are used to visualize the dependence of D_f on the used parameters of f: $f(x) = az^2 + bz + c$ graphically and to give reasons for the behavior of D_f.

3.2 Polynomial Z-functions

The concept of Z-functions can be extended to polynomial functions of a higher degree, as the respective difference functions can be obtained algebraically in an equally simple manner. For the family of Z-functions

$$f(z) = az^3 + bz^2 + c\,z + d,$$

for example, we get the (family of) difference-Z-functions D_f with

$$D_f(z) = 3az^2 + (3a + 2b)z + a + b + c.$$

We can see in particular that D_f is a quadratic function, which is also apparent in the graph.

As an example we look at the Z-function $f(z) = 0.1z^3 - z + 1$, with the corresponding difference-Z-function $D_f = 0.3z^2 + 0.3z - 0.9$ and their respective graphs.

The calculations can easily be extended to difference functions of higher order Z-functions, especially by using a CAS.

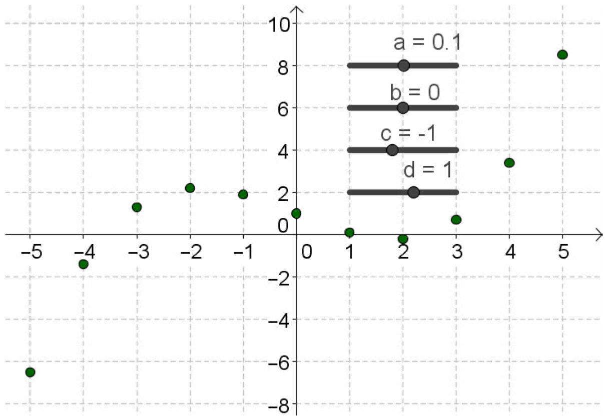

Figure 3: Z-function $f(z) = 0.1z^3 - z + 1$

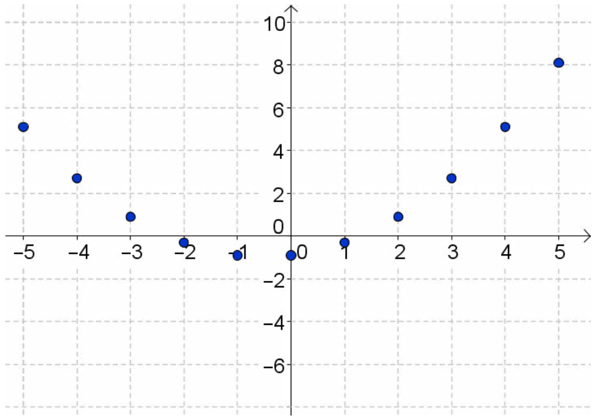

Figure 4: $D_f(z) = f(z + 1) - f(z)$

The advantage of working with these discrete functions is the possibility of obtaining the *rate of change of discrete polynomial functions* only through algebraic transformations and the possibility of *step-by-step argumentations* concerning the properties of the function, especially concerning the rate of change and the difference function.

3.3 Exponential Z-functions

Using the graphical representation of the Z-function $E(z) = a^z$, $a \in \mathbb{R}^+$, $z \in \mathbb{Z}$, and its difference-Z-function, we obtain the graphs shown in Figures 5 and 6.

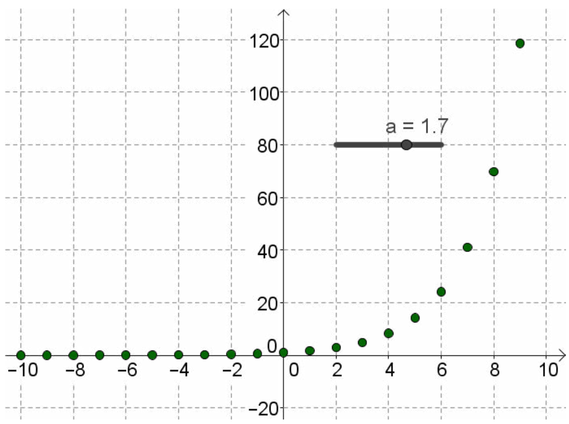

Figure 5: The Z-function $E(z) = 1.7^z$

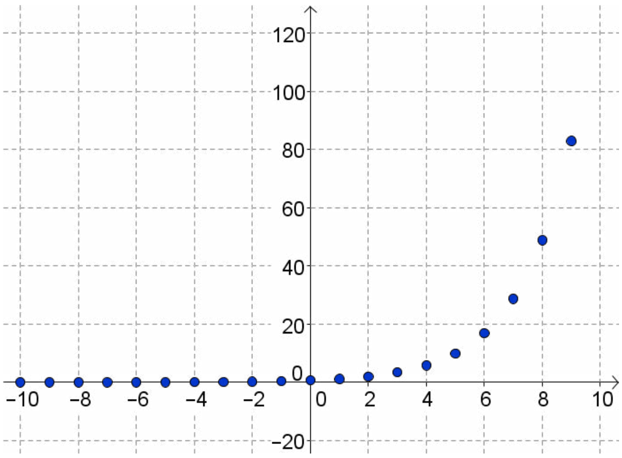

Figure 6: Difference sequence

It is striking how similar the "shapes" of the graphs of the Z-function and difference-Z-function are. A calculation on the formal level gives the following result:

$$D_e(z) = E(z+1) - E(z) = a^{z+1} - a^z = a^z(a-1) = E(z) \odot (a-1).$$

The value $D_e(z_0)$ is obtained by the difference sequence for a given value z_0 by multiplying the value $E(z_0)$ by the factor $(a - 1)$. Geometrically, the graph of the Z-function is the result of an orthogonal affinity of the difference-Z-function with the z-axis as the axis of affinity. For $a = 2$, the two graphs match exactly! There is therefore a value for which the difference-Z-function equals the Z-function!

Level 4: The function of the difference quotients

The difference function D_f with $D_f(z) = f(z + 1) - f(z)$ of a Z-function f can be interpreted as the slope of the Z-function f concerning the points $(z, f(z))$ and $(z + 1, f(z + 1))$ of the graph of f.

The next step of an expansion of the Z-function is considering a domain with non-integer values, but

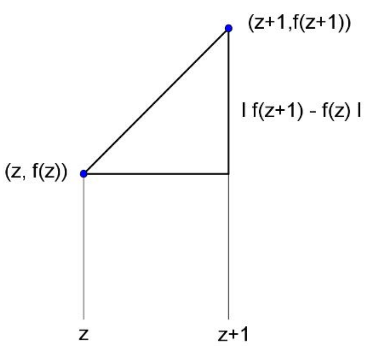

Figure 7

we will still remain within discrete domains. The idea of the difference function as well as the calculation of the slope can be used as long as the domain consists of discrete values.

In a first step the domain $\mathbb{Z}$ of the Z-function f is expanded by considering the values $z_{10} = \frac{z}{10}, z \in \mathbb{Z}$. This means $z_{10} \in \mathbb{Z}_{10} = \{..., -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, ...\}$ and we obtain the Z_{10}-function $f_{10}: \mathbb{Z}_{10} \to \mathbb{R}$. To get the rate of change of successive values, we restrict the calculation to an interval of the length , and get the *difference-quotient-Z_{10}-function*

$$D_{f10}(z_{10}) = \frac{f(z_{10} + \frac{1}{10}) - f(z_{10})}{\frac{1}{10}},$$
$$z_{10} \in \mathbb{Z}_{10} = \{..., -\frac{2}{10}, -\frac{1}{10}, 0, \frac{1}{10}, \frac{2}{10}, ...\}.$$

For example, the difference-quotient-Z_{10}-function belonging to the cubic Z_{10}-function $f(z_{10}) = 0.1z_{10}^3 - z_{10} + 1$ is

$$D_{f^*}(z_{10}) = \frac{f(z_{10} + \frac{1}{10}) - f(z_{10})}{\frac{1}{10}} = 0.3z_{10}^2 + 0.03z_{10} - 0.999$$

This can be generalized to an interval of the length $\frac{1}{n}, n \in \mathbb{N}$, and the difference-quotient-Z_n-function (see Weigand, 2014).

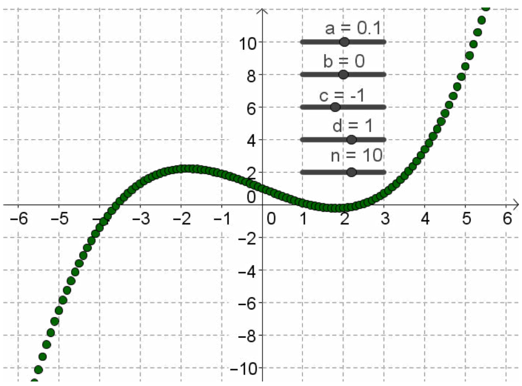

Figure 8: The Z_{10}-function $f(z_{10}) = 0.1z_{10}^3 - z_{10} + 1$

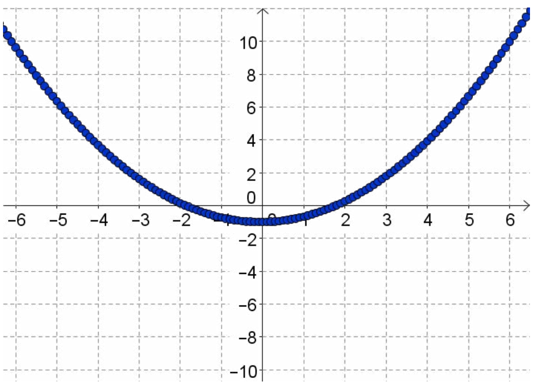

Figure 9: The difference-quotient-Z_{10}-function of f

It emphasizes the *global view of a function* and its (discrete) *difference-quotient-function*, and from the beginning—while working with Z-functions—it draws the attention to the relation between *function* and *difference-Z-function*. Thus, it prepares the understanding of the relation between a function and its derivative function.

Level 5: The local rate of change

The preceding steps to the access to the derivative emphasized the *global view* of the function and the difference-quotient-Z_n-function. The next step will be the concentration on the local view of a function while seeing the relation to the local rate of change of a function.

We continue with any real function f, choose a fixed value $z_0 \in \mathbb{Z}_n$, or even a generalized value $x_0 \in D \subseteq \mathbb{R}$, and consider the sequence of the difference quotient for a real-valued function f with respect to the value of x_0 for $n = \{1, 2, 3, ...\}$:

$$n \to D_n(x_0) = \frac{f(x_0 + \frac{1}{n}) - f(x_0)}{\frac{1}{n}}.$$

Thereby, we obtain a—special—sequence of difference quotients for the function f at the point $(x_0, f(x_0))$. For $f(x) = ax^2 + bx + c$ we get:

$$D_n(x_0) = 2ax_0 + b + \frac{a}{n}.$$

Now, the sequence $D_n(x_0)$ can also be interpreted—considering the graph of f—as the sequence of the slope of the secants through the point $(x_0, f(x_0))$.

Seeing the construction of the derivative of a function f in a special point of the graph of f as a sequence of slopes of secants, the discrete-Z_n-functions with growing n provide a basis for the calculation of the *local rate of change*.

CONCLUSION

The way presented here can be seen as a preliminary stage of the—nowadays already traditional—intuitive limit concept approach with continuous functions. The advantage of the discrete way described here is that working with continuous functions–subsequently after the discrete approach and the working with sequences or Z-functions–and the development of the concept of derivative can be based on a content-oriented level of understanding of the limit concept. To develop this level–beyond an intuitive level of understanding–is the main reason while explicitly working with sequences or discrete functions. Applying this idea of discrete actions or calculations, the concept of derivative of continuous functions is developed on mathematical aspects and properties of the limit or the approximation process and not only on intuitive perspectives.

It is expected that the proposed strategy prepares the concept of *local derivative* of a function at one point and gives a chance of a better understanding of this approximation process because of the possibility of the stepwise construction of this process. But it is also expected that the parallel presentation of sequences (or discrete functions) and their difference sequences (or functions) allow also a well-founded understanding of the concept of a *(global) derivative* function. The aim of the proposed concept is the better understanding of the concept of derivative. It is part of the project *"ABC – A discrete Approach to the Basics of Calculus" (see Weigand 2014)*. It is partially—concerning the first levels—empirically evaluated, theoretically extended to a global concept regarding the access to the derivative, and now needs to be evaluated in an authentic classroom setting. It does not—and cannot—avoid the "cognitive conflict" (Tall, 1992) or the necessity of a conceptual change from the discrete thinking used in working with sequences, difference sequences, and rates of change or difference quotient sequences to working with limits and derivatives of real functions. But it successively develops and explains the approximation processes for an understanding of the derivative concept between the intuitive level, nowadays widely used in classrooms, and the formal mathematical level at university. It emphasizes the processes of understanding the concepts; it is a "procept" (Gray & Tall, 1994; Tall, 2013) for an access to the derivative.

The next step in the frame of this project is the construction and development of classroom, learning or teaching units and the empirical evaluation of the results. This will be done in the near future.

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TWG16

Posters

Using slider tools to explore and validate

Mats Brunström and Maria Fahlgren

Karlstad University, Department of Mathematics and Computer Science, Karlstad, Sweden, mats.brunstrom@kau.se

Technologies in mathematics education provide different tools. Researchers have elaborated on students' use of some of these tools. For instance, students' use of dragging tools (Arzarello, Olivero, Paola, & Robutti, 2002) and measuring tools (Olivero & Robutti, 2007) have been categorized into different modalities. However, research on different ways of using slider tools seems to be sparse. The aim of this study is to identify and categorize different uses of slider tools when students investigate how different values of parameters influence the graphical representation of functions.

Keywords: Slider tools, parameters.

THEORETICAL FRAMEWORK

The theoretical framework combines two different aspects. The first aspect concerns students' use of dragging tools. In the context of dynamic geometry software, Arzarello and colleagues (2002) introduced a hierarchy of different uses of dragging, i.e. dragging modalities. Mainly, two different purposes with dragging were noticed: *drag to explore* and *drag to validate* (Olivero & Robutti, 2007). These two different ways to use dragging have been subdivided into different dragging modalities (Arzarello et al., 2002).

The second aspect concerns the concept of parameters and the associated tool, known as the slider tool or slider bar. The slider tool has been highlighted as a means to examine the visual effect on a graph while changing the value of a parameter (Drijvers, 2003; Zbiek, Heid, Blume, & Dick, 2007). However, Zbiek and colleagues (2007) raised the concern that the slider tool, besides the symbolic and graphical representation, could be regarded as a third representation which might obscure the connection between the parameter value and its visual effect.

METHODOLOGY

In this study, six upper secondary students worked in pairs with tasks designed for a dynamic software environment, in this case *GeoGebra*. The mathematical topic under consideration was the sine function, $y = A \sin (Bx + C) + D$. The students were supposed to investigate how different values of the parameters influence the graph.

To collect data, both video and screen recordings were used. The students worked in pairs in front of *one* computer. The position of the camera was at an angle behind the students to capture how they pointed at the screen when explaining their positions. The screen recordings made it possible to see in detail how students used the software. Both the recordings and transcripts were used to identify the different ways in which students used the slider tool.

RESULTS

The study students used slider tools to *explore* mathematical situations and make conjectures. Further, when they had formulated a conjecture they used the slider tool to *validate* the conjecture. These results correspond to earlier research concerning different dragging and measuring modalities (Arzarello et al., 2002; Olivero & Robutti, 2007). Thus, we suggest that the different uses of slider tools could be categorized into *Exploring* and *Validating*.

Furthermore, we could discern different 'slider modalities' within each category (exploring and validating). We identified the following three exploring modalities: *wandering sliding, exploring systematically,* and *maintain a property. Wandering sliding* could be compared to *wandering dragging* (Arzarello et al., 2002) and *wandering measuring* (Olivero & Robutti, 2007). *Exploring systematically* is reminiscent of *guided dragging* and *guided measuring* since the students examined particular cases in an intentional way.

In the second category, i.e., validating, we identified three different modalities: *checking local hypothesis, slide to validate*, and *validating systematically*. While students use the first modality to check hypotheses about special cases, they use the other two modalities to check more general conjectures. These two modalities could both be compared to *validation measuring* (Olivero & Robutti, 2007).

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Feedback from dynamic software supports creative mathematical reasoning

Jan Olsson

Umeå University, Applied Educational science, Umeå mathematical educational research center (UMERC), Umeå, Sweden, jan.olsson@umu.se

Previous studies have shown that students learn mathematics better when they are engaged in creative reasoning, but that students in regular teaching mainly undertake imitative reasoning. It is therefore important to develop and examine didactical designs that support creative reasoning. These studies examine a didactic design where students work in pairs solving a complex task supported by dynamic software. The studies show that dynamic software supports creative reasoning by providing timely feedback closely connected to the students' activities.

Keywords: Dynamic software, mathematical reasoning, problem solving.

Even though studies has shown that creative reasoning is more efficient for learning than imitative reasoning (Jonsson, Lithner, Norqvist, & Liljekvist, 2014) there is a wide range of research reporting that regular teaching is guiding students into imitative strategies (Hiebert & Grouws, 2007; Lithner, 2008). Lithner (2000) found imitative strategies are a main obstacle when students are solving tasks for which they don´t know a solving method in advance. Imitative strategies are associated to imitative reasoning, IR, i.e., students are trying to recall remembered procedures and facts that are usable to solve the task. Creative mathematical reasoning, CMR, is characterized by creating new (for the student) solving methods supported by argumentation anchored in intrinsic mathematic components (Lithner, 2008).

The didactical situation of the study was designed to invite students to engage in CMR. Pairs of students were solving a task to which they didn´t know a solving method in advance and there were no hints like, e.g. predicting outcome. They had access to dynamic software, GeoGebra. Features of dynamic software like multiple synchronized representations (e.g., algebraic and graphic), immediate response, no explicit indications of right or wrong answer for specific tasks, have been suggested to support students´ reasoning (Barwise & Etchemendy, 1998). The instructions of the tasks meant an intellectual challenge (Schoenfeld, 1985) and the students were responsible to create a solution method (Brousseau, 1997). Students working in pair have been found engaging in discussions and mutual explanations during problem solving (Mullins, Rummel, & Spada, 2011). Data were collected through screen- and video recordings. The main object of analysis was students´ reasoning associated to their use of dynamic software. Lithner´s framework of imitative and creative reasoning (2008) was used to categorize students´ reasoning. The use of feedback was analyzed through Shute´s theories of formative feedback theories (2008).

The study shows that GeoGebra supports CMR by providing neutral immediate feedback. The feedback becomes the object for students' evaluation and argumentation, the latter is an important component of CMR. Furthermore, students who predicted the outcome before submitting the algebraic formula used the given feedback to elaborate on their problem solving and engaged in CMR. Students that did not predict the outcomes solely used the feedback to state if they were right or wrong and by that merely engaged in IR.

The poster presentation will be made up of illustration and descriptions of the didactical design, examples of tasks used, method, CMR an IR-reasoning, and the results of the study.

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Learning trajectory for conceptualizing the fundamental theorem of calculus using dynamic and multiple linked representations tools

Osama Swidan

University of Haifa, Faculty of Education, Haifa, Israel, osamasw@gmail.com

This study was designed to evaluate and elaborate a hypothetical trajectory for learning the fundamental theorem of calculus (FTC) by high school students using dynamic and multiple linked representations tools. The study focuses on 13 pairs of 17-year-old students. Data analysis identified the learning foci involved in conceptualizing the FTC, based on which I suggest a trajectory for learning the FTC.

Keywords: Learning trajectory, Fundamental theorem of calculus, technological tools.

INTRODUCTION

How to promote students' development of new mathematical topics, especially topics whose development is unsure, is one of the significant problems facing mathematics education. One of these topics is the fundamental theorem of calculus (FTC). Many students miss out on the conceptual learning of the FTC (Thompson & Silverman, 2008; Bressoud, 2011). I suggest a hypothetical learning trajectory (HLT) for conceptualizing the FTC using multiple linked representations and dynamic tools that can help students construct an understanding of mathematical concepts. The proposed learning trajectory can make the mathematical learning process more profitable (Steffe, 2004). My aim is to evaluate and elaborate the HLT for learning FTC by high school students learning with a dynamic and multiple linked representations tool.

THE HYPOTHETICAL LEARNING TRAJECTORY

I construct the HLT for FTC based on the functional approach for learning calculus notions. In this approach, the derivative is considered to be a rate of change function and integration is considered to be an accumulation function. At the first level of the HLT, students are asked to conceptualize the Riemann accumulation function as a sum of products. Next, they are asked to conceptualize the accumulation function as a convergence of Riemann accumulation functions. At the third level, they are asked to conceptualize the accumulation function as a rate of change function. Finally, they are asked to conceptualize the evaluation part of the FTC.

THEORETICAL FRAMEWORK

Simon (1995) offered the HLT as a way to explicate an important aspect of pedagogical thinking involved in teaching mathematical understanding. In particular, he described how mathematics educators can conceive the design and use of mathematical tasks to promote mathematical conceptual learning. An HLT consists of three components: (a) the students' learning goal, (b) a hypothesis about the process of the students' learning of mathematical concept, and (c) tasks to be used to promote the students' learning (Simon, 1995). The HLT perspective provides no framework for thinking about the learning process. Therefore, I used the mechanism of reflection on activity-effect relationships to explain the relationship between conceptual learning and mathematical tasks (Simon & Tzur, 2004). This mechanism provides a description of how the learner's goal-directed activity can lead to the generation of new and more sophisticated conceptions for the learner.

METHOD

The present study explores approximately 55 hours of learning by 13 pairs of 17-year-old students. The students volunteered to participate in four after-school meetings. To study the progression processes of the

FTC, we asked students to explain and explore the possible connection between two given function graphs and the table of values. We designed the tasks to be learned by the students based on the mathematical structure of the FTC. The data for the present paper were collected from all the students who participate in the study. All the pairs of students in each session were video-recorded as they engaged in solving the task assigned to them.

PRIMARY FINDINGS

The data analyses suggest that the HLT constructed based on the mathematical structure of the FTC helps students conceptualize the FTC. Based on the data analysis process, I detected the learning foci used by the students to find the possible connections between the graphs, and the mathematical relationships involved in the FTC. These foci can be used to elaborate the HLT for conceptualizing the FTC. In this poster, I present the elaborated learning trajectory for FTC and discuss the role of the learning focus in conceptualizing the components of the FTC. I also present the ways that students used to connect between the differentiation and integration.

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TWG17
Theoretical perspectives and approaches in mathematics education research

Introduction to the papers of TWG17: Theoretical perspectives and approaches in mathematics education research

Marianna Bosch[1], Yves Chevallard[2], Ivy Kidron[3], John Monaghan[4], Hanna Palmér[5], Erhan Bingolbali[6] and Anna Shvarts[7]

1 IQS School of Management, Ramon Llull University, Barcelona, Spain
2 Aix-Marseille University, Marseille, France, y.chevallard@free.fr
3 Jerusalem College of Technology, Jerusalem, Israel, ivy@jct.ac.il
4 University of Agder, Kristiansand, Norway, john.monaghan@uia.no
5 Linnæus University, Kalmar, Sweden, hanna.palmer@lnu.se
6 Gaziantep University, Gaziantep, Turkey
7 Lomonosov Moscow State University, Moscow, Russia

INTRODUCTION

The 'theory working group' (under various names) has been a feature of CERME since CERME4. An early and constant focus has been 'networking theories', exploring ways of using different theories in mathematics education research (MER) into learning and teaching mathematics. The CERME9 'Call for papers' included:

— The need to go beyond a specific theory when researching a phenomena

— Benefits and/or strategies and/or difficulties in connecting theories

— Conditions for a productive dialogue between theorists

— Difficulties and strategies when gathering results from different frameworks

— Linking theoretical and methodological approaches

— The epistemological dimension in theories

— Steps towards (local/global) theoretical convergence in MER

In our 12 hours together at CERME9 we discussed 19 research papers and two posters; our task in these five pages is to introduce you to the ones (almost all of them) which have been accepted for these Proceedings. We arrange the papers into five groups in the next section. These groups are not 'strongly defined' but are, we believe, useful for communication. The closing section examines issues arising from the papers as a whole and future possibilities. For reasons of space, we refer to the papers by the surname(s) of the author(s) alone.

SUMMARY OF THE PAPERS AND POSTERS

The two papers, by Chevallard, Bosch & Kim and by Dudley-Smith, are grouped in relation to the question 'What is a theory?' Yves Chevallard is the founder of the Anthropological Theory of the Didactic (ATD) and Chevallard and colleagues address the above question directly via the ATD:

> ATD conduces to focus the research effort on examining the implicit, unassuming or even wanting parts of technologies and theories. It then appears that a theory is made up of two main components, that we may call its "emerged part" and "immersed part" … a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is

> in truth the current state of a dialectic process of theorisation ...

Dudley-Smith does not directly address the above question but uses 'Social Activity Method' to interpret (deform/recontextualise) theories in order to explore theoretical networking. It is difficult to summarize this paper, but the following words of the author express a key idea:

> well-formed research activities are incommensurable – they are emergent and not graspable as such, even by themselves. The term "continuity" between theories can refer only to those metonymic chains of signifiers that are of interest *to the recontextualising regard of the theory in question*– hence also the possibility of discontinuity.

The papers by Castela, by Zaragoza and by Roos and Palmér are grouped under the heading 'theories in mathematics education'. Castela considers theoretical diversity and networking theories from the points of view of the ATD and of Bourdieu's theory of social fields. The processes of developing theoretical knowledge is shaped through praxeologies that take place in a community. Further to this "A field is characterised by a game that is played only by its agents, according to specific rules". Using these two approaches, Castela argues that networking will result from researchers from different paradigms working together on the same objects. Zaragoza presents a structuralist definition of a theory as a net of 'theory-elements' connected via a 'specialisation'. A *theory-element* is determined by: the portions of reality it conceptualizes; the *laws* which apply; and potential and actual models. Specialisation concerns the models and laws related to theory-elements. Zaragoza applies these ideas to networking theories and the ATD. Roos andPalmér explore the use of Wenger's 'communities of practice' construct in ten published mathematics education studies. The paper documents differences over these ten papers with regard to: foci on pre-existing or designed communities of practice; foregrounding/backgrounding individuals/groups; constructs (e.g., practice, identity, ...) used. Roos and Palmér conclude, "if a researcher says that (s)he has been using Wenger's social theory of learning, we can be quite sure that we do *not* know exactly what that use of Wenger's theory might imply".

The papers by Holm, by Monaghan and by Şay and Akkoç and the poster by Shvarts and Zagorianakos are grouped under the heading 'connecting theories'. Holm reports on an attempt to use both the SOLO taxonomy and the ATD in order to better understand the advantages of peer collaborative learning exercises for group investigation. The analysis shows that these two frameworks evaluate different dimensions of students' behaviour and relating SOLO-levels to characteristics of ATD praxeology was not possible but the two theories are complementary in terms of understanding student activity. Monaghan focuses on tool use in mathematics and how different theories in mathematics education view tool use. Tool use is important in activity theory (AT) but the consideration of tool use in AT studies varies with the unit of analysis. AT places human agency at the centre of activity and, in contrast to actor network theory (ANT), undervalues non-human agency. Monaghan attempts to 'synthesise' AT and ANT with regard to tool use. Şay and Akkoç examine teachers' social and social-mathematical norms and their instrumental orchestrations in technology-enhanced learning environments in a study designed to investigate how orchestration types and norms affect each other. They found teacher-centred orchestrations in classes where the dominant norm was 'teacher as the mathematical authority' and student-centred orchestrations in classes where the endorsed social norms put students into the centre. Shvarts and Zagorianakos explore the complementarity of activity theory and phenomenology through a detailed analysis of perceptual action by an eye-tracking methodology. While activity theory predicts the development of the perception of visual models through involvement into cultural practice, the data showed that it is the child who makes sense from the presented practice, at the levels of her operative intentionality and the intentionality of act.

The papers by Florensa, Bosch and Gascón and by Kidron are grouped under the heading 'epistemological aspects of theories'. Florensa and colleagues argues that didactics involves both the problem of the development of knowledge and the problem of the diffusion, the use and the transposition of knowledge. Using the ATD, the paper considers means to analyse learning and teaching practices within real institutional environments. The construct 'reference epistemological model' is used to explore extant and new praxeologies through the elaboration of alternative mathematical organisations that could be close to or very distant from the institutional contents that are taught and learned. Kidron considers "mathematical objects not as absolute objects, but as entities which arise from the practices of given institutions" and this "leads us to analyze the role of both, the epistemological dimension and the socio cultural dimension, in theories". The paper provides "an example of networking that demonstrates how the social dimension might influence the epistemological analysis".

The remaining papers (and one poster) have been grouped under the heading 'issues in mathematics education related to theories'. The papers by Bingolbali and Bingolbali and by Godino, Batanero, Cañadas and Contreras focus on teacher-learners. Bingolbali and Bingolbali argue that student-centred teaching (SCT) consists of two components: mixed teaching methods and principals. They state six principles of SCT: valuing students' prior knowledge into consideration; handling students' difficulties with appropriate methods; developing students' skills; providing effective feedback; creating communicative classrooms; integrating assessment into instruction. Godino and colleagues argue that the inquiry-transmission polarity of instructional models is a simplification of a complex reality. The paper outlines semiotic, epistemological and cognitive assumptions of the onto-semiotic approach to mathematical knowledge and instruction which recognizes a key role to both inquiry and transmission models. The paper by Kent and Foster also challenges a polarity in mathematics education, conceptual versus procedural understanding in mathematics. The paper asks if it would be appropriate to describe a learner in possession of an algorithm for responding satisfactorily to such prompts as displaying conceptual understanding. They relate this question to Searle's 'Chinese Room' thought experiment and draw on Habermas' theory of communicative action to develop implications for addressing the problem of interpreting learners' mathematical understandings.

The papers by Ertas and Aslan-Tutak and by Perez focus on the teacher. Ertas and Aslan-Tutak report on tests of mathematics content knowledge (MCK) and mathematics pedagogical content knowledge (MPCK) given to senior student mathematics teachers and senior mathematics students. The performance of the student teachers was significantly higher than that of the mathematics students in the test on MCK. The paper discusses the reasons behind this unexpected finding and notes the challenges to measure MPCK. Perez presents the notion of adaptive conceptual frameworks employed to conduct design-based research with the aim of developing ICT supported mathematics instruction. The paper employs three frameworks: one used when the researcher engages with the teachers; one used to understand outcomes and to plan the next design cycle; one for organizing and supporting the teachers' professional development. Perez uses ideas from 'networking theories' to consider interactions between the frameworks.

The papers by Koichu and the poster by Seidou focus on the learner. Koichu focuses on problem solving and introduces a 'confluence framework' which "consolidates ideas taken from several frameworks", mainly John Mason's theory of shifts of attention. The central premise of the framework is that a key solution idea to a problem can be constructed by a solver as a result of shifts of attention that come from individual effort, interaction with peer problem solvers or interaction with a source of knowledge about the solution. Seidou's framework is Brandom's 'inferentialism' which prioritises inference over reference or representation. The paper reports on students' language 'moves' (how claims are put forward) while reasoning in a geometric sorting activity, "The open-ended aspect of this task creates favorable conditions for a fruitful game of giving and asking for reasons".

The papers by Siller, Bruder, Hascher, Linnemann, Steinfeld and Sattlberger and by Lindenskov, Tonnesen, Weng and *Østergaard* focus on policy. Siller and colleagues report on a project that developed a competency grid to assess the quality of Austrian end of school examination questions. The competency grid has three dimensions (operating, modelling and reasoning) and four levels related to students' mathematical actions: "activity theory forms the background for the didactic interpretation of such initially pragmatic levels". Lindenskov and colleagues report on efforts to develop early "interventions for marginal student groups". The work was inspired by critical theories in mathematics education and practical intervention approaches from various countries. The paper investigates possible contextual influences on networking theories. The paper presents a 'program logic model' for early interventions.

ISSUES ARISING FROM THE PAPERS AND FUTURE POSSIBILITIES

To highlight the productivity and limitations of the work done in the TWG, we propose to consider the questions that have been raised by the papers, their reviews during the discussions, and also issues that have been disregarded or overlooked.

The issue of networking theories was omnipresent in previous 'theory' TWGs since 2005. It has been approached, here, in a different (maybe more mature?) way. The efforts have focused on some basic epistemological and methodological reflections (the nature of theorising, for example) rather than on the description and study of networking strategies. Some new questions have also been opened to a broader debate, with a view to developing research topics in the years to come. One such issue can be called "the question of questions": it relates to the way in which teachers' and students' difficulties can be made sense of by different theories and how the research problems thus arrived at depend on the approach taken. In this context, the issue of mathematics education as a discipline seems especially relevant: what is the place of the didactics of mathematics as a discipline in the arts and sciences realm? How is it related to the didactics of other disciplines and to the other sciences? Can it lead the development of teaching and of teacher education?

Another "big" question that deserves to be addressed is the relationship between local and global theorizations. Many proposed "theories" in (mathematics) education seem to be content with trying to account for a deliberately limited number of didactic phenomena: in this respect, they can be termed "local" constructs for which we have to make clear the didactic dimensions they take into account as well as those they (necessarily or not) overlook. This is especially important since what is called a "theory" or a "theoretical approach" in a given research vein may vary in its degree of development, from limited models to more extensive theoretical constructions. For this reason, it is essential to be constantly aware that theories are living entities in continuous development, and that most "global" theories started with a local or limited scope. The main aim, purpose or ambition adopted by a given theorisation process may result, in this context, as a crucial variable to be taken into account, an issue closely related to the notional and methodological tools elaborated to control and ensure the solidity and productivity of the intended theoretical construction.

The balance between group and individual advancement in the development of research areas is also an interesting topic to approach, in terms for instance of the cultures and craft methodologies elaborated within communities, which are often not easy to disseminate through the traditional channels of science communication (papers, surveys, doctoral dissertations, conference proceedings, etc.). Finally, and related to these last issues, the contrast between findings across different approaches is a question that has never been directly addressed, especially when these findings appear to be, if not contradictory, at least not directly compatible. More generally, a more straightforward, frank, and even antagonistic approach to the problem of theoretical diversity could prove fruitful.

TWG17

Research papers

Principles of student centred teaching and implications for mathematics teaching

Erhan Bingolbali and Ferhan Bingolbali

Gaziantep University, Gaziantep, Turkey, erhanbingolbali@yahoo.co.uk

This paper aims to present principles of student-centred teaching (SCT) and provide some implications for mathematics teaching. We have determined six main principles of SCT: i) Taking students' prior knowledge into consideration. ii) Handling students' difficulties with appropriate methods. iii) Developing students' skills (e.g., reasoning). iv) Providing effective feedback. v) Creating communicative classroom environment. vi) Integrating assessment into instruction. We first present the rationale of the study and note the ambiguity regarding student-centred related terms. We then propose that STC approach consists of two main components: mixed teaching methods and principals. The paper ends with discussion and implications of SCT approach for mathematics learning and teaching.

Keywords: Student centred teaching, mathematics, mixed teaching methods.

INTRODUCTION

The emphasis on individual learning has paved the way for the emergence of new terminology regarding the learning and teaching both in education as a whole and mathematics education in particular in the last three decades. One such term is that of student-centred teaching. Intuitively although it might appear to be a straightforward term, it seems that not only the term is not well defined but also what is attributed to the term is not clear. As teacher educators, our experience with pre-service and in-service teachers has also revealed that the term student-centred teaching often is attributed to only constructivist approach (e.g., discovery learning) and students' physical activeness in the classroom, whilst cognitive activeness was regarded as secondary if not disregarded at all. The vagueness regarding the meaning of the term has been the rationale for the emergence of this study. With this in mind, this paper attempts to examine the term SCT and aims to propose some principles in order to contribute to its conceptualisation especially for the practitioners working in the field.

Considering that the term SCT is wide-ranging, any attempt to determine its principles requires an examination of multiple theories. To this end, an eclectic literature (e.g., behaviourist, cognitivist, constructivist, sociocultural perspectives on learning and teaching) has been examined. Six main principles have been determined to characterize the SCT. These principals develop from both the relevant literature as detailed below and our interpretation of what the teachers and candidates might need to know for conducting a SCT approach. Although we do not claim that they are sole principles of SCT, we argue that they provide an overall aspect of what the SCT might include. The determined principles are as follows:

1) Taking students' prior knowledge into consideration
2) Handling students' difficulties with appropriate methods
3) Developing students' process skills
4) Providing effective feedback
5) Creating communicative classroom environment
6) Integrating assessment into instruction

In what follows, we first explain why we chose to examine the term SCT and present our stance on it. We then explain each principle in light of the relevant literature and relate them to SCT. We conclude the paper with discussions of the principals.

THE TERMINOLOGY

Dissatisfaction with teacher-centred approach (often known as traditional teaching) and behaviour-oriented perspective in learning and teaching has directed educators to pay more attention to students and their cognitive needs. This shift in attention has resulted in generating new terms and concepts to capture the new phenomenon. Student-oriented terms that have been commonly used amongst educators are the result of such undertaking. As a result of such endeavours, the terms such as student-centred learning, student-centred pedagogy, child-centred learning, student-centred education, learner-centred learning and student-centred teaching come into use. Common to all these terms is the students and their individual learning.

A close examination of these terms reveals several problematic issues though. First, it appears that student-centred terms have sometimes been reduced to ideas popular to Piaget's constructivist developmental theory and hence "discovery learning". Second, the terms have mainly been associated with students' physical activeness rather than cognitive ones. Third, sometimes a passive role is attributed to the teachers since the students are construed be more active. Fourth, the terms have been loosely used and it is not exactly clear what meaning is actually attributed to them. Lastly, it seems that since the terms have mainly been used by the practitioners for practical reasons and have hence been not the foci of the systematic research, it has been difficult to provide a research-informed operationalization of them for the teaching activities.

Given that the terms are commonly being utilized in the field, we as the researchers cannot be incognizant of their uses and need to make contribution into their clarification. In this study, we particularly prefer to use the term SCT for two reasons. First, we think that the term student-centred learning or similar ones have some shortcomings. This is because all learning, passive or active, is student-centred in nature. Besides, whilst examining different approaches to learning and teaching (e.g., behaviourism, constructivism), although the quality of learning may show variation, what mainly differs is indeed the teaching or the teaching methods. That is why we prefer to use SCT, not student-centred learning. Second, as the teachers are responsible for the teaching, they need to know how to conduct student-centred teaching and hence we take the teachers as the main addressee. In what follows, we present our position on SCT.

OUR STANCE ON STUDENT-CENTRED TEACHING

We use the term student-centred in the sense that students and their learning needs should be prioritised in the learning and teaching activity. For instance, if a teacher takes students' difficulty with a concept into account and teaches accordingly, this suggests that students' needs are prioritised and the teaching has a student-centred feature. Determination of students' needs, however, is not a simple endeavour. This, of course, depends on the teacher competency regarding the subject matter they teach. Moreover, the needs of students can show variability. The nature of concepts and the student competency are just only two factors that can cause the variability. For example, in teaching group concept in abstract algebra, to us, what the students need is the definition in the first place as it is very difficult for them to discover the group concept through such approach as problem-based learning. The concept's nature hence determines what the students' needs are and that affect the teaching. On the other hand, if a teacher values conceptual/meaningful understanding, arousing the need for learning and developing reasoning skills etc., then teaching, for instance, "triangle inequality fact" via problem-based learning method and hence providing the students with opportunity to discover or at least attempt to discover the fact can be more fruitful. Given that all these aspects (e.g., reasoning) are important for the learning, this type of teaching is also considered to have a student-centred feature.

One problematic issue that may arise with "discovering" the inequality fact is that: what happens if a student or students cannot "discover" the fact even though the guidance is provided? If one is concerned with students' needs, it is then possibly acceptable that sharing the formula of "$|a-b| < c < a+b$" with students is more reasonable. That is to say, teachers should (sometimes have to) provide the formula or the fact for the benefit of the students. In teaching, teachers hence may sometimes use a mixed instructional approach (e.g., both traditional and constructivist ones) depending on the concepts and students' needs. This is, to us, what makes the teaching student-centred. In fact, Godino and colleagues (2015) also note that there is a need for mixture of construction/inquiry and trans-

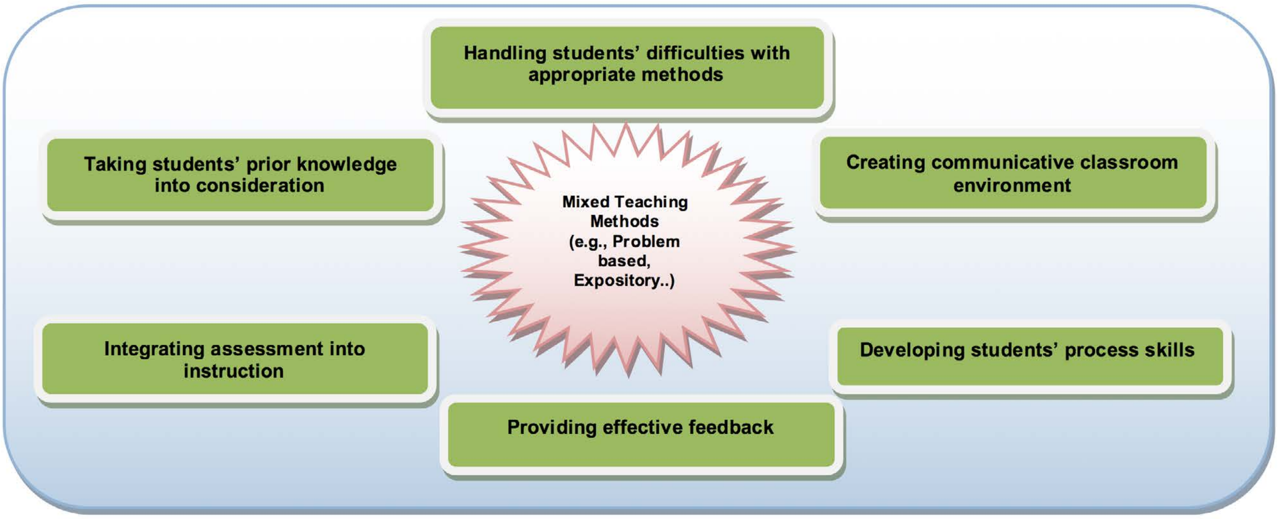

Figure 1: SCT, teaching approach and its principals

mission of knowledge that might optimize learning. They are also critical of basing the instruction solely on "Inquiry-Based Learning" (IBL) or "Problem-Based Learning" (PBL) methods and note that these methods might be more suitable for only gifted students and that these methods generally disregard heterogeneity of the students and the variety of knowledge to be learnt.

In this paper, even though SCT is often associated with constructivist approach in education, we argue that this view is problematic and student-centred teaching needs reconceptualization. We also think that having a practical method (we name it as mixed teaching method) as we presented above is not sufficient to conduct the SCT either, and that is why we propose its principals as well (see, Figure 1). In practice, there is a need for both principals and the mixed teaching method.

As can be inferred from Figure 1, our position is that SCT approach consists of two main components: *mixed teaching methods* and *principals*. In teaching, a teacher might employ mixed methods, that is, the teacher may use both problem-based and expository teaching methods in the same lesson. Yet, to conduct the mixed methods effectively and to take students' needs at the centre, a teacher also needs some principals. The principals guide the methods and enable their implementations. We now turn our attention to principals, their underpinnings and where they stem from.

Taking students' prior knowledge into consideration

Prior knowledge is essential for any learning and teaching activities. Learning theories (e.g., cognitivism, cognitive and social constructivism), particularly the ones shaping the current learning and teaching experiences in many classrooms, emphasise the role of prior knowledge in the learning processes. For instance, as a cognitive learning theorist, David Ausubel put forward the following view on the role of prior knowledge in learning:

> If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly. (Ausubel, 1968, p. 18).

Ausubel's comments can be construed as a radical reaction to behaviourists' view of learning. Prior knowledge draws explicit attention in Piaget's works as well. To Glasersfeld (1995, p. 18), one of the two basic principles (radical) constructivism is that "knowledge is not passively received but built up by the cognizing subject". In Piaget's constructivist theory of knowing, since knowledge is actively constructed, not passively received, then the prior knowledge becomes indispensable in the learning process. For instance, in explaining the notion of assimilation, Piaget (1976, p. 17, cited in von Glasersfeld, 1995, p. 18) notes the importance of prior knowledge:

> ...no behaviour, even if it is new to the individual, constitutes an absolute beginning. It is always grafted onto previous schemes and therefore

amounts to assimilating new elements to already constructed structures (innate, as reflexes are, or previously acquired).

Prior knowledge is not only essential to assimilation but also fundamental to the other two components (accommodation and equilibrium) of Piaget's theory. From a Piagetian perspective, it is thus vital that the teacher takes the prior knowledge into account in teaching. This stance of course requires an examination of students' readiness for the teaching. For instance, in teaching the area of parallelogram, it is important to determine what the students know about the area and the concept itself. The previous experience that the individual brings to learning settings has hence important effects on what he/she is going to learn. We thus take this as a principle of SCT as it is concerned with students' needs. We think that any teaching method with students' needs in mind should begin with determining learners' current knowledge level, types of experience they have and needs analysis.

Handling students' difficulties with appropriate methods

The issues of how students learn and why some have difficulties in learning have always drawn the attention of researchers. Many learning theories (e.g., APOS, Cottrill et al., 1996) have been put forward for the former. For the latter, it is known that students' learning difficulties, misconceptions and errors are the reality of classrooms. Nesher (1987, p. 33) appears to even value the existence errors and notes that "the student's "expertise" is in making errors; that this is his contribution to the process of learning". If students are experts of making errors, then any instructional consideration has to take them into account and teachers need to have an expert approach of handling them. Students' difficulties in learning are also important in the sense that they have been the cause for the emergence of many innovations, including new learning theories, teaching materials and new approaches to teaching etc.

Difficulties generally manifest themselves as errors in the classroom settings. It is critical for teachers to be able to notice the underlying conceptions that cause the errors to emerge. Diagnosing the errors and the causes are hence crucial. Following that, it is essential to have a plan of how to handle the difficulty. This plan might include selecting the appropriate materials and method of handling. For instance, the relevant literature proposes many different ways of handling the difficulties. Such handling methods as cognitive conflict, giving correction, ignoring are just some examples of teachers' dealing with errors (e.g., Santagata, 2004). Deciding which method to use might depend on the nature of the errors and the teacher's competency. Students' learning difficulties are hence one of the most influential factors that influence the learning and teaching. To us, SCT must take this issue into account and acts accordingly. We think that the teaching concerned with students' difficulties has the characteristic of SCT and has a better chance of getting over students' difficulties.

Developing students' process skills

Traditional teaching has mainly been concerned with the knowledge (e.g., fraction, function, derivative) and its transmission to the students. However, the aim of schooling is not only to transmit the knowledge or teach concepts. One of the essential goals of schooling is to teach students to think (Padilla, 1990) in general and to reason, justify and make connections in particular. As Padilla (1990) notes "all school subjects should share in accomplishing this overall goal."

In addition to teaching concepts, equipping students with basic skills has also become a goal for many curricula. For instance, in science education these skills are named as basic process skills and six such skills are targeted: i.) observation; ii.) communication, iii.) classification; iv.) measurement; v.) inference; vi.) prediction. In mathematics education, NCTM (2000) names the skills as process standards and notes that mathematics instruction should aim to develop such skills as i.) problem solving, ii.) reasoning and proof, iii.) communication, iv.) connections, and v.) representation. In addition to conceptual understanding, procedural fluency and productive disposition, Adding It Up (NRC, 2001) document also propose strategic competence and adaptive reasoning as a part of mathematical proficiency. All these suggest that skills have become an essential goal of the curricula in that the teaching should be concerned not only with concept teaching but also with skills acquisition.

The development of these skills may have many advantages. First, they enable students to think, justify and make connections. Second, skills can help students have conceptual understanding and therefore meaningful learning (Ausubel, 1968; Skemp, 1978). Without the skills, concepts in mind may stay disconnected and compartmentalised. Third, the skills may help the students be better problem solvers and hence apply their concepts to real life settings. With all these advantages in mind, we think that the teaching concerned with students' intellectual development must also aim to develop students' process skills. We therefore take the teaching process skills as a main principle of SCT and argue that conceptual and meaningful learning is more plausible through teaching them.

Providing effective feedback

Students' learning is complete with interesting experiences from showing an exemplary performance to making errors, having fundamental misconceptions and not having a sense of direction of what to do under some particular circumstances. An examination of what the students know, where they show good or poor performance and what to do next is sometimes needed for instructional decisions. All these are somehow related to effective feedback and its conduction.

Feedback is regarded as "one of the most powerful influences on learning and achievement" (Hattie & Timperley, 2007, p. 81). Feedback is defined as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (ibid, p. 81). Winner and Butler's (1994) conceptualisation of the feedback is also helpful.

> "feedback is information with which a learner can confirm, add to, overwrite, tune, or restructure information in memory, whether that information is domain knowledge, meta-cognitive knowledge, beliefs about self and tasks, or cognitive tactics and strategies" (p. 5740, cited in Hattie & Timperley, 2007).

As the quotation suggests, feedback can be provided by different agents and in many distinctive forms. A conceptualisation of feedback in the sense of Winner and Butler requires a careful examination of what task to choose, what kind of discourse to create and what method to use to handle students' learning outcomes on the part of the teacher. The teaching concerned with student needs is hence expected to pay attention to the quality of the feedback that the students get and acts accordingly. We therefore argue that one of the basic characteristics of the SCT lies at the quality of feedback provided to the learners.

Creating communicative classroom environment

As students participate in the learning activity as groups and since teaching students in groups is an indispensable reality of the schooling, the teaching cannot solely be reduced to the teaching of an individual and that it needs to address the classroom as a whole. In such situations, the issue of how the teaching, which takes students at the centre, can be conducted also needs to be examined and discussed. To us, a communicative classroom environment can be like an open society so that students can freely express their answers, make arguments and explanations. That is to say, a democratic classroom environment is needed so that students express their opinions. In this connection, Yackel and Cobb's (1996) notions of social and socio-mathematical norms can be employed as a guide for creating such a classroom environment. In such classrooms, different solution methods, reasoning, justification can be encouraged for all students. In such an environment, it is then more possible for students to obtain different perspectives and develop a critical habit of mind. We therefore take communicative classroom environment as a principle of SCT to guide the teacher concerned with student-centred teaching. It should be noted that the application of this principle in the classroom helps the teacher gain insight into the other principles as well. For instance, a communicative classroom environment may pave the way for the expression of free speech and that might help to diagnose the learners' difficulties. The teacher can hence employ this principle to have an overall picture of the instruction with regard to other principles as well.

Integrating assessment into instruction

Traditionally, assessment follows the instruction. This type of assessment is termed as summative and is concerned with cumulative evaluations. It is currently proposed that assessment needs to be built up into and integral to the instruction. This type of assessment is termed as formative one and is concerned with regular control of students' conceptions and understanding (Van De Walle et al., 2010). This type of assessment shapes spontaneous decisions regarding the instruction and the findings reveal that effective formative assessment can increase students' speed of learning by giving the effective feedback (Wiliam, 2007).

As far as SCT is concerned, it is proposed that assessment and instruction need be intertwined.

Assessment should not be something to be done at the end of instruction. Assessment concerned with students' development, difficulties and learning has to be in time and based on students' needs. In this regard, rather than evaluating students through one method (e.g., test) students' performances need to be assessed through different methods. Assessment also should not only be concerned with concept mastery but also with process skills proficiency. As a result, we think that SCT needs to be student-centred in terms of assessment as well. Moreover, as the Assessment Principle in Principles and Standards stresses: "(1) assessment should enhance students' learning, and (2) assessment is a valuable tool for making instructional decisions" (Van de Walle et al., 2010, p. 76). When the assessment is carried out in this respect, we think that it can contribute to the development of the SCT instruction.

DISCUSSION AND CONCLUSIONS

We have attended to the ambiguity of the term SCT and noted that what is attributed to the term is often not clear. We have also stated that SCT has been mainly associated with constructivist approach and argued that reducing it to this approach is misleading. A functional SCT approach does prioritise the students and their needs rather than a particular instructional approach per se. In the light of the relevant literature, alongside the mixed teaching methods, we have provided six principles that might contribute to conceptualisation of SCT. We are aware that the proposed SCT principles are generic in nature. This is particularly due to both the nature of the term and the teaching itself. Although this is the case, we hold the belief that for the practitioners it is important to have a general perspective of SCT as well. This is because; having a broad perspective can help the teacher put a particular learning objective into practice. Therefore, although the proposed principles are generic; they might help the teacher have a broad perspective on SCT and to put it into practice.

Most of current educational reforms suggest student-centred teaching and the chief addressees are teachers and teacher candidates. Although they are expected to conduct SCT, they generally do not have a guideline of how to do that. We believe that these principles as a totality might act as a guide for teachers and candidates to practice SCT. For instance, the principles can be used to design and implement lesson plans. We also think that these principles can be used to develop or assess in-service and pre-service teachers' competencies and knowledge bases. For instance, a teaching programme addressing methods of handling students' difficulties may contribute to the development of teachers' pedagogical content knowledge base. In addition, the SCT principles can be employed as theoretical framework to analyse the classroom discourse and determine whether the teaching is SCT or not. For example, it can be utilized to determine the extent to which the teaching values the process skills. Similarly, the framework can enable one to see how students' difficulties are handled and to show which the types of feedback are provided in the classroom.

As mentioned above, we are aware that these principles are generic and that is why they cannot be specific to any discipline. The nature of disciplines and their concepts will shape how each principle is put into practice. For instance, whilst handling a difficulty or error, one needs to know the nature of the concept and teach accordingly. More specifically, let's take division of two fractions as an example. If the concept is to be taught in an SCT manner, in the light of SCT principles, the teacher first has to take learners' prior knowledge of fraction and division into account. Knowing students' difficulty with division of fractions can help the teacher make necessary preparation, which would improve the instruction. These all suggest that the nature of concept in a discipline itself can affect how SCT is perceived and conducted.

Process skills can play an important role in making SCT approach specific to a particular discipline or carries its distinctive characteristics. Reasoning, justification or representation of the concepts, for example, can differ from one discipline to another. For instance, the function can be represented in many forms (e.g., numeric, graphical, algebraic, verbal). When the teacher teaches this function with its multiple representations alongside with their interconnections to enrich students' understanding, this would suggest that the teaching has a student-centred feature.

Finally, as this work is still in progress, we suggest that further research needs to be carried out to see how functional the proposed principals are and examine them in the real classroom settings. There is also a need for making each principal more explicit. Further research is also needed to examine practitioners' conceptions (e.g., values, beliefs) of SCT and how these conceptions play role in its implementation.

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Considering theoretical diversity and networking activities in mathematics education from a sociological point of view

Corine Castela

Normandie Université, LDAR, Caen, France
Université de Rouen, LDAR, Mont-Saint-Aignan, France, corine.castela@univ-rouen.fr

The paper focuses on the social dimensions of the issues addressed in this working group, social being considered at different levels: interactions, culture, and institutions. It addresses the following questions: what is a theoretical framework? Why are theories so numerous in mathematics education? Is it necessary to reduce this multiplicity? Why or why not? The reflection is based on the anthropological theory of the didactic (ATD) and on Bourdieu's theory of social fields. Assuming that the latter is not necessarily well-known in the mathematics education community, and that it offers an interesting potential to enrich the debate within the networking semiosphere, I devote a substantial part of my text to give an idea about the way Bourdieu applies his theory to science.

Keywords: Praxeology, paradigm, institutional determination, symbolic capital.

INTRODUCTION

Addressing the topic of theoretical diversity in mathematics education from a social point of view is not something new in the European research community. The central preoccupation in this WG has been, since CERME4, the barrier to effective communication created by the multiplicity of theories, be it communication within the field or with external partners from policy makers to educative professionals. Radford (2008, p. 318) suggests considering the networking practices as located in a *semiosphere*, that is, "an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other". It is quite representative of the interaction dimension in networking activities. Among the social aspects this paper considers, some have been more erratically present in the discussions. For instance, the WG11 leaders' introduction at CERME4 (Artigue et al., 2006, p. 1240) refers to a theoretical "more intrinsic diversity linked to the diversity of educational cultures and to the diversity of the institutional characteristics of the development of the field in mathematics education in different countries or global areas." This issue of theoretical multiplicity linked to cultural diversity has not recently been discussed in CERME. Yet the influence of cultural contexts on the research in mathematics education has been discussed in the second plenary talk of CERME9. This confirms the need to address such topic in this WG. My position is that our reflection about theoretical diversity is obstructed by some beliefs that should be deconstructed and that, in order to do so, we need theoretical tools from inside and outside the mathematics education field. In this paper, my objective is to present some tools, borrowed from ATD and from Bourdieu's field theory, which I find helpful to go forward. I briefly show how I use them to consider the social dimension of theoretical multiplicity and to discuss the unifying-theories injunction, thus developing a rational discourse (*logos*) with social concerns about the issues addressed. The adjective "sociological" in the heading must be understood in this etymological meaning, this paper does not avail itself of the sociology scientific field.

Before going any further, I emphasise that, in my opinion, a valuable discussion about connecting theories relies on the participants having minimum knowledge about the theories at stake in the papers. Aside for some well-known theories, I believe it is the author's responsibility to provide the readers with a first understanding of the invoked theories. I try to do so regarding Bourdieu's field theory, assuming that ATD is familiar enough in our research community. Hence, the second part of this text encompasses large quotes intending to provide the readers with a direct, though limited, access to the key elements of Bourdieu's analysis of science which I draw on later. Other theories appear as examples in the discussion for which I can only provide references.

THEORY, RESEARCH PRAXEOLOGY, RESEARCH PARADIGM?

In this part, I recall and connect, especially for the newcomers to theory networking like myself until recently, crucial models elaborated by the first participants in this group to address the issue of what is actually a theoretical framework. This gives me the opportunity to address the issue of what is a theory in the ATD and to discuss some points of Chevallard, Bosch and Kim's contribution to this TWG (2015). Then I propose to encompass into the theory modelling the contribution of well-identified research communities, thus considering the social dimension of networking theories.

What will we consider as networking theories in this 2015 session?

An eight-year-long joint work in CERME as well as in research projects like Remath has largely evidenced that what is at stake cannot be reduced to networking of theories considered as "organized networks of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain…" (Niss, 2007, p. 1308). Other research aspects are involved in the interconnection activities. Two directions have been proposed to model this complexity. Radford (2008) describes the concept of theory using the triplet (P, M, Q) where P is a system of basic principles, including implicit views and explicit statements, M a methodology, and Q a set of paradigmatic research questions. Hence, connecting two theories means connecting two triplets. Artigue, Bosch and Gascón (2011) use the notion of praxeology to model research theories and practices. Introduced by Chevallard as a general model for all human activities (see Bosch & Gascón, 2014, for an introduction to ATD), a punctual praxeology is a quadruplet $[T/\tau/\theta/\Theta]$ with only one type of tasks T and one associated technique τ, θ being the technology of τ, i.e. a rational discourse accounting for this technique. "The fourth component is called the "theory" and its main function is to provide a basis and support of the technological discourse" (ibid, pp. 67–68). Moreover, ATD considers more complex levels of praxeological organisations gathering punctual praxeologies which have a common technology (*local* praxeology) or a common theory (*regional* praxeology). A regional research praxeology is described through a set of research questions considered relevant when others are not, correlated techniques, their technologies and a theory. Artigue, Bosch and Gascón (2011) consider that this is the proper level to address networking issues.

What is a theory in this model? In the case of well-developed research praxeologies, the theory may fit with Niss' definition. However, not all such theories operate as identifier of their associated praxeology, because some are not recognised as "a Theory" in the research field. For instance, let us consider the so-called "double approach" (of the teachers' practices) developed by Robert and Rogalski (2002). A regional "double approach" praxeology obviously exists in mathematics education. Its theory, in both ATD and Niss' meaning, is well developed, coordinating elements from several identified theories like Theory of Conceptual Fields and Activity Theory with some more isolated concepts or results from didactics and cognitive ergonomy. Yet, there is no "Double Approach Theory", the praxeology access to social existence in the research field relies on other means, like the publication of a collective book gathering different studies (Vandebrouck, 2008) and its English translation (2013).

Now, let me emphasise that, within ATD, most praxeologies' theories are not this developed (see Chevallard et al., 2015, to go further on this issue); they may not fit with Niss' definition. One strength of this modelling of research activities is that it may be used to account for the research praxeological dynamics as Artigue and colleagues (2011, p. 2382) do: "Research praxeologies can appear as different kinds of amalgams, more or less organized depending on the maturity of the field". They highlight the part played by the technological discourse in such a stage of praxeology, when the theory of the amalgam is underdeveloped and unable to organise through a coherent whole the first results produced by the research practices. I will focus on the social dimension of the development process: the emerging praxeological organisation would not strengthen and access a certain form of social existence in the research field without the setting up of a group of researchers with common concerns, collaborating towards the development of the praxeology.

In the case of the double approach, such a group was first created around A. Robert and J. Rogalski within the Parisian laboratory Didirem, especially through the completion of several PhD theses. In 2015, the double approach community still exists; it is disseminated far beyond its original laboratory. This idea that there is no research praxeology recognised in the mathematics education field (or in some subfield) without an associated community of researchers is not accounted for by the praxeological model. Thus, I propose an extended model, called a research paradigm [1], composed of a praxeology and a correlated social organisation, working as an institution.

Connecting the three models

The praxeological model and Radford's model appear as efficient tools to account for the fact that connecting theories is not only connecting conceptual structures. They share several aspects: Q is the set of T, M the set of $[\tau/\theta]$, the explicit part of P belongs to Θ, such as does a fourth component, the set of key concepts K, added to Radford's triplet in (Bikner-Ahsbahs & Prediger, 2014). Yet each model highlights an aspect the other one overlooks. With regard to methodology, the praxeological twofold description $[\tau/\theta]$ provides an appropriate tool to consider what is happening in the case of methodological exchanges between theories (with Radford's meaning of the term), an issue addressed by (Radford, 2008, p. 322). The technique may or may not change, but certainly a new technological discourse will be produced to justify that the imported technique is consistent with the importing theory and its principles. Regarding principles, there is no place in a praxeology for the implicit part of P. This claim needs some discussion. Chevallard and colleagues (2015) argue that "a theory is made up of two main components, that we may call its "emerged part" and "immersed part. [...] In ATD, a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory." This recognises the need to encompass an implicit dimension in the human activity modelling. Yet, I dispute the idea that implicit views may be considered as parts of the praxeological *logos* component. According to the etymological meaning of this Greek term, the *lógos* is an explicit discourse. In my opinion, the praxeological model must carry this meaning where the $[\theta/\Theta]$ block refers to explicit, socially legitimised knowledge, to the *savoir* in French. However, referring to ATD and its institutional dimension, I assume that the way a praxeology lives in a given institution is determined by a set of constraints, among which culturally shared incorporated norms, many of them being implicit. Studying this implicit praxeology environment is a condition to furthering the process of developing the praxeology, as highlighted by Chevallard and colleagues (2015). As for research, the paradigm model I propose provides a tool to take into account both emerged and immersed parts: within a given paradigm, researchers' actions are regulated by the reference to the research praxeology and through the influence of the associated social organisation.

In summary, the research paradigm model presents three strong points: incorporating the different aspects of the (P, M, Q, K) and $[T/\tau/\theta/\Theta]$ models; including in the modelling project the contribution of the research community that in some cases or times plays a decisive role in the scientific identification of the research praxeology; and considering social interactions between communities within the networking issue.

LOOKING AT MATHEMATICS EDUCATION RESEARCH FROM OUTSIDE

I now present tools that I use in the last part of the paper to interpret the paradigm multiplicity in mathematics education and the injunction to unify theories.

Institutional determinations

An ATD important contribution has been to introduce the notion of ecology in mathematics education in order to fight the pedagogical voluntarisms. The mathematical and didactic praxeologies are subjected to a complex system of conditions "that cannot be reduced to those immediately identifiable in the classroom" (Bosch & Gascón, 2014, p. 72). They are constrained by a whole scale of institutional determinations among which ATD considers at the highest generic levels the influence of Civilisation and Society (ibid, p. 73). This is only one example of the crucial part given to institutions by ATD, it aims to show that this theory always immerses the addressed questions in the whole anthropological reality, with a special focus on the social organisations and the way they determine human activities. In what follows, I apply this approach to mathematics education research.

Bourdieu's field theory applied to science

A field is a structured social space, relatively autonomous from the wider social space and strongly differentiated from other fields. According to Bourdieu, science is a field. The field theory focuses on the 'closed field' dimension of these spaces, providing analysis of what is going on inside; this is the interesting contribution for our group since ATD provides adequate tools to consider external influences.

A field is characterised by a game that is played only by its agents, according to specific rules. The agents are individuals and structured groups, in science they are isolated scientists, teams or laboratories. The conformity of agents' actions to the game rules is partly controlled by objective visible means, but the key point of the theory, through the concept of *habitus*, is the inculcation of the field social rules into the agents' subjectivity. This individual system of dispositions, partly embodied as unconscious schemes, constitutes an individual's right of entry into the field.

The field game is twofold. Firstly, it is productive of something that is the field legitimised goal in the social space. The rules, and therefore the individual dispositions, are fitted to achieve the goal that every agent considers desirable. In the case of science, the goal is epistemic: accepting tacitly the existence of an objective reality endowed with some meaning and logic, scientists have the common project to understand the world and produce true statements about it. Bourdieu further adds a social dimension to the Bachelardian conception of the scientific fact construction:

> In fact, the process of knowledge validation as *legitimation* [...] concerns the relationship between the subject and the object, but also the relationship between subjects regarding the object [...]. The fact is won, constructed, observed, in and through [...] the process of verification, collective production of truth, in and through negotiation, transaction, and also homologation, ratification by the explicit expressed consensus – *homologein*. (Bourdieu, 2001/2004, pp. 72–73).

Despite this social nature, scientific homologation produces objective statements about the world thanks to specific rules of the scientific critical scrutiny, "the reference to the real, [being] constituted as the arbiter of research" (ibid, p. 69). Bourdieu also emphasises that constructed facts are all the more objective as the field is autonomous and international.

Secondly, the game is a competition between the agents, resulting in an unequal distribution of some specific form of capital, source of advantage in the game, source of power on the other agents. Thus, a field, including the scientific one, appears as:

> a structured field of forces, and also a field of struggles to conserve or to transform this field of forces. [...] It is the agents, [...] defined by the volume and structure of the specific capital they possess, that determine the structure of the field [...This one] defined by the unequal distribution of capital, bears on all the agents within it, restricting more or less the space of possible that is open to them, depending on how well placed they are within the field... (ibid, pp. 33–34)

The capital includes several species, for instance, in science, laboratory equipment, funding, and journal edition. I focus on the symbolic capital, especially on its scientific modality:

> Scientific capital is a particular kind of symbolic capital, a capital based on knowledge and recognition. (ibid, p. 34)

> A scientist's symbolic weight tends to vary with the distinctive value of his contributions and the *originality* that the competitor-peers recognize in his distinctive contribution. The notion of *visibility* [...] evokes the differential value of this capital which, concentrated in a known and recognized name, distinguishes its bearer from the undifferentiated background into which the mass of anonymous researchers merges and blurs. (ibid, pp. 55–56)

This theory of science as a field challenges an idyllic vision of the scientific community, disinterested and consensual. However, through the hypothesis of embodied dispositions, it avoids considering the scientists' participation to the capital conquest in terms of personal ambition or cynicism.

In summary, I will focus on the fact that scientific strategies are considered twofold.

> They have a pure – purely scientific- function and a social function within the field, that is to say, in relation to other agents engaged in the field. (ibid, p. 54)

> Every scientific choice is also a strategic strategy of investment oriented towards maximization of the specific, inseparably social and scientific profit offered by the field. (ibid, p. 59)

One can see a true correspondence between the triplets (institution, subjects, *assujettissements*-subjugation) of ATD (Chevallard, 1992) and (field, agents, *habitus*) of the field theory. In what follows, I consider mathematics education research as an institution immersed in and determined by a complex system of other institutions, and as a field of forces, subfield of the scientific global field.

EXTERNAL DETERMINATIONS OF THE "THEORIES ISSUE"

Research in didactics as externally determined in its questions and answers

I now consider the fact that the realm of reality of mathematics education research studies is determined by various economic, political, cultural institutions of different sizes. No one may dispute the vast distance that separates the following two objects of study: on the one hand, the passing down of arithmetic techniques in the Aymara villages of northern Chile, whose culture developed specific calculation praxeologies, and on the other hand, the use of software in the French education system to promote the learning of algebra. Is looking for universal regularities the epistemic priority of mathematics education research when, unlike physics for instance, the studied reality is so diverse? Assuming that such common phenomena exist (the didactical contract is often cited as such), which part of the two aforementioned complex realities are they able to account for? Moreover, given that the research intends to act upon the mathematics education reality, a more crucial question would be: to what extent can these regularities support engineering projects? In this paper, I will consider that adapted tools must be designed to address the problems raised by the diverse educational institutions around the world, in order to understand the dysfunctions and to produce solutions that are acceptable to these institutions and their subjects. The research questions as well as the produced answers are determined by local characteristics. The paradigm multiplicity therefore appears as a result of the epistemology of a science intending to act upon the studied reality. To take only one example, the ethnomathematics paradigm has been developed in South America as well as in Africa, as a response to a massive failure in mathematics education within educative systems that are still based on the colonial vision and present "mathematics [...] as an exclusive creation by the white race" (Gerdes, 2009, p. 31, my translation). Ethnomathematics follows as a paradigm from the need to "multiculturalise the curricula of mathematics to improve the quality of education and increase the social and cultural-self-confidence of all students". (ibid, p. 21)

Research in didactics as externally determined in its workings

Obviously, research depends on national and inter-national political and economic institutions which provide the material and the human resources. From this derives the existence of mathematics education research sub-institutions we partly find in the ICMI structure. But other institutions influence the research activities through less evident ways and means, such as cultures with more or less extended spheres of influence, up to civilisation. In spite of their scientific specific *habitus*, researchers with common culture also build upon this culture to address the research issues. That is one among other sources of some tacit principles of a paradigm. In other words, the paradigm multiplicity also results from the cultural multiplicity of the agents within the mathematics education research field. The researchers' cultural specificity may echo the educative local reality they study, hence resulting in a form of coherence and perhaps of efficiency. At the same time however, several paradigms may coexist in the same society, in the same country, investigating the same education system with different philosophical, ideological positions. As an example, let me consider ATD and the double approach that are strongly differentiated by their conception of the human being: ATD highlights the multi-institutional building of the framework within which the individual develops and acts (Chevallard, 1992, p. 91), the double approach focuses on the individual variations (Vandebrouck, 2008, p. 20). This second viewpoint is more present in the Western education research paradigms than the first one. I hypothesise that this is deeply correlated with the societies' characteristics and that it is not mere coincidence that ATD emerged in France.

Another example of external determination is the project of reducing theoretical diversity itself. This project is epistemologically founded within Bourdieu's theory since, as seen above, communication between researchers at the most international possible level is crucial in the construction of the scientific facts. However, it also comes from the requirements of political institutions, the Babel Tower aspect of research in mathematics education affecting its credibility. The proposed solution is unifying theories. Policy makers refer to the exact sciences model, and so does, rather surprisingly, mathematics education research itself, still (over)determined by its *alma mater*, mathematics. This reference neglects the diversity of educative reality. It forgets the exact sciences very long lifetime conducive to the unifying process, and that with the colonial expansion many local paradigms have simply been ignored, the occidental ones being imposed to the defeated countries. So the present homogenous theoretical landscape results as much from domination as from unification.

At this point, I have argued that the paradigm diversity is in some sense epistemologically legitimate in mathematics education and results from some social determinations of research. I have also noted that the unifying injunction might be considered as introduced into the field from outside for questionable reasons.

MATHEMATICS EDUCATION RESEARCH AS A POWER GAME

In this part, I build upon Bourdieu's statement that every scientific strategy has a social function within the field, i.e. has something to do with the distribution of power among the agents. In such a framework, the production of independent theories as well as the call for their integration in new entities are taken as contributing to the contestation and conquest of positions. For a researcher, being recognised as the creator of an identified theory clearly increases his scientific capital, much more than a less visible participation to the collective development of an existing paradigm would. This "visibility factor" fosters the paradigm multiplication, especially at the theory level; it should certainly be controlled when individual positions are at stake. However, let me now consider an emergent research community: in this case, developing a specific paradigm is an asset to free from the domination by older communities, generally tending to impose their own paradigms as ready tools which are adapted even for new problems. I will mention here the socioepistemology (Cantoral, 2013), deliberately developed by a group of Mexican researchers with the dual intent of creating tools adapted for the educative reality in South America and putting an end to what was considered as an extension of colonisation through the exclusivity of Western paradigms in didactics research.

I have already put forward that the need to unify paradigms could be epistemologically challenged by virtue of the diversity of the didactic reality depending on the societies and countries involved. Now, I question it as an obstacle to an autonomous organisation of didactical research in countries where the latter is just emerging. To finish, I reverse this point of view: if developing a paradigm is empowering for a community in the field, the call for reducing the paradigm multiplicity has something to do with relative positions of the research institutions incorporated to the paradigm. It is an aspect of the social game in the field, certainly determined by other levels of power struggles outside the scientific field as well.

CONCLUDING REMARKS

In this paper, my intent was not to contest the importance of interactions between mathematics education researchers. I recognise the crucial part of the broadest possible communication in the construction of scientific facts and the major difficulty deriving from the paradigm multiplicity in the field. My aim was to bring to light some aspects of the multidimensional complexity of this well-documented phenomenon, so far almost unexplored in this TWG. Multiplicity is an epistemological adaptation to the diversity of educational realities and a social result of symbolic power struggle within a recent research field, somehow less submitted to colonial and capitalistic rules to determine the power repartition than have been (and perhaps are) the oldest basic sciences. Hence, if reducing the number of paradigms appears as a direct solution, which favours communication thanks to a common conceptual language, this shortcut may be epistemologically inadequate for mathematics education research. Moreover, from the social point of view, it should be considered as the current hidden form of the exercise of power conquest in the field.

Unifying theories in order to produce a common discourse is not the appropriate way to scientificity for mathematics education research in its present state: that is the opinion I have tried to convey through this text. Building on the Remath project experience among others (Artigue & Mariotti, 2014), I suggest that collaborating which brings together researchers who refer to different paradigms might be more relevant; theory networking will result from working together on the same objects. The challenge is to develop scientific research collaboration praxeologies.

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ENDNOTE

1. Using the term paradigm may be provisory. It refers to Kuhn's notion of scientific paradigm (1962). Yet, in the postscript to the second edition (1970), Kuhn writes that: "Paradigms are something shared by the members of such groups [scientific communities]" (p. 178). It seems that he does not include communities within the paradigm model.

What is a theory according to the anthropological theory of the didactic?

Yves Chevallard[1], Marianna Bosch[2] and Sineae Kim[1]

1 Aix-Marseille University, Marseille, France, y.chevallard@free.fr

2 Ramon Llull University, IQS School of Management, Barcelona, Spain

The question tackled here centres on the notion—or, more precisely, the many notions—of theory often used in discussing scientific matters. The analysis that we attempt develops within the framework of the anthropological theory of the didactic (ATD). It purports to show that current usage refers mostly to the "emerged parts" of so-called theories and largely ignores their "immersed parts", which are the correlate of their intrinsic implicitness and historical incompleteness. This leads to favour open theorization over entrenched theory.

Keywords: Theorization, praxeology, knowledge, human activity, institutions.

INTRODUCING THE NOTION OF THEORY

In this study we examine the meaning and scope of a key concept of ATD which, paradoxically, since the inception of this theory, seems to have been consistently overlooked: that of *theory*. A word akin to English "theory" exists in many European languages [1]. According to John Ayto's *Dictionary of Words Origins* (1990), the history of *theory* goes as follows:

> **theory** [16] The etymological notion underlying *theory* is of 'looking'; only secondarily did it develop via 'contemplation' to 'mental conception.' It comes via late Latin *theōria* from Greek *theōríā* 'contemplation, speculation, theory.' This was a derivative of *theōrós* 'spectator,' which was formed from the base *thea-* (source also of *theā́sthai* 'watch, look at,'·from which English gets *theatre*). Also derived from *theōrós* was *theōreīn* 'look at,' which formed the basis of *theṓrēma* 'speculation, intuition, theory,' acquired by English via late Latin *theōrēma* as *theorem* [16]. From the same source comes *theoretical* [17]. (p. 527)

A paper by a classical scholar, Ian Rutherford, gives more information on the uses of the word *theoria* in Ancient Greece:

> The Greek word *theoria* means "watching," and has two special senses in Greek culture: first, a religious delegation sent by a Greek city, to consult an oracle or take part in a festival at a sanctuary outside its territory, and second, philosophical contemplation. *Theoria* in the first sense is attested from the sixth century bce until the Roman Empire, but the sources are particularly rich in the Hellenistic period. Sacred delegates were called *theoroi*, were often led by a so-called *architheoros*, and if they went by sea, the vehicle was a *theoris*-ship. (Abstract)

The first of these two senses has almost disappeared from modern usage. The second sense opened the way for our common uses of *theory*. In the following, we concentrate on "modern" meanings of this word, which dictionaries usually condense into a small number of categories, as does for example the English *Wiktionary*. The entry dedicated to *theory* in this dictionary begins classically with the etymology of the word, then passes on to the uses of it that it does retain:

> **theory** (*countable and uncountable*, *plural* theories)
>
> 1) (obsolete) Mental conception; reflection, consideration. [16th-18th c.]
>
> 2) (sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed; a hypothesis confirmed by observation, experiment etc. [from 17th c.]

3) (uncountable) The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]

4) (mathematics) A field of study attempting to exhaustively describe a particular class of constructs. [from 18th c.] *Knot theory classifies the mappings of a circle into 3-space.*

5) A hypothesis or conjecture. [from 18th c.]

6) (countable, logic) A set of axioms together with all statements derivable from them. *Equivalently*, a formal language plus a set of axioms (from which can then be derived theorems). *A theory is consistent if it has a model.*

In what follows we shall draw upon such semantic summaries in order to suggest that the notion of theory developed in ATD can account for the diversity of usages that exist today.

SOME BASICS OF ATD

In ATD, the basic "entities" are *persons x* and *institutions I*. These notions are close to their ordinary counterparts, although they are more general: in ATD, a newborn infant is a person; and, to take just one easy example, a class, with its students and teachers, is an institution. An institution *I* comprises different *positions p*— in the case of a class, that of student and of teacher. To every person *x* or institutional position *p* is assigned a "praxeological equipment", which is the system of "capacities" that, under appropriate conditions, enables the person *x* or any person *x'* occupying position *p* to act and think through one's actions.

Any praxeological equipment, be it *personal* or *positional*, is made up of, among other things, "notions". Most persons and institutional positions thus have a certain notion of theory—if only through the overused phrase "in theory". The present study could then be said to be partly about the notion of theory in ATD (taken as an institution). However that may be, it is essential to detach oneself from the seemingly undisputed belief that there would exist a unique, shared *notion* of theory of which the meaning would simply vary according to the context of use. In ATD, every person, every institutional position is supposed to be endowed with a peculiar notion of theory, that notion being shaped by the *constraints* to which the person or position is currently subjected. This phenomenon is at the origin of the processes of institutional transposition, of which didactic transposition is but a particular case (Chevallard, 1992). In order to make headway, we shall now delineate the "anthropological" notion of theory—which, at the start, is only one such notion amongst others.

THE NOTION OF PRAXEOLOGY IN ATD

ATD posits a theory of human activity that hinges on an essential and founding notion: that of *praxeology* (Chevallard, 2006, 2015; see also Bosch & Gascón, 2014). The word *praxeology* has been around for (at least) two centuries in the sense recorded by most dictionaries, in which it is held to refer to the "study of human action and conduct", to the "study of practical or efficient activity", or to the "science of efficient action". The use made here of the word pertains properly to ATD and departs decisively from this old-established, though infrequent, use. A key tenet of ATD is that when a person x acts purposely and knowingly, her doings can be analysed into a (finite) sequence of *tasks* t_1, t_2, ..., t_n. Contrary to the common meaning of the term (which has a ring of unpleasantness about it), *task* is taken here in a very general sense, irrespective of its volume or pettiness: *to open this door* and *to smile to this neighbour* are tasks; *to scratch this person's back*, *to write this sonnet*, *to save this polar bear*, *to prove this theorem*, and *to play this guitar chord* are tasks as well. Any task t is regarded as a "specimen" of a *type* of tasks T. In order to execute the task t of type T, a person x draws on a determined *technique*, denoted τ_T, that is to say a (more or less precise) way of accomplishing (at least some) tasks t of type T. No technique τ can cope with the totality of tasks of a given type T—its range of success is usually called the *scope* of τ. If, for example, it is clear that elementary techniques for factoring numbers all have a limited scope, it is true also, for obvious reasons, that *any* technique whatsoever eventually reaches its limits.

Let us take another example, that of a technique for finding the quotient of number a by number b (with $a, b \in \mathbb{N}^*$), which we make explicit on a specimen. Considering that $12 = 2 \times 2 \times 3$, in order to arrive at the quotient of 417 by 12, we first determine the quotient of 417 by 2, which is the quotient of 416 by 2, i.e. 208. We then calculate the quotient of 208 by 2, which is simply 104; and finally we determine the quotient of 104 by 3, which is the same as the quotient of 102 by 3, or 34. The quotient of 417 divided by 12 is "therefore" 34. (Indeed, $417 = 34 \times 12 + 9$.) The inverted commas that surround *therefore* hint at the fact that many people—including mathematics teachers—will highly doubt the validity of this technique, on the grounds that it leads one to carelessly get rid of successive remainders. This paves the way for another key notion that ATD hinges on: the notion of *technology*. This word is used in ATD with its etymological value: as the suffix *-logy* indicates, a technology is a "discourse" on a given technique τ. This discourse is supposed, at least in the best-case scenario, both to *justify* the technique τ as a valid way of performing tasks t of type T and to throw light on the logic and workings of that technique, making it at least partially intelligible to the user. As concerns the technique of division shown above, it seems difficult to hit upon a full-fledged technology that would justify it, let alone *explain* it—if the technique is duly valid, *why* is it so? For lack of space, we shall leave these two mathematical tasks—justify and explain the aforementioned technique—to the perplexed reader.

A key point must be stressed. Owing to the presence of the suffix *-logy*, the word *technology* carries with it the idea of a *rational* discourse (about some *tekhne*—a Greek word meaning "a system or method of making or doing", that is, a technique or system of techniques). In the universe of ATD, there is no such thing as *universal* rationality. Every person x, every institution I, and every position p has its own rationality, afforded by the technologies present in its "praxeological equipment." Of course, persons and institutions strive to indulge their "rationality" or even to impose it upon others. The interplay between competing rationalities is a major aspect of what it is the mission of didactics to explore.

We have now arrived at a crossroads. It appears that no technological justification is self-sufficient: it relies on elements of knowledge of a higher level of generality, which, whenever they do not go unnoticed—they often do—, sound more abstract, more ethereal, oftentimes abstruse, as if they expressed the point of view of a far removed, pure spectator—a *theoros*. In ATD, such items of knowledge, sometimes dubbed "principles" (or "postulates", etc.), compose the *theory* that goes with the triple formed by the *type of tasks* T, *technique* τ, and *technology*. This theoretical component is denoted by the letter Θ ("big theta") while the technology is denoted by (small) θ. We thus arrive at a quadruple traditionally denoted by $[T / \tau / \theta / \Theta]$. It is this quadruple that we call *a* praxeology; it is called a *punctual* praxeology because it is organised around the type of tasks T, considered as a "point".

It should be clear that, by its very definition, ATD's notion of theory already subsumes case 3 of the English *Wiktionary*'s definition of *theory*: "The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]." Let us take a step forward. A central tenet of ATD is that all "knowledge" can be modelled in terms of praxeologies. The "praxeological equipment" of a person x or institutional position p is defined to be the more or less integrated system of *all* the praxeologies that the person x or a person x' in position p can draw upon to do what this person is led to do. A praxeology can be denoted by the letter $\wp$ (called "Weierstrass p"). It can be construed as the union of two parts or "blocks": the *praxis* part $\Pi = [T / \tau]$, also called the *practico-technical* block, and the *logos* part $\Lambda = [\theta / \Theta]$ or *technologico-theoretical* block. One can write: $\wp = \Pi \oplus \Lambda = [T / \tau] \oplus [\theta / \Theta] = [T / \tau / \theta / \Theta]$. The operation $\oplus$ is sometimes called the *amalgamation* of the *praxis* and *logos* parts. The amalgamation of Π and Λ should be interpreted as a *dialectic* process of "sublation" [2] through which the *praxis* and *logos* parts are at the same time negated as isolated parts but preserved as partial elements in a synthesis, which is the praxeology $\wp$. Let us for a moment relabel "knowledge part" the *logos* part and "know-how part" the *praxis* part of a praxeology $\wp$. The dialectic sublation of "knowledge" and "know-how" that $\wp$ is supposed to achieve is hardly ever actualized. More often than not, the *praxis* and the *logos* observable in a person's or institutional position's praxeological equipment do not fit well together. The *praxis* block may be poorly developed while the *logos* part seems to be ahead of the game—a state of things often expressed by saying something like "he knows the theory, but can't apply it." Or the *praxis* part seems to be going smoothly but the *logos* part is so poor that it fails to substantially explain or justify the featured technique, which is consequently turned into a mere "recipe." The failure to arrive at a "well-balanced" praxeology is the rule, not the exception—a key phenomenon that we will now dwell upon.

INCOMPLETENESS AND IMPLICITNESS IN PRAXEOLOGIES

When it comes to discussing praxeological matters, people are prone to using metonymies or, more precisely, synecdoches [3]. This synecdochic bent generally selects as a derived name some (supposedly) "noble" part or feature of the thing to name. The widely shared propensity to metonymize shows up in particular in the use of the word *knowledge*—which is the "lofty" part of a praxeology—to name the *whole* praxeology. It is even more manifest in the generalized use of *theory* as including not only what ATD calls *technology*, but also the *praxis* part and, therefore, the whole praxeological matter. In common parlance, *theory* refers usually, though somewhat fuzzily, to a complex of praxeologies sharing a common "theory" (in a sense acknowledged by the naming institutions). Such a "body of knowledge" can be denoted by the formula $[T_{ij} / \tau_{ij} / \theta_i / \Theta]$ with $i = 1, ..., n$ and $j = 1, ..., m_i$, where the theory Θ "governs" all the technologies θ_i, each technology θ_i "governing" in turn the techniques τ_{ij}. Such a praxeology goes by the name of *global* praxeology. It is this generic analysis that ATD offers when one comes to speak of, for instance, "group theory" or "number theory" or "chaos theory" or "knot theory," etc. It is to be observed that, in doing so, the praxeological complex to which one refers is defined "in intension" rather than "in extension." It allows one to identify conceptually the *possible* content of the praxeological complex, while its real "extension" remains somewhat unspecified. Of course, it is risky to be so unmethodical when it comes to describing praxeological organisations. Naming a part to mean the whole leads to forget or neglect other parts. Therefore, the resulting praxeologies cannot be efficient tools for action—just as a car stripped down to the engine is of little avail to travel (even if, again metonymically, "motor" can be used to refer to the whole car).

This is however one aspect only of the problem of incompleteness in praxeologies. Any praxeology whatsoever can be said to be incomplete, be it technically, technologically or theoretically. And it is the fate of all praxeologies to continually go through a process which can further the development of any of their constituent parts: the technique can be further "technicized", the technology "technologized", and the theory "theorized". Consider the following easy example relating to the century-old "rule of three", that of the so-called "unitary method", which L. C. Pascoe in his *Arithmetic* (1971) introduces as "helpful to those who initially have difficulties with the ideas of ratios" (p. 64). Traditional arithmetical techniques were essentially *oral*: to do mathematics, one had to *say* something, in order to arrive at the sought-for result. For instance, if it is known that 132 tickets cost £165, how much will be paid for 183 tickets? The right "saying" goes somewhat as follows [4]: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £$\frac{165}{132}$; and 183 tickets cost 183 times *more*, or £$\frac{165}{132} \times 183$, that is £228.75." Here the type of tasks T is clearly delineated; and so is the propounded technique τ_0. As is often the case with arithmetic, the technology θ of τ is essentially embodied in the "technical discourse" above, that both activates τ and explains—makes plain—its logic, thereby justifying it. As always, the "justifying efficacy" of θ depends much on the apparent "naturalness" of the supposedly self-evident reasoning conveyed by the technical discourse recited (if n cost p, then 1 costs p/n, etc.). There exist, of course, other techniques. Some centuries ago, people would have said something like "132 is to 165 as 183 is to price p", writing down the "proportion" 132:165::183:p. Using the (technological) assertion that, in such a proportion, the product of the "means" (i.e. 165 and 183) equals the product of the "extremes" (i.e. 132 and p), they would have arrived at the equation $165 \times 183 = 132 \times p$, which gives $p = \frac{165 \times 183}{132}$. This formula appears to agree with the one found using τ_0, provided one knows the (technological) equality $\frac{a \times c}{b} = \frac{a}{b} \times c$. But this age-old technique τ_{-1} was technologically—not technically—*more demanding*, because the reason why the key technological assertion (about means and extremes) is true remains hidden—which, for most users, turns τ_{-1} into a recipe.

The technique τ_0 can be modified in (at least) two subtly different ways. One consists in introducing an easy technological notion from daily life, that of *unit price*, which leads to a technical variant of τ_0: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £$\frac{165}{132}$, that is £1.25; and m tickets will cost m times *more*, or £$1.25 \times m$." This technical variant τ_{01} is a little bit more complex technically (by contrast, τ_0 skips the calculation of the unit price, though the technological concept of unit price is already implicitly present); but it provides more *technological comfort* to the layman. Another variant results from a decisive *theoretical* change. While people generally understand the expression "number of times" as referring to a *whole* number of times, as was the case in the tickets problem, a major advance in the history of numbers consisted in regarding *fractions* as true numbers, on a par with what came to be called *natural* numbers—fractional numbers being called by contrast *artificial* numbers. A second step forward, not yet taken by so many people, consists in extending the scope of the expression "number of times" to include *fractional* numbers, so that, for instance, 183 is $\frac{183}{132}$ times 132 (i.e. $183 = 132 \times \frac{183}{132}$), from which it follows that the price of 183 tickets is $\frac{183}{132}$ times the price of 132 tickets, or $\frac{183}{132}$ times £165, that is £165 × $\frac{183}{132}$ (which is yet another resolvent). As long as one accepts to think in terms of *fractional* number of *times*, we have a new technique, τ_{02}, much more powerful and comfortable than τ_0 or τ_{01}. Knowing for instance that the price of 2988 tickets is £3735, we can now say that the price of 2012 tickets will be $\frac{2012}{2988}$ times the price of 2988 tickets, i.e. £3735 × $\frac{2012}{2988}$; etc. While the variation leading to τ_{01} only called for a rather easy modification in the technique's technological environment, here the change affects the *theory* itself, which in turn leads to a new technological concept, that of a fractional number of times.

In mathematics as well as the sciences, praxeologies turn out to be no less incomplete than in other fields of human activity. Many aspects of a praxeology's incompleteness are in fact linked to the impression of "naturalness" that so many people feel when they use (or even observe) this praxeology. Of course, the notion of naturalness undergoes institutional variations—let alone personal interpretations. But it is too often assumed that what is natural is, by definition, an unalterable given that does not have to be "justified." This, of course, runs contrary to the scientific tradition, of which it is the ambition to unveil the figments of institutional or personal imagination. Thus the French mathematician Henri Poincaré (1902, p. 74) regarded the principle of mathematical induction as "imposed upon us with such a force that we could not conceive of a contrary proposition." However, almost at the same time, progress in mathematics showed that this supposedly self-existent principle could be derived from the well-ordering principle [4]. The same phenomenon had happened more than two centuries earlier. The leading character was then John Wallis. According to Fauvel, Flood, and Wilson (2013), here is what happened:

> On the evening of 11 July 1663, he lectured in Oxford on Euclid's parallel postulate, and presented a seductive argument purporting to derive it from Euclid's other axioms. As Wallis observed, his argument assumes that similar figures can take different sizes. Wallis found this assumption very plausible, and if it were true then the parallel postulate would be a consequence of the other axioms of Euclid. It does, however, imply a remarkable result: in any geometry in which the parallel postulate does not hold, that similar figures would have to be identical in size as well as in shape, and so scale copies could never be made. (pp. 129–130)

Seventy years later, Girolamo Saccheri was to observe that Wallis "needed only to assume the existence of two triangles, whose angles were equal each to each and sides unequal" (Bonola, 1955, p. 29). Wallis's proof of the parallel postulate [5] opened the way to a major change that we can subsume under a broader historical pattern. By making *explicit* a theoretical property of Euclidean space—"To every figure there exists a similar figure of arbitrary magnitude" (Bonola, 1955, p. 15)—, Wallis reduced the incompleteness (in ATD's sense) of Euclidean geometry as a praxeological field. But he contributed much more to the mathematical sciences: he discovered a *constraint* that, until then, had been taken for granted (and thus ignored) and which turned out to be crucial in the development of geometry, in that it drew a clear demarcation line between Euclidean geometry and the yet to come non-Euclidean geometries.

At this point, we must introduce another key notion of ATD: that of *condition*, stealthily used in the behavioural sciences (through the idea of conditioning or being conditioned) and akin to more widespread notions such as *cause*, *variable*, and *factor*. Didactics is defined in ATD as the science of the conditions of diffusion of knowledge to persons and within institutions. More generally, ATD views any science—including mathematics—as studying a certain kind of conditions with a bearing on human life and its environments. In this respect, given an institutional position p, it is usual (and useful) to distinguish, among the set of conditions considered, those that could be *modified* by the people occupying position p, and those which cannot be altered by these people (though they could be modified by those in some position $p' \neq p$). Any science seeks to accrue knowledge and know-how in order to make the most of prevailing conditions and, in the case of constraints, to create new positions for which these constraints become modifiable conditions. Now, before doing so, it is necessary to *identify* such conditions and constraints, and this is precisely what happens in the Wallis episode, where the Euclidean constraint of invariance by similarity is brought out as a key theoretical property. At the same time, revealing some constraint usually brings forth alternative conditions that had gone unnoticed until then—non-Euclideanism, in the case at hand—and which become new objects of study. It must be stressed here that a science does not know in advance the complete set of conditions and constraints it has to cope with: constructing this set is, by nature, a never-ending task. All these considerations extend to any field of activity, whose praxeological equipments are the outcomes of facing sui generis conditions and constraints. We have now arrived at a position where it makes sense to revert to the question from which we started.

WHAT IS A THEORY?

It must be emphasized here that the interrelated notions of technique, technology and theory do not refer so much to "things" as to *functions*. A technique is a construct which, under appropriate conditions, performs a determined function—the technical function. The same may be said about technology and theory, which respectively perform the technological and theoretical functions. Up to a point, these last two functions look weakly distinguishable—indeed, any contrastive definition is sure to be plagued with counterexamples. Obviously, there are some general criteria allowing one to discern the technological from the theoretical: the first of them is regarded as more concrete, more specific and straightforward, while the second one is approached as being more abstract, more general, more meditative and far-fetched, as if it were reminiscent of its origins. In addition, as has been already highlighted, in an intellectual tradition that has persisted to this day, the second one is valued more highly than the other is. However, these considerations may impede the recognition of an essential phenomenon: the use which is often made of words like *theory* refers to the *explicit* aspects of an entity which we described as definitely subjected to inexplicitness and incompleteness.

From the point of view of ATD, it appears that the technological and theoretical components of a praxeological organisation—that is to say, its *logos* part—are almost always misidentified, because the usual view of them tends to focus on their "explicit" part, which looks generally pretentious and assumptive. This tendency clearly shows through the case 2 of the definition of *theory* given by the English *Wiktionary*: "(sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed." This of course is representative of a dominant theory about... theories. Moreover, *theory* is often liberally used to label what boils down to a few guidelines or precepts which, taken together, do *not* function as the theory of any clearly identified object; for a theory should always be a theory *of something*, built around the scientific ambition to study this "something".

The metonymic use of *theory* is no problem in itself: when one says that ATD is a theory of "the didactic", *theory* refers, as is usual in mathematics for example, to the whole of a praxeological field. But it is a symptom of our propensity to give the word free rein with the uneasy consequence that the debate on theory is deprived of its object. By contrast, ATD conduces to focus the research effort on examining the implicit, unassuming or even wanting parts of technologies and theories. It then appears that a theory is made up of two main components, that we may call its "emerged part" and "immersed part". To avoid engaging here in a titanic work, we summarize in two points the constant lesson that praxeological analysis consistently teaches us. Firstly, the immersed part of a theory—in mathematics and, as far as we know, elsewhere—is replete with inexplicit tenets that are necessary to keep the emerged part afloat. Secondly, these tenets have surreptitious, far-reaching consequences, which often go unnoticed and usually unexplained at both the technological and the technical levels. What people do and how they do it owes much to "thoughts" unknown to them—unknown, not unknowable.

In ATD a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is in truth the current state of a dialectic process of theorisation of which it offers an instantaneous and partial view that may prove delusive. The study and exploration of a theory is tantamount to furthering the very process of theorisation. One main feature of this process is that it allows for the expansion of too often ad hoc, punctual praxeologies $[T / \tau / \theta / \Theta]$ into deeply-rooted global praxeologies $[T_{ij} / \tau_{ij} / \theta_i / \Theta]$. The process of theorisation, as well as the networking of theorisations, has thus a liberating effect, in which, by the way, the use of well-chosen terms and symbolic notations helps achieve mental hygiene and theoretical clarity in bringing about what Bachelard once called the asceticism of abstract thought.

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ENDNOTES

1. See for instance the list proposed on the page at http://www.collinsdictionary.com/dictionary/english/theory.

2. The word *sublation* is the traditional rendering in English of Hegel's notion of *Aufhebung*. According to *Wikipedia* ("*Aufheben*", n.d.), "in sublation, a term or concept is both preserved and changed through its dialectical interplay with another term or concept. Sublation is the motor by which the dialectic functions."

3. A synecdoche is a phrase in which a part of something is used in order to refer to the whole of it.

4. See at http://en.wikipedia.org/wiki/Mathematical_induction#Equivalence_with_the_well-ordering_principle

5. For Wallis's proof in modern form, see, for example, Martin, 1975, pp. 273–274.

Discriminatory networks in mathematics education research

Russell Dudley-Smith

University College London, UCL Institute of Education, London, UK, rdudleysmith@ioe.ac.uk

This paper is written in an organisational language developed in the context of mathematics education by Dowling (2009, 2013) – social activity method (SAM) – as a commentary on Radford's (2008, 2014) discussion of theoretical networking. An exemplar is given of SAM's approach of recontextualising, and thus learning from, what it finds of interest elsewhere – here, Chevallard's Anthropological Theory of the Didactic (ATD). The approach puts emphasis on the autonomy and emergent quality of well-formed research activity. SAM is not, however, solipsistic: it is designed to recursively self-organise in relation to what it encounters elsewhere but on the explicit basis of its own principles. By biasing a reading of ATD, SAM's organisational language develops in the form of a discriminatory research network.

Keywords: Anthropological Theory of the Didactic, deformance, discriminatory research networks, recontextualisation, Social Activity Method.

INTRODUCTION

Writing about theoretical networking presents a formidable challenge. This paper looks at the relation between two research programmes in the domain of mathematics education research, Social Activity Method (Dowling, 1998, 2009, 2013 – hereafter SAM) and the Anthropological Theory of the Didactic (hereafter ATD; Bosch & Gascón, 2014) together with one meta-theory of theoretical networking (Radford, 2008, 2014). This already involves three specialised assemblages of principles and tacit knowledges: to introduce all three would exceed the space available. This limitation is addressed by considering the other approaches as an illustration of how, *from SAM's point of view,* theoretical dialogue might be achieved. For this reason, it is the principles of SAM that are given most emphasis: these are then used to *select* principles from the other approaches. This means that the principles of ATD and Radford's meta-theory must, fundamentally, be misread – what I shall refer to as a (I hope, productive) *deformance* (Dowling, 2009) of them.

SAM has in common with some other research in mathematics education an interest in the specificity of social activity in the context in which it is produced and reproduced (see especially Dreyfus & Kidron, 2014, p. 87). Its focus is on the strategies that lead to emergent alliance; ordered relations of action in which people may come to recognise themselves as working together e.g. as *in SAM* or *ATD.* I first introduce the central *Domains of Action Schema* of SAM. This provides principles for further application of the method in forming a regard on both ATD and Radford's work. One part of this schema – the esoteric domain – is then considered in greater detail to allow a discussion of the continuities and discontinuities between SAM and ATD. A new schema is then generated to bias a reading of ATD from the regard of SAM.

The question I address is: what can a strongly institutionalised research programme in mathematics education, SAM, make of another such strongly institutionalised approach, ATD? How does this allow *SAM* to learn and thus deform itself? It needs the greatest emphasis that SAM *makes no assumptions at all about what ATD might or might not learn* because SAM assembles only its own principles. From the point of view elaborated here there can be no literal connection of similars: any metonymic chain between signifiers of two research programmes involves *recontextualising work.* A secondary question is: what light does this shed on the need for meta-theories to conceptualise theoretical networking such as the one proposed by Radford?

For the purpose of clarity and to summarise the position and rationale of the paper: well-formed research activities are incommensurable – they are emergent and not graspable as such, even by themselves. The

	Content (signifieds)	
Expression (signifiers)	I+	I-
I+	*esoteric*	*descriptive*
I-	*expressive*	*public*

Figure 1: Domains of Action (from Dowling 2009, p. 206)

term "continuity" between theories can refer only to those metonymic chains of signifiers that are of interest *to the recontextualising regard of the theory in question* – hence also the possibility of discontinuity. To claim otherwise, I argue, is counter to a fundamental socio-semantic principle: that sense is made locally in the context of an assembled practice not outside of it. There is, therefore, no possibility of "connection" in terms of similar "component parts". Such a claim would also involve an infinite regress: the notion of similarities or points of contact between theories begs the question of what is the theory that allows such similarity to be discerned. I formalise this as a *general argument* later in this paper.

INTRODUCING THE DOMAIN OF ACTION SCHEMA

The Public Domain

Radford's (2008, 2014) discussion of "*networking theories*" in mathematics education research recontextualises some aspects of Lotman's (2001) *semiotics* to introduce "the *semiosphere* as a theory networking space". Of particular interest is the resulting delimitation of theoretical work as "*bounded*" by the principles that grant its "*autonomy*". Radford (2008, p. 319) produces a description of the mathematics research semiosphere that is in "constant motion"; accelerating as information is transmitted and received with new technologies. Autonomy of a theory within the semiosphere is given by a *hierarchical* order of *principles, methodology* and *research questions* in which the *system* (Radford, 2008, p. 320) of principles is in regulative control. The potential for networking theories is then a question of their *closeness of principle*. Some theories are too far apart to work well together, others may have surprising affinities yet to be articulated. Generally, we may be experiencing a drifting apart: networking might stabilise this, at least for a time.

This paper is written in SAM: the selection of, and extracts from, Radford's paper are motivated by its common interest in the terms given emphasis in the paragraph above such as boundedness and system. But these are expressions not specialised in SAM; and neither is their content – see the axes of Figure 1 below. This schema relates expression to content but moves beyond semiotics by further taking into account whether this is specialised or unspecialised in the *institutionalisation* of the activity. My summary of Radford's position is in the *public domain* of SAM – involving unspecialised, weakly institutionalised (I-) expression and content (Dowling, 2009, p. 206) *from the regard of SAM*. Radford's language is a highly specialised one in its own terms; but these specialised terms – and the way in which they interlink – are not recruited in the institutionalisation (denoted I+) of SAM. Figure 1 expresses SAM's *self-reference*: as a research activity it articulates specialised expression and content in its *esoteric* domain, for example "domain of action".

The Esoteric Domain

A central concept in Radford's work is that of autonomy. I will retain this word but *recontextualise it* from SAM's regard in the *esoteric domain* of SAM. Figure 1 schematises this as a socio-semantics rather than a semiosphere – *institutionalisation* (recognisable regularity of practice) occurring as research *activity* where flows of strategic semiosis (gestures, images, words) are *assembled* in more or less stabilised emergent *alliances*. The principles of action in the esoteric domain regulate what can be recognised/realised in the public domain. Weakly institutionalised terms such as autonomy and semiosphere are alienated in favour of I+ terms such as those given emphasis in this section. This is a *deformance*: the "encounter" (Radford, 2008, p. 317) read through the principles of SAM. Yet the *expressive domain* ensures that self-reference need not become solipsism: the "identity" (Radford, 2008, p. 319) of the self-reference changes in its engagement with the other.

The Expressive Domain

The deformance involved in expressive domain action can be illustrated with respect to the expression "networking theories". *(a) Network*. Eco (1984, p. 81)

	Semiotic Mode	
Mode of Action	Discursive	Non-Discursive
Interpretative	*theorem/enunciation*	*template/graph*
Procedural	*procedure/protocol*	*operational matrix/operation*

Figure 2: Modalities of the Esoteric Domain Apparatus (Dowling, 2013, p. 333)

characterises the semiosphere (in his terms the *global semantic universe*) as a *labyrinthine rhizomatic net.*

> The main feature of a net is that every point can be connected with every other point, and where the connections are not yet designed, they are, however, conceivable and designable. A net is an unlimited territory [...] the abstract model of a net has neither a center nor an outside. (Eco, 1984, p. 81).

A network is not a net (fishing, internet, tuber or any other). The metaphoric expression nonetheless points to potentially productive specialised content. Perhaps its most significant aspect is that a network cannot be described as a whole or from a global point of view; because any attempt at such a description is immediately re-inscribed as new connectivity. The concept of connectivity here is semiotic: the deferred and anticipatory action of one signifier on another. This occurs *even if they are the same.* For example, the signifier <institutionalisation> in SAM points to the schematised content of Figure 1. The same signifier is part of ATD; but its sense there – forged in dialogue with Mary Douglas' *How Institutions Think* – is not organised as a relational space. No *literal* connection of this similar is then possible, only a transformative one. (*b*) *Theory.* At the nodes of the network, Radford has "theories". Radford is careful not to impart undue reliance on the discursive by putting emphasis on the composition of the "triplet" to include principles, methodology and the "template" of research questions. Yet from SAM's regard there is still some danger of the term being read as implying potential representational adequacy (the global all-seeing net). For this reason the phrase *research activity* or *approach* has been preferred to allow full room for the "practice turn" in theory.

ASSEMBLAGES OF MATHEMATICAL MODES

In its most recent development SAM has considered the esoteric domain of school mathematics to be constituted as an *assemblage* of strategies, a term recontextualised from Deleuze (Deleuze & Parnet, 2007 [1997], p. 69; Turnbull, 2000, p. 44). As a sociology, SAM is concerned with the distributional consequences of the ways alliances emerge through strategic action in the social: these indicate (never quite fix) the norms of who can say, think, or do what; here in school mathematics.

An assemblage is specified by SAM as a relational schema – Figure 2 – that can be contingently recruited in the (re)production of school mathematics. The dimension *semiotic mode* distinguishes discursive (explicitly articulated principles, methods and symbols, for example formulae) from non-discursive modes of mathematical engagement (diagrams, or equipment such as a pair of compasses). The dimension *mode of action* opposes interpretative and procedural activity: in the former case where there is work to be done in making sense of the semiotic mode (formulae, diagrams), in the latter case where there are rules or sequences to be followed (discursively ordered algorithms, non-discursive techniques for manipulating the compasses or computer software appropriately). This establishes four *general* strategies: *template, operational matrix, procedure and theorem.* Further, the second term of each strategy in the table denotes *local* rather than generalising action.

The schema suggests competence in that discipline (or anything else) is not *acquired as such* but is constituted by the development of a pragmatic ability to *contingently* deploy an effectively inter-linked mixture of strategies in local context – upon which action the assemblage and those whose alliances will be distributed by it will develop or change. SAM therefore has no "epistemological" concerns in contrast, for example, to ATD. Figure 2 is an introduction to the technology for generating empirical description in SAM – see the many further schemas in (Dowling, 2009). These pin down *modes of action.* This is not a speculative space: it arose from an empirical engagement with a number of mathematical settings (Dowling, 2013). For recent further work in SAM see (Burke, Jablonka, & Olley, 2014; Dowling, 2014; Dudley-Smith, 2015; Burke, 2015).

	Discursive Saturation	
Mode of Action	DS-	DS+
Operationalising	*technique* (τ)	*technology*(θ)
Orientation	*type of tasks* (*T*)	*theory* (Θ)
	skill	***discourse***

Figure 3: Praxeological Modes

A RECONTEXTUALISATION OF ATD

Dowling (2014, p. 528) has noted that Chevallard's work also makes use of a "complementary" concept of recontextualisation – didactic transposition – although with a primary focus on the contextualisation of *cultural* sense-making in pedagogic settings. The schema of the assemblage is potentially in dialogue with ATD's vision of schools as providers of *discoveries* along the way of *research and study paths* (Chevallard, 2015) contingent to the opening up of a body of questions found to be of interest as the research unfolds. In what follows the "amalgam" of *praxeologies* (Artigue et al., 2011, p. 2) is recontextualised by SAM in a *deformative re-ordering*.

Consider the praxeological components [T/τ/θ/Θ] of type of tasks, technique, technology and theory (Artigue, Bosch, & Gascon, 2011; Chevallard & Bosch, 2014). ATD notices a key relation between *praxis* and *logos*: one of both imbrication and tension. Yet empirical studies have shown that this is sometimes denied: thus, for example, in the university some action (Bosch, 2014) is seen to hive off [θ/Θ] from [T/τ]. This has proved a highly fruitful distinction: thus, for example, Job and Schneider (2014) use this framework to make a productive separation of the pragmatic praxeology of the development of calculus and the rather monumentalising deductive praxeology of analysis imposed on mathematics undergraduates – with school mathematics very much a hotchpotch of both. However, in ATD the amalgam [T/τ/θ/Θ] is conceived as containing "ingredients" (Artique et al., 2011, p. 3) – suggesting to a casual reader elements to be enumerated. From SAM's regard a prophylactic against such a misreading is to suggest that the idea of a praxeology can be schematised.

First, it is possible to distinguish what I will call *operationalising* and *orientation.* Orientation concerns what one is about in a specific context: practically as embodied in *type of tasks*, logo-centrically as informed by a *theory*. The former involves low discursive saturation (DS-) as it is embedded in the situated interests or (Maussian) habitus of context. The latter is discursively saturated (DS+), i.e., context free. Operationalising involves techniques – in SAM's terminology "DS- skills" or ways of doing – as well as DS+ "technological discourse" (Bosch & Gascón, 2014, p. 69).

In Figure 3 this produces four *strategies* rather than components. In SAM's research activity the development of schemas such as Figure 3 allows a *particular kind* of regulated engagement with the empirical (without exclusion of others such as ATD). One orienting strategic mode of this is given discursively by the *theory*-logos Θ; self-referentially in SAM's case, particularly the semiotics imbricated in the *raison d'être* of the operationalising *technology*-logos θ of its schemas. Yet much is tacitly acquired: the DS- orientation of SAM's emergent *type of tasks T* – a concern with emergent alliance, the strategies achieved that enable a stabilised commonality of action – is difficult to explain to novitiates outside a context of apprenticeship. Operationalising is also composed of strategies of practical *technique* τ. Certainly these can be aggregated in homology with ATD: the DS- modes identified by ATD as [T/τ] can be identified as *skill,* the DS+ strategies of [θ/Θ] as *discourse* (Dowling, 2009, p. 95); but the recontextualisation now sees each as a strategic mode rather than an element of an amalgam.

Once relationised in this way, SAM and ATD (from the deforming regard of SAM) have the same objective: to describe the empirical deployment of strategies in the assemblage of Figure 2 *and* in the praxeological modes of Figure 3. These common objectives are not *translatable* but they are *transformable.* Specifying the schema of Figure 3 allows a development *of SAM.* The insistence that this is a recontextualisation preserves the autonomy of ATD. Both may then point – in a dialogue of potential complementarity – to the principles for a resistance to the closed and syncretic esoteric domains typical of school disciplinary subjects precisely of the kind Job and Schneider (2014) identify. In learning it is then both *operationalising* and the *orientation* of the student to the regularities of practice *in both the DS- and DS+* that would establish apprenticeship

In ATD the theory of didactic transposition acknowledges that school is a specific context of pedagogic relations. In SAM this is expressed as a matter of recontextualising action conceived as a general socio-semantic process of structuration, i.e. in constituting the esoteric domain of a specialised social activity such as school mathematics as a cultural arbitrary. Both approaches put great emphasis on an interrogation of how mathematics is institutionalised differently according to circumstance. Thus in recent programmes for ATD (Chevallard & Bosch, 2014) the T of *the current milieu of the student* (in reference to its sociality outside the school) is given appropriate emphasis – this is so often tragically downplayed by policy makers. As Radford (2008, p. 322) observes, research questions derive from the principles that allow their articulation. The focus in ATD is on the provision of appropriate activity (and the elimination of the inappropriate) to open to the student the possibilities of what mathematics *might become* for the student in their specific context. To ATD the school may (and often does) block this possibility but this is incidental to the possibility. SAM also sees content as constituted through institutionalisation; within the research programme identified by Jablonka, Wagner and Walshaw (2013), the content of school mathematics is itself always-already recruited in processes of social reproduction – the *particular alliances* (and, of course, oppositions) formed in the schoolroom always *different* to those formed in research (for example, mathematics research).

GENERAL ARGUMENT

This paper has considered the way in which SAM might stand in productive relation to other theoretical frameworks and to itself. From the autonomous and self-referential regard of SAM this must be a matter of the principle of recontextualisation, as that is what organises its regard. The self-reference is fundamental; but it is not a solipsism unless foolishly demanding that its categories replace all others to totalise the net. Both development and renewal are possible via an openness to the empirical and to theoretical antecedents. The following *general argument* rejects the idea that there is a "landscape of strategies for connecting theoretical approaches" (Prediger, Bikner-Ahsbahs, & Arzarello, 2008, p. 170) in favour of the deformative determination of autonomous self-reference. To formalise the situation, let the operator → refer to the recontextualising regard of an approach, ABC, to mathematics education research. Further, let Δ refer to its development, ⇒ denote consequence and let ES_i be a particular empirical setting:

> If one has SAM → ES_i and, elsewhere, ABC → ES_i then recognition of commonality would require a *general unifying framework*, GUF, such that GUF → (SAM → ES_i, ABC → ES_i) to integrate an answer to "perspectives *of what?*". This would deny that ES_i is constituted as an artefact of SAM or of ABC (a refutation of this denial is the many (justified) observations in *Networking of Theories as a Research Practice in Mathematics Education* (Prediger & Bikner-Ahsbahs, 2014) that the data was not collected appropriately for the theoretical framework concerned). Rather networking occurs as SAM → (ABC → ES_i) ⇒ ΔSAM with possible *answerability* of the form ABC → (SAM → (ABC → ES_i)) ⇒ ΔABC *& etc*. In each case the recontextualisation is either misrecognised through literalised equivalence (including elements of "similarity") or constituted as a deformative chiasmus (Merleau-Ponty, 1968), that is, realised as (re)new(ed) embodied practice in response to the objectifying regard of the other. For obvious reasons SAM cannot totally catch its own tail: SAM → (SAM → ES_i) also ⇒ ΔSAM; hence the importance of the dialogic (even if with yourself), a potentially unlimited recursion (or freedom).

In terms of their key diagram (Prediger & Bikner-Ahsbahs, 2014, p. 119), there is no role here for understanding, comparison, synthesis or integration, no "relationships between parts of theoretical approaches" (ibid, 118). It is not a question of attempting to find "similarities and differences" (ibid, 119) but to be open to deformative encounters – allowing these to prompt further self-organisation. It is the possibility of *complementarity*, not commonalities, that defeats "isolation", and the principle of *recontextualisation* that annihilates "global unifiers" who put forward GUFs. In Lotman's (2001, p. 143) semiotics, as in SAM's social-semantics, the principle of *asymmetry* is paramount – information-enriching activity *deforms*.

CONCLUSION – SOME GENERAL THEMES

The main theme of the paper has been to provide a framework which allows a discussion of the continuities and discontinuities between SAM and ATD. It sets out an agenda for a commentary SAM → (ATD → ESi) in invitation of a counter-commentary from ATD. The framework claims *autonomy* in the regards of ATD and SAM, but also the possibility of dialogue as described in the *general argument*.

I have to this point left one consequence of this implicit. There is a need to *take mathematics out of the theoretical framework of mathematics education research*. From SAM's regard, mathematics (however institutionalised) is the *empirical setting* of research. Yet, contrary to both SAM and ATD, many research programmes seem to wish to make it part of their *theoretical framework* by including considerations from a (notional) *mathematics-itself*. For SAM, the separation is required because the truth claims of a particular practice (for example, the often rather strange modalities of school mathematics) have their *own specificity*. A further consequence arises from the relationality of SAM's approach. The coherence of a theoretical framework is not a matter of the signification of individual theoretical terms; as if these can be translated by single substitutes to stand on their own account – and thus be 'connected' *as such*, or be absorbed into another theory. A theory's coherence rests on the relationality of its content, not on a collection of atomised concepts. The *general argument* above suggests the importance of dialogue between the esoteric domains of autonomous research activities. The development takes place as a coherent deformance of the principles that enabled a particular position in argument. Above all, therefore, we should see theoretical frameworks as a space for the becoming of the subjectivity of the individuated researcher. As such they must *de-stabilise* existing identities in order to forge new ones. The development of a good research programme will offer the potential subject of research action an on-going deformance of their own certainties.

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Secondary mathematics teacher candidates' pedagogical content knowledge and the challenges to measure it

F. Güneş Ertaş and Fatma Aslan-Tutak

Bogazici University, İstanbul, Turkey, gunesertas@gmail.com, fatma.tutak@boun.edu.tr

In this paper, the authors will discuss pedagogical content knowledge of secondary mathematics teacher candidates in Turkey. The discussion is based on comparisons between senior students from secondary mathematics education and mathematics departments in terms of their pedagogical content knowledge measured by Teacher Education and Development Study in Mathematics (TEDS-M) released items. In addition to comparison of two groups, there will be a discussion on the challenges to measure pedagogical content knowledge.

Keywords: Pedagogical content knowledge, TEDS-M, measuring PCK, secondary school, mathematics teacher candidates.

INTRODUCTION

Teacher knowledge and its components have been described and modelled in different ways by different researchers (Shulman, 1986; Ball, Thames, & Phelps, 2008; Franke & Fennema, 1992; Tatto et al., 2008). However, it can be said that many teacher knowledge approaches have been influenced by the Shulman's (1986) model of teacher knowledge. Shulman made an important contribution by categorizing teacher content knowledge as Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). As Petrou and Goulding (2011) stated, in the Shulman model, the most influential category was the new concept of PCK. Shulman (1986) described PCK as "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 9). According to him as the requirement of PCK, teachers need to know using representations, illustrations, analogies, and demonstrations and also giving examples and explaining concepts in order to make them understandable.

Shulman's conceptualization of teacher knowledge provided a basis for research field of mathematics education. The knowledge that mathematics teachers need to acquire for teaching was described with the Mathematical Knowledge for Teaching (MKT) model, which is the refinement of Shulman's categorization (Ball et al., 2008). MKT model categorizes SMK and PCK into six subcomponents. Ball (2003) defined the subcomponents of PCK by reconsidering Shulman's categorization. The components are Knowledge of Content and Students, Knowledge of Content and Teaching and Knowledge of Curriculum.

Although the MKT model has been widely used, there is some criticism about it. This model was developed considering elementary and middle school mathematics teachers, but not secondary. Therefore, it is argued that the components of MKT do not meet the mathematical need for secondary mathematics teachers (Zazkis & Leikin, 2010). The claim is that "the higher the level taught, the more the teacher needs to know" (Usiskin, 2001, p. 86), so the nature of mathematics that secondary teachers need to know is at the much higher level than elementary teachers. According to Zazkis and Leikin (2010), Advanced Mathematical Knowledge (AMK), which is defined as knowledge of subject matter acquired during undergraduate studies at universities, is necessary knowledge for teaching mathematics at secondary level. It can then be said that, since generally SMK is prerequisite for PCK (Shulman, 1986) specifically at secondary level, AMK is also necessary for PCK. However, it is not sufficient because PCK includes the knowledge of content and teaching, the knowledge of content and students, and the knowledge of curriculum (Ball et al., 2008). Therefore, classroom experiences and practices are also important for the development of PCK. Researchers argue that there is an interaction between SMK, PCK, beliefs and practices (Franke & Fennema, 1992; Walshaw, 2012; Türnüklü, 2005). However, PCK has a special importance because it is influenced by all the others: SMK, practice and belief. It can be said that PCK has a multidimensional nature. Wilson (2007) claims that this complex nature makes it difficult to investigate PCK by using efficient measures. Even though developing scalable efficient measures for content knowledge for teaching is difficult (Wilson, 2007), researchers tried to develop rigorous, effective and valid instruments to measure mathematics teachers' knowledge (Hill, Schilling, & Ball, 2004; Krauss, Baumert, & Blum, 2008; Tatto et al., 2008).

One of the instruments to study mathematics teacher knowledge is the Teacher Education and Development Study in Mathematics (TEDS-M) measure. TEDS-M is a cross-national study in which 17 countries participated, even if Turkey was not involved. The characteristics that differentiate TEDS-M measure from others are to consider both primary and secondary levels and to be designed for international usage and national adaptations. Differences in students' achievement level in Trends in International Mathematics and Science Study (TIMSS) encouraged researchers to study on teacher education internationally in order to investigate how mathematics teaching quality differs across countries. Therefore, the TEDS-M measure was developed to examine the mathematical knowledge for teaching of future mathematics teachers, based on TIMSS 2007 framework of content areas and cognitive domains. By considering such characteristics of the measure, in this study, TEDS-M secondary released items were used for the investigation of the mathematical knowledge for teaching of secondary mathematics teacher candidates.

METHODS

Participants

In Turkey, both graduates of secondary mathematics teacher education departments and mathematics departments (after completing teaching certificate program) have chance to be mathematics teachers in secondary schools. Therefore, the participants of the study were senior students from secondary mathematics teacher education departments (n = 47) and senior student from mathematics departments (n = 48) of two universities in Istanbul. Totally, 32 females and 15 males senior secondary mathematics education students (the mean age is 24) and 35 females and 13 males students (the mean age is 22) from mathematics departments participated in this study. These two universities were ranked as first and second among the secondary mathematics education departments in the national university entrance exam. In the first ranked university, students enrol mathematics and secondary mathematics education programs by getting similar scores from university entrance exam. In the second university, the minimum score of secondary mathematics education department is a little higher than the one of the mathematics department.

These two programs have different curriculum in undergraduate education programs. The secondary mathematics education program includes 50 % of content knowledge and skills, 30 % of professional teaching knowledge and skills and 20 % of general knowledge courses (YÖK, 2007). However, the undergraduate program in a mathematics department consists of 70 % of content knowledge and 30 % of general knowledge. Moreover, participants of the study were asked to explain whether they had an informal teaching experience like tutoring or teaching in cram school. As they stated, 76 % of secondary mathematics education students and 70 % of mathematics students had informal teaching experiences.

Instrument

The instrument was designed by TEDS-M researchers considering the framework of Trends in International Mathematics and Science Study (TIMSS) 2007 (Tatto et al., 2008). MCK items comprised of four content areas: number, algebra, geometry and data, and three cognitive dimensions: knowing, applying and reasoning. Furthermore, MPCK items consist of two parts: knowledge of curricula planning, and interactive knowledge about how to enact mathematics for teaching and learning. These were aligned with the PCK domains in literature. Table 1 and Table 2 show the distributions of MCK and MPCK items according to their content, cognitive domains and PCK components. (In the appendix, Figures 1 and 2 are examples of MCK items and Figures 3 and 4 are examples of MPCK items.)

These items include 23 mathematics content knowledge (MCK) and 9 mathematics pedagogical content knowledge (MPCK) items with three different item

Cognitive Domain	Content Domain				
	Algebra	Geometry	Number	Data	Total
Knowing	-	2	4	-	6
Applying	5	4	-	1	10
Reasoning	2	1	4	-	7
Total	7	7	8	1	23

Table 1: MCK Secondary Items

	Content Domain				
	Algebra	Geometry	Number	Data	Total
Curriculum and Planning	4	-	-	-	4
Enacting	1	-	3	1	5
Total	5	0	3	1	9

Table 2:. MPCK Secondary Items

formats: multiple choice, complex multiple choice and open constructed response.

In order to compare MKT of participants who were studying in different departments, Turkish translated versions of TEDS-M secondary level released items were used. The method which was used while translating the instrument consists of three phases. Firstly, items were translated in Turkish by the researcher who is fluent in English. The translated items were reviewed by a mathematics educator who is an expert in the content area and fluent in English, a three-year experienced mathematics teacher who is fluent in English, and a professional translator. At the second phase, the original tests were administered to a group of pre-service mathematics teachers who are native in Turkish and fluent in English. The same group took the translated versions of the tests three weeks apart. At the last phase, the method of back translation was used to check the quality of translation and to investigate the linguistic or conceptual errors in translation. It was also used to consider particular attention to sensitive translation problems across cultural correspondence of the two versions.

Data collection and analysis

The data was collected from participants in a single point in different times. Instrument administered to senior students during the last two weeks of the spring semester of the 2012–2013 academic year just before they graduate.

After data collection, all items were scored according to the scoring guide of TEDS-M Secondary Items. Participants' scores acquired from 23 MCK items were calculated and called as MCK scores and scores obtained from 9 MPCK items were calculated and called as MPCK scores. Total scores of participants were also calculated by the summation of MCK and MPCK scores.

The scores of these two groups of participants were compared by using appropriate statistical methods. For total scores and MCK scores comparisons, an independent sample t-test was used since all the assumptions were met. For the comparison of MPCK scores, a non-parametric Mann-Whitney U test was used since the normality assumption was violated.

RESULTS

Participants' scores obtained from the 47 senior students from the mathematics teacher education department and the 48 senior students from the mathematics department were compared. Table 3 shows means and standard deviations of the two groups of participants.

		M	*SD*
Total	Math Teacher Education	26.83	3.96
	Math	23.63	4.42
MCK	Math Teacher Education	20.45	3.35
	Math	17.50	3.80
MPCK	Math Teacher Education	6.38	1.19
	Math	6.13	1.35

Table 3: Means and Standard Deviations

The results of the t-test indicate that the mean of the total score of mathematics teacher education students is significantly 3.2 points higher than those from mathematics department, $t\,(93) = 3.72$, $p < .001$ and Cohen's $d = .76$ with the marginal large effect size (Cohen, 1988). Furthermore, the independent sample t-test results show that students from the mathematics teacher education department have significantly higher MCK scores than those of mathematics departments: $t\,(93) = 4.00$, $p < .001$, and Cohen's $d = .82$ with the large effect size. Moreover, according to the non-parametric Mann-Whitney U test, there is no significant difference between them in relation to the MPCK scores: $Z = 1.00$, $p > .05$.

DISCUSSION AND CONCLUSION

The study aimed at comparing the mathematical knowledge for teaching of students who will graduate from a mathematics teacher education department and others who will graduate from a mathematics department. In Turkey, graduates of both departments have a chance to become mathematics teachers at secondary level, but graduates of the mathematics departments need to take a teaching certificate. However, the knowledge and skills that graduates are able to acquire through these programs are different from each other. For example, the contents of undergraduate education programs of these departments are notably different. The mathematics departments' program does not include any pedagogy or education course, but more advanced mathematics courses than the mathematics education departments' program does. Therefore, the result was unexpected: mathematics students, who were not required to take any teaching related courses, were not significantly different from students of the mathematics teacher education in terms of MPCK scores.

This unexpected result may be explained by discussing the nature of PCK for secondary level mathematics teaching. Even though teacher education programs are the most influential factors that affect PCK of teacher candidates, there are other factors when the nature of PCK is considered. PCK includes knowledge of "the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986, p. 9). It may be conceptualized as not only knowledge of students' thinking and conceptions, but also knowledge of explanations, representations and alternative definitions of mathematical concepts, and knowledge of multiple solutions to mathematical tasks (Shulman, 1986; Ball et al., 2008; Krauss, Baumert, & Blum, 2008). Therefore, teaching experiences play an important role in the development of teachers' PCK (Ball et al., 2008). Because of this, teacher education programs include many teaching experiences opportunities like field experience and practicum. Moreover, both groups of students who were studying mathematics teacher education in a mathematics department had informal teaching experiences like tutoring and teaching in cram school. Having this kind of teaching experience may explain the result. However, this may not be the only rationale. Measuring and assessing PCK is another issue that should be considered by focusing on its nature in order to explain the results of study.

Achieving the specialized knowledge for teaching mathematics at secondary level requires Advanced Mathematical Knowledge (AMK) which is defined as the knowledge of the subject matter acquired at the university (Zazkis & Leikin, 2010). Mathematics departments' students take many advanced mathematics courses and they develop AMK. It should be noted that AMK is a necessary but not sufficient condition for achieving the specialized knowledge for teaching at secondary level (Zazkis & Leikin, 2010).

Therefore, as it is seen, according to the PCK's multidimensional nature, deep mathematical knowledge plays an important role because it can provide teachers to use effective explanations, representations and alternative definitions. These components may contribute to make an explanation for the unexpected result of the study. For example, when MPCK items were examined according to required knowledge, and skills were needed to provide a correct answer, the need for AMK might be observed. For instance, one of the questions of the instrument (see Figure 3 in appendix) asks to determine what knowledge is needed to prove the quadratic formula. This question measures knowledge of content and teaching, but without knowing how to prove quadratic formula it is not possible to give a correct answer. Therefore, it is not easy to differentiate and measure this kind of knowledge and skills. Difficulty in measuring PCK may explain the unexpected result that there is no difference in MPCK scores between two groups of students.

Moreover, in this study, PCK was tried to be measured by few items (4 questions, 8 items). Therefore,

Questions (Items)	Content Domain	PCK Domain	Intended Ability
1 (b)	Algebra	Enacting	Analyse why one word problem is more difficult than the other.
6 (a, b, c)	Number	Enacting	Determine whether student's response is valid proof.
9 (a, b, c, d)	Algebra	Curriculum and Planning	Determine what knowledge is needed to prove the quadratic formula.
12 (b)	Data	Enacting	Explain student's thinking about histogram.

Table 4: TEDS-M Secondary PCK Items' Characteristics

only some domains of PCK and some abilities were able to be measured with these items. However, as Shulman (1986) and Ball and colleagues (2008) stated, PCK requires different kinds of knowledge, tasks and skills. This instrument can only address some of them. Table 2 shows the distribution of content and PCK domains of items, and Table 4 above shows the intended abilities for each of them.

The reactions of the two groups of participants to these PCK items are different. For example, item 9b (Figure 3 in the appendix) were answered correctly by 97 % of the mathematics department students and by 86 % of the students from the secondary mathematics education department. On the other hand, 72 % of the students from the secondary mathematics education department answered item 1b (Figure 4 in appendix) correctly, while only 52 % of the mathematics department student gave a correct response.

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APPENDIX

Determine whether each of the following is an irrational number always, sometimes or never.

Check one box in each row.

		Always	Sometimes	Never
A.	The result of dividing the circumference of a circle by its diameter.	☐ 1	☐ 2	☐ 3
C.	The diagonal of a square with side of length 1.	☐ 1	☐ 2	☐ 3
D.	Result of dividing 22 by 7.	☐ 1	☐ 2	☐ 3

Figure 1: An example of TEDS-M Secondary MCK items (Number, Knowing)

Prove the following statement:

If the graphs of linear functions

$$f(x) = ax + b \text{ and } g(x) = cx + d$$

intersect at a point P on the x-axis, the graph of their sum function

$$(f+g)(x)$$

must also go through P.

Figure 2: An example of TEDS-M Secondary MCK items (Algebra, Reasoning)

A mathematics teacher wants to show some <lower secondary school> students how to prove the quadratic formula.

Determine whether each of the following types of knowledge is needed in order to understand a proof of this result.

Check one box in each row.

		Needed	Not needed
A.	How to solve linear equations.	☐ 1	☐ 2
B.	How to solve equations of the form $x^2 = k$, where $k > 0$.	☐ 1	☐ 2
C.	How to complete the square of a trinomial.	☐ 1	☐ 2
D.	How to add and subtract complex numbers.	☐ 1	☐ 2

Figure 3: An example of TEDS-M Secondary MPCK items (Algebra, Planning)

(b) Typically Problem 2 is more difficult than Problem 1 for <lower secondary> students. Give one reason that might account for the difference in difficulty level.

Figure 4: An example of TEDS-M Secondary MPCK items (Algebra, Enacting)

The epistemological dimension in didactics: Two problematic issues

Ignasi Florensa[1], Marianna Bosch[2] and Josep Gascón[3]

1 Escola Univ. Salesiana de Sarrià (EUSS), Departament de Mecànica, Barcelona, Spain, iflorensa@euss.es
2 Universitat Ramon Llull, IQS School of Management, Barcelona, Spain, marianna.bosch@iqs.url.edu
3 Universitat Autònoma de Barcelona, Departament de Matemàtiques, Barcelona, Spain, gascon@mat.uab.cat

This paper presents some theoretical considerations concerning the relationships between epistemology and didactics. We distinguish two big issues that show the mutual enrichment of both fields. On the one hand, considering teaching and learning phenomena as part of the empirical basis of epistemology enables proposing new epistemological models of mathematical bodies of knowledge. On the other hand, these epistemological models provide guidelines for the design and analysis of new teaching proposals, which, in turn, show the constraints coming from the spontaneous epistemologies in school institutions. Some critical open questions derived from these issues draw up the guidelines for a future research programme.

Keywords: Epistemology, didactics, ATD, praxeologies.

PEDAGOGY, EPISTEMOLOGY AND DIDACTICS

What distinguishes didactics of mathematics (or of any other field of knowledge) from general education or pedagogy is the status given to the knowledge or "content" that is taught and learnt. Pedagogy considers the knowledge to be taught as a given, and focuses on the best, conditions or practices to teach and learn it: the knowledge is not problematic, the relationships of the students to it are (Chevallard, 2000). In contraposition, didactics locates the *epistemological problem* at the core of the analysis. A double assumption is meant by this. First, the phenomena underlying teaching and learning processes, at school as well as in other social institutions, are closely dependent on the content that is designed to be taught, actually taught and learnt, and also on how this content is considered by the participants of the teaching and learning process. Second, that the study of these phenomena is also strongly dependent on the way knowledge is considered and modelled by researchers in didactics. In fact, the main point of the paper is to describe a research programme that seeks to clarify the problem of how to teach (and learn) mathematics and its relationship with the problem of what is considered as mathematics.

The pedagogical dimension of teaching and learning phenomena refers to generic practices, discourses, strategies and regularities that can be described regardless of the content to be taught. The didactic dimension is reached when the concrete mathematical activities organised by the teacher and carried out by the students, as well as any other fact affecting the delimitation, construction, management, evolution and assessment of these activities are considered.

The need to integrate both the pedagogical and the epistemological dimensions was one of the main motivations for Guy Brousseau to promote the construction of a new field of knowledge called didactics of mathematics and which he contributed to with the first formulations of the theory of didactic situations during the decade 1970–1980 (Brousseau, 2002). We are here pointing out two main reasons for this integration that will later on be at the basis of the problematic issues we wish to raise. The first one is the dependence between the dominant *epistemologies of mathematics* (or of any other field of knowledge) at an educational institution and the way teaching is organised in this institution. In other words, the way mathematics and its specific bodies of knowledge are considered in a given institution, usually as implicit assumptions, affects the conditions established for its learning. In this sense, we can say, rephrasing Brousseau (2002), that *teaching organisations* are supported by *spontaneous epistemologies* appearing to the subjects of the school institution as the unquestionable and transparent way to conceive the content to be taught.

The second main reason, also put forward by Brousseau, is related to the implementation of research results wishing to improve teaching and learning. Whatever general strategies or conditions we may find at the pedagogical level, teachers will always have to specify them in terms of what ties them to the students: the knowledge-based learning activities. Of course, it is possible to delimit general pedagogical phenomena affecting any content to be taught and to propose general pedagogical actions in order to improve teaching and learning processes. However, eventually, these actions will need to be concretized and converted into didactic facts and strategies, that is, to specific ways of organizing mathematical contents and designing mathematical activities for the students.

Once the necessity to integrate the "epistemological problem" into the "teaching and learning problem" is assumed, there are different levels to take the epistemic dimension of the teaching and learning process into account. In some cases, the focus can be on a given piece of knowledge ("proportionality", "limits of functions", "linear equations"), or a whole area ("algebra", "calculus", "statistics"), thus considering specific models relying on a more or less explicit conception of what mathematics is and how it can be described. Therefore it can be said that the consideration of the *didactic problem* needs to include, in one way or another, a specific answer to the *epistemological problem*. To interpret the interrelation between epistemological and didactical problems, we now present a historical development of the object of study of each discipline and their respective empirical basis.

THE EVOLUTION OF EPISTEMOLOGICAL AND TEACHING PROBLEMS

In a previous study, Gascón (2001) describes a rational reconstruction of the evolution of the *epistemological problem* and, in parallel, the evolution of the *didactic problem* showing that a certain convergence exists between them. The evolution of the epistemological problem can be interpreted as a successive expansion of what is considered as the object of study and of the consequent empirical basis used to approach it. Briefly speaking, this work shows that the nature of the epistemological problem began as a purely logical problem (EP^1), became a historical problem (EP^2) and ended up being considered, at the end of the last century, as an essentially cognitive problem (EP^3). Its successive formulations together with its corresponding tentative answers can be outlined as follows:

> EP^1: How to stop infinite regress to get a logical justification of mathematical theories?
>
> EPA^1: Euclidean models: logicism (Russell), formalism (Hilbert) and intuitionism (Brouwer).
>
> EP^2: What is the logic of the development of mathematical discovery?
>
> EPA^2: Quasi-empirical models (Lakatos)
>
> EP^3: What are the tools and mechanisms found in history and psychogenesis of the development of mathematical discovery?
>
> EPA^3: Constructivist models (Piaget & García, 1982).

This evolution of the epistemological problem can be interpreted as a progressive detachment from logical procedures and an approximation to empirical sciences such as history and psychology. This expansion continues since the 70s, with the inclusion of sociological data. Indeed, sociologists such as Barry Barnes and David Bloor, and later others like Bruno Latour, heavily influenced by the ideas of Thomas Kuhn, tried to highlight the essential social nature of scientific research. Let us notice, however, that, apart from Kuhn's mention of the "textbooks epistemology" (Kuhn, 1971), none of these approaches seem to consider empirical phenomena related to the teaching, learning and disseminating of mathematics. The division between pedagogy and epistemology appears to be taken for granted in this research domain also.

All of the epistemological models above can be related to general teaching models, ranging from *theoricism* (organizing the teaching of mathematics following the logic construction of concepts) and *technicism* (exerciting the main techniques in a given domain without many theoretical tools), to *constructivism*, which aims to enable students to construct knowledge according to certain predetermined stages.

Gascón (1993) shows limitations of the empirical basis used by constructivism to address the epistemological problem. Taking into account personal psychogenesis data, in some sense completed with those provided by the history of science, it does not integrate *didactic facts* and, can thus hardly explain institutional-depending phenomena as the so-called "personal" construction of knowledge. In other words, and according to Chevallard (1991), the study of the *genesis and development* of knowledge (traditional object of epistemology) cannot be separated from the study of the *diffusion, use and transposition* of knowledge (object of study of didactics).

It is at this point where both problems, the epistemological and the didactic one, converge, with the consequently significant expansion of the object of study of both disciplines. Historically, this time corresponds to the first formulations of the theory of didactic situations (TDS) proposed by Guy Brousseau in the early 1970s (Brousseau, 2002). It is no coincidence that at this early stage of didactics of mathematics, Brousseau initially considered to name this new discipline "experimental epistemology". In particular, didactics of mathematics accepted the responsibility to elaborate and use epistemological models of mathematical bodies of knowledge as a new way to study didactic phenomena, thus turning the *pedagogical problem* into an *epistemological-didactic* one.

New questions arise from this perspective: What new general epistemological theories, based on which empirical data, may serve to support new teaching organizations in order to overcome the limitations of the current ones? To what extent and by what means can the dominant spontaneous epistemologies in a teaching institution be changed in solidarity with the teaching models based on them?

AN ANSWER TO THE EPISTEMOLOGICAL-DIDACTIC PROBLEM

The anthropological theory of the didactic (ATD), following the research programme initiated by the theory of didactic situations, considers a specific model of mathematical knowledge and its evolution formulated in terms of a dynamical sequence of *praxeologies*. Praxeologies are entities formed by the inseparable combination of a *praxis* or know-how made of types of tasks and techniques, and of a *logos* or knowledge consisting of a discourse aiming at describing, explaining and justifying the *praxis* (Chevallard, 2000). In didactics research, mathematical praxeologies are described using data from the different institutions participating in the didactic transposition process, thus including historical, semiotic and sociological research, assuming the institutionalized and socially articulated nature of praxeologies. Furthermore, a dialogue with the APOS theory shows how data interpreted as the different *levels of development of schemes* by psychogenetic developments, can be reformulated in ATD in terms of the *institutional evolution of praxeologies* (Trigueros, Bosch, & Gascón, 2011).

Reference epistemological models as sequences of praxeologies

To describe and analyse the specific contents that are at the core of teaching and learning processes, the general model in terms of praxeologies is structured in an articulated set of specific models of the different areas of the mathematical activity at stake called *reference epistemological models* (REM) (Barbé, Bosch, Gascón, & Espinoza, 2005; Bosch & Gascón, 2006). The Reference Epistemological Model of a body of knowledge is an alternative description of that body of knowledge elaborated by researchers in order to question and provide answers to didactic facts and problematic aspects taking place in a given institution. This REM prevent researchers to take for granted how this body of knowledge is conceived in the institution considered. For instance, Ruiz-Munzón (2010) and Ruíz-Munzon, Bosch and Gascón (2013) present a REM about elementary algebra which is used to analyse the status and role of this area of school mathematics in relation to arithmetic and functional modelling. The model takes into account the processes of didactic transposition to explain what is currently taught as algebra at school and provides a rationale to this area that does not coincide with the official and more limited one assigned by the educational system. Some of the difficulties in the teaching and learning of elementary algebra can then be referred to these limitations and new teaching proposals can be designed to overcome them (Ruiz-Munzón, 2010; Bosch, 2012).

In this REM, algebra is interpreted as a *tool for modelling* any type of (mathematical and extra-mathematical) systems and the process of *algebraization* is divided into three stages. The first one concerns the passage from the execution of computation programmes (sequences of arithmetic operations on numbers like the ones carried out when solving an arithmetic problem) to the written or rhetoric description of their structure; the second stage requires the symbolic manipulation of the global structure of written computation programmes (not only simplifying and developing, but also "cancelling", etc.); at the third stage, the whole manipulation of formulas is reached.

It is important to note that this REM is not a static description of a piece of mathematical knowledge, it also suggests a dynamical process to introduce elementary algebra: starting from the study of arithmetic computation programmes (CP) in order to motivate the entrance into the second stage of algebraization by the limitations of the rhetorical formulation of CPs in the first stage. Encountering problematic questions in this arithmetical work with CP may generate the need to build a written symbolic *formulation* of these CP to globally manipulate their structure, thus promoting the need to establish symbolic codes (hierarchy of operations and bracket rules).

In a similar way, different REM of other specific areas of mathematics have been proposed, all formulated in terms of sequences of related praxeologies: limits of functions (Barbé et al., 2005), proportionality (García, Gascón, Ruiz Higuera, & Bosch, 2006; Hersant, 2001), measure of quantities (Chambris, 2010), real numbers (Bergé, 2008; Rittaud & Vivier, 2013), among others. In general, the organisation of a teaching process based on the REM of a given mathematical content is called *research and study activities*.

From teaching of contents to enquiry processes: Study and research paths

These reference epistemological models correspond to previously established bodies of mathematical knowledge: algebra, limits, proportionality, etc. They provisionally assume the delimitations of mathematical knowledge provided by the school and the scholarly institutions, which are then often redefined. In order to also take into account enquiry processes that start with the consideration of problematic questions to be solved (instead of pre-established contents to be learnt), REM have been enriched with the proposal of the *Herbartian schema* (Chevallard, 2006; Chevallard, 2015). This scheme is a useful tool to observe, analyse and evaluate existing and potential didactic processes that start with the consideration of a generating question and evolve with the search of partially available answers ("contents" to be learnt) and the construction of new answers through the interaction with a milieu. The study of a specific *question* leads to a rooted-tree of derived questions and provisional answers, which outlines the generating power of the initial question and the possible paths to follow. We thus obtain new reference epistemological models assigned to problematic questions instead of pre-established praxeological contents. Winsløw, Matheron, and Mercier (2013) provide several examples of this kind of rooted-tree REM, such as the dynamics of a population or the trajectory of a three-point shot in basketball. The enquiry process of a particular generating question materializes in an open didactic organisation called a *study and research path* (SRP). During the development of SRP, the need for new knowledge to solve some of the derived questions found in the path usually leads to the activation of study and research *activities*.

Didactic praxeologies emerging from reference epistemological models

The previous section briefly outlined how the design, implementation and analysis of study and research paths and study and research activities call for the activation of specific didactic techniques and creates new types of didactic tasks. For instance, in the case of elementary algebra illustrated above, the didactic technique proposed by Ruiz-Munzon (2010) consists in introducing the study of "mathemagic" games of the sort "Think of a number, apply these calculations [...], you get 73" as generating questions. How do you explain the magician's trick?" These games generate the need to look for new pieces of answers, in the manipulation of the calculation programmes proposed or in their transformation and generalisation through algebraic symbolism. Questions based on "mathemagic" games allow producing an important number of computation programmes economically. They are presented to the students without much artificiality and their first contact with computation programmes is not problematic. Moreover, the limitations of the rhetorical and numerical formulations of computation programmes inevitably appear and they do so soon enough.

PROBLEMATIC ISSUES

The aim of this paper is to formulate some problematic issues at the crossroads of epistemology and didactics. We will initially explain them within the context of the ATD before extending the questioning to other didactic approaches. If we try to characterize a didactic approach by how "pedagogy" and "mathematics" are integrated, in the case of the ATD such integration can be formulated in terms of two movements. They appear in the design, management and evaluation of teaching and learning processes and can briefly be described as follows:

1) Starting from the analysis of teaching and learning processes at school and considering an empirical basis of study that is large enough to include the processes of didactic transposition, all this empirical work provides tools to design specific REMs for the main mathematical contents or areas that are designed as knowledge to be taught. We can define this movement as "using didactic facts and phenomena to produce epistemological models".

2) Conversely, the principles and criteria that have guided the construction of a REM for a specific area of school mathematical activity and, in particular, the contrast between the rationale assigned by the REM to this area and its official (explicit or tacit) role in school mathematics, all provide some mathematical and didactic tools to design, manage and evaluate teaching and learning processes based on study and research paths sustained by that REM. This movement can be defined as "using the epistemological model as the core of didactic tools".

This double movement raises different open issues which are at the starting point of the research programme we want to propose in this paper.

New didactic needs

We have seen how previously elaborated REM on mathematical contents or problematic questions (obviously complemented with other methodological design tools) can provide criteria for the design and implementation of teaching and learning processes that are considerably different from the existing ones. In principle, they aim at organising activities that should allow the students to carry out new mathematical tasks and techniques in a more autonomous, functional and justified way. The "mathemagic" games in the case of elementary algebra (Ruiz-Munzón, 2010) or the different enquiry processes described in Winsløw, Matheron and Mercier (2013) are good examples of this enrichment. Obviously, these new didactic organisations should be made available to the study community and their viability in different school institutions should be tested.

It is important to emphasize that all didactic approaches and theories are also based on general models of mathematical-didactic activities. These general models are a particular way to interpret the mathematical activity and to conceptualize the study process of mathematics (teaching, learning, diffusion and application). Even though these models are not always clearly spelled out, they remain an essential feature of theoretical approaches, as they strongly affect the type of research problems this approach can formulate. Two crucial questions arise:

1) In the case of ATD, how to transform the REM into possible didactic organisations that could live in current school institutions? How to take into account the interrelation between the REM and the didactic phenomena appearing in the implementation of these new didactic organisations? How to make this process available to the school institutions, especially to the profession of teachers?

2) How is this mutual enrichment between the epistemological and didactic proposals taken into account in other theoretical frameworks?

New epistemological needs

The empirical analysis of the study processes taking place in various institutions (for example, but not exclusively, in schools) clearly shows that the didactic praxeologies are closely related to the epistemological tools available in the institution to describe and manage the mathematical praxeologies. For example, in the institutions where the dominant model is Euclidean, teaching and learning processes are conceived and described in terms of didactic activities around "definitions", "concepts", "theorems", "proofs" and "applications". In addition, these didactic activities tend to be hierarchically structured according to the logical construction of mathematical concepts (real numbers before limits, limits before derivatives, etc.).

If, instead of analysing traditional teaching processes, we look at those based on didactic research, the situation is very similar: how didactic processes and the dynamics of mathematical praxeologies are designed, described and managed also depends on the tools provided by the epistemological model which upholds, more or less explicitly, the didactic approach considered. The further this research-based epistemological model is from the dominant epistemological model at schools and scholarly institutions, the more difficult it becomes for teachers to carry out innovative teaching proposals designed within this frame.

In all these cases, the most remarkable feature is the shortage and inadequacy of tools available in the teaching institution to describe, manage, and evaluate the dynamics of mathematical activity. This lack of tools could in the first place be attributed to the scarcity of spontaneous epistemological models and, in particular, to the shortage of the Euclidean epistemological model of mathematics whose supremacy is still present, to a greater or lesser extent, in most institutions.

Which new notions or tools are needed to describe and manage the dynamics of the mathematical activity that will take place in study processes? How to describe these tools depending on the role addressed (didactic researcher, teacher and students)? How to make them available in the teaching institution and to the participants of the didactic process?

The evolution of didactic-epistemological models

In order to establish an alternative and rich REM of a specific mathematical domain or questioning, it is necessary to take into account the didactic phenomena taking place in teaching institutions. This leads to an enrichment of the spontaneous epistemological model during the first design of the REM. However, it is important to keep the process running during the implementation and the evaluation of teaching proposals based on this REM. The consequent evolution of the REM is a clear example of the dynamic and provisional nature of the epistemological models elaborated by didactics, evolving from its initial proposals through the analysis of empirical facts. From a mathematical perspective, these continuous evolutions of the REMs can be seen as the incorporation of new notions and organisations into the field of knowledge. This phenomenon can be related to the transformation of some paramathematical notions into mathematical concepts, as happened with concepts (such as "set", "function", "continuity", "graphs", etc.), a transformation which takes place as long as researchers deal with new problems. For instance, in the case of elementary algebra, the notion of "computation programme" is a new and crucial element of the proposed REM. In the experiences described by Ruiz-Munzón (2010), this notion played a very ambiguous role in the management of the teaching and learning processes, given the fact that it did not belong to the official mathematics to be taught and the teacher did not feel at ease with it. A similar phenomenon happened when implementing SRP on population dynamics with notions such as "quantities", "model", "system", "mixed and separated generations", etc.

Another important and difficult question is the degree of explicitness that should be adopted with the new epistemological models necessary to design, implement and evaluate new teaching and learning processes depending on the participants of the study communities addressed (students, teachers, mathematicians, etc.). Also, what kind of similar experiences can be learnt from other approaches? Did they find similar difficulties? These open questions establish a new research programme where the results of previous investigations carried out within the ATD should be analysed together with analogous research from other perspectives. In all cases, the status given to the epistemological dimension in didactics analysis seems to appear as a crucial question to take into account.

ACKNOWLEDGEMENT

This research has been funded by the project EDU2012-39312-C03-01 of the Spanish Ministry of Economy and Competiveness.

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Linking inquiry and transmission in teaching and learning mathematics

Juan D. Godino, Carmen Batanero, Gustavo R. Cañadas and José M. Contreras

University of Granada, Granada, Spain, jgodino@ugr.es

Different theories assume that learning mathematics should be based on constructivist methods where students inquire problem-situations and assign a facilitator role to the teacher. In a contrasting view other theories advocate for a more central role to the teacher, involving explicit transmission of knowledge and students' active reception. In this paper, we reason that mathematics learning optimization requires adopting an intermediate position between these two extremes models, in recognizing the complex dialectic between students' inquiry and teacher's transmission of mathematical knowledge. We base our position on a model with anthropological and semiotic assumptions about the nature of mathematical objects, as well as the structure of human cognition.

Keywords: Mathematical instruction, inquiry learning, knowledge transmission, onto-semiotic approach, mathematical knowledge.

INTRODUCTION

The debate between the models of a school that "conveys knowledge" and others in which "knowledge is constructed" currently seems to tend towards the latter. This preference can be seen in the curricular guidelines from different countries, which are based on constructivist and socio-constructive theoretical frameworks (NCTM, 2000):

> Students learn more and learn better when they can take control of their learning by defining their goals and monitoring their progress. When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere (p. 20).

In the case of mathematics education, problem solving and "mathematical investigations" are considered essential for both students' mathematical learning and teachers' professional development. Constructivist viewpoints of learning shift the focus towards the *processes* of the discipline, practical work, project implementation and problem solving, rather than prioritizing the study of facts, laws, principles and theories that constitute the body of disciplinary *knowledge*.

Nevertheless, this debate is hiding the fact that students differ in skills and knowledge, and most of them need a strong guidance to learn; even when some students with high skills and knowledge can learn advanced ideas with little or no help. The issue of the type of aid needed, depending on the nature of what is to be built or transmitted is also missed in this debate. Consequently of this situation, the question of the kind of help that a teacher should give to a usually heterogeneous class, when we want students acquire mathematical knowledge, understandings and skills, also arises.

The family of "Inquiry-Based Education" (IBE), "Inquiry-Based Learning" (IBL), and "Problem-Based Learning" (PBL) instructional theories, which postulate the inquiry-based learning with little guidance by the teacher, seem not to take into account the described reality, namely the students' heterogeneity and the variety of knowledge to be studied. These models may be suitable for gifted students, but possibly not for the majority, because the type of help that the teacher can provides could significantly influence the learning, even in talented students.

In this paper, we analyse the need to implement instructional models that articulate a mixture of construction/inquiry and transmission of knowledge to achieve a mathematical instruction that locally optimize learning. The basic assumption is that the moments in which transmission and construction of knowledge can take place are *everywhere dense* in the instructional process. Optimization of learning involves a complex dialectic between the roles of teacher as instructor (transmitter) and facilitator (manager), and student's roles as active constructor of knowledge and receivers of meaningful information. Hiebert and Grouws (2007) state that "because a range of goals might be included in a single lesson, and almost certainly in a multi-lesson unit, the best or most effective teaching method might be a mix of methods, with timely and nimble sifting among them" (p. 374).

We support this mixed model of mathematical instruction in cognitive (architecture of human cognition) and onto-semiotic (regulative nature of mathematical objects) reasons.

Below we first summarize the main features of instructional models based on inquiry and problem solving and secondly of models that attribute a key role to transmission of knowledge. We then present the case for a mixed model that combines dialectically inquiry and transmission, basing on the epistemological and didactical assumptions of the onto-semiotic approach to mathematical knowledge and instruction (Godino, Batanero, & Font, 2007). Finally we include some additional reflections and implications.

INQUIRY AND PROBLEM BASED LEARNING IN MATHEMATICS EDUCATION

As indicated above, the acronyms IBE, IBL, PBL designate instructional theoretical models developed from several disciplines, which have parallel versions for the teaching of experimental sciences (IBSE) and mathematics (IBME). They attributed a key role to solve "real" problems, under a constructivist approach. In some applications to mathematics education it is proposed that students construct knowledge following the lines of work of professional mathematicians themselves. The mathematician faces non-routine problems, explore, search for information, make conjectures, justify and communicate the results to the scientific community; mathematics learning should follows a similar pattern.

Using problem-situations (mathematics applications to everyday life or other fields of knowledge, or problems within the discipline itself) to enable students making sense of the mathematical conceptual structures is considered essential. These problems are the starting point of mathematical practice, so that problem solving activity, including formulation, communication and justification of solutions are keys to developing mathematical competence, i.e. the ability to cope with not routine problems. This is the main objective of the "problem solving" research tradition (Schoenfeld, 1992), whose focus is on the identification of heuristics and metacognitive strategies. It is also essential to other theoretical models such as the Theory of Didactical Situations (TDS) (Brousseau, 1997), and Realistic Mathematics Education (RME) (Freudenthal, 1973; 1991), whose main features are described below.

Theory of Didactical Situation (TDS)

In TDS, problem-situations should be selected in order to optimize the adaptive dimension of learning and students' autonomy. The intended mathematical knowledge should appear as the optimal solution to the problems; it is expected that, by interacting with an appropriate *milieu*, students progressively and collectively build knowledge rejecting or adapting their initial strategies if necessary. According to Brousseau (2002),

> The intellectual work of the student must at times be similar to this scientific activity. Knowing mathematics is not simply learning definitions and theorems in order to recognize when to use and apply them. We know very well that doing mathematics properly implies that one is dealing with problems. We do mathematics only when we are dealing with problems—but we forget at times that solving a problem is only a part of the work; finding good questions is just as important as finding their solutions. A faithful reproduction of a scientific activity by the student would require that she produce, formulate, prove, and construct models, languages, concepts and theories; that she exchange them with other people; that she recognize those which conform to the culture; that she borrow those which are useful to her; and so on. (p. 22).

To allow such activity, the teacher should conceive problem-situations in which they might be interested and ask the students to solve them. The notion of *devolution* is also related to the need for students to consider the problems as if they were their own and take responsibility for solving them. The TDS assumes a strong commitment with mathematical epistemology, as reflected in the meaning attributed to the notion of fundamental situation, which Artigue and Blomhøj (2013, p. 803) describe as "a situation which makes clear the raison d'être of the mathematical knowledge aimed at".

Another important feature of the TDS is the distinction made between different dialectics: action, formulation and validation, which reflect important specificities of mathematical knowledge.

Realistic Mathematics Education (RME)

In RME, principles that clearly correspond to IBME assumptions are assumed. Thus, according to the "activity principle", instead of being receivers of ready-made mathematics, the students, are treated as active participants in the educational process, in which they develop themselves all kinds of mathematical tools and insights. According to Freudenthal (1973), using scientifically structured curricula, in which students are confronted with ready-made mathematics, is an 'anti-didactic inversion.' It is based on the false assumption that the results of mathematical thinking, placed on a subject-matter framework, can be transferred directly to the students. (Van den Heuvel-Panhuizen, 2000).

The principle of reality is oriented in the same direction. As in most approaches to mathematics education, RME aims at enabling students to apply mathematics. The overall goal of mathematics education is making students able to use their mathematical understanding and tools to solve problems. Rather than beginning with specific abstractions or definitions to be applied later, one must start with rich contexts demanding mathematical organization or, in other words, contexts that can be mathematized. Thus, while working on context problems, the students can develop mathematical tools and understanding. The guidance principle stresses also the same ideas. One of Freudenthal's (1991) key principles for mathematics education is that it should give students a "guided" opportunity to "re-invent" mathematics. This implies that, in RME, both the teachers and the educational programs have a crucial role in how students acquire knowledge. According to Artigue and Blomhøj (2013, p. 804), "RME is thus a problem-solving approach to teaching and learning which offers important constructs and experience for conceptualizing IBME".

TRANSMISSION BASED LEARNING IN EDUCATION

We consider as models based on knowledge transmission various forms of educational intervention in which the direct and explicit instruction is highlighted. A characteristic feature of strongly guided instruction is the use of worked examples, while the discovery of the solution to a problem in an information-rich environment is similarly a compendium of discovery learning minimally guided.

For several decades these models were considered as inferior and undesirable regarding to different combinations of constructivist learning (learning with varying degrees of guidance, support or scaffolding), as shown in the initiatives taken in different international projects to promote the various IBSE and IBME modalities (Dorier & Garcia, 2013). Transmission of knowledge by presenting examples of solved problems and the conceptual structures of the discipline is ruled by didactical theories in mathematics education with strong predicament, as mentioned in the previous section.

The uncritical adoption of constructivist pedagogical models can be motivated by the observation of the large amount of knowledge and skills, in particular everyday life concepts, that individuals learn by discovery or immersion in a context However, Sweller, Kirschner and Clark (2007) state that

> There is no theoretical reason to suppose or empirical evidence to support the notion that constructivist teaching procedures based on the manner in which humans acquire biologically primary information will be effective in acquiring the biologically secondary information required by the citizens of an intellectually advanced society. That information requires direct, explicit instruction. (p. 121)

This position is consistent with the argument put forward by Vygotsky; scientific concepts do not develop in the same way that everyday concepts (Vygotsky, 1934). These authors believe that the design of appropriate learning tasks should include providing students an example of a completely solved problem or task, and information on the process used to reach the solution. As Sweller, Kirschner, and Clark (2007) observe, "we must learn domain-specific solutions to specific problems and the best way to acquire domain-specific problem-solving strategies is to be given the problem with its solution, leaving no role for IL [inquiry learning]" (p. 118). According to Sweller et al., empirical research of the last half century on this issue provides clear and overwhelming evidence that minimal guidance during instruction is significantly less effective and efficient than a guide specifically designed to support the cognitive process necessary for learning. According to (Kirschner, Sweller, & Clark, 2006):

> We are skilful in an area because our long-term memory contains huge amounts of information concerning the area. That information permits us to quickly recognize the characteristics of a situation and indicates to us, often unconsciously, what to do and when to do it. (p. 76).

STUDYING MATHEMATICS THROUGH AN INQUIRY AND TRANSMISSION BASED DIDACTICAL MODEL

In the two previous sections we described some basic features of two extreme models for organizing mathematics instruction: discovery learning versus learning based on the reception of knowledge (usually regarded as traditional whole-class expository instruction). In this section, we describe the characteristics of an instructional model in which these two models are combined: the students' investigation of problem-situations with explicit transmission of knowledge by the "teacher system" [1] at critical moments in the mathematical instruction process. We consider that it is necessary to recognize and address the complex dialectic between inquiry and knowledge transmission in learning mathematics. In this dialectic, *dialogue* and *cooperation* between the teacher and the students (and among the students themselves), regarding the situation-problem to solve and the mathematical content involved, can play a key role. In these phases of dialogue and cooperation, moments of transmitting knowledge necessarily happen.

The onto-semiotic complexity of mathematical knowledge and instruction

The semiotic, epistemological and cognitive assumptions of the Onto-semiotic approach to mathematical knowledge and instruction (OSA) (Godino, Batanero, & Font, 2007) are the basis for our instructional proposal, which recognizes a key role to both the inquiry and the transmission of knowledge in the teaching and learning of mathematics (and possibly other disciplines). This model takes into account the nature of mathematical objects involved in mathematical practices whose students' competent performance is intended.

The way a person learns something depends on what has to be learned. According to the OSA, students should appropriate (learn) the onto-semiotic institutional configurations involved in solving the proposed problem-situations. The paradigm of "questioning the world" proposed by the Anthropological Theory of Didactics (TAD) (Chevallard, 2015), and, in general, by IBE models is assumed, so that the starting point should be the selection and inquiry of "good problem-situations."

The key notion of the OSA for modelling knowledge is the *onto-semiotic configuration* (of mathematical practices, objects and processes) in its double version, institutional (epistemic) and (cognitive). In a training process, the student's performance of mathematical practices related to solving certain problems, brings into play a conglomerate of objects and processes whose nature, from the institutional point of view is essentially normative (regulative) (Font, Godino, & Gallardo, 2013) [2]. When the student makes no relevant practices, the teacher should guide him/her to those expected from the institutional point of view. Thus each object type (concepts, languages, propositions, procedures, argumentations) or process (definition, expression, generalization) requires a focus, a moment, in the study process. In particular regulative moments (institutionalization) are *everywhere dense* in the mathematical activity and in the process of study, as well as in the moments of formulation / communication and justification.

Performing mathematical practices involves the intervention of previously known objects to understand the demands of the problem-situation and implementing an initial strategy. Such objects, its rules and conditions of application, must be available in the subject's working memory. Although it is possible that the student him/herself could find such knowledge in the "workspace", there is not always enough time or the student could not succeed; so the teacher and peers can provide invaluable support to avoid frustration and abandonment. These are the moments of remembering and activation of prior knowledge, which are generally required throughout the study process. Remembering moments can be needed not only in the exploratory-investigative phase, but also in the formulation, communication, processing or calculation, and justification of results phases. These moments correspond to acts of knowledge transmission and may be crucial for optimizing learning.

Results of mathematical practices are new emerging objects whose definitions or statements have to be shared and approved within the community at the relevant time of institutionalization carried out by the teacher, which are also acts of knowledge transmission.

Inquiry and transmission didactical moments

Under the OSA framework other theoretical tools to describe and understand the dynamics of mathematics instruction processes have been developed. In particular, the notions of *didactical configuration* and *didactical suitability* (Godino, Contreras, & Font, 2006; Godino, 2011). A didactical configuration is any segment of didactical activity (teaching and learning) between the beginning and the end of solving a task or problem-situation. Figure 1 summarizes the components and the internal dynamics of a didactical configuration, including the students' and the teacher's actions, and the resources to face the joint study of the task.

The problem-situation that delimits a didactical configuration can be made of various subtasks, each of which can be considered as a sub-configuration. In every didactical configuration there is an epistemic configuration (system of institutional mathematical practices, objects and processes), an instructional configuration (system of teacher and learners roles and instructional media), and a cognitive configuration (system of personal mathematical practices, objects and processes) which describe learning. Figure 1 shows the relationships between teaching and learning, as well as with the key processes linked to the onto-semiotic modelling of mathematical knowledge (Font, Godino & Gallardo, 2013; Godino, Font, Wilhelmi, & Lurduy, 2011). Such modelling, together with the teachers and learners roles, and their interaction with technological tools, suggest the complexity of the relationships established within any didactical configuration, which cannot not be reduced to merely inquiry and transmission moments.

SYNTHESIS AND IMPLICATIONS

In this paper, we argued that instructional models based only on inquiry, or only on transmission are simplifications of an extraordinarily complex reality: the teaching and learning processes. As Hiebert and Grouws (2007) write, "classrooms are filled with complex dynamics, and many factors could be responsible

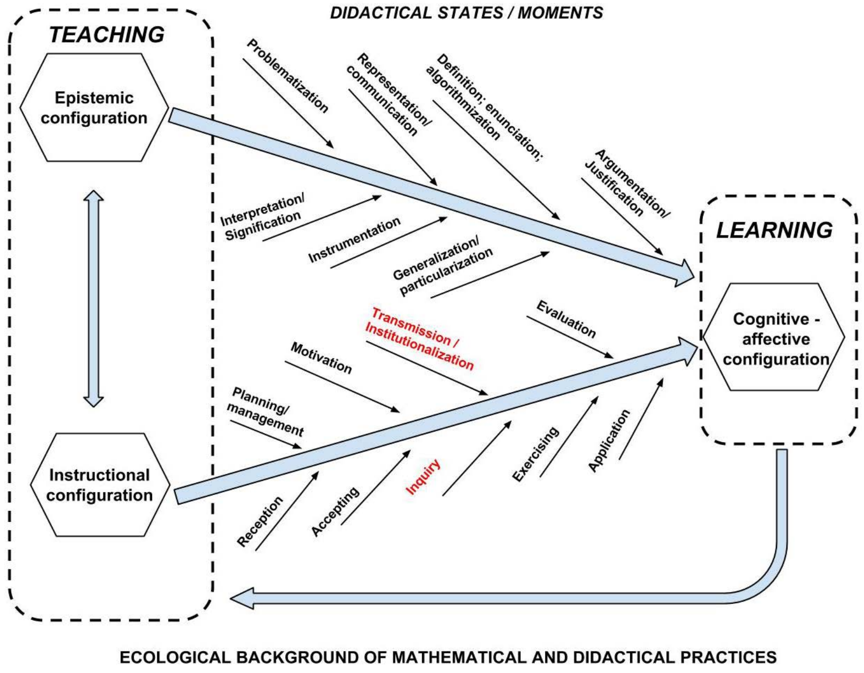

Figure 1: Components and internal dynamic of a didactical configuration

for increased student learning.... This is a very central and difficult question to answer" (p. 371).

Although we need to establish instructional designs based on the use of rich problem-situations, which guide the learning and decision-making at the global and intermediate level, local implementation of didactical systems also requires special attention to managing the students' background needed for solving the problems, and to the systematization of emerging knowledge. Decisions about the type of help needed essentially have a local component, and are mainly teacher's responsibility; he/she needs some guide in making these decisions to optimize the didactical suitability of the study process.

We also have supplemented the cognitive arguments of Kirschner, Sweller, and Clark (2006) in favour of models based on the transmission of knowledge in the case of mathematical learning, with reasons of onto-semiotic nature: What students need to learn are in a great deal, *mathematical rules*, the circumstances of its application and the required conditions for a proper application. The learners start from known rules (concepts, propositions, and procedures) and produce others rules that should be shared and compatible with those already established in the mathematical culture. Such rules (knowledge) must be stored in subject's long term memory and put to work at the right time in the short-term memory.

The scarce dissemination of IBE models in actual classrooms and the persistence of models based on the transmission and reception of knowledge can be explained not only by the teachers' inertia and lack of preparation, but by their perception or experience that the transmission models may be more appropriate to the specific circumstances of their classes. Faced with the dilemma that a majority of students learn nothing, get frustrated and disturb the classroom, it may be reasonable to diminish the learning expectations and prefer that most students learn something, even only routines and algorithms, and some examples to imitate. This may be a reason to support a mixed instructional model that articulates coherently, locally and dialectically inquiry and transmission [3].

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ENDNOTES

1. This system can be an individual teacher, a virtual expert system, or the intervention of a "leader" student in a team working on a collaborative learning format.

2. This view of mathematical knowledge is consistent with that taken by Radford's objectification theory. Radford (2013) writes: "Knowledge, I just argued, is crystallized labor – culturally codified forms of doing, thinking and reflecting. Knowing is, I would like to suggest, the instantiation or actualization of knowledge" (p.16). He adds: "Objectification is the process of recognition of that which objects us – systems of ideas, cultural meanings, forms of thinking, etc." (p. 23). In our case, such crystallized forms of work are conceived as cultural "rules" fixing ways of doing, thinking and saying faced to problem-situations that demand an adaptive response.

3. The research reported in this paper was carried out as part of the research projects EDU2012-31869 and EDU2013-41141-P (Spanish Ministry of Economy and Competitiveness).

Mathematics communication within the frame of supplemental instruction – SOLO and ATD analysis

Annalena Holm and Susanne Pelger

Lund University, Faculty of science, Lund, Sweden, annalena.holm@varberg.se

Teaching at Swedish primary and secondary schools is often combined with collaborative exercises in a variety of subjects. One such method for learning together is Supplemental instruction (SI). Several studies have been made to evaluate SI in universities throughout the world, while at lower levels hardly any study has been made until now. This study aimed at identifying learning conditions in SI-sessions at two Swedish upper secondary schools. Within this study, a combination of ATD (Anthropological theory of the didactic) and the SOLO-taxonomy (Structure of the Observed Learning Outcome) was successfully tried as an analysis strategy.

Keywords: SOLO, ATD, networking, mathematics communication, SI.

INTRODUCTION

The teacher's choice of education methods has a high influence on what students learn (Hattie, 2009), and education research has shown to add to a better understanding of the prospects of successful teaching (Good & Grouws, 1979; Hattie, 2009). In spite of previous educational research, there is no clear answer to the question whether one method has advantages over the other, or if whole-class teaching is more successful than "dialogue-teaching".

To strengthen the findings, researchers have argued that there is a need for more sophisticated research methods (Jakobsson, Mäkitalo, & Säljö, 2009). There is also a need for more systematic connection between various education research theories – so-called networking (Prediger, Bikner-Ahsbahs, & Arzarello, 2008). According to (Prediger et al., 2008), the reasons that theories in mathematics education research have evolved differently are (1) mathematics education is a complex research environment, and (2) various research cultures prioritise different components of this complex field. Different theories and methods have different perspectives and can provide different kinds of knowledge. Thus, different theories and perspectives can connect in different ways.

An educational concept that needs to be explored, and systematically connected with various theories, is the so-called Supplemental instruction (SI). SI is a method where groups of students are provided peer collaborative learning exercises at meetings led by SI-leaders (Hurley, Jacobs, & Gilbert, 2006). The method is used worldwide both at the university level and lower levels. To strengthen students' knowledge in mathematics, a number of upper secondary schools in Sweden have introduced SI as a complement to regular teaching.

AIM

Within the present study, SI-sessions were analysed in upper secondary school. The purpose was gaining more insight into the conditions that may facilitate mathematics learning. For the analyses two frameworks were chosen and tested: (1) the Anthropological theory of the didactic (ATD) with focus on the development of mathematical activities defined in terms of praxeologies (Chevallard, 2015; Winsløw, 2010), and (2) Structure of the Observed Learning Outcome (SOLO), which instead focuses on students' learning outcome quality (Biggs & Collis, 1982). An aim of the study was to explore whether a combination of these two frameworks could contribute to deepen the analysis of the students' discussions. The research questions of this paper are, hence: *To what extent is a combination of SOLO and ATD a suitable strategy for analysing SI-sessions? Are these two frameworks compatible and complementary?*

THEORY AND CONNECTING FRAMEWORKS

Research needs theoretical frameworks. This was stated by (Lester, 2005), who argued that a theoretical framework provides a structure when designing research studies, and that a framework helps us to transcend common sense when analysing data. Below, the two frameworks that have been important for the study are discussed. First, the concept Supplemental instruction is presented. Then follows a section where SOLO-taxonomy and the ATD-praxeology are presented. Finally, possibilities and challenges with combining frameworks are discussed.

Supplemental instruction, or SI, is an educational method used at universities in many countries. Groups of students discuss and solve problems together, and SI is a complement to regular teaching. No teacher is present at the meetings (Malm, Bryngfors, & Mörner, 2012). The groups are instead guided by an older student, who is supposed to provide peer collaborative learning exercises (Hurley et al., 2006). SI has lately been introduced in some upper secondary schools in Sweden. First year students form the groups, while second and third year students serve as SI-leaders (Malm, Mörner, Bryngfors, Edman, & Gustafsson, 2012).

Biggs and Collis (1982) developed the *SOLO-taxonomy* for evaluating learning outcomes for students at tertiary level. SOLO names and distinguishes five levels according to the cognitive processes required to obtain them. The authors argued that SOLO is useful when categorising test results in closed situations with formulated expectations. They used five levels, SOLO-1 to SOLO-5, when categorising student responses (Biggs & Collis, 1982). Later Brabrand and Dahl (2009) used the SOLO-taxonomy for analysing (1) what curricula focus on and (2) what students actually learn. By using so-called active verbs (as shown hereafter), the authors state that it is possible to understand on which level of knowledge the learning outcome is:

SOLO 1 *(pre-structural):* student misses the point

SOLO 2 (*uni-structural*): define, count, name, recite, follow instructions, calculate

SOLO 3 (*multi-structural*): classify,describe, enumerate, list, do algorithm, apply method

SOLO 4 (*relational*): analyse, compare, explain causes, apply theory (to its domain)

SOLO 5 (*extended abstract*): theorize, generalize, hypothesize, predict, judge, reflect, transfer theory (to new domain)

Brabrand and Dahl (2009) conclude that SOLO can be used when analysing science curricula, but they question whether SOLO is a relevant tool when analysing mathematics curricula. They write:

> For mathematics it is usually not until the Ph.D. level that the students reach SOLO 5 and to some extent also SOLO 4. The main reason is that to be able to give a qualified critique of mathematics requires a counter proof or counter example as well as a large overview over mathematics which the students usually do not have before Ph.D. level. (Brabrand & Dahl, 2009, p. 543)

Other researchers, however, claim that SOLO is useful in various contexts. Pegg (2010) has described three studies where SOLO has been used to analyse primary and secondary students' learning mathematics. In addition, Pegg (2010) states that SOLO helps to describe observations of students' mathematics performance. Hattie and Brown (2004) also describe SOLO as a useful tool in mathematics education. They use a strategy where mathematics exercises are formulated by using SOLO, and they claim it is possible to use SOLO when analysing children's mathematics knowledge and when describing the processes involved in asking and answering a question on a scale of increasing difficulty or complexity.

The Anthropological theory of the didactic (ATD) is a research program for analysing and developing mathematics education, which offers a handful of tools (Chevallard, 2006; Winsløw, 2010). One of these is *the notion of praxeology,* and one of the overarching perspectives is *the paradigm of questioning the world*.

While *the paradigm of questioning the world* defines the perspective of the curriculum, the ATD-praxeology makes a helpful tool for analysing the content that is taught. A praxeology can be described as a four-tuple explaining the *components* of activities or knowledge that are taught. This four-tuple consists of: a *type of tasks* (T), a *technique* (τ), a *technology* (θ) and a *theory* (Θ) (Winsløw, 2010). These four constituents, if fully understood and used, can help to analyse what is done at school. The *type of tasks* and the *technique* form the *practice block* or the *know-how*. The *technology* and the *theory* constitute the *theory block* or the *know-why*. Hence, a technique is used to solve a task of a given type, while the technology justifies the technique, and the theory gives a broader understanding of the field. When used to describe bodies of knowledge, praxeologies can refer to "small" as well as "big" fields. Hence, a *point praxeology* is a single type of tasks that is solved by a technique; several point praxeologies can be combined into a local praxeology when they share the same technology and several local praxeologies sharing the same theory can be combined to form a regional praxeology (Winsløw, 2010).

The ATD-praxeology can be applied at various levels of education. Winsløw (2006), for example, discusses how to use the praxeology when studying advanced mathematics, while Barbé, Bosch, Espinoza, and Gascón (2005) suggest how to use ATD when studying classroom activities at upper secondary school. All together ATD is described as a theory which analyses what is taught and thus *showing the shortcomings* or even paradoxes of didactic practices. Winsløw (2010) also states that ATD is useful when proposing ambitious ways to *transform* education.

Different theories have different perspectives and can provide different kinds of knowledge. Looking at the same data from different perspectives can give deeper insights (Prediger et al., 2008). In this study, the ATD and SOLO frameworks were combined in order to study the conditions and outcomes of students' learning through SI. The purpose of combining two frameworks was to catch the advantages of each of them, and hence, to contribute to mathematics education research and networking.

METHOD

This study bases its statements on classroom observations. The phenomenon being studied was students' discussions of mathematics. The context was small groups in upper secondary school. We used a qualitative case study approach (Cohen et al., 2007) to provide an analysis of how the students in the groups dealt with the mathematical problems. (Cohen, Manion, & Morrison, 2007) describe the purpose of a case study to portray, analyse, and interpret situations through accessible accounts. As such, the case study method provided a systematic way of looking in depth, analysing and reporting how students discuss mathematical problems and how the discussions might facilitate learning.

Meetings at two upper secondary schools in south-western Sweden were observed with groups from the humanist, technology and natural science programs. The main criterion for choosing schools was their different experiences of support from the university. Another difference between the two schools was the implementation of SI. The criterion for choosing SI-groups to observe was availability. Not all groups wanted to be observed. Meetings were *videotaped* and the tapes were transcribed. The documents were coded by closed coding, i.e. a deductive *analysis* with codes from theoretical frameworks. During the whole study, the analysis strategy was developed and revised. Due to limited space not all observations can be presented here. For more comprehensive insight in the study see Holm (2014).

The first students to be observed were one group from the technology program and one group from the humanistic program. Both groups discussed the same exercise (see Table 1). The exercise was part of a former national test from the Swedish national agency for education, which in 2010 had been intended for all students in the first grade of Swedish upper secondary school. At these two particular group-sessions no SI-leaders were present as this was a first test of the frameworks. The observed sessions lasted 40 minutes at one school and 60 minutes at the other. The students were not told anything about the SOLO- and ATD-classification of the exercise.

The exercise was *pre-classified* by SOLO and the ATD-praxeology. The intention was (1) to test if it was possible to do this classification in advance before giving the exercise to the students, (2) to decide whether the two frameworks were a suitable choice when analysing student learning outcome, and finally, (3) if it was possible to correlate every SOLO-level to a specific dimension of the ATD-praxeology.

Three different ways of using the SOLO-taxonomy were found in the literature, and initially all three of them were used when classifying the exercise. One of the three was part of the original method defined by Biggs and Collis (1982), with instructions for how to analyse student achievements in elementary mathematics. The authors recommended that the children's solutions were to be analysed by deciding inter alia whether the child can handle several data at the same time and whether the child shows the ability to "hold off actual closures while decisions are made".

A second method was described by Hattie and Brown (2004). They grouped the exercises in advance, so that if a student answered a certain question the student was considered to reach a certain SOLO-level. Finally, Brabrand and Dahl (2009) used the SOLO-taxonomy by the active verbs once formulated by Biggs (2003) and compared university curricula with the table of verbs. Certain verbs were considered to point at certain "intended learning outcomes" in the curricula. Notice that the verb "calculate" and "do simple procedure" are added to SOLO 2. These verbs are mentioned in (Brabrand & Dahl, 2009) and in (Biggs & Tang, 2011). In the results section we explain why not all the three ways of using the SOLO-taxonomy were suitable for the present study.

Although the SOLO-taxonomy is widely used, in different ways, the work done by Biggs and Collis (1982) was based on closed situations, and not open situations, which are one of the main ideas of SI. Thus, it was decided that a complementary framework was needed for this study, specifically designed for mathematics education and also for open situations. Here, the ATD was found a suitable complement to SOLO.

The ATD is widely used, especially within the French, Spanish and Latin-American mathematics education research traditions (Bosch & Gascón, 2006; Chevallard, 2015). It is developed to fit education research in mathematics and other disciplines, and calls for more open situations and open questions at school in general and in school mathematics in particular (Chevallard, 2015). In this study, the analysis and development of open mathematics learning situations was, thus, done by using the ATD-praxeology, while the SOLO-taxonomy was used for the analysis of student learning outcomes.

RESULTS

The initial exercise about the volume of a cylinder was coded before it was given to the students (see Table 1). The SOLO-coding was based on the three methods described above. First, the "Hattie-Brown-method" was used, as it appeared to be near to practice.

Exercise: A roll of paper (statement)	SOLO	ATD praxeology
A rectangular sheet of paper can be rolled to make a tube (cylinder) as shown in the figure.		
Such a tube is made by rolling a square piece of paper with side length 10 cm. **The diameter of the tube will be about 3.2 cm. Find the volume of this tube (cylinder).*	2 /later chang-ed to 3	Technique $\tau 1$ (calculate the volume of a cylinder given its diameter and height)
*Show that the diameter of the tube will be about 3.2 cm if the side length of the sheet of paper used is 10 cm	2/3 /later 3	Technique $\tau 2$ (calculate the diameter of a circle given its perimeter)
If the length and width of the paper are different, you can make two different tubes (cylinders) depending on how you roll the paper. **Starting with rectangular sheets of paper with dimensions 10 cm × 20 cm, two different tubes are made. Find the volumes of the two tubes (cylinders).*	3	Combination of techniques $\tau 1$ & $\tau 2$ (first calculate diameter, then the volume)
*Compare these two volumes and calculate the ratio between them. **Investigate the ratio between the cylinder volumes using sheets of paper with other dimensions. What affects the volume ratio between the tall and the short cylinder?*	4 4	Technique $\tau 3$ (calculate the ratio of the volumes found) Technology (general statement about the ratio)
**Show that your conclusion is true for all rectangular papers.*	5	Technology (variation of $\tau 3$ using parameters, proof the general case)

Table 1: An exercise was pre-classified SOLO and the ATD-praxeology

It seemed to be easy to decide whether one or two aspects were involved in the question. However, when it came to higher SOLO-levels, it was more difficult to judge whether the aspects were "integrated". Here, the "Biggs-Brabrand-Dahl-method" was helpful as it offered additional verbs, alternative to "integrate", e.g., "compare" and "analyse", which could be used for the coding.

An example of the use of active verbs in the coding is the sub-task where students should first calculate two volumes and then compare these two volumes (Table 1):

> "Starting with rectangular sheets of paper with dimensions 10 cm × 20 cm, two different tubes are made. Find the volumes of the two tubes (cylinders)."

> "Compare these two volumes and calculate the ratio between them."

In both sub-exercises several aspects are involved. A volume is calculated by multiple parameters. But the active verbs separate the two sub-tasks, as the first requires only an algorithm: "find" (the volume), while the second requires that the student goes one step further and makes a comparison: "compare" (these two volumes). Finally, it was important to compare the coding with the "Biggs-Collis-method", as Biggs and Collis (1982) had formulated the original recommendations for how to use SOLO. In their book, however, the mathematics examples were fetched from elementary mathematics, and it was not obvious how to apply the method in the present study.

To conclude, the active verbs were found to be the most appropriate method when dealing with mathematics exercises. By using SOLO, a clear borderline could be drawn between the active verbs "do algorithm" (SOLO 3) and "explain causes" (SOLO 4), and the active verbs made it possible to identify these structural differences between exercises. The initial exercise about the volume of a cylinder was also coded using the ATD-praxeology (Table 1). This coding was based on the work done by Mortensen (2011), who has coded museum exhibition exercises – the so-called "intended praxeology". In the exercise about the cylinder, each sentence was coded. It was for example decided whether the students were supposed to deal with available "know-how" to solve a problem (the dimensions type of task & technique) or if they were supposed to deal with "know-why", i.e., a special way to justify the technique (the dimensions technology and theory).

At first, in the analysis of the described exercise (Table 1), SOLO and the ATD-praxeology were laid side by side. The exercise was coded both by SOLO and ATD. The strategy to try to correlate every SOLO-level to a specific dimension of ATD- praxeology caused problems. ATD and SOLO evaluate different dimensions. Thus, the strategy was abandoned at this early stage in the study. From now on, the two frameworks were used for different purposes: SOLO to analyse the quality of student learning outcomes, and the ATD-praxeology to analyse the didactic situations. In other words, they were considered answering different questions: *what qualities does the student outcome show?* and *which dimensions does the learning situation contain?* During the rest of the study it was discovered that the two frameworks often did not correlate.

The next step of the study was to code the group discussions about the cylinder. The sentences of the discussions were coded by the active verbs, and by the praxeological analysis. There were occasions when SOLO and ATD did correlate and there were other occasions when they did not. Table 2 shows part of one discussion and how the discussion can be analysed by SOLO and ATD. The students discussed the volume of the cylinder. They did not remember the formula and therefore they tried different strategies. Finally one student managed to solve the first exercise.

According to the analysis of the discussions of this first exercise, the SOLO-active verbs clarified the *learning outcome*. SOLO 4 for example told that students may have "explained" and/or "analysed". If an element of the situation was classified by ATD as "technology", it means that the student *dealt with a discussion concerning* "knowing why" a technique was being used. *Hence, it was possible to use the two frameworks within one study (compatible).* However, when entering into detail, the two approaches *lead to different characterisations* of students' mathematical activities (*complementary*).

— Students follow instructions "how": SOLO 2 & Technique

- Students use an algorithm: SOLO 3 & Technique
- A discussion about single tasks (point praxeologies) develops into a situation about knowing why (regional praxeologies), students may then explain why a method works: SOLO 4 & Technology
- A problem can develop into a situation that deals with knowing why, but students use the algorithm without discussing why: SOLO 3 & Technology
- The situation deals with knowing how to solve a problem by using an algorithm and students compare different solutions: SOLO 4 & Technique

Finally, it was concluded that the pre-classification did not hold. When the students did not remember the formula, they had to discuss the problem more thoroughly and thus reach other SOLO-levels and ATD-dimensions (Table 2).

DISCUSSION AND CONCLUSIONS

ATD and SOLO were combined to deepen the analysis of students' mathematics discussions. Such networking of frameworks is supported by Lester (2005) and Prediger and colleagues (2008), who argue that networking does not have to imply a total integration or unifying between frameworks. Lester (2005, p. 466) even advocates the adaptation of ideas from a range of theoretical sources to suit goals both for research and for developing practice in the classroom in a way that "practitioners care about".

The initial intention was to correlate specific SOLO-levels to specific ATD-praxeology dimensions. If this had been possible the conclusion would have been that one of the frameworks had been eliminated from this study. However, it was found that the two frameworks were both compatible and complementary. The present study thus succeeded in adapting theoretical models for analysing empirical material and in contributing to the development of strategies for analysing students' learning.

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Quotes	SOLO	ATD	Comments
e) It is the diameter times the length or height ... a) Is that so? (e) I think so. a) But no. It does not become square ... a) It is supposed to be CM3. It just gets CM2. It does not work.	1 3/4	Technique/ Technology	Student (a) and (e) try to find a relevant technique to calculate the volume of the cylinder. However, the technique is erroneous. Student (a) notices that their technique does not work. (a) tries to discuss "knowing why". They try to question the technique.
d) How do you count ... We were supposed to have the area of the circle. b) Wait what are we supposed to figure out? (reading task) d) Volume ... then we need the area of the base b) What?	3	Technique	A parallel discussion goes on between student (d) and student (b). Student (d) comments what (a) just said.
a) Yes exactly b) The area of the base? ... d) Is not the radius times the radius times pi?	3	Technique	The two groups start to discuss with one another. Student (d) takes the command and finds the technique – the "knowing how".

Table 2: Quotes from group discussion analysed by SOLO and by ATD-praxeology. Quotes are translated from Swedish and commented by the observer

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Re-conceptualising conceptual understanding in mathematics

Geoff Kent[1] and Colin Foster[2]

1 University College London, UCL Institute of Education, London, UK, G.Kent@ioe.ac.uk

2 University of Nottingham, Nottingham, UK

In this theoretical paper, we explore interrelationships between conceptual and procedural understanding of mathematics in the context of individuals and groups. We question the enterprise of attempting to assess learners' mathematical understanding by inviting them to perform a (perhaps unfamiliar) procedure or offer an explanation. Would it be appropriate to describe a learner in possession of an algorithm for responding satisfactorily to such prompts as displaying conceptual understanding? We relate the discussion to Searle's "Chinese Room" thought experiment and draw on Habermas' Theory of Communicative Action to develop potential implications for addressing the problem of interpreting learners' mathematical understanding.

Keywords: Conceptual, Habermas, procedural, Searle, understanding.

INTRODUCTION

The quest to help learners develop a deep and meaningful understanding of mathematics has become the holy grail for mathematics educators (Llewellyn, 2012), particularly since Skemp's (1976) seminal division of understanding into "instrumental" and "relational" categories. Relational (or conceptual) understanding is seen as more powerful, authentic and satisfying for the learner, representing true mathematical sense-making. But how can we know whether or not a learner has this relational understanding in any particular area of mathematics? The short, closed questions which dominate traditional paper-based assessments are unlikely to elicit this information. Hewitt (2009, p. 91) comments that "it is perfectly possible for a student to get right answers whilst not knowing about the mathematics within their work", and offers an example in which a learner aged 12–13 was finding the areas of triangles by multiplying the base by the height and dividing by 2, but admitted that he had no idea why he was multiplying or dividing by 2. This same example is used by Skemp (1976) to exemplify his distinction between instrumental and relational understanding of mathematics. Yet inviting learners to go further and explain their mathematics is also problematic. An invitation to "explain" an answer may be experienced as yet another request for "a performance": the "right" explanation that will satisfy a teacher or examiner may be memorised or produced algorithmically, just like the answer itself.

We might ask what it means for learners to have relational understanding of factorising a quadratic expression, for instance (Foster, 2014). If they can perform the procedure fluently (i.e., quickly, accurately, flexibly and confidently) then would we be satisfied (Foster, 2013)? We might argue that relational understanding involves adapting what is known to novel, non-straightforward problem-solving situations. Yet a robust enough algorithm will dispose of a very wide range of scenarios, including unanticipated ones, and a comprehensive enough set of algorithms might successfully deal with any situation likely to be encountered in any assessment (MacCormick, 2012). If the learner's performance continued to be faultless would we wish to probe their thinking further? To some extent mathematical fluency entails withdrawing attention from the details of why and how the procedure works so as to speed up the process and allow cognitive space for focusing on wider aspects of the problem (Hewitt, 1996; Foster, 2013). A mathematician does not want to have to differentiate $3x^2 - 2x + 4$ from first principles every time, although they are capable of doing so. Perhaps relational understanding involves an ability to deconstruct the procedure *if required* rather than an expectation that this is going on every time it is carried out? But deconstructing a procedure could *itself* be regarded as a procedure, and presumably one that can be prepared for – even memorised, just as proofs can be memorised. So is there something more to relational understanding than expert procedural fluency, and if so how might this be conceptualised? Is there a difference between being able to manipulate syntax and being able to understanding meaning?

PROCEDURAL AND CONCEPTUAL KNOWLEDGE

Skemp's (1976) famous distinction between instrumental and relational understanding characterises relational understanding as "knowing both what to do and why" (p. 20), whereas instrumental understanding is merely "rules without reasons" (p. 20). While acknowledging that "one can often get the right answer more quickly and reliably by instrumental thinking than relational" (p. 23), he nonetheless criticises instrumental learning as a proliferation of little rules to remember rather than fewer general principles with wider application. More recently, the terms procedural and conceptual learning have been widely adopted, and theoretical interpretations of these in mathematics education have increasingly highlighted their interweaving and iterative relationship (Star, 2005; Baroody, Feil, & Johnson, 2007; Star, 2007; Kieran, 2013; Star & Stylianides, 2013; Foster, 2014).

The most commonly-used definitions of procedural and conceptual knowledge in the context of mathematics are those due to Hiebert and Lefevre (1986). They see conceptual knowledge as knowledge that is rich in relationships, where the connections between facts are as important as the facts themselves, whereas procedural knowledge is rules for solving mathematical problems. This distinction parallels Skemp's (1976) conclusion that there are really two kinds of *mathematics* – instrumental and relational – dealing with different kinds of knowledge. More recently, Star (2005, 2007) distinguishes between *types* of knowledge (knowledge about procedures or knowledge about concepts) and *qualities* of knowledge (superficial or deep), and complains that these are frequently confounded. He highlights the way in which "procedural" is often equated with "superficial", and "conceptual" with "deep", and draws attention to the possibility of "deep procedural knowledge" and "superficial conceptual knowledge" as valid categories. Kieran (2013) goes further in declaring the dichotomy between conceptual understanding and procedural skills a fundamentally false one. Other researchers have also explored the interplay between procedural and conceptual knowledge (Sfard, 1991), with Gray and Tall (1994) integrating processes and concepts into what they term "procepts" (Tall, 2013). But there remains the question of what precisely it is that conceptual knowledge consists of beyond confident procedural knowledge.

THE CHINESE ROOM

Searle's (1980) famous thought experiment about a "Chinese Room" was an attack on the "strong" artificial intelligence claim that a computer is a mind, having cognitive states such as "understanding". Searle imagined a native English speaker who knew no Chinese locked in a room with a book of instructions for manipulating Chinese symbols. Messages in Chinese are posted through the door and the English speaker follows the instructions in the book to produce new messages in Chinese, which they post out of the room. Unknown to them, they are having a conversation in Chinese, a language which they do not speak a word of. Searle argued that syntax does not add up to semantics; behaving "as if" you understand is not the same as understanding. But it is very difficult to pinpoint exactly where the difference lies (Gavalas, 2007). Searle does acknowledge that "The rules are in English, and I understand these rules as well as any other native speaker of English" (1980, p. 418), but it remains mysterious exactly what test could distinguish a competently performing machine from a real mathematician. A learner performing a mathematical procedure may be making mathematical sense to an observing mathematician, such as a teacher, without apparently knowing much themselves about what they are doing.

The focus here has now changed from whether the computer (or the mind as a computer) understands mathematics to the question of whether some computer could be such that it is indistinguishable from a real mathematician. It may be that, whether or not you could tell them apart, they would perform the tasks of producing syntactically correct mathematics in importantly different manners. Thus the issue becomes the *sense* in which rules are being followed. If rules are followed in a meaningful sense and their semantic content is well defined and connected within constellations of schemas, then test item responses could be strong evidence of mathematical understanding. But this requires that those items are designed so that they engage procedural knowledge in a sophisticated manner which takes into account all of the aspects of the concept image that is the object of assessment. We could specify an additional requirement that the test be administered to a human being and not a computer. While this may seem flippant, it points to the heart of Searle's argument, which is that humans follow rules through semantic causality that is more or less part of the "hardware" of our brains; that there is no (or minimal) "software" layer (Searle, 1984). So does this imply that truly instrumental understanding is an impossibility for a human being?

MATHEMATICAL UNDERSTANDING

Searle's later articulation of social theory addresses how language can be used to create a social reality which is iterative and generative (Searle 1995, 2010). Further, Searle articulates an analysis of language that points towards strong connections between the structure of language and the structure of intentional states. In some ways this leads us back to the idea of the mathematician as performing *as though* merely in command of a complex constellation of algorithms that are triggered and brought to bear in a purely syntactical manner. In light of the argument put forth by Searle, we should rather say that the mathematician employs an array of mathematical understandings which have semantic content. While this seems unsatisfying, as though Searle is saying "it is semantic when humans do it", it bears strong connections with Sierpinska's articulation of procedural understanding and its relationship to conceptual understanding. Procedural understandings, according to Sierpinska (1994):

> are representations based on some sort of schema of actions, procedures. There must be a conceptual component in them – these procedures serve to manipulate abstract objects, symbols, and they are sufficiently general to be applied in a variety of cases. Without the conceptual component they would not become procedures. We may only say that the conceptual component is stronger or weaker. (p. 51)

Hence, it is reasonable for a mathematician to see many elements of their understanding as arrays of algorithms that allow them to address wide categories of mathematical problems. Yet this is fundamentally different from how a digital computer would operate in a purely syntactical approach.

Gordon, Achiman and Melman (1981, p. 2) define *rules* as "statements of the logical form 'In type-*Y* situations one does ... *X*'". For Wittgenstein (1953), it is not possible to *choose* to follow a rule: "When I obey a rule, I do not choose. I obey the rule *blindly*" (p. 85, original emphasis). Otherwise it is not a rule. It is in this sense that Searle raises a question fundamental to this discussion: Should understanding mathematics be understood as sophisticated algorithmic arrays which are akin to complex computer programs? Searle's (1984) critique of this and related ideas has several facets, the most pertinent of which is that there is an ambiguity in what is meant by rule following and that humans and computers do not follow rules in the same sense. In essence, Searle argues that humans follow rules in as much as they understand the *meaning* of the rules (which is thus semantic and about intentional states), whereas computers are purely syntactical in their rule following; they can be said to "*act in accord with formal procedures*" (ibid, p. 45, original emphasis).

Returning to the question of relational versus instrumental understanding, it seems that if we follow Searle's arguments we can say that mathematical understanding is probably not effective human understanding if it is primarily instrumental (in the sense of syntactical rule following). However, it is clear that procedural, syntactical and algorithmic practices and concepts form an important part of the background to meaningful mathematical understanding. Thus from a perspective of assessment we would expect it to be important to assess algorithmic fluency while also seeking to assess the strength of the conceptual content associated with the procedural performance.

So in contrast to the kinds of digital computers that Searle and Hiebert and Lefevre are talking about, algorithms exist within a semantic framework. Perhaps it is as though a digital computer (syntactical machine environment) is being modelled using a semantic machine environment (the brain). If so, the potential problem for mathematics education relating to instrumental learning in mathematics may be that the seeming simplicity of rule following is made vastly more complicated by its need to run in a sort of virtual syntactic machine running on essentially semantic hardware. On the other hand, the generation of correct syntactical content is a power of certain constellations of semantic knowledge (relational knowledge). It seems that the teaching of algorithms and procedures is crucial for the development of sophisticated mathematical understanding, but also that *how* they are taught is critical to supporting the development in learners of mathematical understanding that goes beyond procedural understandings with weak conceptual content (Foster, 2014).

Habermas' theory of communication, partly based in and complementary to Searle's theories, can point towards models of understanding and how to assess it. In communicative action, as defined by Habermas (1984), action is coordinated intersubjectively through achieving understanding. The theory of communicative action (TCA) analyses communication as having an inherent rationality focused on the goal of achieving understanding. Using speech act theory and argumentation theory, Habermas identifies categories of validity claims that are raised in any communicative interaction and also identifies implicit preconditions for successful communication. The former is referred to by Habermas as "discourse", but might better be termed "validity-discourse", in order to differentiate it from other uses of that term in social sciences. The preconditions for communicative action are referred to collectively as the "Ideal Speech Situation" by Habermas and constitute a set of counterfactual norms identified abductively as necessary for successful communication. These norms are focused on equitable conditions for participation in communication where the "unforced force of the better argument" has the opportunity to motivate agreement. This is a bit tricky, as Habermas claims that such conditions must be assumed by participants as in operation in order to communicate, despite representing more of an ideal horizon that never completely obtains. Society is power-laden, and all communication occurs within a social context. Thus the breakdown of communication is all too common, and intersubjective understanding is seen as a fleeting and fallible goal that is ever approached but seldom attained.

The claim that Habermas's TCA and Searle's speech act theory are complementary and can be productively networked is based on the specific arguments made by Habermas in the TCA, his use of speech act theory to develop his ideas of communicative action and also upon analysis of similarities and departures between the principles, methodologies and questions of each author:

> Analytical philosophy, with the theory of meaning at its core, does offer a promising point of departure for a theory of communicative action that places understanding in language, as the medium for coordinating action at the focal point of interest. (Habermas, 1984, p. 274)

While it might be possible to argue that Searle's theories depart somewhat from the kinds of analytic theories that Habermas wants to make use of, this is mistaken, since their focus is on incorporating theories of intentionality. Searle beings with the structure of linguistic expressions and then deals with intentionality, and importantly in his later work he introduces the idea of *collective* intentionality, which is focused on the coordination of speakers, and which is closely related to Habermas' ideas about the importance of intersubjectivity in communicative action:

> For a theory of communicative action only those analytic theories of meaning are instructive that start from the structure of linguistic expressions rather than from speakers' intentions. And the theory will need to keep in mind how the actions of several actors are linked to one another by means of the mechanism of reaching understanding. (Habermas, 1984, p. 275)

Searle's ideas add rigour and detail at the level of social ontology and may allow for a more sophisticated operationalising of concepts and constructs based in Habermas' TCA. These ideas could be used to further network critical theory, cognitive science, neuroscience and other approaches to the study of mathematics education so that they may inform one another without reducing one to the other. Thus the issue of theoretical incommensurability may be navigated without theoretical insights becoming "siloed" within various sub-cultures of theory which do not communicate with one another. A common theoretical language might allow researchers to disagree with greater clarity without running the risk of becoming an over-arching "grand theory". More broadly, Searle's ideas could serve as tools for building rigorous analysis of particular instances of theoretical networking, allowing productive discussion between theoretical perspectives.

These ideas can be operationalised to analyse small-group problem solving and in this manner interpret the mathematical understanding of participants (Kent, 2013), which could serve as the basis for the development of interactive assessment techniques, activities and protocols. Understanding from this perspective is about being able to identify what reasons, arguments and evidence could be legitimately raised to justify a claim. This emphasis on the identification of shared bases for validity can serve as a pragmatic approach to the analysis of human understanding in mathematics. Thus when we speak of assessing mathematical understanding we can begin to identify as a community of mathematicians and mathematics educators (with due consideration of developmental and disciplinary appropriateness) the claims and the appropriate reasons that justify these claims. We can consider how to engage participants in communicative actions around mathematical goals that require the articulation of arguments and justifications that show evidence that the participants can explain why certain mathematical claims are true.

Returning to the Chinese Room, this turn to the social does not suggest that there need be two people in the room, but rather that the person in the room must share requisite background knowledge or be able to develop it contextually with the Chinese speakers outside the room. The idea of communicative competence is key: sharing the contextual background knowledge that allows a language to have semantic meaning is the basis for "understanding". This is different from quickly and accurately manipulating the symbols in a language in a syntactic fashion: no *shared* understanding entails from such activity. Now it is possible that meaning could be attributed to rules or symbols by the person in the Chinese room, but, without the ability to test these against another person who has semantic understanding of the symbols, no interpersonal communication or shared understanding is achievable. The meaning so developed would be a private language that would not necessarily correspond to that of the interlocutor. Thus the person in the Chinese room might imagine that they were having a discussion about a family's vacation outing when in fact the interlocutor interpreted the exchange of symbols as being a mathematical discourse on the solution to an algebraic problem (or vice versa).

CONCLUSION

These ideas about the nature of the relationship between syntax and semantics, procedure and concept, and instrumental and relational understanding do not undermine the importance of procedural fluency. Pimm (1995) addresses the issue in depth and identifies some of the important features of fluency in mathematics education:

> For me, fluency is about ease of production and mastery of generation – it is used also in relation to a complex system. 'Fluent' may be related to efficient, or just no wasted effort. It is often about working with the *form*. Finally, it can be about not having to pay conscious attention. (ibid, p. 174, original emphasis)

Thus fluency, including syntactical fluency, can serve as partial evidence of understanding in a communicational context. Mathematical fluency, as in non-mathematical communication, is a sign of communicative competence, which is a prerequisite for interpersonal understanding according to the hermeneutic/communicational tradition (Habermas, 1984; Sierpinska, 1994). Thus when we say that a human being does not follow rules in the same sense as a computer, we mean that the symbolic rule following (or algorithmic manipulation of syntax) is done in the context of mathematical communication, and thus has semantic framing.

Habermas' articulation of rational behaviour in discursive practices has been identified as productive for the analysis of shared cognition in mathematics education (Boero et al., 2010). In communicative action participants achieve shared goals by coordinating action (including speech action) through the development of a shared understanding. Thus, establishing shared goals and coordinating action around an appropriately designed mathematical task could serve as an interpretive basis for the researcher (or other virtual participant) to make a judgement about the understanding of the participants in collaborative learning of mathematics (Kent, 2013).

We suggest that consideration of Searle's (1984) critique of cognitive science allows for ongoing productive insight into what mathematical thinking is and its relation to education. An important problem faced by the mathematics education community is how we can use ideas of relational understanding and instrumental understanding in a sophisticated manner to promote the learning of mathematics. Learners of mathematics should gain genuine experience of real mathematical sense-making rather than engage in a charade of imitating what they think such behavior should look like. The increasing focus on fluency in policy in the UK (DfE, 2013) suggests the need for tools and practices to be developed which coordinate ideas of cognition, mathematical understanding and educational practices of teaching and assessment. Our consideration of Searle's Chinese Room argument has sought to highlight the nuance involved in these issues and the kinds of practices and theoretical frameworks that could be leveraged to address the problem of interpreting learners' mathematical understanding.

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The epistemological dimension revisited

Ivy Kidron

Jerusalem College of Technology, Jerusalem, Israel, ivy@jct.ac.il

Epistemology and networking was discussed in the last CERME working group on theory. This paper aims to continue the discussion. I reflect on epistemological analysis and the cultural dimension of knowing and present examples which demonstrate how the changes in the cultural context influence the epistemological analysis. Then, I reconsider the epistemological dimension and the networking of theories. In some cases, the epistemological dimension permits the networking. In other cases, we notice how by means of networking, strong epistemological concerns in one theory might be integrated in another theory in a way that reinforces the underlying assumptions of this other theory. I end the paper with an example of networking that demonstrates how the social dimension might influence the epistemological analysis.

Keywords: Cultural dimension, epistemological analysis, networking theories, social dimension.

EPISTEMOLOGY AND NETWORKING THEORIES IN THE PREVIOUS CERME WORKING GROUPS ON THEORIES

The present paper aims to continue the work done at the previous CERMEs in relation to the epistemological dimension in theories. At CERME8, the focus on networking and epistemology was stronger than in the previous working groups on theory. For example, the role of epistemology in the networking of theories was an explicit focus in the paper by Ruiz-Munzón, Bosch, and Gascón (2013). The idea of a "reference epistemological model" (REM) was introduced for networking Chevallard's Anthropological Theory of the Didactic (ATD) and Radford's Theory of Knowledge Objectification (TKO). The authors analyzed how each approach addresses the nature of algebraic thinking. The point of view of the ATD was presented with its own REM about elementary algebra as well as the kind of questions addressed by this approach, in relation to the TKO.

In their paper, presented at CERME8, Godino and colleagues (2013) analyzed two approaches to research in mathematics education: "Design-based research" (DBR) and "Didactic engineering" (DE), in order to study their possible networking. DE (closely linked to Brousseau's theory of didactical situations) focuses on epistemological questions; DBR does not adopt a specific theoretical framework, nor does it explicitly raise epistemological questions. In the working group (Kidron et al., 2013) interesting questions arose like the following one: "is the epistemological focus only a question of 'cultural and intellectual context' or is an epistemological reference necessary for each theoretical approach used in design based research in math education?"

Artigue (2002) wrote that the anthropological approach shares with the socio-cultural approaches the view that mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. These practices are described in terms of tasks in which the mathematical object is embedded, in terms of techniques used to solve these tasks and in terms of discourse which both explains and justifies the techniques. It is interesting to note that the nature of mathematical objects was a theme that appears at CERME4 in the context of the need to be aware of the underlying assumptions of each theory and that underlying assumptions also concern ontological or epistemological questions such as the nature of mathematical objects. This theme reappears in the next CERMEs especially at CERME7 while networking was needed in order to analyze the emergence and nature of mathematical objects. This was well demonstrated, for example, in the paper presented by Font and colleagues (2011). The authors asked "What is the nature of the mathematical objects?" They explored this question by the use of a synthesis between the onto-semiotic approach (OSA), APOS theory (with its four components, Action, Process, Object, and Schema) and the cognitive science of mathematics (CSM) as regards their use of the concept of "mathematical object". APOS theory and CSM highlight partial aspects of the complex process through which, according to OSA, mathematical objects emerge. OSA extends APOS theory by addressing the role of semiotic representations; it improves the genetic decomposition by incorporating ideas of semiotic complexity, networks of semiotic functions and semiotic conflicts; it offers a refined analysis due to the way in which it considers the nature of such objects and their emergence out of mathematical practices. Considering mathematical objects not as absolute objects, but as entities which arise from the practices of given institutions, leads us to analyze the role of both, the epistemological dimension and the socio cultural dimension, in theories.

EPISTEMOLOGICAL ANALYSIS AND SOCIO CULTURAL DIMENSION

The following question was asked by Luis Radford at the colloquium at Paris in honour of Artigue (2012): "How can epistemological analysis take into account the social and cultural dimension of knowing?" In the last decades the increasing influence of sociocultural approaches towards learning processes is well recognized. Therefore, the question is essentially *how* the social and cultural dimensions are taken into account in the epistemological analysis. In this section, I will consider this question in relation to the cultural dimension of knowing, I analyze the changes in the cultural context and their influences on the epistemological analysis. In the section about epistemological dimension and networking theories, I will reconsider Radford's question in relation to the social dimension of knowing.

Changes in the cultural context and their influences on the epistemological analysis

In the last decades we face the changes of our cultural environment as well as the changes of the context in which our theory emerged. I will give an example from my own research on students' conceptual understanding of central notions in calculus like the notion of limit in the definition of the derivative. In my previous research, using essentially theories that privilege epistemological and cognitive dimensions, I was aware of the cognitive difficulties relating to the understanding of the definition of the derivative as the "limit of the quotient $\Delta y/\Delta x$ as Δx approaches 0". In my epistemological analysis, my first thinking was that these cognitive difficulties are inherent to the epistemological nature of the mathematics domain. I realized that students viewed the limit concept as a potential infinite process and I understood that this was a possible source of difficulties. Moreover, previous researches (Tall, 1992) expressed students' belief that any property common to all terms of a sequence also holds of the limit. I therefore realized that this natural way in which the limit concept is viewed might be an obstacle to the conceptual understanding of the limit notion in the definition of the derivative function $f'(x)$ as $\lim_{\Delta x\to 0} \Delta y/\Delta x$. In particular, the derivative might be viewed as a potentially infinite process of $\Delta y/\Delta x$ approaching $f'(x)$ for decreasing Δx. As a result of the belief that any property common to all terms of a sequence also holds of the limit, the limit might be viewed as an element of the potentially infinite process. In other words, $\lim_{\Delta x\to 0} \Delta y/\Delta x$ might be conceived as $\Delta y/\Delta x$ for a small Δx. I therefore looked for a counterexample that demonstrates that one cannot replace the limit "$\lim_{\Delta x\to 0} \Delta y/\Delta x$" by $\Delta y/\Delta x$ for Δx very small. "Finding such a counterexample.... was crucial to my research focus. Such a counterexample demonstrates that the passage to the limit leads to a new *entity* and that therefore omitting the limit will change significantly the nature of the concept. It demonstrates that the limit could not be viewed as an element of the potentially infinite process" (Kidron, 2008, p. 202). In Kidron (2008), I explain that such counterexample exists in the field of dynamical systems which is considered as a new field in mathematics. In the counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. The essential point is that using the analytical solution, the students use the concept of the derivative as a limit $\lim_{\Delta x\to 0} \Delta y/\Delta x$ but, using the discrete approximation by means of the numerical method, the students omit the limit and use $\Delta y/\Delta x$ for small Δx. Students reactions are analyzed in (Kidron, 2008), in particular how students reach the conclusion that passing to limits may change the nature of a problem significantly. The essential point is that the changes in the cultural context permit the new settings for the learning experience. More precisely, the changes in the cultural context permit modern results in research Mathematics which influenced my own research in mathematics education by means of changes in the didactical designs. The didactical design described in (Kidron, 2008) was possible by means of the epistemic status of the new artifacts used in the research study. The way the students interacted with the software demonstrates that the artifact used in this study should not be considered only as an aid for the students. It had a deep cognitive role while learners interacted with it. The artifact was conceived as co-extensive of thinking: the students act and think *with* and *through* the artifact as described by Radford (2008). In another study (Kidron & Dreyfus, 2010) we also notice this specific epistemic status of the artifact as co-extensive of thinking while the computer is considered as a dynamic partner. Kidron and Dreyfus consider the influence of a CAS (Computer Algebra System) context on a learner's process of constructing a justification for the bifurcations in a logistic dynamical process. The authors describe how instrumentation led to cognitive constructions and how the roles of the learner and the computer intertwined during the process of constructing the justification.

Another example describing how epistemological analysis takes into account the cultural dimension of knowing is described in Artigue (1995, p. 16) in which the author describes her mathematical research in differential equations and the way she notes the epistemological inadequacy of teaching in this area, for students in their first two years at university. By means of epistemological analysis, Artigue described how historically the differential equations field had developed in three settings: the algebraic, the numerical and the geometric settings. For many years, teaching was focused on the first setting due to epistemological and cognitive constraints. Reflecting on these constraints was a starting point towards building new teaching strategies which better respect the current fields' epistemology. By means of the epistemological analysis, Artigue could see the epistemological evolution of the field towards new approaches, the geometrical and numerical approaches. The essential point in this example is that the epistemological evolution is a consequence of the changes in the mathematical culture and the epistemological analysis highlights the crucial role of the cultural dimension.

THE EPISTEMOLOGICAL DIMENSION AND THE NETWORKING OF THEORIES

Epistemological sensitivity

A new view on the epistemological dimension is offered in Kidron and colleagues (2014) by means of the networking between three theories, TDS, the Theory of Didactic situations (Artigue, Haspekian, & Corblin-Lenfant, 2014), ATD, the Anthropological Theory of the Didactic (Bosch & Gascón, 2014), and AiC, the theory of Abstraction in Context (Hershkowitz et al., 2001; Schwarz et al., 2009; Dreyfus & Kidron, 2014). The foci of the three theoretical approaches are different. In particular, AiC focuses on the learner and his or her cognitive development, while TDS and ATD focus on didactical systems. The three theoretical approaches are sensitive to issues of context but, due to these differences in focus, context is not theorized and treated in the same way. The authors expected some complexity in the effort of creating a dialogue between the three theories in relation to constructs such as context, milieu, and media-milieus dialectic. However, they observed how the dialogue between the three theories appears as a progressive enlargement of the focus, showing the complementarity of the approaches and the reciprocal enrichment. A new term was introduced in this research study: *epistemological sensitivity*.

The authors explain the meanings of the terms context (for AiC), milieu (for TDS) and media-milieus dialectic (for ATD), each of them being a cornerstone for the theory while all of them try to theorize specific contextual elements. The three theories share the aim to understand the epistemological nature of the episode described in the paper but in each of the three theories different questions were asked. Questions for analyses in AiC stressed the epistemic process itself, whereas researchers in TDS and ATD asked how this process is made possible. Nevertheless, these questions indicated that the researchers were able to build on the other analyses in a complementary way. The dialogue between the different approaches was possible because a point of contact was found. In this case, we may talk about a common *epistemological sensitivity* of AiC, TDS, and ATD, which can be noticed in the a priori analyses provided by each frame. This initial proximity was essential for the dialogue to start and become productive, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific to each one. The three concepts, context, milieu and media-milieus dialectic were accessed by different data or different foci on data in a complementary way sharing *epistemological sensitivity*, which facilitated establishing connections and reflecting on them.

Epistemological concerns as a consequence of networking

It is not by chance that the common *epistemological sensitivity* of AiC, TDS, and ATD, was noticed in the a priori analyses provided by each frame: the reason is that the a priori analyses take into account the mathematical epistemology of the given domain. In the last years, the AiC researchers decided to implement the idea of a priori analysis in an explicit way. This happened as a consequence of the networking experience with the TDS researchers. An example of such a networking experience is described in Kidron and colleagues (2008). Three theories were involved in this case of networking: TDS, AiC and IDS, the theory of Interest-Dense Situations (Bikner-Ahsbahs & Halverscheid, 2014). Kidron and colleagues (2008) focus on how each of these frameworks is taking into account social interactions in learning processes. The authors wrote that

> In a more general way, the different views the three theoretical approaches have in relation to social interactions force us to reconsider these approaches in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks. (p. 253)

The authors identified not only connections and contrasts between the frameworks but also additional insights, which each of these frameworks can provide to each of the others. In this paper, we only focus on a specific kind of insights: the epistemological concerns which were highlighted as a consequence of the networking of theories. We first characterize the epistemological dimension in each of the three theories before the networking experience:

— TDS provides a frame for developing and investigating didactical situations in mathematics from an epistemological and systemic perspective. TDS combines epistemological, cognitive, and didactical perspectives. TDS focuses on the epistemological potential of didactical situations;

— IDS, the theory of interest-dense situations, is "a social constructivist theory that cannot say much about cognitive processes of individuals and does not provide tools for epistemological analyses" (Bikner-Ahsbahs & Halverscheid, 2014, p. 102);

— AiC analysis focuses on the students' reasoning; mathematical meaning resides in the verticality of the knowledge constructing process and the added depth of the resulting constructs. An epistemological stance is underlying this idea of vertical reorganization but AiC analysis is essentially cognitive.

Focusing on epistemological concerns as mentioned earlier, we will only characterize the insights offered by TDS to AiC as described by Kidron and colleagues (2008):

> According to Hershkowitz et al. (2001), the genesis of an abstraction originates in the need for a new structure. In order to initiate an abstraction, it is thus necessary (though not sufficient) to cause students' need for a new structure. We may attain this aim by building situations that reflect in depth the mathematical epistemology of the given domain. This kind of epistemological concern is very strong in the TDS, and the notion of fundamental situation has been introduced for taking it in charge at the theoretical level. It could be helpful for AiC. (p. 254)

This was an invitation for AiC researchers to build an a priori analysis that reflects in depth the mathematical epistemology of the given domain. In the same vein the a priori analysis of TDS offers another perspective to IDS to think about the building of situations reflecting in-depth the mathematical epistemology of a given domain and the consequence of such reflection on the analysis of the social interactions.

The social dimension and its influence on the epistemological analysis

In the following, I analyze a case of networking between AiC and IDS which demonstrates mutual insights in the process of networking. In particular, we will observe how the epistemological analysis carried by the AiC researchers is influenced by the social dimension of knowing which characterizes IDS. This case of networking illustrates how the epistemological analysis might take into account the social dimension of knowing.

Kidron and colleagues (2010) focus on the idea of networking and on two theoretical concepts: the need for a new knowledge construct, and interest. IDS considers social interactions as basis which constitutes learning mathematics. Interest-dense situations provide motivation for processes of in-depth knowledge construction. AiC is a theoretical tool to investigate such processes. As already mentioned, in the AiC analysis, the first stage of the genesis of an abstraction is the learner's need for a new construct. Such a need might arise when the learner's existing knowledge is insufficient to solve a task or to understand a new concept. This individual need is related to the specific mathematical situation at hand. Analyzing this need is a part of AiC epistemological analysis. For IDS the situation is different: interest constitutes a psychological source to gain more knowledge. This need is nested in the situational interest rather than shaped by the epistemic nature of the topic. The aim of the networking was to relate these two concepts: need and interest. As mentioned earlier, the AiC researchers implemented the idea of a priori analysis. Their analysis was based on an a priori analysis of the knowledge elements intended by the design. The AiC analysis focused on the students' reasoning and mathematical meaning resided in the verticality of the knowledge constructing process. The AiC researchers identified students' constructs of the intended knowledge elements. They expected to identify students' need for the new constructs before or during the process of knowledge construction. However, the researchers found it difficult to identify a need for a specific new construct. Networking the two approaches was helpful: The IDS analysis focuses and reconstructs the whole situation sequentially on the basis of utterances that show intense social interactions, whereas the AiC analysis focuses on segments that appear relevant to the constructing process. In fact, the excerpts ignored at first by the AiC researchers did contribute to the constructing process thanks to the social interaction analysis provided by IDS which allowed the AiC researchers to focus on and incorporate these seeds of construction in their analysis. The networking helps AiC researchers realize that there are situations in which constructing actions can occur on the basis of a general epistemic need rather than on the basis of specific needs for new constructs. The benefit of networking was mutual thanks to the epistemological nature of AiC a priori analysis which makes the researchers sensitive for the mathematics at stake and implicit mathematical ideas were identified very early. This was very helpful towards IDS re-analyzing of the epistemic actions in the research study.

CONCLUDING REMARKS

In the last CERME we discussed cases in which the epistemological dimension permitted the networking. This was done, for example, by means of the idea of "reference epistemological model". In this paper, we notice how by means of networking, strong epistemological concerns in one theory might be integrated in another theory in a way that reinforces the underlying assumptions of this other theory. This was illustrated by the insights offered by means of a priori analysis. We also analyzed examples that demonstrate the influence of the cultural context as well as the influence of the social dimension on the epistemological analysis. The cultural context in which the different theories emerged is changing all the time. As a result of these changes, a new view on the epistemological dimension is offered. This new view should be further discussed.

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Towards a confluence framework of problem solving in educational contexts

Boris Koichu

Technion – Israel Institute of Technology, Haifa, Israel, bkoichu@technion.ac.il

An exploratory confluence framework for analysing mathematical problem solving in socially different educational contexts is introduced. The central premise of the framework is that a key solution idea to a problem can be constructed by a solver as a result of shifts of attention that come from individual effort, interaction with peer problem solvers or interaction with a source of knowledge about the solution. The framework consolidates some existing theoretical developments and aims at addressing the perennial educational challenge of helping students become more effective problem solvers.

Keywords: Problem solving, networking theories, shifts of attention.

RATIONALE

It has been repeatedly asserted that problem solving is an activity at the heart of doing and studying mathematics. Its central feature is that problem solving requires the engaged person(s) to invent a solution method rather than to recall and implement a previously practiced method (e.g., Kilpatrick, 1982; NCTM, 2000). Accordingly, a central challenge associated with the use of mathematical problems in educational contexts – how to help learners to become effective or successful problem solvers – can be worded as the challenge of supporting learners' mathematical inventiveness in ways that preserve their problem-solving autonomy and self-efficacy.

For the last 50 years this challenge has been approached through various conceptual frameworks and models (see Carlson & Bloom, 2005; Schoenfeld, 2012; Törner, Schoenfeld, & Reiss, 2008, for comprehensive accounts of the state of the art). Each framework has aimed at addressing specific queries of pragmatic and theoretical importance. Some of the queries were:

- How do mathematicians solve problems? What phases and cycles are they going through while solving problems? (Carlson & Bloom, 2005; Pólya, 1945/1973). Can, and if yes, how problem-solving heuristics be taught? (Schoenfeld, 1985; Koichu, Berman, & Moore, 2007).
- What are the attributes of mathematical problem solving besides heuristics? (Schoenfeld, 1985). What is the role of affect in problem solving? (DeBellis & Goldin, 2006).
- How do the problem-solving attributes come to cohere? (Schoenfeld, 1992). How does decision making occur when an individual solves a problem? (Schoenfeld, 2012).
- What sociomathematical norms should be promoted for supporting learners' intellectual autonomy in problem solving? (Yackel & Cobb, 1996).

Some of these and such queries have been addressed. For instance, we know a lot about phases and cycles involved in problem solving by experts and by some categories of students. The use of problem-solving attributes and phases as a research tool has proven to be particularly helpful for analysing the phenomenon of unsuccessful problem solving. For example, if there is evidence that a particular belief about mathematics is depriving an individual from persisting when solving a problem, then that belief might provide a sufficient explanation for the problem-solving failure (see Furingetti & Morselli, 2009, for an elaborated example). When, however, one problem is solved and another is not by an individual who possesses all needed mathematical, cognitive and affective resources for solving both problems, the explanation of the success and the failure can sometimes be sought outside of the existing problem-solving models and frameworks (Koichu, 2010).

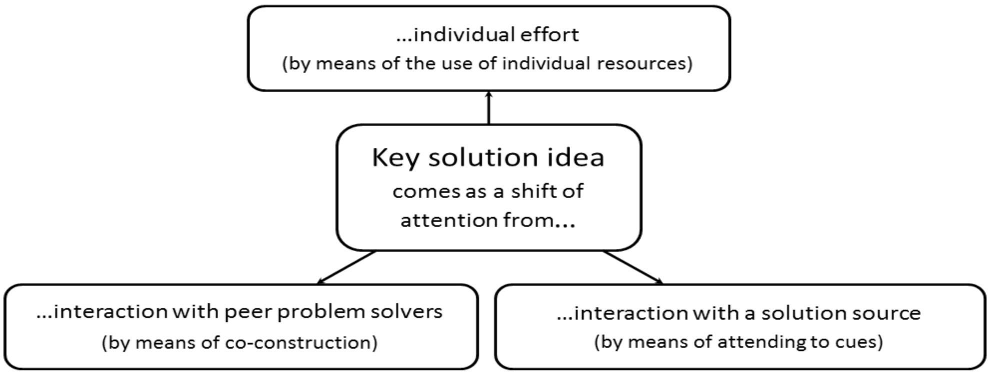

Figure 1: Confluence model of mathematical problem solving

Some of the queries about problem solving have proven to be hard nuts to crack (e.g., Schoenfeld, 1992, 2012). For example, Schoenfeld's (1992) question about how the problem-solving attributes – knowledge, heuristics, control and beliefs – come to cohere has been under research scrutiny for more than two decades. Furthermore, different problem-solving frameworks and models have emerged from different contexts and situations. As a result, it is sometimes difficult to use one model outside its original context. A recent example of extending the scope of a particular problem-solving model to another context is given by Clark, James and Montelle (2014) (their work is discussed in more detail below), but it is rather an exception than a trend (cf. Koichu, 2014, for a collection of views on the recent trends in research on problem solving). As a rule, problem-solving frameworks and models co-exist with little coordination. This is one of the reasons for which, in terms of Mamona-Downs and Downs (2005), a clear identity for problem solving in mathematics education has not yet been developed.

The goal of this article is to present an exploratory problem-solving framework that has the potential to consolidate some of the previous frameworks and can serve as a research and pedagogical tool in different educational contexts. The framework is, in a way, a tool for better understanding the process that Pólya (1945/1973) might term as a heuristic search embedded in the planning phase of problem solving. The central query of the framework is, simply stated, "Where can a solution to a problem come from?" A more precise formulation of the query is as follows: "Through which activities and resources can a chain of shifts of attention towards an invention of a key solution idea to a mathematical problem be constructed by a problem solver in socially different educational contexts?"

CONFLUENCE FRAMEWORK

The confluence framework is schematically presented in Figure 1. A *key solution idea* notion is in the core of the framework. It is a solver-centered notion. Along the lines defined by Ramon (2003), *a key solution idea* is a heuristic idea [1] which is invented by the solver and evokes the conviction that the idea can be mapped to a full solution to the problem. The full solution is a solution, which, to the solver's knowledge, would be acceptable in the educational context in which problem solving occurs.

Examples of key solution ideas include: an auxiliary construction that enables the solver to see a chain of deductions connecting the givens of a geometry problem with the claim to be proved, a way of reassembling the terms of a sophisticated trigonometric equation so that the solver begins to see the equation as a quadratic one, a way of representing a word problem (e.g., Euler Seven Bridge Problem) as a graph that makes the solution to the problem transparent. One can see connections between the notions of a key solution idea and of an illuminating or insightful idea. An insight, however, is frequently defined as restructuring the initial representation of the problem followed by so-called aha-experience. A key solution idea does not necessarily emerge at once and accordingly its invention is not necessarily accompanied by an aha-moment.

The framework relies on three premises. *First premise:* Even when a problem is solved in collaboration, it has a *situational solver*, an individual who invents and eventually shares its key solution idea. *Second premise:* A key solution idea can be invented by a situational solver as a shift of attention in a sequence of his or her shifts of attention when coping with the problem. *Third premise:* Generally speaking, a solver's pathway of the shifts of attention can be stipulated by: (i) individual effort and resources, (ii) interaction with peer solvers who do not know the solution and struggle in their own ways with the problem or attempt to solve it together, (iii) interaction with a source of knowledge about the solution or its parts, such as a textbook, an internet resource, a teacher or a classmate who has already found the solution but is not yet disclosing it. The possibilities (i)-(iii) are intended to embrace all frequent situations of problem-solving. These possibilities can be employed in separation or complement each other in one's problem solving.

The framework is a *confluence* framework because it consolidates ideas taken from several frameworks and theories by means of a strategy that has been introduced at CERME8 as *networking theories by iterative unpacking* (Koichu, 2013). Mason's theory of shifts of attention (Mason, 1989, 2008, 2010) serves as the overarching theory of the framework. Additional theories are embedded. Each of the next four sub-sections begins with a brief introduction of a particular theory and proceeds to show how the theory contributes to the confluence framework.

Invention of a key solution idea as a shift of attention

Mason's theory of shifts of attention had initially been formulated as a conceptual tool to dismantle constructing abstractions (Mason, 1989) and then extended to the phenomena of mathematical thinking and learning (Mason, 2008, 2010). Palatnik and Koichu (2014, submitted) adopted the theory as a tool for analysing insight problem solving.[2] Mason (2010) defines learning as a transformation of attention that involves both "shifts in the form as well as in the focus of attention" (p. 24). To characterize attention, he considers not only *what* is attended to by an individual but also *how* the objects are attended to. To address the how-question, Mason (2008) distinguishes five different *ways of attending* or *structures of attention*.

According to Mason (2008), *holding the wholes* is the structure of attention, where the person is gazing at the whole without focusing on particular. *Discerning details* is a structure of attention, in which one's attention is caught by a particular detail that becomes distinguished from the rest of the elements of the attended object. Mason (2008) asserts that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships* between the discerned elements is a development from discerned details that often occurs automatically; it refers to specific connection between specific elements. *Perceiving properties* structure of attention is different from recognizing relationships structure in a subtle but essential way. In words of Mason (2008), "When you are aware of a possible relationship and you are looking for elements to fit it, you are perceiving a property" (p. 38). Finally, *reasoning on the basis of perceived properties* is a structure of attention, in which selected properties are attended to as the only basis for further reasoning. Palatnik and Koichu (2014, submitted) added a *why*-question to Mason's *what*- and *how*- questions: Why does an individual make shifts from one object of attention to another in the way that he or she does? Possible ways of addressing this query are related to the obstacles embedded for the solver in attending to a particular object and to continuous evaluation of potential "gains and losses" of the decision to keep attending to the object or shift the attention to another one (Metcalfe & Kornell, 2005).

The process of inventing a key solution idea is seen as a pathway of the solver's shifts of attention, in which objects embedded in the problem formulation or *problem situation image* (this notion is used in the meaning assigned to it by Selden, Selden, Hauk, & Mason, 2000) are attended to and mentally manipulated by applying available schemata. The process at large is goal-directed, but particular shifts can be sporadic. A pathway of the shifts of attention depends on various factors, including: the solver's traits, his or her mathematical, cognitive and affective resources and a context in which problem solving takes place. I now turn to discussing the specificity of the process in three socially different educational contexts.

Shifts of attention in individual problem solving

The lion's share of the data corpus that underlies the development of the foremost problem-solving frameworks (e.g., Schoenfeld, 1985; Carlson & Bloom, 2005) consists of cases of individual problem solving. Carlson and Bloom (2005) consider four phases in individual problem solving by an expert mathematician: orientation, planning, executing and checking. The model also includes a sub-cycle "conjecture-test-evaluate" and operates with various problem-solving attributes, such as conceptual knowledge, heuristics, metacognition, control and affect. Generally speaking, Carlson and Bloom's framework offers a kit of conceptual tools that can be used for producing thick descriptions of individual problem-solving effort. These conceptual tools enter the suggested confluence framework as tools for addressing *how*- and *why*-questions about the shifts of attention.

For example, when solving a challenging geometry problem, a solver can direct her attention to proving similarity of a particular pair of triangles, and then shift her attention to another pair of triangles. The pre- and post-stages of the shift can be described as two "conjecture-test-evaluate" sub-cycles within the planning phase. The shift itself can be viewed in terms of the mathematical, heuristic and affective resources of the solver (see Palatnik & Koichu, 2014, for an elaborated example).

Shifts of attention when interaction with peers is available

While studying problem-solving behaviours in small groups of undergraduate students, Clark, James and Montelle (2014) extended Carlson and Bloom's (2005) taxonomy of problem-solving attributes by introducing two new categories/codes. They termed them *questioning* and *group synergy*. The former category was introduced in order to give room in the data analysis to various questions (for assistance, for clarification, for status, for direction) that the participants had asked. The latter category appeared to be necessary in order "to capture the combination and confluence of two or more group members' problem-solving moves that could only occur when solving problems as a member of a group... A key characteristic of this group synergy code is that it leads to increased group interaction and activity, sometimes in unanticipated and very productive ways." (pp. 10–11).

Indeed, when a possibility to collaborate with peers is available to a solver, his or her shifts of attention can be stipulated by inputs of the group members, especially when the inputs are shared in some common problem-solving space (e.g., a small-group discussion or an internet forum) in a non-tiresome way. Here I would like to stop on the word "sometimes" in the above quotation. The possibility to collaborate can increase one's chances to produce a key solution idea, but can also be overwhelming or distracting. When nobody in a group knows how to solve the problem, the other members' inputs of potential value are frequently undistinguishable for the solver from the inputs of no value. Consequently, it can become too effortful for the solver to follow and evaluate the inputs of the others.

Schwartz, Neuman and Biezuner (2000) deeply explored, in laboratory setting, the cognitive gains of two children, who fail to solve a task individually, but who improve when working in peer interaction. They characterized the situations, in which (in their words) *two-wrongs-make-a-right* vs. *two-wrongs-make-a-wrong*. The mechanisms of co-construction behind *two-wrongs-make-a-right* phenomenon were: the mechanism of disagreement, the mechanism of hypothesis testing, and the mechanism of inferring new knowledge through challenging and conceding. These mechanisms might be involved in those cases of collaborative problem solving, in which group synergy led the participants in Clark, James and Montelle's (2014) study to "very productive ways" (ibid) of solving the given problems.

The confluence framework seeks to consolidate the theoretical insights of the aforementioned studies. In particular, to further explore the phenomenon of group synergy, it seems me necessary to acknowledge that the above mechanisms can become active on condition that at least sometimes a solver shifts his or her attention from an object that he or she is being exploring to an object attended to by the peer.

An example of the processes of co-constructing a key solution idea as pathways of students' shifts of attention is drawn from an on-going study on high-school students' long-term geometry problem solving. The example concerns a situation, in which a group of 16 10th grade students were engaged in solving the following problem:

> Two circles with centres M and N are given. Tangent lines are drawn from the centre of each circle to another circle. The points of intersection of the tangent lines with the circles define two chords, EF and GH (see Figure 2). Prove that segments EF and GH are equal.

The students could solve the problem individually, but also share their ideas in a closed forum in one of the social networks. Interestingly, all the students indicated in the reflective questionnaires that they had worked collaboratively only for about 40% of time that had been devoted to the problem. (On average, the students worked on the problem for 3 hours). The

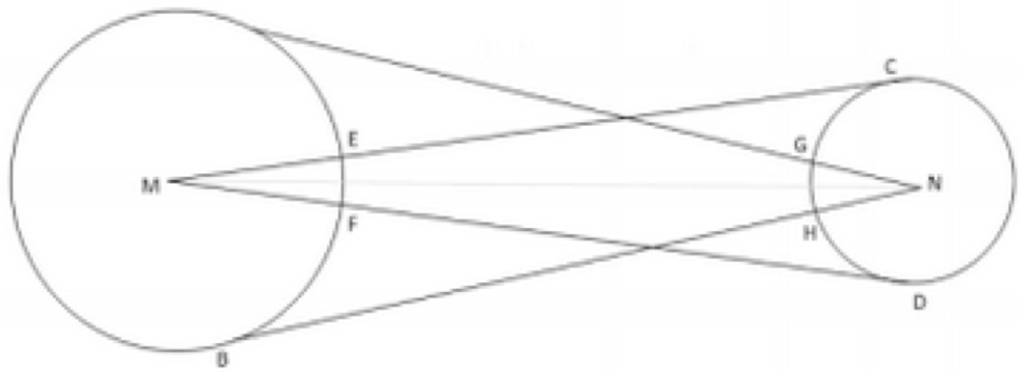

Figure 2: A problem about two chords

shifts of attention of individual problem solvers were stipulated either by individual or by shared problem-solving resources. As a rule, the students chose to collaboratively work in the forum when they were stuck and sought for new ideas or for the feedback on their incomplete ideas. In a few cases the students chose to shift their attention to the ideas of the other students.

Shifts of attention when interaction with a solution source is available

The option to interact with a source of knowledge about a key solution idea to a problem can drastically change a pathway of one's shifts of attention, up to the point that the entire process can stop being a problem-solving process and become a solution-comprehending process. The suggested framework seeks to encompass only the situations in which a solution source is present as a provider of cues to the solution or as a convenient storage of potentially useful facts, but not as a source of *telling* the solution. Such situations are common, for instance, when a teacher orchestrates a classroom problem-solving discussion.

When a source of knowledge about the solution is present but does not tell the solution, the solvers may attempt to extract the solution from the source (e.g., see *questions for assistance* and *questions for direction* in Clark, James, & Montelle, 2014; see also Koichu & Harel, 2007). In some cases, the solver's shifts of attention may occur as a result of a conflict that emerges when more knowledgeable and less knowledgeable interlocutors assign different meanings to the same assertions (cf. Sfard, 2007, for commognitive conflict).

For example, the assertion "Triangle similarity is a good idea" can either pass unnoticed in the group discourse or be a trigger for the solver to shift his or her structure of attention. The effect of the assertion would depend on who it has come from, a regular member of the group or a teacher or a peer who acts as if she has already solved the problem. The occurrence of the shift in one's attention as a result of another person's assertion depends not only on that person's status in the group. It is mainly the matter of different meanings that can be assigned to the assertion by different individuals. In one case, the assertion about triangle similarity may be perceived as, "It is possible that similarity helps," in another, "I've tried it and it helps," and in yet another, "This is the direction approved by the authority." Stimulated by Sfard (2007), I suggest that such a conflict of meanings can first be unnoticed, then it can hinder the communication, and then (when the assigned meanings are explicated), it can help the less knowledgeable solvers to progress.

SUMMARY AND FURTHER DEVELOPMENT

Developing a confluence framework of mathematical problem solving that would be applicable to different educational contexts is motivated by several causes. First, with few exceptions, the existing problem-solving frameworks utilize different conceptual tools for exploring problem solving in socially different educational contexts. Second, the foremost frameworks are comprehensive within the problem-solving contexts from which they have emerged but it is sometimes difficult to apply them to additional contexts. Third, in spite of the comprehensive nature of the existing frameworks, the central problem-solving issue of inventing (as opposed to recalling) a solution method is still not sufficiently understood. At the same time, theoretical tools that can help to progress the state of the art are available from the other sub-fields of mathematics education research. Hence, a *confluence* framework.

In this article, a particular way of constructing a confluence framework is presented. The confluence effect is pursued by considering common roots of problem solving in three socially different contexts. This is done in terms of Mason's theory of shifts of attention, which initially had been constructed for other reasons. Simultaneously, the specificity of the attention shifts in different problem-solving contexts is considered by means of additional theoretical constructs. The use of the model as a research tool for understanding heuristic aspects of problem solving is stipulated by availability of research methodologies for identifying and characterizing shifts of attention in socially different problem-solving contexts. In part, such methodologies are available from past research (e.g., Mason, 1989, 2008, 2010; Palatnik & Koichu, 2014) but they should be further developed. Our research group currently works in this direction and explores long-term geometry problem solving supported by online discussion forums.

As mentioned, the framework is only *exploratory*. The outlined mechanisms of attending to, proceeding of and shifting between objects of attention should be further unpacked. At this stage, it seems that further unpacking would require the adaptive use of selected theories that have been developed outside of the field of mathematics education. For instance, research on learners' decisions about how to allocate study-time (e.g., Metcalfe & Kornell, 2005) can be a source of insights about why some objects of attention are short-living, and the others are long-living. Research on hypothetical thinking and cognitive decoupling (Stanovich, 2009) can be useful for understanding how the attended objects are mentally manipulated. The hope is that, eventually, the confluence framework would have power not only to usefully describe, but also explain the emergence of problem-solving ideas in different educational contexts.

ACKNOWLEDGEMENT

The research is supported by the Israel Science Foundation (grant 1596/13, PI Koichu). I thank Royi Lachmy for valuable comments on a draft of this article.

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ENDNOTES

1. Ramon (2003) explains what a heuristic idea of a proof is as follows: "This is an idea based on informal understandings, e.g. grounded in empirical data or represented by a picture, which may be suggestive but does not necessarily lead directly to a formal proof." (p. 322). Note that not any heuristic idea is a key idea.

2. The next 10 sentences consist of an abridged version of the description that appears in Palatnik and Koichu (2014).

Theories to be combined and contrasted: Does the context make a difference? Early intervention programmes as case

Lena Lindenskov[1], Pia Beck Tonnesen[2], Peter Weng[2] and Camilla Hellsten Østergaard[2]

1 Aarhus University, School of Education, Aarhus, Denmark, lenali@edu.au.dk

2 Metropolitan University College, Department of School and Lerning, Copenhagen, Denmark

The constructivist paradigm is the primary underpinning of the Mathematics Recovery Programme (MRP) in Australia, the UK, the USA and Canada. The critical mathematics education paradigm is the primary underpinning of the Early Mathematics Intervention Programme for Marginal Groups in Denmark (TMTM for Tidlig Matematikindsats Til Marginalgrupper). The paper tells the story of how networking strategies have informed the design of TMTM: one strategy is combining elements from the two paradigms; another strategy is contrasting elements from the paradigms. The two constructs 'the six stages-construct' and 'the math holes-construct' have been decisive for the networking. Possible influences from the Danish educational contexts on the networking processes are put forward for further discussion.

Keywords: Networking theories, early intervention in mathematics, math holes, constructivism, critical mathematics education.

CULTURAL CONTEXT AND OUR CULTURAL CONTEXT

A main issue at CERME9 was cultural contexts in European research in mathematics education. What is meant by cultural contexts and what are the mechanisms by which cultural contexts influence research and practice in mathematics education were displayed in the plenary by Barbara Jaworski, Mariolina Bartolini Bussi, Edyta Nowinska and Susanne Prediger.

This paper primarily focuses at issues in primary school mathematics in a certain cultural context: the Danish educational culture. The development of school mathematics in Denmark is in no way isolated from Nordic, European or global tendencies and policies. For instance the 2004 OECD review of the Danish primary and lower secondary school emphasized the need to support failing pupils in mathematics in the first school years (Mortimore, David-Evans, Laukkanen, & Valijarvi, 2004). It is evident that the OECD review has influenced Danish policy, and so have Danish teachers' and researchers' initiatives to improve the school support for pupils at risk of falling behind in their mathematics learning.

It seems natural to Danish mathematics teachers and educational researchers to get inspirations from abroad, also when it comes to interventions for pupils at risk. Generally speaking, inspirations from theory and practice in school mathematics in especially the Netherlands have been strong since 1970s, and Realistic Mathematics Education from the Freudenthal Institute is still influential.

Still, some characteristic aspects of the Danish school mathematics seem evident and are visible in national goals and aims despite influences from international politics and tendencies. This paper will look into the tension between national and international trends for educational research and practice, by looking at the development processes behind the Danish early mathematics intervention programme for marginal groups (TMTM for Tidlig Matematikindsats Til Marginalgrupper) as a case. Especially the theoretical networking in the development processes is to be explored. We are ourselves central persons in these development processes, which we aim to document and critically assess.

We generally argue that how theories are being interpreted and networked locally is influenced by local contexts as for instance Danish school mathematics cultures. It is shown that culture does matter for values and practices in mathematics education. As an example, teachers in London and in Beijing hold different views on mathematical learning and teaching. Teachers in London see syllabus and textbooks as less important in determining the content taught than interest and meaningfulness, which is again opposite to teachers in Beijing (Leung, 2006). Similarly, teachers in London see students' ability as more important for their learning than effort, which is the opposite of teachers in Beijing. We claim that it is by now a well-known fact that culture matters for values and practices in mathematics education, but how culture matters in networking theories, is not yet well addressed.

CONSTRUCTIVIST THEORY UNDERPINNING MRP

Among the most common implemented programmes internationally for early intervention in mathematics the Mathematics Recovery Programme (MRP) is to our knowledge the most described and researched one. It is broadly implemented in Australia, the UK, Ireland and the USA (Wright, Martland, & Stafford, 2000) (Wright, Ellemor-Collins, & Tabor, 2011). We have studied programme materials and background papers and we have made personal contacts to leading figures like Robert J. Wright and like Noreen O'Loughlin, University of Limerick.

MRP is an intensive one-to-one tutoring offered to first graders falling behind and at risk of mathematics difficulties. Since 2006 the material *Teaching number – Advancing children's skills and strategies* has been suggested for whole-class teaching, too. Tutor training is a mandatory part of the programme. A learning framework and an instructional framework are described in detail. The tutoring is meant to be diagnostic, so that instruction can be adapted to pupils' reactions.

The mathematical content in MRP focuses on numbers and arithmetic. The MRPMP covers knowing and understanding numbers, names and symbols, basic strategies in the four operations (Wright, Martland, Stafford, & Stanger, 2006) and supplemented with basic understanding of part-whole concept (Wright et al., 2011).

Important for this paper are the theories behind the MRP, which is drawn primarily from Leslie P. Steffe and colleagues. Steffe was engaged as early as 1976 in establishing the research program, Interdisciplinary Research on Number (IRON). As documented in (Steffe, van Glasersfeld, Richards, & Cobb, 1983), (Steffe, Cobb, & van Glasersfeld, 1988) and (Steffe, 1992), a tremendous empirically based work has been going on to explore how children reason and meaningfully grasp processes and concepts related to numbers and arithmetic. The research has led to insights into many diverse aspects of children's mathematical knowledge and how this knowledge is developed.

The constructivist underpinning manifests itself by maintaining that knowing is not passively importing other peoples' knowledge as if you were receiving a birthday gift. It is maintained and underlined that knowing is an active endeavour for the person, who is acquiring/constructing knowledge through genuine problem solving.

THE SIX STAGES CONSTRUCTS

The idea of development through stages runs as a general structuring idea through the work of Steffe et al. Like the construct of the five stages of geometrical reasoning from van Hiele (1985/1959), it seems that an underlying idea of stage has been transformed into several constructs of several and often six stages by Steffe et al. and by their followers.

While the descriptions of the counting stages used in solving addition and subtraction problems is grounded through the work of Steffe et al., the stage constructs are further developed by Wright (1991, 1994) and by Wright and colleagues (2000, 2006, 2011). The stages of children's development of different areas of mathematical knowledge are put forward in the learning framework and in the instruction framework of MRP. The organisation of how tutors' come to observe pupils' development and how tutors adapt instruction to the pupils' present knowledge is facilitated by the lenses of stages.

In Wright and colleagues (2006), the idea of six early arithmetical strategies is used as the main organising principle. The six stages go from stage zero to perceptual, to figurative, to initial number sequence, to implicitly nested number sequence and end with explicitly nested number sequence. Each stage is described. At the last one, the explicitly nested number sequence, the child uses a range of non-count-by-ones strategies. The child also uses a known result, adding to ten and commutativity, and the child knows that subtraction is the inverse of addition (p. 9).

The Wright and colleagues' book from 2006 presents a network of stages in each of the mathematical area covered. Also for *forward number word sequences and number word after* six stages are described. The same for *backward number word sequences and number word before,* while *number identification* has five stages, and *base-ten arithmetical strategies* has three stages.

CRITICAL MATHEMATICS EDUCATION

The Nordic part of critical mathematics education has primarily been dominated by the Danish scholars Mogens Niss (1984) and Ole Skovsmose (1994, 2001) and the Norwegian scholar Stieg Mellin-Olsen (1987, 1991). Niss underlined that it is an important but difficult task for research and practice in mathematics education to relate to authentic applications of mathematics from a critical stance of view, and to let mathematics education address fundamental social functions of mathematics and school mathematics. For school mathematics this implies mathematical modelling and project work. Skovsmose added the philosophical view of learning mathematics as investigation of mathematical landscapes. For school mathematics this implies a critique towards exercise paradigm and a focus on communication between teachers and pupils. Skovsmose also invented the concept of pupils' foreground as just as important as pupils' background. Mellin-Olsen (1991) added the project work about authentic applications should be for all. It should be for less able and less motivated pupils as well as for all other pupils. He also added the metaphor of travelling as a tool to understand teachers' thinking about instruction. He recognised that teachers talked about their mathematics instruction as being the drivers in the busses full of pupils.

THE MATH HOLES-CONSTRUCT

Based on Nordic critical mathematics education we have researched special needs education in Denmark. We found that the ideas from Niss, Skovsmose and Mellin-Olsen of mathematics instruction and learning were all promising. For instance we found the idea of mathematics instruction and learning as a common teacher-pupil investigation of mathematical landscapes promising. Since 2003 Bøttger, Kvist-Andersen, Lindenskov and Weng (2004), we have therefore been involved in demonstrating mathematics learning as a journey in landscapes, which evolve with hills and holes as you travel. Many routes can be appropriate for the teachers and students involved, depending of their backgrounds and foregrounds and depending on available authentic (in Freudenthal's sense *realistic*) materials and activities for problem solving and for modelling.

Closely connected to the general idea of investigating or constructing mathematics landscapes, we needed to add a specific idea of pupils at risk of becoming low-level performers. Therefore we invented the math holes-construct.

The math holes-construct metaphorically describes pupils in difficulties as pupils, which for the moment are stuck and are not progressing further into the mathematics landscape. It is as if the pupil has fallen into a trap or – as we finally decided to name it – it is as if the pupil has fallen into a hole.

Sure, it is not our intention with the math holes-construct to see learning mathematics in a negative biased light. Neither is it our intention to avoid mathematical elements, which may cause obstacles, as for instance learning the rules of the positional decimal number system. Our intention is to develop pedagogical constructs to help teachers recognise and support every child's mathematical learning. Therefore it is important to identify when a child is stuck and to provide intervention models, materials and teacher training for how schools and teachers can help getting the pupil moving again.

Within the math holes-construct pupils in difficulties are not pupils lacking behind or pupils with special neurological characteristics as implied by some definitions of dyscalculia. Within the math holes-construct pupils in difficulties are pupils who stopped progressing learning. Gervasoni and Lindenskov (2011) describe the construct with the following:

> When mathematics is seen as a landscape it means that whenever students stop learning and feel stuck it is as if they 'fall into a hole'. There are several ways for a teacher to cope with a student's 'fall'. First, a teacher can invite the student to move to another type of landscape, maybe far away from the hole in which the student was stuck; this means that even when students fail to thrive in one area of mathematics there are still many other mathematics landscapes to experience and learn. Second, teachers can help students 'fill up' the hole from beneath with mathematical building stones; or third, teachers can 'lay out boards over the hole' in order to let the student experience new and smart mathematical approaches. (p. 317)

It is, in our view of the math holes-construct, crucial to draw attention to the individual pupil: Much still needs to be done, following Ginsburg's (1997) more than 15-years-old call for teaching experiments focusing on pupils with learning difficulties, as children today are exposed to physical and social environments that are rich in mathematical opportunities. In Denmark as in many other countries, children today are exposed for instance to even very big numerals in computer and board games and in family activities. This questions the generality of the existing cognitive models of children's reasoning and of the theories of instruction. For instance it questions the motivating effect as well as the learning effect of splitting instruction on numbers into first 1–20, and then 1–100, as it is done in MRP.

What children are exposed to rapidly develops, and no one can predict what will happen just a few years from now. This means that the generalisation of identified mathematics learning trajectories in research studies may not be generalizable to all children in the world. On the other hand, the learning trajectories, which are found in research, and upon which MRP have built their stages, do show valuable insight into children's learning, which is not visible for teachers even with profound training. We do advocate for teachers to obtain extended knowledge on researched learning trajectories.

NETWORKING THEORIES TO INFORM DEVELOPMENT OF TMTM

Our design processes towards the Danish model for early intervention for marginal groups TMTM were highly informed by networking theories from constructivism and from critical mathematics education. Tutor material is published in a 191 pages book (Lindenskov & Weng, 2013), but lots of work were done before.

Certainly some authors see the two paradigms as contradictory, and Mellin-Olsen is a prominent example. He disliked the focus on the cognitive development of the individual pupil in the constructivist paradigm.

But as researchers have showed for the last decade networking theories are much more than rejecting or approving specific theories. Bikner-Ahsbahs and Prediger (2010, p. 492) suggested the following illustration of a landscape of strategies for connecting theoretical approaches (Figure 1).

We chose to let the development processes for TMTM be informed by *contrasting* some of the above mentioned theories and constructs and by *combining* others.

The *combining* part includes the deep insight into individual pupils' problems and development coming from Steffe and colleagues' constructivistic research *combined* with the math holes-construct.

Also in the Danish programme tutor training is a mandatory part of the programme, and also the tutoring is meant to be diagnostic, so that instruction can be adapted to pupils' reactions.

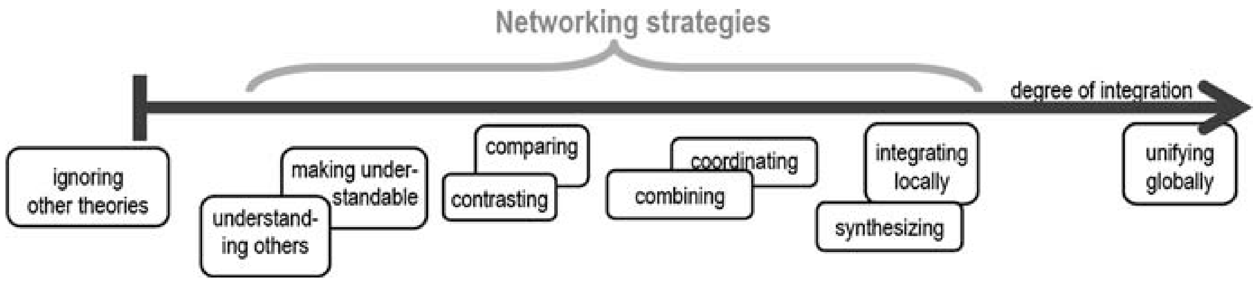

Figure 1

The *contrasting* part includes that Steffe and colleagues' focus solely on numbers and arithmetic *contradicts* the broader goals and aims in critical mathematics instruction. It is not enough for a Danish model to focus on numbers and arithmetic. We find them necessary, but not sufficient.

The *contrasting* part also includes that the stage constructs *contrast* the idea of learning mathematics as a journey where parts of the mathematical landscapes are being constructed as pupils and teachers travel together. The paths are chosen by teachers and pupils collaboratively, and the teachers take into account children's background and foreground in order to motivate the children. The stage construct does not allow that the teachers also take the pupils' foreground into account.

CONTEXTUAL INFLUENCE ON THE THEORY NETWORKING

School cultures in Denmark are influenced by the German/continental tradition in educational philosophy. Danish language has, like German language, two main concepts for 'Education': one is 'Dannelse' (German: Bildung), another is 'Uddannelse' (German: 'Erziehung').

At the level of school mathematics this tradition emerges in the common aim for primary and lower secondary school mathematics by pointing at everyday life, citizen life, creativity, problem solving, and democratic responsibility and impact:

> The aim is that students develop mathematical competences and acquire skills and knowledge in order to appropriately engage in math-related situations in their current and future everyday, leisure, education, work and citizen life.
>
> Subsection 2. Students' learning should be based upon that they independently and through dialogue and cooperation with others can experience, that mathematics requires and promotes creative activity, and that mathematics provides tools for problem solving, reasoning and communication.
>
> Subsection 3. Mathematics as a subject should help the students experience and recognize the role of mathematics in a historical, cultural and social context, and that students can reflect and evaluate application of mathematics in order to take responsibility for and have an impact in a democratic community. (Undervisningsministeriet, 2014, our translation)

On this background it was evident, that the constructivist paradigm had to be combined with the critical mathematics education. The constructivist paradigm gave a too narrow picture of what should be learnt.

According to the choice of subject matter, the school mathematics tradition emerges in broad scope throughout all school grades and for all pupils. Danish pupils are not streamed before Grade 10. This means that skills, conceptual understanding and authentic applications are taught to all pupils. Mathematical competences as well as mathematical conceptual fields like numbers and algebra, geometry and measurement, and statistics are included for all pupils from the very start of primary school.

Besides it is our impression that Danish teachers have a relatively high self-confidence and a relatively strong wish to influence. From the European comparative perspective it seems as if Danish teachers more than teachers in England prioritise the pupils' personal development and see the pupils' mathematical development as a means for personal development (Kelly, Pratt, Dorf, & Hohmann, 2013).

This led us to include a group of teachers, in a decisive way, in design cycles in the development towards a Danish model (Lindenskov & Weng, 2014), and it led us to make the material much open for teachers' adaptation. This also led us to *confront* the stage construct with the math holes-construct. With a stage construct less is up to the tutor to decide, and with a math holes-construct pupils' emotional and motivational background and foreground are easier to take into account.

CONCLUSION

Thus, the Danish developmental and research projects on early intervention are based on an original construct, the math holes-construct. But the whole development processes are very much informed by *combining* and *contrasting* theories.

With Skott (2015), in our view networking theories is very much depending on what we mean by theory, constructs and conceptual framework. Skott points to decisive elements as a) preliminary understanding of concepts involved, b) theoretical stance on interpretation of these concepts, and c) the overall rationale for engaging in the field of inquiry.

The math holes-construct implies differences compared to MRP concerning all three elements, although similarities between MRP and TMTM exist.

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Tool use in mathematics: A framework

John Monaghan

University of Agder, Kristiansand, Norway, john.monaghan@uia.no
University of Leeds, Leeds, UK

In the course of research into the interpretation of tools in the didactics of mathematics I found both voids and conflicts. This paper presents the results of my research and a resultant statement on tool use in mathematics education. The statement incorporates constructs from several theoretical frameworks and I consider the consistency of my statement on tool use with regard to activity theory.

Keywords: Activity theory, agency, artefacts, mediation, tools.

INTRODUCTION

In the course of work on tool use in mathematics I examined literature which I summarise in this paper. The literature sit in various theoretical frameworks and this paper, in the language of Prediger, Bikner-Ahsbahs and Arzarello's (2008), can be considered as an attempt to 'synthesise' frameworks with regard to a statement on tool use in the didactics of mathematics. This synthesis, however, does not aim at synthesising complete theories but synthesising activity theory with principles from other theories. This paper has the following structure: a definition of tools; a survey of theoretical frameworks with regard to tools; an exposition of activity theory with regard to tools; actor network theory ideas that augment an activity theoretic account of tools; an activity theoretic statement on tool use in mathematics education which incorporates ideas from outside of activity theory, and a consideration of the consistency of this statement with regard to networking theories.

TOOLS: A DEFINITION

I define a tool via four action-related distinctions, the first of which is between an artefact and a tool. An artefact is a material object which becomes a tool when it is used by an agent to do something; a compass becomes a tool when it is used to draw a circle (its intended purpose) or to stab someone. This establishes that tool use cannot be separated from the animal using the tool and the purpose of use. After being used as a tool (for whatever purpose), the compass returns to being an artefact. The materiality of an artefact is not just that open to touch. An algorithm, e.g., for adding two natural numbers, is an artefact. It is material in as much as it can be written down or programmed into a computer. My second distinction is between an artefact/tool and ways of using the artefact/tool. For example, I could use a calculator to perform 45 + 67 by typing in '45 + 67=' or I could imitate the standard written algorithm (adding the units, storing the result) and adding the tens and adding on my stored results. My third distinction is between the mental representation of a tool and material actions in tool use. This distinction comes with an interrelationship: to carry out material actions with an artefact you need some form of mental representation, which may be quite crude, of how to act with the artefact-tool, but actions with the artefact-tool will provide feedback to the user which may change the mental representation. My fourth distinction is between signs and tools. Signs, like tools, are artefacts but a sign points to something whereas a tool does something. Having said this, signs or systems of signs, can function as tools. Similarly representations can function as tools but they may also have non-tool functions, e.g., to point to an object.

Is there such a thing as a 'mathematical tool'? – only in use, a compass is a mathematical tool when it is used to draw a circle but not when it is used to stab someone. When artefacts are used for mathematical purposes they generally incorporate mathematical features, e.g., a compass encapsulates the equidistant relationship between the centre and points on the circumference of a circle.

A SURVEY OF FRAMEWORKS WITH REGARD TO TOOLS

I conduct an historical tour of theoretical frameworks employed in Western mathematics education. I select papers from the 1960s to the present which reflect dominant 'grand theories' over this time that address or ignore tools. Behaviourism regarded artefacts as a means of stimulating a response in a subject. Suppes (1969), for example, considers computers as tutorial systems that can provide:

> individualized instruction [where the] intention is to approximate the interaction a patient tutor would have with an individual student ... as soon as the student manifests a clear understanding ... he is moved on to a new concept and new exercises." (ibid, p.43).

Suppes does not consider the environment in which the tool is used. During the period when behaviourism ruled two psychologists, E. and J. Gibson, ventured on a non-behaviourist route to the theory construct of affordances (and constraints):

> The *affordances* of the environment are what it *offers* the animal, what it *provides* ... If a terrestrial surface is nearly horizontal ... nearly flat ... and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface *affords support*. (Gibson, 1979, p. 127)

There is no mention of tools in this quote but mathematics educators have learnt that the construct 'affordances' is useful in considerations of the relevance of artefacts and digital software environments to students' mathematical learning.

The demise of behaviourism in mathematics education saw the rise of cognitive studies and Piaget was the Guru. The interesting thing about Piaget's extensive output with regard to tool use is that he says nothing at all about the role of tools in cognitive development. Piaget's work inspired several 'local theories' in mathematics education: Brousseau's theory of didactical situations (TDS), constructionism and constructivism. TDS was developed over decades starting in the 1960s. The influence of Piaget in Brousseau's work is explicit. An important construct of TDS came to be called the 'milieu' which includes the teacher, the materials and the designed learning strategies. I know of no explicit consideration of mathematical tools in 20th century TDS but tools are a part of the milieu. Papert, who spent several years with Piaget, experimented with children using the computer language *Logo*. Constructionism views that learning occurs through the construction of meaningful products. *Logo* is integral to constructionism but, despite statements that these languages equip students with tools to think with, there is no clear statement as to what a tool is in Papert (1980) and a clearer constructionist view of tools did not emerge until Noss and Hoyles (1996) – by which time constructionism had relinquished its Piagetian roots and embraced socio-cultural viewpoints.

Piaget's ideas inspired constructivism, which focused on the ontogenic development of the individual child but developed to include a focus on microgenetic (child-environment) development (social constructivism). Yackel and Cobb (1996) is a developed form of social constructivism which examines teacher-student discussions and argumentation in a classroom context. This paper introduced the construct 'sociomathematical norms'. The classroom considered in the paper had various resources (centicubes and an overhead projector) but the paper does not mention tools. This neglect has been noticed by others, e.g., Hershkowitz and Schwarz (1999, p. 149) "... socio-mathematical norms do not arise from verbal actions only, but also from computer manipulations as communicative non-verbal actions."

In summary, 20th century mathematics educator frameworks influenced by Piagetian ideas had little to say on tools in learning and teaching but outside of mathematics education, deep ideas, published in the 1970s, on tools were in circulation.

Wartofsky (1979) includes an essay on perception, "an *historically* evolved faculty ... based on the development of historical human practice" (ibid, p.189). Practice is "the fundamental activity of producing and reproducing the conditions of species existence ... human beings do this by means of the creation of artefacts ... the 'tool' may be *any* artefact created for the purpose" (ibid, p. 200). Wartofsky extends the concepts of artefacts to the skills required to use artefacts as tools:

> Primary artefacts are those directly used in this production; secondary artifacts are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artefacts are therefore representations of such modes of action (ibid, p. 202)

Vygotsky (1978), published posthumously, was to have a profound influence on mathematics education. Vygotsky was interested in language, signs and mediation. His interest in tools was in their mediating qualities, "the basic analogy between sign and tool rests on their mediating function that characterizes each of them" (1978, p. 54). The difference between signs and tools rests on:

> The tool's function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to a change in objects ... The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is *internally* oriented. (ibid, p. 55)

The reader, however, may note the influence of both Wartofsky and Vygotsky in my definition of tools above. I now move on to the work of B. Latour and P. Rabadel[1]. Latour is a sociologist and, around 1980 and with others, established what is now called actor network theory (ANT); Latour (2005) is a fairly recent exposition. ANT is a theory about how to study social phenomena – by following the actors, where an actor is "*any thing* that does modify a state of affairs by making a difference" (ibid., 71). ANT symmetrically views both society and nature as being in a state of flux and looks to the performance of the actors in situations. Objects (artefacts/tools) can make difference in performance and so can be actors, exerting agency, in the playing out of social situations. Pickering (1995), who is 'almost ANT' in my opinion, examines practices of 20th century elementary particles physics. He accepts ANT's human and material agencies and adds 'disciplinary agency' (in our discipline $a+a=2a$ regardless of what we might want it to be). He proposes a 'dance of agency' where, in the performance of scientific inquiry, human, material and disciplinary agencies "emerge in the temporality of practice and are definitional of and sustain one another" (ibid., p.21). I see this in 'dance' in techno-mathematics lessons – a myriad of influences between students, teachers, computers and mathematics.

Rabadel introduced the 'instrumental approach' which distinguishes between an artefact, as a material object, and an instrument as a psychological construct. An instrument is an emergent entity that begins its existence when a person appropriates an artefact to do something; this has influenced my distinction, above, between an artefact and a tool. The instrumental approach has been well known in mathematics education since Guin and Trouche (1999). This views an instrument as a composite entity composed of the artefact and knowledge (knowledge of the artefact and of the task constructed in using the artefact). Artefact and the agent(s) are interrelated: the artefact shapes the actions of the agent, *instrumentation*; the user shapes the use of the artefact, *instrumentalisation*. The process of turning an artefact into an instrument is called 'instrumental genesis'. The agent brings her/his knowledge and the artefact brings its potentialities and constraints to the artefact agent interaction.

I leave my historical tour at this point with the observation that a lot of the frameworks used in mathematics education pay scant regard to the nature of the tools used in doing mathematics but frameworks initiated by Wartofsky, Vygotsky, Latour, Pickering and Rabardel provide interesting, though diverse, insights into the role of tools in activity. I now turn to the focus framework of this paper, activity theory.

ACTIVITY THEORETIC CONSIDERATION OF TOOLS

I briefly outline activity theory (AT), trace its genesis into mathematics education research (MER) and consider differences in approaches.

AT is an approach to the study of human practices. It sees constant change (flux) in practice. Activity became a focus for Vygotsky in his conviction that consciousness originated in socially meaningful activity. In AT 'object orientated activity' is the *unit of analysis*, that which preserves the essence of concrete practice. 'Object' here refers to *raison d'etre* of the activity. Educators employing AT must take care that they do not merely employ the word 'activity' without considering the object and the unit of analysis. Vygotsky's AT is often presented via a triangle with 'subject', 'object' and 'mediating artefacts' at its vertices. Leont'ev (1978) developed Vygotsky's work by considering individual and collective *actions* (usually with tools) and *operations* (things to be performed or modes of using tools) involved in socially organized *activity*. Engeström (1987) extends Vygotsky's and Leont'ev's frameworks to 'activity systems' and extends the focus on mediation through signs and tools to multiple forms of mediation including the community and social rules underlying activity. Activity systems research often examines interactive activity systems with a focus on the objects of activity in the two systems; the place of tools in such research usually emphasises tool use in the context of the whole system. I now turn to the influence of activity theory in MER.

I was curious of AT's introduction into Western MER literature and I traced its introduction into the journal *Educational Studies in Mathematics* (ESM). Two AT papers appeared in ESM in 1996. Crawford (1996) is an exposition of Vygotskian AT and asks "What difference does the use of tools such as computers and calculators make to the quality of human activity?" (ibid, p. 47) but does not explore the nature of tools further. Bartolini Bussi (1996) reports on a teaching experiment on geometric perspective. The word 'tool' has two uses in the paper: Leont'ev's theory as a tool for analysis; 'semiotic tools', which are defined via examples. In 1998 two ESM AT papers considered tool use in different ways to Bartolini Bussi (1996). Chassapis (1998) focuses on the processes by which children develop a formal mathematical concept of the circle by using various instruments to draw circles: by hand; using circle tracers and templates; and using a compass. "The process of learning to use a tool ... involves the construction of an experiential reality that is consensual with that of others who know how to use [the tool]" (ibid, p. 276). Pozzi, Noss and Hoyles (1998) focuses on nursing and ask "how do resources enter into professional situations, and how do they mediate the relationship between mathematical tools and professional know–how?" (ibid, p. 110) The paper states that AT provides evidence that "acts of problem solving are contingent upon structuring resources, including a range of artefacts such as notational systems, physical and computational tools" (ibid, p. 105). Radford (2000) focuses on early algebraic thinking "considered as a sign-mediated cognitive *praxis*" (ibid, p. 237):

> to accomplish actions as required by the contextual activities ... The sign-tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification (ibid, p. 241).

The first mention of Engeström in ESM is in Jaworski (2003, p. 249). This outlines "*insider* and *outsider* research and *co-learning* between teachers and educators in promoting classroom inquiry" and is not concerned with tool use in mathematics.

Thus, although AT is quite an old theory, it is a fairly recent theory in terms of Western MER and there is wide variation with regard to the meaning of tools in ESM AT papers from 1996 to 2003. After 2003 a considerable number of ESM papers used AT as a theoretical papers but I do not have room to summarise. To get a handle on contemporary AT conceptions of tools in MER I go to a special edition of *The International Journal for Technology in Mathematics Education* devoted to AT approaches to mathematics classroom practices with technology. For reasons of space I focus on three (of 11) papers which illustrate a range of approaches.

Chiappini (2012) focuses on the learning and teaching of algebra with software with a visual 'algebraic line' and conventional algebraic notation, to draw students' attention to the culture of mathematics. Chiappini is interested in 'cultural affordances', which, "allow students to master the meanings, values and principles of the cultural domain" (ibid, p. 138). With regard to tools, Chiappini's focus is the evaluation of software designed to exploit visuo-spatial and deictic affordances and allow teachers to consolidate student learning. Ladel and Kortenkamp (2013) focuses on the design and use of a multi-touch-digital-table to engage young children in meaningful work with whole number operations, "We want to restrict the students' externalizing actions to support the internalization of specific properties ... mediation through the artefact is characterized by restriction and focussing." Artefacts are the focus of attention and the word 'tool' is not mentioned in the paper. They hold that "the artefact itself does not have agency and is only mediating ... [but] the artefact changes the way children act drastically and in non-obvious ways" (ibid, p. 3). Mariotti and Maracci (2012) outline the Theory of Semiotic Mediation (TSM) with regard to "the use of artefacts to enhance mathematics learning and teaching, with a particular focus on technological artefacts" (ibid, p. 21); like Ladel and Kortenkamp (2013) above, the word 'artefact' is favoured over the word 'tool'. This paper continues the work of Bartolini Bussi (1996) considered in the previous section and is critical of research where "the mediating function of the artefact is often limited to the study of its role in relation to the accomplishment of tasks" (ibid). TSM views that "teaching-learning ... originates from an intricate interplay of signs... mathematical meanings can be crystallized, embedded in artefacts and signs" (ibid) The paper presents a rather strange (to me) take on mediation, "The mediator is not the artefact itself but it is the person who takes the initiative and the responsibility for the use of the artefact to mediate a specific content" (ibid, p. 22). To mediate the learning of mathematics the teacher has to design specific circumstances, a didactical cycle, aimed at fostering specific semiotic mediation processes.

Differences in the papers outlined above include the unit of analysis, cognition, the words used, mediation and agency. Some papers explicitly state the unit of analysis, e.g., Chassapis (1998), but many do not. Chassapis' unit of analysis is 'quite small' compared to Engeström's, the activity system itself. I think the 'size' of the unit of analysis impacts on the extent to which the AT analysis permits a study of microgenetic learner development with tools (i.e., Chassapis' unit of analysis allows a focus on cognition and tool use but details of cognitive development are easily 'lost' when the focus is on activity systems). With regard to the words used it is clear that some scholars use 'artefact' for what I refer to as a tool. This seems unimportant but the difference between sign and tool is important and the fact that this difference is sometimes blurred does not downplay this importance; some of the papers do not consider signs vs tools. With regard to mediation the biggest difference is between Ladel and Kortenkamp, where artefact mediation is central, and Maracci and Mariotti, which holds that people and not artefacts mediates. My final consideration concerns agency. Only Ladel and Kortenkamp comment on this, to claim that artefacts do not have agency. The differences noted above show that AT in MER is a collection of approaches in which there are many ways to view tools within AT.

ANT IDEAS THAT AUGMENT AN AT ACCOUNT OF TOOLS

I am drawn to AT as a framework as it mirrors my view that tools are important but tool use is not an activity in itself though tool use and activity are interrelated. But I detect an anthropocentric position in AT – even though AT recognises that people think through/with tools, people are at the centre, they appear as 'the' agents. This anthropocentrism is explicit in Maracci and Mariotti's view that artefacts are not mediators and Ladel and Kortenkamp's statement that artefacts do not have agency. I think tools can be powerful things and I am drawn to an ANT view on material agency, but can ANT ideas be brought into AT? I first look at a potential major obstacle to networking these theories and a difference between Latour and Pickering.

Miettinen (1999) considers ANT and AT as approaches to studying innovations and locates the main division between these approaches as ANT's generalised principle of symmetry which states that the same "vocabulary must be used in the description and explanation of the natural and the social ... no change of register is permissible when we move from the technical to the social aspects of the problem studied" (ibid, pp. 172–173). This is a problem for AT because the object (of activity) is generated from human needs. OK, humans do generate the object but once the object is established the agency which follows in the activity can be distributed. Indeed, Latour (2005) states that he abandoned most of the symmetry metaphor because what he had in mind was a "joint dissolution of both collectors" (ibid, p. 76). Pickering (1999, p. 15) also considers the generalised principle of symmetry to be problematic, "As agents, we humans seem to be importantly different from nonhuman agents". With the generalised principle of symmetry 'put in to perspective' I now look to two commonalities in principles between Latour and Pickering: focus on performance; don't restrict agency to animals (humans) alone.

Latour (2005) mentions performance with to regard groups, social aggregates. Classical sociologists are accused of making *ostensive* definitions of groups – there's a group of teachers – and focusing on stability but, from an ANT point of view, "the rule is performance and what has to be explained, the troubling exceptions, are any type of stability over the long term [and this cannot be explained] without looking for vehicles, tools, instruments, and materials able to provide such a stability" (ibid, p. 35). This focus on performance is akin to flux in AT. A sketch of a performative view of science is presented early in Pickering (1995, p. 6), instead of a world where scientists only generate knowledge from facts, he sees a world filled with agency:

> The world ... is continually doing things, things that bear upon us not as observation statements upon disembodied intellects but as forces upon material beings ... Much of everyday life ... character of coping with material agency, agency that comes at us from outside the human realm and that cannot be reduced to anything within that realm.

Later, in Pickering (1995), 'disciplinary agency' and the 'dance of agency', as described above, are introduced. Neither Latour nor Pickering are concerned with mathematics education but their 'multi-agent' stance resonates with my experience of mathematics classrooms. When a teacher uses a tool in a mathematics class, then s/he is only one of the agents in the activity, other potential agents are: other teachers; students; the curriculum; the institution; other available artefacts; and the tool itself.

I now consider mediation and what mediates: language, signs, artefacts or people? I think the problem here can be viewed via the ostensive-performative distinction. Scholars have different interests and tend to point to something and say "that (those) is (are) the mediator(s)" whereas the mediator in a specific situation exists in relation to what is actually done (the activity/performance). I am, for instance, interested in artefact/tool-mediation but two learners may be involved in ostensibly similar activities with a mathematical tool but one learner may be heavily reliant on the tool whereas the use of this tool to the other learner may be peripheral; mediation by the tool comes down to the actual use of the tool. Similarly Mariotti and Maracci (2012) may expect the mediator to be the teacher but I doubt if this is always the case. Latour (2005, p. 39) appears to present a similar idea in distinguishing between mediators and intermediaries, "An intermediary ... is what transports meaning or force without transformation ... Mediators transform, translate, distort, and modify".

A STATEMENT ON TOOL USE IN MATHEMATICS EDUCATION

The considerations above, together with those in the previous three sections, provide a basis for the following statement (in italics) on tool use in mathematics education.

AT provides a framework to interpret tool use in practice but the level of detail on tool use will depend on the 'size' of the unit of analysis. An AT account of tools would benefit from being augmented by constructs from instrumentation theory and the theory of affordances. Activity is mediated by human and non-human mediators but this mediation cannot be stipulated in advance of the performance of the activity. Human and non-human agents impact the activity; as with mediation, the impact of these agents cannot be stipulated in advance of the performance of the activity.

I now state my networking argument. The theories of affordances and of instrumentation have few assumptions and a lot of application. Recognition of the relationship between learners and their environments is important in AT as is the process by which an artefact becomes a tool for learners. Both theories can be used in MER to shed light on the action and operation aspects of AT without compromising any tenets of AT. With regard to taking ideas from Latour and Pickering I focus on the two principles outlined above. The 'focus on performance' principle is entirely consistent with the concept of flux in AT. AT focuses on describing practice and tools (and, I add, other things) are used as they are used (or not) – there is no pre-ordained plan. As for not restricting agency to humans alone, well, this is a problem for many activity theorists because the object of an activity is generated by humans. But if the principle of non-human agency is weakened to restrict non-humans from initiating activity, then I don't think there is a problem.

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ENDNOTE

1. I would have liked to have considered tool-focused work within the Anthropological Theory of Didactics as well but this was not possible in the length restrictions for this paper.

Structuralism and theories in mathematics education

Pedro Nicolás

University of Murcia, Murcia, Spain, pedronz@um.es

We present the structuralist conception of scientific theories as a Deus ex Machina which allows to resolve the entanglements of theories in Mathematics Education. We illustrate with examples how this conception, which forms a solid and solvent body of knowledge in Philosophy of Science, provides us with tools to perform a careful analysis of a theory, both by itself and in connection with other theories.

Keywords: Praxeology, theory, model, law, networking.

RECONSTRUCTION OF SCIENTIFIC THEORIES

As it is the case in many other disciplines, in Mathematics Education there are several theories living together: Theory of Didactic Situations (Brousseau, 1997), Anthropological Theory of the Didactic or ATD (Chevallard, 1999; Bosch et al., 2011), APOS[1] theory (Dubinsky & McDonald, 2002), Onto-Semiotic Approach (Godino, Batanero, & Font, 2007), Theory of Abstraction in Context (Dreyfus, Hershkowitz, & Schwarz, 2001), Theory of Knowledge Objectification (Radford, 2003)... Whereas the cohabitation of theories is perfectly normal, efforts aiming to connect some of them, especially from the CERME working team "Theoretical perspectives and approaches in mathematics education research" (CERME8, 2013), are also very natural and desirable.

We defend in this work that, for a better understanding of the possibility of connection of two theories, we must reconstruct them by using the same language. The reconstruction of a theory can be carried out from different conceptions. When we speak of 'conceptions' we mean ways of giving an account of what a scientific theory is, and not of how a scientific theory (in particular, a scientific law) is constructed. Thus, a priori these conceptions do not pay attention to methodological aspects.

The one favoured here is the so-called *structuralist conception* (Balzer, Moulines, & Sneed, 1987). This is an elaboration of the *semantical conception* (initiated by Suppes and Adams in the 1970s), and it seems to reconcile the most important aspects of the *syntactical conception* (advocated by Reichenbach, Ramsey, Bridgman, Campbell, Carnap in several works from the 1920s to the 1950s) and the *historicist conception* (advocated by Kuhn, Lakatos, Laudan in several works in the 1960s), while it avoids their problems (Diez & Moulines, 1997).

Now we will give a brief explanation of the main points of the structuralist conception. For a more extensive treatment, see (Balzer et al., 1987).

According to the structuralist conception, a scientific theory is a net of many nodes (which will be called *elements of the theory* or *theory-elements*) connected in several via *specialization*, see Definition 2 below. Of course, such a net does not appear out of the blue, but it is developed little by little along the time. This is how the structuralist captures the diachronic character of a theory. The synchronic character of a theory appears in the description of the theory-elements.

Definition 1: To determine a *theory-element* one has to specify:

1) The portions of reality the theory-element conceptualizes, i. E. The portions of reality the theory can speak of, called *potential models*. These potential models are described as portions of reality which can be modelled by using a *structure* (that is to say, a tuple $(D_1, D_2, ..., R_1, R_2, ...)$ of sets D_i and relations R_j between these sets) and a list of properties applicable to the structures of the former type. We call M_p the set of potential models.

2) The *laws* with which the theory-element aims to enlighten reality. Each law is a property applicable to the structures of the specified type. The laws distinguish the so-called *actual models* among the potential models. We call M the set of actual models.

3) The *partial potential models*, which are these portions of reality which can be checked to be potential models without assuming the laws of the theory-element. Notice that to verify that a portion of reality is a potential model we check, in particular, that the relations R_j appearing in the type of structure are satisfied. In this checking we use some method and this method might, or might not, assume the laws of the theory-element. A relation R_j is *theoretical with respect to a theory element T* (or, in short, *T-theoretical*) if every method of determination of R_j assumes the validity of the laws of this theory-element. Thus, a partial potential model of a theory-element is nothing but a potential model in which we omit the theoretical relations. We call M_{pp} the set of partial potential models.

4) Those partial potential models that are expected to be actual models. These partial potential models are, after all, the *intended applications* of our theory-element. We call I the set of intended applications of our theory-element.

Thus a *theory-element* is an ordered pair $T = (K, I)$ where I is the set of intended applications and $K = (M_p, M_{pp}, M)$ is the *core*, formed by the set of potential models, the set of partial potential models, and the set of actual models.

The *empirical claim* of a theory-element is just the statement which asserts that the intended applications are actual models, $I \subseteq M$, that is to say, that in certain portions of reality, which can be detected without assuming the laws of the theory-element, these laws actually hold.

In the next section we will give several examples of theory-elements but, unfortunately, we will not point out a theoretical relation in any of them. It is an important open question whether there are theoretical relations in the current theories of Mathematics Education. In Classical Mechanics (CM), the relations of *position* or *time* are not CM-theoretical, since you can determine them without assuming any proper law of Classical Mechanics. However, the relation *mass* is CM-theoretical, since any method of determination of the amount of mass of an object assumes a law proper of the CM. For examples in other disciplines see (Balzer et al., 1987).

NETWORKING THEORIES

In what follows we use the structuralist approach to present different kinds of possible connections between theory-elements.

Definition 2: A theory-element T' is a *specialization* of another theory-element T, and we write $T' \sigma T$, if:

1.

   1.1. $M'_p = M_p$, that is to say, both theory-elements conceptualize the world in the same way.

   1.2. $M'_{pp} = M_{pp}$, that is to say, both theory-elements consider the same theoretical relations.

   1.3. $M' \subseteq M$, that is to say, every law in T is also a law in T'.

2. $I' \subseteq I$, that is to say, every portion of reality aimed to be explained by T' is also a portion of reality aimed to be explained by T.

In short, to specialize consists of increasing the amount of laws without changing the conceptual architecture.

Definition 3: A *net-theory* is a pair $N = (\{T_i\}, \sigma)$ where $\{T_i\}$ is a non-empty set of theory-elements and σ is a specialization relation on $\{T_i\}$.

Next we are defining the notion of *theorization*, but first we need the following:

Definition 4: Given two structures (see Definition 1) $x = (D_1, ..., D_m, R_1, ..., R_n)$ and $y = (D'_1, ..., D'_p, R'_1, ..., R'_q)$, we say that y is a *substructure* of x if:

1. $p \leq m, q \leq n$.
2. Every D'_i is a subset of some D_j.
3. Every R'_i is a subset of some R_j.

Definition 5: A theory-element T' is a *theorization* of a theory-element T if:

1. Every intended application of T' admits an actual model of T as substructure.

2. There are potential models of T' which are not substructures of potential models of T (because they contemplate new domains and/or new relations).

The first condition says that every portion of reality T' aims to explain satisfies the laws of T. The second condition says that T' includes new (not necessarily T'-theoretical) concepts not contemplated by T.

Next I will show in examples some tentative structuralist descriptions of some elements of the ATD.

Example of theory-element: Our first example is inspired in the so-called *Herbartian scheme* (Chevallard, 2015), which is probably the most general structure proposed by the ATD to deal with situations of study. In this structure there are things like a task or question which requires some answer, a series of partial answers, and a final answer. Therefore, the structure corresponding to our theory-element T_1 will be the tuple $(\{1, ..., n\}, P, s)$ where $\{1, ..., n\}$ is the set of the first n natural numbers, P is a non-empty set P, and s is a map from $\{1, ..., n\}$ to P. The image of 1 is said to be a *generating question*, the image of n is said to be a *final answer* and the other images are said to be *partial answers*. Since no law is stated, there is no distinction between potential, partial potential and actual models. Notice that every temporal sequence of n events fits in this structure, but, of course, not every such sequence is an intended application of T_1. This is why it is important to explain which are our intended applications, namely, those sequences of events consisting in finding an answer to a question.

Example of theorization: If, moreover, in each of the partial answers of T_1 we distinguish between tasks, techniques and *logos* elements, that is, if we look at the constituent parts of the so-called *praxeologies* (Chevallard, 1999), we would have reached a theorization, T_2, of T_1. The structure corresponding to T_2 will be a tuple $(\{1, ..., n\}, S_T, S_t, S_L, s)$ where S_T, S_t, and S_L are non-empty sets whose elements are called *tasks*, *techniques* and *logos-elements*, respectively, and s is a map from $\{1, ..., n\}$ to $S_T \times S_t \times S_L$. Since no law is stated there is no distinction between potential, partial potential and actual models. Now not every temporal sequence of n events fits in the structure of T_2. Not even every temporal sequence of n events consisting in finding an answer to a question! In fact, our intended applications are temporal sequences of events consisting in finding an answer to a question such that in each of these events we find three components and such that, moreover,

— all the first components of the events are "of the same nature" (this is encoded in the fact that they belong to the same set), namely, tasks;

— all the second components of the events are of the same nature, namely, solutions to the task specified in the corresponding first component, and

— all the third components of the events are of the same nature, namely, explanations of why the corresponding second element solves the corresponding first element.

Example of theorization: If, moreover, we take into account the dynamics of each of these praxeologies, recognizing the so-called *study moments* (Chevallard, 1999), we would have a theorization, T_3, of T_2. The structure corresponding to T_3 will be a tuple $(\{1, ..., n\}, S_T, S_t, S_L, \{0,1\}, \{*\}, s)$ where

– $\{1, ..., n\}$, S_T, S_t and S_L are as before;

– s is a map from $\{1, ..., n\}$ to $S \times (S \cup \{*\}) \times (S \cup \{*\}) \times \{0, 1\} \times \{0, 1\} \times ([0, 1] \cup \{*\}) \times (\mathbb{N} \cup \{*\}) \times (\mathbb{N} \cup \{*\})$, where S is the union of S_T, S_t and S_L, called the *study sequence map*, and its images are called *events*.

The structure is now more complicated because it has to model more ambitious intended applications. Indeed, in the events of the sequence we still look at tasks, techniques and logos, but we also pay attention to the way they are related:

– The first (respectively, second, third, fourth and fifth) component of an event refers to the first (respectively, second, third, fourth and fifth) study moment (Chevallard, 1999).

– The last three components of an event refer to the sixth study moment, namely, to the evaluation moment. More precisely, the sixth component refers to the scope of the technique (it is a bounded magnitude which reaches the value 1 if the technique covers all the possible cases of the task), the seventh component refers to its economy and the eighth component refers to its reliability (see Sierra, Bosch, & Gáscon, 2013)[2].

For example, an event which is an element of $S_T \times S_t \times S_L \times \ldots$ is regarded as a task followed by an elaboration of a technique followed by an explanation of why this technique works, whereas an element of $S_t \times S_T \times \{*\} \times \ldots$ is regarded as a technique followed by a task which is solved by the technique followed by no explanation of why the technique works. We use * to express absence of activity in the second, third, sixth, seventh and eighth components, and we use 0 (respectively, 1) to express absence (respectively, presence) of activity in the fourth and fifth components. We can add some axioms devoted to prevent us from considering impossible events, for example:

Axiom 1: There are not events starting with a task and continuing with a *logos* element.

Axiom 2: If an event starts with a technique, then it cannot continue with a *logos* element.

Axiom 3: In an event there are not two tasks, two techniques or two *logos* elements. Thus, for instance, there are not events which are elements of $S_T \times S_T \times \ldots$

Axiom 4: If in an event there is no task, then the last three components of the event are (*, *, *).

Axiom 5: If the fourth component of an event is *, then the last three components are (*, *, *).

Examples of specialization: Imagine we create a new theory-element T_4 by adding the following law to the theory-element T_3:

Law: The last three components of every event are (*, *, *).

The new theory-element T_4 is a specialization of T_3. Indeed, there are actual models in T_3 which are not actual models in T_4, namely, those study sequences having at least an event in which the last three components are not (*, *, *). After the axioms, it is clear that the former law holds for those study sequences in which each event $s_i = (s_{i1}, s_{i2}, s_{i3}, s_{i4}, s_{i5}, s_{i6}, s_{i7}, s_{i8})$ satisfies that none of the s_{ij} are a task or that $s_{i4} = *$. Hence, those study sequences would be actual models of our theory-element T_4.

The notion of *didactic contract* (Brousseau, 1986) is a good source of laws for theory-elements dealing with study sequences. Indeed, a didactic contract can be regarded as a special family of *clauses* or *conventions*, and, inspired in Lewis (1969), we could express a convention as a law stating that a certain regularity in the events of a study sequence holds (see for instance the law above).

Remark: In Chevallard (1988b), there is a sketch of the possible sets and relations of the structures an anthropological theory of the didactics would deal with. It would be interesting to compare them with the ones used in our examples above.

Remark: Brousseau (1986), inspired among others by Suppes (1969, 1976)[3], used finite automata to give a structuralist formulation of the notion of *situation*. Our structuralist formulations of notions of the ATD are more in the spirit of the Stimulus-Sampling Theory (Estes & Suppes, 1959). It is worth noting that, as proved in Suppes (1969), given any finite connected automaton there is a stimulus-response model that asymptotically becomes isomorphic to it.

Finally, let us consider the relation of *reduction* between theory-elements.

Definition 6: A theory-element T is *reducible* to a theory-element T^* if there exists a relation $\rho \subseteq M_p(T) \times M_p(T^*)$ such that:

1. If $(x, x^*) \in \rho$ and $x^* \in M(T^*)$, then $x \in M(T)$.

2. If $y \in I(T) \cap M(T)$ then there exists $y^* \in I(T^*) \cap M(T^*)$ such that $(y, y^*) \in \rho$.

The underlying idea is to regard the elements (x, x^*) of ρ as pairs of portions of reality so that x^* is the T^*-version of x. The first condition says that the laws of T can be derived from those of T^*. The second condition says that all the successful applications of T admit T^*-versions which are also successful applications of T^*. In other words, the successes of T can be explained in virtue of those of T^*. Notice that, in contrast to what happened with the *theorization* (Definition 5), reduction does not require an increase in the conceptual map, that is to say, the kind of structures contemplated as potential models. Indeed, the conceptual map of T^* might be completely different to the conceptual map T.

Examples of reduction: The *classical mathematics education* (see Gascón, 1998) explains certain phenomena with laws involving cognitive or motivational concepts. Indeed, these would be the kind of concepts used by the classical mathematics education to explain the kind of phenomena presented in IREM de Grenoble (1979). One can use Chevallard (1988a) to sketch how part of this classical mathematics education can be reduced to a theory-element including among the laws the clauses of the *didactic contract*. On the other hand, one can also use (Chevallard, Bosch, & Gascón, 1997) to reduce part of this classical mathematics education to a theory-element with laws stating the *incompleteness* of scholar study processes (this incompleteness can be expressed in terms of the study moments, for example, by saying that *the moment of the construction of the technological-theoretical frame* or *the moment of the work of the technique* is lost).

In Bikner-Ahsbahs and Prediger (2010) the following "networking strategies" are presented: to ignore other theories (as an extreme strategy of non-connection), to make your own theory and foreign theories understandable, to compare/contrast, to coordinate/combine, to integrate locally/synthesize and to unify globally (as an extreme strategy of total connection). Next, let us clarify this taxonomy by presenting, in a brief and simplified way, possible translations of these strategies to the structuralist language:

- To ignore other theories: not to consider the possibility of (even partial) specialization, theorization or reduction (see Definitions 2, 5 and 6) as a relation among two theory-elements.

- To make your own theory and foreign theories understandable: to accomplish this, as we said at the beginning of this paper, one need to translate both theories to the same language. What we suggest here is to use the structuralist language. So, in a sense, in the present work we take seriously this second networking strategy.

- To compare/contrast: to check which are the potential models shared by two theory-elements. Thus, when comparing/contrasting we could be performing a theorization.

- To coordinate/combine two theory-elements T and T' consists in saying that a common intended application is both an actual model of T and an actual model of T'. It is important to notice that, for this to happen, T and T' must share the partial potential models. This last sentence explains the meaning of the following statement of Bikner-Ahsbahs and Prediger (2010): "Whereas all theories can of course be compared or contrasted, the combination of (elements of) different theories risks becoming difficult when the theories are not compatible."

- To integrate locally/synthesize two theory-elements T and T': to find a third theory-element T'' to which we can reduce the theory-element derived from T when considered just some sub-structures z of the structures x of T, as well as the theory-element derived from T' when considered just some sub-structures z' of the structures x' of T'. Notice that the structures x'' of T'' should admit both z and z' as sub-structures.

- To unify globally: to find a theory to which any other theory could be reduced.

CONCLUSION

Here we suggest to use the structuralist formalization of scientific theories to the benefit of the questions about the theoretical status of different approaches in Mathematics Education. Needless to say, we do not mean one cannot work properly in theory unless this is formalized. For example, it is not reasonable to say that Newton was not doing Mechanics just because he did not have at hand a strict formalization. On the other hand, theories in Mathematics Education are still far from being formalizable, being this (even partial) formalization a long-term project in any case. Concerning this, it is important to point out that the degree of resistance of a theory to be formalized is inversely proportional to its degree of development. For example, if we cannot distinguish the actual models among the potential models, then we cannot identify any law of the theory (and at this point we should wonder whether this forces us to accept this theory is nonexistent...). Anyway, regardless of the difficulty of a complete formalization, we defend that:

- The framework offered by the structuralist conception of scientific theories is illuminating to the extent that it provides us with high order tools which allow a better understanding of the theoretical scene in Mathematics Education.

— Even if we were not interested in networking theories, the attempt to formalize a theory in the structuralist way forces us to consider extremely interesting questions about this theory. For instance: which are the underlying structures? Which are the laws? Which are the theoretical relations?

Among many other things, it is still an open question which are the links between our structuralist approach, the definition of *theory* by Radford (2008) and the notion of *research praxeology* by Artigue,Bosch, and Gascón (2011a, 2011b).

ACKNOWLEDGEMENT

I thank Josep Gascón for his reading of a previous version of this article and his helpful comments.

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ENDNOTES

1. This is the short form for "Action, Process, Object and Scheme".

2. For the sake of simplicity, we do not distinguish between *task* and *type of task*, and between *technological* and *theoretical* elements among the logos elements, even if they are important distinctions in the ATD.

3. Actually, the last two components should be interpreted as evaluations of a technique in comparison with another technique. Indeed, we typically speak of a technique as being more or less economic or reliable than another technique. However, for the sake of simplicity, we do not take into account this aspect here.

4. It is a remarkable fact that Suppes was the main promoter of the semantic conception, direct precedent of the structuralist conception.

Adaptive conceptual frameworks for professional development

Miguel Perez

Linnaeus University, Faculty of Technology, Växjö, Sweden, miguel.perez@lnu.se

In this paper, the notion of adaptive conceptual frameworks is presented. These frameworks have been used to conduct educational design research aiming at developing ICT supported mathematics instruction. In this approach, empirical data is connected with various theories in an adaptive and iterative process. Differentiation is made between conceptual framework for development (CFD) and conceptual framework for understanding (CFU) depending on how the frameworks are used in the design process. The use of adaptive conceptual frameworks contributes to the transparency in the design process by making explicit the levels at which different theories operate and how the design process is evaluated.

Keywords: Conceptual framework, educational design, professional development.

INTRODUCTION

During the last decades, several similar methodologies have emerged that address the desire to conduct educational research with relevance for school practices. For example, design-based research aims explicitly at developing theories that could do "real work" by providing theoretically underpinned guidance on how to create educational improvement in authentic settings (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; McKenney & Reeves, 2012). A common feature of these approaches is the design of teaching activities in an iterative design process that shares many similarities with teachers' daily work.

This paper contributes to research by describing how the design process may be co-determined by the interaction between different stakeholders such as researchers and teachers, disciplinary knowledge, theoretical frameworks, and other resources. This approach has been inspired by co-design, as a design methodology that highlights the importance of involving different stakeholders such as teachers in the design research process in order to address the issue of ownership of innovation (Penuel, Roschelle, & Shechtman, 2007). Furthermore, working in close collaboration with teachers deepens our knowledge about pragmatic issues and promotes development of "*innovations that fit into real classroom contexts*" (ibid, p. 52). Following the conceptualization of knowledge proposed by Chevallard (2007) in the Anthropological Theory of the Didactic (ATD), the two different perspectives of understanding and development could be viewed as two inseparable aspects of knowledge, integrating a practice that includes the things teachers do to solve different educational tasks (*praxis*) with a discursive environment that is used to describe, explain, and justify that practice (*logos*). The adaptive conceptual frameworks presented in this paper explicitly address both perspectives.

The purpose of this paper is to describe the development of adaptive frameworks and how they were used to meet the emerging needs in a design process of ICT supported mathematics instruction during one design cycle. The empirical data presented in this paper is therefore only used to motivate the development of the adaptive conceptual frameworks and not analyzed with respect to the intended learning objectives.

ADAPTIVE CONCEPTUAL FRAMEWORKS

In this approach the researcher connects empirical data with various existing theories that are chosen in retrospect and that are used to generate additional empirical data in an iterative, incremental and adaptive process. In other words, theory is not applied onto practice, it is more about a "progressive interaction between theory and practice, by means of appropriating existing theoretical tools" (Bartolini Bussi, 1994, p. 127). Furthermore, the adaptive conceptual frameworks are considered in a state of flux and changeable according to the different challenges that might emerge when conducting design-based research in authentic settings. Thus, the adaptive conceptual frameworks should be regarded as tentative and a result of a research work that has similarities with research that sometimes is portrayed by the "bricolage" metaphor (Kincheloe, 2001), particularly regarding the efforts of embracing methodological flexibility and plurality of theories. From this perspective, this research approach aligns with the Singerian inquiry system (Churchman, 1971; Lester, 2005).

The workflow of the formal stages of a design cycle is illustrated in Figure 1. Each design cycle starts with a planning phase, followed by an implementation phase involving the teachers. The cycle is completed with an evaluation of the outcomes. Three different frameworks are distinguished depending on their role in the different phases:

— methodological framework for professional development (MFPD),

— conceptual framework for development (CFD),

— conceptual framework for understanding (CFU).

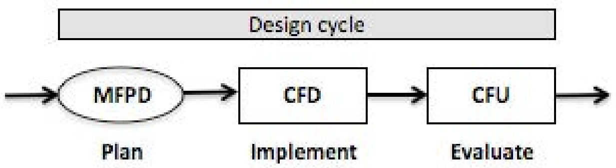

Figure 1: The adaptive frameworks for research and professional development

The researcher uses the methodological framework for professional development (MFPD) to plan and support the teachers' participation in the design process. The conceptual framework for development (CFD) is used to describe and justify the different activities that the researcher engages in together with the teachers. Finally, the conceptual framework for understanding (CFU) is used to understand the outcomes of an intervention in order to decide how proceed in the next cycle. While the conceptual frameworks for development and understanding (CFD and CFU) naturally share similarities, since they both put focus on the design process, the methodological framework for professional development should be regarded as a separate framework for organizing and supporting the teachers' professional development.

The different frameworks in this approach consist of multiple theoretical components. In order to consider how they interact, the categorization presented by Prediger, Bikner-Ahsbahs, and Arzarello (2008) was used. In their landscape of different levels of integration, the authors present a scale ranging from one extreme of ignoring other theories to the other extreme of unifying theories globally. Those strategies that are intermediate are called networking strategies and include strategies such as *comparing*, *combining*, *coordinating* and *integrating locally*. According to Prediger and colleagues (2008), the strategies of *coordinating* and *combining* are mostly used for a networked understanding of an empirical phenomenon or a piece of data and are typical for conceptual frameworks that, as in our case, not necessarily aim for a coherent theory. While *comparing* and *contrasting* always are possible the strategies of *coordinating* and *combining* can be a more difficult task especially if the theories are not compatible relative a specific purpose. The *coordinating* strategy is in turn used when a conceptual framework is built on well-fitting theoretical elements (ibid). The networking strategies used in this study were *comparing* and *coordinating*.

THE BACKGROUND OF THE CASE STUDY

The participating teachers were involved in a developmental project in their school related to how ICT could enhance their students' learning of mathematics. As part of this project, the teachers participated in a one-day event with lectures and hands-on learning activities developed by researchers from media technology and mathematics education. One specific learning activity was designed to stimulate students to communicate, collaborate and generate general problem solving strategies (Sollervall & Milrad, 2012). Mobile phones were used in this activity to bridge between formal and informal learning spaces. During the discussions about this activity the teachers seemed to be more worried about the practical issues (e.g., handling the mobile phones) rather than the didactical issues. It seemed that the teachers perceived the didactical challenge of connecting between students' activities outdoors and a mathematical content as unproblematic. In fact, a successful orchestration would depend on the quality of the student-generated artifacts as well as the teachers' ability to orchestrate this remaining part of the activity performed indoors.

Later on, two of the mathematics teachers from the school and a researcher from mathematics education met to discuss the prospects of developing new activities supported by ICT. The teachers expressed their concerns about their students' inability to use the distributive law and wanted to address this issue. The teachers had themselves completed the above-mentioned activity, which also could be used to address students' conception of the distributive law by connecting multiple representations (ibid). Using the activity with this particular focus would not require any modifications of the activity itself but would require the teacher to orchestrate the activity towards this content. None of the teachers seemed to perceive this opportunity and the continued discussions revealed that they did not know about possible geometrical representations of the distributive law. These events influenced the direction of design process. Based on the overall goal of creating educational improvement, it seemed important for the researcher to address the teachers' ability to use ICT for different goals and purposes. This issue was perhaps more important than developing new activities with the teachers. With this pre-understanding the planning phase of the design was initiated.

METHODOLOGICAL FRAMEWORK FOR PROFESSIONAL DEVELOPMENT

The methodology of collaborative design based research is at the same time a process of professional development (Penuel et al., 2007) that should be regarded as gradual and difficult for the teachers (Guskey, 2002). In this case, the teachers' insufficient understanding of mathematical representations was taken as a constraining factor for the teachers' participation in the design process. To address this issue, two complementary theories were used to guide and plan for the teachers' professional development. One of the frameworks specifically focuses on knowledge for teaching mathematics: Mathematical knowledge for Teaching (Loewenberg Ball, Thames, & Phelps, 2008) and the other framework focuses on the affordances provided by ICT and on the integration of ICT in different subject areas: Technological Pedagogical Content Knowledge (Koehler & Mischra, 2008).

The strategy of *comparing* (Prediger et al., 2008) was used to identify common principles in these two theories related to the use of ICT for supporting students' learning of mathematics. Based on this comparison, the researcher decided to specifically support teachers' understanding of the affordances for representation provided by ICT. The idea was to use the dynamic geometry software GeoGebra (www.geogebra.org) to develop an application, with focus put on providing affordances for representation, that the teachers could use later in a learning activity to address their students' conception of the distributive law. Thus, depending on the user, the software was used with different purposes.

CONCEPTUAL FRAMEWORK FOR DEVELOPMENT

Inspired by the work of Duval (2006) the dynamics of GeoGebra is used in the application to illustrate how numerical expression can be interpreted and represented geometrically (Figure 2).

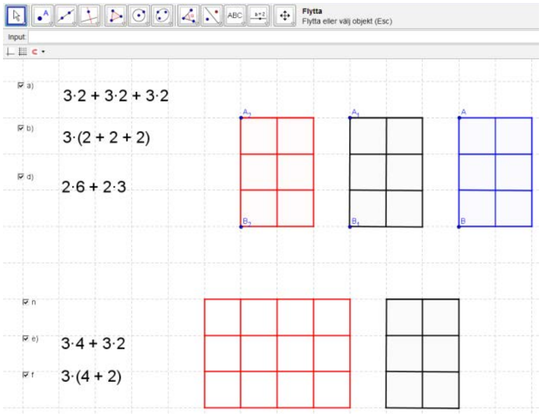

Figure 2: Snapshot of the application, implemented in GeoGebra

The teachers were not familiar with the software so the application was designed for them as end-users to operate only by using "click and drag" features. Even if the teachers were provided with the application, the teachers needed to create a *hypothetical learning trajectory* (HLT), i.e. "the consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" (Simon, 1995, p. 133). In other words, while the researcher took responsibility for the didactical design (the application), the pedagogical design was intended for the teachers to decide.

When the application was presented to the teachers they wanted immediate access to it. They seemed to recognize the limitations of the explanations that they normally used that were exclusively based on instructions on how to manipulate different variables. They also agreed on using the application with their students but they never did. Therefore, there was an additional meeting where the researcher demonstrated a possible way to use the application in a learning activity. The demonstration was followed by a discussion about possible ways to orchestrate the interplay between different representations and the dynamical affordances (dragging mode, show/hide figures) supported by the application. By discussing related pedagogical issues and offering the teachers opportunities to adapt the application according to their needs, the researcher wanted to challenge the teachers to create their own hypothetical learning trajectory (HLT). The teachers were offered additional support on how to adapt or use the application but the teachers did not use this possibility. In the following section, the crosscutting features of the enacted lessons will be presented.

The teachers used different interpretations of multiplication simultaneously and alternately without making explicit why and when an interpretation was preferable in some situations and not in others. This lack of explicitness resulted in vague connections between the numerical and geometrical representations. Justifications were based on computations or algebraic manipulations instead of referring to the available geometrical representations in the application. When the teachers became uncertain on how to proceed with the activity they tended to rely more on the more familiar numerical and algebraic representations to maintain the flow of the lesson. A significant part of the lessons was also dedicated to what seemed to be other more familiar activities such as formulating expressions for area and perimeter. Furthermore, the teacher-initiated communication with the students did not seem to support a discussion on how and why things work the way they do. Occasional misinterpretations of students' responses, not acknowledging their responses as correct, and not connecting their responses to the available representations, further contributed to the activity not proceeding as intended.

In summary, the CFD was developed by using the networking strategy of *coordinating* (Prediger et al., 2008) theoretical components (i.e., mathematical *representations*, *GeoGebra* and *HLT*) for practical reasons without aiming for a deeper integration. The purpose of this framework was to outline the theoretical underpinnings of the activities with the teachers. In contrast to the other components in the CFD, the notion of hypothetical learning trajectory (HLT) was not presented explicitly to the teachers.

CONCEPTUAL FRAMEWORK FOR UNDERSTANDING

The enacted lessons were also different compared to the suggestions the teachers had themselves when discussing different ways to orchestrate a lesson supported by the application. During the first two phases of this design cycle, the focus was on different types of teacher knowledge but the crosscutting features of the lessons revealed another dimension. How does teacher knowledge come into play in the moment of teaching? Why did the teachers not make use of the ICT-supported affordances for connecting representations? To address these emerging questions the researcher decided to go beyond the theories of representation and different categories of teacher knowledge used previously and focus on the teachers' overt orchestration of the lessons. In other words, a different representation was chosen to evaluate the design process and in particular the teachers' professional development.

Developing the CFU

A different conceptualization of knowledge was found in the Anthropological Theory of the Didactic (ATD). In this theory a body of knowledge, a *praxeology*, consists of two inseparable blocks, the *praxis* and the *logos*. The praxis block refers to the kind of given *tasks* that you aim to study and the different *techniques* used to face these problematic tasks. In this sense, the praxis block represents the "know-how" of the praxeology. The logos block provides a discourse that is structured in two levels with the purpose to justify the praxis. The first level of the logos is the *technology*, which provides a discourse about the technique. The second level of the logos is the *theory*, which provides a more general discourse that serves as explanation and justification of the *technology* itself (Chevallard, 2007) by providing a framework of notions, properties and relations to organize and generate technologies, techniques and problems (Barbé et al., 2005). The *praxeology* is the minimal unit of human activity

The ATD includes the study of didactic transpositions processes, which concerns the transformation of knowledge through different institutions. The transposition is a process of de-constructing knowledge and rebuilding different elements of knowledge into a more or less integrated whole with the aim of establishing it as "teachable knowledge" while trying to keep its character and function (Bosch & Gascón, 2006). It consists of the four following steps; *scholarly knowledge*, *knowledge to be taught*, *taught knowledge* and *learned knowledge*. The notion of *didactical transposition* provided a new way of putting the intervention into perspective. As the researcher provided the teachers with competence development in terms of the didactical value of multiple representations, the focus of this design cycle was on the step between intended and enacted knowledge, that is, between knowledge to be taught and taught knowledge (shaded in Figure 3). The researcher could therefore use the recordings of the enacted lessons to understand the teachers' professional development from this new perspective.

Moreover, teaching is a didactic *type of task* that teachers can solve in a complex process of *didactic transposition* by using a set of available resources (*didactical techniques*), both external resources (curriculum, textbooks, tests, ICT-tools, colleagues, manipulatives, etc.) and teachers' internal resources that in our case of ICT-supported instruction could be related to technological-pedagogical content knowledge (Koehler & Mischra, 2008). The logos block of a didactical praxeology then serves as means to describe and justify teaching and learning practices in the considered institution (Rodríguez, Bosch, & Gascón, 2008).

A specific adaptation to the new empirical data was to replace the notion of *HLT* by the notion of *routines* (Berliner, 2001) with focus on the *IRE* sequence (Initiate, Response, Evaluate). The IRE sequence is a three-part pattern where the teachers ask a question, students reply, and teachers evaluate the response or gives feedback (Mehan, 1979; Schoenfeld, 2010). In its most basic form the teacher initiates the sequence by posing a question to a student to which the teacher already knows the answer. The student then replies and the teacher evaluates by using phrases such as "yes" or "that's fine" and continues with the next question or next problem. Communicational exchange patterns, such as the IRE sequence, can be regarded a didactical technique that teachers use in the creation of a mathematical praxeology. This adaptation was made in order to better describe the teachers' overt orchestration of the lessons and especially the communication patterns between teachers and students. This theoretical component was further developed in a second design cycle into a didactical resource (Perez, 2014).

The role of *representation* is multifaceted. From one perspective it is a generic property of many ICT tools (Koehler & Mischra, 2008). From a second perspective, mathematical representations have important didactical affordances (Ainsworth, 1999), and finally representations are essential to mathematics as a discipline (Duval, 2006). Thus, mathematical representation is closely related to both praxis and logos of a mathematical praxeology. In addition, instructional strategies that systematically focus on knowledge about representations could be conceptualized as an element of a didactical technique and consequently a part of a didactical praxeology. Thus, depending on the purpose in which representations are used, the role of representation for a discipline as mathematics could be attributed to both a mathematical and a didactical praxeology. Thus, r*epresentations* were placed within the notion of praxeology instead of being treated as a separate theoretical component as in the CFD. These adaptations allowed the researcher to provide a more comprehensive description of the crosscutting features of the enacted lessons and to understand the teachers' professional development.

In summary, the conceptual framework for understanding (CFU) consists of several theoretical components where the ATD is used as overarching perspective. The CFU was developed by using the strategy of *coordinating* different theoretical components (Prediger et al., 2008). To achieve this, the theoretical components of representations and routines (the IRE sequence) were interpreted as knowledge resources in accordance with the ATD and its focus on the epistemic dimension of teaching and learning processes in different institutions.

Evaluating the first design cycle

The theoretical notions provided by the CFU allowed the researcher to capture the essence of this part of the design process and to better understand the

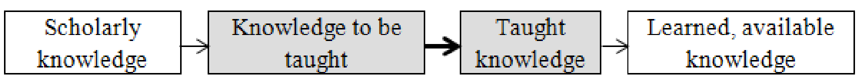

Figure 3: The transposition of knowledge

crosscutting features of the enacted lessons. The researcher's intention was to introduce the geometrical representation as a technological element in a mathematical praxeology. Instead, the teachers used the application to allow the students to work with more open-ended tasks. The result was that only some aspects of knowledge (praxis) were addressed during the lessons. Furthermore, the didactical techniques used by the teachers did in many cases not support the creation of a mathematical praxeology including a well-developed logos discourse. In other words, the underlying principle-based learning objectives did not survive the transposition from how the researcher intended the application to be used and how it was actually used by the teachers. The step from knowledge to be taught and taught knowledge in the didactical transposition proved to be a greater challenge requiring more scaffolding than the researcher had anticipated and planned for (in the MFPD). With this understanding, a new design cycle could be initiated.

SUMMARY

In this research the possibility of viewing design research as incremental and adaptive has been considered. For example, the common goal of designing a new activity supported by ICT was adapted to include the teachers' ability to use ICT for different purposes. Thus, the researcher decided to address the teachers' understandings of the affordances for representation and communication provided by ICT and used the software GeoGebra for this purpose. Towards the end of the design cycle, additional adaptations were made in order to meet the emerging challenges originating from the teachers practices. New theoretical notions were introduced and others were replaced in order to evaluate design process from a prospective view – what could be done differently in the next cycle? This resulted in a more comprehensive conceptual framework for understanding (CFU) based on the Anthropological Theory of the Didactic (ATD). The ATD served as an overarching theoretical perspective although only some aspects of the ATD have been particularly highlighted in this research project. The flexibility of the ATD allowed the researcher to attend to all the didactical issues that the researcher decided to pursue during the research process.

It is important to stress that adaptability should not be interpreted as a matter of searching for whatever works in the current situation. Instead, it is about the problematic task of assuring that the activity of inquiry is meaningful relative to the research objectives, i.e. the problem of developing systems guarantors (Churchman, 1971). This is a basic problem for any researcher but in this case, the problem of guarantors was not settled a priori and once and for all. The design problem of knowing when and how to revise becomes therefore even more difficult because there is no a priori authority to rely on. Instead, the decision to pursue a revision depends on an ambition to improve the performance of the system according to a specific measure and relative to the purpose (Churchman, 1971). Furthermore, when the system has been revised a new measure may be adapted to the new system. In order to make such tactical decisions, the researcher must be prepared to consider a "whole breadth of inquiry in its attempt to authorize and control its procedures" (ibid. p. 196). In other words, the development of adaptive conceptual frameworks could be understood as a modeling process that aims at developing system guarantors while preserving a high level of complexity in order to achieve high ecological validity.

Finally, the use of adaptive conceptual frameworks specifically affords transparency in the design process by making explicit the levels at which different theories operate and how the system performance is evaluated. In other words, it allows the researcher to cast light on how theory and practice interact throughout the design process.

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Communities of practice: Exploring the diverse use of a theory

Helena Roos and Hanna Palmér

Linnaeus University, Växjö, Sweden, Helena.Roos@lnu.se, hanna.palmer@lnu.se

The social learning theory of communities of practice is frequently used in mathematics education research. However, we have come to recognise that the theory is used in diverse ways, regarding both the parts that are used and the ways in which those parts are used. This paper presents an overview of this diverse use of the theory based on three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? The aim of the paper is twofold: to make visible the diverse possibilities within one single theory, and to make visible how, even though we might think we know what a theory implies in research, if we look beneath the surface we may find that "the same" theory can imply many different things.

Keywords: Communities of practice, theory, social, learning, Wenger.

INTRODUCTION

Since Etienne Wenger published his book *Communities of Practice: Learning, Meaning, and Identity* in 1998, the notion of communities of practice has become common in mathematics education research as well as in other areas of educational research. Both authors of this paper have been using Wenger's social theory of learning in research within mathematics education. In reading other researchers' work we have discovered that the theory of communities of practice is frequently used in mathematics education, but there are many differences regarding both *which* parts are used and *how* those parts are used. In this paper, we will explore some of the ways in which the theory of community of practice is used in different mathematics education studies. The aim of this is twofold: to make visible the diverse possibilities and uses of one single theory, and to make visible how we in research may think we know what using a specific theory in a study implies, but when we look beneath the surface we may find that "the same" theory can imply different things to different researchers.

The notion of communities of practice has been investigated and discussed before, for example by Kanes and Lerman (2008). They investigated similarities and differences in how the notion is used by Lave and Wenger (1991) and by Wenger (1998), respectively. (However, we find Kanes and Lerman's (2008) description of Wenger's communities of practice very different from our own interpretation and the interpretations we found when preparing this paper.) In this paper, we focus only on research referring to Wenger's 1998 book, in which he writes that his aim is to present a conceptual framework where learning is placed "in the context of our lived experience of participation in the world" (p. 3). In this paper, we will not present Wenger's theory more than that, in order to avoid imposing our own interpretations of which concepts are the main ones in his theory. Instead, the use of communities of practice will be explored according to the differences we found when reading other researchers using Wenger's theories. Hence, the exploration is divided based on the following three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? These three themes will be presented under each heading followed by a concluding discussion.

SELECTION OF STUDIES

Our selection of studies to explore was limited to those focusing on mathematics teaching or learning and/or mathematics teachers' professional development. We searched 19 databases, using the search words *communities of practice, mathematic** and/or *teach**;

the search was limited to peer reviewed journals or books. From this selection, consisting of more than 8000 articles, we limited the search to *communities of practice* and *mathematic** and/or *Wenger*; although that reduced the number of articles, there were still too many in some of the databases. We then removed "or" *teach**. Thereafter we were able to browse through all the titles and keywords to find a selection of research articles using *communities of practice*. This selection is not at all comprehensive, however, the purpose is not to generalise but to illustrate some of the differences we have found. Wenger's theory is also used frequently in studies within economy and management, but such studies are not explored in this paper.

Due to space limitations, this paper cannot present all the articles we have read; instead, we present articles that together illustrate the differences we found based on our three themes. The following ten studies will be discussed in relation to the three themes in the paper: Bohl and Van Zoest (2003); Corbin, McNamara and Williams (2003); Cuddapah and Clayton (2011); Cwikla (2007); Franke and Kazemi (2001); Goos and Bennison (2008); Graven (2004); Hodges and Cady (2013); Pratt and Back (2009) and Siemon (2009).

DESIGNED OR PRE-EXISTING COMMUNITIES OF PRACTICE

Some studies using Wenger's social theory of learning view communities of practice as pre-existing. In some other studies, for example, Bohl and Van Zoest (2003), Cuddapah and Clayton (2011), Goos and Bennison (2008), Hodges and Cady (2013) and Franke and Kazemi (2001), communities of practice are designed by the researcher(s).

In the study by Goos and Bennison (2008), a web-based community of practice is designed within teacher education. After graduation, interaction in the community of practice continues through the web-based tool developing an "online community" (p.41). In their article, Goos and Bennison discuss the issue of emergent versus designed communities of practice. Although, in their study Goos and Bennison design the external frames for the community of practice, their interest is in whether or not the web-based community develops into a community of practice. To give the community the best chance to develop into a community of practice on its own, the researchers provide only a minimum of structure concerning how community members are to communicate using the web-based tool. As such, they design a community, but it is its emergence as a community of practice they investigate in their study.

Hodges and Cady (2013) seek to expand on the work of Goos and Bennison (2008) by investigating the development of communities of practice within a professional mathematics teacher's development initiative. In this study a web-based tool is used to "foster the development of communities of practice" (p. 302). Hodges and Cady design a virtual space in order to see the emergence of communities of practice. However, unlike Goos and Bennison (2008), Hodges and Cady do not highlight the issue of an emergent or a designed community, even though the emergence of potential communities of practice is in focus.

Cuddapah and Clayton (2011) design a community of practice by arranging physical sessions with a group of novice teachers. They focus on one of several groups of novice teachers that, within a university-sponsored project, meet every second week. The novice teachers meet 15 times during the study. Every session has a theme and the sessions are planned and led by experienced educators. Cuddapah and Clayton write that the group of novice teachers "itself was a community" (p. 69) and they use Wenger's theories to analyse the development of the group and its function as a resource for new teacher support. In their analysis they present how the "community was observed throughout and between the data" (p. 72). As such, the group of novice teachers being a community of practice was both a precondition and a result of their analysis.

A fourth example of researchers who design communities of practice is Franke and Kazemi (2001). In their study they design communities of practice with mathematics teachers with the purpose of providing teachers with opportunities to learn about mathematics teaching and learning. The teachers in this study do mathematical tasks with their students in their classrooms and then they meet and discuss their experiences. The researchers take part in the discussions and they also visit the teachers at their schools several times. Franke and Kazemi do not describe why or how the group of teachers is a community of practice, but they analyse and describe the interactions in the group connected to teacher professional development.

Examples of studies in which communities of practice are treated as pre-existing, developed before the study began and without the influence of the researchers, are studies by Bohl and Van Zoest (2003), Corbin and colleagues (2003), Cwikla (2007), Graven (2004), Pratt and Back (2009) and Siemon (2009). In some studies the communities of practice are identified in the research process based on concepts from Wenger's theory, whereas other studies do not explain how they are identified as communities of practice.

Bohl and Van Zoest (2003), Graven (2004), Corbin and colleagues (2003), Cwikla (2007) and Pratt and Back (2009) are examples of studies where communities of practice are viewed as pre-existing at the start of the study, where the researchers do not explain how the communities have been identified as such.

Bohl and Van Zoest (2003) analyse how different communities of practice in which novice teachers participate influence their mathematics teaching. They give an empirical example of one novice teacher, in relation to whom they discuss differences in the role of novice teachers in different communities of practice, but they do not present how they identified these as communities of practice, nor do they explain how they identified the novice teacher's membership in these communities.

Graven (2004) investigates teacher learning in a mathematics in-service program. In this study an in-service program is considered to be a community of practice, but it is not explained how this community of practice has been identified as such. This is also the case in the study of Corbin and colleagues (2003), who investigate numeracy coordinators in an implementation of a national numeracy strategy. They use the notion of communities of practice as a tool to describe the participation of the coordinators in different communities, but they do not explain how they define the communities.

Pratt and Back (2009) investigate participation in interactive discussion boards designed for mathematics students. They simply state that "two idealised communities of practice" (p. 119) were adopted as a means to understand the discussion boards. How these communities were created and why they can be seen as such is not explained. They even describe the communities of practice as "hypothetical communities" (p. 128). Cwikla (2007) uses the concept of communities of practice in her study of the evolution of a middle school mathematics faculty. The concept of communities of practice is used to identify boundary encounters, but the article does not present any definition of communities of practice, nor does it specify which communities of practice are identified within the study.

Siemon (2009) is an example of a study where communities of practice are viewed as pre-existing at the start of the study, but where the researcher explains how the communities of practice have been identified as such. Siemon (2009) investigates improvements in indigenous students' numeracy skills after they worked on key numeracy issues in their first language. Three pre-existing communities of practice are described and it is explained, using Wenger's concepts, why these are considered to be communities of practice. In the study, the intersection between the acknowledged pre-existing communities of practice is investigated. The members of these communities are not described in detail, only as, for example "members of the local Indigenous community" (p. 225), or "all those that by virtue of their responsibilities are concerned in some way with school mathematics" (p. 225). The intersection between the communities of practice is not highlighted, although the author states that the edges of the communities took time to emerge.

FOCUS ON INDIVIDUALS OR GROUPS

Wenger's theory makes it possible to foreground groups (communities of practice) or individuals (learning and/or identity) or both. Since Wenger's theory is very broad and yet detailed, it is not surprising that either groups (communities of practice) or individuals are foregrounded in the studies. Wenger explains that this is not a "change of topic but rather a shift in focus within the same general topic" (p. 145). Franke and Kazemi's (2001) study is an exception, however, and an example of "both" since they analyse both the interaction within the community of practice and the identity development of individual participants.

In the studies by Cwikla (2007), Cuddapah and Clayton (2011), Goos and Bennison (2008), Hodges and Cady (2013) and Siemon (2009), groups of teachers are in the foreground and individuals are in the background or are not mentioned as individuals at all. Bohl and Van Zoest (2003), Corbin and colleagues (2003), Graven (2004) and Pratt and Back (2009), however, foreground the individuals, trying to understand how they are influenced by the different communities of practice in which they participate.

The issue of communities of practice or individuals being foregrounded in the studies as presented in this section is connected to which parts or concepts from Wenger's theory are used in the analyses, which is the focus of the next section.

WHICH PARTS OF THE THEORY ARE MAINLY USED?

Another consequence of Wenger's theory being very broad and yet detailed is that researchers focus on and use smaller parts of the theory, selecting just some of the concepts within it.

Graven (2004) uses the concepts of *practice, meaning, identity,* and *community* to describe and explain teacher learning. These four concepts are, according to Wenger "interconnected and mutually defining" (p. 5). Graven also mentions Lave and Wenger's (1991) concepts of *co-participation* and *participation*, but these are not used in her analysis. Even though Graven describes communities of practice in her study, the "three dimensions" (p. 72) that according to Wenger are the source of a community of practice, *mutual engagement, joint enterprise* and *shared repertoire*, are not used. However, Graven instead wants to add *confidence* as a supplement to *practice, meaning, identity,* and *community.*

Cuddapah and Clayton, like Graven (2004), initially refer to Lave and Wenger (1991) but to the concept of *legitimate peripheral participation.* They discuss this concept as one that can be used when analysing novice teachers as newcomers in teaching. However, as all novice teachers in their study are new members of a new community of practice designed by the researchers, they instead, like Graven (2004), use *practice, meaning, identity,* and *community* when coding their empirical material. They briefly mention the concepts of *mutual engagement, joint enterprise* and *shared repertoire,* but they do not use them in their analysis.

Those three concepts, *mutual engagement, joint enterprise,* and, *shared repertoire*, are used by Goos and Bennison (2008), Hodges and Cady (2013) and Siemon (2009) in their studies. As shown in the last section, these three studies have communities of practice in the foreground. Goos and Bennison (2008) use the three concepts when they analyse the emergence of their designed web-based community of practice. To investigate mutual engagement they count the number of interactions in the web-based tool. By analysing the content in these interactions they also investigate the joint enterprise and the shared repertoire that develops. Siemon (2009) uses the three concepts by making lists of what it is in the different communities of practice identified in the study that indicates joint enterprise, mutual engagement and a shared repertoire. Consequently, in her study communities of practice are pre-existing, but she defines them by mutual engagement, joint enterprise and shared repertoire. Three communities of practice are acknowledged this way. Hodges and Cady (2013) use the three concepts in the same way, but their approach is somewhat different. They use the concept in order to find and/or see development of communities of practice in a designed web-based tool. In their analysis they look for evidence of joint enterprise, mutual engagement and a shared set of ways of interacting in order to see if a community of practice has been developed. As such, the concepts of *mutual engagement, joint enterprise* and *shared repertoire* are used to identify both designed (Goos & Bennison, 2003; Hodges & Cady, 2013) and pre-existing (Siemon, 2009) communities of practice.

In addition to mutual engagement, joint enterprise and a shared repertoire, Siemon (2009) also uses Wenger's concept of *negotiation of shared meaning* when referring to a space where the participants in the different communities of practice can meet. This space is used both as a place to negotiate meaning and as a research tool to "explore the processes involved in building community capital" (p. 226). Furthermore, Siemon uses the concept of *boundary objects* when defining Probe Tasks as a boundary object in the negotiation described above (a Probe Task is described in the paper as a specifically chosen or designed task to support indigenous teacher assistants as they teach key aspects of number). Cwikla (2007) also uses the concept of *boundary objects.* In her investigation of the evolution of a middle school mathematics faculty, she uses this concept together with the concept of *brokers*, which is also from Wenger. She mentions communities of practice, but she does not define them. When using the concept of brokers, she refers to Wenger's definition, stating, "a broker can serve as a conduit for communication and translation between communities of practice" (p. 558). Corbin and colleagues (2003) also use the concept of *brokering* when investigating numeracy coordinators in an implementation of a national numeracy strategy. The concept is used to theorise tensions in the work of the coordinators. Corbin et al. find signs of brokering in their analysis by using three more of Wenger's concepts: the *modes of belonging: engagement*, *alignment* and *imagination*. Pratt and Back (2009) also use the concepts of *engagement, alignment* and *imagination* in their analysis. They also use Lave and Wenger's (1991) concept of *legitimate peripheral participant* as well as *peripheral and central participation* in their analysis. These concepts are used to describe a person's participation, and changes in participation, in two different communities of practice.

Bohl and Van Zoest (2003) mention that communities of practice develop through mutual engagement, joint enterprise and shared repertoire, but in their analysis they use two other concepts of Wenger's: *modes of participation* (their term for what Wenger refers to as *modes of belonging*) and *regimes of accountability*. They use these two concepts to analyse how novice teachers have different roles in different communities of practice and how this influences their mathematics teaching.

As mentioned, Franke and Kazemi (2001) analyse both the interaction in one community of practice and the identity development of individual participants. However, they do this without explicitly using any of Wenger's concepts. The artefacts they mention are not identified explicitly as artefacts used by Wenger but as used in sociocultural theories in general. They also mention *identity* and *negotiation of meaning*, both of which are thoroughly elaborated by Wenger, but they do not refer explicitly to how the concepts are used by Wenger. As such, Franke and Kazemi refer to, and use, Wenger's social theory of learning, but not explicitly or solely; rather, they present it as part of a general sociocultural view of learning.

Overall, several of Wenger's concepts are used in the studies presented in this paper, including *practice, meaning, identity, community, mutual engagement, joint enterprise, shared repertoire, modes of belonging, engagement, alignment, imagination, identity, brokering, negotiation of meaning, boundary objects, regimes of accountability, co-participation* and *participation*. However, seldom are more than three or four concepts used in the same study. Since the theory is broad and yet detailed, it is not surprising that researchers focus on and use only parts of it. Even so, none of the articles referred to in this paper draws attention to the fact that only certain parts of Wenger's theory will be used. Neither do they discuss the eventual consequences of not using the theory in its entirety. Hence, anyone reading only one of the articles may easily believe that the whole of Wenger's theory is used.

DISCUSSION

As seen in the examples in this paper, Wenger's social theory of learning is used in different ways in different studies. Wenger (1998) terms his work a "conceptual framework" (for example, p. 5), a "social theory of learning" (for example, p. 4) and/or a "perspective" (for example, p. 3). According to Eisenhart (1991), there are three kinds of research frameworks: theoretical, practical and conceptual. Eisenhart distinguishes these as theoretical frameworks based on formal logic, practical frameworks based on practitioner knowledge and conceptual frameworks based on justification. Somehow Wenger's social theory of learning comprises all three of these features. According to Niss (2007), theories are stable, coherent and consistent systems of concepts that are organised and linked in hierarchal networks. Those criteria apply to the content of Wenger's book. However, when researchers use only some of Wenger's concepts the criteria are no longer met. Furthermore, Niss (2007) writes that one purpose of theories "is to provide *a structured set of lenses* through which aspects or parts of the world can be approached, observed, studied, analysed or interpreted" (p. 100). The diverse uses of Wenger's social theory of learning presented in this paper show that the *structured set of lenses* used in these studies differ substantially.

According to Lester (2005), a framework provides structure in research when it comes to the questions being asked and the concepts, constructs and processes being used. Connected to the overview in this paper, the use of Wenger's social theory of learning appears to coincide with the first (questions), but not the rest. Even though the use of Wenger's social theory of learning differs in the studies presented in this paper, one similarity is the type of questions asked. These questions imply that the theory is considered suitable for studies of mathematics teachers', novice teachers', student teachers' and/or students' learning. Furthermore, in several of the studies (for example, Bohl & Van Zoest, 2003; Siemon, 2009) the social dimension of learning provided by Wenger is emphasised as its main strengths. As such, the use of Wenger's theory in mathematics education research seems to be part of the "turn to social theories in the field of mathematics education" (Lerman, 2000, p. 20). According to Lerman (2000), social theories make it possible to foreground individuals (practice in person) or practice (person in practice). However, both elements (person and practice) are always present and part of the analysis, which is in line with Wenger's "shift in focus within the same general topic" (p. 145).

As shown in this paper, there are differences in the presented studies in terms of communities of practice being viewed as pre-existing or designed as well as communities of practice being identified based on Wenger's concepts or not. In his book Wenger actually writes that since communities of practice are about content and negotiation of meaning – and not form – they are not "designable units" (p. 229). That is, according to Wenger, it is possible to design the outer limits but not the practice that may, or may not, emerge. As presented above, there is also diversity with respect to whether individuals or (communities of) practice are in the foreground. As also shown, there are differences regarding which of Wenger's concepts is used, even when the same perspective (individuals or communities of practice) is in the foreground. In terms of the concepts used, we were surprised by the rare presence of *reification* and *negotiation of meaning*, as these two concepts recur frequently throughout Wenger's book. Furthermore, there are many other concepts of Wenger's that are not used in any of the studies we read, including *local/global, identification, economies of meaning, ownership of meaning* and *trajectories*.

Finally, what can be learned from this overview of how Wenger's social theory of learning is used in different ways in mathematics education research? Well, often we (think that we) know what researchers imply when they say they have been using a specific theory in their research. However, from the overview presented in this paper, we know that if a researcher says that (s)he has been using Wenger's social theory of learning, we can be quite sure that we do *not* know exactly what that use of Wenger's theory might imply. In this paper, we have highlighted some of the diverse uses of Wenger's social theory of learning based on three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? Probably further comparisons based on other themes will reveal other diversities. Further, based on the breadth and wealth of details in Wenger's social theory of learning, the list of themes and diversities may become quite long.

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Beyond orchestration: Norm perspective in technology integration

Rüya Şay and Hatice Akkoç

Marmara University, İstanbul, Turkey, sayruya@gmail.com, haticeakkoc@yahoo.com

The aim of this study is to bring a socio-cultural dimension to "instrumental orchestration" framework. Our claim is that social and socio-mathematical norms endorsed by teachers are crucial for their pedagogies. A case study was designed to investigate how orchestration types and norms affect each other in technology-enhanced learning environments. Participants are five pre-service mathematics teachers. Data were collected through lesson plans, semi-structured interviews and observations. Analysis of data indicates that there is a two-way interaction between norms and orchestration types. In some cases, norms are determinants of orchestration types used by participants. In other cases, orchestration types challenge participants' endorsed norms.

Keywords: Socio-cultural approach, Social norm, Socio-mathematical norm, Instrumental orchestration.

INTRODUCTION

Recently, mathematics teaching in technology-enhanced environments has been widespread and mathematics teachers are faced with a large number of resources (Drijvers, 2012). Various official curriculum documents around the world emphasise the importance of using technology to support learning (NCTM, 1999, 2000; DfES, 2013a, 2013b). This requires certain knowledge and pedagogy. For example, International Society for Technology in Education describes technology standards and performance indicators for teachers. Teachers should be able to "plan and design effective learning environments and experiences supported by technology" (ISTE, 2000, p. 9).

Teachers play a key role in effective use of technology in the classroom and the way they integrate technology into teaching affects the way students learn mathematics (Ely, 1996, as cited in Besamusca & Drijvers, 2013). Therefore, mathematics teachers and teacher educators should be guided for the design of learning environments using technological tools and resources (Şay, Kozaklı, & Akkoç, 2013).

One of the theoretical frameworks to investigate the use of technological tools in the classroom is "instrumental orchestration" which is based on the framework of instrumental genesis (Trouche, 2004). Considering the literature on orchestration, it can be claimed that this theoretical framework focuses on classroom organisation but fall in short to explain psychological and sociological development of teachers. Teachers and pre-service teachers have different pedagogical approaches and go through different professional development phases. Therefore, an investigation of technology integration purely based on physical organisation of technology-enhanced learning environments and certain teacher behaviours is only one part of the whole picture. Teachers might have different norms and these affect the way they integrate technology into their lessons. However, there is little research in the literature on how teachers' activities in the classroom are shaped by their norms and very few of them investigated this in the context of technology. The aim of this study is to bring a socio-cultural dimension to instrumental orchestration. Socio-cultural theory aims to investigate human action and its relationship with cultural, institutional and historical situations. Therefore, it focuses on social interactions and the effects of culture on psychological development (Wertsch, del Rio, & Alvarez, 1995; Lerman, 2001). Technological tools can turn into effective instruments for learning mathematics via effective classroom interaction. Social and socio-mathematical norms, as one of the aspects of socio-cultural theory, might take a role in shaping student-teacher-tool interaction in the classroom. Furthermore, they are also shaped by this interaction. Therefore, one could elucidate how teachers use technological tools by embracing a norm-perspective within socio-cultural approach. Our claim is that social and socio-mathematical norms (Višňovská, *Cortina*, & Cobb, 2007) endorsed by teachers are crucial for their pedagogies and their choices for different orchestration types. To support this argument, this study investigates the interaction between orchestration types used by pre-service mathematics teachers and social and socio- mathematical norms.

INSTRUMENTAL ORCHESTRATION

An instrumental orchestration is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a learning environment in a given mathematical task situation, in order to guide students' instrumental genesis (Trouche, 2004).

Drijvers (2012) distinguishes three elements within an instrumental orchestration: a didactic configuration, an exploitation mode and a didactical performance. "A didactical configuration is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it" (p. 266). An exploitation mode is defined as the teacher's decisions on the way she or he configures a task by providing certain roles for the artefacts to achieve his or her didactical intentions.

> A didactical performance involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals (Drijvers, p. 266).

Drijvers and his colleagues (2010), Drijvers (2012) and Tabach (2013) distinguish ten types of instrumental orchestrations as seen in Table 1 (The last three orchestration types are not in the original table and were added from the literature). In this study, pre-service teachers' lessons will be analysed considering the orchestration types in this table.

The orchestration types	Didactical configuration	Exploitation mode
Technical-demo (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher explains the technical details for using the tool.
Explain-the-screen (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher's explanations go beyond techniques and involve mathematical content.
Link-the-screen board (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher connects representations on the screen to representations of the same mathematical objects that appear either in the book or on the board.
Sherpa-at-work (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The technology is in the hands of a student, who brings it up to the whole class for discussion.
Not-use-tech (Tabach, 2011)	Whole-class setting, one central screen	The technology is available but the teacher chooses not to use it.
Discuss-the-screen (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	Whole class discussion guided by the teacher to enhance collective instrumental genesis.
Spot-and-show (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher brings up previous student work that he/she had stored and identified as relevant for further discussion.
Work-and-walk-by (Drijvers, 2012)	Students work individually or in pairs with computers	The teacher walks among the working students, monitors their progress and provides guidance as the need arises.
Discuss-the-tech-without-it (Tabach, 2013)	Every students have own laptops or laptops bring classroom with wheeled vehicles	The teacher uses mobile transport system if he/she needs computers in teaching
Monitor-and-guide (Tabach, 2011)	----	The teacher uses a learning management system by giving guidance to students

Table 1: Orchestration types (Tabach, 2013, p. 3)

SOCIAL AND SOCIO-MATHEMATICAL NORMS

In mathematics education literature, it is widely recognised that social interaction promotes learning opportunities. Norms construct how students learn mathematics and how they become mathematically autonomous (Cobb & Bowers, 1999; Pang, 2000). Norms regulate the way teachers and students participate in learning and teaching activities within a classroom culture (Cobb & Yackel, 1996). While norm emerges from social interaction; belief, value, opinion and attitude are concerned with the individual.

Cobb and Yackel (1996) propose social and socio-mathematical norms to investigate how students' mathematical values and beliefs develop within the classroom culture from the psychological and socio-cultural perspectives. Cobb and his colleagues (2007) also investigated teachers' professional development through social and socio-mathematical norms (Visnovska, Cortina, & Cobb, 2007). Social norms apply to any subject matter area. Students' cooperation when solving problems or privileging a logical explanation over other correct answers are examples of social norms (Hershkowitz & Schwarz, 1999). Another example is the way teachers promote students' thinking and value different ideas. On the other hand, socio-mathematical norms are specific to mathematics and are concerned with the way mathematical beliefs and values develop in the classroom. For example, acceptability of a mathematical explanation or a justification is a socio-mathematical norm (Yackel & Cobb, 1996; McClain & Cobb, 2001).

METHODOLOGY

This study embraces the interpretive paradigm to investigate how orchestrations types and norms affect each other in technology-enhanced learning environments. A case study was designed to answer the research question. Participants are five pre-service mathematics teachers who were enrolled in a teacher preparation program in a state university in Istanbul, Turkey. It was a four-month program which will award its participants a certificate for teaching mathematics in high schools for students aged between 15 and 18. The program accepts graduates who have a BSc degree in mathematics. There were two kinds of courses in the program: education and mathematics education courses. The study was conducted during "Instructional Technologies and Material Development" and "Teaching Practice" courses. The former course focused on six software, namely Geogebra, Graphic Calculus, Derive, Geometry Sketchpad, Excel and Probability Explorer. Participants were involved in hands-on activities in front of a computer and prepared teaching materials. Participants also taught lessons in partnership schools during "Teaching Practice" course.

There were thirty-six participants in the program. They were all interviewed on their approach to the use of technology for teaching mathematics. Five participants were purposefully selected. Two of them (one male and one female) had positive attitudes and two of them (one male and one female) had negative attitudes towards the use of technology. One participant was selected because she had neutral attitude.

The data collection methods are observation and semi-structured interviews. Each participant taught a total of five lessons in a partnership school. At least two of these lessons were technology-based. Each participant taught at least two same classes of students. They were interviewed after their lessons. During the semi-structured interviews they were asked what kinds of norms they endorsed, how they used technology in their lessons and differences between their lessons with or without technology. Their lessons were video recorded. The first author of this paper observed lessons using an observation form. The aim of the observation form was to reveal social and socio-mathematical norms endorsed by pre-service teachers. Interviews and lesson videos were verbatim transcribed. Data from different sources such as interviews, observations and field notes were triangulated. Common themes emerged from verbal discussions among pre-service teachers and students, patterns in pre-service teachers' behaviours and statements about their endorsed norms during the interviews. For instance, the socio-mathematical norm "Answers which are logical are acceptable" was determined considering pre-service teachers' discussion with students and how they defined "an acceptable answer" during the interview.

FINDINGS

This section presents orchestration types and social and socio-mathematical norms used by participants. First we demonstrated how participants used orches-

Orchestration types	Pre-service teachers
Technical-demo	Nil, Orkun, Melek
Explain-the-screen	Mahir, Orkun, Melek
Discuss-the-screen	Melek
Sherpa-at-work	Orkun, Melek, Nil
Not-use-tech	Oya, Mahir, Nil, Orkun, Melek

Table 2: The orchestration types used by pre-service teachers

tration types and then how norms and orchestration types affected each other.

The analysis of the data indicated that participants mostly used *technical-demo, explain-the-screen, link-the-screen-board, discuss-the-screen, Sherpa-at-work and not-use-tech* orchestration types as seen in Table 2.

Explain-the-screen promoted the social norm "the authority is the teacher". For example, Mahir taught a lesson on parabolas using Geogebra software. He started his lesson by explaining how to draw a parabola and adding a slide which is defined as the determinant. He then moved the slide and explained what happened to the graph of parabola. At this stage he did not questioned the mathematical meanings behind what the software performed, but just explained how to use the software.

When participants used *link-the-screen*, they explained a concept or a mathematical idea on the board followed by an elaboration using the software. For example, Orkun taught a lesson on how to draw the graph of $y = \sin x$ using Geometry Sketchpad. He first plotted a few points and then drew the graph on the board. However students claimed that points should be joint using straight lines. He then moved to the software and constructed a unit circle. He defined a point A on the unit circle and a point B which defines the sine function. Using "trace" feature, he obtained the graph of $y = \sin x$. Up until now, the authority was the teacher. Therefore, it can be claimed that *link-the-screen* orchestration type promoted this social norm. Afterwards he asked students how to draw $y = \cos x$ and $y = \tan x$ themselves. His question is an indication of a social norm "Students are challenged with the questions of why and how". This social norm required using *discuss-the-screen* orchestration type.

Another participant who used *discuss-the-screen* discussed with their students the actions they performed using the software. For example, Melek used Geometry Sketchpad to explain how to draw trigonometric functions. She first drew the graph of $y = \sin x$ and then asked one of her students to draw $y = \cos x$. Later she discussed with her students how to draw $y = \tan x$ using the software and tried to reach a common ground ($\tan x = \sin x/\cos x$):

Melek:	Is there anyone who wants to draw the tangent line?
Student A:	This time, we will construct a point with x and y (on the unit circle)...
Student B:	Slope
Melek:	What else? What is slope? One of the definitions is opposite over adjacent. It's the ratio of opposite side over adjacent side or what is $\tan x$?
Student C:	$\sin x$ over $\cos x$
Melek:	Isn't it $\sin x$ over $\cos x$. That's the expression that everybody knows. Therefore, when we want to find the ratio of $\sin x$ over $\cos x$, that is when we think graphically (showing the point on the unit circle) if we vertically projected this point onto x-axis, we say opposite over adjacent to find the tanjant

When pre-service teachers were using *discuss-the-screen* they endorsed the socio-mathematical norm "Answers which are logical are acceptable". For example, Melek who used *discuss-the-screen* aimed at having her students discuss mathematical meanings behind what the software perform. When doing this, she considered different student answers and did not impose her answers or solutions. She encouraged her students discover their own solutions which were meaningful for them. As can be seen in this case, the orchestration type used by this pre-service teacher affected her endorsed norm. In other words, a norm has emerged which support *discuss-the-screen* orchestration type.

Pre-service teachers chose not to use technology (not-use-tech orchestration type) at least once out of five lessons they each taught. Before participants had teaching experiences with using technology in the classroom, they had the socio-mathematical norm which gives the teacher the mathematical authority and believed that technological tools were not necessary for teaching mathematics:

> Oya: I'm quite conservative. I believe that mathematics should be taught using the blackboard. I think that maths would be better understood this way

Oya was a unique case who's social or socio-mathematical norms did not change after she started using technological tools in her lessons. On the other hand, Orkun who has negative attitudes towards using technology in a mathematics lesson changed his endorsed norms and this situation is illustrated with the following excerpt:

> Orkun: In my first lesson (which he did not use any technological tool) I wished that student would not ask me any questions. Because I was teaching inverse trigonometric functions and the questions I prepared were very difficult ones... students in this school were very clever and I was worrying about receiving different questions. And there was no help from technology. I had to teach on the blackboard. But on the next lessons when I used technology, I wasn't afraid of their questions. When I'm stuck on the board I knew that I could switch to technology

As can be seen from the excerpt above, he sees technological resources as a helpful tool which gives him confidence. This confidence changed his norms and pedagogy.

Another orchestration type observed in this study is *Sherpa-at-work*. Participants in this study used this orchestration type in a different way when compared to the related literature. In the literature, when using Sherpa-at-work students work in front of a computer individually or in pairs and "the technology is in the hands of a student, who brings it up to the whole class for discussion" (Tabach, 2013, p. 3). However, there was a lack of technological resources in partnership schools and students did not get the chance to use their own computers during the lessons. Participants had their own computers which were projected on to a screen. This situation prevented active participation of students. Orkun, Melek and Nil tried to resolve this problem by having students use the teacher's computer. This corresponds to *Sherpa-at-work* orchestration type which emerges as a result of "students who answers correctly go to the blackboard" social norm.

DISCUSSION

This study investigated how orchestration types and norms affect each other in technology-enhanced learning environments. The findings indicated that pre-service mathematics teachers used some of the orchestration types frequently such as *link-the-screen-board* and *not-use-tech*. On the other hand, some of the orchestration types such as *spot-and-show*, *work-and-walk-by*, *discuss-the-tech-without-it* and *monitor-and-guide* were not used because of lack of technological resources in the partnership school. All classrooms in this school have smart boards but students did not have their own computers or tablets. Therefore, some of the orchestration types were not observed.

Drijvers and his colleagues (2010) claimed that teachers make pedagogical choices based on their views about how to teach mathematics. This study has similar findings by illustrating how orchestration types and norms support each other. Social and socio-mathematical norms endorsed by participants affected their choices of technological tools for teaching mathematics and as a result orchestration types they used.

Our claim was that there was a two-way interaction between orchestration types and social and socio-mathematical norms. This study attempted to justify this claim. As a matter of fact, Drijvers (2012) described *technical-demo*, *explain-the-screen* and *link-the-screen-board* orchestration types as teacher-centred and *Sherpa-at-work*, *spot-and-show* and *discuss-the-screen* orchestration types as student-centred. Therefore, participants who used teacher-centred orchestration types endorsed socio-mathematical norms accept the teacher as the mathematical authority. On the other hand, participants who used student-centred orchestration types endorsed social norms which puts students into the centre.

Findings of this study also revealed that instrumental orchestration framework fall in short in explaining the socio-cultural aspect of technology integration. In terms of teacher-student-tool interaction, technological tools provide a language which supports communication between students and teachers (Noss & Hoyles, 1996). Examining the micro-culture of the classroom provided by this kind of language and social and socio-mathematical norms required by that culture expanded our understanding of orchestration framework. Integrating instrumental orchestration framework into norm perspective provided an insight on the question of why and how particular orchestration types are used.

This study suggests some implications for researchers and teacher educators. First of all, as mentioned above, there is not satisfying research which explicitly investigates norms in the context of technology integration. In this study, this was investigated in the context of a short-term teacher preparation program in Turkey. There is a need for further studies. Second, teacher education programs which aim successful technology integration should develop an awareness of social and socio-mathematical norms and monitor pre-service teachers' development with regard to their endorsed norms.

ACKNOWLEDGEMENT

This study is supported by Scientific Research Project Commission (BAPKO) in Marmara University (Project number: EGT-D-140115-0021).

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Competency level modelling for school leaving examination

Hans-Stefan Siller[1], Regina Bruder[2], Tina Hascher[3], Torsten Linnemann[4], Jan Steinfeld[5] and Eva Sattlberger[5]

1 Universität Koblenz-Landau, Campus Koblenz, Koblenz, Germany, siller@uni-koblenz.de
2 TU Darmstadt, Darmstadt, Germany
3 Universität Bern, Bern, Switzerland
4 Paedagogische Hochschule FHNW, Solothurn, Switzerland
5 BIFIE, Wien, Austria

A project group was commissioned to develop a content- and action-related competency grid in order to enable quality assessment and comparability of mathematics examination questions in the Austrian Matura (final examination at the end of the Secondary School Level II). Based on theoretical grounds, in the competency grid the three dimensions operating, modelling and reasoning are distinguished and described on four levels.

Keywords: Competency level model, school, final exam, operating, modelling, reasoning.

Obtaining information on the development of mathematical competency is a central concern of mathematics education (e.g., Leuders, 2014) and empirical educational research (e.g., Hartig, 2007). In Austria, an approach with the goal of a standardized competency-based written final examination – the so called 'Matura' at the end of Secondary Level II – (cf. AECC Mathematik, 2009; BIFIE, 2013a) in the context of mathematics as a general education subject (cf. Fischer, 2001; Fischer & Malle, 1985; Klafki, 1985; Winter, 1975, 1996) was applied. Examinees are expected to have both mathematical (basic) knowledge and (basic) ability, as well as general mathematical skills such as reasoning skills, problem solving skills, and also the ability to use mathematics in different situations, i.e. modelling skills. However, in PISA 2000, a lack of modelling competency was observed, when students failed to solve (real-life) problems with the help of models in a satisfying way (cf. Klieme et al., 2001). Based on this result, modelling competence was crucial for competency orientation in the curriculum enhancement of mathematics education in the German-speaking region (thus also for the *Matura* in Austria). With reference to Weinert's definition of competencies (2001, p. 27) as

> the cognitive skills and abilities which the individual possesses, or which can be learned, to solve certain problems, as well as the associated motivational, volitional and social readiness and skills in order to successfully and responsibly use problem solutions in a range of situations.

In an iterative process we developed a competency level model for the written final exam in mathematics at the end of Secondary School Level II. The process consisted of four elements: the discussion of competency specifications and developments, the discussion of mathematical tasks, task rating in due consideration of the competency model and the discussion of these ratings. Against the background of theoretical and also experience-based ideas about the current development of mathematical skills in school learning, we described the following three domains of mathematical competencies: operating, modelling, and reasoning[1] (O-M-A) on four levels.

In close cooperation with the Federal Institute of Educational Research, Innovation and Development of the Austrian School System ('BIFIE'), we developed a competency level model facilitating the description and comparison of the exam requirements, especially with regard to examination questions in the final examination in mathematics (Siller et al., 2013).

COMPETENCY LEVEL MODEL

In the competency models of the German-speaking countries Austria, Switzerland and Germany (AECC Mathematik, 2008; HarmoS, 2011; KMK, 2012), content areas (such as geometry or arithmetic), general mathematical competencies (such as reasoning) and skill levels (usually three-stage) are considered. The elements of the model in each country are therefore different when compared to one another. The competency levels are somewhat vague. Therefore, they can only be described on the basis of empirical task difficulty. To put our competency level model in a wider scientific context, we follow Leuders (2014, p. 10): "A model is discussed which (i) a priori postulates levels in acquiring a certain competence, is describing (ii) through stepped task situations and (iii) hierarchically ordered categorical latent ability variable. This allows (iv) determination about which competency pupils possess at each level."

In comparison to earlier statements, the development that has taken place in this area is evident. For example, Helmke and Hosenfeld stated in 2004 (p. 57):

> Neither are the currently available versions of the educational standards derived at the time from comprehensive and didactic accepted competency models (...) nor is there already in all relevant areas of content expertise models which meet the abovementioned requirements, particularly theoretically coherent developmental and learning psychology based levels concepts.

Thinking in (competency) levels is common in schools since curricula and teaching materials are based on this view (cf., e.g., Kiper, Meyer, Mischke, & Wester, 2004). Competency level models contribute to the diagnosis of the learners' levels of competency by the assessment of their achievements. Moreover, the models aim at describing the development of competencies. Their weaknesses, however, are embodied in the fact that it usually remains undetermined how a change to the next level can be accomplished and what conditions are necessary for this. Furthermore, a fixed sequence is assumed, which implies that neither can any steps be skipped nor regressions occur, but which assumes steady, cumulative learning.

For the present competency level model we have agreed on four stages, which can be identified in a manner analogous to Meyer (2007), who described the following four levels (Meyer, 2007, p. 5):

1) Execution of an action, largely without reflective understanding (level 1)

2) Execution of an action by default (level 2)

3) Execution of an action after insight (level 3)

4) Independent process control (level 4)

The activity theory forms the background for the didactic interpretation of such initially pragmatic levels (cf., e.g., Lompscher, 1985) with the corresponding concept of different cognitive actions and their specific dimensional structure. Nitsch and colleagues (2014) developed and empirically verified a competency structure model that describes relevant student actions when translating between different forms of representations in the field of functional relationships. For example, they could show that the two basic actions of acquirement *Identification* and *Implementation* (Construction) and the basic cognitive actions *Description* and *Explanation* differ in their cognitive demands, i.e. they are based on different facets of competency. Therefore, we used the theoretical model of hierarchical structure of cognitive actions (Bruder & Brueckner, 1989) for the description of competency levels.

DEVELOPMENT OF A COMPETENCY LEVEL MODEL

Currently existing competency models are primarily based on empirical analyses: Based on the solution probabilities of tasks (items), competency levels are modelled in the context of large-scale studies. An alternative approach is to primarily derive a model from theoretical concepts. This also requires the recognition of central instructional goals such as a sustainable understanding of mathematical relationships, which in turn presupposes a high level of cognitive activation in the teaching processes (cf. Klieme et al., 2006). This can, for example, be achieved by the following measures:

— the preparation of relationships for basic knowledge and skills learned;

— the challenge to describe mathematical relationships or solutions in application contexts;

— the creation of occasions for reasons or reflections.

Such criteria of demanding instruction should also be appropriate to form a competency level model.

THE COMPETENCY LEVEL MODEL O-M-A

Competency level models that are empirically based indicate to what extent tasks differ in their level of difficulty in terms of processing. Evidence of existing difficulties can be obtained by carefully analysing potential and actual solutions. Normative stipulations of difficulty levels imply that it is not possible to successfully process the task on a lower level. The levels of the competency model postulate what skills are needed to solve them. This does not exclude that there are multiple solution strategies, particularly for complex task definitions.

For designing the domains of mathematical competencies, we follow an orientation to Winter's basic experiences (cf. Winter, 1996, p. 37):

1) To perceive and understand phenomena of the world around us that concern or should concern all of us, from nature, society and culture in a specific way,

2) to learn and comprehend mathematical objects and facts represented in language, symbols, images and formulas as intellectual creations as a deductive-ordered world of its own kind,

3) to acquire task problem-solving skills that go beyond mathematics (heuristic skills).

While the first basic experience corresponds to mathematical modelling as a fundamental action area in learning mathematics, there are the other two basic experiences "operating" and "reasoning", which serve the second fundamental experience as well as "problem solving" for the third basic experience. In various competency models „communicating" is included to emphasize the linguistic aspects, as well as other domains of mathematical competencies.

"Problem solving" is not separated as an independent domain in the Austrian requirements for the final examination (BIFIE, 2013). "Problem solving" is defined as a more complex aspect of action and therefore includes the domains of the mathematical competen-

	Domains of mathematical competency		
Level	**Operating**	**Modelling**	**Reasoning**
1	Identify the applicability of a given or familiar method; Implementing/executing a given or familiar rule	Implementation of a representation change between context and mathematical representation Using familiar and directly recognizable standard models for describing a given situation with appropriate decision	Perform basic technical language reasoning Examine the application of a relationship or method and the fit of a term for a given (intra-mathematical) situation
2	Implementing/executing multi-step methods/rules, possibly with the use of computers and use of control options	Description of the given situation by mathematical standard models and mathematical relationships Recognizing and setting general conditions for the use of mathematical standard models	Understand, comprehend, explain mathematical concepts, principles, methods, representations, reasoning chains and contexts
3	Determine whether a particular method/specific rule is appropriate for a given situation, make and perform the appropriate method/rule	Apply standard models to novel situations, find a suitable fit between suitable mathematical model and real situation	Examine and complete mathematical reasoning, perform and describe multi-step mathematical standard reasoning
4	Develop/form macros[1] and join together macros already available	Complex modelling of a given situation; reflection of the solution variants or model choice and assessment of the accuracy or adequacy of underlying solution methods	Form independent chains of reasoning, technically correct explanation of mathematical facts, results and decisions

[1] aggregated mathematical rules

Figure 1: O-M-A Grid

cies Operating, Modelling and Reasoning, especially in higher levels of performance. “Communicating” is seen as an important domain of mathematical competencies for teaching mathematics, but cannot be specifically differentiated from Operating, Modelling and Reasoning and is therefore included in the other aspects.

The domain “Reasoning” is related to the suggestions of Bruder and Pinkernell (2011), who also pick up on considerations of Walsch (1972). “Modelling” served as the basis of the fundamental work of Niss (2003) and other ideas, e.g. of Boehm (2013) or Goetz and Siller (2012). There are relatively few preparations for a levelled conception of competencies in the mathematical domain “Operating”.

For example, Drueke-Noe (2012) shows that complex algorithms are required already in early grades. But for a high level of expertise, it is not only necessary to use complex algorithms, but also to find the right algorithm to apply in a given situation and to combine different algorithms where appropriate.

The result of our considerations as part of this project is a model with three domains of mathematical competencies (cf. Figure 1) that substantially captures the key aspects of mathematical work at school. The competency level model is aimed at fulfilling all essential requirements with regard to the conception of mathematical learning outcomes in Austrian mathematics education of the Secondary School Level II (cf. BIFIE, 2013a). Complex problem solving situations can be described by the interaction of the three domains of mathematical competencies.

EMPIRICAL EVIDENCE FOR THE COMPETENCY LEVEL MODEL

The question about an empirical verification of the theoretical competency level model with respect to the separation of domains of mathematical competencies and the gradation can be answered only in the context of a sufficient number of processed tasks for each area of expertise.

Data were taken from the so-called “school experiment” in 2014. Before the central final examination throughout Austria will be implemented in the school year 2014/2015, secondary academic schools and mathematics teachers were invited to voluntarily take part in a pilot study on graduating students’ math competences. In this study, the math tasks were processed under the same conditions as they would be processed at the mandatory central final examination. It is important to note that the performance in the tasks contributes to students’ final grade. For the school experiment whose data are being reported here, there were 803 students (m = 345, f = 458) from 42 classes from 9 districts in Austria. The examination consisted of two separated parts with so-called type 1 and type 2 tasks (cf. BIFIE 2013b).

Type 1 tasks “focus on the basic competencies listed in the concept for written final examination. In these tasks, competence-oriented (basic) knowledge and (basic) skills without going beyond independence are to be demonstrated.” (cf. BIFIE, 2013a, p. 23). They are coded as solved against non-solved. The various bound response formats such as multiple-choice format and a special gab-fill format enable accurate scoring. For the award of points in tasks with open and semi-open response format, solution expectations and clearly formulated solution keys are specified.

The characterization of type 2 tasks presents serious challenges to the basic principles of modern test theory. The tasks are considered “for the application and integration of the basic competencies in the defined contexts and application areas. This is concerned with extensive contextual or intra-mathematical task assignments, as part of which different questions need to be processed and operative skills are, where appropriate, accorded greater importance in their solution. An independent application of knowledge and skills is necessary” (cf. BIFIE, 2013a, p. 23). These tasks are also consistently structured in design and presentation, as well as in terms of scoring (cf. BIFIE, 2013).

A total of 16 (type 1) tasks in the competency domain of operating, 2 tasks (type 1) in the competency area of modelling, and 4 tasks (type 1) in the competency area of Reasoning were tested in the 2014 school experiment. Thus, no level analyses could be conducted.

There is a relatively high variation of the solution frequency within competency domain Operating (cf. Figure 2), which can be explained by the heterogeneity of tasks presented, especially with regard to high profile / over-training. Variation of solution frequency was also observed for the competence do-

Task	Competence domain	level of difficulty (% solved)
3	Operating	0.81
4	Operating	0.84
7	Operating	0.98
8	Operating	0.94
9	Operating	0.84
10	Operating	0.88
12	Operating	0.87
13	Operating	0.59
14	Operating	0.88
15	Operating	0.56
17	Operating	0.84
18	Operating	0.57
21	Operating	0.80
22	Operating	0.49
23	Operating	0.81
24	Operating	0.58

Figure 2: Difficulty level of the tasks for the competence domain Operating (n = 803)

Task	Competence domain	level of difficulty (% solved)
19	Modelling	0.51
20	Modelling	0.91

Figure 3: Difficulty level of the tasks for the domain Modelling (n = 803)

Task	Competence domain	level of difficulty (% solved)
1	Reasoning	0.80
5	Reasoning	0.73
6	Reasoning	0.94
11	Reasoning	0.64

Figure 4: Difficulty level of the tasks for the domain Reasoning (n = 803)

main Modelling (cf. Figure 3) as well as Reasoning (cf. Figure 4). The parameter "difficulty" was not measured, only the percentage of solution as an indicator for the level of difficulty of a task.

Two of the type 1 tasks are positioned on competency level 2 and could be analysed. A heterogeneous picture emerged for these two tasks: While task 2 could rarely be solved, task 16 was easily mastered by the students.

Can the pre-defined four levels be confirmed empirically in all the three areas of competency? This question can be answered in a first approximation only on the basis of type 2 tasks for levels 1 and 2 due to the fact that higher graduations did not appear in these exam booklets.

As can be seen in Figure 5 (in general) and Figure 6 (separated among O-M-A), the level 2 tasks seem to be more difficult in general. Thus the competency level of the task gives us a good statement about the level of difficulty.

SUMMARY AND OUTLOOK

The provided model with the three domains of mathematical competencies Operating, Modelling and Reasoning (O-M-A) distinguishes three basic mathematical operations on four levels. It is based on considerations from educational sciences and

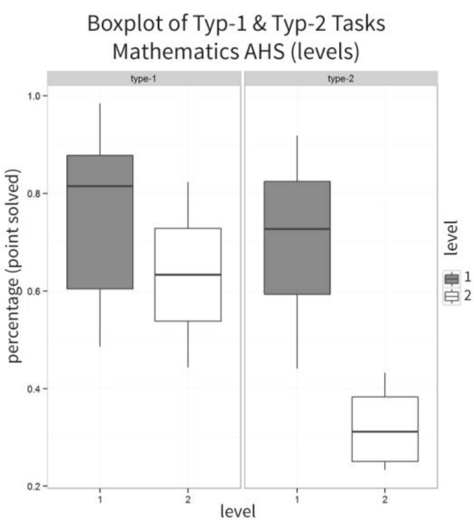

Figure 5: Empirical difficulties of type 2 tasks separated by type and level (n = 803)

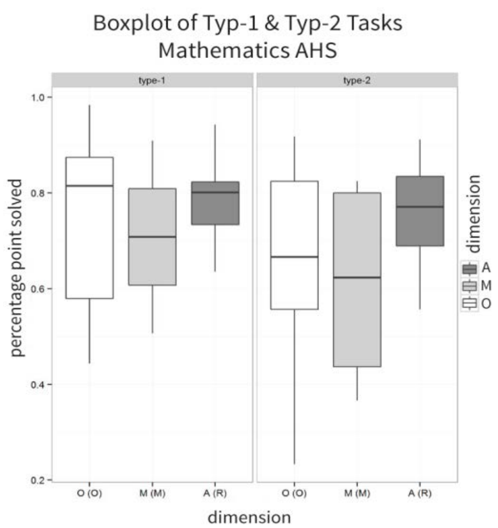

Figure 6: Empirical difficulties of the type 1 and type 2 tasks among O-M-A (n = 803)

learning theories as well as insights and experiences with regard to relevant factors for learning mathematics in school. It is part of a complex effort to gain a sound basis for competency diagnostics and performance assessment in mathematics in the German-speaking countries. It differs from other models by its consistent theoretical foundation and by the focus on potential lines of development for long-term competency building. The model O-M-A provides a normative setting for relevant levels of requirement in the three domains of mathematical competencies. This facilitates a certain comparability of type 1 and type 2 tasks provided for the final examination.

The added value of the developed model lies in several areas:

— It provides guidance both for the assessment of (written) performance and for the learning tasks in the classroom.

— It serves the purpose to reveal potential for development in the classroom.

— It allows for the identification of development potential in the task structure.

Limitations of the competency level model O-M-A lie in the coarseness of the approach. Neither can all the differences between the test tasks relevant to their level of difficulty be considered in detail (such as linguistic complexity), nor can individual developmental trajectories be mapped in learning processes. Further restrictions of the model are also indicated by the fact that of all the mathematical content and activities implied in each task only a basic competency referring to the list of basic skills (cf. BIFIE, 2013a) can be adopted. The specific situation of each school class or priorities of teachers cannot be reflected. Thus, many tasks can prove to be easier, but also more difficult than in the rating.

The competency level model O-M-A aims at describing levels of competencies by identifying the qualitative differences of each competence. The growing body of research on mathematics learning served as the theoretical background. The data and results presented so far are preliminary and did not account for not controllable influence factors such as training effects. However, they can be interpreted as a first clue that the O-M-A can be rudimentarily verified empirically. Therefore, further research is necessary to empirically test the levels of the model and to test the model against level 3 and level 4 tasks.

The model O-M-A is indefinite in explaining the attainment of the next higher level. For this reason we define it as a competency *level* model and not a competency *development* model. To answer the question as to whether this model could map potential lines of students' long-term competency development, more theoretical and empirical work is needed. So far, it cannot be applied to the development of a math learning process.

ACKNOWLEDGEMENT

The project "Competency Level Modelling" was funded with support from the Federal Institute of Educational Research, Innovation and Development of the Austrian School (BIFIE). This publication reflects the views of the authors, and the BIFIE cannot be held responsible for any use that may be made of the information contained therein. We acknowledge the contribution of Martin Schodl, who has been working in the project as a coordinator and who has helped selecting certain data.

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ENDNOTE

1. The German word 'Argumentieren' is synonym to reasoning.

TWG17

Posters

Inferentialism in mathematics education: Describing and analysing students' moves in sorting geometrical objects

Abdel Seidou

Örebro University, Örebro, Sweden

This poster presents a language for describing and analysing students' language "moves" while reasoning in an open-ended sorting activity. A close analysis of one individual student's language moves in a collaborative activity is supposed to shed light on individual contribution to the collaborative reasoning process. Furthermore, these moves give indications on what pupils decided to be relevant in the simultaneous enterprise of reasoning in collaboration, and even the prior knowledge available in the classroom.

Keywords: Inferentialism, deontic score-keeping, mathematical reasoning, collaboration.

THEORETICAL FRAMEWORK

The purpose of this poster is to present an inferentialist language for describing reasoning in terms of moves in language game. Inferentialism is introduced by Robert Brandom (1994), and it advocates a new order of explanations. Inference is to prioritize over reference or representation, and it is set at the heart of human knowing. Inferentialism identifies the meaning of an expression by its inferential relationship to other expressions. Brandom (2000) stated:

> To grasp or understand [...] a concept is to have practical mastery over the inferences it is involved in – to know, in the practical sense of being able to distinguish (a kind of know-how), what follows from the applicability of a concept, and what it follows from. (Brandom, 2000, p. 48, his italics)

The introduction of inferentialism to mathematics education research is recent. Nevertheless, ideas based on inferentialism have already been used in different ways in mathematics education research. Schindler and Hußmann (2013) used the status of claims (commitments and entitlements) to investigate 6th grade students' individual learning process and concepts formation in the topic of negative numbers. Bakker and Derry (2011) draw upon inferentialist epistemology, to design tasks in teaching statistics inferences. Based on inferentialism, this poster will present a language to describe and analyze young learners' collaborative mathematical reasoning.

Geometrical objects (2D) of different sizes and shapes were presented to groups of four first grade young learners (6–7 year olds) by the teacher. They were challenged to collaboratively come to an agreement on ways of sorting. The open-ended aspect of this task creates favorable conditions for a fruitful game of giving and asking for reasons.

Brandom (1994; 2000), used the term deontic scorekeeping to name a process embedded in the game of giving and asking for reasons. During this process, different competent interlocutors keep track of their own and others' linguistic performance. It describes the course by which different interlocutors converge toward the same meaning, in search of agreement or/and objectivity. It comprises both collaboration and the reasoning. The analysis of the deontic scorekeeping, especially the "moves" will serve as tool to characterize and analyze young learners' collaborative mathematical reasoning.

The "moves" express how claims are put forward: *Attributing*, *acknowledging* and *undertaking*. Attributing is just a kind of reporting, and it does not indicate an understanding or knowing. Acknowledging is to take something to be true. Undertaking a claim is to be aware of the premises and consequences of it.

The moves are interrelated and depend on each other. For instance undertaking a commitment is something that makes it appropriate for others to attribute it (Brandom, 2000).

I believe the "moves" could also show signs of participating norms/rules in a classroom if they are investigated with appropriate quest. To illustrate the analytical points of the proposed theoretical framework, the poster presents video recorded data from a Swedish classroom.

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Crossroads of phenomenology and activity theory in the study of the number line perception

Anna Shvarts, Andonis Zagorianakos

Lomonosov Moscow State University, Moscow, Russia, shvarts.anna@gmail.com
Manchester Metropolitan University, Manchester, UK

The research is a study of the productivity of a dual analysis of the same phenomenon from the cultural-historical activity theory (CHAT) and from the Husserlian phenomenological perspective. It analyses the eye-movements of children and their parents while they were operating with the number line. We also were focused on the interaction between children and parents while each parent was teaching the child and we analyzed it in correspondence with eye-movements. The implementation of the dual theoretical perspective to our eye-tracking data sources disclosed the subtle synchronization of intersubjective communication, already starting in the embodied, pre-reflective area of the cognitive experience. The two theoretical perspectives exerted a significant complementarity in bringing about the social and the subjective – intentional aspects of the phenomenon of learning.

Keywords: Activity theory, phenomenology, intentionality, eye-tracking, number line.

Radford (2010) considers the development of perception in mathematics education from a cultural-historical activity theory (CHAT) perspective and argues that only through social practice the "domestication" of a perceptive organ (an eye) can occur, and that the phenomenological approach towards perception as a system of intentional acts cannot explain acquiring of the new, culturally specific ways to approach objects. Our research shows the productivity of a dual analysis of the same phenomenon from CHAT and phenomenological perspectives. We analysed the eye-movements of participants while they operated with the number line. An SMI RED eye-tracker was used with sample rate 120 Hz. The task was to specify the point on the number line from 0 to 10 a grasshopper was sitting. There were 6 pairs of adults and pre-schoolers. We collected three sorts of data: adults' perception; interaction between children and parents when we asked each parent to teach their child; and children's perception. The results in our poster from qualitative analysis presented as series of pictures, which represent synchronized data of short time interval (0,5–5 seconds) tracks of eye-movements, speech oscillograms, audio transcriptions and pictures of gestures, taken by an external camera. Here is a space-saving summary of interweaving ways to address results from the two perspectives.

(1) Adults' perception and teaching strategies revealed a vivid difference in how the parents detected the point on the number line themselves, and how they taught the children to do it. All adults either immediately grasped the answer by one fixation, or by a couple of fixations while they were counting from the midpoint or the last point. While teaching, most of our adults showed the child a strategy of counting from zero up to the point, making arc movements with their finger from point to point, and rhythmically counting or making pauses for the child to count. From the CHAT perspective that focuses on cultural practices and artefacts (Vygotsky, 1981) we need to interpret the adults' kind of perception as *mediated* by previous knowledge and by the number line itself—which is a visual semiotic means that has sedimented the activity of counting. We can judge the way that adults performed counting as a developed perception that has a form of mental action that keeps only a general form of real action and misses the intermediate parts (Davydov, 1959). Following the Husserlian phenomenology by adopting the first person perspective in our analysis, we see the adults' perception as *immediate*, where a number line is "taken for granted" (Husserl, 2001, p. 338), i.e., approached as a natural object of their living world without treating it as a product of previous mathematical work.

(2) In the teaching stage the attention of the children was strongly driven by a system of means used by the adult. Each pointing movement of the finger of the adult corresponded to a prosodic stress and to a fixation of the child's glance on a point on the number line; the adult made a theoretical perception possible through involving the child into *social practice*, "through an intense recourse to pointing gestures, words, and rhythm", as Radford (2010, p. 5) puts it. Now let us look more closely at the children's eye-movements. In one case the child followed the adult's movements precisely in counting from zero up to ten, but at the moment when her finger was passing the point where the grasshopper was sitting, the child found time to look at the written question of the task. In another example a child misperceived an adult's gestures: she was moving her eyes one point ahead of where the adult was pointing. But being in contact with the original task she managed to correct herself at the end of the counting: she was making two fixations on the same point where the grasshopper was, corresponding to two separate arc movements of her father's finger. Thus, being "moved" by the adult activity the children caught up the meaning of it through the goal, which was retained at a grounding level of the children intentionality. So, our data show that it is exactly a complex system of *intentionality* that makes possible the "crucial form of communication in which two consciousness meet in front of the cultural mathematical meaning" (Radford, 2010, p. 6) within the social practice. From a phenomenological perspective this meaning should not be perceived as a "ready-made entity" which a child is expected to follow; instead, only a serious and genuine move back to the intentional origins of this meaning gives us a real understanding of its constitutive potential (Husserl, 1970, 2001).

CONCLUSIONS

CHAT analysis focuses on cultural means and social practices, which necessarily mediate the transformation of perception, while phenomenological analysis aims towards understanding of the role of intentionality in acquiring new forms of immediate perception. Hence these two perspectives attempt to grasp two important aspects of the educational/learning complexity, and seem to be neither contradictory nor reducible to each other, but rather essentially complementary.

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TWG18

Mathematics teacher education and professional development

Introduction to the papers of TWG18: Mathematics teacher education and professional development

Stefan Zehetmeier[1], Marc Bosse[2], Laurinda Brown[3], Alena Hošpesová[4], Nicolina Malara[5] and Bettina Rösken-Winter[6]

1 University of Klagenfurt, Faculty of Interdisciplinary Studies, Klagenfurt am Wörthersee, Austria, Stefan.Zehetmeier@aau.at
2 University of Duisburg-Essen, Faculty of Mathematics, Essen, Germany
3 University of Bristol, Graduate School of Education, Bristol, UK
4 University of South Bohemia Ceske Budejovice, Faculty of Education, České Budějovice, Czech Republic
5 University of Modena & Reggio E., Faculty Education and Human Sciences, Modena, Italy
6 Humboldt-Universität zu Berlin, Professional School of Education, Berlin, Germany

RATIONALE

The study of mathematics teacher education and professional development has been a central focus of research over recent decades. Research activities have focused on topics such as reflection, collaboration and teachers' professional growth. In particular, models and programmes of professional development, as well as their respective content, methods, and impacts have been described and analysed. Research has increasingly focused not only on the participating teachers, but also on the role of teacher educators and academic researchers. So far, the research community has attempted to develop theoretical and methodological frameworks to both *describe* and *explain* the complex topic of mathematics teacher education and professional development.

The goal of TWG18 was to offer a communicative, collegial and critical forum for the discussion of these and other related issues, which would allow diverse perspectives and theoretical approaches to contribute to the development of our knowledge and understanding as researchers, educators and practitioners.

PARTICIPANTS

39 papers underwent a peer review process in TWG18: during this process, all papers were revised, according to reviewers' remarks, by authors. 34 papers were accepted as contributions and five were re-submitted for poster presentation. Four of the accepted papers were withdrawn. Finally, 30 papers were presented during the TWG sessions.

19 posters (14 original submissions and 5 former paper submissions, see above) underwent a peer review process in TWG18: all authors revised their posters, according to the reviewers' remarks. All of the posters were accepted. Four of the accepted posters were withdrawn. Finally, 15 posters were presented during the conference poster session.

ORGANISATION

Due to the high number of participants and presentations, the TWG sessions comprised both plenary and sub-group working phases. During the plenary phases, three (or four) papers were presented for a maximum of five minutes each, in which the authors provided their respective paper's central message(s) and challenging questions for discussion. These plenaries were followed by parallel sub-groups, which were each managed by one of the presenting authors. Participants were free to choose and join one subgroup, where they discussed the paper for 20-30 minutes. Afterwards, the TWG's participants met in plenary to hear reports of each sub-groups' central topics and to summarise emerging issues.

TOPICS

The presentations were categorised into seven main topics:

— Models of Teacher Education Programmes

— Pre-Service Teacher Education

— Lesson Study and Videos in Teacher Education

— Tasks, Problems and Assessments

— Mathematics Teacher Educators

— Reflecting Teaching Practices

— Cooperative Communities

Within the topic "Models of Teacher Education Programmes", the papers dealt with teacher education programmes, teacher educators, teachers' professional growth, training models, teachers of levels K-8, teacher empowerment, problematisation of knowledge, meta-didactical transpositions and praxeologies.

The topic "Pre-Service Teacher Education" comprised teacher preparation, pre-service primary and secondary mathematics teachers, inquiry based mathematics education, students' mathematical thinking, scaffolding, one-to-one interactions, deficits pertaining to core mathematics, positive attitudes towards mathematics, analyses in one-on-one interviews, diagnostic strategies, subject matter knowledge and knowledge base for teaching.

The session on "Lesson Study and Videos in Teacher Education" focused on lesson study, video-based professional development, mentor teachers, pre-service and in-service teachers, pedagogical content knowledge, theory-practice problems, initial teacher education, classroom situations, teacher expertise, peer discussions, anthropological theory and mathematical knowledge for teaching.

The papers of "Tasks, Problems and Assessments" dealt with task design, learning scenarios, problem posing, problem solving, teacher professional competencies, a four-step dynamic model, characteristics of a good (mathematics) teacher, and levels of cognitive demand.

The topic "Mathematics Teacher Educators" comprised practices of teacher educators, prospective teachers, mathematical knowledge, awareness, students' errors, non-standard reasoning, instructional coherence of teacher educators, teaching-learning environments, impact on teacher and student learning, professional development and social network sites.

The focus of "Reflecting Teaching Practices" was on self-reflection, beginner teachers' practices, content and methods of professional development courses, mathematical quality, didactical analysis competency, pedagogical content knowledge, didactic transposition, anthropological theory of didactics, content representation, spatial visualisation ability, teaching experiments, common games and childhood education.

Within the topic "Cooperative Communities" the papers dealt with communities of practice, inquiry communities, developmental research on tasks, identity and social perspectives on learning.

EMERGING ISSUES

This section provides several questions and issues which emerged during the sessions of TWG18:

Models of Teacher Education Programmes

— How do we deal with the differences between the role of the mentor and the mathematics educator?

— How can we evaluate teachers' changes from their classroom actions?

— How can we assess the stability of change induced in the teachers?

— If we analyse ourselves when we work as teacher educators, what are possible problems, or advantages?

Pre-service Teacher Education

— How does our take on the nature of mathematics affect our studies?

— Perspectives are hard to change and student teachers learn to answer what their teacher educators want – do you consider this when designing research?

— How are student teachers able to identify and imagine key aspects of development and the learning trajectories of their students?

— How can we measure the effectiveness in the development of prospective teachers of 1-1 interactions with students? How do we research this?

— How do we measure immediate and long-term effects?

Lesson Study and Videos in Teacher Education

— How does Lesson Study literature fit into wider perspectives such as teacher knowledge?

— Lesson study across countries and across different subjects is different due to cultural background. How can we map the territory?

— How to make change visible?

— How to keep the balance between quantitative and qualitative aspects within conference papers with reduced length?

Tasks, Problems and Assessments

— There is a job to be done in mapping and bringing together, rather than separating mathematics, didactics and pedagogy. After the mapping, what is our vision, what would we propose?

— Theoretical frameworks that are too general are not helpful for teachers. Which tasks could illustrate more specific points of frameworks?

— Notions and definitions: what is problem-posing, problem-solving; is problem-posing a part of problem-solving? How clear are we with notions and definitions?

— It is important to work with teachers in PD programmes. But the question remains: What will the teachers do with the PD in their classrooms?

— Mathematics Teacher Educators

— Thinking through our roles as teacher educators we are also researchers – how can we manage this?

— How to reflect on philosophy of mathematics and the philosophy of mathematics education? How do they meet?

— How can we assess a chain of effects in teachers' professional development?

— How to conduct research on the internet with social communities? There are ethical issues and how do we deal with these? How to intervene in a group as a researcher?

Reflecting Teaching Practices

— How to deal with being (at the same time) a researcher, a teacher, and an educator?

— How to access practice if we want to use that as a research object?

— How to optimise teaching time? How to focus during that time on the deep questions teachers face about mathematics?

— Teacher education is heterogeneous across one country. How can we make comparisons between countries?

— Culture is so important in comparing countries. What are the valid lenses we can use in comparisons?

Cooperative Communities

— Difference between 'community of practice' and 'community of inquiry' – what makes an inquiry community? How will we recognise it in the data?

— What is relevant to the teachers within the community?

— The roles of researchers and teachers when we do research on practice – symmetry, asymmetry?

— How can we actually observe identity development in terms of practice?

TWG18

Research papers

Developing mathematics teachers' pedagogical content knowledge through iterative cycles of lesson study

Aoibhinn Ni Shuilleabhain

University College Dublin, Dublin, Ireland

This research presents features of knowledge of content and students (KCS) and knowledge of content and teaching (KCT) as empirical evidence of mathematics teachers' pedagogical content knowledge (PCK) utilised and enhanced through their participation in iterative cycles of lesson study. Over the course of one academic year, twelve teachers in two secondary schools engaged in this research as a double case study of teacher learning within a lesson study community. Qualitative data was generated through audio recordings of teacher meetings and through multiple teacher interviews. Dialogue within the lesson study communities was mapped to a framework of PCK as proposed by Ball, Thames and Phelps (2008). Results of this study find empirical evidence of the features of KCS and KCT in teachers' planning and reflection conversations and demonstrate teacher learning over iterative cycles of lesson study.

Keywords: Pedagogical content knowledge, lesson study, reflection, teacher knowledge.

INTRODUCTION

Lesson study is growing in popularity as a form of professional development for Mathematics teachers (Dudley, 2013). Although much research has shown that teacher knowledge is developed through participating in lesson study (Fernandez, Canon, & Chokshi, 2003; Lewis, Perry, & Murata, 2006; Murata, Bofferding, Pothen, Taylor, & Wischnia, 2012) this learning has not yet been explicitly mapped to a framework of knowledge for teaching. Furthermore, lesson study research has mainly focused on primary mathematics teachers and on single cycles of lesson study (e.g., Corcoran, 2011; Lewis, Perry, & Hurd, 2009; Murata et al., 2012).

In this research the development of secondary mathematics teachers' pedagogical content knowledge (PCK) is investigated through their participation in iterative cycles of lesson study. Twelve mathematics teachers in two schools were introduced to this model of professional development and participated in multiple cycles over the course of one academic year. As teachers' participation in lesson study continued, they began to incorporate and develop more elements of PCK in their planning and reflection meetings around research lessons.

Utilising Ball and colleagues' (2008) framework of PCK, qualitative data generated through teachers' conversations and interviews were analysed in terms of knowledge of content and students (KCS) and knowledge of content and teaching (KCT). Features of KCS and KCT emergent from the data are presented here as empirical evidence of PCK in teachers' planning and reflection conversations around research lessons and also as evidence of teacher learning in lesson study.

LESSON STUDY

Lesson study is a systematic inquiry into teaching and learning where teachers collaboratively plan, examine, conduct, observe, and reflect on research lessons (Fernandez et al., 2003; Lewis et al., 2006; Lewis et al., 2009; Murata et al., 2012). The aim of lesson study is not to construct a "perfect" mathematics lesson, nor is it to study lessons in detail, but rather it aims to engage teachers in dialogue around their pedagogical practices. Lesson study also incorporates many of the features of teacher community advocated as a form of sustainable professional development (Grossman, Wineburg, & Woolworth, 2001).

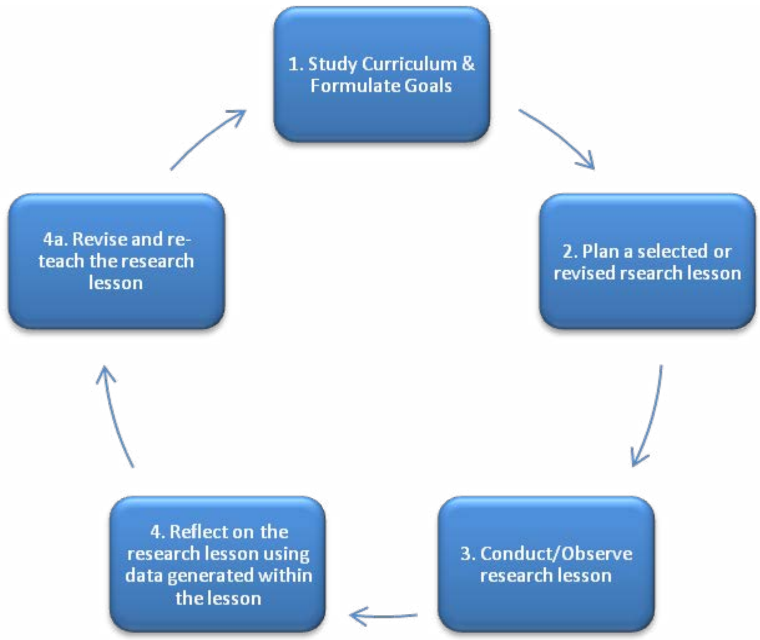

Figure 1: Lesson study cycle – adapted from (Lewis et al., 2006)

A modelled description of the lesson study cycle can be seen in Figure 1 (adapted from Lewis et al., 2006):

1. Study curriculum and formulate goals
2. Plan a research lesson
3. Conduct or observe research lesson
4. Reflect on research lesson and planning process

4a. Option to revise and re-teach the research lesson.

Teacher knowledge is enhanced through participation in lesson study (Lewis et al., 2006; Murata et al., 2012) and this research aimed to consider such learning relative to a particular framework of teacher knowledge. In this paper, detailed analysis focused on fine-graining the features of KCS and KCT evident within teachers' planning and reflection conversations in lesson study as a lesson study community.

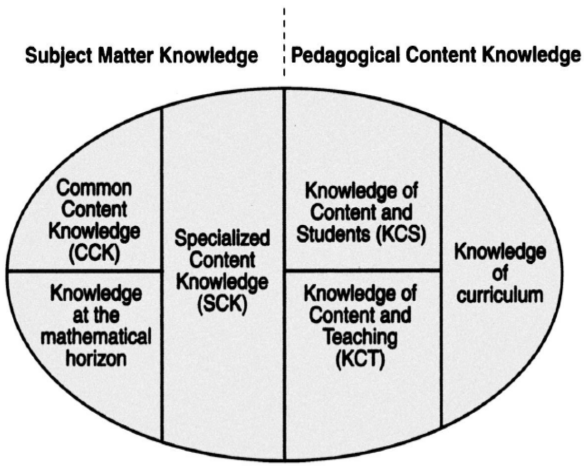

Figure 2: Mathematical Knowledge for Teaching (Ball et al., 2008)

PEDAGOGICAL CONTENT KNOWLEDGE: KCS AND KCT

In attempting to define teacher learning in this research, teacher knowledge was mapped to a particular framework of PCK proposed by Ball and colleagues (2008) (Figure 2).

KCS and KCT are important elements of PCK which combine teachers' knowledge of mathematics with their knowledge of students and of mathematical didactics respectively. KCS and KCT incorporate elements of knowledge specific to mathematics teachers from developing an awareness of students' mathematical thinking (Carpenter, Fennema, Peterson, Chiang, & Loef, 1989; Simon, 1995), to more effectively sequencing learning trajectories (Ball et al., 2008; Simon, 1995). Features of KCS and KCT utilised and developed in teachers' planning and reflection conversations in lesson study were identified as part of this research.

DATA GENERATION

This research took place in two urban secondary schools, Doone and Crannog (all pseudonyms), over the course of the 2012/2013 school year. Teachers were invited to participate in the research with 5 teachers in Doone and 7 teachers in Crannog (varying in teaching experience from 1 to 33 years) agreeing to take part.

Each lesson study community was taken as a case study within which the development of teachers' conversations around the teaching and learning of mathematics could be analysed. Data was generated through audio recordings of each of the lesson study meetings in both schools (3 cycles in Crannog and 4 cycles in Doone) and through individual teacher interviews held at three stages during the research.

Teachers had autonomy over the content they taught, the class group involved, their overall lesson study goal, the construction of their lesson plans, and how they reflected on their students' learning. As participant-observer the researcher was present in each of the meetings and research lessons as an additional member of each lesson study community and participated as lesson study facilitator in the first cycles in both schools. In each of their subsequent lesson study cycles teachers rotated roles of conducting teacher and of facilitator.

In total, 38 hours and 17 minutes of teacher discussions around planning and reflection were recorded (over 18 hours in Crannog and over 20 hours in Doone). Teacher interviews with all participating teachers served as an additional data source in providing teachers with opportunity to self-report on their own learning and on any changes to their classroom practices that may have changed as a result of their participation in lesson study.

DATA ANALYSIS

The data was transcribed and analysed over four phases as a chronological evolution of teachers' planning and reflection conversations over iterative cycles of lesson study. A framework of analysis of KCS and KCT derived from the literature (Ball et al., 2008; Hill et al., 2008) and incorporating codes emergent from the data was utilised in the analysis. Analysis involved reading all of the transcript text, identifying if the text qualified as a legitimate code, and deciding if the text was relevant to the codes within the framework of analysis.

A unit of analysis was defined as any episode of conversation which a) was relevant to the lesson study cycle and b) was relevant to constructing content of a lesson from either the perspective of the student or from a pedagogical perspective. This parsing approach of conversation excerpts, also utilised by Cjakler and colleagues (2013), aimed to encapsulate elements of teachers' conversation where teachers introduced elements of KCS and KCT within their planning and reflection phases. For example, the following conversation excerpt identifies an episode of learning for Lisa who, prior to participating in lesson study, had not recognised how her students might interpret variables within Pythagoras' theorem. This discussion occurred as part of the final cycle in Doone where teachers planned a series of lessons introducing students to Pythagoras' theorem.

Lisa	The thing about learning for the students is that they can learn the theorem but then it is confusion when the diagrams are labelled in any given way... We think it's saying $a^2 + b^2 = c^2$ but it's meaningless to them when you give them a thing and 'a' is the hypotenuse and then you go, $a^2 + b^2 = c^2$ ... They don't actually understand.
Owen	So that's rote learning.
Kate	Yeah, concept rather than formula...
Lisa	*We* know it. We know that this is the formula but we don't look at it from the kids [perspective]. And it's only that you talking about it today – if they label the hypotenuse 'a' – I hadn't actually realised that that is what's causing the problem.

This conversation higlights Lisa's realisation of seeing the mathematical content through the eyes of a student (Fernandez et al., 2003), where she made sense of students' common conceptions – developing her KCS as part of this lesson study community's planning dialogue.

The final two phases of analysis led to a further determining of the categories of KCS and KCT relevant to these elements of lesson study such as: noticing students' mathematical strategies, developing contextualised questions, and reflecting on student talk. These features, linked to existing literature on teacher learning, formed the basis of codes for a final phase of analysis and are presented below as indications of teacher learning through participation in iterative cycles of lesson study.

FINDINGS

Features of KCS and KCT are presented as empirical evidence of PCK in teachers' planning and reflection conversations in lesson study and also as evidencing teacher learning, since the frequency of these features increased as teachers' participation in iterative cycles of lesson study continued. As part of both teacher communities' initial engagement with lesson study, not all of these features were present in the initial cycles of lesson study but began to be incorporated as teachers began to observe research lessons, focus on students' mathematical thinking, and plan subsequent lessons.

Teacher learning through the development of these features of KCS and KCT in iterative cycles of lesson study may be modelled as proposed in Figure 3. As an example of developing KCS, following the observation of students within research lessons (noticing and interpreting students' mathematical responses) teachers became more cognisant of anticipating how their students would mathematically engage in planning subsequent lessons. From an increased focus on anticipating students' thinking, teachers became more aware of the value of engaging students in their own learning and began designing relevant, contextualised questions (KCT) which, in turn, focused teachers on noticing and reflecting on students' mathematical strategies in attempting these activities (KCS).

This model is included as suggested theoretical frame of teacher learning in iterative cycles of lesson study and further research is required in refining and developing this model.

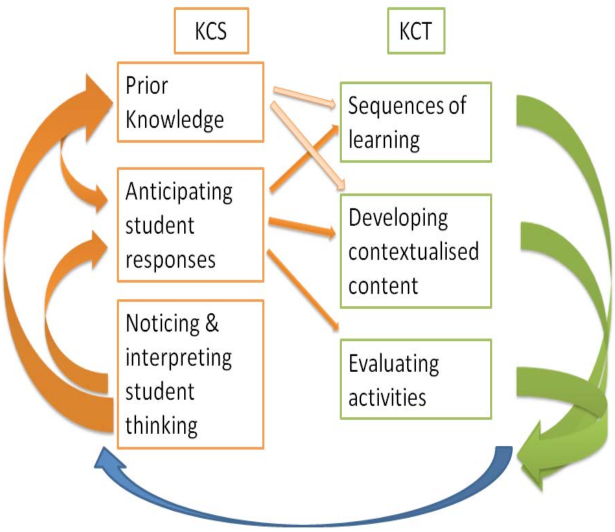

Figure 3: Features of KCS and KCT developed in iterative cycles of lesson study

Findings: Features of KCS

From the analysis of the data three features of KCS were identified as being utilised and enhanced through teachers' participation in iterative cycles of lesson study:

- Identifying students' prior knowledge (Ball et al., 2008)
- Anticipating students' mathematical responses (Ball et al., 2008; Hill et al., 2008)
- Noticing students' mathematical thinking (Carpenter et al., 1989; Jacobs et al., 2010; van Es & Sherin, 2008)

A number of narrative samples of each of these features will now be explored.

Identifying students' prior knowledge

While at the beginning of their engagement in lesson study teachers in Doone did not incorporate students' prior knowledge in their planning, this became more and more important to them as their planning of research lessons continued. From their observation of research lessons, teachers reflected on the need to correctly identify students' prior knowledge in being able to then anticipate students' responses (KCS) and plan a relevant sequence of learning (KCT). In their final research lesson, teachers identified and incorporated the mathematical content students had already met and used this information to build on students' mathematical thinking over a series of lessons. This development of KCS in turn benefitted teachers' KCT in outlining sequences of learning over a number of lessons.

Anticipating students' mathematical responses

From observing particular students within the first research lesson, teachers in both schools began to anticipate how students might engage with and respond to mathematical tasks within subsequent lessons. Within their planning, teachers began to anticipate and identify various strategies which students might employ for particular mathematical activities and also began to articulate how students might think about particular topics. In Crannog's first research lesson, teachers began to anticipate how students might respond to a planned activity exploring quadratic patterns:

Stephen	Because some would look as if it is 1, 4, 9. Some will look at it as being squared. Some will look at it as being –
Fiona	Add 3 and 5 –
Stephen	And 7, and that is...if you give them a pattern, if you give them a list of numbers of 1 to 9, that is what they do. They will actually see how much it is going up whereas some don't relate it to being anything squared.

This anticipation of students' mathematical responses began to also impact on teachers' practices outside of lesson study as reported by Eileen (a newly qualified teacher) in her mid-point interview:

Eileen	Yeah, I probably would ask myself a bit more how would they react to this or what questions will they have. Pre-empt their questions or pre-empt their confusion. Yeah, I would think about that a little bit more.

Noticing students' mathematical thinking

Through the detailed planning and anticipation of students' mathematical responses within teacher meetings, teachers began to focus on and notice more of students' mathematical thinking during their observation of research lessons. This noticing of students' mathematical thinking was an important element of developing teachers' perspectives on their own pedagogy (Carpenter et al., 1989; Corcoran, 2011; Jacobs et al., 2010; van Es & Sherin, 2008) and as the cycles continued teachers began to explicitly reflect on elements of student strategies and student talk in their post-lesson discussions. Teachers self-reported that this noticing and interpreting of students' mathematical thinking also impacted on their classroom practices outside of lesson study, such as extending the 'wait time' for students to answer questions in.

Findings: Features of KCT

From the analysis of the data three features of KCT were identified as being utilised and enhanced through teachers' participation in iterative cycles of lesson study with their colleagues:

— Sequencing learning trajectories (Ball et al., 2008)

— Designing contextualised questions (Schoenfeld, 2011)

— Evaluating mathematical activities (Ball et al., 2008)

A number of narrative samples of each of these features will now be explored.

Sequencing learning trajectories

It was a surprising result that in both schools, as teachers continued their participation in lesson study they began to plan series of lessons along a learning trajectory within which the research lesson was incorporated. In Crannog's second cycle teachers realised that planning a number of lessons would be more beneficial to students in revising and developing important mathematical concepts.

Fiona	I suppose what we have to do in a pre-runner class, we have to go back with them over the concept of factors: "What are factors?" and then they need to look. Because we kind of gloss over that a bit when we go into factorising usually.
Stephen	That we don't make up two numbers in algebra.
Fiona	Yeah, so go back into factors.
Gerald	Yes and I think the discussion of factors should start with prime numbers because they have only got two factors to talk about then.

In their final research lesson teachers planned a series of 6 lessons which incorporated students' prior knowledge and guided students towards a necessity for differentiation in Calculus.

Similarly in Doone, their final research lesson was planned as part of a series of lessons which teachers felt was far more valuable to both the lesson study community and to students in building their mathematical understanding.

Kate	This is the first time we've actually kind of planned a little scheme.
Nora	For the whole thing. Take you, follow it – follow it through. Because I think you know exactly where you stand or where they [the students] should stand.

Designing contextualised questions

As teachers continued to participate in lesson study it became more and more important to them to design mathematical activities that were relevant and context-based for their students. In Doone's first cycle students' activities were of a traditional textbook format (O'Sullivan, Breen, & O'Shea, 2013) which were not particularly relevant to their group of secondary students. In their subsequent cycles teachers developed activities that were both context and content based but were also of interest to these students, such as a rugby based problem designed for a particular class of 15 year old male students.

Evaluating mathematical activities

As a further feature of KCT, teachers also began to critically analyse and evaluate mathematical activities during planning. This evaluation of activities during planning impacted on how they taught or introduced such activities during the research lesson as exemplified in the following conversation excerpt where teachers modified a question in order to necessitate students multiplying two fractions together:

Lisa	The Ireland rugby squad: 1/5 of these have eye problems. Of this 1/5, ½ wore contact lenses. What fraction of the players wore contact lenses?
Kate	But – they're going to get 30 players.
Lisa	30 multiplied by 1/5 is 6.
Kate	Well, they're going to divide by 5. So unless we said "a squad" instead of saying 1/5 of the "players". Don't give them a number of players because they'll divide by 5 and get 6.
Lisa	I think that's what we're doing wrong. We just want a fraction –
Owen	So it's one whole squad.
Lisa	So it's "a squad". Brilliant!

This evaluation of mathematical activities also encouraged teachers in developing and designing their own activities instead of their traditional reliance on textbook questions (O'Sullivan et al., 2013).

CONCLUSION

Over a number of lesson study cycles, teachers developed in their perspectives of and approaches to teaching and learning mathematics through planning research lessons, observing students' mathematical responses during those lessons, and reflecting on students' interactions and responses. This research maps the learning of these teachers in their lesson study community to a framework of PCK suggested by Ball and colleagues (2008). Furthermore, the research provides empirical evidence of KCS and KCT in the context of teachers' planning and reflection conversations of research lessons in lesson study.

In this paper, three features of both KCS and KCT were identified within the data as part of teachers' planning and reflection conversations in lesson study. While these features were not all present in the data in initial lesson study cycles, they began to be incorporated as teachers' participation in cycles continued where their PCK was developed through structured conversations with their colleagues. While some of these emergent features of PCK were expected from literature on teacher learning and lesson study (such as highlighting students' prior knowledge), others (such as designing contextualised questions and interpreting students' mathematical responses through reflection) were added to this proposed framework of PCK as part of teacher learning in lesson study.

The presentation of these features of KCS and KCT as empirical evidence of PCK in planning and reflection phases of lesson study represents a contribution to the literature in identifying teacher learning through participation in iterative cycles of lesson study.

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Applying the structured problem solving in teacher education in Japan – A case study

Yukiko Asami-Johansson and Iiris Attorps

University of Gävle, Gävle, Sweden

In this paper, we examine the implementation of a Japanese teacher educators' lesson, where he applies and, at the same time, inform the students about "structured problem solving". We describe a specific lesson titled "Quantity and Measurement" for elementary school teacher students and we show how the educator make the students aware of the didactic transposition of the material and how he makes the students experience and learn about applying "structured problem solving" in practice. We also show how the Japanese curriculum influences the scale of the mathematical praxeology to be learned and how the students are given opportunities to develop their insight into the PCK during their education in mathematics.

Keywords: Teacher education, pedagogical content knowledge, didactic transposition, anthropological theory of didactics, content representation.

INTRODUCTION

In recent years, the need to train teachers' and teacher students' skills for teaching mathematics has been strongly emphasized by both politicians and mass media in Sweden. One of main reasons for this is that the performance of Swedish students in international surveys of education, such as the TIMSS and Pisa study in mathematics has radically declined since 1990's. According to Brown and Borko (1992), one of the most important purposes of teacher education is the acquirement of pedagogical content knowledge (PCK). It is recognized, that such knowledge forms the essential bridge between the academic subject matter knowledge (SMK) and the teaching of the subject matter. Furthermore, it includes an understanding of which representations are most appropriate for an idea, which ideas are difficult and which are easy for learners and what conceptions and preconceptions students of different ages hold about an idea. Specifically, if the preconceptions are erroneous conceptions, teachers need to know about strategies for reorganisation of the learners understanding (Shulman, 1986).

This is a part of a future comparative study project between Japan, Finland and Sweden concerning teacher education that aims to identify and analyse differences of institutional settings in several countries (Artigue & Winsløw, 2010). Our intention is to illuminate teacher educators' perception of the SMK and PCK, by analysing the mathematical and didactical organisations in the countries' primary school teacher education. In this paper, we present our first study from Japan, where the focus is on how the teacher educator applies *the structured problem solving* in his lecture. Structured problem solving has the emphasis on creating learning opportunities for students by using challenging problems and to stimulate students' corrective reflection on their solutions.

Shimizu (1999) explains some pedagogical terms which are used daily by Japanese teachers in mathematics class: *hatsumon*: *asking a key question, kikan-shido*: *teachers' instruction at students' desks, neriage*: *whole-class discussion, matome*: *summing up.* Having such common didactical terms indicates that Japanese teachers have acquired an institutionalised perception about the teacher's role in the classroom. Many Japanese teacher educators in mathematics apply and instruct on this teaching pattern in classes for their students. We observed and analysed the mathematical and didactical organisation of a lesson concerning "Quantity and Measurement" in a course named "Arithmetic Education" for prospective elementary school teachers in Japan.

THEORETICAL FRAMEWORK

Teacher knowledge

The importance of teacher knowledge both in teaching and in teacher education has been cogitated by researchers in several articles (see, e.g., Shulman, 1986; Kind, 2009). A number of models of teacher knowledge have been generated in this field. Although researchers differ in their definitions of various components in teacher knowledge, three areas of teacher knowledge can be seen as the cornerstones of the emerging work on professional knowledge for teaching: subject matter knowledge (SMK), general pedagogical knowledge and pedagogical content knowledge (PCK) (e.g., Shulman, 1986). PCK is a term to describe 'the ways of representing and formulating the subject that make it comprehensible to others' (ibid, p. 9). It has been found that PCK is a useful tool for understanding the professional practices of teachers (Kind, 2009). Investigating PCK in teaching practice is a difficult process, but Kind (2009) points out that using Content Representation (CoRe), developed by Loughran, Mulhall and Berry (2006), might give a unique awareness into the teachers' PCK and their practices relating to specific topics and subject areas. CoRe focuses on different parts of PCK and offers a way to give an overview of the teaching approaches for a specific subject area and to motivate teaching decisions.

The didactical transposition theory and the anthropological theory of didactics

Chevallard developed the conceptualisation of a didactical transposition: how the knowledge content is adapted for the purpose to be taught within a given institution. It means a transposition from *scholarly knowledge* (Bosch & Gascón, 2006), which is produced in the community of mathematicians, into the knowledge for teaching at different levels within the teaching system. Bosch and Gascón (2006) illustrated the steps of a didactical transposition process through different institutions as *Scholarly knowledge → Knowledge to be taught → Taught knowledge → Learned, available knowledge* (ibid, p. 56). Chevallard's attempt to describe the mathematical knowledge in an institutional context extended into "*the anthropological theory of didactics*" (ATD) (ibid). There, mathematics learning holds to be modelled as the construction within social institutions of *praxeologies* (ibid). A praxeology supplies both methods for the solution of a domain of problems (*praxis*) and a framework (*the logos*) for the discourse regarding the methods and their relations to a more general setting. The block of "praxis" is usually described as "know-how" and the "logos" is described as "know-why". The praxis can be described by the set of *tasks* and *techniques* and the logos is constituted of a *technology* that informs and describe techniques and a *theory*, which is used to motivate and establish the technologies. A praxeology that describes some mathematical knowledge is also called *mathematical organisations* (*MO*) (Barbé, Bosch, Espinoza, & Gascón, 2005). In the same way, *didactical organisations* (*DO*) (ibid) is a praxeology that describes the knowledge and know-how used by teachers to teach the subject matter knowledge to their students.

Table 1 A CoRe Template

Year level for which this CoRe is designed:____________	Important Science ideas/concepts					
Content Area:___________	Big Idea A	Big Idea B	Big Idea C	Big Idea D	Big Idea E	Big Idea F
What do you intend the students to learn about this idea?						
Why is it important for students to know this?						
What else do you know about this idea (that you do not intend students to know yet)?						
What are the difficulties/limitations connected with teaching this idea?						
What is your knowledge about students' thinking that influences your teaching of these ideas?						
Are there any other factors that influence your teaching of these ideas?						
What are your teaching procedures (and particular reasons for using these to engage with this idea)?						
Specific ways of ascertaining students' understanding or confusion around this idea (include a likely range of responses).						

Figure 1: CoRe Template (Bertram & Loughran, 2012, p. 1029)

METHOD

We applied two methods for data acquisition in this study: Firstly, we used classroom observation with video recordings and, secondly, an interview, using a Content Representation (CoRe) template as a reflection document. CoRe, was developed, by Loughran and colleagues (2006), to help to focus on different parts of the PCK. An illustration of a Core Template is given here below.

CoRe is originally developed for science teachers' practice. It used before a lesson is conducted as a collaborative tool helping teachers to identify important aspects of the content within the specific area. After the lessons, the experiences by the teachers' can be documented in a Pedagogical and Professional-experience Repertoire (PaP-eRs), which are linked to the CoRe, and illuminate the decisions underpinning the teacher's actions intended to help the students better understand the content (Loughran et al., 2006). In our study we have used the CoRe template in a different way: After the lesson, the teacher educator answers to the items on the CoRe template to reflect and to consider on the content of the lesson. Thereby making explicit his different conceptions and decisions about teaching the specific topic. The reason we chose to apply CoRe as an interview format is that it is a convenient way to map how the teacher perceives the association between *scholarly knowledge* and the *knowledge to be taught*. Furthermore, by conducting the interview afterwards, it is possible to analyse the association with *the taught knowledge*. Thus we may illustrate the teacher educators take on the discourse on techniques and technologies – the praxis and the knowledge, in the sense of ATD, into account when he/she designs the lessons.

"Quantity and measurement" in Japanese curriculum

The guideline of the Japanese curriculum "The Course of Study" for primary school describes determination of area and volume in the domain "Quantity and Measurement". It states that (Ministry of education, culture, sports, science and technology (MEXT), 2008, pp. 23–26), pupils need to learn to "*compare* length, area and volume of different objects" in grade 1, learn about "standard units of length and volume (e.g., meters and liters) and measurement" in grade 2, "the units of area and its measurement" in grade 4 and they are obligated to learn "to determine area of triangles and parallelograms" in grade 5. Thus, area and volume determination is considered as a part of learning "Quantity and Measurement". Hence, it is not located in the domain "Geometry" in the Japanese Curriculum. The introduction to "Quantity and Measurement" in Japanese elementary schools usually consists of four phases (Miyakawa, 2010): 1. Direct comparison of two objects. 2. Indirect comparison of two objects with a third object, having the same kind of quantity. 3. Comparison of two objects with arbitrary object as a unit (e.g., a pencil). 4. Comparison using standard units (e.g., meters).

RESULTS AND ANALYSIS

The lecture "Quantity and Measurement" in "Arithmetic Education"

A main focus for the course "Arithmetic Education" is the content of primary school mathematics and how to teach such content. The lecturer of this course Mr. Matsui, has himself worked as a mathematics teacher in lower secondary school for 14 years. We observed 55 teacher students in Matsui's class.

Mr. Matsui begins the lecture by instructing the students to read the description of the domain "Quantity and Measurement" in the "Guidelines of the Course of study". He remarks that the comparison of two objects' areas and volume is a new addition from 2008 in the Course of Study for grade one. Furthermore, he refers to the Guidelines of the Course of study and explains the four phases in the process of pupils learning about quantities. Mr Matsui picks up two pens and asks his students "How do you compare the length of these two pens?" He requests that a student show the class how to put the pens together in a way so that the difference in lengths demonstrates clearly and he explains the term *direct comparison* of two quantities. Secondly, Mr. Matsui clarifies an example of the *indirect comparison* of two quantities by comparing the length and the depth of his desk. A student answers that one can use an object such as paper tape and so on to compare the length of the two sides of the desk. Accordingly, Mr. Matsui takes up a pencil and describes how to *use an arbitrary object as a unit* to measure and compare the two sides of the desk. He illuminates the disadvantages of this method using an arbitrary unit, (the length of the pencils varies), and he explains the reason why *standard units*, like meters and centimetres, are finally introduced in order to get exact measurements. Finally, Mr. Matsui writes down these four phases on the blackboard and then shows the class a digital textbook that shows how the first three phases are handled in the textbook.

In the textbook for grade 4, the author uses a small grid of $1cm^2$ squares to determine the area of a rectangle by counting and then to demonstrate that it is obtained by multiplication of the width times the length. Moreover, the textbook describes that the area of a square is determined in same way and defining what it calls a *formula* that determines the area of *any* kind of rectangles and squares. Mr. Matsui explains that the area of triangles, parallelograms, trapezoids and rhombs will be learned in the fifth grade and mentions that most textbooks nowadays consider the determination of the area of parallelograms before that of triangles; it used to be taught in the opposite order.

Here, Mr. Matsui's actions indicate that he is making the students aware of the process of didactic transposition: That is, the transformation from the *knowledge to be taught* to the *taught knowledge*. The knowledge to be taught, i.e. the curriculum as stated in the "Course of Study", is designed by Chevallard calls the *noosphere* (1992), which is a non-structured set of experts, who have a big influence within the educational system. In Japan, the contents of the textbooks are controlled by MEXT before publishing. Therefore, the contents of the textbooks can be treated almost as the knowledge to be taught. Mr. Matsui's students verify how the textbook treats the area of rectangles and how to think of a formula as a generalised computation. The textbook treats the determination of the area of rectangles as an initial task and where some techniques are eventually justified by algebraic reasoning. It is clear that the theory, which justifies this technology, is both algebra and geometry and, in that sense, the praxeology of the textbook is large since it follows the stipulated praxeology in the Course of Study.

The *taught knowledge* is created by the teachers' praxis when conducting lessons in their classrooms. By referring to the four phases of learning "Quantity and Measurement" in the curriculum guidelines and relating it to the *taught knowledge* – by referring to textbooks – the students are given an opportunity to reflect deeply upon the SMK (or MO) and PCK (or DO). Thereby, the teacher students can avoid the "*illusion of transparency*" (Chevallard, ibid), i.e., that one believes that the mathematical knowledge is fixed and known and that one does not question the form it is presented in the curriculum since one feels that it is already known.

The praxeology of the lecture

The students are now going to find out several different methods for the determination of the area of parallelograms with the intention of teaching pupils of grade five. For this reason Mr. Matsui distributes grid papers with parallelograms and reminds that grade five pupils have learned the how to determine the area of rectangles and squares but not of triangles. He gives his students several minutes to reflect, and starts to walk between the students' desks (kikan-shido) and gives them hints and decides which students' solutions will be presented later, due to the variation of their methods for the solutions. Mr. Matsui lets four students draw and explain their solutions on the blackboard. Student A has combined some incomplete grid squares with the corresponding incomplete squares on the opposite side of the parallelogram. Thus transforming the parallelogram to a rectangle (see Figure 2 in the middle) without changing the area.

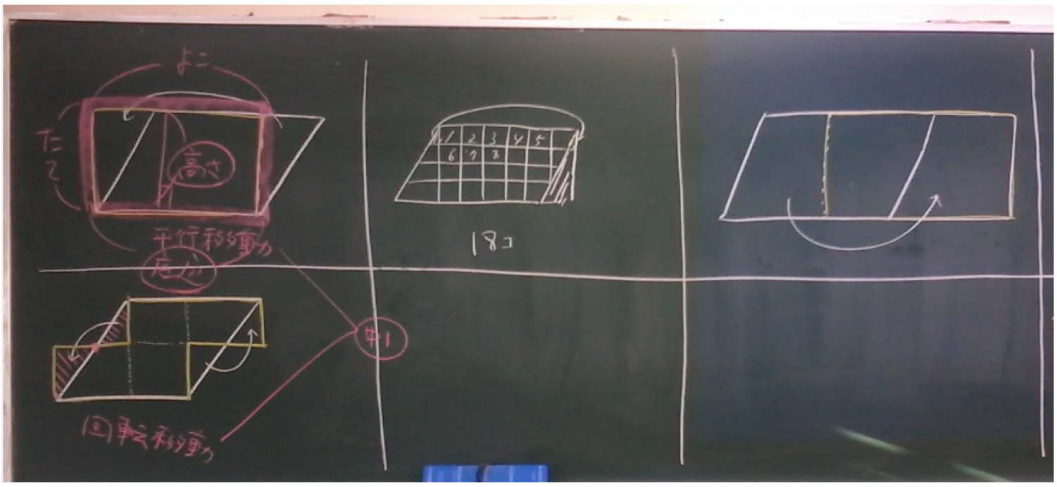

Figure 2: Students' presented methods for determination of area of parallelogram

Matsui remarks that student A is using an arbitrary unit to determine the area.

Student B has divided the parallelogram into a triangle and a trapezoid to shift the triangle to the other side to make a rectangle (Figure 2 in the top left). Student C has divided the parallelogram into two small rectangles, two trapezoids and two triangles. Then she rotates the triangles in order to construct two rectangles (Figure 2 in the bottom left).

Finally, student D has divided the parallelogram in the middle and shifted one of the two trapezoids to the other side (Figure 2 in the top right). Mr. Matsui points out the different kinds of "shifts" used by student B, D and C. Student B and D used parallel *translation* and student C also used *rotation*.

Mr. Matsui notes that the operations of translation and rotation will be covered in more detail in grade seven. He continues by writing "previous knowledge" on the blackboard. He goes on to explain the formula for the area of the parallelogram as $w \times l$ (width times length), since the geometric transformations shows that the width (or height) and length of the parallelograms exactly corresponds to those of rectangles. He mentions that once pupils have learned how to determine of area of a parallelogram, they are able to consider the area of any triangles.

Thereafter, Mr. Matsui gives as task to find out methods for determining the area of trapezoids, using same didactical approach. He chooses 7 students. Three of them shift parts of the trapezoid in different ways so to transform it to a rectangle. Two of them divide the trapezoid in different ways. One student adds a small triangle on the top of the trapezoid to make a big triangle. Finally, the last student doubles the trapezoid so as to transform it into a big parallelogram. From the last method, Mr. Matsui establishes the formula for the area of trapezoid, which is $(a + b)h/2$.

To finish, Mr. Matsui shows an article, written by a teacher in service, about a case study of teaching the area of trapezoids using the structured problem solving approach. By doing this, he institutionalises the conception of the structured problem solving as a general didactical approach.

Mr. Matsui's didactical task is to make his students consider how pupils would reason about such tasks concerning area determination. At the same time, he is letting the students *experience*, how the taught knowledge in domain of "Quantity and Measurement" might *look like*. In this sense, the knowledge disseminated in the lecture has a double focus – one is for the students to construct the didactical praxeology based on structured problem solving and the other is to discuss the viability of different mathematical organisations to be taught in grade five.

The mathematical organisation for the educator/teacher students (MO_1); *Types of tasks*: comparison of lengths and areas of different objects. *Techniques*: measuring with direct comparison, indirect comparison, comparing with arbitrary units and standard units, transformation of shapes, using formulas. *Technology*: comparison, figures, translation, rotation, formulas. *Theory*: Quantity and Units, Euclidean geometry.

The didactical organisation for the educator to be used in teaching teacher students (DO_1); *Tasks*: determine how pupils in various grades would reason during a class with area determination of polygons, by considering the pupils' previous knowledge. *Technique*: make the student participate in an example lesson using the structured problem solving approach, and follow it up with discussions. *Technology*: statement of previous knowledge, mathematical textbook and curriculum used as reference. *Theory*: structured problem solving.

The mathematical organisation for the teachers/pupils of grade five (MO_2) of this area determination: *Task*: to derive a formula for the area of parallelogram/trapezoid, *Technique*: transformation of shapes, using formulas for rectangles. *Technology*: figures, parallel shift, rotation. *Theory*: Quantity and Units, Euclidean Geometry.

The didactical organisation for teachers of grade five (DO_2): *Task*: making the pupils participate in the lessons and to reason about the determination of area of parallelogram and trapezoids. *Technique*: questioning, giving the task (hatsumon), and using graph paper (grid of 1cm) to draw their ideas on. *Technology*: group discussion, whole-class discussion (neriage). *Theory*: Structured problem solving.

Interview with Core template

As the Big ideas in CoRe template for the theme "determination of the area of a parallelogram and trapezoid", Mr. Matsui named: 1. Area of geometrical figures, 2. The concept and properties of geometrical figures, 3. Formula to generalize the calculation of the area. These ideas show how Matsui understands how the knowledge to be taught is derived from the scholarly knowledge. He considers that "determination of area of parallelogram and trapezoid" originates from the domain "Quantity and Units" (according to the big idea 1), "Euclidean Geometry" (according to the big idea 2) and "Algebra" (according to the big idea 3). In order to illustrate his perception of these ideas, the outcomes of the interview, is presented below.

Mr. Matsui's intention for students learning about this idea is that *Area of parallelogram, triangle, trapezoid and rhombus* can be determined in various ways by using the previously learned knowledge – the area of a square and rectangle by dividing those shapes into standard areas. Matsui mentions also that his students should be able to apply some mathematical terms such as *tosekihenkei* (*same area transformation*: transformation of the shape without changing the quantity of the area), *baisekihenkei* (*double area transformation*: transforming the shape of with a duplication of the area). These terms describe the various ways for determination and helps in understanding the methods. Mr. Matsui intends that the students should learn the process of finding out the formulas for area of various geometrical figures, rather than memorising those formulas.

As a starting point for the planning of teaching, Mr. Matsui takes up the pupils' previous knowledge. It is an important component in the teacher PCK; the pupils' previous knowledge has a strong impact when choosing techniques. The sharing of terms like *tosekihenkei* and *baisekihenkei,* (for which we could not find exactly corresponding terms in English), indicates that the technology is institutionalised among teacher educators in Japan. Emphasising the learning process of finding out formulas, shows that Mr. Matsui acknowledges the importance of developing conceptual knowledge, rather than procedural knowledge. He also emphasises the necessity for his students to understand the meaning of the algebraic generalisation.

Matsui names some concepts which the students do not yet need to know within this area: 1. additivity, 2. *bunriryo* (discrete quantity, where the range of possible values are not continuous), 3. *gaienryo* (extension quantity with additivity, e.g., length, time and area), 4. *naihoryo* (inclusion quantity, a quantity that does not have additivity, e.g., temperature, velocity and density), 5. other remarks as Cavalieri's principle.

The mathematical concepts named by Mr. Matsui above, shows his awareness of the technology within the praxeology MO_1. He links these concepts to the domain "Quantity and Units". This shows that MO_1 is strongly influenced by the knowledge to be taught, which are denoted by the curriculum.

Concerning teaching procedures listed in the CoRe template, Mr. Matsui names several detailed methods. As preparation, he proposes to use graph papers (grid of 1cm), an article written by a teacher in service and to refer to corresponding pages of in digital textbooks of the subject matter.

He states that the teacher educator should stress the importance of the children's perspective during the lecture, so that when the students perform their own lessons in the future, they are able to confirm previously learned items and acknowledge various ways of solutions (including wrong answers) during their lessons. According to Mr. Matsui, the teacher students know intimately the flow of working with tasks: reason individually → discuss with neighbors → present the solutions in class → respond to comments from the lecturer.

Mr. Matsui uses different didactical methods in order to transform SMK to PCK, making the subject understandable to the teacher students. For instance, without using graph paper, student A would never come up with the idea for her solution to determine the area of parallelogram. The didactical contract (in this case, both the teacher and the students are aware of the basic flow of working with tasks) is required in order to apply the structured problem solving approach in the classroom. This holds both for the lessons instructing teacher students and classroom lessons for pupils in elementary school.

CONCLUSION

Our aim was to analyse a lesson titled "Quantity and Measurement". Our impression is that the act of didactical transposition from the scholarly knowledge to the knowledge to be taught is insightfully done by the Japanese noosphere. For example, the area and volume determination is located within the topic of

"Quantity and Measurement" and not in "Geometry", which is the case in many other countries (e.g., Sweden). Compared to the Swedish curriculum, the Japanese curriculum provides a relatively detailed fundament for the large mathematical (and also didactical) organisations that are also replicated in the textbooks. As a consequence, the transposition from the knowledge to be taught to the taught knowledge becomes more explicit and more uniform. It also makes the progression of the content visible. For instance, the "four phases of comparison" in different grades is clearly institutionalized within the discourse of the teachers in Japan. Furthermore, the curriculum sets the scale of the mathematical praxeologies in the textbooks, which in turn influences the complexity of the mathematical organisations that are discussed in the teacher education classes. The scale is also enhanced by the use of the structured problem solving when reviewing the mathematical organisations in these classes, since it gives a natural setting to discuss and enrich the associated technologies. It is also a way to consolidate the didactical organisations implied by the problem solving approach. By combining an analysis of the interview with the analysis of the didactical transposition and ATD, we have illuminated the teacher educator's perception of the PCK and the SMK. This case study indicates that prospective teachers in Japan are given good opportunities to develop their insight of the PCK during their teacher education in mathematics.

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When working together to plan a lesson in a Swedish professional development initiative

Anna Bengtsson

Linnaeus University, Faculty of Technology, Kalmar, Sweden, anna.bengtsson@lnu.se

This paper is based on a study that draws on Wenger's Communities of Practice perspective and accounts for the coherence of mutual engagement, joint enterprise and shared repertoire in a community of Swedish upper secondary mathematics teachers participating in a professional development initiative. The aim of this paper is to describe and understand practice when teachers are working together to plan a lesson. An overall characteristic of practice is that it develops in a teaching culture, and as the community lacks of awareness of how it organises their teaching, practice becomes resistant to change when planning the lesson. Also, this paper addresses further research considering teaching culture when teachers work together to plan a lesson as a way to obtain, and maintain collegiality.

Keywords: Collegiality, lesson, teaching culture, community of practice.

INTRODUCTION

The present trend aimed at improvements in schools is through collegial collaboration. Today "communities of practice" fill the air (Putnam & Borko, 2000). However the shift towards collegiality is a new setting for many teachers. Teachers in secondary education primarily feel responsibility for their own classroom practices, resulting in largely autonomous and isolated work and private learning activities (Hodkinson & Hodkinson, 2004). The discussions in many staff development sessions are characterised as "style shows". These sessions provide few opportunities for meaningful reflection and growth and maintain the individualism and isolation of teaching (Ball, 1994). Collegiality is de-privatising the work of teaching, and it means being able to disagree constructively about professional practice (Evans, 2012). It is more than simply sharing ideas, it means confronting traditional practice – the teacher's own and that of his or her colleagues.

The Swedish upper secondary school was reformed 2011 and a new curriculum was formulated. It emphasises that teachers should cooperate with other teachers in order to achieve the educational goals (National Agency for Education, 2013). This paper is based on a case study that captures the characteristics of a community of four upper secondary mathematics teachers in a professional development initiative. A version of the Japanese lesson study – learning study – gave access to empirical data of the study. Learning study involves teachers and researchers working together to plan a lesson. The lesson is taught by the teachers in one or several cycles, and is observed, evaluated, and modified by the team before the next cycle is taught (Marton & Lo, 2007).

The aim of this paper is to describe and understand practice when Swedish upper secondary mathematics teachers work together to plan a lesson.

BACKGROUND

In order to understand practice I will review literature regarding a Swedish lesson and a Japanese lesson, as the idea of working together to plan a lesson was imported from Japan.

In Japan teachers work in collegiality and Japanese lesson study was chosen as model for Swedish learning study.[1] The premise behind lesson study is simple; if you want to improve teaching, the most effective place to do so is in the context of a classroom lesson.

1 The difference between learning study and lesson study is that the former comes with a theory of learning. Most often learning study draws on the theoretical assumptions of the variation theory. However, in this paper, the variation theory is not in focus.

Yoshida (2004) writes that Japanese teachers spend hours planning a single lesson in a lesson study. The teachers first engage in the problem from which the lesson will be launched, as the Japanese mathematics lessons are based on structured problem solving. Then the anticipated solutions, thoughts and responses that students might develop as they struggle with the problem will be explored. This is in relation to the kinds of questions that may be asked to enhance student thinking during the lesson, as the type of guidance that could be given to students who show misconceptions in their thinking. The end of the lesson, the moment at which students understanding can be advanced, is carefully considered in the lesson study (Yoshida, 2004; Stiegler & Hiebert, 1999).

Swedish lessons however, are largely synonymous with solving of exercises. The teacher presents a few tasks on the whiteboard, while the students are listening. Similar exercises will then follow and the remaining time of the lesson is spent for individual work in the textbook. The exercises are solved with a specific method and have a correct answer. It is not necessarily that the students are practicing the same skills in their individual work in the textbook as those the teacher presented at the beginning of the lesson (Lundin, 2008). Swedish teaching resemble in many ways with U.S. teaching. Evans (2012) writes that teaching is highly personal in the U.S., and over time, every teacher develops a unique instructional repertoire, a set of personal, artful, assumptions and responses.

Stiegler and Hiebert (1999) stress that teaching is a cultural activity, it is composed of elements that interact and reinforce one another. The methods teachers use, are not determined by their qualifications as much as by the culture in which they teach and the role of the teacher will follow his/her assumption of the nature of learning. When a lesson takes the form of following the teacher's directions by practicing a procedure during seat-work, the teacher believes his/her responsibility is also to keep students engaged and attending. Moment to moment attention is fundamental. Teaching in this typical culture is about enhancing students' interest by increasing the pace of the activities, by praising students for their work and behaviour, by the cuteness or real-lifeness of tasks and by their own power of persuasion through enthusiasm, humour and "coolness". This practice should be relative error-free, as the importance of the feeling of success is not underrated in a learning situation. The teacher acts as if confusion and frustration are signs of them not succeeding at their jobs (Stiegler & Hiebert, 1999). The Japanese teaching culture on the other hand is reinforced by that learning occurs by first letting the students struggle to solve mathematical problems.

Stiegler and Hiebert (1999) write about challenges in importing lesson study to another teaching culture. Trying to improve teaching by changing individual features usually makes little effect, positive or negative, especially when the feature is imported from another teaching culture. Lewis (2009) however, argues that there is evidence that lesson study can be used effectively outside Japan. She reports changes in teachers' professional community in terms of, capacity to improve instruction, shared language, processes, and frameworks for analysing instructions.

Collegiality is not a feature in the Swedish teaching culture, and in this case, the model of working together to plan a lesson is imported from Japan. It is above given that planning a Swedish lesson is different from planning a Japanese lesson.

THEORETICAL FRAMEWORK

Within the paradigm of social practice theory, Wenger (1998) conceptualised Communities of practice as a social theory of learning.

> Practice is doing in historical and social context that gives structure and meaning to what we do. [...] In this sense, practice is always social practice. (Wenger, 1998, p. 47)

Wenger (1998) writes that communities of practice are groups of people who share a concern or a passion for something they do and learn how to do it better as they interact regularly. A community of practice has a shared domain of interest. Membership therefore entails a commitment to the domain and a shared competence distinguishes members from other people. In pursuing their interest in their domain, members engage in joint activities and discussions and share information. They develop a shared repertoire of resources: experiences, stories, tools and ways of addressing recurring problems. We belong to several communities of practice, in some we are core members, in many we are merely peripheral.

The framework is not about whether the practice is right or not. It is about the active involvement and how it takes place; what is brought to the table in a community of practice (Wenger, 1998). The teachers in the case are an active part of their community but at the same time they are influenced by a teaching culture. The teachers are accountable to the quality of the community of the practice. Their experience of teaching and learning mathematics will be negotiated in the community of practice and validated as competences. The tension between competences and experience is very important for the dynamic in a community of practice. When the core is too strong there is a lack of tension between competences and experience and the community of practice may become static and stand in the way of learning (Wenger, 1998).

The framework of Communities of practice has been used in previous mathematics education research examining teacher learning, with different focus, and in different ways. My approach to the framework is neither attempting to design, nor analysing if a community of practice emerges or not. *I see the mathematics teachers as a community of practice; it is my unit of analysis*.

Framing this case as a community of practice pays attention to the teachers' negotiation of meaning. Meaning is defined as an experience of everyday life and is located in a process; in the negotiation of meaning[2] (Wenger, 1998). That is the negotiation of their experiences of teaching and learning mathematics. A community has dimensions of source of coherence through mutual engagement, a joint enterprise and a shared repertoire. Mutual engagement defines a community and being engaged gives a sense of belonging. It can give rise to differentiation as to homogeneity, as it involves competences and competences of others (Wenger, 1998). The teachers' practice draws on what the teachers know, and their ability to negotiate what they do not know. The joint enterprise is what is being negotiated and reflected upon in the community. It does not imply that everybody agrees with everything. This reflects the complexity of mutual engagement (Wenger, 1998).

> Communities of practice are not self-contained entities. They develop in larger contexts – historical, social, cultural, institutional – with specific resources and constraints. Some of these conditions are explicitly articulated. Some are implicit but are no less binding. (Wenger, 1998, p. 79)

An explicit condition for the practice of this case is that it takes place in a setting; the teachers are to plan a lesson. The process of defining a joint enterprise is keeping the practice in check, just as it also pushes it forward (Wenger, 1998). A history of teaching culture may also be a binding condition, as it is so fully integrated into teachers' worldview. A shared repertoire is the development of the joint enterprise, it is the words, tools, concepts that are produced or adopted throughout the community of practice.

My approach to Communities of practice and the aspects of practice have been characterised by concepts from the theory. This is for the analysis of empirical data. Next methods will be discussed.

METHODOLOGY

This research has arisen in response to the shift towards collaborative work in schools. This is a search for understanding rather than establishing explanations and looking for causes. This is a distinction between qualitative and quantitative research (Stake, 1995).

Stake (1995) writes that we study a case when it itself is of very special interest, when we look for details of interactions with their context. This case constitutes four upper secondary mathematics teachers in a setting of learning study, taking place at an upper secondary school in Sweden. The four teachers have been teaching mathematics in upper secondary school for 4–12 years and they have been employed at the school for 3–12 years. An external advisor, based at a university, is also participating in the learning study. The teachers and advisor met on 7 occasions and in between the teachers were set up for work. Each meeting had a purpose and the work in between was also defined. Their meetings focused discussions on what and how to teach the mathematical concept of slope aiming to plan a lesson.

2 The negotiation of meaning involves the interaction of two constituent processes, participation and reification. Participation is defined as active social involvement but also as personal membership. Reification is defined as a shortcut for communication, a focus, a projection of what they mean. Participation may refer to the active involvement in planning the lesson (Wenger, 1998).

The choice of method in this case study is based on it being qualitative research as well as on the nature of the selection of case. Through the setting of the learning study there was access to 14 two-hour meetings. Empirical data was therefore generated through observation of these meetings. Previous research regarding learning study has mostly aimed at developing practice and the advisor and the researcher is then the same person. In this case study, I was a strict observer (Bryman, 2001) meaning that I did not interact with the respondents. Field notes were taken during these observations, and transcribed as soon as possible after the observation. The field notes did not follow a structure or include any categories. I was writing down my immediate reflections, trying to make sense of the case as it unfolded in practice (Flyvberg, 2006). The meetings were also video-recorded. The video-recordings have not been important for the purpose of hearing the exact words; it was the meaning that was important. It gave access to the source of empirical data again, and again. Hence I would not capitalise on making sense of the case (Stake, 1995).

An interview took place once the learning study was conducted, a month later. Interview questions were used to confirm the empirical data (Bryman, 2001). The interview was also held to provide a complement, to find out what was not understood or not heard through the observations. The interview was consequently semi-structured, i.e., a set of questions had been prepared but there was also space for further questions. The themes focused on in the interview were the teachers' expectations and experience of collegiality. The interview was also aimed at complementing and confirming issues of the setting of learning study.

The case study has an abductive approach, rather than a deductive or an inductive approach. Eriksson and Lindström (1997) say abduction is a way to discover meaningful underlying patterns. It makes possible to connect surface and deep structures. The abduction has a starting point in interpreted knowledge. The interpretation is made in a wide sense, including literature, conceptual analyses and historical sources. The perspectives determine which of the deep structures that are tangible. From this approach new knowledge is established (Eriksson & Lindtröm, 1997). By focusing on different aspects, the interpretations have helped to define the unit of analysis. Alvesson and Sköldberg (2000) define abduction as entailing a commutation between data and theory in a scientific and systematic way to look for answers to research questions of interest. They continue that the researcher is minimising the risk of interpreting what they think they are seeing in light of their own unreflected preunderstandings or to reinvent the same theory but in new words and concepts. A chronologically analysis and an attempt has been made to capture the case through short, impressionistic scenes that focus on one moment or give a particular insight into meaning and community (Stake, 1995).

WHEN THE LESSON IS THE UNIT OF ANALYSIS

The teachers have focused discussions on teaching and learning slope, which has resulted in a mapping of the concept of slope and relating mathematical concepts. The lesson will take the form of the teacher presenting the concept, increasing the difficulty and pace of the activities. They have identified that understanding the meaning of a coordinate in the Cartesian coordinate system is crucial for the students to fully understand the rate of change. The following scene captures the teachers planning an initial part of the lesson concerning coordinates:

Teacher 1: So now I have drawn four points here. [...] Then they will be named A, B, C and D.
Teacher 3: Will you name them A, B and C?
Teacher 1: They can be named anything. Or?
Teacher 3: I was thinking that you have the points and that you fill out the coordinates, the coordinates should not be given. [...] Or will you display all the three coordinates at the same time?
Teacher 1: Four [points]. But it might be a smart idea to present one point at a time.[...] I did not plan to write the coordinates out, but of course you can do that as well.
Teacher 2: It becomes clear if you write them out.
Teacher 3: Why did you not want to write them out?
Teacher 1: I can write them out!
Teacher 3: I think there is a value in introducing one point at a time.

The community negotiate how the points should be labelled, if they should be introduced one by one, if they are to write them up on the whiteboard. It is typical for this case that as the community plans the lesson they start to negotiate teaching techniques. The mutual engagement concerns how to create a presentation maximizing the students' attention and understanding of the content. It gives rise to differentiation, as it involves competences of three teachers. The teachers' practice draws on different techniques of presenting content, developed in larger implicit contexts. As history of how to maximize the students' attention may be a binding condition, as it is so fully integrated into teachers' worldview. The joint enterprise is what is being negotiated and that is different techniques introducing coordinates. The process of defining a joint enterprise when planning the lesson is keeping the practice in check, just as it also pushes it forward (Wenger, 1998). The scene captures that it is hard to coordinate practice to move forward.

To give a dimension to the above their previous experience of collegiality, will be included. I asked them about their experience of working together:

> Teacher 2: We have never experienced anything like this [learning study] together. We are very traditional, those who teach the same courses in parallel classes might construct tests, mark tests and assess students' grades together.

Asking to what extent they have experience of planning lessons together, they say:

> Teacher 1: Yes, sometimes, as an outline of a lesson. It is more often the activity we plan, rather than the lesson. We plan the courses together, in term of its schedule; let's cover this chapter by then, let the students take a test then and so on. In addition we also talk about what we have done today as in how far [in the textbook] we have come. We have done this for a long time, more or less. When it suits us.

The teachers have no previous experience of planning lessons together; they are rather collaboratively engaged in more organisational matters in the faculty.

Discussing the model of working together to plan a lesson, the teachers reflected:

> Teacher 2: It does not feel that the primary goal of this learning study is to plan a perfect lesson. What is important to me is that I have got something from this. When I teach my lessons later, that are not in a learning study, then I take this with me. Those lessons will not be ruled by manuscript.
>
> Teacher 4: That is also my experience, that it was everything around that gave me that good feeling when processing the lesson. The lesson was very tightly structured and I felt by the end, as I was teaching, that the students were quite exhausted. Normally I would have cut it, or done something different. It rarely happens that you have such a controlled lesson for 60 min. The last 20 min you often let them work on their own.

As the teachers reflect about the role of the lesson they reflect that the lesson is not the primary goal of the learning study, it is the teachers' professional development that is their mutual engagement. Still the lesson has a value, for them to imagine and to engage around. The analysis gives that it is the joint lesson that keeps the practice check, as it moves forward. However the shared repertoire is not defined by a lesson as a product. The projection of the joint enterprise is what was negotiated as they planned the lesson.

When planning the lesson further they negotiate to let the students discern the relation between distance and change:

> Advisor: We want the students to discern what a distance is and what a positive and negative change is.
>
> Teacher 3: But then.....I don't know. As I do, I always let Δx be positive, should we talk about negative and positive change on the x-axis then. (...)
>
> Advisor: I have experienced that students do not have the meaning of positive and negative change. I always say to them to follow the direction of the axis. If you go with the axis then it is a positive direction and if you go in the other direction then it is negative.
>
> Teacher 2: That is a way of going through the structure of the coordinate system! (...)

Teacher 3: What comes with this is that Δx can become negative and then we have the negative sign in the denominator to handle. If we instead always treat Δx as positive then if Δy is negative the m-value ($y = mx + c$) is also negative. (...)

Teacher 2: It is still the structure of the coordinate system.

(...)

Teacher 3: We create a natural structure for the student. We say we call them point one and two. It says in their formula-booklet.

Advisor: Yes, but this could be point one and this point two, it doesn't matter which point is point one.

Teacher 2: I think, (...) we should not decide which point that is the first and the second. It is rather the structure [of the coordinate system].

Teacher 3: You think? I do not! I think the students will drown in minus signs and they need to consider going left or right.

Teacher 2: Why should we be afraid of minus signs?

Teacher 3: Because it becomes wrong. Minus signs are shit. [Laughing]

This scene rather captures a lack of discontinuity; it shows a static core in the community. The challenge in the community is to allow discontinuity, to keep the tension between competence and experience (Wenger, 1999). The competence is formulating questions from your experience, but from a new perspective.

The core is very static regarding the negotiation of Δx; students' difficulties with negative numbers will cause frustration and confusion in the classroom, thus it is better to avoid it in the teaching. The idea is to always let the leftmost point be point number one and hence the right most point number two. Then the students can use the algorithm in the formula booklet, without any risk of ending up with a negative denominator. It saves students from "drowning in minus signs". The community does not agree when they are negotiating how to teach the content in the lesson. A shared repertoire does not imply shared as in a common view on what is negotiated (Wenger, 1998).

Wenger (1998) writes that practice develops in historical, social and cultural contexts that give structure and meaning to what we do. The analysis will be discussed in relation to the background.

DISCUSSION

An overall characteristic of practice is that it develops in a teaching culture. As the community lacks awareness of how it organises their teaching, practice becomes resistant to change when planning the lesson. This case captures the fact that the teachers value the lesson and negotiates its importance, but their mutual engagement is in regard to their teacher professional development. They say they were engaged around everything that was learned as were working together to plan the lesson; it was not an engagement to produce a perfect lesson in itself. Stiegler and Hiebert (1999) write that the unit of the lesson has validity for the teachers, as it does not lack of generalisation to real life experience. The lesson is also a part within a teaching culture and coincides with teachers' thoughts on the nature of mathematics and how learning takes place. The Japanese mathematics lesson tells a story, it is tightly connected with a beginning, a midpoint and an end. They are different from the Swedish mathematics lesson, which are described to be more modular with fever connections. Yoshida (2004) writes that a lesson is highly sharable among teachers in Japan. Teachers plan these lessons in collegiality and all work in lesson study is done after school. According to both Evans (2012) and Stiegler and Hiebert (1999), U.S. teachers find professional communities as "more work" and they would rather go home early to plan tomorrow's lessons. Even so:

> Teaching can only change the way cultures change: gradually, steadily, over time as small changes are made... (Hiebert & Stigler, 2004, p. 13)

Working together to plan a lesson in professional development initiatives might be challenged in certain teaching cultures; when teaching is highly personal, underpinned by a unique instructional repertoire; if the lesson is more modular with fewer connections and not a unit in itself. In the introduction I have problematized that; collegiality is a new setting for many teachers; collegiality requires structure that goes beyond simply sharing ideas, that sustains the individualism and isolation of teaching and collegiality requires de-privatising of the work of teachers to start to engage critically with issues of practice. This paper addresses further research considering teaching culture when teachers work together to plan a lesson as a way to obtain, or maintain collegiality.

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How to improve spatial visualization ability of preservice teachers of childhood education: A teaching experiment

Ainhoa Berciano and Guadalupe Gutiérrez

University of the Basque Country, Faculty of Teachers Training, Bilbao, Spain, ainhoa.berciano@ehu.eus, guadalupe.gutierrez@ehu.eus

In this paper, we describe a teaching experiment designed and implemented in a classroom of preservice teachers of Childhood Education with the main purpose of measuring the improvement of their spatial visualization ability. Indeed, taking into account some common materials and games, we have designed a four-sessions experiment in mathematical classroom of an Undergraduate Degree of Childhood Education of the University of the Basque Country. After the implementation of the experiment in a classroom with 27 students, the analysed results assert that the spatial visualization ability of the students is better than before, implying that this ability can be achieved and improved even in adults.

Keywords: Spatial visualization ability, teaching experiment, common games, pre-service teachers, childhood education.

INTRODUCTION

Nowadays, we get more and more information through symbols and images that has to be analysed and interpreted in order to extract the meaning.

Understanding symbols is not the only application of the visual-spatial ability that we use in daily life, moreover we are actually surrounded by actions that need it like parking a car, where ideally we need to be able of visualizing the parking spot and calculate if our car fits in. This involves making decisions about how we see objects in a three-dimensional world and how they behave in it. That means we are constantly using our natural ability to position objects, project them on a plane, moving or rotating them mentally. But how can [these abilities] be acquired? And more important, are they susceptible of being taught?

Among the objectives of the book "principles and standards for school mathematics" of the National Council of Teachers of Mathematics (NCTM, 2000) we can find we need to develop the spatial sense and appreciate the geometry as a way to describe and model the physical world. These goals are completely related with the orientation and spatial visualization and, as "Curriculum Focal Points" of geometry points out, they should be studied from childhood Education to 8th course.

In our case, the curriculum of the Childhood Education of the Basque Country has as one of its main goals the "Identification of flat and three-dimensional elements of the environment" (therefore, the visualization skill is required) (EJ / GV, 2010).

These facts imply the visualization ability should be taken into account in the mathematics classroom, developing the necessary activities to improve the skills related with spatial visualization. In this sense, many studies have been done with the purpose to research the factors involved in the teaching-learning of visualization (Bishop, 1983; Gaulin, 1985; Gutiérrez, 1996; Gonzato, 2013; Hershkowitz, Parzysz, & Van Dormolen, 1996; Presmeg, 2006) and one of the open questions is the influence of teacher's ability in this process.

SPATIAL VISUALIZATION

According to Eurydice and Culture Executive Agency (EACEA, 2011), mathematical competence is one of the key skills for personal development that facilitates social and labour inclusion of citizens satisfactorily. After a thorough analysis of the results of the Programme for International Student Assessment (PISA) and Trends in International Mathematics and Science Study (TIMSS) studies at European level, training and professional development of teachers of mathematics highlights as one of the key issues in education.

About mathematical competence, one of the sections analysed by PISA is the "space and form" section, in which the phenomena and geometric and spatial relationships are examined. Indeed, the development of this competence requires observe similarities and differences, analyse the components, recognize shapes in different representations and dimensions and understand the properties of objects and their relative positions. Summarizing it, we must learn how to visualize objects in space and understand the two-dimensional representation thereof, but what does "*visualize*" mean?

Spatial visualization can be defined as "the ability, the process, and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technology tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings" (Sarama & Clements, 2009, p. 183).

Furthermore, Presmeg specifies spatial visualization ability as a collection of processes involved in generating and manipulating mental images, as well as guiding the drawing of figures or diagrams on paper or computer screens (as cited in Sarama & Clements, 2009, p. 184).

These processes and skills of subjects have been extensively studied to perform certain tasks that require spatial visualization ability. In fact, these investigations figure out the visualization throughout focusing on different tasks depending on mathematical content: planar representations of 3D-objects (Gutiérrez, 1996; Hershkowitz, Parzysz, & van Dormolen, 1996), planar developments of 3D-objects and 3D-constructions of planar developments (Cohen, 2003; Fischbein, 1993), classification of figures, comprehension of concepts and properties, geometrical transformations (rotations: Battista et al., 1982), compose and decompose 3D-objects in their parts (Bishop, 1983).

About visualization in education, according to NCTM and Basque government, spatial thinking can be learned and should be taught at all educational basic levels. Regarding this, Bishop (1980) proposes questions about teaching that, nowadays, some remain still unanswered: a) Should experimental teaching methods in this area take into account the spatial abilities of the teacher; b) How much responsibility should mathematics teachers take for the training and teaching of spatial abilities? Is this perhaps an area like language, which is every teacher's responsibility?

Hershkowitz, Parzysz and van Dormolen (1996) underline that the nature of mathematics is the search of patterns and therefore visualization is a fundamental tool to recognize them, but even though this relevance, the visual education is often neglected in the curriculum.

In the same way, but some years later, Presmeg (2006, p. 227) gives a big perspective of the researches done in mathematics education and, again, she proposes 13 open questions, some of them related with visual ability: 5. What conversion processes are involved in moving flexibly amongst various mathematical registers, including those of visual nature, thus combating the phenomenon of compartmentalization? 9. How may use of imagery and visual inscriptions facilitate or hinder the reification of processes as mathematical objects? 10. How may visualization be harnessed to promote mathematical abstraction and generalization? 13. What is the structure and what are the components of an overarching theory of visualization for mathematics education?

Actually, different studies have been published giving partial answers to these questions, for example, Gaulin (1985) outlines some activities connecting coding and decoding of spatial information through representations. One of the conclusions was that even the teachers had problems to interpret some graphical representations.

Likewise, Battista, Wheatley and Talsma (1982) show that a specific design of a course based on manipulative materials and some concrete models (symmetry of polygons and polyhedral, paper folding, tracing and using a Mira), improves the specific part of the spatial ability of prospective teachers related with rotations (the test used was "Purdue Spatial Visualization").

In this sense, the work presented here tries to analyse the influence of a specific teaching experiment designed for prospective teachers of Childhood Education, where the methodology is based on common materials, games and working as collaborative teams (see, for example, Gutiérrez & Berciano, 2012).

METHODOLOGY

Context of the study and objectives of the research

The research showed here is included in a bigger project where the main objective is to improve the skills of our students with respect to their didactical competency in spatial visualization as teachers of childhood education (this project has the restriction of time: in 60 hours a big curriculum related with childhood mathematics has to be given); but, first of all, we will analyse if their spatial visualization ability can be improved and how. For this purpose, we will focus on the first question, measuring the improvement of the spatial visualization ability of our students after the implementation of a designed teaching experiment.

Participants

The group chosen for the teaching experiment was a Spanish group of the 3rd level of the Undergraduate Degree of Childhood Education of the University of the Basque Country of the topic "Mathematical thinking and its Didactics", formed by 27 persons, 26 women and 1 man. All of them usually went to classroom and participated actively. This group was divided in small groups of 2 or 3 persons to work in a collaborative way during the entire course.

Instruments to measure the spatial visualization ability

The instruments used to analyse the results have been mainly two: the diaries of the students, where their evolution could be seen, and a test to measure their competency about spatial visualization (used as a pre-test and a post-test).

About this second tool, we have seen that different authors focused their research in a specific task related with spatial visualization ability, but in our case, to measure it, we have taken into account all of them, that is, we have focused on studying the tasks of interpreting perspectives of three dimensional objects (activities requiring recognize and change views (change of perspective)), rotate objects mentally, interpret different planar representations of three-dimensional objects (perspectives, views,...), turn a planar representation into another, build objects from one or more planar performances, ...

For this purpose, we have used the test called "Test of Three-Dimensional Objects" (VOT), designed and validated by Gonzato (Gonzato, Fernández, & Godino, 2011; Gonzato, Godino, & Neto, 2011; Gonzato, 2013). In this test, the mathematical knowledge and the knowledge of teaching Spatial Visualization in elementary education are measured. As the author shows in her PhD thesis, the tasks proposed in the test involve the following actions: *change the representation type* (planar or three-dimensional object); *rotate* the object or parts of the object, or equivalently, *change* mentally *the perspective* of it; *fold* a planar development to create a three-dimensional object (physical or represented), or vice versa, *expand* the object for obtaining one of its developments; *compose* and *decompose* in parts; and, given a solid, *count* the component parts (units of volume, faces, edges, vertices, etc.).

In fact, the test has 5 items (with sub items), each of them dedicated to a previous action. Let's see the first item and its parts (extracted from (Gonzato, 2013), question 1b') has been adapted to Childhood Education):

> *1a)* From which positions have been taken the pictures you can see at right side? *1a')* Justify the answer. *1b)* Identify the knowledge involve in the resolution of *1a). 1b')* Indicate how should you change *1a)* to be able to work it at Childhood Education classroom.

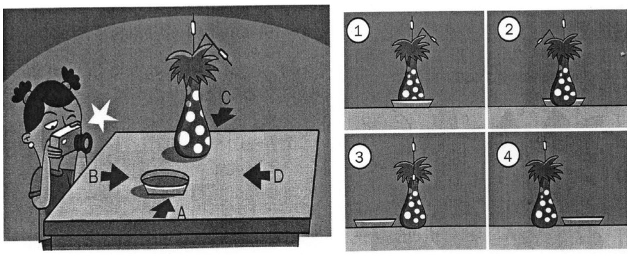


Figure 1

> *1c)* The following figure shows a building drawn from front-right angle. Draw the view from back. *1c')* Justify your answer.

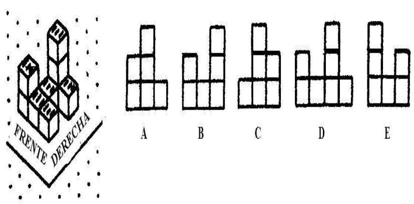


Figure 2

The way to correct this test is:

1) To measure the skills our students have about mathematical knowledge, we have analysed the results given by them in the first sections of each item. In this case, the correction is right, partially right or wrong (1a, 1c). In the same way, it is possible to do a qualitative analysis of the errors given by the students to obtain more information about the error types.

2) To study the skills that the students have about didactical knowledge, we have examined the answer given in sub item 1b' (not included in this paper).

We have used this test as a pre-test and post-test to, on one hand, study the starting point of our students and, on the other hand, see the variance of the ability of our students after the implementation of the teaching.

Mathematical task

In all the activities of the experiment, the task requested to the students is *to determine which mathematical concepts and properties are needed* to realize a specific exercise where visualization skills are involved. These are some questions:

1) Which properties of a 3d-object should be used to do its planar development (and the reverse)?

2) Given a 3d-object, how many different projections can be done, and which mathematical properties are involved?

3) Which are the mathematical concepts involved in the movement of a 2d-object in the plane, which properties do they have and how can be they composed?

Design of the experiment, schedule of the materials used and the activities done

Previously to the experiment, the students filled out the pre-test. The experiment has been realized inside the topic "Geometry" in 3 sessions of 2 hours each and another hour and a half of a fourth session has been used to evaluate the results after the implementation using the post-test. The way to design the session has been the same, that is, first the students should become familiar with a selected material (a specific game or common material); second, they should realize different activities to explore the characteristics of the material from mathematical point of view; third, they should formulate a hypothesis about a given question; fourth, they must compare the hypothetical results with the real answer; fifth, they had to give an explanation about the result with mathematical arguments. All of these activities were done in small collaborative teams and the students had to talk to each other at all times.

Next, we describe each session with the given instructions to the students:

Session 1: *Visualization and spatial representation (01/04/2014)*

The activities done: VOT test and playing with boxes, where the main objective is to learn how are the cardboard boxes constructed from their planar development, and vice versa.

This activity has two different parts, but the way to work with the material is always the same. Each pair of students should complete a diary where the next steps should be done:

1) Observe the given object (3d or 2d).

2) Describe it verbally and graphically.

3) Make a hypothesis about its planar development/ the 3d object that can be done with it (3d and 2d object respectively).

4) Do the planar development/ the 3d object.

5) Describe verbally and graphically the planar development/ the 3d object.

6) Compare the hypothesis with the experimental result.

The materials selected were a commercial box for the first activity and a common planar development for the second activity.

Next, let's see a small part of the diary realized by a student with respect to the first exercise, the planar development of a cardboard box that contains coffee capsules.

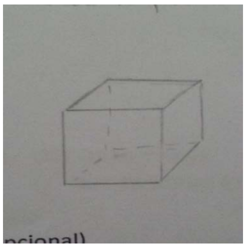
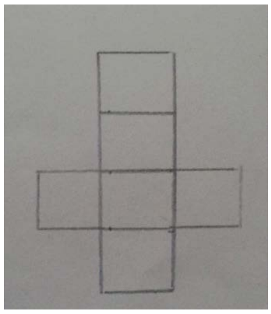
Take the box	*Describe it verbally*	*Describe it graphically*	*Make an hypothesis about its planar development*
	It is a cube-shaped box that has all the edges equal (where two faces meet), which is measuring the same. It also has 6 faces.		

Table 1: Theoretical planar development of a given box and part of the diary of a student

Session 2: *Identification of 3d objects and their component parts* (08/04/2014). The activities done: constructions with multilink cubes and constructions of buildings with the game called "*Skyscraper*".

The way to give the instructions is the same than in session 1. Two different parts, but with the same line of argue.

With the multilink cubes, the students should construct different structures, describe them verbally to other colleagues of the same collaborative group and the other students should replicate the construction only with the explanations, without see it.

In the second part of the session, students should play with Puzzle Skyscraper. Each puzzle consists of a 4x4 grid with some clues along its sides. The object is to place a skyscraper in each square, with a height between 1 and 4, so that no two skyscrapers in a row or column have the same number of floors. In addition, the number of visible skyscrapers, viewed from the direction of each clue, is equal to the value of the clue. Higher skyscrapers block the view of lower skyscrapers located behind them.

When the construction is done, the students should describe all the projections of the construction verbally and graphically.

Session 3: *Geometric transformations* (symmetries, turns) (15/04/2014). In last session, the main purpose is to recognize the differences between symmetries, translations and turns. For this end, the material used was: game reflections, game "the prince and Munster" and mirrors. The instructions were given following the next steps:

1) To distinguish between symmetries and translations, the students should find examples at the classroom.

2) Next, taken into account the symmetry, a classification of alphabetic letters was done.

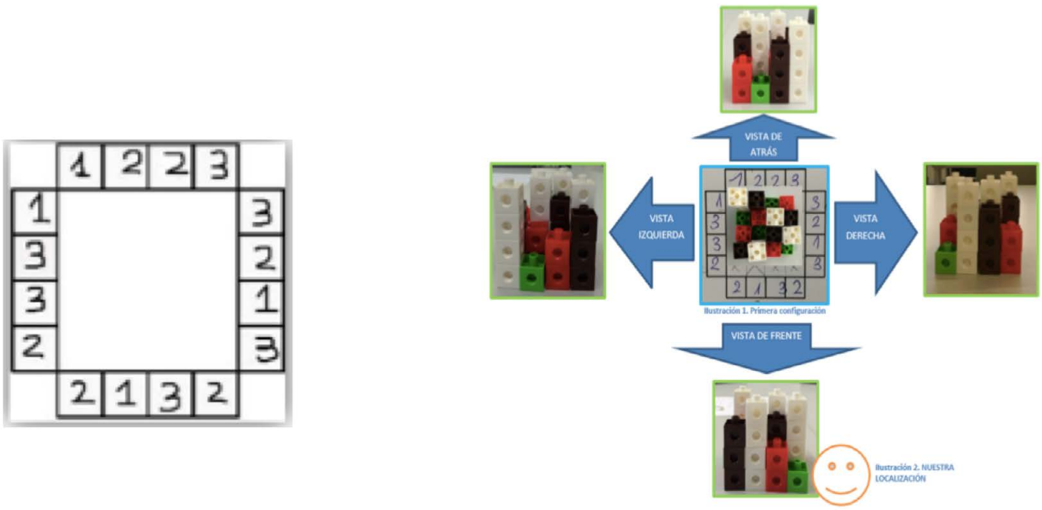


Figure 3: A given square and its construction with all the perspectives done by a student (left and right respectively)

3) Find the differences between a photo and an image in front of a mirror.

4) Finally, the students should anticipate the hypothetical answer of the reflections of the mirrors to create a copy of the asked image (reflection) or the face of the prince (prince and Munster). Again a verbal explanation should be given to their colleagues.

Session 4: *Evaluation of the spatial visualization ability* (29/04/2014). Again, VOT test was used to evaluate the results after the experiment.

RESULTS

To show the main results, first of all we've focused in the analysis of the differences between the results in the pre-test and the post-test considering that the maximum score is 20.

As Figure 3 shows (left graphic), it is clear that in a big percentage, the students have improved their results in the test. Only 4 students have obtained worse result in the post-test than in the pre-test (right graphic), but this can be biased because two of them showed no interest in doing right both the pre and the post-test. Furthermore, the average in the pre-test is 10.04 and 11.56 in the post-test, the median is 10 and 12 respectively and the standard deviations are almost the same (3.5 and 3.6). All this implies that their spatial visualization ability is better, which is the main objective of this teaching experiment.

On other hand, if we count the students who have done successfully each task in pre-test and post-test (Figure 4, left graphic), we can see that, apart from task 1A and 5A, this number has increased, implying again that the students have developed their spatial visualization (see Figure 4, right graphic).

CONCLUSIONS

As we have shown along the paper, we have designed and implemented a new didactical proposal to work visualization with preservice teachers of childhood education. The main part of the design is the use of common materials and games, which normally are known by the students, but with the main purpose of focusing their attention in activities related with visual skills. The activities have the same structure, beginning with an experimental part with verbal and

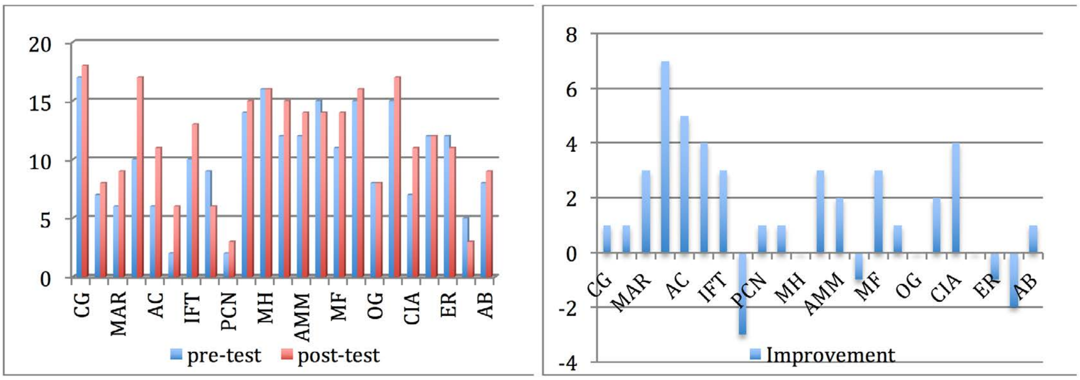


Figure 3: Quantitative results in pre-test and post-test about total punctuation of spatial visualization and the improvement about it (left and right respectively)

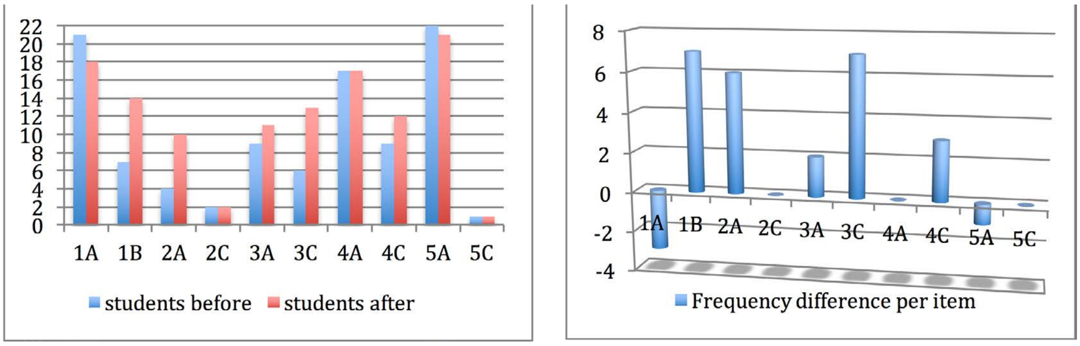


Figure 4: Number of students with correct answer per item and the frequency difference after the experiment (left and right)

graphical descriptions; later, they ask for a hypothesis about a question and a verification of the veracity of the hypothesis and finally end with a question about the description of mathematical properties and concepts involved on it. Regarding to results, our students have significantly improved their spatial visualization.

This new perspective allows us to go from a simple activity to abstraction, opening a new research line where Realistic Mathematics Education is the basis of the methodology for designing new teaching experiments, where the goal is to evaluate the impact and the possible improvement of mathematical competencies.

ACKNOWLEDGEMENT

The authors of the paper thank the financial support given by the research project EHUA11/13 of the University of the Basque Country.

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What questions do mathematics mentor teachers ask?

Raymond Bjuland, Reidar Mosvold and Janne Fauskanger

University of Stavanger, Faculty of Arts and Education, Stavanger, Norway, raymond.bjuland@uis.no

Lesson study was originally a professional development initiative from Japan. In some of the previous attempts to introduce it into initial teacher education, the role of the teacher educator has been highlighted. In this study, we analyse the questions that two mathematics mentor teachers ask in mentoring sessions, one in a lesson study intervention and one in a regular practice period in teacher education. Our findings indicate that the mentor teacher's questions in the lesson study intervention were more focused on planning, observation and pupil engagement and less focused on a deep understanding of the mathematical content.

Keywords: Lesson study, initial teacher education, mentoring sessions, questioning.

INTRODUCTION

A main goal in teacher education is to develop reflective practitioners who are able to carry out the work of teaching mathematics with high quality and proficiency. Lesson study has a focus on teachers' critical reflection about the content and organisation of lessons in order to develop more high-quality teaching and learning. A decisive feature of lesson study is that groups of teachers conduct focused observation of lessons along with collection of data necessary to collectively analyse the lesson (Lewis, Perry, & Murata, 2006). Although originally used in systematic professional development in Japanese schools, lesson study has lately been applied also in teacher education (Hart, Alston, & Murata, 2011). When Murata and Pothen (2011) implemented lesson study in mathematics methods courses for student teachers, they devoted 8–9 weeks to preparation and lesson planning, and they also underlined the importance of the written guidelines in creating lesson plans as well as carrying out the entire lesson study. Dudley and Gowing (2012) argue that lesson study is relevant for student teachers because it enables them to learn from detailed micro-level practices and allows them access to the tacit knowledge of their experienced mentor teachers.

The teacher educator has an important role when lesson study is applied in teacher education (Potari, 2011). In the school-based part of initial teacher education in Norway, the mentor teachers have the role of teacher educators (Nilssen, 2010), and the interaction between the mentor teacher and student mathematics teachers is important in this respect. Recently, lesson study has been implemented in a Norwegian teacher education context, and the role of the mentor teacher is a natural focus of investigation. Initial teacher education in Norway is organised as a four-year bachelor programme with 20 weeks of field practice. For this research project, the student teachers were in their second year (fourth semester), preparing for a three-week long field practice.

In this paper, we focus our attention in particular on the questions posed by two mathematics mentor teachers in the pre- and post-lesson mentoring sessions. We approach the following research question:

> What kind of questions does the mentor teacher ask in mentoring sessions in a lesson study intervention compared with the questions asked in a regular period of teaching practice?

As our initial attempt to investigate this question, we analyse two cases: one mentor teacher and a group of student teachers in the lesson study intervention, and another mentor teacher with her group of student teachers in a regular teaching practice with no lesson study intervention.

MENTOR TEACHERS ASKING QUESTIONS

In a review of research on the role of mentor teachers in mentoring dialogues, Hennissen and colleagues (2008) found that mentor teachers are normally directive and focused on organising activities. This coincides with analyses of mentoring sessions in Norway (Helgevold, Næsheim-Bjørkvik, & Østrem, 2014). In this paper, we investigate the questions that mentor teachers pose to student teachers when mentoring them in connection with the teaching practice that is part of their initial teacher education. Asking questions is related to characteristics of mentor teachers with non-directive supervisory skills, highlighted as important for student teachers' learning (Hennissen et al., 2008).

Gadamer (2004) proposes that questions are imperative to the development of knowledge, but this requires that the questions are "true". He refers to questions where the answer is already settled as apparent questions; questions where the answer is not settled are referred to as true questions. Following Gadamer, it can be argued that mentor teachers should ask true questions, which guide the student teachers toward subject matter knowledge (SMK) and pedagogical content knowledge (PCK) in the mentoring sessions (Johnsen-Høines, 2011).

Posing questions that guide the student teachers toward SMK and PCK relates to Shulman's (1986) early categorisation of teacher knowledge. His categories have been important for the development of several frameworks for teacher knowledge. Shulman divided teachers' content knowledge into three domains: SMK, PCK, and curricular knowledge. PCK relates to instruction, integrating teachers' knowledge of content with their knowledge of pedagogy. SMK, on the other hand, relates to content knowledge only. Shulman's work, in particular PCK, has created debate and has given rise to new categorisations (e.g., Graeber & Tirosh, 2008), all including knowledge of instructional strategies and knowledge of pupils' understanding (e.g., Ball, Thames, & Phelps, 2008). Both types of content knowledge are highlighted as important for high quality teaching (e.g., Ball et al., 2008). A focus on SMK and PCK is therefore emphasised as important in mentoring of student teachers (Johnsen-Høines, 2011). Therefore, it is of importance to study the content of questions posed by mentor teachers while mentoring student teachers.

METHODS

The present study is a part of the larger TasS ("Teachers as Students") project. This project aims at investigating student teachers' learning in field practice through a time-lagged design experiment (Hartas, 2010). Two groups of student teachers from each of four subject areas (mathematics, science, English as a foreign language, and physical education) participated in a control group situation; the same number of groups participated in an intervention situation. The control group is referred to as the "business as usual condition" (BAU), and the intervention group is referred to as the "lesson study approach condition" (INT). In BAU, data collection included video observations from student teachers' planning lessons with their mentor teacher (pre-lesson mentoring sessions), from carrying out lessons and from mentoring sessions after carrying out the lessons (post-lesson mentoring sessions). In INT, data collection included video observations from pre-lesson mentoring sessions, from carrying out lessons, from mentoring sessions after carrying out the lesson for the first time, carrying out the lesson for a second time, and from post-lesson mentoring sessions. In mathematics, which is in focus here, data were collected from four groups of student teachers (two BAU and two INT groups) altogether. In one of the INT groups, the mentor teacher was replaced by a colleague during the lesson study cycle due to sick leave. In this paper, we therefore analyse the four mentoring sessions from one BAU group and one INT group. The duration of the mentoring sessions varied from 18 to about 46 minutes (see Table 1). Both the mentor teacher from the BAU group (we have called Rut) and the mentor teacher from the lesson study intervention (referred to as Ina) are experienced mathematics teachers.

The mentor teachers in the lesson study intervention participated in three workshops on lesson study. In the first workshop, the mentor teachers were introduced to important ideas concerning lesson study and the different phases of the lesson study cycle. The aim of the second and third workshops was to develop a draft version of a "Handbook for Lesson Study". Inspired by Munthe and Postholm (2012), this handbook provided suggestions for possible questions to ask throughout the lesson study cycle. More specifically, crucial elements of the handbook were to highlight questions that could help the student teachers to make a detailed lesson plan, emphasising careful planning and focused observations with a clear content goal for the research lesson. The handbook also stressed the importance of posing a research question for the research lesson. This research question would normally have a focus on increasing pupils' learning of the mathematical content. An aim of the planning and observation throughout the lesson study cycle is to answer this question. The aspects stressed in the handbook are closely related to SMK and PCK.

The unit of analysis is the mentor teachers' questions as posed in the mentoring sessions. The analytical approach is directed content analysis (Hsieh & Shannon, 2005), and the coding was inspired by important elements emphasised in the handbook. In order to increase the reliability of the coding, the first and third authors coded the questions independently. The codes were discussed and agreement reached in the few instances where there was a mismatch. The second author then revised and ensured that the coding was consistent.

Initially, we identified three main categories: *Observation*, *Planning* and *Other*; these were further split into sub-categories of questions (see Table 1, first column). The mentor teachers' questions that are related to more general comments, concerning observation in the classroom were coded as *Observation* (Obs). The subcategory Obs-Goal is related to the goal for the lesson. Obs-Content focuses on mathematical observations from the classroom. The subcategory Obs-Pupil highlights observations about pupils' learning. The subcategory Obs-Teaching is related to observations based on incidents from the student teachers' teaching in the classroom.

The questions that are related to the planning of a lesson were coded as *Planning* (Plan). The subcategory Plan-Goal focuses on the planning of a lesson with a more general focus on the goal. The subcategory Plan-Content is related to a focus on the mathematical content. The subcategory Plan-Pupil Engagement highlights questions, emphasising how the chosen problems or activities could lead to increasing or decreasing pupil involvement and motivation. Plan-Prediction refers to the subcategory of questions that is related to possible teaching problems or pupil difficulties that may arise in a lesson. The subcategory Plan-Teaching focuses on questions that are related to practical considerations about the organisation of the teaching activities.

The third category, *Other*, refers to questions that do not fit into any of the other categories, for instance, a question about practical issues and clarifications. Two particular consecutive questions, challenging the student teachers to reflect on what they have learned about the pupils' learning, were also included in this category.

In Table 1, MS1 and MS2 refer to the mentoring sessions before and after the first (research) lesson that was recorded; MS3 and MS4 refer to the mentoring sessions before and after the second lesson. The four student teachers of the INT group had written in their lesson plan document that the plan for the research lesson was to teach the pupils in this particular tenth grade class about algebraic factorisations and how to simplify algebraic expressions. In the BAU group, the three student teachers' plan for the first lesson was to teach the pupils in this eighth grade class about the equal sign and the unknown, helping the pupils understand the balance in a simple equation like box (x) + 3500 = 5000. In the second lesson of the BAU group, the student teachers planned to teach the pupils about the difference between an unknown and a variable.

RESULTS

In our analysis, we mainly focus on the qualitative differences between the two mentor teachers' questions. When applying content analysis, however, a combination of counting the frequency of particular words or content and a more qualitative interpretation of the content is often used (Hsieh & Shannon, 2005). In our study, the counting of frequencies was useful to discover patterns in the data that were later investigated more qualitatively. Table 1 displays the comparison of relative frequencies of the different categories of questions posed by Ina and Rut.

Planning for pupil engagement

In lesson study, there is a focus on planning for pupil engagement, not only on how to deliver the content (e.g., Dudley & Gowing, 2012). Rut, the mentor teacher in BAU (Table 1, shaded columns), did not ask any questions related to planning for pupil engagement. She was more focused on questions related to prediction and planning the teaching. Ina, the mentor teacher in the intervention (Table 1, white columns), on the other hand, asked a number of questions about planning for pupil engagement. In the first mentoring session, when discussing pupils who were not active,

	MS1		MS2		MS3		MS4	
	BAU	INT	BAU	INT	BAU	INT	BAU	INT
Observation								
- General	0,08	**0,28**	0,07	**0,22**	0,10	0,05	0,00	0,07
- Goal	0,00	0,00	0,00	0,03	0,00	0,00	0,00	0,00
- Content	0,00	0,00	0,12	0,08	0,07	0,00	0,14	0,00
- Pupil	0,00	0,00	0,10	0,16	0,00	0,16	**0,28**	**0,17**
- Teaching	0,00	0,00	0,12	0,03	0,00	0,03	*0,27*	0,07
Planning								
- General	0,08	0,14	0,06	0,14	0,02	0,11	0,00	0,00
- Goal	0,03	0,07	0,05	0,03	0,00	0,00	0,02	0,10
- Content	0,15	0,10	0,05	0,03	**0,26**	0,16	0,08	0,00
- Pupil Eng.	0,00	**0,20**	0,00	0,11	0,00	0,14	0,00	0,07
- Prediction	**0,36**	0,10	0,07	0,03	0,10	0,05	0,02	0,03
- Teaching	**0,31**	0,03	0,16	0,08	**0,21**	**0,22**	0,17	0,00
Other	0,00	0,07	**0,21**	0,08	**0,26**	0,08	0,03	**0,50**
Duration	18:17	33:43	40:36	33:33	19:45	33:53	46:10	38:12

Table 1: Relative frequency of questions in the four mentoring sessions

Ina asked: “Yes, what do you think we as teachers can do then, in order to include these pupils?” Later in the same mentoring session, they were getting more practical and discussed tasks that are suitable for a diversity of pupils: “Do you feel these tasks are suitable for all levels? I mean, to start with what is known in order to move them along.” As a third example from the same mentoring session, Ina addressed the issue of motivating pupils to learn algebra: “Yes, how are you going to motivate all pupils to learn algebra? Because that should be the goal, right?” Later on she asked about the potential use of manipulatives and games to motivate pupils.

In the second mentoring session, after the teaching of the first research lesson, Ina asked: “Could you have done it differently in order to get more pupil engagement?” Later, when commenting on how two of the student teachers reported that they had ignored some initiatives from the pupils, she asked: “Do you think you managed to activate the pupils? Could they have been even more active, participated more, contributing more verbally?” Towards the end of this mentoring session, she commented on the diversity of pupils in the class, and she asked: “Looking back at this group of pupils, I’m thinking: how could we ensure that everyone is following us?” She returned to these questions in the next mentoring session, where the student teachers were planning for the second teaching of the research lesson. One of the student teachers commented that she would observe whether or not the pupils are active, and Ina challenged her on this: “Yes, you say that you will observe if they are active, but there are some pupils who never participate or say anything. How are you going to ensure that they participate?” This question was followed by some questions about how they were going to observe the pupils during the research lesson. This leads to another interesting difference between the two groups. In the intervention, the mentor teacher posed more questions related to observation, which is also highlighted in lesson study research (e.g., Dudley & Gowing, 2012).

Focus on observation

A feature of lesson study is that teachers use observation as a means to collect data in order to analyse a lesson (Lewis et al., 2006). We find examples of such questions about observation in the first mentoring session in the intervention group. Having grasped this aspect of lesson study, Ina challenged the student teachers with regards to observation: “How are you going to observe to see that the pupils actually learn something from the teaching, in relation to what you have been planning and the goals you have?” One of the student teachers responded that they are going to observe which pupils are active and which not. Ina followed up by asking: “Yes, but what about pupils who are not active in the lesson, who you don’t observe raise their hands, would that imply that they have not received anything, or that there is no learning?” When the student teachers responded that they plan to ask the pupils after the lesson, Ina continued to challenge them on this: “Yes, what do you think is the reason that they – if it appears that they have really learned something from the lesson – that they are not active; what is the reason for that?"

In the second mentoring session, Ina reminded the student teachers about what they had planned to observe, and she asked: "What issues were the observations supposed to focus on? [...] [W]hat questions did you pose yourselves there?" Later on in the same mentoring session, she asked: "How did you plan this thing about observation before [the lesson]? Did you ask yourselves how, what you were going to observe, how, and why?" When planning for the second teaching of the research lesson, in the third mentoring session, Ina returned to this and asked: "And how are you then going to see and evaluate whether or not the pupils learn?" She followed up with questions about how they were going to plan for pupil engagement, before she posed some more questions about how they were going to observe the pupils.

Rut also posed some questions concerning observation in the mentoring sessions in the BAU group, but her questions were different from Ina's. Whereas Ina focused on challenging the student teachers about how they were planning to observe, using observation as a tool to collect data (e.g., Lewis et al., 2006), Rut mainly asked questions related to observation in the sessions after the lessons. Her questions were mainly focusing on what the student teachers asked and what the pupils responded. As an example, they were discussing some observations from the first lesson (in the second mentoring session), and Rut said: "And then you underlined this and said: Yes, so you thought the opposite, how many ice creams do you need?" This question was related to an observation of pupils, but Rut was mainly referring to the student teacher's question rather than asking about the pupil's learning. Shortly after, she asked a question related to an observation concerning the content: "Is this a correct use of the equal sign?" This leads to another observed difference between the two mentor teachers. Whereas Ina asked more questions about planning and observation, Rut asked more questions that focused on content. This is interesting, since a deep reflection about content is emphasised in lesson study research (e.g., Murata & Pothen, 2011).

Focus on content

In the third mentoring session, prior to the second lesson, Rut asked several questions about how the student teachers planned to present the content. At the beginning of the session, she asked the following question to clarify the focus of the lesson: "So, the lesson is really about variables then?" Shortly after, she posed another question about the content: "Are you going to end up with some kind of definition, or are you only going to teach them about variables?" After having discussed this for a while, they started to discuss the concepts of 'unknown' and 'variable'. Rut asked: "Why do we use two different words [variable and unknown] in this situation?" When one of the student teachers responded, Rut continued to challenge them about the connection between these two concepts by asking: "Because we end up with letters in both, don't we?"

Later, in the third mentoring session, Rut posed some more questions about planning and observation related to content: "It is about an expression that contains parentheses, but we mentioned yesterday that we are missing a small revision really, about what a parenthesis really is? And then one of the pupils also asked: What is it really, what does a parenthesis mean?" She followed up by challenging the student teachers about the meaning of a parenthesis by asking: "How would we respond to that, what does a parenthesis really mean?"

In comparison, Ina asked fewer questions that focused on content. Her questions seemed less focused on challenging the student teachers about their understanding of the content and more focused on the pupils' learning of the content. Some examples from the second mentoring session, after the first research lesson, illustrate this: "If you had asked her [referring to one of the pupils] if she knew prime factorisation, do you think she would have known what to do then?" In the third mentoring session, she posed some more questions about prime factorisation, but this time with a focus on the planning of the second research lesson: "And then you mentioned about prime numbers, because in the previous lesson you started off directly with factorisation and didn't mention prime numbers a lot. And I observed that, and if this is what is important, that they should be able to do prime factorisation, then you should probably mention it?" Later she followed up by asking another question about their planning of this: "Yes, and then you are going to explain factorisation and reducing?" When comparing the two, it appears that Ina focused less on asking questions about the content than Rut, and when she did, her focus was more on how the student teachers were going to plan their teaching of the content in order for the pupils to learn. Rut was more focused on challenging the student teachers' own understanding of the content.

CONCLUDING DISCUSSION

Teachers who participate in lesson study are supposed to have a strong focus on specifying a clear goal for the lesson. In addition to this, a deep reflection about content as well as pupil engagement and prediction of pupils' responses is often emphasised in lesson study groups (Munthe & Postholm, 2012; Murata & Pothen, 2011). When implementing lesson study in teacher education, the role of the teacher educator has been stressed (Potari, 2011). In our study, the mentor teacher had the role of a teacher educator, and it is thus interesting to observe the differences in emphasis in the mentor teachers' questions in these two groups. We notice that the mentor teacher in the business as usual condition asked far more questions regarding the mathematical content and teaching (Table 1, shaded columns). The questions on teaching were, however, mostly related to practical issues such as organisation. Contrary to what one might believe, we observe that this mentor teacher also had more questions relating to prediction of pupil learning than the mentor teacher in the lesson study intervention. More specifically, the analyses of the BAU-sessions have revealed that the mentor teacher (Rut) posed several questions on Plan-Content and Obs-Content, and these questions have the potential of being true questions (Gadamer, 2004). We suggest that these types of questions, emphasising the mathematical content to be learned by the pupils in the classroom, could be an important affordance that might inform future implementations of lesson study (Table 1, white columns). Our analyses have also illustrated that these types of questions were not that prominent in the INT sessions.

Murata and Pothen (2011) emphasise the importance of spending a lot of time on preparation and lesson planning, discussing the mathematical theme in focus. The student teachers in this study did not have the same opportunity to work on their lesson planning for several weeks and future implementations of lesson study might consider this. The student teachers in our study had also finished their course work in mathematics before their period of teaching practice and the subsequent lack of involvement from the teacher educator at the university could have led to a lack of focus on the mathematical content in the lesson study cycle. The lack of time for preparation and lesson planning as well as the lack of involvement by the teacher educator could represent serious constraints for the lesson study implementation. It is also possible that the "Handbook for Lesson Study", developed during the TasS workshops, was too focused on more general questions related to planning and observation since it was designed for use across subjects. In that respect, the handbook could function both as an affordance (initiating pedagogical questions), but at the same time it could function as a constraint (too little emphasis on the subject matter).

Lesson study seems to have several benefits when implemented in the right way and one benefit is the focus on important aspects of SMK and PCK such as planning and student engagement, observation as a means to collect data in order to analyse a lesson (Lewis et al., 2006) and content. The handbook inspired by Munthe and Postholm (2012) used in this study provided suggestions for possible questions to ask throughout the lesson study cycle highlighting these important aspects of SMK and PCK. Our study indicates that implementing lesson study in teacher education is far from straightforward. The mentoring dialogues in the intervention had a stronger focus on planning and pupil engagement, as well as observation, but a weaker focus on content. The questions on content in the lesson study intervention were also more related to planning for pupils' learning of the content, whereas the questions in the business as usual condition had a stronger focus on the student teachers' own understanding of the mathematical content. Further studies are needed to investigate whether or not this is a general tendency, or if a stronger focus on the content can be achieved by revising the handbook.

ACKNOWLEDGMENT

The TasS project is supported by the Research Council of Norway (project number: 212276).

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Teachers' professional development in terms of identity development – A shift in perspective on mathematics teachers' learning

Marc Bosse and Günter Törner

University of Duisburg-Essen, Faculty of Mathematics, Essen, Germany, marc.bosse@uni-due.de

The theory of Communities of Practice (Wenger, 1998) provides a model of social learning whose assumptions differ from those made in the common theoretical frameworks of teachers' professional competencies. This shift in perspective on learning takes into account the concept of identity and connects it to the concept of practice. In this theoretical paper, we will clarify how teachers' learning works in terms of selected elements of Wenger's theory. Simultaneously, we will refer to empirical research findings on teachers' professional development and check whether they are in line with the theoretical considerations derived from the social perspective on learning.

Keywords: Teacher identity, teacher learning, professional development, identity development, communities of practice.

INTRODUCTION AND MOTIVATION

Facilitators of mathematics teachers' professional development are looking for effective ways of promoting teachers' learning. The term *learning* normally refers to a psychological-individual process of gaining professional competence, i.e., the capacity, skills or abilities to be more effective when teaching. A targeted aim of professional development programs is therefore not only to improve teachers' knowledge but also to change teaching practice on this basis and thereby to improve students' competencies (Lipowsky & Rzejak, 2012).

Schoenfeld (2011) considers that teachers' implementations of goal-oriented decisions – that make up an important part of teaching practice – do not only base on knowledge (and other resources) but also on context-dependent *orientations*. This summarizing term comprises beliefs, attitudes, dispositions etc. and can be regarded as a notion that describes the affective-motivational characteristics of a teacher's professional competence. In this respect, Schoenfeld's term *orientation* frames, extends and in particular refines belief theory, since especially beliefs are very often discussed isolatedly from the complex surrounding framework.

Current research on teachers' professional competencies (Döhrmann, Kaiser, & Blömeke, 2012) as well as guidelines for professional development programs (Deutsches Zentrum für Lehrerbildung Mathematik, 2013) include both, cognitive competencies *and* motivational-affective characteristics of professional competence in the respective theoretical frameworks. This extension also marks a progress since the older postulate 'teaching should be cognitively enhancing' lacked the axiom that mathematics lessons should also be emotionally inspiring.

In accordance to Zehetmeier and Krainer (2011), teachers' learning can also be described in terms of knowledge, beliefs, and teaching practice. However, they conclude that further categories like "in-school levels" and "beyond-school levels" have to be considered when analyzing teachers' professional development: Which roles do colleagues and principles play for teachers' learning? Which impact is induced by students and the teachers' experiences in the classroom? And how does personal experience beyond professional practice in school constitute hindering or fostering factors for learning?

Particularly in our studies dealing with the professional development of so-called *out-of-field* teaching mathematics teachers, i.e. teachers without formal qualification for teaching mathematics, we realized that considering cognitive competencies is not enough if we want to understand their actual professional practice (Bosse & Törner, 2014). In an ongoing analysis of interviews with this group of mathematics teachers, we can observe that identity-related aspects (e.g., experiences made with mathematics when the teachers were students themselves) influence their present teaching.

On this account, we want to change the perspective on teachers' *learning* from a psychological-individual transformation to a more holistic socio-cultural process in terms of *identity development*.

We will delineate teachers' learning as identity development as follows:

Firstly, we will give a short overview of the relevant literature locating teachers' learning as a social process in the field of theories of teachers' professional development.

Secondly, we will define the terms *identity*, *identity development* and *learning* by using the theory of *Communities of Practice (CoP)* (Wenger, 1998).

Thirdly, some important components of the CoP framework will be selected and discussed with the aim of explaining research findings and illuminating considerations of why mathematics teachers' professional development may be sustainable and effective or not.

Finally, we will critically reflect on our theoretical considerations and derive reasonable conclusions for designing effective professional development courses for mathematics teachers

TEACHERS' LEARNING IN COMMUNITES OF PRACTICE

The socio-cognitive approach and identity in practice

Rösken (2011) comes to the conclusion that recognizing professional development as identity development means integrating "personality as a relevant variable in the classroom" (p. 25). In the same way, Alsup (2006) postulates that teachers should be seen as intellectuals in a lifelong process of learning but not as technicians whose possible shortcomings in knowledge, beliefs, and practice have to be overcome via professional development programs.

This understanding of teachers' learning entails a *temporal-historic dimension:* Learning can be seen as the holistic process of turning an actual identity into a designated identity (Sfard & Prusak, 2005). Moreover, the notion of identity is directly connected to the role of *contexts* of teachers' learning. Beijaard, Meijer, and Verloop (2004) highlight that context is an important factor for analyzing and understanding identity development while Zehetmeier and Krainer (2011) even claim that communities and socio-cultural contexts have an impact on the sustainability of teachers' learning. Similarly, Beauchamp and Thomas (2009) come to the conclusion that information about contexts as well as communities and their influence on shaping teacher identities should be integrated into professional development courses.

Bohl and van Zoest (2002) have picked up the idea of using identity as an analytical and descriptive category of teacher development as well. In terms of our introduction, identity from their point of view is not exclusively located either "In-the-brain" or fully in the "Social". In fact, they assume identity being a "combination of aspects of self-in-mind and aspects of self-in-community" (p. 142).

Having this in mind, mathematics teachers' identities are therefore located within a socio-cognitive continuum: On the one hand, the aspects of self-in-mind relate to the areas of knowledge and beliefs, involving the domains of content, pedagogy and professional participation. This is in line with the perspective on teachers' professionalism and teachers' professional development in terms of competencies referring to individuals' psychological realms. On the other hand, Bohl and van Zoest (2002) consider identity as the "self-in-community", referring to processes of participation and perception within communities of practice.

In this paper, we want to focus on the social perspective of this mediating model. Of course, identity can be seen at various positions within the continuum, depending on the researchers' own identities (Grootenboer, Lowrie, & Smith, 2006). However, Grootenboer and colleagues (2006), reasonably sympathize with the idea that "identity is always connected to activity or practice" (p. 614), and *therefore* a "useful construct for understanding the formation of mathematical teaching" (Grootenboer et al., 2006, p. 614). We want to follow this assumption when illustrating teachers' learning under the notion of identity development.

In the same respect, Skott, Moeskær Larsen, and Hellsten Østergaard (2011) argue in favor of shifting the research perspective on teachers' learning from the "underspecified" concept of beliefs (Skott, 2013, p. 548) to the patterns-of-participation framework. By doing so, it is possible to consider the relationships between teachers' identities and their participation in social practices (Skott, 2013, p. 511) which naturally comprise both, teaching in classrooms and teachers' learning in professional development courses.

Wenger's theory of Communities of Practice

In the theory of Communities of Practice (=CoP) (Wenger, 1998), identity development can be positioned at the very 'social end' of the above mentioned continuum. By doing so, Wenger describes learning from two points of view: The perspective of practice, in which learners are characterized as individuals taking part in socially-determined, community-based practices, and the perspective of identity, in which learners are considered as individuals, each being a member of different communities.

The crucial point of the CoP theory is a complex process called *Negotiation of Meaning* (=NoM), comprising the process of participation and the interconnected process of reification. It is a fundamental statement of the theory that NoM does not necessarily involve direct interaction with others but is a constant process in terms of experience in everyday life. Therefore, CoP theory can be applied to all mathematics teachers who are able to take part in a specific process of NoM, even though they are not able to interact directly because of geographical distance. Certainly, the theory can also be used naturally when referring to the learning of a group of mathematics teachers who attend a particular professional development course.

CoP-definition of identity: The assumption is that identity is the fully lived and negotiated *experience* of engagement in the practice of a community. In this sense, the definition of identity can be derived from the extent and quality of participated experiences and corresponding reifications through the process of NoM. Accordingly, identity exists as a constant work of negotiating and therefore, identity is not a static entity but a constant "becoming".

CoP-definition of identity-development: Because of the definition of identity alone, the historic-temporal dimension of a development process is already considered. Identity in the CoP theory is neither a core of personality that already exists at a particular beginning of a development nor a final state. Identity development occurs when so-called *trajectories* are formed due to the participation in communities of practice. These trajectories connect the past, the present and the future; however, they are not predefined paths but continuous motions occurring through the negotiated experiences which are made in the process of NoM.

CoP-definition of learning: In CoP theory, learning is characterized as "the engine" of practice and practice is described as "the history" of learning. This means that practice as an ongoing, social, and interactional process of NoM goes hand in hand with learning processes. As a consequence, educational processes based on actual participation in practice are effective in promoting learning since they are "epistemologically correct". As a further consequence, Wenger (1998) underlines that there was a "match between knowing and learning, between the nature of competence and the process by which it is acquired, shared, and extended" (pp. 101–102). This assumption requires a competence term that basically differs from the individual-psychological one: Competence means knowing how to engage in practice, understanding the enterprise of a community of practice, and being able to use the shared repertoire of resources that is available to a community for using it in practice. In this sense, learning is becoming a member of a community by being familiar with its practices. However, since being a member of a community and being engaged in its practices is tantamount to the work of identity formation, learning and identity development are two interconnected, inseparable processes in the CoP theory.

MAPPING THE THEORY TO MATHEMATICS TEACHERS' PROFESSIONAL DEVELOPMENT

The coherence of a community and translations from community membership to identity

According to the theory, the practice of a community is based on three dimensions: *Mutual Engagement*, a *Joint Enterprise* and a *Shared Repertoire.* As a community membership is defined by these dimensions, they are important when newcomers are supposed to become a new member of the community or peripheral members turn into core members. Applied to mathematics teachers' professional development, the extent of how these three dimensions are proportioned is an indicator of how likely the practice in a training course actually becomes a practice of teaching in a classroom – or in terms of identity – of how likely participating in a course transcribes identity as a form of competence into a teacher.

The mutuality of engagement in this manner is a crucial factor for teacher learning (Goos & Bennison, 2008). Are the teachers able to interact with each other and take part in the process of NoM? Is it possible that the teachers are actually in a position to be involved in the practices desired by the teacher educators? Or is the training course designed in a way that the teachers have no opportunities to engage at all? Research shows that participant-orientation is an important and effective element of teachers' professional development (Clarke, 1994).

In the CoP theory, a community additionally bases on a joint enterprise, i.e. the negotiated goals and the corresponding ways to achieve them. Here it is also a matter of how and how far the members of a community are actually accountable for the enterprise. According to the theory, teachers attending a professional development course will be more likely to develop their identities by gaining new perspectives on interpretations, choices and values if they are actually included in the process of negotiation. This issue is discussed in literature as the question of "ownership" of professional development (Zehetmeier & Krainer, 2011).

Becoming a member of a community in terms of the theory also means that newcomers have to learn how to use and interpret the repertoire of practice (styles, artefacts, actions, tools, concepts, discourses, stories, etc.). Therefore, effective professional development programs manage to translate the shared repertoire of the training course into a personal repertoire as a feature of a teachers' identity. Only by doing this, the teacher is able to apply the repertoire to new practices outside of or peripheral to the practices of the course. Hence, the question is if there should just be exposure to the repertoire and the related practices or if the teacher should actually participate in the practice using the repertoire. Effective professional development is characterized by opportunities to try out practices in the classroom and by referring to the practices or cases of the teachers' actual teaching (Lipowsky & Rzejak, 2012; Timperley, Wilson, Barrar, & Fung, 2007). Therefore, one can further conclude that teachers need sufficient time during the professional development program in order to make use of the new repertoire of the course (Penuel, Fishman, Yamaguchi, & Gallagher, 2007) even if it is negotiated and not only presented.

In terms of the theory, mutual engagement, a joint enterprise and a shared repertoire are conditions for a coherent and robust community. Such communities or networks contribute to a sustainable and effective teacher learning. (Cochran-Smith & Lytle, 1999; Lipowsky & Rzejak, 2012; Zehetmeier & Krainer, 2011). If facilitators of mathematics teachers' professional development intend fostering collaboration in this sense, they should reflect on whether the three dimensions have actually been considered and how they can be integrated into the program.

Boundaries and multi-membership

Boundaries between different communities are defined by practice, not by institutional or organizational structures. All participants of professional development courses for mathematics teachers are teaching mathematics in school; however, they each have each experienced different practices of mathematics teaching in their individual professional life. It is therefore the usual case that a training course for teachers is an encounter between different communities whose respective members form a new community via practices negotiated within the frame of the course.

As a consequence, a professional development course implicates several boundaries between the practices of the respective teachers and a further significant boundary between the teachers and the teacher educators. The aim of a successful course in terms of the CoP theory is to allow crossing the boundaries by allowing the teachers to engage in the practice of other communities and especially in the teacher educators' practice. In this way, teachers should be able to use practices of other communities for the NoM at their own schools (Lipowsky & Rzejak, 2012). In other words, boundary crossing means the construction of an identity that includes different forms of practice.

On the one hand, this "multi-membership" requires the work of reconciliation: Teachers attending professional development programs must often cope with conflicting forms of practice, because they are supposed to develop identities and gain competences defined by teacher educators or even by outsiders. On the other hand, various forms of practice can offer new learning trajectories for the teachers and therefore constitute a basis for identity development.

The CoP theory considers two mechanisms of boundary crossing. Firstly, so-called *brokers* are able to transfer elements of one practice into another. In our case, the teacher educators themselves are brokers as they are able to translate, coordinate and align between different perspectives. Furthermore, teacher educators are capable of opening new possibilities for the process of NoM. The question is: Who is actually an effective broker, being able to allow boundary crossing and to promote identity development? In our opinion and with respect to the theory, successful brokers are teacher educators and mentor teachers who have strong experience in multi-membership themselves. This means that they have made experiences in teaching the content of the course in school as well as in educating teachers in the topic. This also means that these educators have experienced the content in a perspective of practice as well as in a perspective of theory. In conclusion, when it is not possible to find one single teacher educator who has experienced that multi-membership, the tandem approach could be helpful for initiating boundary crossing (Rösken, 2011, p. 81).

The second mechanism for initiating boundary crossing is the use of boundary objects like artefacts, documents, terms, concepts, styles and so forth. An object can become a boundary object if it allows different interpretations, each interpretation being based on the perspective of the respective community of practice. Such objects are particularly suitable for discussions: In which ways can a mathematical concept like derivations be taught? What kinds of activities are possible in order to teach the aspect of covariation between two variables? In which manner can the graphing calculator be a reasonable tool for teaching stochastic? Each answer and each corresponding perspective represents a specific practice and therefore individual experiences and identities teachers bring with them into professional development courses. We also know from literature that discussions and collaborative reflection are effective elements of teachers' professional development leading to processes of change (Zehetmeier & Krainer, 2011). Furthermore, students' documents, actually used textbooks, and videotaped lessons are suitable for being used as boundary objects in training courses since they connect the practice in school with the practice in the professional development program.

Modes of Belonging

As a part of his conclusion, Wenger (1998) introduces the three *Modes of Belonging* to a community and therefore having access to its practices: *Engagement*, *Imagination* and *Alignment*. Since engagement as an "active involvement in mutual processes of negotiation of meaning" (Wenger, 1998, p. 173) has already been discussed, we want to draw attention to the remaining two.

Imagination: This mode refers to the process of extrapolating from one's own experiences to future images of the world, future practices and future identities. Since imagination in terms of the CoP theory is not exclusively considered as an individual process but also as an integral feature of the collective process of NoM, imagination can be used as a source of identity development in a professional development program. In order to initiate processes of imagination, teacher educators should enable discussions and activities of self-reflection. By doing so, new learning trajectories are generated: How do teaching practices that are different from the experienced ones improve students' learning? What is it like to adopt teaching practices that vary from one's own teaching style? Lipowsky, Rzejak, and Dorst (2011) suggest that showing teachers videos of teaching practices that lead to an improved students' learning generate such trajectories. Teachers having observed these scenarios rather change their practices (Lipowsky & Rzejak, 2012) as they are able to see their own identities in the light of a different practice by means of imagination.

Alignment: Professional development programs usually intend aligning teachers to intended curricular requirements. Alignment in terms of the CoP theory comprises activities which lead to the adjustment of practices to broader structures and enterprises. In this context, Wenger (1998) emphasizes that the "challenge [...] is to connect local efforts to broader styles and discourses" (p. 186). In other words: Curricular requirements are commonly political or administrational specifications which are not in line with the mathematics teachers' needs on a local level in the sense of a community of practice. Hence, the task for the facilitators of teachers' professional development is to negotiate the different perspectives by "convincing, inspiring and uniting" (Wenger, 1998, p. 186). A means for doing this might be coordinating boundary practices and reconciling diverging perspectives with the help of brokers and boundary objects.

CRITICAL REFLECTION AND CONCLUSIONS

With respect to educational designs, Wenger (1998) claims: "The primary focus must be on the negotiation of meaning rather than on the mechanics of information transmission and acquisition." (p. 265). This is in a particular way in line with the postulation of avoiding deficit-oriented approaches in teachers' professional development: Instead of analyzing deficits and filling the holes in teachers' professional competence profiles, philosophies of teachers' growth and empowerment are named (Rösken, 2011). We want to join the attitude of valuing the teachers' practice and the experiences they make in their classrooms every day. Therefore, we have the opinion that the CoP theory provides a framework of learning that respects the teachers' individual backgrounds they bring along into the professional development programs. In addition to that, we have shown that the shifts in perspective on teachers' learning are in line with the empirical findings concerning the effectiveness of professional development programs.

Nevertheless, we still have to think about how our considerations concerning teachers' learning in terms of identity development might be implemented in practice. A next step in this sense could be pondering appropriate teaching methods based on Wenger's ideas of learning and respecting the specific situation in mathematics teachers' training courses.

To be honest: Modeling professional development is a never completed, challenging task which involves many, partly cultural and administrative variables. We have learned that what is working in country x may be an unrealistic measure in country y. Last not least any theory has to handle these eventualities. It seems to us that Wenger's theory possesses this potential; however, this quality finally complicates its structure. To put it in another way: Wenger's contribution is not just a straight and lean theory.

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Some characteristics of learning to notice students' mathematical understanding of the classification of quadrilaterals

Ceneida Fernández[1], Gloria Sánchez-Matamoros[2] and Salvador Llinares[1]

1 Universidad de Alicante, Alicante, Spain, ceneida.fernandez@ua.es, sllinares@ua.es

2 Universidad de Sevilla, Sevilla, Spain, gsanchezmatamoros@us.es

The goal of this study is to examine how prospective secondary school mathematics teachers learn to notice students' mathematical thinking about the process of classification of quadrilaterals. Findings point out that when prospective teachers identified the inclusive classification of quadrilaterals as a "key developmental understanding" (Simon, 2006), they modified what they considered evidence of secondary school students' understanding of quadrilaterals classification process.

Keywords: Noticing skill, classification of quadrilaterals, students' mathematical thinking, prospective teachers learning.

INTRODUCTION

Previous research focused on prospective teachers education has underlined the importance of learning to notice what it is happening in a classroom (an important teacher's skill) (Mason, 2002). Furthermore, research has shown that this skill could be developed in teacher education programs in some contexts designed ad hoc (Coles, 2002; Fernández, Llinares, & Valls, 2012; Sánchez-Matamoros, Fernández, & Llinares, 2014).

A particular focus of this skill is how prospective teachers notice students' mathematical thinking. Magiera, van den Kieboom and Moyer (2013) showed that prospective teachers demonstrated a limited ability to recognise and interpret the overall algebraic thinking exhibited by students in the context of one-to-one interviews. Furthermore, research has underlined the relationship between mathematical content knowledge (MKT, Ball, Thames, & Phelps, 2008) and the process of recognising evidence of students' understanding (Sztajn, Confrey, Wilson, & Edgington, 2012). For instance, Bartell, Webel, Bowen and Dyson (2013) examined the role mathematical content knowledge plays in prospective teachers' ability to recognise evidence of children's conceptual understanding. After an instructional intervention (based on lessons where prospective teachers had to examine many examples of students thinking), their ability to analyse children's responses improved. Fernández, Llinares and Valls (2012) indicated that discriminating between proportional and non-proportional situations was a key element in the development of prospective mathematics teachers' abilities to identify evidence of different levels of students' mathematical understanding in the domain of proportionality. Sánchez-Matamoros, Fernández and Llinares (2014) examined the development of prospective teachers noticing skill of students' mathematical understanding of the derivative concept. This study indicated that a key element in this development was prospective teachers' progressive understanding of the mathematical elements that students use to solve problems in the domain of the derivative. However, difficulties in developing the skill of noticing students' mathematical understanding make more research in this line necessary, and particularly, in the different mathematical domains. In this study, we are going to identify key elements in the development of this skill in the domain of the classification of quadrilaterals.

Learning how to classify quadrilaterals causes difficulties for secondary school students. These difficulties are related to the relationship between inclusive and exclusive classifications of quadrilaterals since students recognise the different quadrilaterals by means of prototype examples without considering the inclusion relations associated with the classification processes (De Villiers, 1994; Fujita, 2012). Inclusive classifications result when the application of a classifying criterion to a specific set creates subsets in which it is possible to establish an inclusion relation (hierarchical chain) among its elements. For example, in an inclusive classification of parallelograms, the square can be considered a special type of rhombus; while in an exclusive classification (partition) the square and the rhombus belong to separate groups. Understanding the role inclusive and exclusive classifications plays when classifying the quadrilaterals in order to define different types of quadrilateral (Usiskin & Griffin, 2008) is important in learning about students' mathematical understanding. In this context, understanding inclusive classifications and how they are related to the process of defining geometric figures can be considered as a *key developmental understanding* (KDU) (Simon, 2006) since the understanding of inclusive classifications implies a conceptual advance for students that enables them to understand inclusive definitions (for example, that a square is a special type of rhombus).

Taking these aspects into account, our research questions are: (1) how do prospective teachers use their knowledge of quadrilateral classification in order to identify evidence of students' understanding? (2) What teaching decisions do prospective teachers take in order to support the development of students' understanding?

METHOD

Participants and design principles

The participants of the study were six Spanish science graduates (mathematics and engineering) enrolled on an initial training programme that provided them the skills needed to teach mathematics in the secondary school (we will refer to the prospective teachers as PSTs). The programme included subjects such as school organisation, psychology of instruction, mathematics education and teaching practice in secondary schools. In the part corresponding to mathematics education, the prospective teachers were studying a subject aimed at learning the characteristics of secondary school students' mathematical understanding. This subject was taught for 4 hours a week, for 13 weeks, and focused particularly on students' understanding and how to select tasks that would promote a conceptual understanding. One of the topics was students' understanding of the classification of quadrilaterals.

The module focused on the classification of quadrilaterals consisted of 3 sessions of two hours, and the participation in a week-long online debate. The design of the module incorporated a socio-cultural perspective (the spiral of knowing, Wells, 2002) and considered four aspects: Experience, Information, Knowledge Building and Understanding. "Experience" is the prior knowledge that prospective teachers have constructed during their participation in learning and teaching situations. "Information" consists of our understanding (as a scientific community) of the quadrilateral classification processes (theoretical information) that we provided to prospective teachers. "Knowledge Building" is related to how prospective teachers engage in meaning-making with others in an attempt to extend and transform their understanding of a student's mathematical thinking and their own understanding of mathematics. Finally, "Understanding" constitutes the interpretative framework in terms of which prospective teachers make sense of new situations, that is, what they mobilise to identify students' mathematical thinking in order to anticipate and monitor student response, select and sequence tasks and make connections with students' responses.

In the module, firstly, PSTs had to answer a task where they had to anticipate, individually, the way in which students' answers to the classification problems reflected evidence of understanding and had to take decisions to promote students' understanding ("Experience"). Next, the teacher trainer presented information about the characteristics of the quadrilaterals classification process and about students' understanding of quadrilaterals classification (inclusive and exclusive classifications, De Villiers, 1994; Usiskin & Griffin, 2008), and discussed this with the

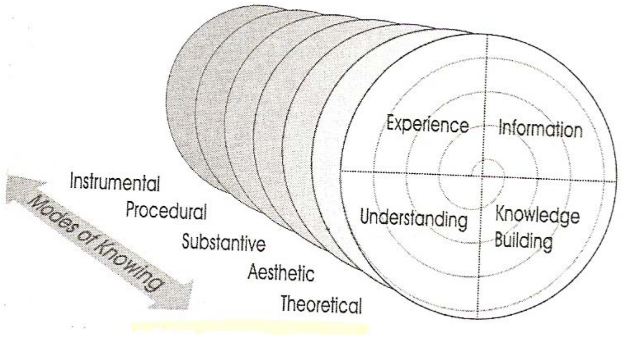


Figure 1: The spiral of knowing (Wells, 2002, p. 85)

PSTs ("Information" in the spiral of Wells). Finally, the PSTs compared their answers in pairs in order to explore differences and similarities in the way they recognised evidence of students' understanding of the classification process (Knowledge building) and take teaching decisions to promote students' understanding ("Understanding" in Wells' spiral of knowledge).

The task (instrument)

Prospective teachers had to answer a task consisted of two quadrilateral classification problems (ages 14–15) from secondary school textbooks (Figure 2), and six professional questions aimed at prompting prospective teachers to anticipate the response of students with different levels of conceptual understanding and propose tasks to improve their understanding:

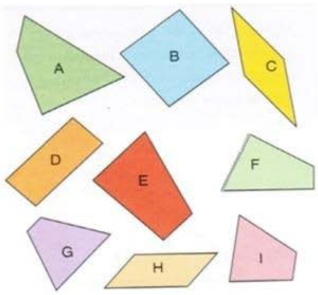
A1. Indicate exactly what Maria, a 3rd year secondary school student (aged 14–15), would have to do and say in each problem in order to demonstrate that she has achieved the learning objective assigned for the problem (Classify the quadrilaterals according to different criteria).

A2. Explain which aspects of Maria's answer to each problem make you think that she has understood the classification of quadrilaterals. Explain your answer.

B1. Indicate exactly what Pedro, another 3rd year secondary school student (aged 14–15), would have to do and say in each problem in order to demonstrate an understanding of certain elements of the classification of quadrilaterals while remaining unable to achieve the learning objective. Explain your answer.

Problem 1

Which of these quadrilaterals



have the properties described below:

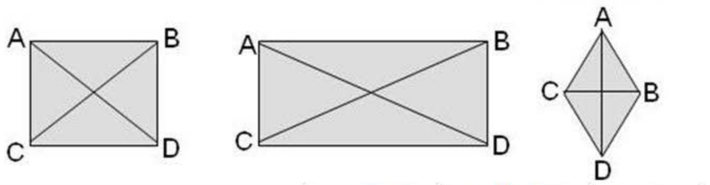
a. Only have two parallel sides

b. Have two parallel sides and two right angles

c. Their sides are equal, but their angles are not

Problem 2

2.1 Look at the figures and complete the table below, putting YES or NO in the empty boxes:



	Square	Rectangle	Rhombus
The four sides are equal			
The four angles are equal			
The opposite sides are equal			
The diagonals are equal			
The diagonals intersect at the middle point			
The diagonals form right angles			

2.2. Indicate the similarities and differences of the diagonals when comparing:

a) rectangle and square

b) rhombus and rectangle

c) rhombus and square

2.3. How are the parallelograms classified when using the diagonals as criteria? Justify the answer.

Figure 2: The two quadrilateral classification problems of the task

> B2. Explain which aspects of Pedro's answer to each problem makes you think that he has not achieved the intended learning objective. Explain your answer.
>
> C. If you were the teacher of these students,
>
> How would you modify/extend the task in order to confirm that Maria has achieved the intended learning objective? Explain your answer.
>
> How would you modify/extend the task so that Pedro achieves the intended learning objective? Explain your answer.

The first four questions refer to the teacher's ability to anticipate possible answers to the problems that reflect different levels of secondary school students' understanding of the process of classifying quadrilaterals. The last two questions (section C) are related to the teaching decisions; the decisions that the teacher should take in order to promote student progress.

The aim of problem 1 is to classify a set of nine quadrilaterals using three different criteria. The different sections of the problem could be solved by identifying the figures that met the criterion and grouping them together. The problem 2 requires the use of some elements of the geometric figures (sides, angles and diagonals of the square) to classify the parallelograms on the basis of the diagonals.

Analysis

We analysed PSTs answers to the individual task (Experience) and the modifications introduced when solving the task in pairs (knowledge building) on the basis of the theoretical information related to students' understanding of the classification process (Information). Initially, we focused on how the PSTs considered that the hypothetical answers they gave individually to problems 1 and 2 indicated different levels of students' understanding of the classification process. We then identified the decisions they took to help students consolidate or improve their understanding. In the analysis of the task resolution carried out in pairs we tried to identify how the previously discussed theoretical information modified what the PSTs understood as evidence of secondary students' understanding of the classification process (taking into account how PSTs considered the understanding of inclusive and exclusive classification as a key development understanding). We also identified modifications in the decisions taken to help students to improve or consolidate their understanding.

The data was analysed by four researchers creating categories. The initial categories were redefined as new data was added. Points of agreement and disagreement were discussed, with the aim of reaching a consensus on the inferences from the data by means of a process that looked for evidence that did or did not confirm the characteristics initially produced.

RESULTS

The results section is organised in two parts. In the first part, we identify the changes in how the PSTs characterised students' understanding and, in the second part, the changes in how they decided to support students understanding.

Changes in how the PSTs characterised students' understanding

The 6 PSTs, initially, considered the use of just one criterion as evidence of the understanding of the classification process: the one that generated exclusive classifications. For example classifications in which a square is a parallelogram but is not a special type of rectangle or rhombus. One PST considered as criteria whether the diagonals were congruent and if they formed a right angle (item 2.3 of problem 2). From this perspective the parallelograms formed three groups: squares (parallelograms with congruent and perpendicular diagonals), rectangles (parallelograms with congruent diagonals that do not form a right angle), and rhombuses (parallelograms with non-congruent diagonals).

On the other hand, PSTs considered evidence of an incomplete understanding of the classification process of parallelograms when the secondary school student was not capable of generating exclusive classifications. Anna, one of the PST, justified this fact by stating that:

> (the student with an inadequate understanding of the classification process) does (would do) problem 1 correctly but not problem 2. The student's error comes (would come) from considering that the diagonals of the square do not intersect at right angles – perhaps because of a

printing error- (referring to Figure B in problem 1), which means that the classification of item 2.3 only creates two large groups: those that have equal diagonals (square and rectangles) and those that do not (rhombus).

After the discussion of the theoretical information ("Information" in the Wells' spiral of Knowing) focused on the relations between the inclusive and exclusive classifications and definitions, PSTs started to consider the inclusive relation as a key developmental understanding when they tried to identify evidence of students' understanding. Putting the relation between inclusive and exclusive classifications in the focus of their noticing allows them to accept that a square can be considered a special type of rhombus or rectangle and this understanding should be considered as an advance in the student understanding.

For example, a pair of PSTs anticipated using a criterion based on diagonals (item 2.3 in problem 2, Figure 1) and the possibility of generating an inclusive relation. They used the criterion "congruent diagonals", which leads to a classification in which the rectangles and squares are in the same set and the rhombuses and rhomboids in another. Then, in both sets they considered the criterion "the diagonals intersect at a right angle". From this way, they obtained an inclusive classification that allowed them to define, for example, that "*a square is a rectangle whose diagonals intersect at a right angle*". To demonstrate an inadequate understanding of the classification of parallelograms, they considered that the student was not capable of generating inclusive classifications that could allow him/her to get definitions where a square can be considered a rectangle. So, for example, they evidenced the difficulties involved in handling inclusive relations assuming that the student could define squares and rectangles with no relationship: a square is a parallelogram with equal diagonals that form a right angle; a rectangle is a parallelogram with equal diagonals that do not form a right angle.

Changes in the PSTs' teaching decisions to support students' understanding

The changes in teaching decisions were linked to how PSTs understood the inclusive classification as a key factor in the understanding of classification processes. Initially, when PSTs considered that the understanding of the classification process was linked with exclusive classifications, they supported their teaching decisions in helping students to identify and apply classification criteria that generate singleton subsets. For example, to consolidate students' learning of the classification process, Anna proposed the following problem:

a) In problem 1 find a square, a rectangle and a rhombus, and check the characteristics in the table (section 2.1. of problem 2).

b) Analyse the characteristics in the table for all the figures in problem 1. What do you observe?

c) Give definitions for the parallelograms

Furthermore, to help better understanding of the classification process, Anna tried that the student was capable of recognising a second criterion "diagonals intersect at right angle" that produce an exclusive classification with singleton subsets. Anna stated:

> First I would ask the student about the differences between the square and the rectangle in their classification (observing the answers given in activity 2.1); and then I would make the student draw different squares with their diagonals and measure the angles with a protractor.
>
> If it is possible, in a computer room, I would also make sure that the student correctly draws and measures the angles the diagonals form with enough squares so as to be sure that they fully understand that this characteristic is common in all squares.

After the discussion of the theoretical information ("Information" in the Wells' spiral of Knowing) the PSTs understood that inclusive classifications were a key factor in the understanding of classification processes and they took different teaching decisions. For example, to consolidate understanding of the classification process in students that accepted inclusive classifications, Anna and Robert proposed *extension* activities aimed at establishing the equivalence of different definitions.

a) Analyse the characteristics in Table 2.1. in the figures in problem 1.

b) Give two different definitions of parallelograms

With students that hypothetically only generated exclusive classifications, they recognised that inclusive classifications are a key factor in helping them to understanding the classification of quadrilaterals. To this end, they designed tasks that serve, through analysis and reflection, to underline the possibility of seeing a figure as a special case within a larger group. For example, Anna and Robert modified sections 2.1 and 2.2 in problem 2, incorporating the figure of a rhomboid and asking *Classify Figure H in Table 2.1. Is it a parallelogram?* and *Classify the parallelograms into only two groups. Can you do it in another way?* In this way, in their view, the new problem 2 forces the student to choose classification criteria for the parallelograms in two groups and thereby creates an opportunity to generate an inclusive classification.

These changes in the PSTs' teaching decisions can be linked to the knowledge building (an aspect from the Wells' spiral of knowing) in which the inclusive classification was considered as a key developmental understanding in the task of attempting to recognise evidence of students understanding of quadrilaterals classification.

DISCUSSION

This research aims to examine some characteristics of PSTs learning. Our focus is how the changes in the way that PSTs understand the classification process of quadrilaterals influence on what they consider evidence of students' understanding, and on how they decide to assist students in their development. Initially, the PSTs only considered exclusive classifications, which implied that the definition of the different parallelograms was a succession of properties without the establishment of relationships between them. This meant that the way in which they were able to help students was to propose tasks in order to identify the largest possible number of characteristics in the figures, but without establishing relationships between them. The recognition of inclusive classifications of quadrilaterals as a key mathematical factor for conceptual development enabled them to modify the way in which they characterised students' understanding. This led the PSTs to focus their attention on helping students to identify relationships between the characteristics of figures as a means to generate inclusive definitions. To some extent, the prospective teachers set the development of the understanding of inclusive classifications (a "key developmental understanding", Simon, 2006) as a learning objective. For example, to define a rhombus as a parallelogram with four congruent sides and the opposite angles congruent two by two, and to consequently be able to define a square as a rhombus with four congruent angles. This result is in line of previous studies which shown the importance of prospective teachers' mathematical knowledge when attempting to interpret students' understanding (Fernández et al., 2013; Magiera et al., 2013; Sánchez-Matamoros et al., 2014; Son, 2013). To summarise, participation in this module enabled prospective teachers to recognise the understanding of inclusive classification as a conceptual advance in the development of classification and definition of geometric figures. As a consequence, prospective teachers identified the understanding of inclusive classification as a learning objective, recognising it as a necessary qualitative transition in the ability of students to think about and perceive relationships between elements of geometric figures. This recognition was demonstrated in the way they posed new tasks to support the understanding of classifications and the process of defining geometric figures.

Although more research is still needed to help us to identify the factors that influence the development of prospective teachers' noticing of teaching and learning, our data provide characteristics of the learning of knowledge needed to teach and its use in noticing evidence of students' understanding. Although the intervention might be considered short in terms of time, these results provide ideas that can help in the design of sequences of learning activities for prospective teachers, aimed to make explicit what they considered "key knowledge" when notice students' understanding and make teaching decisions in order to support students in their learning.

ACKNOWLEDGMENT

The research reported here has been financed in part by the Project I+D+i EDU2011-27288 of the Ministerio de Ciencia e Innovación (Spain) and in part by the Project I+D para grupos de investigación emergentes GV/2014/075 of the Conselleria de Educación, Cultura y Deporte de la Generalitat Valenciana.

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Pre-service teachers' growth in analysing classroom videos

Marita Friesen, Anika Dreher and Sebastian Kuntze

Ludwigsburg University of Education, Ludwigsburg, Germany, friesen@ph-ludwigsburg.de, dreher@ph-ludwigsburg.de, kuntze@ph-ludwigsburg.de

Analysing classroom situations belongs to the everyday requirements of mathematics teachers. From the perspective of professional development, empirical evidence related to the growth of this aspect of teacher expertise however is still scarce. Consequently, this study focuses on developments in the way pre-service teachers analyse classroom situations. In a first approach, we concentrated on pre-service teachers' analysis of how representations are dealt with in the classroom, which is a key for fostering students' understanding. The study examined whether it is possible to develop the pre-service teachers' analysis through a specific university course. The results suggest that the quality of the participants' analyses increased significantly according to several relevant criteria.

Keywords: Teacher education, video analysis, noticing, representations.

INTRODUCTION

What expert teachers should "see" in classroom situations does not only consist of noticing in the sense of identifying and describing relevant events in classroom interaction, but it also encompasses linking their observations to criteria, giving arguments and discussing the interaction against this background. As several studies have shown, differences in noticing between samples of expert and novice teachers (van Es & Sherin, 2011), it is widely accepted in the scientific community that a development in this aspect of expertise takes place over a long-term process of professional development. However, the base of empirical evidence whether pre-service teachers' analysis can be developed, in the framework of a one-semester university course, still needs to be broadened.

This study aims to respond to this research need by focusing on pre-service teachers' analyses of how representations are dealt with by students and their teachers in classroom videos. What we understand by *growth in analysing* is the pre-service teachers' development in analysing the use of representations in classroom videos and their use of theory as fostered by a university course. We build on prior work related to research methods (e.g., Dreher & Kuntze, 2013) and examine whether it is possible to develop the pre-service teachers' analysis through a specific one-semester university course (Dreher & Kuntze, 2012). To our knowledge, this is the first study that combines quantitative and qualitative methods in order to describe pre-teachers' growth in this specific context.

We will, in the following, introduce the theoretical background of this study (1), derive the research interest of this study (2), inform about the sample and methods (3), present results (4) and discuss their implications for theoretical and practical contexts (5).

THEORETICAL BACKGROUND

There is a broad consensus that expert mathematics teachers should be able to foster learners by providing them with challenging learning opportunities and help according to their needs. Both of these requirements can be considered to be closely linked to activities of analysing – in this case analysing learning opportunities and analysing the learners' thinking or potential difficulties, for instance. Such analysis makes use of professional knowledge (e.g., Shulman, 1986; Ball, Thames, & Phelps, 2008; Kuntze, 2012), for example, when referring to criteria that are necessary for the analysis. For the case of analysing classroom situations, the concepts *teacher noticing* or *professional vision* (van Es & Sherin, 2008; Sherin, Jacobs, & Philipp, 2011) describe elements of teachers' analysis of such situational contexts in the classroom. These elements are identifying relevant events for learning mathematics, describing the events from a criteria-oriented perspective as well as reflecting about their role and about potential implications connected with the analysis of classroom situations. These aspects make it almost obvious that noticing should be considered as knowledge-based (van Es & Sherin, 2008).

In the following, we would like to give an example of such criterial knowledge – and how this knowledge may enable teachers to analyse classroom situations – namely professional knowledge related to dealing with representations of mathematical objects. Representations stand for mathematical objects (Goldin & Shteingold, 2001), and they are the only possibility to approach these abstract and 'invisible' mathematical objects (Duval, 2006). This highlights the importance of dealing with representations in the mathematics classroom. Connecting different representations in the learning process can enrich a learner's concept image (Ainsworth, 2006; Lesh, Post, & Behr, 1987), however, changing between representations is often very demanding for learners and can hence also be an obstacle for the learning process (e.g., Duval, 2006; Dreher & Kuntze, 2015).

Expert teachers (cf., e.g., Kunter et al., 2013, for the notion of 'expertise') are expected to be able to use such criterial knowledge when analysing relevant classroom situations regarding the use of representations. For instance, whether representations are carefully connected in the classroom by explicitly emphasising how properties of a mathematical object can be seen in different representations, whether the learners are encouraged to reflect on representations, on their use and on changes between representations, or whether a learner's difficulty is connected with the interpretation of a specific representation (Dreher & Kuntze, 2015).

This discussion is mainly in line with the teacher-noticing framework (e.g., van Es & Sherin, 2008). However, diverse conceptualisations of teacher noticing can be found in the literature (Sherin, Jacobs, & Philipp, 2011), so the full spectrum of criterion-based analysing is not always reflected in what is understood by "noticing". Given this situation, it may be useful to include the perspective of *systematic observation* (Schwindt, 2008), in which analysing videotaped classroom situations comprises of aspects such as *describing, explaining* and *evaluating situations*, *highlighting* (*making focused and structured comments*), as well as *articulating critical incidents*.

Bringing together these approaches under the focus of using representations in the mathematics classroom, we define "analysing" in this study to be an interaction of the following processes: 1) Identifying relevant situations concerning the use of representations, which marks the "starting point" of an analysis; 2) evaluating such situations in a critical way based on connecting relevant situations and arguments with corresponding elements of theory regarding the use of representations; and 3) presenting/articulating the results of the analysis. These processes should neither be considered to be ordered nor completely separable, as there may be jumps between processes or simultaneous and interacting processes.

In order to develop the process of analysing with the pre-service teachers as described above, we carried out a one-semester university course. The core of this course was the analysis of videotaped classroom situations and the analysis of tasks from textbooks, both focusing on the learning potential related to using multiple representations. At the beginning of the course, key elements of theory (e.g., Duval, 2006) were introduced and the pre-service teachers developed criteria based on literature about dealing with representations. Such criteria concerned, e.g., changes between different representations (treatments and conversions); possible obstacles to students' understanding connected to representations; or the teachers' support in using and reflecting on representations. The criteria were used in two ways. First, as a framework for analysing and evaluating videotaped classroom situations as well as textbook tasks and second as a basis to further develop the analysed material regarding the use of representations. In case of the textbook tasks, the pre-service teachers were encouraged to change and enrich the tasks concerning the use of representations. In the case of the videos they were asked to conceive improved classroom situations regarding the use of representations. The course sessions provided the opportunity to share and discuss the ideas and material that the pre-service teachers developed throughout the course.

RESEARCH QUESTIONS

For evaluating whether it is possible to develop the process of pre-service teachers' analysing, the study aims at answering the following research questions:

- How do pre-service teachers who participated in a specific university course analyse videotaped classroom situations before and after the course?
- Is there a growth in linking relevant situations with criteria related to the use of representations?
- To what extent do the pre-service teachers examine the observed classroom situations critically against the background of corresponding elements of theory before and after the course?

SAMPLE AND METHODS

In order to answer these research questions, a video-based test was developed which comprised of two videotaped classroom situations and a corresponding questionnaire with different questions (see Table 1). The test was completed by 18 pre-service teachers (14 of them female) in a pre-post test design at the beginning and the end of a four-month university course (one 90-minute-session per week). The participants were all advanced or last-year students at Ludwigsburg University of Education, preparing to teach mathematics at primary or secondary schools.

The first video clip lasted about six minutes and shows a classroom situation with individual work. A student working on a problem in her book has difficulties creating three-digit numbers where the sum of the digits would be nine. The teacher asks her to draw three columns in order to represent the hundreds, tens and units on a place-value board. He then gives her nine chips, asks her to arrange them in the three columns and to tell him the number she had arranged. As the girl appears to have problems in translating the chips into numbers, the teacher shows her how to write down the number of chips under each column. At the end of the video, the teacher leaves the student with the instruction to arrange and write down more numbers.

The second video clip lasted about seven minutes and shows a whole-class activity where a teacher works out the solution to an age word problem with her class. First she lets the students read the problem in their book and they explain in their own words the relation between the age of a father and his son. Then she draws a table on the board to set up the variables and asks the students to label them with their definitions. At the end of the clip, the teacher tells the students to use the information from the table to set up equations and reminds them of important solution steps.

Although the two classroom videos seem to be quite different, they share important aspects concerning the use of representations that appear suitable to evoke pre-service teachers' attention and elicit analysing in this context. In both videos, the teachers use different registers of representations in order to support their students' understanding. However, both teachers miss important opportunities to offer their students help in dealing with those representations, for example, in the sense of translating between them, connecting them or reflecting on their use.

Table 1 shows the open-ended questions of the questionnaire the pre-service teachers had to answer at the beginning and the end of the university course. Item 1 aims at eliciting pre-service teachers' analysis without specifically prompting them to look at representations. Items 2 and 3 prompt the pre-service teachers to look at representations and motivate evaluations of corresponding events.

The answers to the open-ended questions were coded by two researchers, according to a top-down coding manual. The inter-researcher agreement was found to be good (κ = .83). For item 1, it was coded whether representations were mentioned, whether critical

Item 1	In what way was the students' understanding (not sufficiently) supported in the video clip? Please describe.
Item 2	Please evaluate the support given by the teacher regarding the use of representations.
Item 3	Please evaluate the support given by the teacher regarding the translation between different representations.

Table 1: Open-ended questions for each videotaped classroom situation

evaluations of the teachers' support were made, and whether there was a reference to elements of theory regarding the use of representations. Critical evaluations were defined as evaluations that also included negative judgements of the teachers' support as shown in the videos. Regarding elements of theory related to the use of representations, it was coded whether an answer was related to at least one of the following five aspects: support in using representations, translating/changing between representations, enhancing reflections on the use of representations, clarification why certain representations were used, and how learning was fostered by the way representations were dealt with.

Before answering items 2 and 3 in which the participants were prompted to look at representations (see Table 1), the pre-service teachers were given a definition of representations in the mathematical context in which examples of different registers of representations such as symbolic or iconic were mentioned. Accordingly, it was coded whether the answers contained both a reference to the corresponding elements of theory (teachers' support in using representations or translating between representations) and critical evaluations of the teachers' support.

RESULTS

We would like to start with two sample answers, which may also illustrate the coding process (see Figures 1 and 2).

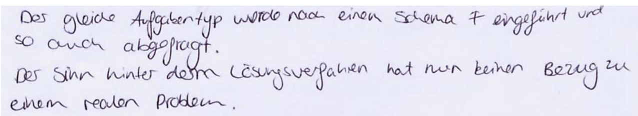
Already in the pre-test answer, the pre-service teacher in Figure 1 (see below) appeared to give critical evaluations, but these evaluations are not connected to representations and the way they are dealt with (e.g., "The way of problem solving has no relation to a real-world problem"). The criterion of using or dealing with representations is not referred to. However, in the post-test the pre-service teacher does not only attend to the representation registers from the video clip ("word problem", "table", "equation"), but also manages to analyse their use according to different elements of theory (e.g., "But what about the connections?"). Further, both positive and negative evaluations of corresponding events are given (e.g., "Chance unfortunately not optimally taken"), which was coded as critical evaluation.

Comparing the answers in pre-test and post-test, the analysis of this case suggests a progression in the quality of the analysis.

In the sample answer for item 2 (see Figure 2 below), the pre-service teacher's analysis of the video clip remains somewhat incomplete both in pre-test and post-test.

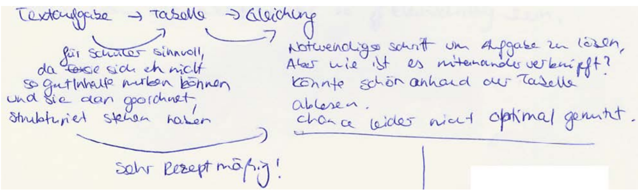
Although referring to the teachers' support in using representations in the pre-test as prompted by the question, there is no critical evaluation of this support. The pre-service teacher mainly sticks to mere descriptions of events (e.g., "student was asked to draw the place-value board herself; numbers were written directly under the board"). In the post-test, the pre-service teacher suggests alternatives for the teacher's action. The expressions used (e.g., "could have explained") suggest that the participant sees the video clip critically with respect to the suggest-

Pre-test



The same type of problem has been introduced before and is now recalled from the students in a very schematic way. The way of problem solving has no relation to a real-world problem.

Post-test



word problem → table → equation

useful for students as they often cannot keep all information in mind and so they get it sorted and structured	Necessary step to solve the problem. But what about the connections? Could be nicely done with the help of the table. Chance unfortunately not optimally taken.

Figure 1: Sample answer and translation for item 1, video 2, questionnaire #9

Pre-test	
• Schülerin sollte die Tafel selbst malen und Nägel hineinlegen • Zahlen wurden direkt darunter geschrieben um den Bezug zwischen Stellenwerttafel und Zahl zu zeigen • Anregungen durch Fragen: Sollen wir das mal machen?	— student was asked to draw the place-value board herself and to arrange the chips — numbers were written directly under the board to show the connections between the place value and the number — stimulating questions: Should we try to do this?
Post-test	
• L hat Material eingebracht • hat die Schülerin handeln lassen • hätte evtl. dieses Material ausführlicher behandeln & nutzen können	— he brought in material — [he] let the student take action herself — [he] could have explained and used the material — in more detail

Figure 2: Sample answer and translation for item 2, video 1, questionnaire #16

ed alternatives. However, the coding yielded no clear direct reference to the criterion of using representations beyond rather general remarks (e.g., "could have explained and used the material in more detail"). In the case of these answers, we could thus not detect a progression in the quality of analysis according to our coding categories.

On the basis of the coding of answers in the pre-test and post-test, we could count how often the corresponding codes were assigned to the pre-service teachers' answers. Table 2 shows the results of all 18 pre-service teachers for the first item by displaying the relative frequencies of the codes in per cent. The data mainly suggests an overall growth in analysing for this item. In the post-test, nearly all pre-service teachers mentioned representations when asked about the teachers' support of students' understanding in the video clips and there is also a growth in referring explicitly to the use of representations as analysis criterion.

Figure 3 shows the results for the code frequencies related to items 2 and 3. The data reveals a strong growth of the pre-service teachers' criterion-based analysing consisting both of the reference to the teachers' support in using representations or translating between them, and of the critical evaluation of corresponding

	video 1		video 2	
	pre	post	pre	post
mentioned representations	72%	94%	78%	100%
gave critical evaluations of the teachers' support	50%	67%	61%	39%
referred to the use of representations/elements of theory	50%	56%	50%	67%

Table 2: Code frequencies related to pre-service teachers' analysis in item 1

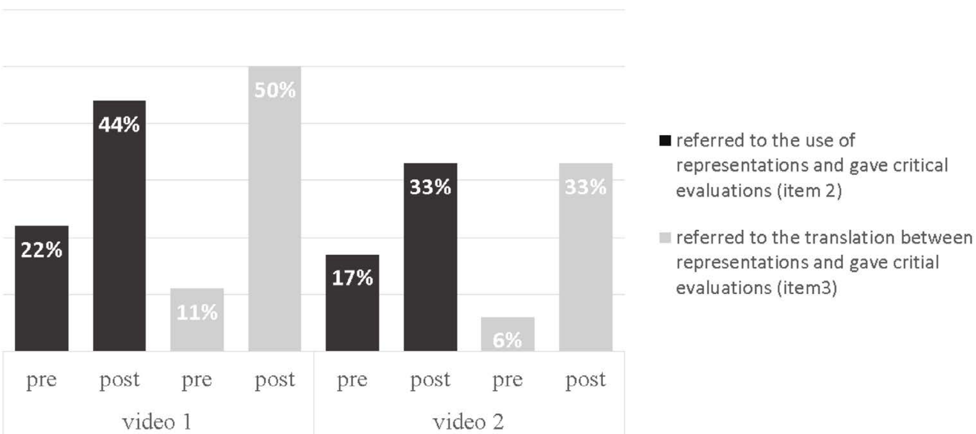


Figure 3: Code frequencies related to pre-service teachers' analysis in items 2 and 3

events in the video clips. However, even in the post-test, not more than half of the students carried out an analysis satisfying the code definition.

Concerning the pre-service teachers' connections to elements of theory, data analysis shows a growth in the number of references to *translations between representations* from pre-test to post-test (see Table 3). While the majority of the pre-service teachers mentioned only up to one translation in the pre-test, most answers contained two or even three connections to this element of theory in the post-test. Comparable to these findings, there was a growth in the number of references to additional elements of theory in the answers to items 2 and 3: nine pre-service teachers made additional connections to the elements of reflection or goal transparency in the pre-test, whereas 17 pre-service teachers additionally referred to elements such as reflection, goal transparency or the fostering of learning processes in the post-test.

In a further step in our data analysis, we made use of indicator scores, which can be seen as an exploratory first step towards quantitative forms of describing how criteria-based pre-service teachers analyse classroom situations. For this explorative quantification approach, code-based scores were created as follows.

In the case of the answers to item 1, one point was given if the answer mentioned representations. In this case, an additional point was obtained if not only the mere existence but also the *use* of representations was mentioned explicitly, and another point corresponded to the code of "critical evaluation given". Like this, the participants could reach a range from 0 up to 3 points for each video, resulting in a total score ranging from 0 up to 6 points. In the case for the answers to items 2 and 3, one point was given only if the answers contained both critical evaluations and relations to the teachers' support in using representations (item 2) or the teachers' support in translating between representations (item 3), resulting in a total score ranging from 0 up to 4 points.

Data analysis for the score based on item 1 showed that the pre-service teachers had on average higher scores in the post-test (M = 4.22, SE = 0.25) than in the pre-test (M = 3.33, SE = 0.38). This increase is significant and corresponds to a medium-sized effect (T = -3.06, df = 17, $p < .01$, $r = 0.60$, $d = 0.55$).

For items 2 and 3, the comparison between the scores yielded comparable results: the pre-service teachers reached on average higher scores in the post-test (M = 1.73, SE = 0.32) than in the pre-test (M = 0.53, SE = 0.19). This increase is also significant and corresponds to a strong effect (T = -3.52, df = 14, $p < .01$, $r = 0.69$, $d = 1.62$).

DISCUSSION AND CONCLUSIONS

Before discussing the results in detail, we would like to recall that the data has to be interpreted with care. We do not have data from a control group so far and the sample size restricts the possibility of making any broader generalisations from the results.

However, the research questions of the study could be answered and the findings show the pre-service teachers' development in analysing classroom videos throughout the university course. Even if not all pre-service teachers have reached the level of in-depth analysis with an extensive reference to aspects of relevant theory, the overall results suggest a deeper and more careful analysis of the videotaped classroom situations after the university course. On average, more pre-service teachers referred to representations in their analyses after the course, more critical evaluations of the teachers' support in the videos were given and the pre-service teachers made more connections between the classroom situations and criterial knowledge related to the use of representations. A particular growth could be detected in the pre-service teachers'

	video 1		video 2	
number of references to the translation between representations	pre	post	pre	post
0	67%	22%	28%	17%
1	28%	17%	50%	11%
2	6%	28%	17%	33%
3 or more references	0%	33%	6%	39%

Table 3: Pre-service teachers' connections to theory in items 2 and 3

analysis of relevant situations where the translation between different representations played an important role, which is a key to students' understanding.

At first sight, a possible explanation of the observed growth might be the pre-service teachers' increased awareness of representations and their use in the post-test. Taking a closer look, however, this interpretation of the findings appears too simplified. Already in the pre-test, the teachers were prompted to the criterion of dealing with representations in items 2 and 3, which was not reflected in their answers. Moreover, in the post-test, many pre-service teachers still were not able to analyse the classroom situations in depth as the corresponding coding only showed frequencies of up to 50% (Figure 3), indicating that the potentially higher awareness of dealing with representations was not sufficient for carrying out successful analysis. This underpins the significance of both components of the analysing process: the identification of relevant classroom situations as well as their critical and theory-based evaluation.

The quantitative results suggest that it is possible to describe the quality of teachers' analyses by quantitative measures and thus to tap into a key aspect of their expertise in follow-up research. Corresponding instruments are currently being developed.

For practice in teacher education, the findings clearly highlight needs of professional development. The analysis of classroom situations offers challenging learning opportunities for pre-service teachers and corresponding competencies should be fostered. This study can offer a first orientation to how criterion-based analysing of classroom situations could be the subject of a focused university course (cf., Dreher & Kuntze, 2012) in which pre-service teachers showed growth in this aspect of professional expertise.

We thus conclude, that pre-service teachers' growth in analysing classroom videos can be encouraged in the framework of a university course, but that this growth should be seen as one step in a longer chain of ongoing professional development.

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Mathematical tasks for preservice primary teachers

Mercedes García, Victoria Sánchez and Rocío Toscano

University of Seville, Department of Didactics of Mathematics, Seville, Spain, rtoscano@us.es

This paper tries to delineate principles and frameworks for task design in primary mathematics teacher education. The reflection process developed both for mathematics teacher educators and members of a community of inquiry led us to identify a four-step dynamic model relating theory and practice, which is used here in a specific part of mathematics teacher educators' practice: the design of tasks. The identification and characterization of two dimensions, related to what to teach during a course in mathematics for preservice primary teachers, allowed us to design tasks that seek to consider specific features of each one. For us, task design is a process in which mathematics teacher educators are involved through the development of their professional activity.

Keywords: Task design, four-step dynamic model, task design as a process.

INTRODUCTION

Some years ago, with the aim of discussing our ideas within our community of practice, we presented in CERME3 an ongoing research related to the knowledge and the learning process of preservice elementary school teachers. The objective of the research was to determine how student teachers used conceptual tools provided in mathematics methods courses (García, Sánchez, Escudero, & Llinares, 2003). The results were later published (2006) and provided us with a theoretical framework leading with the relationship between theory and practice (García, Sánchez, & Escudero, 2007). The study of learning processes led us to focus on the specific mathematical background that these teachers could need, leading us to focus on the key aspects of the mathematical activity. Six years later, in CERME6, we focused on the characterization of secondary students' justifications and their persistence (or not) when making decisions related to tasks that involve defining, proving and modelling, considered as metaconcepts that constitute a background to advanced mathematical thinking (García, Sánchez, & Escudero, 2009). Now, we focus on the design of tasks that can incorporate theory in practice and develop a specific teachers' mathematical knowledge.

THEORETICAL FRAMEWORK

The reflection process developed both for mathematics teacher educators and members of a community of inquiry led us to identify (in previous research) the 'skeleton' of a process of relationship between theory and practice (García, Sánchez, & Escudero, 2007). From there, we have been able to build the model schematised in the Figure 1.

In the figure, the model starts from the practice (P). As a first step (Step 1), we seek the theoretical aspects more closely related to the practical problem under consideration (TE) in the general theory (T). In a second step, we are aware of some implications of adopting the theoretical framework provided by TE in our practice. The analysis of this new way of considering practice led us to a third step in which, from the identification of practical problems, some research questions are formulated. In the final step (Step 4) there is a return to the theory, which is now enlarged.

This four-step dynamic model relating theory and practice is used here in a specific part of teachers' practice: the design of tasks. Design of tasks has been considered by several authors with very different aims. Special issues of journals (e.g., *Journal of Mathematics Teacher Education*, 10(4–6), December 2007); books (e.g., the edited collection by Clarke, Grevholm and Millman, 2009), or specific research projects (e.g., the QUASAR project see (Silver & Stein, 1996; Stein, Smith, Henningsen, & Silver, 2000)) have

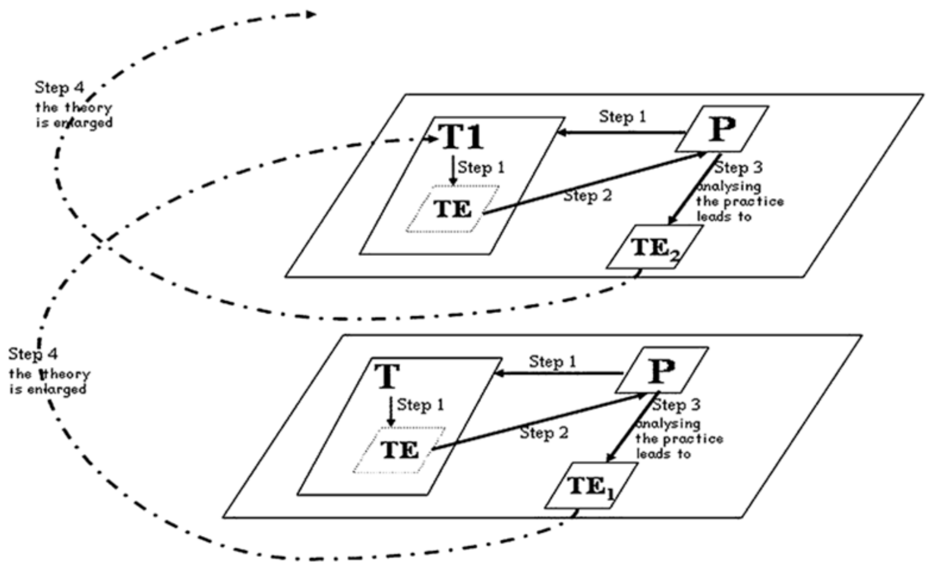

Figure 1: A four-step dynamic model relating theory and practice (García, Sánchez, & Escudero, 2007, p. 15)

emphasised the importance of the design of tasks in mathematics education.

Here we consider the design of tasks as a part of mathematics teacher educators' practice, focusing on the connection between theory and practice in this particular context.

SPECIFYING THE FOUR-STEP MODEL IN THE DESIGN OF PRESERVICE MATHEMATICS TEACHER TASKS

Starting from the practice (P)

Some years ago when we planned the organisation of our courses in our Primary Teacher Education program, we started to take into account the fact that, in the future, Primary Teachers will have to use mathematical ideas in the classroom from the perspective of mathematics and from consideration of them as teaching/learning objects. As mathematics teacher educators, we looked for key ideas that were the background to our decisions. We were aware of the need for theoretical perspectives that could be the base for our decisions. Thus, we looked to theory.

A look to theory (T)

At that time, researchers such as Brown, Collins, and Duguid (1989), Cobb (1994), and Lave and Wenger (1991) had started to develop theoretical ideas that have been summarized by different authors (García, 2003, 2005; Putnan & Borko, 1997). These ideas enabled us to understand the characteristics of cognition in an educational context. Our approach was rooted in some of them, in particular, the situative perspective (Brown et al., 1989), which allows us to characterize the activity of teaching mathematics, the specific knowledge and skills that are needed for this activity, and the learning processes that allow student teachers to develop this knowledge. At that point, we asked ourselves the following question: What should a primary teacher know about mathematics?

From general theory to specific aspects to be used (TE)

At that moment, we particularised the general theory (T).

We started by considering that primary teachers' mathematics knowledge should be related to contexts and situations where that knowledge was used, i.e., with activities, educational goals and contexts of teaching of mathematics. In addition, the identification of authors such as Shulman (1986), Llinares (1991), and Bromme (1994) of different mathematics teachers' domains of knowledge, allowed us to infer components of content of the curriculum of primary teacher education related to mathematics (García & Sánchez, 2002). In particular, we started to consider that a component of such a curriculum should be knowledge "of" and "about" mathematics (Ball, 1990). Thus, we went back to practice (P).

Going back to practice: designing tasks (P)

The identification and characterization of the two aspects in the component mentioned above allowed us recognition of specific features, such as problem solving and its consideration as a deductive science with empirical tools, and to incorporate processes, concept and procedures in its conceptualization. At that moment, we tried to design tasks that sought to consider these new aspects (García, Sánchez, 2002).

At that point, we asked ourselves the following question: Are these tasks appropriate to generate specific mathematical knowledge in primary teachers? That is to say, is the activity generated by the task valid for this purpose? We turned to theory (TE1), in search of some theoretical elements that allowed us to answer those questions. At that time, we reinitiated the process, moving on to a new plane and reformulating the practice question into a research question: What specific mathematics does a primary teacher have to know? What are the characteristics of that knowledge?

Starting another time

Assuming that, even with different characteristics and characterizations, and regardless of the theory that is taken into account, a knowledge of mathematics exists among the different domains of knowledge that are part of the knowledge base of mathematics teachers, from our point of view, this knowledge of mathematics has two subdomains, which are complementary as well as autonomous and necessary. On the one hand, there is a subdomain related to the particularities of mathematical content when it becomes a mathematical content to teach at the considered level. We consider this knowledge analogous to Ball's (2008) *specialized content knowledge*. On the other hand, we identify a knowledge related to the particular characteristics of the subject matter itself, which is essential to ensure the teacher can generate mathematical knowledge in his/her future students. We have called this knowledge *intrinsic mathematical knowledge*. To consider the features that *intrinsic mathematical knowledge* entails, we have taken into account the works of some authors that enable the identification of aspects that are indispensable in mathematical knowledge (Anderson, 1976; Hiebert & Lefevre, 1986; Lesh & Landau, 1983; Skemp, 1976; Thompson, 1985).

From general theory to other specific aspects to be used (TE1)

We focused on the characterization of mathematical intrinsic knowledge, considering it as a knowledge that considers mathematical elements from a dual point of view, integrating "operational"/"structural" aspects (Sfard, 1991), and also the mathematical competence of analysing said elements, allowing "packing and unpacking" of the mathematical elements. Other people who understand and use mathematics in other contexts and situations do not necessarily have to share a particular teacher's competence. Specificity adds to their knowledge and expands the common content knowledge in Ball's sense (2008).

For instance, when, from a mathematical point of view, we talk about operations, we may know, for instance, their meaning, use, properties, and algorithms related to them. A teacher also needs to identify the mathematical elements involved in these operations, as well as the role these elements have in the different ways of establishing relationships between, for example, different properties and different algorithms.

In some cases, intrinsic mathematical knowledge can be confused with the mathematical knowledge coming from other levels or contexts. The lack of recognition of the existence of that specific knowledge has led, on many occasions, to a repetition of topics corresponding to previous levels, or has contributed to generating a certain inaccuracy between what future teachers need to know and what their students have to know.

As researchers, this led us to look for the dimensions that must become operative in practice (Sánchez & García, 2008, 2009). These dimensions were:

- On the one hand, we consider activities of mathematical practice such as defining, justifying and modelling, among others (Rasmussen, Zandieh, King, & Teppo, 2005). These activities underlie any mathematical content and are part of what we consider to be "doing mathematics" (García, Sánchez, & Escudero, 2009).

- On the other hand, we consider mathematical content to be organized into areas, taking into account those that have traditionally been considered part of mathematics. These areas include, among others, analysis, geometry, algebra, statistics, and probability. The manner in which the content of these areas is considered will depend on the specialization of the teachers (primary or secondary school teachers).

In the Figure 2 we show the dimensions and the spaces generated by them.

Mathematical practices' activity / Mathematical content areas	Defining	Justifying	Modelling	....
Analysis				
Geometry				
Algebra				
Statistic/Probability				

Figure 2: Dimensions related to what to teach in a course of mathematics for primary teachers. (Sánchez & García, 2008, p. 286)

Designing new tasks (P)

As teacher educators, we will start again to put the theoretical ideas into practice, looking for new tasks that incorporate the above mentioned dimensions. In Figure 3, we include a task that we use as a starting point (basic task). This task corresponds to justifying in Algebra and is based on the Remainder Theorem (extracted from García & Sánchez, 2012).

We think that in these tasks preservice teachers could:

— analyse the different elements of a theorem carefully and in detail,

— think about the algebraic elements that intervene,

— identify axioms, definitions and theorems, and distinguish between them,

— notice that justifying is a characteristic of theorems, and not a characteristic of axioms or definitions.

In the process of solving the task, preservice teachers could:

— verbalize their ideas, recognize explicitly what they "see" and give reasons to support their comments;

— "unpack" characteristics and particularities that give form to a specific theorem and

— "pack" them only in a property that has been justified.

TASK

a) Give the axiom, theorem, or definition justifying each step in the following proof, indicating the mathematical elements that intervened.

Proof	What you say
a) $p(x) = g(x)q(x) + r(x)$	
b) the degree of $r(x)$ is less than that of $g(x)$	
c) if $g(x) = (x-a)$ as the divisor	
d) degree of $r(x)$ as 0	
e) $r(x) = r$	
f) $p(x) = (x-a)q(x) + r$	
g) setting $x=a$ in the above equation	
h) $p(a) = (a-a)q(a) + r$	
i) $p(a) = r$	

b) Identify premise/statement/proposition, and the different roles they play; state the property that is proven with that proof.

Figure 3: Task corresponding to justifying in Algebra (García & Sánchez, 2012, p. 25)

This process of unpacking and packing favours the characterization of the different parts of theorems and the role of premises, statements, and propositions.

From our point of view, this type of task can contribute to the generation of specific mathematical knowledge that allows preservice teachers to construct and deconstruct mathematical elements.

If we consider the task (Figure 3) as a 'basic task', its characterization as a task for concept development could be associated with the incorporation of the following questions: Can you give an example of any concept/idea in which they are part? Can you characterize any other concept/idea/mathematical property in which some of the identified elements take part? If our aim is a generalizing task, we can ask for the way in which the identified process can be generalized.

CONCLUSIONS

On the basis of the selection from the many different proposals related to tasks for mathematics teacher education that were sent to the above mentioned special issue of JMTE, Zaslavsky (2007) pointed out that "they provide insight into different theoretical foundations and discourses which are used to inform and justify choices in task structure, task presentation, and exploitation of learner experience arising from the consequent activity" (p. 433). She schematised the information obtained in a diagram, showing the dynamical nature of task design for teacher education that emerged from the analysis.

In this work, we have enlarged this information, considering in addition the design of tasks as a mean of connecting theory and practice in mathematics teacher educators' professional activity.

For us, task design is a process, not a product. It is a process in which a teacher is involved through the development of his/her professional activity, in which he/she is going to incorporate aspects originating from several sources: for example, his/her practice, interaction with colleagues, textbooks, and information from specialized journals. Nevertheless, unlike what happens in other scientific fields (for example, scientific research in the field of medicine), where results (once validated) are quickly incorporated into professional activity, the results of mathematical education research are not incorporated so readily into task design activities. Fostering their incorporation is our task.

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A prospective secondary mathematics teacher's process of developing a progressive incorporation perspective

Gulseren Karagoz Akar[1], Ali Delice[2] and Emin Aydın[2]

1 Boğaziçi University, İstanbul, Turkey, gulserenkaragoz@yahoo.com

2 Marmara University, İstanbul, Turkey

In this study, during a secondary mathematics methods and practice-teaching course, we investigated one prospective secondary mathematics teacher's, Denise's, development of a progressive incorporation perspective from a traditional perspective on mathematics, mathematics learning and mathematics teaching. We analyzed data from Denise together with data from classroom discussions with 28 prospective secondary mathematics teachers. Results showed that the awareness of the quantitative and numerical operations, tasks focusing on logico-mathematical and empirical learning processes, conceptual analysis, and clinical interviewing afforded Denise's shift towards a progressive incorporation perspective by the end of the study.

Keywords: Teacher preparation, prospective mathematics teachers.

INTRODUCTION

Preparing teachers to teach mathematics effectively is an ongoing concern for teacher educators. There has been research on mathematics teachers' knowledge (e.g., Ball, Thames, & Phelps, 2008) and beliefs (Ernest, 1989); the relation of teacher's beliefs to their practice (Beswick, 2005) and the role of methods and/or practice teaching courses on prospective teachers' knowledge and/or beliefs (e.g., Ebby, 2000). However, compelling evidence suggests that rather than focusing on prospective teachers' knowledge and beliefs separately, research needs to address the consistencies between knowledge and beliefs prospective teachers hold and practices they engage in so that they can '... learn to use their knowledge base to provide the grounds for choices and actions' (Shulman, 1987, p. 13) during methods and practice teaching courses (Ebby, 2000).

In this respect, some researchers have made important contributions regarding teachers' perspectives (e.g., Heinz, Kinzel, Simon, & Tzur, 2000; Simon, Tzur, Heinz, & Kinzel, 2000). These researchers postulated perception-based (PBP) and conception-based (CBP) perspectives on mathematics, mathematics learning, and mathematics teaching, and contrasted these with the traditional approach (see Table 1). Extending the results of these studies, Jin and Tzur (2011) also proposed a progressive incorporation perspective (PIP). These researchers argued for the need to investigate the promotion of teacher development from a traditional perspective towards a PIP. PIP is a conglomerate of thinking about mathematical construction both dialectically independent and dependent of the knower. The dependency relates to mathematics learning occurring through active mental participation of the knower and the dialectic independence relates to mathematical concepts having commonalities/differences amongst each other. In this respect, PIP regards mathematics teaching involving prospective teachers' frequent questioning (ibid) to take their students' attention to the commonalities/differences among mathematical concepts.

Informed by the aforementioned studies, done mostly with in-service teachers, and coordinating these with quantitative and numerical operations (Thompson, 1994), logico-mathematical and empirical learning processes (Simon, 2003), conceptual analysis (Thompson, 2008) and clinical interviewing (Hunting, 1997), the work reported in this paper is an attempt to go beyond understanding particular knowledge and beliefs in the context of the *practices of prospective secondary mathematics teachers (PSMTs)*. We investigated the following research questions:

How does a prospective secondary mathematics teacher develop a progressive incorporation perspective on mathematics, mathematics learning and mathematics teaching?

What practices in a secondary school methods and practice teaching course are likely to afford changes towards a progressive incorporation perspective on the prospective teachers' part?

CONCEPTUAL FRAMEWORK

Simon and his colleagues (2000) postulated a CBP on mathematics, mathematics learning and mathematics teaching and stated that it dwells on the basic principles of radical and social-constructivism. Therefore, *mathematics* knowledge, compatible with the Problem Solving view, is considered to be "a dynamic, continually expanding field of human creation and invention, a cultural product" (Ernest, 1989, p. 250). By the same token, *mathematical learning* occurs through one's transformation (accommodation) of existing ideas (assimilatory schemes) through their own logico-mathematical mind activities rather than empirical learning processes (Simon, 2003). So, *mathematics teaching* requires that; first, the teacher is aware of her/his current mathematical understandings being qualitatively different from her/his students' understandings (Jin & Tzur, 2011); second, a teacher focuses on what students currently *do* know rather than what they *do not* know (Heinz et al., 2000). Jin and Tzur (2011) stated that developing a CBP on teachers' part is a hard endeavour.

On the other hand, one important difference between a CBP and a PIP is that, in the latter, mathematics is also dialectically independent. This means that mathematics concepts have commonality in themself and it is the teachers' responsibility to bring such commonality to students' attention through focused and frequent questioning. In this regard, a teacher with a PIP might focus on both what the students *do* know and *do not* know. Though similar to a CBP, in this perspective, *mathematics learning* is an active mental process, and, therefore mathematics is also dependent on the knower. That is, although teachers engage in frequent questioning to point to commonalities among mathematical concepts, students are thought to have such ability to re-construct the mathematical knowledge on their own. Jin and Tzur (2011) stated '...a PIP-rooted teacher' practice can endenger students' learning processes envisoned by CBP without requiring the teacher's explicit awareness of such view...' (p. 20). It is in this respect that we think that *effective mathematics teaching* (Hiebert, Morris, & Glad, 2003) corresponds with a PIP and a CBP.

The difference of a PIP and a CBP from a PBP is that, in the latter similar to a traditional perspective, mathematics is an ontological reality independent of the knower, compatible with the Platonist view of knowledge (Ernest, 1989). Thus, *mathematics learning* means, coming to see a *first-hand* experience of mathematical reality shared by all through *discovery*. That is, mathematics is obvious to everyone in engagement in materials interconnecting it on her/his own. That is why teachers with PBP *do not* realize that their mathematics is different from their students mathematics. Also, they focus *only* on what students *do not* know. It is in this respect that researchers postulated a PIP to be more powerful than a PBP. They further stated '... Whereas, fostering teachers' progress toward a CBP was found rather challenging, it seems much more reasonable to expect teachers to adopt teaching stra-

	View of knowing	View of learning	View of teaching
Traditional perspective (TP)	Independent of the knower, out there	Learning is passive reception	Transmission; lecturing; instructor
Perception-based perspective (PBP)	Independent of the knower, out there	Learning is discovery via active perception	Teacher as explainer ('points out')
Progressive Incorporation Perspective (PIP)	Dialectically independent and dependent on the knower	Learning is active (mentally); focus on the known required as start; new is incorporated into known	Teacher as guide and engineer of learning-conducive conditions
Conception-based perspective (CBP)	Dynamic; depends on the knower's assimilatory schemes	Active construction of the new as transformation in the known (via reflection)	Engaging students in problem solving; Orienting reflection; Facilitator

Table 1: Placing PIP within Ernest's (1989) and Simon and colleagues' (2000) frameworks

tegies ...' (p. 20) corresponding with a PIP (Jin & Tzur, 2011). Jin and Tzur (2011) proposed the following table to refer to the differences among these perspectives

METHOD

Participants

This study was done in a teacher education program at a university in Turkey for which the language of instruction is English. Twenty-nine PSMTs (including Denise) participated in the study. They were in their fifth year of undergraduate studies. Denise had a GPA of 3.53, being at the first rank among her classmates, whose GPAs ranged between 2.35 and 3.53. Denise was eager to become a mathematics teacher. She was one of the prospective teachers who have attended classes regularly doing all the tasks. We used data from Denise because it was representative among data coming from the rest of the participants in the study and allowed for a full account on the development of a PIP perspective.

Data collection

Data were collected for a total of eleven weeks during the methods course and the practice teaching course taught by the first author. We used classroom teaching experiment methodology while collecting the data (Cobb, 2000). The first author planned the teaching sessions in advance. However, for each teaching session in the methods and practice teaching courses, the (sub)learning goals depending on the hypotheses about what PSMTs knew were revised. Each of the teaching sessions was videotaped and transcribed. PSMTs also kept weekly journals concerning in-class and out-of-class discussions, due online before midnight of the same day of class. Also, a pre-assessment and two post-assessments, which collected their thoughts on learning and teaching mathematics and assessed their understanding of multiplication of fractions (the topic in the case study), were given at the end of the methods and the practice-teaching courses. Finally, PSMTs' practice teachings and an interview with Denise afterwards were videotaped and transcribed.

In the classroom teaching experiment methodology, prior to the enactment of the study, the teaching sessions are planned in advance. For instance we conjectured that once the PSMTs knew the nature of mathematics, this could help establish the nature of mathematics learning (with understanding) towards a PIP: we used Thompson's (1994) theoretical framework on quantitative and numerical operations, aligned with a CBP and Problem-Solving view of mathematics (Ernest, 1989). For that, prospective teachers read the understanding of improper fractions provided and pointed to the quantitative operations. For the out-of-class activity, they considered the understanding of exponential numbers in three ways.

Similarly, to assist prospective teachers to develop the view of mathematics learning occuring through the abstraction of the regularities in their own logico-mathematical activities, such as subdividing, matching, etc., we brought Simon's (2003) logico-mathematical and empirical learning processes to their attention. Additionally, we included the conceptual analysis framework Thompson (2008) proposed for teachers to provide lenses through which they could analyse a case study (Stein et al., 2000) because it created an opportunity to reflect on both the teachers' and students' focus of attention (Karagöz Akar, 2010). Also, researchers argued for the addition of clinical aspects of teaching (such as interviewing) to (prospective) teachers' knowledge repertoire. Thus, we brought clinical interviewing (Hunting, 1997) to the PSMTs' attention. Finally, prospective teachers did peer-teachings and practice-teachings and reflected on them (Hiebert et al., 2003).

Data analysis

We read each of the data sources line-by-line looking for Denise's explanations for the questions in the within and outside classroom activities regarding her perspectives on mathematics, mathematics learning and mathematics teaching. Using the characteristics of teachers' perspectives given in Table 1, we looked for both Denise's existing and developing meanings on them. For instance, after working on Thompson's (1994) framework, Denise provided the articulation of the exponential numbers. This made us realize that she was able to think through students' mind activities. Similarly, once we had spotted the changes in her meanings in any of the data sources, we also checked other data sources that chronologically provided us further evidence on such change. Then, we went back to the whole data set to challenge our conjectures and check the chronologies. When our conjectures were challenged we modified them to cohere with the whole data. This way we were able to write the narrative on data from Denise.

FINDINGS

We show data in five sections regarding Denise's views on the nature of mathematics, mathematics learning and mathematics teaching.

Section 1 – The beginning: Prior to any instruction, Denise answered the written pre-assessment questions. Data showed that she had a traditional perspective with similarities to a PBP. One prompt was: 'Please draw a model of what effective mathematics teaching means and explain the components'. Denise drew the following:

The first picture represents a teacher. Text beneath: "guides, helps, encourages to ask questions." The second picture represents a student. Text beneath: "research, ask questions, is curious." The third column reads: "Mathematics learning: The ability to analyze and synthesize. The ability to problem solve. Internalizes the mathematical concepts."

Denise's explanation for the components of effective mathematics teaching involves teachers' and students' roles, such as questioning and guiding. These practices Denise mentions seem to fit with a PBP.

To examine how Denise and her classmates think about how connections occur in conceptual understanding, during the first teaching session after the discussion on Van de Walle (2007, pp. 21–25), the first author asked them to think about it again. This generated the following discussion:

R: Okay guys. How do connections occur in conceptual understanding?

A: Could it occur when you remember something? When you see something you remember another thing and then you make a connection.

B: It happens when you observe something. Then you observe something else and then you connect them, relate them.

Denise: I think making connections is not only limited to making observations. If the student is making connections between what she knew before this is important too, the connections might be logical or not logical but what she knew before has an important place. It is like, there is a diagram on page 23 (referring to Van de Walle, 2007, p. 23). It is like showing earlier knowledge and the new knowledge and how we see them related in our minds. It is like making new chains into the already existing ones.

F: What if we learn something totally new? What will happen then?

Denise: Then, we put it somewhere else. For example, its place is somewhere totally different than others.

C: She puts it and then she makes the connection later on.

Denise: Yes, but when she cannot make a relationship this happens, when she makes the relationship then they are connected.

Denise's uttering "then we put it somewhere else" is important because when a "totally new" topic is learned, it might not be related to the students' current knowledge. This suggests that, for Denise and her classmates, some mathematics topics are connected, but others are compartmentalized. This indicates that Denise's perspective on mathematics deviated from a PBP, in which mathematics is viewed as interconnected. In addition, Denise agrees with her classmates that connections in learning occur through remembering and observing. Viewing *learning* this way also leads away from a PBP and towards a traditional one. To ex-

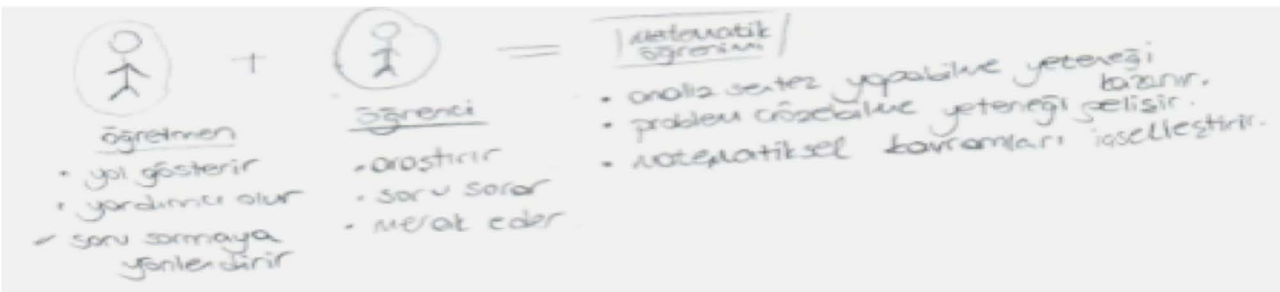

Figure 1: Denise's model for effective mathematics teaching

amine Denise's thinking further, we read her journal after this class. She wrote:

> A mathematical idea is to go one step further in the mathematical knowledge by making some logical inferences from the prior mathematical knowledge. ... Let us think about an example we discussed in class. Assume that a teacher wants to teach proper fractions. He/she gave the definition of the proper fractions and showed some examples about the proper fractions and their representations with shapes. Then a student realized that the shape of ¾ and 12/16 are the same. Then the student thought these two fractions are equivalent.

Denise's statement about a mathematical idea suggests that she views *mathematics* as interconnected. For Denise, *mathematics learning* proceeds with a) teacher giving a definition, (b) students recognizing from the example an equivalence (teacher showing with representations) and (c) students arriving at the conclusion that both fractions are equivalent. That is, for Denise, students *learn mathematics* through observation: a second-hand experience. These data also indicate that Denise thinks of *mathematics teaching* from a traditional approach, although she thinks of using representations.

Section 2 – Quantitative versus numerical operations: Data from this section showed that, after engaging in the activities regarding quantitative and numerical operations, there was a shift in Denise's reasoning on *the nature of mathematics and mathematics learning, the relationship between the tasks, and teachers' learning goals for their students*. Also, she was able to articulate a mathematical idea from a student's perspective. After working on a task during the class on the understanding of exponential numbers, Denise wrote: Figure 2.

Data show that Denise splits the quantity into identical copies, repeating the same process for the newly found whole until she reached the desired power. Also she finds the number of groups using the base as a measuring unit, showing that for her *mathematical understanding* is the result of one's mind activities, splitting and measuring. So, mathematical learning (with understanding) is dependent on the knower, suggesting a shift on her part towards a PIP. Denise also wrote the following.

Here, Denise has constructed a relationship among the teachers' *learning goals for their students, selection of effective tasks,* and *knowledge of students' thinking*: For Denise, once a teacher has a task focusing on quantitative operations, she has access to the quantitative/numerical operations, which in turn provides access to her students' thinking. This then assists in determining the learning goals. Strikingly, Denise started with the task rather than other components. Unfortunately, we failed to ask her why she chose to do so.

At this point in the semester, we discussed the meanings of logico-mathematical and empirical learning processes. First, PSMTs did the two tasks on equivalent fractions in Simon (2003) on their own to determine

I think, understanding the exponential numbers with positive number base requires a quantitative reasoning rather than a numerical reasoning. Now an exponential number with a positive base means for me that there is a positive number which is the base of the exponential number and for each increase or decrease in the power of that base means multiplication or division of the latest result by the base. Maybe it is more understandable to think in such a way: now every positive number base of an exponential number is a whole for me, say 1st whole. If I want to make it gain a power of 1, I should increase the number of 1st whole at my hand to the base many identical 1st wholes. Now, I get a bigger whole, say 2nd whole, than the 1st one. If I increase the power of the base to 2, then I increase the number of the 2nd whole to the base many identical 2nd wholes which forms another whole, say 3rd whole. For each increase in the power, I increase the number of the wholes at hand to form a bigger whole. Similarly, for any decrease in the power, I divide the whole into the positive number base, so I find base many smaller wholes that are identical. Each whole shows me the whole of the exponential number which lost one power. It is clearly seen that increase or decrease of the power causes an increase or decrease of the dimension of the wholes, but the amount of the changes are related the dimension of the whole at hand. There is an exponential growth or decrease in the amount and the positive number base is so important in determining the amount of this growth or decrease. Understanding exponential numbers means that understanding the relation between the geometric patterns and exponents by making firstly quantitative reasoning then the numerical reasoning.

Figure 2

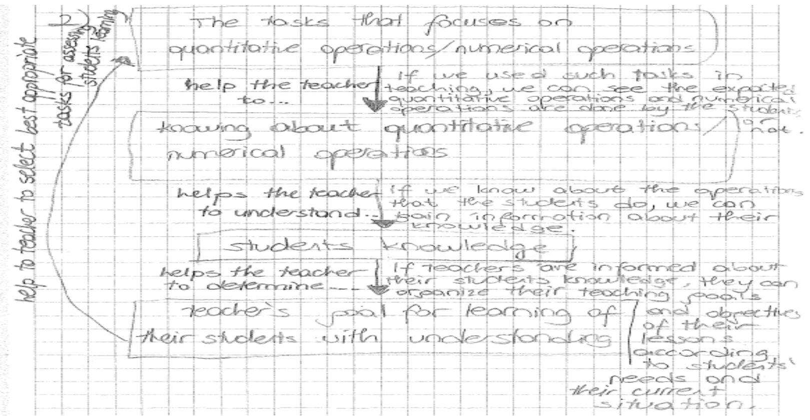

Figure 3: Denise's answer during within-class activity

whether these were cognitively high demand tasks. They concluded that the first task, which focused on finding equivalent fractions from the fraction chart, writing number sentences showing equivalences, and looking for patterns in the number sentences, concerned a task on empirical-learning processes. On the other hand, the task focusing on finding equivalent fractions based on re-drawing (re-partitioning) the given shaded rectangles focused on logico-mathematical learning processes. Then, in her weekly journal, Denise stated 'I think high cognitive demand tasks are tasks that require quantitative reasoning ability...' This suggests that Denise constructed a relationship between the quality of thinking in cognitively high demand tasks and quantitative operations.

Section 3 – Conceptual analysis: Data in this section showed how conceptual analysis might contribute to prospective teachers' awareness of the cyclic nature of effective mathematics teaching. We provide data from one week later after the discussion on conceptual analysis, pointing to the robustness of prospective teachers' knowledge of conceptual analysis.

R: Do you remember what we have talked about conceptual analysis last week?

N: I think you asked about the process of it. Who does it?

R: Yes. Can somebody say one more time what was it, what was the nature of it?

Q: It was like, while choosing the task, if she is choosing or if she is constructing, then we were doing the conceptual analysis. Like, we were thinking where the students might have difficulties, or might make mistakes, as teachers we were thinking how we can overcome those, help our students. Then, while the students are doing the task, the teachers were getting feedback from them. So, she would know how her students think. Also, she was getting feedback about how she could change the task the next time. So, it was like recursively going on this way.

Denise: She is looking at how much they know conceptually before the lesson too.

Strikingly, student Q stated that conceptual analysis can be done before, during, and after the lesson for different purposes, and Denise agreed. Also, the statement "So, it was like recursively going on this way" suggests that student Q is aware of the *cyclic* nature of the teaching. In addition, Denise's statement suggests that she is aware of a teacher's focusing on *what students know* prior to the teaching.

Section 4 – Clinical interviewing: In this section, data from Denise in the Clinical Interviewing Project was shown. Denise and her two group members chose radian, degree and the relationship between them as their topic to interview their classmates. We show Denise's analysis of her classmate, Veli, on the 4th question (see next page).

Our contention here is not to point to Denise's understanding of angle or radian; rather our emphasis here is *what she focuses on*. Denise's focus is *not just* on what her students do not know (he does not know what radian is). Rather, Denise focuses on *how Veli understands* it (he understands it as a length measure rather than openness, from her point of view), a characteristic of a PIP on mathematics teaching.

Then, prospective teachers did the Post-Assessment-2. We include the data to show that Denise's model for

Question 4: What is 1 degree? What is 1 radian?

When I asked him our fourth question, I saw that he could think better than a typical student might think. He thought 1 degree as the measure of the central angle which sees the arc length of the one of the 360 pieces when we divide the circumference of a circle into 360 equal pieces and making a similar connection with 2π radians and 1 radian, he said that 1 radian is the central angle which sees the $\frac{1}{2\pi}$ of the circumference, but he did not understand what 1 radian means. He had a misconception here, he thought that the measure of the central angle of a circle, which is 360 degrees or 2π radians, is equal to the length of the circumference of the circle and when he partitioned the circumference into 360 pieces or 2π pieces, 1 piece would be seen by the central angle of 1 degree or 1 radian. …the circumference of a circle could not be defined as degree, since degree is not a real number or is not a length measurement unit.

Figure 4

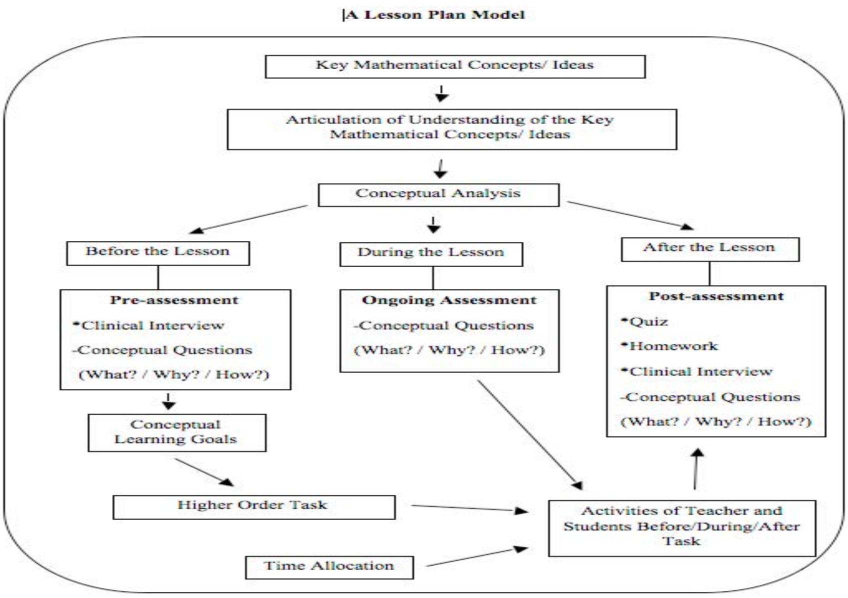

Figure 5

effective mathematics teaching changed at the end of the study. For the second question, Denise drew her model of a lesson plan for effective mathematics teaching.

For this model, Denise also provided her explanation as to why and how this figure models a lesson plan. For lack of space, we are not able to share her analysis. What is important is that Denise started her model with the key mathematical ideas that would yield her to learning goals for her students prior to the teaching. This would also allow her to choose higher order tasks she would engage her students in during the lesson through conceptual questions. This suggested a shift on her part towards a PIP since she was aware of the relationships between the components of teaching.

DISCUSSION

At the beginning of the methods course, Denise viewed mathematics, mathematics learning and mathematics teaching from a traditional perspective although she stated using representations and thought of some mathematical concepts as interconnected. After engaging in the quantitative and numerical operations her thinking has changed such that she was able to articulate a concept, exponential numbers, through her own mind activities. This suggested that she started viewing mathematics and mathematics learning as dependent on the knower. This suggested a shift from a traditional perspective towards a PIP. Also, working on the tasks focusing on logico-mathematical activities of mind, Denise came to know that cognitively high demand tasks require quantitative reasoning on the students' part. Also, Denise was able to think of tasks from a student's point of view. This indicated that she realized the differentiation of her reasoning from her students' reasoning. This was evident in her analysis of one of her classmates' reasoning on radian from the clinical interview project, in which she both focused on what her classmate did not think and how he thought. Denise was also able to realize the cyclic nature of the teaching process and showed it in her lesson model, given that she engaged in conceptual analysis, suggesting that awareness of the quantitative-numerical operations, logico-mathematical / empirical learning processes, the conceptual analysis and the clinical interviewing might afford changes in perspectives on the prospective secondary mathematics teachers' part. Empirical evidence showing the effect of engaging in aforesaid practices on a prospective teacher's perspective from a traditional approach towards a PIP represents a contribution to teacher development in terms of the design of methods and practice- teaching courses.

ACKNOWLEDGEMENT

This research is funded by Boğaziçi University Research Fund; BU-BAP- 6004.

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Dimensions of mathematics teaching and their implications for mathematics teacher education

Andrea Hoffkamp and Elke Warmuth

Humboldt-Universität zu Berlin, Department of Mathematics, Berlin, Germany, hoffkamp@math.hu-berlin.de

Developing the professional competence of mathematics teachers we need to consider different dimensions, for example, the content knowledge, the pedagogical content knowledge, and the pedagogical knowledge. Empirical studies in this area seem to provoke certain "trends" regarding the conclusions about "most effective" characteristics of a good (mathematics) teacher. Our findings with future mathematics teachers show that our students over-emphasize the pedagogical dimension while almost neglecting the importance of content knowledge. We analyse how the different dimensions interdepend and present exemplary learning scenarios for the education of future mathematics teachers focussing on the content knowledge dimension but, at the same time, combining it with pedagogical intentions derived from the special nature of our subject.

Keywords: Dimensions of competency, role of content knowledge, learning scenarios.

INTRODUCTION AND RATIONALE

The professionalization of mathematics teachers is still in the focus of politics and empirical studies like the international TEDS-M study (König & Blömeke 2012). In order to investigate and develop the professional competence of mathematics teachers different dimensions need to be considered. Describing the professional competence of teachers, Bromme (1997) distinguishes between *general pedagogical knowledge, content knowledge, curricular knowledge, the philosophy of the subject*, and *pedagogical content knowledge*. However the description of the complex structure of interweaving conditions between these dimensions is often missing or not addressed.

Especially since the study of Hattie (2009) there is a strong focus on the teacher personality:

> [..] the differences between high-effect and low-effect teachers are primarily related to the attitudes and expectations that teachers have when they decide on the key issues of teaching – that is, what they teach and at what level of difficulty, and their understandings of progress and of the effects of their teaching. This brings me to the first set of attributes [..]: passionate and inspired teachers. (Hattie, 2009, p. 26)

This led to a certain trend: the attempt to promote the pedagogical dimension within the teacher personality by describing and recommending only certain factors. For example, Anthony and Walshaw (2009) describe "characteristics of effective teaching of mathematics". In our opinion some of the descriptions are too little connected to the subject mathematics. Also the special (complex) relationship between the characteristics is not a subject of discussion, which implicitly implies that they can be treated independently.

Hattie also stated that the *content knowledge* dimension seems to have little effect on the quality of student outcomes. This can be similarly found in the study of Bromme (1997). By misinterpreting these results one might underestimate the dimension of content knowledge. Indeed Hattie concluded that "experts possess knowledge that is more integrated" and – of course – the content knowledge is an integral part of it (Hattie, 2009, p. 28). In fact Hattie (ibid) emphasizes the importance of formative assessment and feedback. Of course this requires a strong mathematical background of the teachers since the effect size of feedback referring to the subject or content appears to be high.

An opposite standpoint to the over-emphasis of pedagogical dimensions is the contribution of Wu (2005) – a professor of mathematics in Berkeley. He criticizes a "mathematics avoidance syndrome" at school and analyses how content "opens up the world of pedagogy and offers many more effective pedagogical possibilities".

Helmke (2012) criticizes that there are hardly any empirical studies investigating the professional quality concerning the content knowledge dimension of teaching at schools. Two very important works in this context originate from Ball and colleagues (2008) and Wittmann and colleagues (2001). Both refer to primary school mathematics. Based on Shulman's (1987) categories of teacher knowledge Ball and colleagues (2008) analyse the *content-specific dimension* detached from the *general dimensions* (like the pedagogical one). They characterise the subject matter knowledge, which is specific for mathematics teaching and differentiate it from the pedagogical content knowledge. With a different focus we find a similar approach in the work of Wittmann and colleagues (2001). They describe the *content knowledge as the core of mathematics teaching*. Moreover they develop the pedagogical dimension and the teaching methodology on the basis of mathematical ideas or content. In contrast to Ball et al. they particularly emphasize the role of metacognition ("consciousness"): teachers need to encourage the children to perceive the specifics of mathematics as a subject. This helps children to establish self-regulation mechanisms with regard to the subject.

In a deep theoretical analysis the educational scientist Gruschka (2008) also underlines the importance of content knowledge for teaching processes: "Teaching at school suffers by the shrinkage of content" (Gruschka, 2008, p. 73). In his work Gruschka often refers to mathematics teaching and reflects the role of content in a systemic way. According to Gruschka the professional competence concerning the content knowledge needs to be regarded within the complex of curriculum, pedagogy and philosophy of the subject. In more sophisticated words Gruschka (2008, p. 49) states that the unity of content knowledge and philosophy of the subject manifests itself in convictions about the pedagogical content dimension; whereas the pedagogical content dimension is determined by the anthropology of the students (as the core of pedagogical knowledge) and the attitude of the teacher towards the curriculum.

In our contribution we adopt the positions of Wittmann and colleagues (2001) and Gruschka (2008) for secondary mathematics teachers. We consider the whole complex of dimensions based on the specifics of our subject mathematics in an integrated way. By presenting examples we illustrate that the general dimensions, like the pedagogical or educational ones, are strongly connected to the content knowledge and philosophy of our subject mathematics. Our article is meant as a theoretical contribution to this topic containing illustrating examples. We also draw conclusions for the design of learning scenarios for mathematics teacher education at university. Following the above theoretical considerations we start with an analysis of some statements of future mathematics teachers in this context. These statements show that the beliefs about the dimensions of teacher competence have been shifted in a disadvantageous way.

DIMENSIONS OF TEACHER COMPETENCE FROM THE STUDENTS' POINT OF VIEW

To understand the context we shortly describe the situation of teacher education at Humboldt-Universität zu Berlin. At this university the future mathematics teachers learn their subject mathematics by attending mathematics lectures given by the mathematicians of the institute. Although these lectures are established for the future mathematics teachers only they are usually not practically oriented with regard to their future profession. As we will see later this is a dilemma. The pedagogical knowledge is acquired separately at the department of educational sciences. The pedagogical content knowledge is taught in seminars and lectures of the mathematics education group. The only courses in which the dimensions mentioned in the rationale are explicitly combined are some courses of the mathematics education group called "Stochastics and its pedagogy" or "Algebra and number theory and its pedagogy". Within these courses the content knowledge and pedagogical content knowledge dimensions are combined. Apart from these courses the dimensions are *not* taught in an integrated way. Especially the pedagogical dimension is nearly completely separated from the content knowledge.

The following statements and reflections originate from future mathematics teachers within a seminar in 2014 preparing the educational practical training phase (a 4-weeks period at school) in the master study program. Our students have to complete two practical training phases – one for each studied subject. Most of the students of this course will become secondary school teachers.

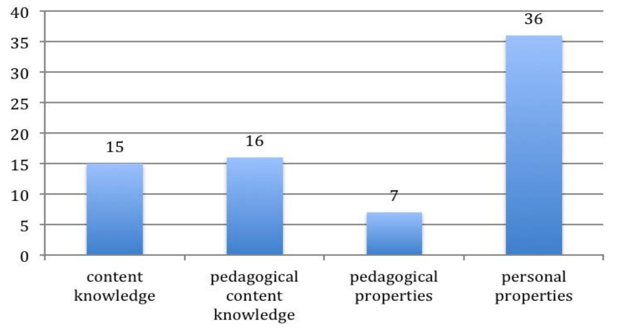

Figure 1: Students' answers to „Name the three most important characteristics of a math teacher": Absolute frequencies sorted by four categories

In the second session of the seminar we asked the students to write down the three most important characteristics of a good mathematics teacher. In the written answers of 24 students four categories could be identified: *content knowledge, pedagogical content knowledge, pedagogical properties and abilities,* and *personal properties*. The absolute frequencies of the answers are presented in Figure 1. Unexpectedly half of the students' answers could be attributed to the personal dimension. These were answers like *personality, fairness, patience, empathy, being not unforgiving, authenticity,* or *spontaneity*. Since the personal dimension is closely related to the pedagogical dimension, we were wondering about the students' beliefs about the connections and interrelationships between the categories.

Students' reflections about mathematics and pedagogy

Considering the mathematical content as the core of teaching we tried to find out which interrelations the students describe between the pedagogical and the content knowledge dimension. We were especially interested if the students deduce any pedagogical or personal factors from the content dimension. Therefore the students were faced with the results of the first questioning in the following way:

> "Comparing and clustering the characteristics named by the students of the course we see that most of the characteristics refer to personal and pedagogical abilities resp. qualities (see list of characteristics attached). On the other hand certain educational goals could be deduced from the list of characteristics.
>
> Please draw up your opinion to the following statements and write it down:
>
> 1) Without a firm mastery of the mathematical content, good pedagogy is impossible.
>
> 2) A firm mastery of the mathematical content opens up the scope for pedagogical actions and reinforces the pedagogical effectiveness."

The statements are adapted from Wu (2005, p. 7). Different from Wu the term "pedagogy" addresses mainly the pedagogical and personal dimension ("Pädagogik" in German) and not (only) the pedagogical content dimension.

The results of this questioning lead to more differentiated categories and allow a deeper insight. The following categories are specified by typical formulations of the students.

— *Content knowledge as the core of teaching:*

A firm mastery of the mathematical content leads to charisma, respect, self-consciousness and self-confidence of the teacher. Without a strong mathematical background a teacher cannot manage a class or will provoke discipline problems. Credibility and authenticity need a high level of mathematical competence.

> "Without a firm mastery in mathematics good pedagogy does not make sense, because the mathematical content and its mediation is the core of teaching and it has to be mediated by using pedagogical abilities."

In the statements within this category the students connect personal properties (self-confidence, charisma) and pedagogical factors (discipline, classroom management) directly with a high competence in the content knowledge dimension. In the students' view a strong mathematical background seems to strengthen personal and pedagogical skills that are important for teaching at school. However none of the students deduced personal or pedagogical properties or goals that are specific for the subject mathematics.

— *Mathematical content and pedagogy are separated areas:*

> "No, a firm mastery of the mathematical content and the scope for pedagogical actions are two totally disconnected 'construction sites'."

— Good Pedagogy is possible with low or no content knowledge:

"I am convinced of the thesis that a teacher with low professional content knowledge can cope well or better than teachers with high mathematical competence. I compare this with the study where an actor without any mathematical content knowledge gives a convincing talk about game theory in front of experts in this field."

This illustrates the dilemma mentioned at the beginning of this section. Mathematics lectures at university are not considered to be relevant for the teaching at school.

— High mathematical competence could hinder good teaching and pedagogy.

"I observed cases in which a high competence in mathematics hindered the establishment of empathy for the children, because these teachers were not able to imagine why and wherewith the children have problems."

The statements in these categories reflect the students' strong need for the human and emotional component of their profession. The implicitly mentioned aim of the students is: They want to be well received and appear likeable when teaching at school. In this sense the pedagogical dimension is over-emphasized by separating it from the content knowledge. Usually the students refer to their own experienced mathematics classes and want to do better. Therefore the students need to build their own pedagogical framework and system of values. We are confident that by considering the subject mathematics as the core for education in an integrated way, we could and have to support the development of the students in the above sense.

Analysis of a teaching professional's introduction of a new mathematics concept

With the following example we want to show how a teacher loses pedagogical effectiveness because of insufficient mathematical competence. We chose this example because it illustrates in an impressive way the interrelationships between the different dimensions concerning the professionalism of mathematics teachers. Besides the discussion of the interrelations we also show that examples like the following offer a valuable pedagogical potential for the education of future mathematics teachers.

One of our master students completed her 4-weeks practical training phase in which she had to teach and observe mathematics classes at school and write a training report. These reports consist of different parts. Aside from two lesson protocols and their reflection, the students describe their own lesson planning and resource development. The planning section especially contains an analysis of the taught subject by the student – as part of the content knowledge. About 30% of our students do *not* succeed in this part in their first attempt.

In her training report, she documented the observation of a "basic math course" in grade 13 (last year of secondary school). In the following protocol of the lesson, the student focussed on the methods of the mathematics teacher to introduce a new concept to the class – the concept of "expected value" in the case of the binomial distribution.

The teacher started with an extrinsic motivation: "What I will do next, will also be important for the next written exam." Having said this the teacher immediately moderates his statement by "But it is not that hard." The teacher uses an inductive approach to the concept and solely uses examples of the following type: "If I throw a fair dice 720 times. What do you think, how often will I get a 4?" The pupils answer "120 times" together with the reasoning "Well, there are six possible results. Thus 720 divided by 6 is 120." is accepted by the teacher with the words "Good, now the example with the tetrahedron. Whose example was that?". After four more examples of exactly this kind the teacher explains "Let's write this down in a mathematical way. $\mu = E(X)$ denotes what we expect. The number we receive is not a probability any more. The number usually doesn't lie between 0 and 1. We can also receive integers, for example if n is very large." After another example, which does not really fit to the binomial distribution, the teacher writes down the formula for the standard deviation by saying: "That's not difficult, you can just learn the formula." After that the class ends after 45 minutes.

This example reveals several dimensions of our subject matter. First the *content knowledge dimension*: The teacher does not have a conceptual understanding of the concept of "expected value" (*low content* *knowledge*). Therefore he *cannot act didactically*. He is not able to address the previous knowledge of his students and cannot use it for the development of the new concept. Therefore the teacher is *methodologically restricted* to direct instruction, since the content is not presented logically coherent and does not allow for pupil-centred methods. Also the teacher does not refer to the relevance of the concept for everyday life. Therefore he *cannot act educationally resp. pedagogically*. *Education to critical use of reason* would mean to discuss the significance and the misinterpretation of mean values as well as random fluctuations specific for stochastic phenomena. Expected values are a mathematical means for structuring and communicating. Since they reduce information they are supplemented by standard deviations. Their legitimacy as a teaching subject is only given if their relevance and limits are experienced (*curriculum dimension*). This is the prerequisite to educate mature people in an intellectually honest way.

Coming back to the education of future mathematics teachers, we take a look at the student's reflection of her protocol: The student criticized some methodological details and the abrupt termination of the class. From a mathematical perspective she did not have any objections. This example and the fact that about 30% of our students fail when working out a subject analysis is of course an alarming feedback for our education at university. It shows that we have to put more emphasis on the linkage between the content, the pedagogical content, and the pedagogical knowledge dimensions, and enable the future mathematics teachers to reflect on these linkages.

LEARNING SCENARIOS INTEGRATING DIFFERENT DIMENSIONS

As we analysed in the last section the different dimensions of teaching and learning mathematics depend on each other in a complex way. Since we cannot assume that the students achieve this view on their own, we need to offer substantial learning scenarios at university that allow them to actively deal with the dimensions in an integrated way. We want to present two exemplary learning scenarios following our theoretical considerations.

Using authentic material to educate reflective practitioners

As Gruschka (2008, p. 59 & 49) writes "if you want to understand teaching you have to understand the content dimension of the subject", and the core of pedagogical knowledge lies in the anthropology of the pupils. Therefore we start from the subject mathematics and use authentic material for the design of the learning scenario. The basis of the following activity is the protocol (presented in the last section) which is used as authentic material. The students receive the whole protocol as working material. The tasks should be worked out in small groups and afterwards discussed and reflected with the whole group.

Give-your-opinion!-task

a) Work out the definition of the "expected value E(X)" of a discrete random variable X with finite range of values. Which information of the distribution of X contains E(X), which information gets lost? Illustrate three different examples by using a graphical representation of the distribution of X.

b) Let E(X)=3. Interpret this value by switching from the level of mathematical model to the real world level. To which previous knowledge do you have to connect to?

c) Assess the approach of the teacher to introduce the new concept of "expected value". Do you agree with the teacher's given view of mathematics? Give reasons for your answer.

d) Give a sketch of your ideas for the introduction of the concept of "expected value".

e) How is the concept of "expected value" connected to the education to the critical use of reason? Where are the limits of the concept and its necessity to complement it by further concepts?

The above task combines the content knowledge dimension (a) as a necessary condition and basis for the following subtasks containing the pedagogical content and learning psychological dimension (b and d). Afterwards (c) the reflection of the situation and of the own view of the subject mathematics is required (philosophical dimension). The last subtask (e) refers to the pedagogical dimension and its connection to the mathematical subject matter. Particularly the pedagogical dimension in this scenario has the potential to develop – by *criticising the authority* of the teacher – *autonomy* and *critical faculties*, which are worthwhile pedagogical aims when teaching mathematics.

Including the metacognitive dimension

With the next scenario the content knowledge is combined with the philosophical dimension. This activity aims at the reflection of the special nature of the subject, since – as we analysed in the rationale – metacognition does play an important role when teaching mathematics and acting pedagogically. In a sense the following can be seen as a continuation of the first scenario, as it makes the *role of definitions within mathematics* a subject of metacognitive and philosophical discussion. The next scenario builds on the work of Hoffkamp and colleagues (2013). We already integrated it in a university course at Humboldt University and will briefly refer to our experiences.

> *Task 1:* (An exercise in defining in number theory.) Define the concept "even number". Also consider how you would define this concept at school and at university on different levels: primary school, secondary level and at the transition from school to university. Discuss the validity of the given definitions.

This seemingly simple task led the students to definitions like "the number 0,2,4,6 and so on", "all twosome numbers" (primary level), "all numbers that can be divided by two without remainder" (secondary level), or "the definition of divisibility leads to the description of the set 2Z" (university level). Then a lively discussion about the validity of the different definitions arose. Especially the definitions at primary level were not accepted by everybody as "being mathematical". With this task the students realized, that a definition is not necessarily unique, but depends on the mathematical context and purpose.

> *Input phase:* The students are confronted with the definitions and propositions of Euclid in the "Elements" (Book VII and IX): *An even number is that which is divisible into two equal parts. An odd number is that which is not divisible into two equal parts, or that which differs by a unit from an even number.*
>
> Using these definitions the following (simple) propositions of Euclid were deduced together with the students: *If as many odd numbers as we please are added together, and their multitude is even, then the sum is even. If an odd number is subtracted from an odd number, then the remainder is even.*

What the students experience in this part of the activity is that definitions change under historical conditions. They perceive the work of Euclid as the beginning of the axiomatic method and realize that Euclid's definitions are descriptive. They also realize that the proofs of the propositions "differ" from each other when using different definitions (like Euclid's or the modern university definition).

> *Task 2:* (An example from geometry.) Is it possible to decompose a square in two congruent parts? (from Fischer & Malle, 1985)

By discussing this task the students realize that the answer to the above question depends on our predefined concepts of "square", "decomposition" and "congruency". One can show that it is actually impossible to decompose a square into two congruent (and disjoint) parts. In fact a square could be mathematically described as a set of points in the plane. Then we have to ask: If we "cut" the square at the "center line", to which part do the "dots of the line" or the midpoint belong? Certainly mathematical definitions abstract from reality (of course we can cut a quadratic sheet of paper with a pair of paper scissors into two equal parts) and create ideal (mathematical) objects. Because of the idealization we need to proof our statements within our theory. The first two tasks lead to a sort of cognitive conflict: Both mathematical objects (even number and square) are familiar terms and the above difficulties are unexpected. This opens the way to discuss the nature and role of definitions from a metatheoretical point of view.

> *Task 3:* Give your opinion to the following statement: *Definitions are abstractions from reality following certain interests/purposes and change under historical conditions.*

Based on the previously made experiences the students discussed this statement philosophically in an explicit way. They started to emancipate from absolute truths and to reveal convictions about their subject.

They especially realized that – if each definition follows a certain purpose – this purpose has to be made explicit at school. This is strongly connected with the pedagogical dimension: as teachers we should take the pupils seriously as partners in a dialogue about mathematics and enable them to decide reasonably in a self-determined way.

CONCLUSION

In our article we analysed the dependency of the different professional dimensions of teaching mathematics forming an integrative entity. Based on our findings with future mathematics teachers we reasoned that the content knowledge dimension should be the core of mathematics teaching. We also derived pedagogical aims connected to our subject and its philosophy: a serious and genuine dialogue with the students and the education of the students at school (and university) to act and reason autonomously and rationally. In other words we developed the dimensions of the teacher competence based on the content and philosophy of mathematics – which defines the special nature of our subject. We claim that by offering learning scenarios (like the described ones) at university we help the students to create their professional system of values concerning educational aims. This could enable the students to build their own pedagogical framework based on the specifics of the subject mathematics. In this sense this is a very important point in the professionalization of future mathematics teachers.

Our present and future work is and will be guided by this approach and more learning scenarios will be developed and evaluated.

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Training primary school teachers through research in mathematics' didactics

Julie Horoks and Brigitte Grugeon-Allys

University of Paris-Est Creteil, Laboratoire de Didactique Andre Revuz, Paris, France, julie.horoks@u-pec.fr

In order to become school teachers, since the last reforms in teacher education in France, the student teachers have to obtain a master's degree, including a research report in a field linked to education, or to the disciplines taught in school. In this study, we analyse the content and methods of an initial course in research in mathematics education, building some tools to assess how such a course can influence the beginner teachers' practices in class during the first years after their training.

Keywords: Teacher education, research initiation, practices, primary school.

In France, teacher training has gone through many changes during the last five years. The main result is that, since 2010, teacher training became part of a university master's degree, including some periods of internship in classrooms, the preparation for the competitive examination which serves to recruit teachers, and, what is new, a research course that leads to a research report on a field linked to education or a discipline taught in school. This initiation into research is supposed to help the prospective teachers in their professional development, reading and understanding results from research articles. Adapting findings for teaching should allow the teachers to reflect on their own practices.

Each university makes its own choices for the contents of the master's program for teachers, within the national guideline for the number of hours for the 2-year training. In the University of Créteil, near Paris, where the researchers of this report are also school-teacher educators, the time allowed for the research part of the training is rather important (120 hours out of 770 hours in total, spread over 2 years), compared to the training programs of the other French universities, but also compared to the time allowed for the rest of the training (professional and disciplinary content). The report requested from the student teachers after 2 years can be considered to be a research report, as it has to follow the same rules as any research report in our research field (a research question and hypothesis, an experiment in class, citation and bibliography norms) even though we probably are less demanding than with students following a research program. This part of the master's represents 34 ECTS out of the 120 needed for the diploma, including 10 ECTS for the report itself.

We are teacher educators for school teacher training (for teaching pupils from 3 to 11 years old) and particularly we run the course for research in mathematics education, which we designed a few years ago, after the reform. As researchers (Nadine Grapin, Brigitte Grugeon-Allys, Julie Horoks, Eric Mounier, Cécile Ouvrier-Buffet, Monique Pézard-Charles and Julia Pilet, all from the LDAR) we are also trying to assess how this course affects the professional development of our students, and find ways to pinpoint the effects of this initiation into research in mathematics education on their practices while teaching mathematics during the early years after the end of their training.

In order to study the effects of the training, we need theoretical and methodological tools to analyse both the content of our research course and the practices of the beginner teachers who were our former students. We present here the different sets of tools that we have developed for this purpose.

RESEARCH QUESTIONS AND THEORETICAL FRAMEWORK

To study teacher training

Some studies in France have already tried to assess the effects of initial training, for school teachers (Butlen et al., 2003) or high-school teachers (Grugeon-Allys, 2008; 2010). But these studies were carried out before the many reforms of teacher training and the introduction of research in the training program. Since 2010, the content for the training has changed in various ways depending on the universities (Grugeon-Allys, 2010). Note that we refer mostly to studies that were carried out in France, as we believe that the institutional and cultural context has an influence on teacher training and teacher practices.

We are trying to build tools to assess the effects of teacher training on professional development, and, in particular, the (potential and real) effects of training teachers through an initiation into research. We define some reference grids, linking the content of the training to the teachers' practices that are aimed for, and being able to relate the evolution in the teachers' practices at the beginning of their career to the dynamics of the training program. We are also looking for consistency among our former students' professional practices that we could link *a priori* to the training that we provided, in terms of knowledge as well as in terms of strategies used to share this knowledge.

The general question that we are asking is: does a course offering an initiation into research in mathematics education allow the students to enter a process of reflection upon their own practices, on the mathematical content taught in class and on their didactics, reflection which could promote their professional development as teachers? In this paper, we will focus on the tools that we are building to witness potential similarities between our former students' practices when they begin teaching in class, and link them to what was aimed for during the training.

Initiation into research in mathematics education, inside a teacher training program

The content of the course "initiation into research in mathematics education", in the Master's program for school teachers in the University of Créteil, is inspired by research results from the French field of *Didactique des Mathématiques*, and particularly results about the content taught in nursery and primary school (pupils aged 3 to 5 and 6 to 10 years old respectively), and about the related teaching and learning (and of course the content offered in this part of the training is influenced by the interests of the researchers participating as educators in this part of the training as well as in other courses). Our priorities are to prepare the students for their 50-page research report but also to bring them knowledge and tools that we consider useful and essential for their practice as future school teachers (for example, the ability to analyse and criticise a textbook on a mathematical topic, which is both a research and a teaching tool). The main theory that we talk about and use for analysis is the Theory of Didactical Situations (Brousseau, 1997), which can also be found among the objects and vocabulary used in other courses in the rest of the training, through some of its conceptual notions (such as *didactical contract* (a system of reciprocal obligation between the teacher and the students that sets their responsibilities, mainly implicitly, in class, during a didactical situation); *didactical variables* (parameters of the situation, with values that affect solution strategies. The effects can be of three kinds: (i) a change in the validity of a strategy, where a strategy that produces a correct answer with a certain value of a didactical variable will produce an incorrect answer with another value; (ii) a change in the cost of the strategy, for example, counting elements one by one is efficient for a small number but much more costly for a larger number; and (iii) the impossibility of using the strategy. (Mackrell et al., 2013); or, *devolution* ("the act by which the teacher makes the student accept the responsibility for an (adidactical) learning situation or for a problem, and accepts the consequences of this transfer of this responsibility" (Brousseau, 1997)) and *institutionalization* (which can take place after a series of activities where a piece of knowledge has been useful in the class to act on, communicate, or validate something, and is then linked by the teacher to a more general and shared knowledge)). Some of the sessions deal with particular mathematical content, its learning (pupils' errors) and its teaching (didactical strategies). Other sessions are more focused on the work with tools for the researcher, depending on the data to be analysed (textbooks, pupils' papers, videotaped sessions) or around the research literature (database, critical review). There are also many sessions dedicated to helping the student teachers with their research report.

Theoretical frameworks: Many levels of analysis

When trying to build some tools to try to assess the effects of the training on professional development, we have to be able first to analyse every stage of the training process, from the analysis of our goals and choices as teacher educators, to the analysis of the practices and choices of the teachers that we trained, during the first two years of their teaching when we are in their classes. This is a complex process, with many sides and points of view (the researcher's, the teacher educator's, the student's, the teacher's), and with many pieces of knowledge (on mathematics, on pupils, on teachers) (Shulman, 1986, although note that we do not use this framework in this research because we do not use the same categorisation of knowledge), some transmitted through the training, and some only influencing our choices without being visible to the students.

Different frameworks are thought to be useful for this study at different levels: to guide our choices for teacher training; to implement this training; and to analyse both its setting and its effects on students. We propose a multi-dimensional approach.

The Theory of Didactical Situations (Brousseau, 1997) is useful to us when we analyse sessions in class, in terms of *didactical variables* and *a priori analysis* of a task or situation. These are among the tools that we are presenting to our students during the initiation into research in mathematics education. They can be used to build and/or analyse situations for the classroom, to teach or to experiment in class with a research question. We also use the concepts of *chronogenesis* (progress of didactic time: description of the evolution of the knowledge proposed by the teacher and studied by the students) and *topogenesis* (change of positions of students and teacher with regard to knowledge (cf., for example, Laborde & Perrin Glorian, 2006).

To analyse and interpret teachers' practices, as well as our own as educators, we use the Theory of the Double Approach (Robert & Rogalski, 2005) defining five components in teachers' practices: cognitive and mediative components (what happens in class in terms of content and teaching/training strategies), but also personal, institutional and social (the curricula, the background, the institution, the colleagues ...). This theory allows us to take into account some constraints, which can explain teaching and training choices (for example, the fact that the content that we teach is linked to our own research can be explained through the personal component of our practice as educators).

Our hypotheses about teacher training also come from the Double Approach: being willing to take into account the constraints of training (not everything is possible for any teacher on any classroom) and the actual practices and needs of the teachers while training them. Regularity observed among teachers' practices might then be linked to their training, and variability of their practices linked to the particular constraints of the profession applied to each teacher.

To refine our analysis of teachers' practices, we use the concept of *didactical vigilance* (Charles-Pézard, 2010): we consider the permanent didactical adjustment made by the teacher in class and outside of class at different levels (local, global, micro practices), mobilising knowledge on mathematics as well as the way they are taught to analyse situations *a priori* and *a posteriori*, using tools from the Theory of Didactical Situations to detect phenomena in the classroom and take decisions regarding them.

To take into account the complexity of the teaching practices, we need to develop a multi-dimensional study, intertwining several frameworks, depending on what we focus on (teachers' or educators' practices), with a more or less wide focus.

METHODOLOGY

Based upon the rich theoretical material that we presented in the previous section, we built several grids of analysis to link the teachers' training and the teachers' practices potentially achieved through training (and later compared to the practice that we actually observed in class):

- a list of the types of tasks actually proposed during the training through the initiation into research in mathematics education;
- a list *a priori* of the expected teachers' practices that could be shaped by the work done during the training, the ones we might expect, based on research results of practices and on the national list of competences for teachers;
- a scale *a priori* of professional development of the school teachers, linked with the activities of the teachers in class.

We will show these grids and how we put them into use to assess and interpret teachers' professional development linked with their training.

The list of the categories of tasks that we proposed during the initiation into research in mathematics education

We analysed the content of the two-year course organised for the initiation into research in mathematics education in the Master's training program of the university of Créteil between 2011 and 2013. This content was not chosen with this research in mind, given that the research only started in 2013, two years after the beginning of the course. We considered the tasks given to the student teachers during the sessions and drew a list of types to characterise what our student teachers might have worked on: different kinds of analysis (mathematical tasks, pupils' productions, textbooks, mathematics sessions) as well as bibliographical work or the construction of the research questions and the methodology to test them. Some of these tasks can be found in the rest of the training but probably dealt with in a different way and with other aims: for example, mathematics sessions' analysis *a priori / a posteriori* are probably also proposed in the professional blocks of the program, but without a research question in mind. Tasks such as selecting, reading and analysing research articles might only be found in the research part of the training.

We consider that answering a research question and answering a professional (teacher) question does not require the same tools and resources. The data analysed might be the same (pupils' productions, textbooks) but with a different question in mind (testing research hypothesis). We believe that the resources used for the initiation into research are different from the ones the student encounters in the rest of the training (research articles / institutional documents). We also think that doing an experiment in class and teaching are not the same activities, even if they both usually involve building and implementing a session. The analyses involved are not the same and will not produce the same effects in the classroom.

The list of expected teachers' practices

The list of expected practices of the teachers when teaching mathematics was drawn up *a priori*, not considering what might or might not have been addressed during the training, but using the French reference of competences for teachers and some results in mathematics education research on teachers' practices that have been linked to potential learning for the pupils in mathematics. This helped us focus on particular activities of the teachers when they teach, or prepare their class, taking into account the mathematical content aimed at by the teaching:

- Choose or build a pertinent situation regarding the learning goals and the progression in learning;
- Know the mathematical content to be taught and their didactics;
- Manage different types of sessions (introduction, institutionalization, training, assessment, ...) and the different moments of a session (devolution, research, comparison of the procedures, validation, ...);
- Evaluate the pupils' learning and manage their heterogeneity;
- Reflect on one's own session afterwards, keep training and innovating.

We can see, in particular, considering the types of tasks proposed during the initiation, which practices we did not train at all (build an entire sequence of sessions on a subject for example) and the ones on which we spent much time (analysing tasks and sessions in class), but of course we cannot isolate the particular effects of the initiation into the global training.

The levels of achievement for every teacher activity

To analyse the evolution of every teacher's practices and compare it with other teachers, we built three levels of achievement for the practices (see Table 1). It takes into account what happens in the classroom, in terms of tasks and management, as in Butlen *et al.*, (2011), but also, on a more global focus, the preparation of the class by the teacher, and the reflection that he or she can have upon his or her own sessions. To illustrate this scale, we give an example, to assess the competence of a teacher for "managing different moments in a session" (see Table 1): level A is the highest one, and includes elements of analysis from the Theory of the Didactical Situations, to characterize the different phases of the session (from the *devolution* to the *institutionalization*).

Level C	Level B	Level A
*The different moments are not clearly identified (collective + individual research, no comparing of the procedures) *Not many initiatives for the pupils, the teacher is in charge of the validation *No moment of recollection of previous knowledge	*Some moments are organized during the session but they do not allow the pupils to engage actively in research or to compare procedures *Shared initiatives but the teacher is still in charge of the validation *The moments of recollection are about tasks and not knowledge, and taken in charge by the teacher	*The "launching" of the activity is organized (re-wording) *There is potential for active research for pupils *Procedures are gathered and compared with a hierarchy in the order in which they are presented, with shared initiatives for their validation, and construction of mathematical meaning *a summarization is made, leading to an institutionalization (*dépersonnalisation*, *décontextualisation*) linking it with the previous activity *Some moments of recollection of previous knowledge are promoted by the teacher

Table 1: An example of the three levels of achievements of teacher activity (managing different moments in a session)

EXPERIMENT AND FIRST RESULTS

To analyse the practices of our former student teacher with as much objectivity as we could, we built the categorization of the teacher practices and defined three levels of achievement for each of them, based upon some *a priori* result from mathematics education, and from national standards. We confronted these grids with the actual practices of our students, once they had started teaching full time in school, after the end of their initial training.

Data collected

We followed 13 students out of the 16 who took part in the initiation into research in Mathematics Education course between 2011 and 2012, who agreed to participate in the study, in order to assess the effect of their training:

— 7 out of 13 answered a questionnaire at the end of their Master's program, about their opinion of their training, with questions on their background before the training, but also on the interest they had taken in each of the subjects proposed in the initiation into research course, and the use they made (or thought they could make in the future) of the content of their training for their teaching.

The answers of our students to the questionnaire filled in at the end of the training showed a certain lack of interest for the theoretical content of the research initiation. The reasons why they chose our initiation into research in the first place vary from: "good in math" to "fear of teaching math", which means that we do not only have the best scientific students in our study. We asked them also to point out links between the research initiation course and the rest of the training, and they did not seem to see much common ground between research and the professional part of the training, not even with the same tasks proposed in both cases. We still have to pursue this study, but the small number of answers does not allow us to draw on the results of this questionnaire much.

— We followed our 13 former students (the ones who volunteered out of the 16) through observations in class and individual interviews by one of the researchers, during their first years as teachers.

We did not record the sessions (as it would have been difficult in this context) but the researcher who did the observation filled in a grid that we all had built before the observations, about the context of the school and the class, the phases organized by the teacher and the mathematical content and tasks proposed during the session, according to the tools we have set in place to analyse teachers' practices in mathematics. The interview allowed us to know if the teacher explicitly used some tools from the research training (through his or her vocabulary for example, or when describing his or her activity with other words than the ones used with us during the training), and to ask questions about how they would describe now the contribution of the training to their practices.

We also followed in class 8 student teachers who did not choose our initiation into mathematics education (but followed an initiation into research in another field). This part of the data has been collected but not analysed yet. It should allow us to pinpoint specificities of our former students, but we will not have as

Expected practices	C	B	A
Choose or build a pertinent situation regarding the learning goals and the progression in learning	1	5	6
Choose a textbook or other resources and use them with a critical view	1	6	5
Know the mathematical content to be taught and their didactics	1	8	3
Manage different types of sessions	1	8	3
Manage the different moments of a session	1	9	2
Evaluate pupils' learning	1	7	4
Manage the heterogeneity of the pupils	1	4	7
Reflect on one's own session afterwards	1	8	3
Keep training and innovating	1	9	2

Table 2: Levels of achievement of our former students at the end of their 1st year in class

much information on these eight students' training, which will be an important limitation.

First results in class

Trainers have to visit teachers twice in their classroom during their first year after training, to assess their work, and the two observations took place in this context. The evidence assessing the level of achievement for each element of the practices comes from the two observations, a few months apart, of half a day of class, including, each time, a session in mathematics, and from the interview that took place at the end of the second visit. The results for our 13 former students can be seen in Table 2.

We can see here that the global level of achievement of the 13 beginner teachers is between B and A, except for one student (who obtained a level C for each activity; it was a student who was in great difficulty, and did not succeed in becoming a teacher at the end of her first year in class). They built rather meaningful and pertinent situations for their class in mathematics, with an *a priori* analysis that allowed them to anticipate the pupils' difficulties in most cases.

They seem particularly efficient in managing the different phases of the session, with long phases of research for the pupils in certain classes. This is a type of task that has been often proposed to the student during the initiation into research course, while analysing videos, or experimenting in class for the research report. Validation of the tasks is still performed by the teacher rather than letting the pupils take the initiative.

Their opinion on the content of their training through the initiation into research has often evolved (from "too much work, too much theory" to "useful for preparing the session and reflecting on it afterwards"). Though they do not use concepts from the didactical theories in their discourse, they still seem to enact them in their practice (*didactical variables*, comparison and *organisation of a hierarchy in the* pupils' procedures, *a priori* analysis). For example, even if almost none of them used the term "*a priori* analysis" during the interview afterwards, they showed some capacity to analyse the content of their sessions, and the potential gaps between what was planned and what actually happened during the session. Incidentally, in the questionnaire at the end of the training, the majority of the students said that some content of the initiation was helpful to anticipate pupils' procedures, and that some of the content also gave them means to analyse their own sessions afterwards.

They seemed to us very reactive when reflecting on their sessions (changing things from their original plan immediately to take into account our remarks) and giving themselves means to keep improving their practices; which could confirm our hypotheses about the value of training through research to help the teacher enter a reflexive attitude for their teaching. Of course we still have to analyse the data collected in the other classes to confirm these results.

CONCLUSION

Even if the first results on teachers' practices during their first year teaching are very encouraging to us, we obviously have to underline the many limitations of our research:

— the small number of students, that does not allow us to generalise our findings;

- the impossibility to totally differentiate the effects of the research training from the effects of the rest of the program (but the questionnaire gave us an idea about the originality of the tasks proposed in our initiation, compared to the rest of the training, in which we also take part);
- conflicting roles of researchers, at the same time teacher educators and evaluators of the students whose practices they are trying to assess (even if we tried to separate the research from the assessment by giving clear protocols to the students) in a training context that was not built as an experiment on teachers' development.

As teacher educators, this study helped us build tools to organize the training, and gave us a clearer view of what we can potentially offer to the prospective primary school teachers through an initiation into research. We already made changes in our initiation program to take into account the needs of our students and address the question of the development of their practices more efficiently.

But we still have a lot of research to do to assess the real effects of training in general and through research in particular.

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Tasks analysis as a mean to reflect and (re)think the pedagogical practice of teachers who teach mathematics

Cristina Cirino de Jesus[1], Márcia Cristina de C. T. Cyrino[1] and Hélia Oliveira[2]*

1 State University of Londrina, Londrina, Brazil, criscirino@gmail.com

2 The University of Lisbon, Institute of Education, Lisbon, Portugal

This study discusses how the analysis of mathematical tasks, within a teachers' in-service education study group, can help teachers who teach mathematics in the early years of Elementary School to reflect and (re)think their pedagogical practice. This research was carried out in a Brazilian public school, with the participation of 14 teachers. Within the in-service study group, the recognized importance of the tasks and their classification according to levels of cognitive demand by the teachers is discussed. The reflections and discussions in this study group allowed the identification of some changes in the teachers' perspectives in regard to tasks choice/preparation, the role of the teacher in the classroom and the recognition of students' capacities.

Keywords: Mathematical tasks, levels of cognitive demand, teacher education.

INTRODUCTION

The tasks proposed to students influence what and how they learn (Doyle, 1983; Stein, Smith, Henningsen, & Silver, 2009); therefore, it is essential for teachers to be prepared to select, in an informed way, the tasks to be proposed to their students and to support them in their work without reducing their complexity.

Several researchers have developed studies in which tasks are the focus of investigation, indicating that there is a relationship between the type of the proposed tasks and type of thinking developed by students (Doyle, 1983; Christiansen & Walter, 1986; Shimizu, Kaur, Huang, & Clark, 2010; Stein & Smith, 1998; Stein, Smith, Henningsen, & Silver, 2009). Thus, it is important to provide teachers with professional development opportunities to help them reflect on the role and relevance of the task for the teaching and learning processes and make them realize that not only tasks play a significant role in these processes but also that the way they are explored by the teacher in the classroom is fundamental.

In this article, we present the results of a research whose objective was to investigate how tasks analysis in a context of in-service education can help teachers who teach mathematics in the early years of Elementary School to reflect and (re)think their pedagogical practice and, more specifically, to understand these teachers' perspectives in regard to the tasks' choice and preparation, proposition and implementation, based on the levels of cognitive demand of each task (Stein, Smith, Henningsen, & Silver, 2009).

THE IMPORTANCE OF TASKS AND THE LEVELS OF COGNITIVE DEMAND

Tasks are used by teachers for different teaching purposes and they may appear in different moments of the lesson as exercises and problems, for instance. In this study, task is viewed as a proposition made by the teacher in the classroom with the objective to focus the student's attention in a determined mathematical idea which implies an activity from his part (Stein, Smith, Henningsen, & Silver, 2009).

When planning their lessons, many teachers are not used to think about the reasons behind their task choices and many times this action is supported merely by the mathematics contents which they are working or those included in the textbook. According to Pepin and Haggarty (2007), tasks can and must be seen as "[...] a process that can potentially help to enhance mathematical understanding rather than simply a vehicle for content" (p. 13). Next, three significant arguments related to the role of the task are presented to help teachers understand the importance of having a set of task selection criteria.

Firstly, "tasks with which students engage constitute, to a great extent, the domain of students' opportunities to learn mathematics" (Stein, Smith, Henningsen, & Silver, 2009, p. 131). The student's work is defined by tasks carried out by him/her, daily; however, some have the potential to mobilize complex forms of thought while others do not. Therefore,

> tasks that ask students to perform a memorized procedure in a routine manner lead to one type of opportunity for student thinking; tasks that require students to think conceptually and that stimulate students to make connections lead to a different set of opportunities for students thinking (Stein & Smith, 1998, p. 68).

A second argument is that tasks are instruments used to connect the students' learning objectives (Stein, Smith, Henningsen, & Silver, 2009). When choosing or preparing a task, it is important for the teacher to have a clear objective to propose to students. To think about the tasks' objectives prior to implement them can help teachers create a classroom environment that motivates the student to get involved in the solving of the proposed tasks.

A third argument is that tasks affect significantly the reasoning process students develop in order to solve them (Stein & Smith, 1998). Therefore, teachers should prioritize cognitively challenging tasks with the potential to engage students in complex thinking forms. This type of task can help students develop reasoning forms and strategies that allow them to go beyond the memorization of facts and procedures. However, the teacher must be aware of the fact that to select or prepare this type of task does not guarantee students' engagement since there are several classroom factors that can contribute to the maintenance or decline of the task's cognitive demand (Stein, Smith, Henningsen, & Silver, 2009).

Tasks can be analyzed from several perspectives such as nature, characteristics, number of strategies used to solve them, levels of cognitive demand among others. In this study, tasks are analyzed based on their levels of cognitive demand since, according to Stein, Smith, Henningsen and Silver (2009) "the cognitive demands of mathematical instructional tasks are related to the level and kind of student learning" (p. 17).

For Stein and Smith (1998), tasks can be grouped in four levels of cognitive demand: memorization; procedures without connections; procedures with connections, and doing mathematics. The first two categories involve tasks with lower-level of cognitive demand and the last two refer to high-level of cognitive demand tasks. With the purpose to provide support to teachers during task analysis according to their levels of cognitive demands, in a context of teachers continuing education, these researchers prepared a Guide to Tasks Analysis (Appendix 1), which includes a list of tasks characteristics in each one of the four levels of cognitive demand that can be used as a parameter for their classification.

The teacher can still use this guide "as lens for reflecting on their own instruction and as a shared language for discussing instruction with their colleagues" (Stein, Smith, Henningsen, & Silver, 2009, p. 2). Thus, by selecting/preparing tasks based on their cognitive demand allows the teacher to look into what the students learn and how they work with the tasks; their actions at the moment of proposing and implementing the tasks; the factors that affect the proposition and implementation of the tasks, which may contribute to their maintenance or decline.

However, in order to choose or prepare tasks based on the levels of cognitive demand, the teacher must not only be informed by some principles for tasks analysis but also know their students deeply and pay attention to aspects such as age, learning pace, level of education, mathematical knowledge and prior experiences so that those tasks constitute a real challenge for students (Stein, Smith, Henningsen, & Silver, 2009).

RESEARCH CONTEXT AND METHODOLOGICAL PROCEDURES

This research was developed within an in-service education program, in a study group context, with the participation of 14 teachers working with the early years of a public Elementary School in Brazil. The teachers had more than eight years of experience and played different functions in their schools.

The study group was coordinated by two researchers, one of them is the first author of this article, from the Mathematics Education and Sciences Teaching Graduate Program– PECEM[1], and was part of an in-service education project outlined by the school within the City Education Department annual planning. One of the goals of this study group was to motivate participants to learn about the levels of cognitive demand of the tasks and how to analyze tasks based on some principles. The study group met weekly, for one hour, for a period of six months and it focused on tasks analysis based on the categories proposed by Stein and Smith (1998).

The group discussions were organized in four phases. In the first, we discussed the tasks brought by the teachers and, they presented and explained the reasons that guided the selection of these tasks and how they worked with the tasks in the classroom. In the second, the teachers studied and discussed the features of tasks in each level of cognitive demand and later they conducted a new analysis of these tasks, classifying them according to the level of cognitive demand. In a third moment, the teachers worked on tasks with different levels of demand cognitive proposed by the coordinator of the study group, they analyzed and classified these new tasks and some teachers have expressed interest to apply them in the classroom (due to space limitations, it is not possible present the tasks in this paper). After this implementation, in the fourth phase, the study group discussed the relevant points and the difficulties faced by the teachers in the classroom, as well as evaluated the importance of task analysis and cognitive demand levels.

The instruments adopted for data collection included transcriptions of the audio recordings of the meetings so that the participants' speeches could be registered in their original form, keeping the integrity of the dialogues; written productions, with comments and reflections of teachers on discussions that occurred in group (free writing) and semi-structured interviews with some teachers. To identify the instrument from which the information was obtained, we use the fictitious name of the participant followed by the first letter of the instrument. So, the letter G was used for group meeting followed by the number of meeting, P for written productions, and I for interviews. For example, the identification of an information given by Cintia in the first meeting, is registered as "(Cintia, G1)". Due to space limitations, in general, only one excerpt was presented as an example of each analysis unit. Analysis units were constructed after several readings of the meetings' transcriptions, written productions and interviews, highlighting relevant excerpts for our study. Data analysis uses the Tasks Analysis Guide as reference (Stein & Smith, 1998).

1 This program is being developed at State University of Londrina, Brazil.

Reasons given by the teachers for their choice of tasks	Tasks allow teachers to approach mathematical content	I choose a task to work with content. If I want to work with the operations, the multiplication table, I choose tasks that allow me to work with them. If the content is measures, for instance, I choose a task that explores this content (Isadora, G1)
	Tasks allow to verify whether the mathematical content was "assimilated".	When I propose tasks to my students it is always with the intent to check whether they have learned the content and also to verify which contents still need more work to be done [...] (Fabiane, P1)
	Tasks allow to work with non- mathematical aspects	I never choose a task which deals exclusively with Mathematics. I choose a problem that involves several areas such as Sciences, Portuguese, Geography, several disciplines, a task that does not involve only calculations. (Carla, G1)
	Tasks allow to relate mathematics with the student's reality	[...] Tasks must always be connected with the child's reality. Children learn mathematics for their life. [...] They are more motivated (it's more pleasurable) when they have to solve real problems. (Mariana, G1)
	Tasks allow to develop reasoning skills	The objective of the tasks must be to contribute with the development of the student's reasoning skills. (Cintia, P1)

Table 1: Summary of the initial reasons given by teachers for their tasks choices

Reasons given by the teachers for choosing tasks

One of the actions developed in the beginning of the study group was a discussion over the tasks teachers had brought upon request of the teacher educators. They were asked to explain the reasons behind their tasks choices which included: to approach mathematical content; to verify whether the mathematical content was assimilated; to work with non-mathematical aspects; relate mathematics to the student's reality and to develop reasoning (Table 1). Some teachers gave more than one reason for their choices of the tasks.

The reasons presented by the teachers show that most of the tasks they selected are based on memorization or on procedures without connections (Stein, Smith, Henningsen, & Silver, 2009), i.e., they require only the reproduction and memorization of contents, rules and formulas learned previously, without establishing connection with concepts or meanings that give support to the contents given, and being more focused on correct answers than on the student's understanding.

Tasks analysis and the levels of cognitive demand relevance

The study of the levels of cognitive demand, the reflection on the tasks' role in the teaching and learning processes and of the high-level of cognitive demand tasks led teachers to think about their initial reasons to choose tasks and develop another view in relation their selection or preparation. Next, we present a table with some points provided by the teachers on the relevance of tasks and levels of cognitive demand (Table 2) during the study group work.

Teachers started to notice that tasks have different characteristics and that this difference can be interpreted through their levels of cognitive demand, bringing implications to the student's activity and to the complexity of the mathematical processes involved. They also became aware that their tasks selection criteria were focused almost exclusively on the coverage or verification of mathematical content.

Signs of changes triggered by discussions, reflections and task analysis

The actions developed by the study group on the reasons for choosing tasks, levels of cognitive demand as a tool to classify them and on tasks analysis allowed us to identify signs of changes in regard to tasks choice, role of the teacher in the classroom and student's capacity (Table 3).

During our first meeting, teachers had difficulty in justifying their choices to the group, considering that the habit of thinking about *why the tasks were selected*; *for they serve*; *what contents they cover, their level of complexity* and *whether the objectives were met*, it was not common practice of most teachers who participated in the study group. *"I never stopped to think why I chose the tasks. Today I become to realize how important it is to think about it". (Ana Lívia, G1). "Great! You know, when I gave the task to my students I did not think about the objective, about the reason behind it; so, I'll pay more attention to it from now on"(Ana Júlia, G1).*

During the group sessions, some teachers started to become more aware of the importance of selecting tasks and aggregating levels of cognitive demand to their criteria, reducing the number of tasks proposi-

Reflections on tasks analysis and levels of cognitive demand	Tasks relevance for the teaching and learning processes	I had never stopped to think about the importance of tasks for the student's learning process; but, now, here with this group, with these discussions and being aware of the difficulties some teachers had to justify their choices I realized how important it is to reflect on it. I realized that we have chosen tasks haphazardly. (Vitória, P1)
	The relevance of learning about the levels of cognitive demand of the tasks	[...] knowing about levels of cognitive demand changed my way of thinking and choosing tasks and helped me think better about how and which task to use and whether it is going to be useful to the public at hand [...]. (Ana Júlia, I)
	The relevance of high-level of cognitive demand tasks	It is extremely important to work with high-level of cognitive demand tasks, since they promote the development of autonomy, self-confidence and critical thinking, argumentation and therefore to think mathematically and to search for solutions to problems. (Denise, P19)

Table 2: Reflections on tasks and levels of cognitive demand

Change signs	Sentences that show the signs of change
Tasks selection by the teacher	These reflections have contributed a lot to my mathematical tasks selection criteria and also to tasks from other subjects. We are becoming more careful, thinking not only on the objectives of the tasks but also on their levels of complexity and the students, trying to imagine how they will react to the task and which are their possible answers. (Cintia, P6) The reasons I had before remain, but now I also select high level tasks which involve reasoning and not only formulas. But, I look for a balance among the tasks. Students still have lots of difficulty, so it is impossible to work only with high level tasks, that's why I stick to my old set of criteria (Fernanda, I) Today I look for tasks that make my students think; not only tasks that focus on the correct answer but on how the student will carry it out, on the process. But, I continue choosing tasks for verification, which are also important (...) I think that the other criteria came up due to my participation in the study group otherwise I would not have learned about them and become aware of their existence (Ana Júlia, I).
The role of the teacher	My posture in the classroom has also changed. Now I try to question the student more to know about his reasoning process (Ana Júlia,G17) [...] This group work not only helped us know the task but also to try to work with tasks in a different way. For example, I never gave the opportunity for my student to explain their work. Now, I always ask them to explain their work process to me and their classmates. (Fabiane, G17)
The students'capacity	I learned to value my students and their capabilities. Before I saw them as individuals coming from a poor neighborhood, with whom I was supposed to teach some mechanical skills and I would be very happy if they were able to learn them. However, I learned that they can achieve more; I did not know their potentials so, when I gave them the task you suggested I saw how wrong I was and that I could go beyond mere mechanical tasks. (Mariana, I) I was surprised by some students who many times have difficulty in carrying out simple mathematical operations, an algorithm; however, they solved very fast the sequence's problem they did very fast without questioning [me] much. Other students, who were always the first to finish a routine task, had difficulty solving this problem. (Denise,G13)

Table 3: Sentences that show signs of change by the teachers

tions focused on memorization or on the realization of a procedure without connections.

In regards to their role, discussions and tasks analysis allowed teachers to understand the impact of their actions on some teaching and learning processes, especially in relation to tasks choice and to promote a change in the way to implement tasks in the classroom. This way, they started to promote teaching and learning processes focused on the students' understanding.

As for the student's capacity, tasks implementation and solution helped teachers to realize that, many times, they underestimate the cognitive capacity of their students, their specific learning pace and difficulties and that the role of the teacher is to help them think on how to solve the task without removing its challenges (National of Council of Teachers of Mathematics, 2000).

FINAL CONSIDERATIONS

To think about the tasks they propose was not a usual practice for most teachers in the study group. Consequently, in the phase of the study group, some of them had difficulty justifying their choice of the tasks to be used in the classroom. However, despite these difficulties, some teachers were able to voice their reasons for choosing the tasks such as to use a task to approach mathematical content or verify whether the mathematical content was assimilated, and these practices are strongly present in the pedagogical practice of these teachers.

To learn about the levels of cognitive demand and analyze tasks was highly significant for most teachers, since they were able to realize the importance of thinking about the tasks and the way to work with them in the classroom. Thus familiarity with the levels of cognitive demand has also helped teachers to understand that tasks with a high-level of cognitive demand have the potential to involve the student in a work focused on reasoning and on understanding, making them choose and propose this kind of tasks.

Teachers became more critical regarding their choices of tasks which became an action based on a lot of thinking. They become more attentive to the objectives of the task, trying to anticipate how students would react to it and their possible solutions.

In addition, the teachers had the opportunity to reflect on their role in the classroom, since they realized that their attitudes and decisions influence their students' learning. Therefore, some teachers considered that they changed the way they teach by questioning more the students during the lessons, to know about their line of thought rather than being only concerned with the correct answers and mainly giving them more opportunities to justify their reasoning processes. Consequently, this change in the teacher's attitude shed a new light on the way they regard the students, since some of them used to underestimate their students' capabilities.

Thus, according to Stein, Smith, Henningsen and Silver (2009), to know about the levels of cognitive demand allowed teachers to differentiate mathematical tasks in order to identify those that offer opportunities for students to think without being led by their superficial characteristics.

However, even acknowledging the potential of tasks with high-level of cognitive demand, teachers have only aggregated them to their previous reasons. The lack of confidence in dealing with these tasks, the fear of not being able to meet the objectives of the lesson and at the same time demotivate students due to their difficulties and the justification that memorization and verification tasks are also relevant, are some of the arguments presented by teachers to justify the low frequency of high-level of cognitive demand tasks in their work. Such resistance must be understood within the context of an instituted professional practice, considering that the knowledge of levels of cognitive demand can help teachers to reflect on their teaching style and focus more on what students learn and how they work on a task as well as on their actions and attitudes when proposing and implementing a task.

ACKNOWLEDGEMENT

* CNPq (National Council for Scientific and Technological Development – Brazil) scholarship holder.

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APPENDIX 1. GUIDE TO TASKS ANALYSIS (STEIN & SMITH 1998)

Tasks Characteristics that involve low-level of cognitive demand	Memorization	Procedures without connections
	- Involve either the reproduction of facts, rules, formulas learned previously or memorization; - They cannot be solved by using procedures because they are either not necessary or the time is short to use them; - They are not ambiguous and involve the exact reproduction of the subject studied previously and what is to be reproduced is clear and directly stated; - No connection with concepts or meanings underlying the facts, rules, formulas or definitions taught or reproduced.	- They are algorithms. The use of one procedure or is specifically indicated or evident due to a previous instruction, experience or question location; - Require limited cognitive demand to resolve it successfully. There is little ambiguity on what needs to be done and how to do it; - There is no connection with concepts or meanings underlying procedures used initially; - They are focused on the production of correct answers instead of on the development of Math comprehension; - Do not demand explanations or whenever needed, they are explanations focused only on the description of the procedure used.
Tasks Characteristics that involve high-level of cognitive demand	Procedures with connections	Doing Math
	- Focus the students' attention on the use of procedures to develop an in-depth knowledge of concepts comprehension levels and Math ideas; - Suggest ways to be followed (explicitly or implicitly), which are general, common procedures which have intimate connection with conceptual ideas; - Usually represented by several ways (such as diagrams, manipulatives, symbols and problem-situations). They make connections among multiple representations that help develop meanings; - Demand some degree of cognitive effort. Although general procedures can be followed, they cannot be followed without being fully understood. Students need to get involved with the conceptual ideas underlying the procedures to be followed in order to fulfill the task successfully and develop comprehension.	- Demand complex thinking skills rather than algorithmic skills, and the tasks do not suggest, explicitly, a predictable way to be followed, instructions for its solution, or an example, which, when well-trained, could lead to their resolution; - Demand that students explore and understand the nature of Math concepts, procedures or relations; - Demand high monitoring or high regulation of the student's own cognitive process; - Require students to mobilize relevant knowledge and experiences make appropriate use of them during the task; - Require students to analyze and examine the task actively and whether it has limited resolutions and solutions limitations; - Demand considerable cognitive effort and may involve some levels of anxiety from the students' part for not having a previous natural list of the processes demanded by the problem.

Video-based peer discussions as sources for knowledge growth of secondary teachers

Ronnie Karsenty, Abraham Arcavi and Yael Nurick

Weizmann Institute of Science, Rehovot, Israel, yael.nurick@weizmann.ac.il

This paper reports on a study conducted as part of a larger project, named VIDEO-LM, which centres on video-based professional development for secondary mathematics teachers. The project aims to facilitate the reflective skills and the Mathematical Knowledge for Teaching (MKT) of secondary school teachers, in particular those teaching advanced mathematics courses. At the core of the project is a 6-component framework developed for analysing videotaped lessons in collaborative discussions with teachers. We describe the rationale and novelty of the project and the framework. Then, we focus on a study which examines the MKT growth of a group of teachers who participated in VIDEO-LM peer discussions, and present some preliminary findings.

Keywords: Video-based professional development, peer discussions, mathematical knowledge for teaching (MKT), secondary mathematics teachers.

INTRODUCTION

The power of videotaped teaching episodes as a vehicle for stimulating discussion and reflection among mathematics teachers has been discussed in recent years from various angles (Borko et al., 2011; Coles, 2013; Sherin & van Es, 2009). Although video has been used as a tool for teachers' professional development for the past 50 years (Sherin, 2004), the rapid advancements of digital video documentation has allowed for significant amplification in this field, which manifests in a host of professional development programs in various countries that include video as a major resource (e.g., KIRA and Mathe sicher können in Germany; the Problem-Solving Cycle and the Learning and Teaching Geometry programs in the US; WMCS in South Africa). Online video resources are now largely available to educators (MET in the US and Teachers Media in UK are prominent examples) and at least two international symposia were dedicated recently to the use of video in mathematics teacher education (see http://www.weizmann.ac.il/conferences/video-lm2014).

The following citations distil the main feature of video that can explain why it is regarded as a valuable tool for teacher development. Sherin and van Es (2009, p. 21) claim that "Teachers benefit from opportunities to reflect on teaching with authentic representations of practice"; Brophy (2004, p. 287) argues that video can introduce "the complexity and subtlety of classroom teaching as it occurs in real time"; and Nemirovsky and Galvis (2004, p. 68) suggest that "because of the unique power of video to convey the complexity and atmosphere of human interactions, video case studies provide powerful opportunities for deep reflection". All these scholars emphasize the role of video as a window to the authentic practice of teaching, which allows teachers to focus on complex issues that may be unpacked through observing, re-observing and reflecting on specific occurrences.

Generally speaking, there are three main trends in using videotaped episodes from mathematics lessons as resources for teacher development. First, video is utilized for introducing new curricula, activities, pedagogical strategies, etc. This target is mainly implemented through supplying teachers with video cases that model how teaching the new curricula, or using the pedagogical strategies, may be enacted (e.g., Seago et al., 2010). A second trend, becoming more and more prevalent (particularly in the US), is using videotaped lessons as a source for feedback and evaluation. Teachers watch videos from their own classrooms and discusses them with colleagues or instructors, often with the use of a pre-constructed standard-based rubric such as the one developed by Hill and colleagues (2008). The third trend is using videotaped episodes to enhance teachers' proficiency to notice, understand and discuss students' mathematical thinking (Sherin et al., 2011), usually in the form of "video clubs" (Sherin & van Es, 2009).

This paper brings forward a different direction for using video as a major resource for professional development of mathematics teachers, as emerges from a new project in Israel, named VIDEO-LM (Karsenty & Arcavi, 2014). In what follows, we describe the project and introduce the framework of analysis which lies at its core. We then report on a study conducted as part of the project, investigating the development of new Mathematical Knowledge for Teaching (MKT) in a group of teachers who participated in VIDEO-LM sessions.

THE VIDEO-LM PROJECT

VIDEO-LM (Viewing, Investigating and Discussing Environments of Learning Mathematics), is a project launched on 2012 at the Weizmann Institute of Science. Its over-arching goal is to improve mathematics teaching, with particular emphasis on the advanced tracks in secondary schools, through enhancing the reflective skills and the mathematical knowledge of teachers. The means to achieve this goal is by creating a pool of videotaped mathematics lessons, which serve as a basis for guided peer discussions with teachers. We use the lessons as "vicarious experiences" for teachers, centering on how the filmed teacher displays multifaceted elements of practice. The videos we use do not necessarily display 'exemplary teaching'; rather, we pick lessons that can potentially trigger fruitful conversations. As opposed to the first trend noted above, teachers are not presented with demonstrative videos focused around new materials or strategies, nor do they engage in evaluative discourses, as in the second trend. They may relate to students' thinking, the aspect centralized in the third trend, but only as part of the whole "teaching picture" revealed on the screen. In other words, the discussions are intentionally *teacher-centered,* and not *student-centered*.

Rationale

It is well known that teaching can be a lonely profession. Despite participation in professional communities, online forums and other forms of communication and collaboration with other teachers, the reality is that the vast majority of teachers are the "solo adult actors" in their classrooms, where they spend the lion's share of their professional life. In many countries teachers seldom get the chance to watch their peers in action once the pre-service period is over. This is not merely a social deficit, but also a barrier to certain processes of professional evolutions embedded in peer learning *in situ*. For instance, watching peers may expose teachers to alternative instructional strategies, which makes it possible to change routine thinking and actions (Santagata et al., 2005; Sherin, 2004). Thus, the VIDEO-LM project aims at creating opportunities for teachers to watch whole lessons given by others. Moreover, we seek to enhance the potential gains from these opportunities, by directing teachers to collectively analyse these vicarious experiences through a systematic use of a 6-component framework.

The framework for analysing videotaped lessons in teachers' discussions

We have developed a unique framework for analyzing videotaped lessons, inspired by the work of Schoenfeld (1998) and Arcavi and Schoenfeld (2008). Schoenfeld's theoretical model of "teaching in context" describes and predicts how teachers' goals, knowledge and beliefs affect their in-the-moment decision-making during lessons. Arcavi and Schoenfeld have taken this model as a basis for creating analytical tools with which mathematics teachers can reflect upon their own practice while watching videotaped lessons of other teachers. In the VIDEO-LM project, we have modified and extended these analytical tools to include six components, which are the building blocks of the framework we use. In the following, we briefly describe these six components.

1) *Mathematical and meta-mathematical ideas*. Given the topic of the lesson, there is a range of relevant concepts, procedures and ideas that may be associated with this topic. For instance, the topic of the square root function may involve the following ideas: the non-negativity of the function's domain; its monotonously increasing graph; continuity and derivability of the function; its relation to the function $y=x^2$; and so forth. Topics may also evoke meta-mathematical ideas, such as what makes a proof legitimate, why is one counter example sufficient to refute a conjecture, the arbitrariness of certain mathematical definitions, etc. Before watching a videotaped lesson, teachers are requested to elicit ideas, in an attempt to gauge the boundaries of this range. Then, once the tape is screened, they can refer to questions such as: Which of these ideas, or oth-

ers, did the teacher bring forward in the lesson? Which ideas were left out? How can this decision be explained? Which meta-mathematical notions were evident in the lesson?

2) *Explicit and implicit goals*. The rich span of mathematical ideas around a given topic enables choices of the goals teachers wish to pursue within a lesson. One of the reasons that lessons of different teachers on the same topic do not resemble one another is that teachers derive different goals from the range of relevant mathematical ideas. While watching a video, teachers try to identify the goals they think the filmed teacher was attempting to achieve, whether explicitly or implicitly. In other words, they ascribe goals to the teacher, just as one would ascribe meaning to a poem or some other piece of art. In this context, our aim is not to scientifically verify any "true situation" (i.e., what were the teacher's "real" intentions); Rather, we encourage the mental exercise of ascribing goals, targeted at (a) promoting the skill of articulating goals; and (b) enhancing awareness to the fact that alternative (sometimes even competing) goals to teaching a certain mathematical subject may exist.

3) *Tasks and activities selected by the teacher*. The tasks, problems and activities presented by the teacher during the lesson are the means by which the teacher's goals are fulfilled, hence reflecting the mathematical ideas chosen by the teacher. The video enables teachers to watch a "task in action"; how it is implemented, the nuances in introducing it and how the teacher addresses students' reactions. This enables quite a different exploration than the one teachers may preform when presented with the task in its written form (i.e., as it appears in textbooks or other written resources). We refer to such an exploration as an *a posteriori task analysis*, which can potentially enrich the discussion, giving an additional angle to that of the *a priory task analysis*.

4) *Interactions with students*. The implementation of the tasks and activities selected by the teacher is carried out through classroom interactions. This component includes generic elements such as positive and negative feedbacks given by the teacher, listening to students, wait time, etc., but also considerations that are more related to the subject matter, for instance how the teacher navigates the students' responses during the mathematical activity and poses subsequent questions. Following Clarke (2014), questions of equity, authority and knowledge construction are also valuable as triggers of productive conversations: Who gets permission to speak? Who is responsible for the flow of ideas? Is the mandate to produce new knowledge distributed or centralized?

5) *Dilemmas and decision-making*. The mathematics education community has learned much about teachers' decision making processes, from the work of Schoenfeld (1998; 2008) and others. However, for many teachers the "diving" into another teacher's decisions is a novel experience. The exercise offered to teachers participating in the discussion is to focus on the filmed teacher's dilemmas as they may be uncovered in the lesson, the decisions taken in order to resolve these dilemmas, and their consequent tradeoffs. The risk is drifting into criticism and judgmental talk, a problem pointed out in the literature on video sessions (e.g., Coles, 2013; Jaworski, 1990). To avoid this, teachers are guided to consider the choices made by the teacher under the assumption that she acts in the best interest of her students (Arcavi & Schoenfeld, 2008). Taking this as a starting point, the constraints and affordances of the teacher's choices can then be examined, and alternative paths can be elicited and explored.

6) *Beliefs about mathematics teaching*. The issue of how teachers' beliefs shape their practice has been widely studied (e.g., Li & Moschkovich, 2013; Schoenfeld, 1998). In fact, all the components (1) through (5) above are likely to be guided by the set of beliefs the teacher brings into the classroom. Facilitating discussion about beliefs is a highly complicated and delicate matter. However, we suggest that such a conversation can be valuable. Teachers are not always aware of messages they convey during mathematics lessons, through direct or latent communications, nor of their considerable influence on how students perceive the domain of mathematics and how they function during the lesson. Thus there is a potential gain in the exposure to explicit and implicit attitudes reflected in another teacher's actions. The discussion focuses on questions such as: What

may be the filmed teacher's views about the nature of mathematics as a discipline? How does the teacher perceive her role? What may be her ideas about what "good mathematics teaching" is? What does she think about the students' role as learners?

THE STUDY

The framework of analysis described above was implemented by the VIDEO-LM team in several courses with in-service teachers. As described earlier, and in line with Sherin's review (2004), different video-based programs set different learning goals for discussions with teachers. Ours was a two-folded goal, strongly linked to two agendas that we found valuable. One is the need to promote teachers mathematical knowledge for teaching (MKT), as defined by Ball and colleagues (2008). The other is the important move from a judgmental or evaluative discourse about the mathematics teaching profession towards a reflective and more constructive discourse, as advocated for instance by Jaworski (1990). In this paper, we limit our focus to the first goal only. We report on an exploration conducted on a group of teachers, who experienced peer discussions using the VIDEO-LM framework. The research question was defined as follows:

— What may be the gains of video-based peer discussions around the VIDEO-LM framework, in terms of the teachers' MKT?

Design, data collection and data analysis

During 2013–2014, a group of teachers participated in VIDEO-LM workshops, as a pilot for a professional development program conducted later on with other groups. The group met once a month throughout the 2013 academic year, and continued to meet monthly, with some change in members, during 2014. Each session lasted about 4–5 hours, a total of 60 workshop hours. For every session we used one videotaped lesson (45 minutes on average) with various modes of watching implemented (e.g., watching together or in small groups, focusing on different components of the framework, watching the whole lesson uninterruptedly vs. breaking it to selected episodes). About half the lessons used were filmed in high track high school classes (grades 10–12), and the others were junior high school lessons (grades 7–9). Data collection means included field notes taken during workshops, video-documentations of all discussions, documentation of e-mail correspondences initiated by the teachers after some of the sessions, and questionnaires administered at the end of the course, focusing on the participants' views about the 6-componant framework used in workshops. The content analysis preformed on the collected data included (a) tracing all of the participants' utterances associated with MKT (i.e., unpacking mathematical concepts or relating to teaching these concepts); (b) grouping utterances into units of analysis that share similar ideas; (c) using the units to form "utterances maps" that convey the development of mathematical, meta-mathematical and pedagogical ideas throughout different parts of the sessions. This process is still ongoing; therefore the results reported herein are preliminary.

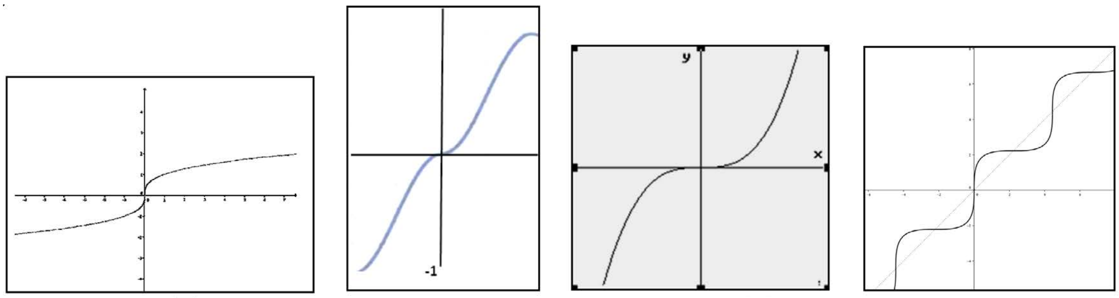
Subjects

The 2013 group comprised of 10 teachers, of which 7 continued to the second year in 2014, with 5 new members joining in. All participants were secondary school mathematics teachers with a teaching record of over ten years, and were well acquainted with the mathematics curriculum of grades 7–12. Nine of them were lead teachers, i.e., holding additional positions such as heads of mathematics departments in their schools, instructors, or principals. The group was diverse in terms of gender and sector (i.e., included religious and secular Jews, Israeli and Palestinian Arabs). None of the subjects had a prior significant experience with watching videotaped lessons.

Preliminary results

The data analysis revealed that discussions focused around the six components of the framework were rich in examples, insights and suggestions brought up by participants. The deep mathematical conversations during sessions and in subsequent e-mail correspondences, although not fully analyzed yet, point to the joint development of new mathematical knowledge for teaching, triggered by the videotaped lesson. We chose to present here two detailed examples, demonstrating processes of collective knowledge growth.

Example I: How do we define an inflection point? The videotaped lesson in this case was given in an 11th grade high track calculus class. The teacher explored with her students the concept of concavity of functions, leading to the definition of inflection points as points where the graph changes from concavity



$f(x)=\sqrt[3]{x}$ $\quad$ $f(x)=(\cos x-1)\cdot \operatorname{sgn}(x)$ $\quad$ $f(x)=x\cdot|x|$ $\quad$ $f(x)=\sin x$ ‘turned’ by 45°

Figure 1: Teachers’ generated examples

upwards to concavity downwards, or vice versa. This was then translated into a “working tool”, associating inflection points of $f(x)$ with the extreme points of $f'(x)$, or the zeros of $f''(x)$. Discussing the mathematical ideas introduced in this video, participants raised the following question: What about an inflection point where the first or the second derivatives do not exist? The group became motivated to find counterexamples where $f(x)$ has an inflection point in x_0 but $f'(x_0)$ or $f''(x_0)$ do not exist, and found a graphic example but not an algebraic representation of such a function. Following the session, in an intense and rich e-mail conversation, teachers found and shared different counter- examples, as described in Figure 1.

In all these examples, $f(x)$ has an inflection point in 0, but $f'(0)$ and/or $f''(0)$ do not exist. Furthermore, one teacher generalized that the product of *sgn(x)* and any even function in which $f'(0)=0$ and $f''(0)\neq 0$ would be a suitable counterexample (e.g., $g(x)=(\cos x-1)\times \operatorname{sgn}(x)$).

As a result, the group reached a consensus about the accuracy of definitions of inflection points that are customarily presented in advanced calculus classrooms. This new collectively generated MKT was explicitly articulated by one of the teachers, as follows: “I think that everything we have seen so far shows that the correct definition of an inflection point is a point where the second derivative changes its sign, that is, there is an opposite sign in the neighborhoods before and after the point. The ‘usual’ definitions are incorrect – (1) a point where the second derivative is zero, and (2) a point where the first derivative has an extremum”.

The process of knowledge development also included valuable pedagogical ideas offered by participants, such as the idea to have students find on their own counter- examples to the “rule” that identifies inflection points with $f''(x)=0$. Another component of the process evolved during the session, when the goals of the videotaped teacher were discussed. Participants attempted to justify the choice of the teacher to present an inaccurate working definition, by ascribing to her two major considerations: firstly, students may not be ready to grasp the correct definition, which requires advanced thinking, and secondly, left/right derivatives and functions such as $x\cdot|x|$ are not included in the curriculum and in the final exams. This part of the discussion opened a debate on a more general question, i.e., when is it legitimate to “sacrifice” mathematical accuracy for the sake of our students’ best interests?

Example II: Are the commutative and associative properties interdependant? In this case, the teachers watched an episode from a lesson on the commutative and associative laws, given in a 7th grade heterogeneous class. Prior to watching the video, they were asked to elicit any mathematical ideas that may be associated with this topic. They suggested a fairly wide range of ideas, from the simple fact that addition and multiplication satisfy both laws, while subtraction and division do not, through various models that demonstrate the laws, to efficient solutions of multi-term exercises using the laws. It appeared that most teachers perceived the topic as natural and intuitive for students, at least in the numerical level. Thus, the lion’s share of the discussion was dedicated to considering the general algebraic forms of these properties (e.g., $a+b=b+a$), and suggesting why and how they should be taught. Some teachers viewed the teaching of the algebraic generalizations as necessary for consolidating students’ intuitive knowledge, while others perceived it as a difficult goal to achieve in 7th grade.

In the video episode screened, the teacher asked the class whether operations that satisfy the commutative law necessarily satisfy the associative law as well, and vice versa. The students’ spontaneous collective

Verbal description:	**Operation #1:** The operation on a given pair of numbers returns the *first* number in the pair.	**Operation #2:** The operation on a given pair of numbers returns the *larger* number in the pair.	**Operation #3:** None
Operation Table presented:	$\begin{array}{|c|c|c|c|c|} \hline \triangle & 0 & 1 & 2 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 1 & 1 \\ \hline 2 & 2 & 2 & 2 & 2 \\ \hline 3 & 3 & 3 & 3 & 3 \\ \hline \end{array}$	None	$\begin{array}{|c|c|c|c|} \hline * & a & b & c \\ \hline a & a & a & b \\ \hline b & a & c & b \\ \hline c & b & b & a \\ \hline \end{array}$
Commutative law	✗	✓	✓
Associative law	✓	✓	✗

Figure 2: The operations discussed in the 'commutative and associative laws' episode

answer was "yes". The teacher then introduced three examples of mathematical operations (see Figure 2), and led a discussion about which property exists in each example, resulting in the conclusion that the properties are not interdependent.

Each pair of teachers was requested to focus, while watching the episode, on one of the components of the analysis framework. Then, in the plenary, findings were described by the teams and discussed by all participants. According to our goal, we focus herein on the mathematical ideas which were elicited and discussed. It should be noted, however, that these ideas were triggered not only by the first component ('mathematical and meta-mathematical ideas'), but also when discussing the videotaped teacher's goals, choice of tasks, interactions with students and beliefs.

On the whole, teachers were surprised by the episode, since the main mathematical idea they have noticed was not included in the span of ideas constructed by the group earlier: they described it as "undermining the perception that an operation can either satisfy both the associative and commutative laws, or none of them". The teachers used representations from set theory to express this idea (see Figure 3), noting that addition and multiplication are in the intersection of the commutative operations and the associative operations sets, while subtraction and division are in the complement of the union of these sets. While students might hold the misconception that the other possible two sets are empty, the lesson demonstrates that operations exist in all possible sets.

A major discussion evolved around the use of operation tables to exemplify operations that satisfy only one of the two properties. Some teachers asserted that operations on small finite groups are not equivalent, both mathematically and pedagogically, to operations defined on the real numbers. The discussion facilitator asked the teachers to consider the advantages and disadvantages of the teacher's choice of operations. The advantages offered by participants were that (a) these examples clearly serve the teacher's apparent goal – challenging what students erroneously perceive as obvious; (b) the process of considering these operations and checking which properties they hold may contribute to the development of critical thinking, which is another goal that can be ascribed to the teacher; (c) using such examples conveys that operations

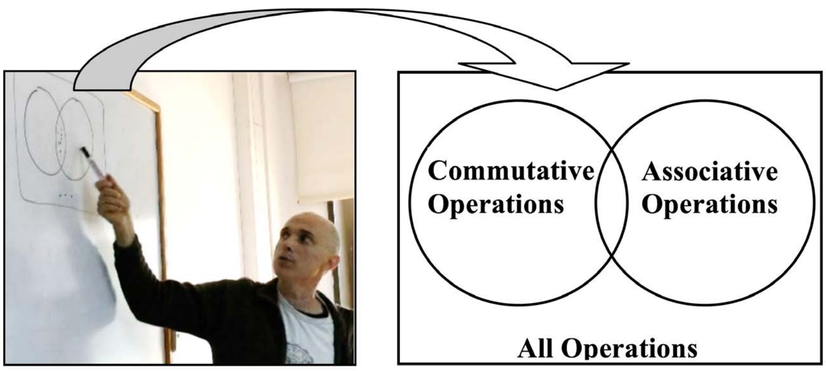


Figure 3: A teacher presenting the mathematical idea of the episode using set theory

can appear in different contexts, for example, real numbers, finite groups, or height of students (which was one way the teacher used to illustrate operation #2), and furthermore, demonstrate that operations can be defined for one's needs. On the other hand, the disadvantages mentioned concerned the same fact that the operations were "made up", i.e., somewhat artificial, and in 2 of the 3 examples were defined only on 3–4 objects. This was viewed by some teachers as limiting, irrelevant to the students' prior or subsequent knowledge and therefore unconvincing; they opined it was problematic to generalize from these examples that the laws are not interdependent. Thus, the teachers were challenged to find an operation, defined on the real numbers and relevant to students' school learning, for which only one of the properties holds. They have eventually found such examples: $a \square b = (a+b)^n$, $a \square b = \sin(a+b)$ and $a \square b = |a+b|$. In all these cases the operation satisfies the commutative law but not the associative law.

CONCLUDING WORDS

The instructional practice for using video with teachers is still underdeveloped (Ball, 2014). Despite a notable progress in this field, essential questions such as how to design and facilitate effective discussions need further exploration (Coles, 2013). This paper reports on a pilot work that may contribute to the development of such practice, by using a unique framework of analysis in video-based teacher discussions. The framework is deeply rooted in the subject matter of mathematics, thus discussions are perceived, alongside their role as promoters of reflective skills, as opportunities to deepen mathematical knowledge. The collaborative discussions support teachers' attempts to unpack the practice observed on the screen, through the implementation of mechanisms such as ascribing goals and weighing alternatives. The two examples presented clearly demonstrate the potential contribution of such video-based discussions to the evolution of a rich and multifaceted mathematical knowledge for teaching. We hope that the continuation of our studies, exploring teachers' use of this framework, will amplify the understanding about the nature and impact of this process.

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Pre-service mathematics teachers' scaffolding practices

Hulya Kilic

Yeditepe University, İstanbul, Turkey, kilichul@yahoo.com

The aim of this study was to investigate pre-service mathematics teachers' scaffolding practices occurred during one-to-one interactions with students. Seven pre-service teachers worked with a pair of sixth grade students for 12 weeks. Their interactions with students were videotaped and they were used the main data source for the analysis. Their interactions with students were analysed according to Anghileri's (2006) scaffolding practices framework. The findings revealed that the pre-service teachers' scaffolding practices were mostly based on reviewing what students did by using prompting and probing type of questions and asking for explanation and justification.

Keywords: Scaffolding, pre-service teachers, mathematics, one-to-one interactions.

INTRODUCTION

Student-centered teaching strategies are accepted to be an effective way of enhancing students' understanding and learning by constructivist theorists (Moore, 2009). In a constructivist teaching environment, teachers scaffold students' learning by arranging classroom settings, supplying appropriate materials, giving meaningful tasks or using effective questioning (Anghileri, 2006; Bliss, Askew, & Macrae, 1996; Hobsbaum, Peters, & Sylva, 1996; Wood, Bruner, & Ross, 1976). The effectiveness of such scaffolding practices depends on how effectively the teacher leads the student-teacher interactions for the sake of students' learning. To promote students' understanding, teachers should encourage students to explore, think and practice rather than simply telling and explaining (McCosker & Diezmann, 2009). As teachers gain experiences in teaching, specifically if they know how students learn, what is difficult or confusing for them and what misconceptions they have, better they begin to manage scaffolding practices (Goos, 2004).

For effective teaching, teachers should know how to teach the given topic to the particular group of students by using appropriate strategies, examples and representations, in other words, they should possess strong pedagogical content knowledge (PCK) (An, Kulm, & Wu, 2004; Ball, Thames, & Phelps, 2008; Shulman, 1986). An and colleagues (2004) defined knowledge of students' thinking is one of the components of PCK such that teachers should build on students' mathematical ideas, address to their misconceptions, engage them in mathematics learning and promote their mathematical thinking. Because scaffolding practices entails direct interactions with students aiming to contribute their understanding and learning, those practices are naturally counted as the signs of teachers' PCK (Appleton, 2008; Rosiek, 2003). In other words, teachers' scaffolding practices reflect their PCK. Furthermore, because the mission of teacher training programs is to prepare pre-service teachers (PSTs) to the profession, they should enable PSTs to begin to build up their PCK through experiences. Therefore, in order to contribute to the development of PSTs' PCK in a teacher training program, they might be given opportunities to experience some scaffolding practices with a small or large group of students and then reflect on their practices.

The aim of this study was to investigate how one-to-one interactions with a pair of students contribute to PSTs' PCK, in particular, their knowledge about how students think and how to scaffold students' understanding and learning. However, in this paper, PSTs' scaffolding practices occurred in three of twelve interactions are discussed in terms of the frequency of using those particular practices because overall analysis of data has not been completed yet.

THEORETICAL FRAMEWORK

Since the constructivist learning theory is the essence of many reformed school curricula and teaching practices, there are more studies on how to train PSTs to address the requirements of such reforms and practices. Although recent studies revealed that PSTs are lack of PCK (e.g., Morris, Hiebert, & Spitzer, 2009), engaging in one-to-one interactions with students aiming to understand their mathematical thinking or analysing videos of such interactions with the same purpose contributes to PSTs' PCK (e.g., Llinares & Valls, 2010; Stockero, 2008). During the interactions with students, PSTs need to provide necessary scaffolding through guidance and effective questioning. There are different scaffolding practices varying in terms of the effectiveness of students' conceptual understanding and learning. Anghileri (2006) defined 3 hierarchical levels of scaffolding practices to enhance mathematical learning (see Anghileri, 2006 for details). In this paper, her classification of scaffolding practices will be used to analyse PSTs' interactions with students.

Anghileri (2006) named the first level of scaffolding practices as *environmental provisions* such that teacher organize the classroom, provide necessary artefacts (manipulatives, tools) and tasks and follow a sequence during the instruction to help students understand the subject matter. At this level student-teacher interaction is not so direct with respect to the other levels. The second level is *explaining, reviewing and restructuring* and it involves direct student-teacher interactions. Explaining involves *showing and telling* and *teacher explaining* where the teacher has the control of the interaction. Reviewing involves five types of interactions: "1) getting students to *look, touch and verbalise* what they see and think, 2) getting students to *explain and justify*, 3) *interpreting students' actions* and talk, 4) using *prompting and probing questions*, and 5) *parallel modelling*" (p. 41). In the first type, the teacher encourages students to tell what they did, repeat the instructions or use manipulatives and reflect on what they observe. In the second type, the teacher asks students to explain their ideas by providing their reasoning so that the teacher gets opportunity to understand students' thinking and catch possible gaps in their reasoning. In the third type, the teacher attempts to make students' actions explicit for other students as well as for that particular student. In the fourth type, either the teacher directs the students to the predetermined solution or the solution in her mind through prompting questions or she attempts to encourage students to think deeply through probing questions. In the last type, the teacher provides a task similar to the one that the students have difficulty to understand and then she solves it. Anghileri (2006) defined four interactions for Restructuring: "1) provision of *meaningful contexts* to abstract situations, 2) *simplifying the problem* by constraining and limiting the degrees of freedom, 3) *rephrasing students' talk*, and 4) *negotiating meanings*" (p. 43). In the first type, the teacher provides real life problems for abstract calculations to help students to understand the meaning of the calculations. In the second type, the teacher gives a simpler problem or decreases the complexity of the problem to make it understandable for the students. In the third type, as different from *interpreting students' actions and talk*, the teacher provides appropriate terminology and encourages students to use appropriate mathematical language. In the fourth type, the teacher attempts to understand whether students have some misunderstandings or they are able to derive correct meanings from the interaction. The third level of scaffolding practices identified by Anghileri (2006) is *developing conceptual thinking*. She described those practices as to be "teaching interactions that explicitly addresses developing conceptual thinking by creating opportunities to reveal understandings to pupils and teachers together" (p. 47). She defined three types of such practices: 1) *making connections*, 2) *developing representational tools*, and 3) *generating conceptual discourse*. Making connections refers to showing how mathematical concepts are related to each other and encouraging students to use those relationships when necessary. In the second type, the teacher uses various representations (symbolic, numeric, graphical, etc.) to promote students understanding and facilitate making meaningful connections. In the third type, the teacher initiates a discussion to revise what have been said so far and what the students get from it eventually.

METHODOLOGY

Sample

Seven female pre-service mathematics teachers and 14 sixth-grade students participated in this study. Two of the PSTs were junior students and the rest were seniors. The PSTs voluntarily participated in the study. The sixth grade students were from the middle school that Faculty of Education had a *university-school col-*

laboration agreement. Those students were determined by their own mathematics teachers amongst the students who volunteered to participate in. The teachers informed us that it was a heterogeneous group in terms of mathematical achievement. The PSTs were randomly matched with a pair of students and they worked with those students during the whole semester.

Data collection

In this study, data was collected through the videos of interactions of PSTs with a pair of students, videos of group discussions before and after the intervention sessions, and PSTs' written reflections about the sessions. The data was collected for 14 weeks of 2014 spring semester but the PSTs worked with students for 12 weeks because the first week was used to introduce the aim of the study and the expectations from the PSTs and the last week was used for PSTs' evaluative feedbacks about the study.

At the beginning of the study, PSTs were told about how to scaffold students' understanding and learning by referring to the related literature (e.g., Anghileri, 2006; Moyer & Milewicz, 2002). They were given examples (in the form of video clips and vignettes) of what to notice during the interactions, how to use effective questioning and how to use representations and manipulatives to enhance students' mathematical thinking and understanding. As they worked with the students, some episodes were selected from their own interactions as an example to discuss in the following weeks.

The researcher prepared the tasks that PSTs would work with students. Those tasks were aligned the sixth grade math curriculum and they involved the topics that had already been covered in the school. The PSTs were asked to work on the task before the pre-group discussion. Each week the researcher and the PSTs discussed the tasks in terms of the expected students' performance. PSTs shared their ideas about possible misconceptions and misunderstandings and how to eliminate them. In some cases they suggested to make some revisions on the tasks and the revised versions were used during the intervention sessions. After 10 weeks, the PSTs were asked to prepare and implement two tasks for their own groups and give a rationale for preparing those tasks.

The intervention sessions took place in a classroom provided by the middle school administration. During the intervention sessions, the students firstly worked on the given tasks individually for 10–12 minutes and then they discussed their answers with each other for 5–10 minutes. During this period, the PSTs did not interact with the students, they just observed and videotaped what students did or said. Then she began to interact with them by asking what they did and why they did so. At the end of each intervention session, whole class discussion was carried out by the researcher to summarize the solutions of the given tasks. After then, the PSTs came back to the Faculty of Education which is located 10-minutes apart from the middle school to discuss how the session went. In this post-group discussion the PSTs talked about whether the students performed in a way that they expected, whether they were able to address students' difficulties or misconceptions effectively, whether that particular difficulty or misconception was common for the majority of the students. Then the PSTs were asked to watch their own intervention videos and write about what happened during the sessions, what they noticed, what questions they asked, etc. In the following week, the researcher gave feedbacks to the PSTs at the beginning of the pre-group discussion after reading their reflections and watching their videos. This procedure was followed for 12 weeks.

Data analysis

The videos of each intervention session of each PST and all written reflections will be analysed to understand how the setting of this study contributed to PSTs' PCK. However, the data analysis has not been completed yet. In this paper, three of intervention sessions were selected and analysed according to Anghileri's (2006) scaffolding practices framework. Three intervention sessions were determined according to the date of occurrence in order to discuss whether the type of scaffolding practices changed over time or not. Therefore, the videos of the third, the sixth and the tenth week were analysed. Each session was analysed in terms of whether the PST used that particular scaffolding practice or not. This analysis did not involve the evaluation of whether they used them effectively or whether it eventually helped students' understanding. Two researchers watched the videos together and determined what scaffolding practices occurred during the interactions. Therefore, the coding according to the framework based on the full agreement of the raters.

FINDINGS

The third, the sixth and the tenth weeks were selected for the analysis of PSTs' scaffolding practices. All scaffolding practices were coded at Level 2. None of them was at Level 1 or Level 3.

In the third week, the task was about divisibility rules. There were two questions in the task such that in the first question the students were asked to construct the smallest or the largest number that is divisible by 3, 4 or 6 by using the given digits. In the second question they were asked to figure out divisibility rule for 15 by filling out the given table. As shown in Table 1, PSTs mostly used *prompting and probing*, *looking, touching and verbalizing* and *students' explaining and justifying* type of scaffolding practices.

In the third week, the majority of the PSTs began discussion by asking to tell what the students did (*verbalisation*). They usually asked for *justification* of students' answers such as "Why do you think that 96 is the largest number that is divisible by 3?" Some of them used *prompting questions* to help students figure out the divisibility rule for 15. For instance, they said "Look at the numbers divisible by 15. Are there any other common numbers that they are divisible by?" Thus, the students were able to notice that those particular numbers were divisible by 3 and 5 at the same time. In some cases to make sure that students understood the divisibility rules, they repeatedly asked for them. For instance, after asking why 96 is divisible by 3 in the first item, they asked how 45 is divisible by 3 or why 80 is not divisible by 3 in the following items (*probing*).

In the sixth week, the task was about equations, inequalities and solving simple equations. There were three questions in the task such that in the first question the students were asked to determine the heaviest or the lightest object by looking at the given system of pan balances. In the second question, they were asked to figure out what to put on the left side of the pan balance by using the relationships in given two other balances. In the third question, they were given a system of balances involving different solids and asked to determine the height of each. As shown in Table 2, again PSTs mostly used prompting and probing, looking, touching and verbalizing and students' explaining and justifying type of scaffolding practices. The majority of the PSTs preferred to begin the interaction by asking what the students did. After listening to their answers if they did something wrong they

	Explaining		Reviewing					Restructuring				
	Showing and telling	Teacher explaining	Looking, touching and verbalizing	Prompting and probing	Students' explaining and justifying	Interpreting students' actions and talk	Parallel modelling	Providing meaningful context	Rephrasing students' talk	Negotiating meanings	Simplifying the problem	**Total**
PST_A	0	1	2	2	2	1	0	0	2	0	0	**10**
PST_B	0	0	2	2	2	2	0	0	0	0	0	**8**
PST_C	0	0	2	2	2	2	0	0	1	0	0	**9**
PST_D	0	1	2	2	1	2	0	0	0	0	0	**8**
PST_E	1	0	2	2	1	0	0	0	0	0	0	**6**
PST_F	0	2	1	1	1	2	0	0	0	0	0	**7**
PST_G	0	2	0	2	1	0	0	0	0	2	0	**7**
Total	**1**	**6**	**11**	**13**	**10**	**9**	**0**	**0**	**3**	**2**	**0**	

Table 1: The frequency of PSTs' scaffolding practices during the third week

	Explaining		Reviewing					Restructuring				
	Showing and telling	Teacher explaining	Looking, touching and verbalizing	Prompting and probing	Students' explaining and justifying	Interpreting students' actions and talk	Parallel modelling	Providing meaningful context	Rephrasing students' talk	Negotiating meanings	Simplifying the problem	**Total**
PST_A	0	1	3	1	1	1	0	0	2	0	0	**9**
PST_B	0	1	2	2	3	1	0	0	1	1	0	**11**
PST_C	0	3	0	2	2	3	0	1	1	1	0	**13**
PST_D	0	1	3	2	3	2	0	0	1	0	0	**12**
PST_E	0	0	2	3	1	0	0	0	0	1	0	**7**
PST_F	0	0	2	1	3	1	0	0	1	0	0	**8**
PST_G	1	1	1	1	0	1	0	1	0	2	0	**8**
Total	**1**	**7**	**13**	**12**	**13**	**9**	**0**	**2**	**6**	**5**	**0**	

Table 2: The frequency of PSTs' scaffolding practices during the sixth week

either encouraged them to *explain their reasoning* or asked *probing questions* to make them realize their mistakes. For instance, PST_F's one of the students thought that if three cubes are in balance with one sphere in a pan balance then the cube is heavier than the sphere. When she asked him to explain his reasoning he said that the amount of cubes is more than the amount of the sphere. Then she said that "numerically 3 is greater than 1 but what characteristics of the objects you need to pay attention when comparing them in a pan balance? That is, when they are in balance... The amount of the objects?" He thought for a while and told that he needs to compare the weight because there is inverse relation between the amount and the weight. That is, if there are more cubes than spheres it means that single cube is not enough to hold one sphere, that is, it is lighter.

In the tenth week, the task was about fractions. There were two questions in the task such that in the first question the students were asked to share the given amount fairly among the given number of people. In the second question, they were expected to solve a problem involving two-step partitioning. As shown in Table 3, PSTs mostly used looking, touching and verbalizing and prompting and probing type of scaffolding practices.

In this task, the PSTs asked students to represent the problems by using fraction tiles (*looking and touching*). Because most of the students failed to solve the problems correctly, such manipulatives helped them to understand their mistakes. Then, the PSTs helped students to find the answers by asking *prompting and probing* questions. For instance, some students thought that when they share 3 sandwiches among 6 kids, each gets 1/6 of the sandwich because when they divide each sandwich into two, they get 6 pieces totally and each kid gets one of the pieces. The PSTs asked students what 1/6 represents as a fraction and what the "whole" is in the given problem. The students realized that one sandwich is the "whole" and each kid is getting the half of a sandwich. That is, each kid is getting one of the six pieces but each piece is half of a sandwich.

DISCUSSION

The findings revealed that the PSTs' scaffolding practices were at Level 2 according to Anghileri's framework. Other studies also support this finding (e.g.,

	Explaining		Reviewing					Restructuring				
	Showing and telling	Teacher explaining	Looking, touching and verbalizing	Prompting and probing	Students' explaining and justifying	Interpreting students' actions and talk	Parallel modelling	Providing meaningful context	Rephrasing students' talk	Negotiating meanings	Simplifying the problem	**Total**
PST_A	0	1	2	1	1	2	0	0	1	1	0	**9**
PST_B	0	1	2	2	1	1	0	0	1	0	0	**8**
PST_C	0	1	1	2	1	1	0	0	0	1	0	**7**
PST_D	0	1	2	2	2	1	0	0	0	0	0	**8**
PST_E	1	0	2	2	2	0	0	0	1	0	0	**8**
PST_F	0	2	2	2	0	1	0	0	0	0	0	**7**
PST_G	0	1	2	1	0	1	0	0	0	1	0	**6**
Total	**1**	**7**	**13**	**12**	**7**	**7**	**0**	**0**	**3**	**3**	**0**	

Table 3: The frequency of PSTs' scaffolding practices during the tenth week

Kaldrimidou, Sakonidis, & Tzekaki, 2011). Probably because the setting was not an actual classroom environment and the PSTs worked with just a pair of students, many of the scaffolding practices at Level 1 (*environmental provisions*) were not observed. However, the setting allows the usage of Level 3 scaffolding practices but none was observed in those three weeks.

A similar pattern in terms of the frequency of scaffolding practices was observed in those three weeks. The PSTs mostly used *reviewing* type of practices. The *parallel modelling* and *simplifying the problem* practices were not observed in any of three weeks. Because the PSTs were told not "to teach" but "to help" students, they might avoid using modelling practice to support their learning. Furthermore, because the tasks were clear and simple for the sixth grade students, they might not need to simplify the problems.

Having similar pattern in scaffolding practices was not surprising because the PSTs were encouraged to try to understand students' mathematics and their thinking. Therefore, they mostly used prompting and probing questions and asked for students' explanations and justifications. Although the discussion of how effectively they used those questions is not the scope of this paper, such questioning seemingly contributed to their PCK in terms of knowledge of students and teaching strategies as noted in the studies of other scholars (e.g., Llinares & Valls, 2010; Stockero, 2008). For instance, as time passed, the PSTs began to estimate the students' performances more accurately. Before the implementation of the task in the tenth week, some of the PSTs claimed that their students would answer the sharing question as 1/6 because they would count the pieces rather than using the definition of fraction and those students did so. However, a further analysis needs to be done to claim how this setting helped the development of PSTs' PCK.

Although it is expected that PSTs would use higher level of scaffolding practices over time, it was not the case for those three weeks. For each session the PSTs inclined to explain some facts, review the discussed concepts or tell what to do. The PSTs mostly used *teacher explanation* towards to the end of the sessions. The reasoning behind this might be to make sure that everything is clear in students' mind and there are no misunderstandings. The PSTs used *showing and telling* when they realized that the students' did not possess/remember prior knowledge to solve the given problems. When the students did not remember the mathematical concepts involve in the tasks, they did not do anything with the task. This "showing and telling" practice commonly occurs whenever the teacher notice that the students are not able to engage in task because of lack of knowledge (Kim & Hannafin, 2011; Stylianides & Stylianides, 2011). Therefore, this tendency of PSTs is understandable.

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Advice and guidance for students enrolled in teaching mathematics at primary level

Nicole Koppitz and Christof Schreiber

Justus Liebig University Gießen, Gießen, Germany, nicole.koppitz@math.uni-giessen.de

In the winter semester 2013/14, the department of Mathematics Education at the Justus Liebig University of Gießen launched a two-year-project offering assistance to students aspiring to become primary school teachers. The aim is to identify and rectify deficits pertaining to core mathematics. Moreover, the objective is to develop a positive attitude towards mathematics. This project applies to all but particularly to students, who have a weak mathematical foundation and require a boost in motivation. Data from the student's initial standard is used to develop practical modules such as tutorials, reflections, learning portfolios, consultations as well as systems that help manage individual learning.

Keywords: MKT, primary school preservice teacher education, reflection, learning opportunities, education support programme.

INTRODUCTION

Various studies have shown that a teacher's mathematical foundation plays a key role in good teaching (Döhrmann, Blömeke, & Kaiser 2010). As such, the German state of Hesse has made German and mathematics mandatory subjects for teacher-students, who are studying to become primary school teachers (Grade 1 to 6). Hence, passing these two subjects is a prerequisite to becoming a primary school teacher. Our investigation is about the skills, beliefs, self-concepts and motivations of teacher-students in the first year of their studies. In this paper, we focus solely on the students' skills and the various support-offers for students.

According to our surveys, only about half of the students, who enrolled into the course of becoming a primary teacher, would have opted to study mathematics by choice (see Table 1). The study groups have heterogeneous motivation and foundation levels. Among students who pass mathematics with ease, there are also struggling students that attend math lectures. Therefore, weaker students need to overcome their deficits. Fears, which may have developed and accumulated over time, are also covered in this project. If the deficits in math are not eliminated, the path of becoming a primary teacher cannot continue despite the level of motivation or performance in German and their third subject of choice. Should this be the case, students would be forced to change to secondary teaching or transfer to a university located in another state of Germany.

PROJECT IDEA

Due to the pressing issue mentioned above, the department of Mathematics Education of the Justus Liebig University of Gießen has launched a two-year-long project since the winter semester 2013/14. The aim of this project is to identify potential deficits of students' with regard to their skills, problems, attitude as well as their self-concepts pertaining to mathematics. Thereupon, supportive arrangements should be developed and evaluated.

All students, including those with weak mathematical foundations and low motivational drive, are supported throughout their course of study. In an attempt to overcome learning difficulties, students learn to reflect and develop a positive attitude towards mathematics through consultation sessions and additional training programmes provided by the department. This is done based on retrieved data from the analyses of the students. The level of math expertise and motivation as well as self-concepts of one's own abilities are to be inquired. In addition, the attitude towards mathematics needs to be identified. The results from these steps are used to customise consultations and training programmes for students in need of assistance.

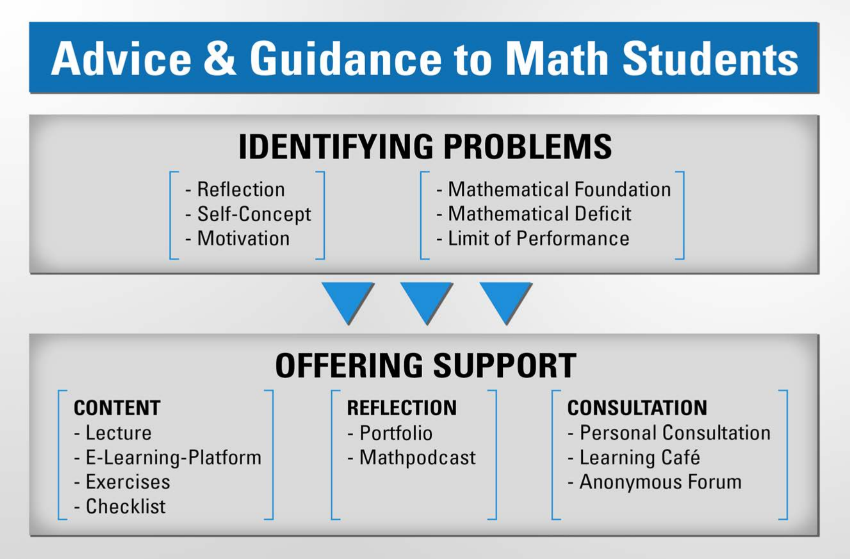

Figure 1: An overview of the project idea

INVESTIGATION

This investigation focuses on primary school teacher-students, who are in their first two semesters of academic study and may experience a drastic change: changing from school to university life. They are now confronted with having to be responsible for their own attendance of lectures and tutorials. Their whole academic study period is made of 7 semesters (3 ½ years) including a semester of examination. It has a total of 180 credit-points (ECTS). In the department of Mathematics Education they have to earn 26 or 38 credit-points, depending on whether they choose to do an internship for German or mathematics. The lecture, pertaining to this project, deals with 'subject matter knowledge' (Ball, Thames, & Phelps, 2008), together with the tutorials constitutes of about 330 hours of work, which are equivalent to 11 credit-points (ECTS). The module consists of two main lectures that run over two semesters and are accompanied by tutorials. A final exam will round up the second semester. The lecture's content covers in particular the basics of arithmetic, stochastic and geometry. Around 190 students attend the first academic year. In the first year, the dropout rate, usually before the second semester's final exam, is about 10%. We gathered data of all these students in order to get a good overview about their skills, their motivation and their beliefs. In future, we hope to continue to evaluate individual students to investigate different cases in greater detail.

At the beginning of their first semester, students completed a mathematical proficiency test and filled out a questionnaire anonymously with regard to their personal attitude towards mathematics (Grigutsch, Raatz, & Törner, 1998), their self-concepts (Dickhäuser, Schöne, Spinath, & Stiensmeier-Pelster, 2000) and level of motivation (Brandstätter, Schüler, Puca, & Lozo, 2013) towards their tertiary studies. The mathematical problems that had to be solved in the test were mostly from junior high school. The math problems were selected according to their mathematical relevance at university level, so as to identify the student's mathematical knowledge boundaries. These results are explicitly of a pilot-study and they are still preliminary in nature. Therefore, we will only describe observed trends and provide a few examples from the tests of October 2014 in the following (n=155):

Pertaining to individual personal assessment, we observed that a majority of students started their studies with overall low levels of motivation. Only about 32% of the students have indicated to be of the opinion that they are good enough to meet the demands of their mathematics course. However, 93% of them believe that they will become great math teachers.

Item	Strongly Agree (%)	Agree (%)	Agree Less (%)	Disagree (%)	No Decla-ration (%)
Having been given the freedom of choosing a subject, I chose mathematics.	18	31	15	33	3

Table 1: Choosing Mathematics freely

Item	Strongly Agree (%)	Agree (%)	Agree Less (%)	Disagree (%)	No Decla-ration (%)
I believe I can meet the demands of the subject mathematics.	32	38	23	7	0
I believe that I can be a good mathematics teachers.	18	75	6	0	1

Table 2: Excerpt of the questionnaire concerning students' self-concepts

We have to consider that these results are based on the students' self-assessments. 70% of the students feel, that they are able to study the math they need for becoming a primary school teacher. Interestingly, the percentage of students who believe they already have the required knowledge, does not match the results of the mathematical proficiency test.

The evaluation of the mathematical proficiency test shows that 80–95% of the students were able to solve questions of the German 'Abitur' (general qualification for university entrance) standard. In particular, applying Pythagoras theorem and mathematical equations. Moreover, commonly used units such as length or currency were converted correctly with ease.

The following results show that one third or even half of the math students cannot apply their knowledge learnt in school. This correlates with our assumption that there is a need to provide support options for the weaker students. Thus, these problem areas are indeed important and relevant to the educational path of studying to become a primary school teacher.

In the following year, we will try to achieve a greater participation of students for the questionnaire, so as to optimise the foundation of our support options. Furthermore, we will evaluate the initial approaches that started in the first year of the project in the following paragraphs.

After having gathered the data mentioned above, we have chosen a broad sample of students to be interviewed. The interviews adopted the problem-centred approach by Witzel (2000). Each student is interviewed 4–5 times. It should be noted, that the focus lies not on the interviews nor on the questionnaires, but rather on the upcoming project approaches.

PROJECT APPROACH

As the project launched in the winter semester 2013/14, only a few approaches to address the identified deficits could be outlined. We have conducted another inquiry, although still ongoing, dealing with the initial circumstances that the students begin their studies with. By the means of this data, measures are customised to fit the needs of the students. The measures, which have already been implemented and those that have yet to be, will be described in the following. We have to consider, that it is an alternating iterative process: There are existing measures and some that have yet to be implemented, but the problems are not yet sufficiently identified in detail. At the same time we have to support the student's learning, we have to identify problems, prevent them from occurring and evaluate measures.

Exercises

Besides lectures, exercises play the most important role in learning. Students may even feel that exercises

Item	Tested skills	Answered Correctly (%)	Answered Wrongly (%)	No Attempt (%)
Solve the following problem: 9475:25=	Basic Arithmetic Operations	65	27	8
In a square, draw all the axes of symmetry.	Concepts of Plane Figures, Symmetry	76	16	8
In a parallelogram, draw all the axes of symmetry.	Concepts of Plane Figures, Symmetry	48	44	8
How many results can be produced by rolling a dice four times consecutively.	Combinatorics, Concepts of Multiplication	47	36	17

Table 3: Examples of the mathematical part of the questionnaire

go before lectures. As the feeling is mutual, this paper will focus on the importance of exercises. As the project began upon the commencement of the winter semester lectures, the already prepared math exercises received only minor adjustments. Just like in previous semesters, students were assigned exercises to deepen their understanding of the lecture content. These exercises were discussed in weekly tutorials lead by students of higher semesters, supervised by a research associate. The exercises can be categorised into five types, namely arithmetic problems, application questions, problem solving and modelling tasks as well as justifying mathematical concepts.

In the following semester, the exercises will be restructured to take the form of "productive practicing" (Leuders, 2014). The exercises should help sustain knowledge and be retrievable at any given point in time. During these exercises, automated calculations should be prevented, as they tend to be forgotten easily. Nevertheless, it is important to strengthen and link mathematical understanding to arithmetic technique. In each series of exercises, the following competences should be covered: 'Securing factual content', 'automation of skills', 'broadening and deepening of perception', 'reflection of concepts and their application', 'applying knowledge pertaining to problem-solving' as well as 'creating an appropriate impression of mathematics' (Leuders, 2014, p. 253; translation by the authors). As such, the homework of the following semester will comprise of three parts:

- The first part deals with tasks in which students are taught to solve problems on a syntax level. This skill should be integrated into structure and problem-solving tasks. Either a structure is investigated to practice applying skills or simple forms of problem-solving tasks can be used. As each student approaches the task on his own terms, they develop a personal heuristic to solving problems using the given arithmetic rules. The tasks remain at a standard that allows students to check if they have understood their calculation method and the arithmetic rules used. Two days before the tutorial, students receive a suggestion sheet providing a solving method for the task. If necessary, the students can receive help using the online-forum or bring their query to the tutorials.

- The second part covers mathematical problems that are in line with the competences mentioned above. These tasks should meet the demands of the math lecture. The students are given one week to solve the problems. These problems aim to teach the students mathematical understanding, justification, problem-solving and derivational skills. After the given week, the suggested answers are only handed out if they have been compared and discussed within the tutorial. The tutorials actively incorporate the student's input and provide a platform for communication to take place.

- The third part occurs during the tutorials, where math problems are done in class. These tasks can cover various types of the targeted competences. Due to the broad spectrum of lecture topics available, various types of questions can be chosen for discussion in the tutorial. Moreover, mathematical topics can be explored and nurture comprehension by using the resources provided by the Math Learning Lab to create practical situations.

Personal consultation

After the mathematical proficiency test had been evaluated in the first part of the project, students were given the opportunity to seek advice and support through consultation sessions. Consultations were visited on a voluntarily basis, in light of having to remove a student's anonymity in order to discuss his test results.

First, the students were asked about their own expectations. The idea was to observe if the students were able to assess themselves and reflect on their performance. Thereafter, the students received feedback pertaining to the test. Having identified their strengths and weaknesses, they were given advice on what topics to focus and brush up on. These suggestions were supposed to help the students to cope with the expectations of the syllabus. If deemed necessary, a follow-up consultation could be arranged to discuss topics of the first session in greater depth.

As the project continues, we plan on implementing consultation sessions addressing topics beyond the proficiency test, so that students are able to receive advice and support anytime. In order to make these consultation sessions more sustainable in the long run, consultation sheets have been introduced to secure results and progress as well as to give students a way to prepare for the sessions (Macke, Hanke, & Viehmann, 2012).

E-learning platform

The University provided an online learning platform called 'Ilias' for the students. Based on this platform we provide tasks, which enable students to practice additional math problems at their own time, while being able to instantly monitor their learning progress. The following types of math problems were incorporated into this system.

- Some problems merely recapped mathematics of secondary education. Algebra, geometry, arithmetic and stochastic theory make up the foundation and prerequisites of studying mathematics at tertiary level.
- The majority of problems were similar to the above-described homework. The tasks are derived from lecture content and serve as additional practice. These tasks are especially relevant for the examination at the end of the lecture-period.
- Some tasks go beyond basics and covered lecture content. This is done, so that students receive a broader spectrum of topics and greater understanding through exploring other topics on their own.

In future, more features will be added to this platform in order to break down complex lecture topics and to be able to tackle them with a variety of methods. For example, relevant Internet links and teaching videos can be useful for post-preparation of lecture content. In addition, the learning platform is undergoing developments to enable the upload of mock exams for additional exam preparation. As a result, students receive a first glance and what might face them during an exam.

Portfolio

Different forms of e-Portfolios used in teacher education are described by Vogel and Schneider (2012). Our paper exclusively refers to a 'reflection-portfolio' (Vogel & Schneider, 2012, p. 136, translation by the authors). Nearly every exercise worksheet has questions incorporated that trigger self-reflection. These are supposed to be worked on individually and filed, so that each student has a portfolio that documents their learning progress over the first year of their studies. Up to now, the students were required to reflect on the following:

- Personal experiences of primary and secondary school
- Their opinion of certain lecture topics being relevant for teaching in primary school
- The importance of mathematics and certain lecture topics to society
- Their factors which motivate them to become a teacher

At the end of the summer semester 2014, the portfolios were collected, looked through and given feedback by the lecturers. On one hand, the portfolios shed light on personal attitudes and each learning progress. On the other hand, they revealed that several students did not take the reflections seriously as they were written rather superficially. Bräuer (2014) experienced the same scenario and demonstrates how students could be supported.

Hence, the next round of portfolios will be integrated into the platform's server. The lecturers can not only monitor entries and strategically upload additional exercises and revision material but also address problems and difficulties in the lecture itself once voiced out in the reflections. If necessary, the lecturer can also leave comments immediately and encourage the student to reflect deeper (Bräuer, 2014). The portfolios can also serve as personal learning diaries or be published on a blog to help others too.

Anonymous forum

From the winter semester 2014/15 onwards, students have the opportunity to post questions onto a second forum. Previously, students had the tendency of asking questions directly to their tutors via email. Thus, interactive discussions on the forum were non-existent and there was no need for other kinds of support. Furthermore, many questions were asked repeatedly. It was not clear if they were general problems or isolated incidents. We assumed that the reason students had refrained from using the non-anonymous forum was due to the fear of humiliating themselves when posting their question. Therefore, the anonymous forum aims to promote the asking of questions through the means of an open and safe platform. As the students are generally familiar with the social-media platform 'Facebook', Kempen (2014) refers to it as an appropriate tool for exchanges between the students. We however, chose 'Ilias' (s. above), an e-learning platform that is more secure and ensures more privacy.

Learning Café

Students can come to the Learning Café located at the institute's Math Learning Lab. They will be given a room in which they can meet to discuss math problems. Additionally, a staff member of the lecture will be present to provide answers to questions students might have. Some math concepts can be visualized through the ready-available resources found in the Math Learning Lab. Over time, study groups may develop, as the Learning Café serves as a conducive meeting point. Students can create "learning communities" (Gunnarsdottir & Palsdottir, 2013, p. 3085) to acquire and develop a professional jargon as well as collaborative competences. This suits reserved students, who may now have the courage to clarify concepts with lecturers, tutors or even students directly. A similar project is described by Zimmermann (2012), in which though, these opportunities are offered on a daily basis and for students of different ages.

MathPodcast

Students can volunteer to participate in a podcast project that runs alongside the lectures. Audio podcasts are created within small groups. They are based on math lectures and trigger students to reflect on mathematical content and their standard of math skills. They need to explain a lecture topic verbally without the use of gestures or other materials. The podcast production starts with the recording of a spontaneous attempt of trying to explain the topic using basic knowledge. This recording will then be fine-tuned through collective research. Thereafter, a script will be designed which will be used to create a podcast ready to publish (Schreiber, 2013). This process is suitable to allow students to reflect on a particular math topic. Tutors will accompany the podcast productions throughout, while the students will receive feedback on their correct and wrong understanding of concepts (Klose, Tebaartz, Schreiber, & Lengnink, 2014).

FURTHER PROSPECTS

In the second part of the project it is important to get data of all the students, not only of 110 out of 160 like we did this semester. The E-Learning-Platform will be available once the lecture-period begins. As such, the students can benefit from its content throughout the semester and we receive access to evaluate the platform's success. Furthermore, the Learning Café and tutorials will be offered from the first week onwards throughout the lecture-period. Hence, we can evaluate the students' reception of both support options and optimize them further.

The project's running time is limited to two years. It serves to develop and implement measures that nurture and foster mathematical understanding. Once the project is over, the measures have to be continued without any additional financial support. Many points mentioned above are easily implemented and complemented in terms of materials. Specific assistance can only be given if the evaluation of a student's learning process requires minimal effort. Therefore, it is vital to carry out an online evaluation, so that lecturers receive an overview by the means of simple statistical data on the current learning standard of the students.

ACKNOWLEDGEMENT

This study is supported by HMWK (Hessisches Ministerium für Wissenschaft und Kunst) titled "Weiterentwicklung von Beratungs- und Begleitangeboten für Studienanfänger/innen im Studiengang L1 (Primarstufe)".

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A protocol for analysing mathematics teacher educators' practices

Ana Kuzle[1,2] and Rolf Biehler[2]

1 University of Osnabrück, Osnabrück, Germany

2 University of Paderborn, Paderborn, Germany, biehler@math.upb.de

Studying practices in a teaching-learning environment, such as professional development programmes, is a complex and multi-faceted endeavour. While several frameworks exist to help researchers analyse teaching practices, none exist to analyse practices of those who organize professional development programmes, namely mathematics teacher educators. In this paper, based on theoretical as well as empirical results, we present a protocol for capturing different aspects of mathematics teacher educators' practices in a professional development setting. Implications for professional development programmes' planning, implementation and evaluations are given at the end of the paper.

Keywords: Mathematics teacher educators, professional development, training practices.

PROFESSIONAL DEVELOPERS OF MATHEMATICS TEACHERS

The teaching-learning quality in schools and, with it, teacher professionalisation has long been a concern of educational organisations, schools, governments and researchers and continues to grow in importance. Many countries, such as England (National Centre for Excellence in the Teaching of Mathematics (NCETM)), Sweden (National Centre for Mathematics Education), Austria (Austrian Centre for Instructional and School Development), set their focus to improve education, and in that effort, are investing vast resources in teacher professional development. These countries established so called CPD institutions, which offer a systematic approach to large scale and sustainable CPD, establishing standards for high quality CPDs. This has been recognised as one driving force to allow reform of school mathematics. Germany is not an exception to this phenomenon. For years already, the German educational system is being challenged with many difficulties, such as considerable shortage of specialised secondary subject teachers, high quota of at-risk students, more and more out-of-field teachers, and a high variability between different federal states (Kramer & Lange, 2014). In 2010, the German Center for Mathematics Teacher Education (DZLM) was founded as a cooperation of seven universities, and is oriented at improving and innovating the German mathematics classroom. The core objectives of the centre are to promote continuous professional development, set nationwide standards for it, and develop quality needs-based CPD programmes.

However, do we have the capacity to reach plethora of mathematics teachers? Gal (2013) in his PME plenary talk in Kiel discussed this issue in a sense of *2.5% rule*. The 2.5% rule denotes the fact that only 2.5% of teachers are novice teachers coming from the university. Thus, this small group might receive "up to date" quality teaching and learning ideas as advocated by current professional organisations and new curricula. Gal further warned that, when we train as many inservice as preservice teachers, a quota of barely 5% can be achieved. This fact raises the need to develop new structures within the professional development institutions to reach the big masses. Along these ideas, DZLM's key mission is to develop comprehensive training programs to educate mathematics teacher educators (MTEs), sometimes shortly called "multipliers", as it would allow for a large scale dissemination of the centre's initiative. Depending on the country, MTEs take upon different roles (e.g., coach, mentor, specialist). They are responsible for strengthening classroom teachers' understanding of mathematics content, and helping teachers develop more effective mathematics teaching practices, and by doing that to better student learning. Hence, MTEs are central for providing opportunities for teachers' professional development.

Elliot and colleagues (2009) contend that this area has been understudied – we know very little as to what MTEs need to know and be able to do – but is growing in its importance (e.g., Kuzle & Biehler, 2015; Rösken-Winter et al., 2015). Understanding MTE's practices is essential for attending their diverse needs – during planning, implementation, and evaluation of the professional development programmes. The more we know about how to support them, the more mathematics teachers can benefit. In order to fill in this gap, we present here a protocol for analysing MTEs' training practices, which was developed on the basis of effective CPD practices. We contend that such protocol may allow scholars with lenses for evaluating MTEs' PD courses and at the same time supporting their further development.

THEORETICAL CONSIDERATIONS WITH RESPECT TO EFFECTIVE CPD FACTORS

In this section, we give a succinct overview of descriptors for effective professional development from the literature (e.g., Cochran-Smith & Lytle, 1999; Garet et al., 2001; Lipowsky, 2004; Lipowsky & Rzejak, 2012; Putnam & Borko, 2000). These are elaborated by outlining DZLM's (2013) six PD design principles, which were used as a theoretical frame in this paper as well as in Kuzle and Biehler (2015). There is a small overlap between the two papers, however, without literal quoting.

Learner-orientation

Effective professional development links directly to teachers' job, namely teachers' curriculum, and their specific needs and concerns. In other words, the training courses focus on the individual, heterogeneous prerequisites and needs of the participants. They encourage and demand active and responsible participation of the participants in design and implementation of the PD. Hence, the participants are not informed, but involved as active learners in this process through which they develop their professional knowledge (Garet et al., 2001). However, PDs that are based solely on this knowledge do not suffice for influencing their actions. They need to, however, address the individual circumstances of the participants, account for participants' daily activities, and capitalise on teacher's prior knowledge and experiences. Ensuring and amplifying learner-orientation can be achieved through designing and administering preliminary inquiry regarding participants' experiences, expectations, and needs (Kuzle & Biehler, 2015).

Case-based learning

Relating CPD practices to the participant, his or her experiences, teaching and/or student learning supports teacher motivation and commitment to the learning process. In other words, the reference to teachers' everyday situations, so called "cases" or "training cases" serve both as a starting point and as a field of application for teaching and learning in the context of the PD. To intensify case-relatedness the MTEs can incorporate their student work or use participants' practical experiences (Kuzle & Biehler, 2015). In either case, subject-specific learning processes and learning outcomes of learners (e.g., pupils) should be diagnosed, interpreted, and direct consequences for the teaching practices should be extracted (e.g., Lipowsky & Rzejak, 2012; Timperley, 2011). Combining individual needs with overreaching goals of the professional development initiative, strengthens teacher commitment to the PD and increases motivation to learn (Timperley, 2011).

Competence development

Effective professional development is coherent; competence and goal orientation are a crucial prerequisite for a didactical and organisational design of PD that satisfy "depth and breadth of impact" (e.g., Garet et al., 2001; Lipowsky & Rzejak, 2012). This construct is multi-faceted and encompasses all resources teachers need to create quality teaching-learning environments. Among others, competence development should address teachers' professional knowledge, including also orientation towards students' learning (Lipowsky & Rzejak, 2012). This competence and goal orientation should be transparent for all concerned parties (Kuzle & Biehler, 2015). Only then can the references to one's own teaching practices become clear and the implementation tangible (Elliot et al., 2009).

Application of various instructional formats

To ensure interactive learning experiences, various instruction formats should be combined in collaboration with the leader(s) and other teachers. A diverse variety of working methods (e.g., blended learning seminars, practice- and collaborative based work, self-study) supports participants in their skill acquisition, and helps accommodate different learning styles and preferences. The participants must also be given time to engage in different activities at different levels and in different settings in order to learn or consolidate their knowledge (Putnam & Borko, 2000). In addition, an intertwinement of input, active learning and reflection phases (so called "sandwich model") in crucial for connection between theory and practice.

Stimulating collaboration

Another essential CPD design aspect is to stimulate cooperation among the participants, and between the participants and the professional developer. This fosters exchange of experiences. Whereas using various instructional formats begins at fostering collaboration (short-term collaboration), these could be used as a starting point for a collaboration going beyond the CPD course itself (long-term collaboration). Thus, beyond just sharing ideas, and reflecting on the learning process, the participants, for instance, work together towards a common goal, jointly plan lesson, and organise mutual classroom visits (Lipowsky & Rzejak, 2012). In that manner, community building and networking can take place, which is together with professional developer-teacher structure, important for sustainability of PDs (Zehetmeier & Krainer, 2011). The extent of effect is dependent on the PD format.

Fostering (self-)reflection

Relevance of a PD and the sustainability of the professional learning can also be attained through reflective activities (Ingvarson, Meiers, & Beavis, 2005). Participants are encouraged to engage in collaborative and self-reflection on covered topics/material as well as on their own teaching, student work, attitudes, conceptions, and other. Through reflective practices the teachers can consolidate their skills and knowledge, and better understand teaching and learning in the classroom (e.g., Cochran-Smith & Lytle, 1999; Putnam & Borko, 2000). Reflective practices are most effective when reviewed throughout the PD (Ingvarson et al., 2005).

PROTOCOL FOR ANALYSING TRAINING PRACTICES

In the previous section and in Kuzle and Biehler (2015) we outlined effective professional development design principles. As seen above each design principle is composed of many different attributes. These were used to create a protocol for analysing what different attributes of each design principle can get implemented (see Table 1). In addition, we assigned to what extent these practices can be addressed in a PD with "yes", "no", and "partly".

This instrument was tested on a sample of PDs, that took place as a part of five month long DZLM's CPD "Competence-oriented teaching and learning of data analysis" for MTEs (Biehler, Kuzle, & Wassong, submitted; Kuzle & Biehler, 2015), which was developed by a team of researchers from the University of Paderborn (Biehler, Kuzle, Oesterhaus, Wassong). The CPD program focused on deepening MTEs' professional knowledge of teaching statistics using digital tools, and developing MTEs' competencies and knowledge for developing and implementing PD in statistics. As a part of that CPD, five MTEs' teams developed and implemented five 4-hour long PDs on teaching data analysis with statistical software. The general structure was prescribed by the course designers and was composed of 4 thematic blocks: (1) introductory block (ca. 1 hour), (2) block 1 (1¼ hours), (3) block 2 (1¼ hours), and (4) reflection and closure (ca. ½ hour). While the general function of the first and last block was clear, the mentors were free to organize and implement blocks 1 and 2, however, they had to select content and activities from the CPD for them and implement DZLM design principles. These were video-taped and then analysed using content method analysis as suggested by Miles and Huberman (1994).

For filling in the protocol, we used a five-step procedure for analysing the MTEs' practices exhibited in each PD. First we divided the first short PD into blocks. Secondly, for each block in the PD 1 we identified whether different design principles occurred at all. Thirdly, after having determined the six design principles and its accompanying attributes for each block, we looked at their quality in each block. For each block, we assigned the three categories, yes, partly, no, to each of the characteristic attributes for each design principle based on the following algorithm:

- A statement got categorised as "yes" when it was thoroughly and thoughtfully addressed within the block.
- A statement got categorised as "partly" when it was either not thoroughly or thoughtfully addressed within the block.
- A statement got categorised as "no" when it was not addressed whatsoever within the block.

DESIGN PRINCIPLES	Addressed level		
	Yes	Partly	No
Learner-orientation			
MTEs' focus of the PD is of relevance to MTs.			
MTE provides opportunities for MTs to share experiences with respect to the PD topic.			
MTE designs the PD in a manner that allows MTs to be integrated into the learning process as active learns through hands-on activities.			
MTE designs the PD in a manner that allows MTs to be integrated into the learning process as active learners through hands-on technology use.			
MTE provides MTs opportunities to actively build their content knowledge on the basis of their existing knowledge and experiences.			
MTE provides MTs opportunities to actively build their pedagogical content knowledge on the basis of their existing knowledge and experiences.			
MTE provides MTs opportunities to actively build their technological knowledge on the basis of their existing knowledge and experiences.			
MTE provides MTs with opportunities to actively build their technological pedagogical content knowledge on the basis of their existing knowledge and experiences.			
MTE provides MTs with opportunities to build an understanding of student's thinking in a specific area.			
Case-based learning			
PD connects to MTs' teaching practices.			
PD combines the MTs' needs with the goals of educational initiative.			
MTE integrates MTs' input with respect to the topic of the PD.			
MTE addresses explicitly specific needs and concerns of MTs with respect to their teaching experiences and daily concerns.			
MTE allows MTs to apply newly learned knowledge into follow-up activities.			
MTE allows MTs to discuss newly learned knowledge with respect to their teaching practices.			
The content of the PD is illustrated on real student work (artefacts).			
MTs analyse student work, interpret it and reflect on their student learning in their own classroom or in the classroom of other MTs.			
Competence development			
MTE's focus of the PD connects to specific curricula and learning standards.			
MTE has a well-defined image of effective classroom learning and teaching.			
MTE makes/describes clearly fostered competencies and/or goals are transparent.			
MTE focuses on developing MTs' content knowledge.			
MTE focuses on developing MTs' pedagogical content knowledge.			
MTE focuses on developing MTs' technological knowledge.			
MTE focuses on developing MTs' technological pedagogical content knowledge.			
MTE engages MTs as adult learners in the learning approaches.			
MTE models pedagogy and various instructional strategies for the whole sequence of lessons designed to support development of conceptual understanding.			
MTE provides MTs with concrete ideas for implementing new materials and/or ideas in own classroom.			
MTE supports MTs in understanding student thinking with respect to the topic.			
MTE emphasises how to improve student learning.			
MTE helps MTs anticipate possible student learning difficulties and/or misconceptions.			
Application of various instructional formats			
MTE accommodates individual learning styles and preferences.			

MTE engages MTs in different learning formats such as sharing and discussion, reflection, solving problems.			
In a PD, input, active learning and reflection phases are intertwined.			
MTE relates different parts of the PD one to another.			
Stimulating collaboration			
MTE provides MTs with opportunities to collaborate with other MTs (share ideas, viewpoints, work together).			
MTE supports MTs to develop their professional expertise and to serve in leadership roles.			
MTE supports MTs to plan together instruction and/or analyse student work with respect to a common goal.			
MTE offers MTs support beyond the PD course itself.			
Fostering (self-)reflection			
MTE provides MTs with opportunities to reflect throughout the PD.			
MTE provides MTs with opportunities to reflect critically on their teaching practices.			
MTE provides MTs with opportunities to reflect critically on the new ideas particularly with regards to their teaching practices, and experiences.			

Table 1: Protocol for analysing MTEs' training practices

In the fourth step, on the basis of step 3 we assigned the three categories, yes, partly, no, to each of the characteristic attributes for each design principle for the PD as a whole. The categorisation was based on the following algorithm:

— A statement got categorised as "yes" when it was overall always or not once thoroughly and thoughtfully addressed (2 points).

— A statement got categorised as "no" when it was never addressed (0 points).

— A statement got categorised as "partly" for all other cases (1 point).

Lastly, we visualised a profile of the PD in a table (see Table 2) on the basis of the sum of the scores of characteristic attributes for each design principle. This process was used for all other short PDs. In addition, Table 2 allows for insights what design principles were more prevalent, and how the PDs differed in their focus and goals.

In addition, another rater coded the data. We checked the inter-rater reliability by using the formula recommended by Miles and Hubermann (1994), in which the coder reliability is calculated in the following manner: coder reliability = number of agreements/total number of agreements + disagreements. The inter-rater reliability was calculated at 96.6%.

DISCUSSION: A 4-DIMENSIONAL MODEL OF TRAINING PRACTICES

Reform of mathematics classroom is an ambitious and very much needed cause. Understanding MTEs' practices is an essential mean to achieving this goal.

PD	Design principles					
	Learner -orientation	Case-based learning	Competence development	Application of various instructional formats	Stimulating collaboration	Fostering (self-) reflection
PD 1	13 (72.2%)	10 (62.5%)	17 (65.4%)	8 (100%)	4 (50%)	6 (100%)
PD 2	9 (50%)	5 (31.3%)	12 (46.2%)	6 (75%)	3 (37.5%)	1 (16.7%)
PD 3	16 (88.9%)	15 (93.8%)	23 (88.5%)	8 (100%)	5 (62.5%)	5 (83.3%)
PD 4	10 (55.6%)	4 (25%)	10 (38.5%)	6 (75%)	2 (25%)	1 (16.7%)
PD 5	11 (61.1%)	7 (43.8%)	15 (57.7%)	5 (62.5%)	4 (50%)	2 (33.3%)
Total	18 (100%)	16 (100%)	26 (100%)	8 (100%)	8 (100%)	6 (100%)

Table 2: Descriptive statistics of protocols for analysing MTEs' practices

The developed protocol for analysing MTEs' practices proved to be a reliable instrument for measuring what different effective CPD design principles and to which extent these got implemented in the PD courses. More particularly, the above presented protocol offers means to evaluating MTEs' PD practices with respect to effective CPD factors. Hence, it offers lenses for insights into their training practices and challenges that seem to impact the quality of their professional development programmes. On this basis needs-based professional development programmes can be developed to support MTEs' diverse needs in the professional development system. On the other hand, it may provide MTEs' with lenses for capturing different aspects of their training practices when planning and implementing their PD courses.

While DZLM principles (2013) and the work of other researchers (e.g., Cochran-Smith & Lytle, 1999; Garet et al., 2001; Lipowsky & Rzejak, 2012; Putnam & Borko, 2000) focus on practices for effective PD and the activities of its participants, in the instrument we focused explicitly on the MTE itself, that is, on their actions to achieve the prescribed practices. No matter the situation, the professional developer (here MTE) is a critical protagonist, as he/she is the one who sets goals for a professional development. For that reason, we contend that their doing – as it is when focusing on teachers – should be made the central focus of theoretical frameworks.

With these considerations in mind, we propose here a model in which four interrelated professional development dimensions with respect to MTE's doing stand in focus (see Table 3). This model focuses on 4 dimensions: (1) general MTE's role, (2) nature of selected materials/tasks and its quality (3) role of the MTE when using manipulatives, and (4) established socio-mathematical norms. The first dimension entails practices aligned with learner-orientation. The second dimension focuses on activities in which MTs engage to make the new ideas problematic, connect

MODEL DIMENSIONS AND ITS FACETS
(1) General role of the MTE
— Engages MTs as adult learners in the learning process (e.g., time on tasks, building on existing knowledge, practices, and experiences)
— Integrates in depth knowledge about assessment, curriculum and how to teach it
— MTE and MTs work together on a specific concern about student engagement, learning, etc.
— Makes target goals transparent (e.g., emphasises the important content to know and understand, and why)
— Uses ample opportunities for an on-going assessment of learning
— Integrates MTs' knowledge into an active learning process
(2) Nature of activities and materials
— Allows MTs to build and/or consolidate their professional knowledge
— Challenges MTs existing practices
— Helps MTs focus on student learning
— Helps MTs understand their students' learning, misconceptions
— Allows integration of theory and practice
— Connects with teacher current knowledge and/or practices
(3) Role of the MTE when using manipulatives
— Allows MTs to construct meaning for the tool with respect to the topic
— Supports MTs to use the tool for a purpose (e.g., to better their teaching practices, to consolidate their knowledge and skills)
— Connects the use of tool with MTs' teaching practices
— Connects the use of tool to the topic(s) of interest
(4) Socio-mathematical norms (professional exchange, collaboration)
— MTE uses various instructional format allowing MTs to engage in a variety of continuous teaching-learning scenarios
— MTs have opportunities for discussions (professional exchange) with the MTE and other MTs in professional learning
— Professional learning focuses on MTs' problems in teaching and learning
— MTs can choose issues of interest and share those
— MTs have ample opportunities to share their results, concerns, etc. and to reflect on those
— Continuous learning and learning opportunities are a part of the PD

Table 3: A 4-dimensional model for analysing MTEs' practices with some descriptors

to where the MTs are with respect to their knowledge or practices, and leave behind a mathematical value. Thus, this dimension entails practise aligned with competence development and case-based learning. The third dimensions focuses on manipulatives used to achieve mentioned dimensions. The fourth dimension entails practices aligned with application of various instructional formats, stimulating collaboration and fostering (self-)reflection. Thus, its focuses on normative aspects of professional development discussions that are specific to MTE's mathematical activity.

In our future work we plan to continue developing instruments for examining MTEs' PD programmes. Our goal is to define and develop a descriptive model – on the basis of the above presented protocol and the 4-dimensional model – that would allow us to assign different quality levels (level 1 to level 5) to the five PDs, and PD programmes in general. This would allow giving MTEs a detailed feedback on their practices, and target those facets of professional knowledge that may be either lacking or need to be further developed.

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Principles and tools for teachers' education and the assessment of their professional growth

Nicolina A. Malara and Giancarlo Navarra

University of Modena and Reggio Emilia, Modena, Italy

We describe here the training model, developed within the ArAl project and characterized by the tight interrelation between contents to be taught (didactics of arithmetic and algebra in the perspective of early algebra) and teacher-educational processes, meant for teachers of the K-8 levels. We show how teachers, tutors and mathematics educators, by reflecting onto the Multi-commented Transcriptions (MTs), attain a shared development of the theoretical frame, of the methodologies and of the teaching materials that shall create the basis for the teachers' professional evolution. Finally, we tackle the question of how to assess teachers' professional growth by showing that MTs contain precious evaluation indicators.

Keywords: Teacher education, professional growth of teachers, assessment.

INTRODUCTION

In the last years, many studies have been devoted to teachers' professional growth. Most of them underline that the teachers' change depends on the intertwining of different teachers 'inner' factors: mathematical knowledge for teaching, beliefs, emotions and awareness at different levels (Jaworski, 2012, Mason, 2008, Schoenfeld, 2013). In particular, Sowder (2007) states that many of a teacher's core beliefs need to be challenged before change can occur. Schoenfeld stresses that the point is not what a teacher knows, believes or says, but rather how his/her knowledge and beliefs play out in the classroom. The change needed to traditional teachers to become non-traditional, inquiry-oriented teachers is not simple to achieve: it is very difficult to deeply affect the complex structures that constitute the basis for teachers' practices and strictly influence what they do, and their way of interpreting everything what happens. Our studies consider these aspects together: they aim at the renewal of the arithmetic and algebra teaching, they take into account both cultural and affective factors influencing teachers' action and develop the implementation of teachers' educational paths that are linked with the class activity and the students learning.

TEACHERS' EDUCATION: OUR APPROACH

As mathematics educators we are involved in activities for pre-service/temporary teachers or in-service teachers. In both cases, we develop studies on teachers' education with the aims: a) to validate our hypotheses about the effectiveness of our approach to favour changes in arithmetic/algebraic teaching; b) to individuate relationships between educational processes enacted and new teachers' awareness, attitudes and behaviours; c) to confirm, extend and generalize results highlighting the essential conditions which determine them. Our research studies are essentially qualitative and develop through a methodology framed in the Italian research model for teaching innovations which is based on a sharp interaction with the teachers and a constant practice of shared reflections on the experimental paths jointly planned and enacted by the teachers about the mathematical content in play, the quality of their behavior, attitudes, knowledge, ways of thinking promoted in the students. Our interventions with/for the teachers are centered on the following key points.

Fostering teachers' reflection on their knowledge and on the coherence between their expressed beliefs and their teaching practice

The teachers involved in our educational programs (K-8 segment) have different cultural backgrounds, due to their different educational paths. Their beliefs about arithmetic and algebra are based on their knowledge but mainly on their emotions and beliefs, fruit of their previous experience (as student and teacher). These aspects constitute an often fragmented and fragile net of reference which underpin their teaching orientations. Most of the teachers believe that the teaching of arithmetic precedes the teaching of algebra; therefore, the procedural point of view prevails over the relational one, the manipulation of mathematical objects prevails over the reflection upon them, products prevail over processes, in problem solving the operational aspect prevails over the representation and interpretation leaving in shadow the control of meanings. As to the teachers' vision of their role in the classroom, the most widespread one is that of a ('soft') director, deriving from the belief that teaching means mainly conveying pieces of knowledge. Trying to make teachers overcome such beliefs through training means, first of all, leading them – through the *practice of the reflection* – towards a deep analysis of their knowledge and beliefs, which makes them call into question and gradually re-formulate their personal epistemology in a new frame oriented towards early algebra.

Developing open and redefined educational paths

Teachers' educational programs often present a structure having a mono-directional motion within work-groups: mathematics educators limit themselves to giving teachers indications on how to improve their teaching, together with a set of tasks to be presented in the classrooms. In such a structure, teachers are basically users of a framework vision that they receive in top-down mode. On the contrary, our model suggests a multidirectional motion, characterized by a synergic network of relationships among teachers, tutors and mathematics educators, aimed at obtaining results that a single teacher would hardly achieve on his/her own. This synergy of relationships allows a continuous development of theory, methodologies and tools. We can therefore define this framework as being built in a bottom-up mode, within a tight interrelation between teaching contents and educational processes. A glossary shared with the teachers supports the educational process. It is a dynamic tool, to which teachers constantly refer. With the aim of constituting a real sharing of these theoretical elements, the glossary is progressively integrated according to the teachers' declared needs. Thus, a real 'formative communication' develops among teachers, tutors and mathematics educators in the inquiry community [1] (Cusi et al., 2011).

Planning teachers educational paths and designing tools for data detection

The teachers addressed by our project are distributed in almost all Italian regions but they are subject to a frequent turn-over (funding decreasing, retirements, change of school directors and teachers etc.) and are fluctuating in duration. For this reason, the educational paths we develop for pre-service/temporary teachers are different from those developed for in-service teachers. In the first case more space is given to the study of classroom processes of *others* teachers with the aim of educating trainee-teachers to develop a good sensitiveness about constructive teaching (Malara & Navarra, 2009; Cusi & Malara, 2011). In the second case the educational paths are focused on the study of the teachers' *own* teaching processes and on the comparison between the attitudes and behaviors of the teachers involved in the same experimentations. This kind of activities are developed with the aim of making the teachers become aware of their own ways of acting and of the possible strategies to adopt in order to refine problematic didactical approaches. These activities also enable to deduce general results and to identify new research problems.

Collecting narratives and discursive data meaningful to teachers' growth

Our results are generated by different kinds of data: *notes about the meetings* during which the teachers and the researchers discuss the potentiality of the activities to be presented in the classes or, a posteriori, analyse the effectiveness of the teachers' ways of proposing the activities; *in-progress didactical materials* (tasks, students protocols, short excerpts of micro didactical situations, terms of the glossary, etc), selected by the teachers for the sharing of experimental results; *teachers interviews or discussions,* where aspects linked with their knowledge or methodological and emotional aspects are highlighted; *multi-commented transcripts* (MTs) [2] of audio (sometime video) recordings of classroom processes. Teachers' meta-reflections are encouraged, since they are asked to analyse the transcripts of class discussions commenting not only on students' interventions but also on their own interventions, trying to refer to the different constructs of the wide theoretical frame (see forward the point 'Methodology of Multi-commented Transcripts'). The collected data are studied to identify categories of teachers' behavior, characterized by set of actions modeling a metacognitive and effective teaching (Cusi et al., 2011).

THE CONTENTS: EARLY ALGEBRA AND THE ARAL PROJECT

Early algebra arises from the need to favour the pupils' construction of meanings in arithmetic in a pre-algebraic perspective across K-8 school levels. Its approach to arithmetic is based on specific principles: the anticipation of generational activities at the beginning of primary school; the social construction of knowledge, i.e. the shared construction of new meanings, negotiated on the basis of the cultural instruments available at the moment to both pupils and teacher; a focus on natural language as main didactical mediator for the slow construction of syntactic and semantic aspects of the algebraic language; identifying and making explicit algebraic relationships and structures within concepts and representations in arithmetic. In this context, our *ArAl Project: arithmetic pathways towards favouring pre-algebraic thinking* (Malara & Navarra, 2003) counters the traditional sequence of arithmetic/algebra teaching and suggests that their teaching is based on the intermingling between two disciplines, in the perspective of a continuity between primary and secondary school (see Figure 1).

We claim that the main cognitive obstacles to the learning of algebra arise in unsuspected ways in arithmetic contexts and may impact on the development of mathematical thinking, mostly owing to the fact that many students have a weak conceptual control over the meanings of algebraic objects and processes, seen as translations of verbal sentences and in their mathematical status. Our hypothesis is that algebra should be taught as a new language from the beginning of primary school, so that one gets to master – through a set of shared social practices (collective discussion, verbalization, argumentation)– modalities that are analogous to the learning of a natural language: gradually appropriating its semantic aspects and putting them in their syntactic structure. For this reason, it is necessary to build up an experimental and continuously redefined environment, capable of informally stimulating the autonomous elaboration of a formal coding for verbal sentences by discussing them with the whole class. This process of construction/interpretation/refinement of 'draft' formulas is what we call *algebraic babbling*.

This approach leads to a sort of Copernican revolution in the teachers' beliefs. It brings about awareness for the meaningfulness of their role, with reference to their position within the educational process. The teaching of arithmetic in an algebraic perspective is fostered by making teachers shift their attention from procedures towards relationships in arithmetic. Let us clear up this point by presenting four key-issues that exemplify our framework.

The equal sign and the duality process-product

The usual reading of 5 + 6 = 11 is '5 plus 6 *is* 11': what's on the left to the equal sign is seen as an *operation,* whereas on the right it is seen a *result*. The two sides of the equal sign are interpreted as *ontologically different entities*. But the algebraic meaning is different: it indicates the equivalence between two representations of the same quantities, i.e. between two entities that are *ontologically equal*. We usually introduce this alternative perspective by discussing with teachers the words of an 8-year-old pupil: "It is correct to say that 5 plus 6 makes 11, but you cannot say that 11 'makes' 5 plus 6; so, it is better to say that 5 plus 6 'is equal' to 11, because in this case, the other way round is also true."

Canonical and non-canonical representations of a number

Similarly to what happens with the equal sign, writings such as $[(3 + 2) \times 4]^2$ are seen as *operations* waiting for a *result*. In order to promote a reflection upon them in an algebraic perspective, we use the strategy of writing on the blackboard a list of facts concerning a specific pupil: name..., daughter of..., owner of a dog called..., and so on. We then explain that the situation is similar with numbers: each number can be represented in many different ways, through any odd equivalent expression. Among these representations, only one (e.g., 12) is its *name* – the so-called *canonical* form, whereas the others (3 × 4, 3 × (2 + 2), 48/4, ...) are its *non-canonical* forms, each of which makes sense with reference to the context and the underlying process. This experience enables to understand that $[(3 + 2) \times 4]^2$ is one of the many non canonical forms

	Arithmetics												
K	1	2	3	4	5	6	7	8	9	10	11	12	13
									Algebra				

Early algebra (EA)													
K	1	2	3	4	5	6	7	8	9	10	11	12	13
EA						EA-AT			AT				
						Algebraic Thinking (AT)							

Figure 1: Towards a new perspective of the teaching/learning of arithmetic and algebra

of the number 400. Being able to recognize and interpret these forms builds up the semantic basis for the understanding of algebraic expressions like $-4p$, ab, x^2y, $k/3$. The concepts of canonical/non-canonical form also allows to reflect upon the possible meanings associated with the equality sign; in $[(3 + 2) \times 4]^2 = 400$ we don't see the operations and results anymore, but rather the equivalence between two representations (non- canonical and canonical) of the same quantity.

The duality 'representing vs. solving'

It is a widespread belief that solving a problem means identifying its *result*; this perspective focuses the attention on the *operations*. In order to bring about a change of perspective, it is necessary to move from the cognitive to the metacognitive level, where the solver *interprets the structure of the problem and represents it through algebraic language.* In the traditional perspective, solving a problem means separating the entities that are known from the entity to be found, then spot out the necessary operations. In the perspective of early algebra, the attention is concentrated on the (known or unknown) *entities*, and on the *relationships* among them. In a 'classical' problem for level 4, 'The sides of a rectangle are 3cm and 4cm long. find the perimeter', *the explicit entities are two* (3 and 4), the implicit one is the 'double' operator applied to each side and *the operations are two*; we obtain the writing $(3 + 4) \times 2$ which gives the result 14. But if the task is expressed as follows: '*Represent* in mathematical language the situation so that you can find the perimeter', *the entities are four*: the length of the two sides, the 'double' operator, the length of the perimeter (p), the *operations are two* but there is also a *relationship*: the equality between p and the representation of the process through which it is obtained. The sentence $p = (3 + 4) \times 2$ expresses all this is. This shift of perspective amplifies comprehension; in order to foster it, we use the principle '*first represent, then solve*'.

The duality transparent-opaque

A representation in mathematical language is made of symbols that convey meanings, the comprehension of which depends both on the representation in itself and on the ability of those who interpret it. One could say that the canonical form of a number is poorer of meanings than its infinite possible non-canonical forms, for example: the tendency of immediately carrying out the calculation $5^2 \times 5^1 \times 5^3$ leads to a result, represented by its canonic form (15 625), but the efficacy produced by the 'intermediate' representation 5^{2+1+3} gets lost, whereas it would allow to build the comprehension of why, in algebraic realm, $ab^2 \times a^2b = a^3b^3$. We can therefore talk of a higher *opacity* for writings such as 15 625, of a higher *transparency* for those like $5^2 \times 5^1 \times 5^3$ and 5^{2+1+3}. Generally speaking, the transparency of the process favours the control of meanings, highlighting the underlying properties; it allows to understand possible errors and to clear up possible misconceptions which may arise.

THE EDUCATIONAL PROCESS: TEACHERS' PROFESSIONAL GROWTH IN THE PERSPECTIVE OF EARLY ALGEBRA

The methodology of multicommented transcripts

In order to fulfil these, and others, key-issues, we involve teachers in an activity of critical analysis, and consequent reflections, of the transcripts of audio and video-recordings of classroom processes, which we call Multi-commented Transcripts Methodology (MTM). It has the aim of highlighting the interrelation between the students' construction of knowledge and the teacher's behaviours in guiding them to perform this construction. The Multi-commented classroom Transcripts (MTs) are sent by teachers, together with their own comments, to e-tutors who make their own comments and send them back to the authors and other members of the team. The e-tutor highlight not only the positive aspects, but also the possible stereotypes, beliefs and behaviour that are often mistaken, and comment them with reference to the theoretical framework that is shared in the community of inquiry and, for some key-elements, also with the pupils. The joint reflection on the MTs strongly influences the development of theoretical, methodological, instrumental, material aspects (Units of the ArAl Series,

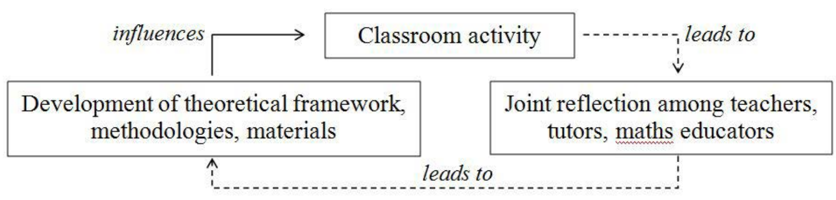

Figure 2: The cycle of teachers' mathematics education

papers, articles, learning objects) and supporting elements (website, blog, Facebook Group) aimed at offering teachers a cultural background that can help them act differently in the classroom (see Figure 2).

HOW TO ASSESS

The problem of how to assess the teachers' change is an open research question. The strategy we adopt is to assess in an indirect and articulated way, through the MTs, the way in which the teacher's classroom action evolves during the training path. The factors we observe are: (a) the pupils' interventions; (b) the modalities in which the teacher interacts with the pupils; (c) the comments expressed by a teacher in the MTs and his/her reflections received in the sharing of his/her MTs within the community of inquiry. With this goal, the intersection of experience between both typologies of teachers (pre-service/temporary and in-service) becomes relevant in the strategies of the project, for example: meaningful excerpts from MTs produced by in-service teachers give vent to tasks for future teachers, aimed at assessing their ability to face specific classroom situations by hypothesizing contingent actions and foresee their development (Malara and Navarra, 2009). Some examples (teachers in service):

How pupils express themselves

One can infer from pupils' sentences whether the teacher works in a pre-algebraic perspective, i.e. fostering the development of algebraic babbling and therefore inducing a 'metacognitive' attitude.

Episode 1(9 years old, grade 4)

Pupils are asked to represent in mathematical language the total number of sweets contained in six bags (each of which contains four chocolates and three candies).

Alessandro: I have written 7×6.
Miriam: I have written $(3 + 4) \times 6$: it is more *transparent*, Alessandro's writing is *opaque*. It means that it is not clear, whereas transparent means that you understand.

Miriam refers to the dichotomy *opaque/transparent* to express *how the non-canonical form of a number helps to illustrate the structure of a problematic situation*. This awareness in the pupil testifies to the fact that the teacher has acquired the ArAl theoretical framework and shares it with the pupils.

How teachers interact with pupils

The way in which a teacher interacts with the pupils, and the role that he/she takes up have a strategic importance, because they influence the quality of the teaching.

Episode 2 (8 years old, grade 3)

The pupils have represented this situation in the mathematical language and reflect on the suggestions made by Alice $n = 5 + 2 \times 8$, Martina $5 + 2 \times 8 = n$ and Ada $n = (2 + 5) \times 8$.

Francesco: I think they are right because 5 +2 represents the marbles that are in a box, by 8 which is the number of the boxes.
Maria: They are the same as Ada's but they don't have parentheses.
Teacher: Let's reflect on the presence of the parentheses. Do they change anything?
Andrea: I think they do, because $5+2\times8$ is equal to 21, while $(2+5)\times8$ is equal to 56.
Bruno: It's true: the teacher said once that in a chain of operations you solve multiplications first.
Maria: This means they are not the same!
Francesco: That's right, the translation with the parentheses is the more correct one.

There are many elements here that let us assess positively, at different levels, the teacher's action: (a) *mathematically*: she has introduced the pupils to the use of letters, to the priority of operations within expressions, to the use of parentheses; (b) *linguistically*: she has fostered the organization of meaningful, complete sentences; (c) *metalinguistically*: she has promoted the reflection on the mathematical writings and their comparison; (d) *socially*: by inviting pupils into discussion, she has let them interact without her influence, listening to each other and having spontaneous dialogues; (e) *methodologically*: she has shared the theoretical framework with the class, by spreading words such as 'represent' and 'translate'. On the other hand, we make the teacher notice and reflect on the fact that (2) Francesco has referred '5 + 2' *to the marbles* and not *to their number*; (2) Andrea speaks of *result*, and pupils should be guided from the level of *calculations* to the level of *representations*.

The teacher's self-reflection and the suggestions offered by other comments

The way in which a teacher interacts with the pupils, and the role that he/she takes up have a strategic importance, because they influence the quality of the teaching.

The comments written by the e-tutor and other experts – even in several diaries by the same teacher – allow to ascertain whether the classroom-leading strategies have changed (and *how*) during the educational project. Among the factors that influence the assessment there are, for example, the following issues: Does the teacher develop a wide range of roles in order to promote a refection onto mathematical processes or objects? Does he/she foster linguistic interactions by encouraging verbalization, argumentation, and collective discussion? Does he/she negotiate and share with the pupils the theoretical framework? Does he/she modify their initial points of view or does he/she seem unsensitive towards meaningful changes in his/her initial attitude?

Example 3 (12 years old, grade 7)

The class is tackling with the teacher the mathematization of a situation that requires the translation of the sentence 'for each 4 sage plants there are 6 rosemary plants'. The pupils attain first the formula $s = (6/4) \times r$, then $s = 3/2 \times$ r. The teacher shifts to the level of interpretation and asks the pupils to determine the number of rosemary plants corresponding to 66 sage plants. Some pupils offer a solution by substituting the value 66 and s in the second formula. Then, another pupil speaks:

Mario: I have done it in a different way: 66:4=16 and the remainder is 2. Then I have done 16×6=96 because for 4 sage plants I have 6 rosemary plants. Then, since 4s=6r, I have divided by two that is 6:2=3 and after that I have added 3 to 96. The result is then 99.

Teacher: Bravo, now let's draw a graph by using the relationships that we know.

To this point, the teacher's reflection and the tutor's comment in the MT are:

Teacher: Here I should have lingered on what the pupil was trying to say, because I think that his reasoning is very interesting and it could have helped his classmates to see the same situation from different points of view.

Tutor: Ok. But before that, one should have underlined the improper use of letters as labels and the argumentation should have been interpreted at relational level, so as to show that his procedure is based on the non-canonical representation of 66 as 4×16+2, on recognizing the multiplicative relationship between 2 and 4 (2 is the half of 4) and on the implicit assumption of the distributive property. The pupil is referring to the non-simplified formula r=(6/4)s and transforms the calculation (6/4)×66 into (6/4)×(4×16+2) = = (6/4)16+(6/4)2=6×16+3=96+3.

CONCLUSIONS

The question of teachers' evaluation is complex also because what we want to evaluate is a process which, in its nature, is continuously becoming and reflects daily changes of mood, energy, personal interest in the topic being dealt with, relationship with the pupils. All these factors interfere with the quality of the classroom activity management and therefore influence what one would like to evaluate: the effect of participating in the training on the strategies of leading a lesson. Our studies concentrate therefore on a formative evaluation of what arises in the MTs, a dynamic evaluation, which pays attention to nuances, apparently unimportant details, to micro-decisions that the teacher makes in just as many micro-situations, in which he/she often shows the co-existence of pre-existing beliefs and possibly hesitating opening attitudes, induced by the training. We hold this way promising, particularly if it is embedded in a educational context that constantly involves teachers and broadens the sharing of the MTs and of their reflections (along with their possible evaluation) to other actors of the community of inquiry, fostering a system of crossed evaluations.

These aims could be reached in a medium/long-term (two/three years or more) under these conditions: (i) enacting an educational project with several supporting tools; (ii) avoiding separation between theory and practice; (iii) building an environment in which effective circular relationship occurs between what happens in the classroom, the joint reflection of teachers-tutors-mathematics educators on classroom events, the shared effort of refining practices by relating it to the theoretical frame and developing materials, tools, as well as the assessment of the progressive change of the teaching through further classroom activity.

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ENDNOTES

1. We use the term 'community of inquiry' in the sense of Jaworski (2012). It involves mathematics educators (in the double role of didacticians and researchers), tutors (in the double role of mentors and teachers/researchers) and teachers.

2. The MTs are shared by e-mail and after critically analyzed during the periodical meetings of the ArAl community. Currently about 300 MTs have been put on the website <http://www.progettoaral.wordpress.com>. Together with other theoretical or practical materials, they constitute an integrated set of tools for teacher professional development".

Development of teachers' mathematical and didactic competencies by means of problem posing

Uldarico Malaspina[1], Albert Mallart[2] and Vicenç Font[2]

1 Pontificia Universidad Católica del Perú in Lima, Faculty of Science, Lima Perú, umalasp@pucp.pe

2 Universitat de Barcelona in Barcelona, Faculty of Education, Barcelona, Spain

This paper presents a didactic experience with problem posing carried out with in-service secondary teachers. We propose a strategy specifically designed to modify a given problem and enrich its mathematical and didactic potential. The starting point is a teacher's class episode, which includes a previously designed problem as well as the reactions of the teacher's students when solving it. We ask the participating teachers to pose 'pre-problems' and 'post-problems', working individually at first and then in groups. The experience shows that problem posing contributes to the development of teachers' didactic and mathematical competencies.

Keywords: Problem posing, problem solving, teacher competencies, teacher training.

INTRODUCTION

In 1989 the National Council of Teachers of Mathematics (NCTM) recommended providing opportunities for students to think mathematically and develop knowledge by creating problems: "Students in grade 9–12 should also have some experience recognizing and formulating their own problems, an activity that is at the heart of doing mathematics" (p. 138). In addition to this, the NCTM recommended offering opportunities to formulate problems from a given situation as well as opportunities to create new problems by modifying the conditions of a given problem (NCTM, 1991, p. 95). In this sense, teachers must obviously develop their problem posing skills in order to work in this way with their students; they should not be limited to using the problems found in books or online (Ellerton, 2013; Singer & Voica, 2013; Malaspina, 2013b; Bonotto, 2013). The cited authors also emphasize the importance of the relationship between problem solving and problem posing. In Bonotto's words (2013): "There is a certain degree of agreement in recommending problem-posing and problem-solving activities to promote creative thinking in the students and assess it." (p. 40).

Some research on problem posing and its relation to the problem solving process has led to new research on the benefits of incorporating problem posing in teacher training programs (Ellerton, 2013; Tichá & Hošpesová, 2013; Malaspina, Gaita, Flores, & Font, 2012; Malaspina, 2013b). We agree with Ellerton (2013) when she says: "For too long, successful problem solving has been lauded as the goal; the time has come for problem posing to be given a prominent but natural place in mathematics curricula and classrooms" (pp. 100–101) and our research shares this idea. We have designed activities with the purpose of motivating pre-service teachers and current teachers to create math problems and reflect on their didactic aspects. The problems were posed starting from a given problem or from a given situation. In this paper, we describe some cases of the first type.

FRAMEWORK

Mathematics teacher's competencies

At the international level we have observed a tendency for convergence among university curricula design. Some countries have opted for a curriculum model organized by professional competencies that differentiates general (or transversal) competencies from specific ones.

Many of the tasks proposed in order to develop and evaluate students' mathematical competencies are problem-based. A teacher must not only be good at solving problems, but also needs to know how to choose, modify and create them with a didactic purpose. A teacher also needs to be able to critically evaluate the quality of the mathematical activity required to solve the problem proposed and, if necessary, to be able to modify the problem in order to facilitate a richer mathematical activity.

Teachers should already have mathematical competence to solve problems, but if they want to select, modify or create them with a didactic purpose, they need to be competent in didactic analysis of the mathematic activity (Rubio, 2012). The first competence is common in many of the professions developed by mathematicians. The second one is, however, more specific to the mathematics teacher.

According to Giménez, Font and Vanegas (2013) and Rubio (2013), we understand the competence of didactic analysis as the ability to design, apply and evaluate learning sequences by means of didactic analysis techniques and quality criteria. In teacher training, this competence has to be developed by proposing tasks to the future and current teachers that require them to carry out didactic analysis. One of these tasks consists of creating problems and thinking about them didactically.

Problem posing

Stoyanova and Ellerton (1996) summarize the meaning of creating mathematical problems from different points of view:

> Problem posing has been viewed as the generation of a new problem or reformulation of a given problem (Duncer, 1945); as the formulation of a sequence of mathematical problems from a given situation (Shukkwan, 1993); or as a resultant activity when a problem is inviting the generation of other problems (Mamona Downs, 1993). Dillon (1982) conceptualized „problem finding as a process resulting in a problem to solve." Silver (1993, 1995) referred to problem posing as involving the creation of a new problem from a situation or experience, or the reformulation of given problems. (p. 518)

As Silver, we consider that posing mathematical problems is a process through which one produces a new problem from a given one (problem's variation) or a new problem from a situation (problem's elaboration), whether it is real or imagined.

In order to develop this perspective of problem posing, it is necessary to identify the four key elements of a problem: Information, Requirement, Context and Mathematical Environment (Malaspina, 2013c). The Information consists of the quantitative or relational data that are given in the problem. The Requirement is what is asked to be found, examined or concluded; it can be quantitative or qualitative, and it can include graphics and demonstrations. With respect to Context, a "contextualized problem" usually relates to any real situation, to everyday life; but we consider that the Context can also be strictly formal or mathematical. In this sense, we affirm that the Context can be intra mathematical or extra mathematical. In the first case, as its name implies, the problem is more linked to an abstract situation and, in the second case, the problem is more linked to a real situation. Finally, the Mathematical Environment refers to the mathematical concepts needed to solve the problem.

Therefore, we understand the problem's variation as a process that builds a new problem by modifying one or more of the four key problem elements.

OBJECTIVES AND METHODOLOGY

We have two objectives:

- To show that an appropriate strategy helps stimulate the ability to create mathematical problems by modifying a given problem and considering its mathematical and didactic aspects.
- To show that problem posing is a means of contributing to the development of teachers' didactic and mathematical competences.

Regarding the methodology, since the research relates to creativity, we have chosen a qualitative methodology that includes a strategy, observations and case studies.

The first step was choosing a topic and designing some easy and motivating problems as starting points to pose new problems.

At the beginning, the problem posing experiences were performed with pre-service primary school teachers as part of the mathematics course in the Faculty of Education of the Pontificia Universidad Católica del Perú. These did not have a context of a specially designed strategy. The positive experiences of both the individual and group works were the basis for designing problem posing workshops, which are summarized below.

A PROBLEM POSING STRATEGY

We give a short presentation on problem posing, including some examples of problems created in previous workshops, in which we emphasize the importance of creating problems that favour learning and mathematical thinking. We present a previously elaborated problem to the workshop participants considering the context of a concrete class episode of Teacher P. In this episode, some of the students' reactions when solving the problem are described briefly. We ask participants to: i) solve the given problem; ii) pose problems by modifying the given problem to make the solution easier and to help clarify students' reactions (these problems are called 'pre-problems'); iii) pose problems by modifying the given problem so as to challenge Teacher P's students beyond correctly solving the given problem (these problems are called 'post-problems'). The problem posing must be carried out individually at first and then in groups with the help of the instructor of the workshop. Moreover, the problems created by a group are also solved by other groups. There is also a socialization phase with all the participants. In this phase, the participants share the rationale behind the individually or collaboratively created problems. In addition, the problem solved by a group (which is not the author group of the problem) is exposed and commented critically. The purpose of this is that the discussion with the authors of the problem as well as the participants' and instructor's comments help to enhance the capacity of posing problems with mathematical and didactic potentialities.

This is the strategy we have followed in several workshops, especially with current secondary teachers. It should be mentioned that the experiences we show and examine in this article are about percentages.

CASES OBSERVED

After explaining and exemplifying some ways for varying a given problem, we applied the described strategy in a workshop with 15 current secondary school teachers. We proposed the following class episode and we asked the teachers to do the tasks (i)–(iii) related to the episode problem. Both, the interaction of the teachers with the instructor and the socialization phase, were very important for obtaining information about the rationale behind the created problems.

The episode

In a class of mathematics, teacher Sánchez asks his students to solve the following problem:

The first week of July, a store called ALFA sold all products at full price; the second week, the store discounted all items 20%; and the third week, the store applied an additional discount of 15% that was announced as the "GREAT DISCOUNT OF 20%+15% ON ALL THE PRODUCTS".

You have to decide whether or not during the third week of July ALFA sold its products at prices 35% less than during the first week of July.

After a few minutes:

— Most of the pupils say "Yes, they did."

— Juan and Carla say "No, because in the third week the discount was less than 35%."

— Maria says that in the third week the discount was 68%.

Some pre-problems posed by teachers in the workshop

Pre-problem 1 (Individual work)

Rosa bought a $100 blouse that was discounted 20% because of end of season sale and with an additional discount of 10% for having the store credit card. What was the total percentage discount that Rosa received?

The author's idea when she posed the problem was to set a price with percentages that are easy to calculate in order to help students focus their attention on the total discount.

Pre-problem 2 (Individual work)

In a clearance sale, a shop applies a 50% discount on all its textiles during a week, and the following week it applies an additional discount of 50%. What is the total percentage discount applied during the second week?

The author of this problem was interested in showing the students that the total discount is not a simple sum of percentages. In order to achieve this objective, the author had chosen discounts of 50% because a total discount of 100% is not intuitive.

Pre-problem 3 (Group work)
(The author of Pre-problem 1 joined this group)

Rosa bought a $100 blouse that was discounted 20% because of end of season sale and with an additional discount of 10% for having the store credit card.

a) How much did Rosa pay for the blouse?

b) What percentage of the blouse´s original price did Rosa pay for the blouse?

c) What is the blouse's total percentage discount?

The author group thought this problem would help students distinguish between the amount paid and the discount. This seemed to be the confusion of student Maria in the episode. Apparently she had done the calculations well, but she did not distinguish between the amount paid (68% of the initial price) and the total discount (100%–68% = 32%). Some of the participants commented that considering $100 as the initial price should be used as a counterexample for illustrating the wrong answers, and they said to be careful because it could reinforce a simplified and not deeply reasoned way of generalizing a particular case.

Some post-problems posed by teachers in the workshop

At the beginning of creating post-problems, the participating teachers created problems that were very similar to the given problem, some with other prices and, in other cases, considering three successive discounts; essentially, they were the same problems as the original, but with quantitative modifications in the information. However, little by little they carried out more creative modifications when they formulated post-problems:

Post-problem 1 (Group work)
Pedro and Juan each bought a shirt. Pedro bought a shirt with a discount of 20% plus an additional discount of 20%. Juan bought one with a discount of 30% plus an additional discount of 10%. Who received a greater discount?

The author group thought this problem would reinforce the fact that the total discount is not a simple sum. When it was solved by other group and socialized, the participating teachers appreciated that there were no specific initial prices for the shirts.

One of the teachers said that he created a similar post-problem considering percentage wage increments: for worker A, 5% in 2011 and 4% in 2012; and for worker B, 6% in 2011 and 3% in 2012. The problem was to determine which of the workers received a better percentage increment in the two years; or did they receive the same percentage increment? In his group, this problem was considered easier than the episode problem and, for that reason, it was not used as a group post-problem.

Post-problem 2 (Group work)
There is a store where you can pay 30 days later, but there is a 10% surcharge. And if you want to pay after 31 days but before 35 days, there is an additional 5% surcharge. If Julio bought something on August 20th and paid on September 23rd, what total percentage surcharge did he pay?

The author group thought it was interesting to pose situations about cumulative percentage, considering surcharges and not only discounts. As in Post-problem 1, initial amounts are not specified in this problem and its solution requires a better understanding of the percentage concept. In the socialization it was commented that this problem was easier than the problem of the percentage wage increments.

Post-problem 3 (Group work)
Celia bought a dress for $ 125.46. If the dress was 15% off with an additional discount of 18%, what was the original price of the dress?

The author group said its intention was to motivate students to use algebra to solve percentage problems. Indeed, the group that solved the problem used the equation:

$$(0{,}82)(0{,}85)x = 125.46$$

One of the comments during the discussion was that all problems maintained an extra mathematical context. However, we also need to create problems with an intra- mathematical context. Generalization allows us to work in this context. The following problem was created with this idea.

Post-problem 4 (Group work)

If the shop called BETA has an end of season discount of p%, plus an additional discount of q%, what is the total percentage discount in relation to the original price?

The problem allowed us to illustrate the total discount using a composition of linear functions that express the sale price of a product whose original price is x and has a discount of r%. That means, functions of the form $f(x) = (1 - \frac{r}{100})$.

COMMENTS

The studied cases show that the proposed strategy contributes to the development of the competency of problem posing and to thinking about problems didactically (e.g., Pre-problem 2, the comments about Pre-problem 3 and Post-problem 1).

In most cases, the primary modification of the initial problem is about its information and its requirement. Post-problem 4 did not arise spontaneously. However, the alterations are not only quantitative (e.g., different percentages), but also qualitative (e.g., the problem deals not only with discounts, but also with increases), relational (e.g., the information is given in a way that it makes it easier to reflect on possible wrong answers, as in Pre-problem 2) and, in some cases, a piece of information is added or a requirement is extended (e.g., Pre-problem 3)

The percentage theme favours the creation of problems in an extra mathematical context and we highlight the great diversity of imagined situations in problem posing workshops. The problems exposed in this paper are only a part of those imagined by the participants in this workshop and there are many others created in other workshops. The processes of reflecting on these diverse new problems – individually, in a small group and with all the workshop participants – contribute to the teachers' advances in knowledge of the mathematical object, in the observation of their reality and in elaborating tasks to go deeper into the subject to solve the problem created. We underscore the importance of working individually at first and the richness of working in groups later. All work is strongly enriched with socialization, in which arguments, opinions and comments that reflect involvement in creating problems arise.

Problem posing has to be designed to promote students' learning or to develop their mathematical thinking. We have studied (Malaspina & Vallejo, 2014) that problem-posing workshops related to one concrete theme allow participants to go deeper into the subject matter and to make mathematical connections. In the present research, Post-problem 4 shows the connection between percentages and linear functions, which was unknown for most of the participating teachers.

Examining the quality of the created problems is not the intention of this article, but we can appreciate that problems created by groups have a higher mathematical and didactic potential – above all if they have been created from episodes in classes. Certainly, the better the teachers' mathematics background and teaching experience, the higher the quality of the problems they create.

FINAL CONSIDERATIONS

We have developed problem posing activities with pre service teachers and current ones. We have considered individual work and group work in dealing with a given problem during an episode, which happens under certain characteristics. The analysis of these activities – in particular the cases considered in this paper, with current secondary teachers – show that they contribute to the development of didactic and mathematical competencies. Problem posing provides opportunities in which the two competencies have the possibility to interact in a creative way. The results are didactically valuable suggestions for their students as well as advances in teacher training.

It is important to break the "enculturation process of accepting problems that others create as those which need to be solved" (Ellerton, 2013, p. 87). We will contribute to this breaking by giving pre-service and in-service teachers good opportunities and orientations for creating problems. When teachers have experiences with didactic analysis and mathematical connections through processes of creating new problems, they improve their mathematical and didactic competencies, and they could induce their pupils to create their own problems.

This article is part of a wider area of research in which problem posing is also considered from a given situation. The corresponding activities have been considered in a next phase. We have interesting didactic experiences, some of which are explained in Malaspina and colleagues (2012) and Malaspina (2013a).

The problem posing strategy exposed and commented in this paper could be a good methodological tool for teacher training. Certainly, from a research point of view, it would be interesting to test it in relation to different mathematics topics at various teaching levels and in different countries.

As a part of the challenges posed by this research on creating math problems in mathematics education contexts, we invite readers to consider the following questions:

How do we measure the influence of the problem posing competency development for teachers on their performance in class with students?

How do we verify or reject the conjecture that assuming the challenge of creating math problems on a given topic activates new learning processes that favour intra mathematical connections with other fields of knowledge and reality?

ACKNOWLEDGEMENT

This study was funded by research project EDU 2012–32644 and research project 53133-IREM-PUCP 2013.

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A development over time of the researchers' meta-didactical praxeologies

Francesca Martignone

Università del Piemonte Orientale, Vercelli, Italy, francesca.martignone@unipmn.it

This paper shows how the study of the meta-didactical praxeologies can highlight different features of tasks, actions and theories that a researcher can develop working also as a teacher educator. The theoretical tools presented are part of the Meta-Didactical Transposition model: a model based on the Anthropological Theory of Didactics adapted to the analysis of teacher education processes. These tools are useful to identify and describe when and how some different components of the researchers' praxeologies changed during the dynamic processes, which involved researchers and teachers in a teacher education project.

Keywords: Teacher education program, Meta-Didactical Transposition model, meta-didactical praxeologies, teacher educators.

INTRODUCTION

The study presented in this paper is part of an ongoing larger research that involves Italian and French researchers (Aldon et al., 2013; Arzarello et al., 2014). It aims at developing theoretical tools in order to describe and analyse the peculiar features of different teacher education programs. The first result was the elaboration of a model that takes into consideration the complexity of the processes evolving in a teacher education program. This model is based on the Anthropological Theory of Didactics (Chevallard, 1992, 1999) adapted to teacher education, and is named Meta-Didactical Transposition model (MDT).

This paper deals with the first step of a fresh study that focuses on one of the aspects that MDT model can investigate: how to study the development over time of researchers' praxeologies during a teacher education program. At first we present MDT model and then we analyse the praxeologies of the researchers, who worked as teacher educators in a particular teacher education program (the MMLab-ER project). The paper shows how the theoretical lenses given by the MDT model can be useful to identify and then to analyse the development of a specific type of praxeologies: the meta-didactical praxeologies.

THEORETICAL LENSES

The knowledge needed for an effective teaching of mathematics is a crucial issue faced by the international research on math education. Drawing from Schulman's categories of knowledge (Shulman, 1986) and using examples from the practice of teaching, Ball and colleagues (Ball & Bass, 2003; Ball, Thames, & Phelps, 2008) identified a special Subject Matter Knowledge that they labelled as Specialized Content Knowledge (SCK): i.e., "the Mathematical knowledge and skills unique to teaching" (Ball et al., 2008, p. 400). This is a set of knowledge and skills that cannot be reduced only to the knowledge of the discipline or to the pedagogical knowledge: for instance, to look for patterns in student errors or to size up whether a nonstandard approach would work in general, require a specialized mathematical work not needed in other settings. The identification of the SKC is useful for analysing the knowledge involved in specific activities, but in order to describe the processes developed over time during a teacher education program, other theoretical tools are necessary. Therefore this paper presents some suitable interpretative tools that can be useful to describe and analyse the actions developed to face particular tasks in a teacher education program. In our research (Aldon et al., 2013; Arzarello et al., 2014), we adapted the Anthropological Theory of Didactics – ADT- (Chevallard, 1992, 1999) to teacher education, and we constructed the Meta-Didactical Transposition model – MDT – (Aldon et al., 2013). The ADT is a model of mathematical knowledge, conceived as a human activity developed for the purpose of addressing specific families of tasks. Its main theoretical tool is the notion of praxeology, which is structured into two levels: the "know-how" (*praxis*) and the "knowledge" (*logos*). The *praxis* includes different kinds of problems to be studied as well as techniques available to solve them. The *logos* includes the "discourses" that describe, explain and justify the techniques used. This theoretical discourse is called technology whose etymology derives from "discourse (logos) on the technique (technè)". The formal argument, which justifies such technology, is called theory. (Garcìa et al., 2006). As already underlined by Huillet (2009), the ADT can also give strong interpretative tools to analyse the mathematical knowledge and competencies needed by a mathematics teacher to teach mathematics in a given institution.

The MDT model gives the lenses for studying the different praxeologies developed during a teacher education program. These praxeologies develop by means of the interaction between the community of researchers and teachers, who can develop both new awareness at a cultural level and new skills in teaching.

This paper argues the researchers' meta-didactical praxeologies development by means of an explanatory example of a teacher education program carried out in Italy, the MMLab-ER project (see next paragraph). The paper focuses on the meta-didactical praxeologies, analysing their development over time at *praxis* level and at *logos* level. The term "meta-didactical" denotes that these praxeologies deal with the actions and the reflections of researchers about the educational activities. The analysis of these praxeologies is useful to show when and how researchers' actions and theories develop by reflecting upon the nature of and reasons for the changes that occur. The study of the meta-didactical praxeology is a fresh way to analyse the processes involved in a teacher education program because it doesn't focus only on PCK (Schulman, 1986), or on SCK (Ball et al., 2008) or on the relationship between researchers and teachers, or on the actions carried out to orchestrate the working group sessions.

This paper shows a germinal study about the developing interlacement of the praxeologies linked to researchers' work as teacher educators, and to the praxeologies developed when they study theories, plan, observe and analyse classroom activities, or construct theoretical lenses to describe and interpret students' difficulties and successes.

THE MMLAB-ER PROJECT

The MMLab-ER project, framed in the Italian research for innovation (Arzarello & Bartolini, 1998), responded to national and international standards about Inquiry Based Science Education (Rochard et al., 2007). The researchers involved in the projects are also the teacher educators. The project aimed at the construction of a network of well-prepared in-service teachers (primary and secondary school teachers) about mathematical laboratory and about the semiotic mediation processes (Bartolini Bussi & Mariotti, 2008). In this project, ancient and new tools were involved: i.e. working reconstructions of ancient mathematical machine and ICT (Bartolini Bussi et al., 2011; Martignone, 2011, Bartolini Bussi & Martignone, 2013). The focuses of the MMLab-ER laboratory approach were the analysis of exploration and argumentation processes (Boero, 2007) and the study of the cultural aspects involved (Bartolini Bussi & Martignone, 2014). During the MMLab-ER activities, the teachers carried out a Cultural Analysis of Content (Boero & Guala, 2008) that consists of working on what they know about a specific content and, above all, on their capacity of analysing the historical, cultural and epistemological aspects of the mathematical content involved. During the project, teachers were engaged in laboratory activities designed and orchestrated by the teacher educators; afterword, the teachers designed and carried out several teaching experiments with their classes. During the meetings, and also by means to e-learning platform, the teachers and the teacher educators shared and discussed the teaching experiments plans and results. Over a period of four years (2008–2012), the project was carried out for the first two years, followed by one year break in which the researchers analysed the project results, and resumed and completed on the fourth year. This development makes the MMLab-ER project a suitable example for studying the development of the meta-didactical praxeologies over time.

THE RESEARCHERS' PRAXEOLOGIES DEVELOPMENT

In this paper, the focus is on the researchers' meta-didactical praxeologies. Some of the researchers involved in the MMLab-ER project (in particular, over the whole four years, the researchers involved were Rossella Garuti and myself) analysed their actions using the MDT model lenses. We analysed our meta-didactical praxeologies (both as teacher educators and as researchers) in three different phases of the MMLab-ER project: at the beginning, during the first two years, and in the last year of the project. The MDT model gives us some interpretative tools to control the overall development of researchers' meta-didactical praxeologies, identifying how these praxeologies changed at different levels: *praxis* level and *logos* level.

At the beginning of the MMLab-ER project

When the MMLab-ER teacher education program started, the researchers had their own meta-didactical praxeologies linked to: 1) their studies on students' activities in the classroom, and 2) their experiences and studies on teacher education. In the former ones, the problem is to study the educational potential of the laboratory activities with mathematical machines. In the second ones, the researchers' praxeologies are linked to their work as teacher educators. These praxeologies are strictly intertwined but the *tasks*, *techniques* and the *theoretical discourses* are different. According to the aims of the MMLab-ER project, the researchers designed the activities to foster teachers' attention on the exploration processes, on the conjecture productions, and on the proof constructions. The *techniques* concerned the development of *tasks for teachers* (Watson & Sullivan, 2008) that include, for example, the selection and the analysis of some teaching experiments (Martignone, 2011). The following tables contain the answers to the questions: which tasks did the researchers face? Which techniques did they use? Which theoretical tools were in the background?

Table 1 describes the different aspects of researchers' meta-didactical praxeologies (P_{R1}), where the *task* is to think over the educational potential of the laboratory activities with mathematical machines. These praxeologies are meta-didactical because they reflect on educational activities. They are developed when the researchers study theories, plan, observe, analyse classroom activities, and construct theoretical lenses to describe and interpret students' difficulties and successes. Table 1 details the techniques and the theoretical discourses involved in these praxeologies.

Table 2 describes the researchers' praxeologies linked to the researchers' work as teacher educators and their studies on teacher education processes (P_{R2}). Also in this case, the praxeologies are meta-didactical, because the *task* is about the reflection on educational activities that involved in-service or pre-service teachers.

In the MMLab-ER project the researchers were also the teacher educators, therefore they directly observed and then reflected on the activities carried out during the teacher education program. The theoretical background was grounded on the research on teacher education (see Table 2).

Tables 1 and 2 show how the praxeologies components developed by the researchers are different when they reflect on classroom activities or on their work as teacher educators. The actions and the theoretical

<table>
<tr><th colspan="2">P_{R1} praxis level</th></tr>
<tr><td>Tasks
To reflect on the educational potential of the laboratory activities with mathematical machines</td><td>Techniques
Design and study of activities for primary and secondary school students. Analysis of students' written worksheets, clinical interviews, and collective discussions.</td></tr>
<tr><th colspan="2">P_{R1} logos level</th></tr>
<tr><td colspan="2">The theoretical discourses that describe, explain and justify the techniques of these praxeologies are based on the studies about laboratory activities with the mathematical machines:
— Semiotic mediation framework (Bartolini Bussi & Mariotti, 2008)
— Educational studies about proof (Garuti, Boero, Lemut, & Mariotti, 1996; Garuti, 2003; Theorems in School (Boero (Ed.), 2007))
— Mathematics laboratory [1]
— Italian standards for grades 1 to 8 [2] and for grades 9 to 13 [3]
— Studies about the mathematical machines carried out by the MMLab research group [4][5]
— Studies on mathematical machines utilization schemes (Martignone & Antonini, 2009)</td></tr>
</table>

Table 1: P_{R1}

<table>
<tr><th colspan="2">P_{R2} praxis level</th></tr>
<tr><td>Tasks
Design and analyse the activities for developing teachers' attention on the exploration processes, on the conjecture productions and on the proof constructions by means of laboratory sessions with mathematical machines.</td><td>Techniques
The development of tasks for teachers: e.g., a priori analysis of tasks, discussion of teaching experiments, etc.
The analysis of teachers' actions during the program and of teachers' reflections by means of logbooks and final reports</td></tr>
<tr><th colspan="2">P_{R2} logos level</th></tr>
<tr><td colspan="2">The theoretical discourses that describe, explain and justify the techniques of these praxeologies are based on studies about teacher education and proving processes:
— Cultural Analysis of Content (Boero & Guala, 2008)
— Research in Mathematics Education and studies about argumentation processes (Gutierrez & Boero, 2006; Boero (Ed.), 2007)
— Research on teacher education (Schulman, 1986; Wood (Ed.), 2008, etc.)</td></tr>
</table>

Table 2: P_{R2}

background are coherent. The study of the *praxis* and *logos* levels of these meta-didactical praxeologies allows us to underline the different features of the work of the researcher, who is also a teacher educator during a teacher education program. As already underlined, the *praxis and logos le*vels of the different praxeologies identified are strictly linked but we can distinguish, according to the different *tasks* faced, the specific *techniques* and *theoretical discourses* involved. The next paragraphs describe how these praxeologies change over time.

During and at the end of the MMLab-ER project

During the MMLab-ER project new praxeologies are generated by the intertwining and sharing of praxeologies developed by researchers and teachers. The teachers and the researchers analysed together the laboratory sessions: in particular focusing on the exploration processes, the conjectures productions, and the proof constructions in the activities with mathematical machines. By means of this work, in which theory and practice are strictly intertwined, new shared praxeologies developed, and also the previous teachers' and researchers' praxeologies improved. The analysis of praxeologies is useful to show when and how researchers' actions and theories develop, by reflecting upon the nature of and the reasons for the changes that occur. Focusing on researchers' meta-didactical praxeologies, we have identified the aspects that do not change and how the *praxis* and *logos* levels of the initial researchers' praxeologies (Tables 1–2) are modified over time. We face these questions: How do the researchers meta-didactical praxeologies change? At what level? Why do they change?

During the first two years of the MMLab-ER project some *techniques* of the researchers' praxeologies were improved by introducing new activities elaborated with the teachers, and by modifying some *tasks for teachers* and classroom tasks. We took into account what was discussed during the teacher education program, and we refined the tools for analysing teachers' and students' worksheets, logbooks, videos, etc. (Martignone, 2011). Also the *theoretical discourse* was improved by refining some interpretative tools: for example, the cognitive studies about mathematical machines were developed by analysing and identifying the argumentation processes involved (Antonini & Martignone, 2011).

During 2011, while waiting for new funds for a new start of the project, the researchers had the possibility to analyse all of the documentation collected: in particular, videos and worksheets about the laboratory activities carried out by teachers and students, the teachers' reflections collected in the logbooks, and the final reports of the teaching experiments, all stored in the MMLab-ER e-learning platform. In the last year of the project, the materials produced in the previous teaching experiments (educational paths, worksheets, final reports [6], logbooks, etc.) became the starting point for new teaching experiments designed and carried out in the classrooms by the teachers. The reflection on the previous activities influenced the design of the tasks and the development of the researchers/teacher educators' *techniques*: for example, some activities were modified, others were removed. Through the lenses of praxeologies development (P_{R2}) we can see that an improvement of meta-didactical praxeologies *techniques* occurred: modifying some *task for teachers*, removing others, introducing new tasks elaborated with the teachers involved in the previous years of the project. Researchers worked with the new teachers in analysing many logbooks, used as tools for reflection, and in sharing the analysis carried out with the previous teachers. There was also an improvement in the *logos* level: developing new studies on cultural and educational aspects involved (Bartolini Bussi & Martignone, 2014).

Thanks to the project, the researchers had the possibility to work with teachers for many years, developing long term collaboration. They carried out design and fine grain analysis of their teaching experiments. Therefore the researchers improved also their praxeologies about the reflection on students' activities with mathematical machines (P_{R1}). They shared reflections and theoretical tools with teachers in order to interpret students' difficulties and successes (Banchelli & Martignone, 2013), and to design new activities to overcome the found obstacles and to investigate new aspects about the educational goals of the activities.

CONCLUDING REMARKS

The paper describes how the identification of the meta-didactical praxeologies can be used to analyse the researchers' actions and reflections developed during a teacher education program. The development over time is described as a sequence of frames where the researchers' praxeologies change. In particular, the overall development of researchers' praxeologies can be checked by studying how the praxeologies changed at different levels: practical-technical and theoretical. The identification of these different levels, the *praxis* level (or "know how") and the *logos* level (or "knowledge"), gave us a fine grain tool to analyse how the praxeologies change over time. We would like to answer the questions: Which are the researchers' meta-didactical praxeologies? How do the researchers' meta-didactical praxeologies change? At what level? Why do they change?

In the paper, the different features of the work of the researchers, who work also as teacher educators, are identified by distinguishing the different *tasks*, *techniques* and *theoretical discourses* developed. The new key (explained by the word "meta") is the identification of reflective actions about the activities carried out by the researchers/teacher educators in a teacher education program. The analysis of the meta-didactical praxeologies is useful at first to identify and then to think about the evolution over time of important aspects linked to the work of researchers involved in a teacher education project. The next step of the research will consist in an improvement of the theoretical tools used to study the praxeologies development: in particular, we will carry out a fine grain analysis of the praxeologies by classifying them as specific, local, regional and global (Chevallard, 1999). These further interpretative tools could be useful to describe the complexity of the meta-didactical praxeologies developed in the teacher education programs.

As a final remark, it is important to stress that this paper deals with a little part of the research that used the MDT model: as a matter of fact, MDT model can give also useful tools to compare the components of meta-didactical praxeologies involved in different teacher education programs carried out in the same country or in different ones (Aldon et al., 2014). Moreover, the research group will work on further developing of the MDT theoretical tools, in order to make them suitable also for planning and designing new teacher education programs.

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ENDNOTES

1. www.umi.dm.unibo.it/downloads/icme10.pdf

2. Italian Standards (grades 1–8): http://www.edscuola.it/archivio/norme/programmi/indicazioni_nazionali.pdf

3. Italian Standards (grades 9–13): http://www.indire.it/lucabas/lkmw_file/licei2010///indicazioni_nuovo_impaginato/_decreto_indicazioni_nazionali.pdf

4. MMLab publications: http://www.mmlab.unimore.it/site/home/pubblicazioni.html

5. Website of the Mathematical Machine Association: http://www.macchinematematiche.org/

6. Final reports written by the teachers involved in the MMLab-ER project: http://www.mmlab.unimore.it/site/home/progetto-regionale-emilia-romagna/risultati-del-progetto/report-delle-sperimentazioni.html

Mathematics educator transformation(s) by reflecting on students' non-standard reasoning

Maria Mellone[1], Arne Jakobsen[2] and C. Miguel Ribeiro[3]

1 University of Naples Federico II, Naples, Italy, maria.mellone@unina.it

2 University of Stavanger, Stavanger, Norway

3 Norwegian University of Science and Technology, Trondheim, Norway

In this study, we present some results stemming from a research work exploring the way in which prospective teachers develop their interpretative knowledge and awareness by discussing students' errors and non-standard reasoning. For this purpose, we designed a particular kind of task that was administrated and discussed in our own lectures. The discussions and reflections associated with this experience allowed us, as educators, to expand our own mathematical knowledge and awareness. Based on the analysis of video-recorded lessons delivered as a part of a course based in Italy, we will argue that work grounded in discussing students' naive ideas/non-standard reasoning represents a core aspect of the mathematics teachers' education field, whereby educators are also viewed as learners.

Keywords: Teacher's knowledge, educator's knowledge, interpretation, pupils' productions.

INTRODUCTION

This work is a part of a wider research project aimed at accessing and developing the knowledge mathematic teachers require for effective instruction. Part of this work pertains to designing and implementing contexts and tasks suitable for promoting the development of such knowledge. In our previous work, the focus was mainly on prospective teachers' answers and reasoning when solving a specific task. In this paper, we expand on this early work and promote a discussion on our own reflections upon the implementation and analysis of this particular task and the way of managing it. This particular task is essentially rooted in asking the (prospective) teachers to provide sense to students' productions, some of which can be considered incomplete, containing errors, or simply being grounded on nonstandard reasoning (e.g., Jakobsen, Ribeiro, & Mellone, 2014; Ribeiro, Mellone, & Jakobsen, 2013b). Our aim is to provide reflection on how this kind of task can promote mathematical knowledge development among (prospective) teachers and teacher educators.

Several historical examples show that the growth of mathematical knowledge often occurs through a dialectic process of "proofs and refutations," where initial, and often partially incorrect, hypotheses are progressively refined through a critical analysis of their consequences (Lakatos, 1976). Moving from a phylogeny to an ontogeny level, we can argue that this is similar to the learning process experienced by an individual. Adopting this perspective, a learner that makes an error can be compared to a person that got lost on his/her journey. Thus, if (s)he had an important meeting, (s)he will likely arrive late and agitated. On the other hand, if (s)he is a tourist who is visiting new places, getting lost may be perceived as an opportunity to discover new places that (s)he wouldn't have known otherwise (Borasi, 1994).

Grounded in some of our previous work (mentioned above), and starting from these reflections, we argue that the work on errors, incomplete answers, or non-standard reasoning should represent a core aspect in and for developing mathematics teacher education. Agreeing with Tulis (2013), we claim that teachers must be sensitive to students' errors and nonstandard reasoning. We also point to the fact that, in everyday classroom, a learning climate in which errors are perceived in a positive way should be established, allowing students to learn from their mistakes. The development of positive attitudes towards errors should be pursued from the beginning of mathematics teachers' professional development (assuming that it starts in teachers' initial training). With this aim, we worked in our own courses with prospective teachers (and in other professional development contexts) using the previously mentioned particular type of task. This task has been used in our lessons as a prompt to orchestrate a mathematical discussion (Bussi, 1996). We used mathematical discussion as main tool to develop, together with our students, mathematical knowledge and new awareness of the opportunities and richness of learning experience that can come from the work on nonstandard reasoning. Such use is thus also aimed at building a co-learning community, throughout an inquiry community perspective (e.g., Jaworski & Goodchild, 2006).

THEORETICAL FRAMEWORK

In mathematics teacher education, only a few approaches are using mistakes and nonstandard reasoning as a resource in Mathematics Education (Tulis, 2013). From our perspective, one of the core aspects of teaching practice should focus on developing teachers' knowledge that can assist them in giving sense to students' productions and perceiving errors as a learning opportunity (e.g., Ribeiro et al., 2013b). Such knowledge would allow teachers to develop and implement ways to support students in building knowledge that is founded in their own reasoning, even when such reasoning differs from that expected by the teacher. Aiming at accessing and developing such knowledge and ability, we have been developing tasks for teacher training that require them to solve problems before trying to give sense to students' productions aimed at answering such problems. Thus far, these tasks have been used as a tool for both observing and deepening the access to prospective teachers' mathematical knowledge for teaching (MKT) (Ball, Thames, & Phelps, 2008). Driven by this prompt, we also aim to support, in our lectures, the development of prospective teachers' MKT. During this research work, we have been focusing on a particular kind of knowledge we refer to as *interpretative knowledge* (Jakobsen et al., 2014). It is essential for teachers to possess, as it entails the knowledge in and for making sense of students' solutions and helps teachers provide productive feedback to them (in the sense discussed by Bruno & Santos, 2010).

In particular, we recognize the peculiar and specific nature of this knowledge and thus consider it a part of the Specialized Content Knowledge (SCK) domain of the MKT model, while also recognizing its links to the Pedagogical Content Knowledge (PCK). Findings of our previous studies indicated that a poor Common Content Knowledge (CCK) compromised the prospective teachers' ability to give sense to students' solutions that differed from their own. Indeed, we found evidence in support of our hypothesis that a lack of common knowledge on a particular mathematical topic hinders the prospective teachers in forming a flexible perception of that topic, making it difficult to move to different visions and their potential use in teaching (Ribeiro et al., 2013b; Jakobsen et al., 2014).

At this time, our research work is moving to another aspect we consider intrinsically involved in the kind of mathematical activity developed in teacher training—our growth and development as mathematics teacher educators. For this reason, we are complementing the previous focus on MKT and on the interpretative knowledge with the Inquiry community perspective (see, for example, Jaworski & Goodchild, 2006). According to this approach,

> Didacticians have designed activity to create opportunity to work with teachers, to ask questions and to see common purposes in using inquiry approaches that bring both groups closer in thinking about and improving mathematics teaching and learning . . . . This design process is generative and transformative. (Jaworski & Goodchild, 2006, p. 354)

The principal aim of this approach is to work with teachers as co-learning professionals, with didacticians and teachers each contributing with their specialist knowledge in order to collaboratively develop new knowledge in practice (e.g., Schön, 1987; Wagner, 1997). By using this type of tasks in our lectures, and paying attention to the response we receive from the attendees, we, as educators, are living a transformative experience, derived from participating in what we consider a co-learning community with our students. The idea of Inquiry Community is rooted in the Activity Theory Framework (Vygotsky, 1978), comprising of a subject, an object, and a mediation between them. In our case, the subjects involved are the community of prospective teachers and teacher educators. The mediation corresponds to the task, whereby the prospective teachers are asked to give sense and feedback to students' solutions. Finally, the outcome is the MKT development of both prospective teachers and mathematics teacher educators. The considered perspective for professional development of teacher educators is grounded, from one side, in reflecting and discussing upon our own practices (e.g., Avalos, 2011). On other side, complementarily, we also assume that being a teacher educator involves much more than applying the skills of school teaching to a different context (e.g., Loughran, 2014). In our view, it requires a specialized type of (complementary) knowledge that the teachers need (e.g., Superfine & Li, 2014) in order to expand teacher trainers' vision of teaching. At this stage, working with students must be different from working with teachers, and should focus on deepening the "hows," the "whys" and the connections among and within topics.

METHOD

The context of this study is the mathematics courses in teacher education, in particular some courses where we (the authors) were the lectures. In this paper, for brevity, we limit our focus to the data pertaining to two classes of the course held in Italy. This course is taught in the third year of a five-year program of the master degree in education, and can be taken upon passing the Foundation of Mathematics exam.

In the course, different tasks were explored, focusing on problem solving aimed at exploring particular mathematics education issues in depth (e.g., arithmetical, algebraic, and geometrical thinking) and the students' MKT pertinent to such issues. Complementarily, the course also focused on more general mathematics education approaches (e.g., sociocultural nature of learning, semiotic mediation process), among other goals. The entire course was conceptualized assuming the Inquiry Community Perspective, with the aim of promoting the development of a co-learning community. Around 100 prospective primary teachers participated in these classes. In addition, one educator/researcher was responsible for delivering the course, while one researcher took responsibility for the data collection (gathering the prospective teachers' written answers, as well as audio and video recording the classes). This was the second course in which this task was implemented and pertinent data gathered.

During the first lesson of the activity, a questionnaire containing the task was given to prospective teachers, asking them to answer it individually within one hour. The task required them to solve a very "simple" problem. They were given the following instructions: *Teacher Maria wants to explore with her students some notions concerning fractions. For this purpose, she has prepared a sequence of tasks involving five chocolate bars. Let us look at one of them: What amount of chocolate would each child get if we divide five chocolate bars equally among six children?*

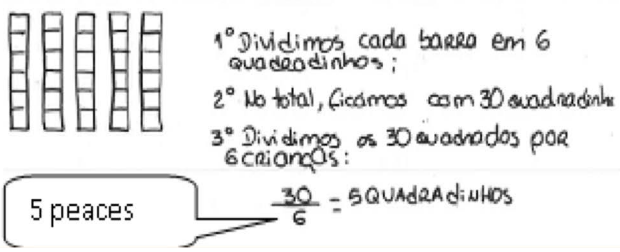

Figure 1: Ricardo's solution "1° We divide each bar in six squares; 2° In total, we have 30 squares; 3° We divide 30 squares among the 6 children. 30/6=5 squares"

After completing the task (i.e., providing their own answer to the aforementioned problem), we asked the participants to consider some students' solutions to the same problem. Some responses contained errors while others were incomplete, and some involved a nonstandard approach to problem-solving. Figure 1 and 2 present two of those responses, while a broader discussion of the productions included in the questionnaire is given in Ribeiro, Mellone, and Jakobsen (2013a). In particular, we asked prospective teachers to give sense to a set of pupils' productions, while following specific requirements: (i) For each pupil's production, decide if you consider it mathematically correct (adequate) or not, and justify the (in)adequacy of the mathematical rationality shown; (ii) Give a constructive feedback to pupils, especially to students whose answers you consider inadequate, and create a set of possible questions in order to promote their mathematical knowledge development.Data from the questionnaire was analyzed in terms of prospective teachers' own solutions to the problem, taking into consideration the different types of answers given,

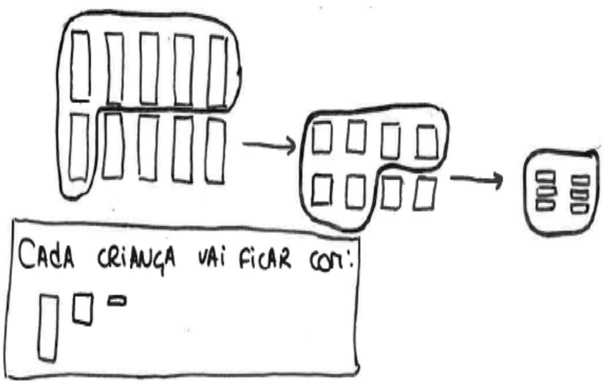

Figure 2: Mariana's solution "Each child will get"

the number of representations used, and their evaluations of the pupils' solutions (for further details, see Ribeiro et al., 2013b). The subsequent lesson, grounded in the collective mathematical discussion (Bussi, 1996) of the task was video recorded and analyzed in terms of prospective teachers' and educators' own reflections, discoveries, and points of turn in the MKT development. The fact that we, as educators, are considered an integral part of the inquiry community, in particular community of learners, is the key point of this paper. Nonetheless, this same fact represents a sensitive point of the method we chose for our analysis, since we analyzed "ourselves." However, in this case, we analyze the Italian experience. Thus, at the beginning of the lesson, the Italian researcher was chosen to provide her point of view on some crucial points of the mathematical discussion that took place. In the next stage, we conducted a joint analysis, followed by joint discussion and reflection.

SOME EVIDENCE OF THE TRANSFORMATIVE EXPERIENCE

The transformative experience is a complex process and there are several considerations to take into account. Here, we will focus on two examples of the kind of reflections that are driving our experience, as well as our reflections and discussions. These examples were chosen because they are perceived as constituting crucial and critical points of the mathematical discussion undertaken. In particular, we focus on some of the mathematical discussion that took place when commenting on Mariana's solution (Figure 2). It should be noted that prospective teachers' answers to the questionnaire revealed that some found interpretation of this solution particularly difficult. This finding is in line with the reports of Norwegian (Jakobsen et al., 2014) and Portuguese experiences. Among the prospective teachers' comments to this solution, the following answers were particularly interesting:

> Mariana's solution is not understandable, so the first question would be: what does this representation mean? After I have listened to her answer, I will try to show her my own representation and we will reach the solution together"; or "She does not understand fractions – she is just dividing the pieces.

Comments such as these reveal the difficulty these prospective teachers had in leaving their own space of solutions. In particular, when this space consists of a single element (Jakobsen et al., 2014), it seems impossible for prospective teachers to appreciate and understand different solution strategies students adopt. As a result, they are unable to exploit them to support children's deeper knowledge development on the subject. Many of the prospective teachers responded in a similar way. A particular case concerns those that provided answers using only natural numbers, similar to Ricardo's solution (Figure 1), or saying "*each will get 5/6 of each chocolate bar.*"

The collective discussion promoted some changes in reasoning and argumentation. Here, it is worth noting the difficulties in orchestrating a mathematical discussion within a community of about 100 individuals. Thus, in order to facilitate constructive discussion, Mariana's production was projected on the wall and was thus clearly visible to all participants, while the educator focused on identifying prospective teachers that wanted to comment on the solution (using a microphone). After some minutes of discussion, in which most prospective teachers expressed their difficulties in understanding Mariana's reasoning (including how she obtained 10 bars), something changed. This is evident in Miriam's contribution to the discussion.

> Miriam: She basically takes the five bars and divides them in half and so she has ten pieces, so she gives six, while four remain. Then she divides the others in half again, and then there are eight and she assigns six, so two remain. The other two are divided into three parts, so six more pieces. Then finally, she says: every child will have a half of bar, plus the half of the half of a bar and a third of a bar (she stopped)
>
> –voices of students who want to intervene–
>
> Educator: Wait a moment, give her time...
>
> Miriam: mmm ... it's as if ... a third of the half of the half.

It is worth pointing out that Miriam, along with the majority of the prospective teachers that took part in the discussion, did not previously understand Mariana's solution. However, by building her own reflections on those of her colleagues, she was able not only to understand something that was not clear during the individual work, but also to recognize in the pieces of Mariana's representation the particular fractions representing parts of the unit (the chocolate bar) involved. In the next step, the educator proposed to all participants to write down the explanation Miriam provided. The aim of this task was to prompt the prospective teachers to understand and verify the equivalence between 5/6 and the sum of these particular fractions. The amazement most prospective teachers felt upon discovering the meaning of Mariana's solution was the prompt to build an interesting discussion about the fact that, just because one does not understand something, it does not mean that it is incorrect. Such prompt also facilitated exploring links between teachers' specialized and pedagogical content knowledge, as well as understanding its importance/role in practice (e.g., Ribeiro & Carrillo, 2011). One of the focal points of the task conceptualization and implementation is on exploring and developing students' awareness on the knowledge involved (nature and type) in and for elaborating constructive feedback. In taking this approach, we were deepening some aspects of Bruno and Santos's (2010) work on written feedback.

While discussing and reflecting upon Mariana's production, prospective teachers experienced some contingency moments (Rowland, Huckstep, & Thwaites, 2005), allowing us to reflect and discuss upon the work we had done thus far. Such reflection is linked to the development of a mathematical knowledge we, as mathematics teacher educators, experienced by sustaining our own professional development. The situation presented here was driven by a comment made by Francesca, another prospective teacher. Indeed, after recognizing the correctness of the subdivision of the bars presented in Mariana's solution, most prospective teachers also saw the possibility to create numerical representations of the quantities indicated within the drawings. However, another issue was raised by Francesca.

Francesca: Yes, I think it is mathematically correct, because there are six children. But, if I had seven children, for example, I don't know if this division into equal parts could work. However, it was a trial and error process and this time it succeeded; but I do not know if, with other numbers, it could work.

Educator: So . . . you are saying that this procedure does not seem to be applicable to other numbers ...

Francesca: I do not know, it amazes me, but perhaps it could not work with other numbers.

At that moment, the educator was not prepared for such a comment and did not prompt a discussion that would explore this issue further. However, Francesca's comment was a focus of discussion and reflection in the scope of the research group, and served as a starting point for developing a new mathematical awareness. Indeed, on one hand, according to Empson, Junk, Dominguez, and Turner (2006), Mariana's solution can be seen as a *progressive parts strategy* considering it without any anticipatory organization of the subdivision. Yet, on the other hand, such solution reflects a peculiar management of subdivisions, with the potential for being generative of a precious mathematical insight. One can argue that Ricardo's solution (Figure 1) is grounded in the understanding/assumption that, in order to equally share five chocolate bars, each bar can be partitioned into 6 parts, thus perceiving $\frac{5}{6}$ as equivalent to $\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}$. Linked to this strategy, in which it is possible to recognize a kind of anticipatory thinking (Empson et al., 2006), there is the view of a fraction $\frac{n}{m}$ as equivalent to the sum of n unitary fractions $\frac{1}{m}$. In contrast, Mariana's progressive partitions strategy suggests an interpretation of $\frac{5}{6}$ as equivalent to $\frac{1}{2}+\frac{1}{4}+\frac{1}{12}$. Thus, it is impressive to see how this particular representation of a fraction as the sum of unitary fractions appears naturally in her reasoning. Her approach reveals the possibility to represent uniquely any fraction as a finite sum of decreasing rational numbers, such that the first element is the integer part of the fraction and each subsequent one is the greatest unitary fraction that is contained in the remaining part, which can be represented as $\frac{n}{m}=i+\frac{1}{q_0}+\frac{1}{q_1}+\ldots+\frac{1}{q_k}$. Such a representation has the strong advantage of showing clearly a sequence of rational numbers, simpler than the one assigned, that are, in a sense, the best lower approximations of it. Thus, after developing these reflections, during the following class, the educator had the opportunity to discuss the findings with the prospective teachers. The aim was to allow them to develop complementary elements to be included in their own space of solutions (e.g., Jakobsen et al., 2014).

SOME FINAL COMMENTS

In this paper, some insights and reflections pertaining to a particular task were presented. The discussions prompted by Mariana's production have provided the opportunity to reflect on prospective teachers' knowledge elaboration on a particular aspect of fractions, as well as on our own role in the development of such knowledge and of our own awareness. However, we want to stress that, although such reflections were grounded in a task on fractions, the mathematical topic itself and the particular mathematical fact of "discovering" the possible link between Mariana's solution and the representation of a fraction as the sum of unitary fractions, are not the focus here. Indeed, as a meta-discourse, we are arguing something more general. The experience, and the way in which we presented it, is perceived as a contribution to the discussions and the associated reflections that highlight the need for a more focused work (and research) on the interpretative knowledge needed by (prospective) teachers and their educators, and the way(s) to promote it. Moreover, assuming that teachers and teacher educators need different "ways of hearing," we argue that different aspects and different natures of professional knowledge need to be taken into account when designing and implementing teacher training. The ultimate aim of this initiative is a joint and intertwined development of all the constituents of the inquiry community (Jaworski & Goodchild, 2006).

A significant number of diverse possibilities and paths for discussion were anticipated when we discussed and prepared the task and its implementation. Fortunately, some unforeseen situations emerged, leading to improvisations, some of which corresponded to contingency moments (Rowland et al., 2005). The reflections upon these situations, and the associated discussions on our own practice, have enabled us to develop a broader perspective on the process of teaching teachers, providing us with a deeper insight into what it requires and entails. Indeed, as we listened to the prospective teachers' comments on the proposed students' solutions, we also had some difficulties in interpreting and giving sense to their reasoning. This has led to mathematical critical moments, allowing us to reflect upon them. In our view, this is an essential aspect in and for promoting professional development for teachers as well as teacher trainers. In this sense, collaboration among the authors was crucial. By reflecting upon the discussion with prospective teachers, we could appreciate the type and nature of possible connections, representations, and navigations that could be made (needed) between and within topics. This has been one triggering event, leading to the awareness of the need to help teachers develop a much deeper knowledge. The work of Superfine and Li (2014) is a good example of an effort aimed at deepening and evolving this knowledge further.

Finally, we want to call attention to inextricable links among the tasks we prepare and implement (type, nature, and focus), the role of research on (prospective) teachers' knowledge and practices, and the learning opportunities we, as facilitators, provide. In order to also bring together theory and practice, we must recognize the importance of the role teacher trainers play in developing teachers' knowledge and practices. Thus, we argue that, if we want to enable prospective teachers to give sense to students' solutions and provide constructive feedback, we, as educators, need to do the same, albeit with a different focus. We posit that the courses specifically designed for teachers must have aims differing from those classes for students are geared towards. In this sense, a practice-based approach, in which prospective teachers can experience similar situations to those we expect them to encounter with their students is essential. We also recommend that teacher training address the role and the attitude of the educator, as one of the key elements. In particular, we recognize the importance and the need for teacher educators to live/work in terms of transformative experience, being sensitive to growing opportunity it offers.

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Characterizing one teacher's participation in a developmental research project

Hélia Oliveira and Ana Henriques

Universidade de Lisboa, Instituto de Educação, Lisbon, Portugal, hmoliveira@ie.ul.pt

This study was developed in the context of a developmental research project with eleven mathematics teachers and two university researchers working collaboratively during part of one school year. The paper analyses the participation of one teacher in the emergent inquiry community devoted to developmental research on tasks targeting students' statistical reasoning. Inquiry and modes of belonging to an inquiry community are present in the analysis of the data that were mainly collected in the collective meetings of the project. Preliminary results show that the teacher's participation is closely connected with an inquiry stance and critical alignment to her practice and in some instances to the project's goals.

Keywords: Inquiry community, statistical reasoning, developmental research.

INTRODUCTION

Assuming the important role of sound tasks and the use of technological tools to promote students' statistical reasoning, a developmental research project that seeks to deepen the understanding about the conditions for the development of that kind of reasoning in the classroom has been carried out by the two authors. Recognizing the teacher's central role in the classroom practice, the project assumed a collaborative nature, involving the authors, as researchers and a group of teachers who wanted to learn and share new ideas about the teaching of statistics in basic education.

The project was originally planned to be, simultaneously, a research and a teacher development project. Teachers had some previous contact with research in education and assumed the shared role of designers and researchers with the university researchers, as it happened in other developmental research projects (Goodchild, 2014). Our first reflections about the process, made us consider the teachers' participation in the group and its nature, as we observed that some teachers expressed doubts, resistance and delayed the implementation of the tasks in the classroom. Addressing this issue is seen by us as an opportunity to question the dynamics foresaw for the project, as well as to understand the conditions for professional development that were created. Preliminary analysis of the work carried out in the group made us conjecture if implicitly were we trying to promote an inquiry community (Jaworski, 2008) since we intended that all participants would assume a role in the tasks' design and in reflecting about their use in the classroom. We did not have readymade tasks for teachers to apply and we did not intend to include them in the action only (Jaworski, 2008). In fact, teachers got involved in designing the tasks, making punctual or important suggestions or proposing alternatives. We, as university researchers, had many questions and few answers and wanted to learn from this experience. However, since this was a funded project, there were certain constraints to the collaboration with the teachers, namely: the sessions should follow a predetermined time schedule and the teachers' written reflections had to fulfil some conditions. All this made us question how teachers would conceal the interests of their teaching practice and the compromise they had with the project.

In order to understand the nature of the teachers' participation in an emergent inquiry community committed to developmental research on tasks targeting students' statistical reasoning, we started by developing an exploratory study focusing on one teacher in the group. Thus, this study is oriented by the following research question: how did one mathematics teacher's participation in an emergent community of inquiry evolve? With this study we also intend to reflect about the conditions created by this developmental project for teachers' professional development.

THEORETICAL BACKGROUND

This study comes from a developmental research project that assumes the perspective of co-learning among teachers and researchers as they design tasks and environments and investigate their students' learning. This is in line with many recent studies that see this collaboration as a favourable setting for developing teachers' knowledge (Potari, Sakonidis, Chatzigoula, & Manaridis, 2010). Specially, when implementing new approaches in their classroom teachers may benefit from working in collaborative settings that involve other teachers and researchers (Ponte, Segurado, & Oliveira, 2003).

In a developmental research project based on collaboration, teachers and researchers both learn in the process since they come together to develop teaching with the goal of improving students' experience of mathematics (Goodchild, Fuglestad, & Jaworski, 2013). Concerning the practice that evolves when teachers and researchers are collaborating, learning occurs through inquiring into it (Potari et al., 2010). According to Jaworski (2008) inquiry is "a mediational tool in social settings enabling development of knowing between people and hence the participative individuals" (p. 327). When inquiry is used as a tool for learning and development, we can talk about an inquiry community (Jaworski, 2008).

According to Potari and colleagues (2010) the difference between a community of practice as conceived by Wenger (1998) and an inquiry community is that in the later inquiry is part of the norms of practice and so it attributes great importance to critical reflection on the practice, as a form of meta-knowing. Inquiry is assumed as a stance in those communities which "means to challenge regular practice when it is ineffective, to reflect on the reasons why an approach might not achieve the intended outcome and to propose alternative approaches" (Goodchild, 2014, p. 313).

Goodchild, Fuglestad, and Jaworski (2013) consider that inquiry can be used "as a tool to be a form of critical alignment, that is engagement in and alignment with the practices of the community while at the same time asking questions, trying out new approaches and reflecting critically" (p. 396), and therefore critical alignment is crucial for the developmental process. Jaworski (2008) contends that in opposition to communities of practice that are more stable, the inquiry community is emergent as "It does not avoid issues, tensions and contradictions, but deals with them as part of emergent recognition and understanding leading to possibilities for expansive learning" (p. 327).

In Wenger's work (1998), to belong to a community of practice involves three different modes: *engagement* – active involvement in mutual processes of negotiation of meaning; *imagination* – creating images of the world and seeing connections through time and space by extrapolating from our own experience; *alignment* – coordinating our energy and activities in order to fit within broader structures and contribute to broader enterprises. According to the author, when these modes work in combination in a community of practice, it can become a learning community. In terms of an inquiry community, critical alignment is also required as participants "engage in existing practices, aligning to some extent with those practices, but in a questioning or inquiry mode" (Jaworski, 2008, p. 314).

THE CONTEXT

The idea for the developmental project grew from our perception that statistics receives reduced attention from mathematics teachers in basic education in our country. They usually consider it an easy theme that does not require much time in its teaching. Recognizing the need to further understand how to develop students' reasoning using the technological tools at our disposal, specifically the *TinkerPlots* software, we envisioned that it would be more productive to involve teachers that are well informed and interested in this issue (Wells, 2007) in a joint enterprise of developing and testing specific statistical tasks in their basic education classrooms (5^{th} to 9^{th} grades), rather than to develop our theoretical ideas and then simply propose them to teachers. Previous experiences in collaborative research projects showed us that collaboration between teachers and researchers can produce important synergies (Ponte, Segurado, & Oliveira, 2003).

The proposal presented to the teachers concerned the participation in the developmental research project where two or three cycles of a sequence of task design, implementation in the classroom, analysis of results and refinement of the tasks would involve everyone in the group. To make collaboration possible in the multiple activities of the project we required that participant teachers were acquainted with research methods in education. Therefore the eleven mathematics teachers who accepted the invitation had diverse experiences in teaching statistics but all of them had a masters' degree in mathematics education or were attending such a course. However, the theme of statistical reasoning and the use of *TinkerPlots* software were new for the majority of the teachers in the group.

The work carried out was inspired in Design Based Research and followed a teaching experiment design (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003). The university researchers were responsible for the overall plan of the project and for conducting the research and the teachers for developing small scale teaching experiments in pairs. The university researchers also assume the ultimate responsibility for the tasks to be implemented in the classroom and the guidelines for teachers to elaborate the reports of the teaching experiments.

The project had financial support that facilitated the support of one research assistance to collect data and extended itself for eight months (from November to June), in a total of 13 meetings (three to four hours each). The first three meetings were dedicated to discuss big ideas concerning statistics reasoning and the required conditions to its development and to explore the software's potentials. The perspective of Garfield and Ben-Zvi (2010) of Statistics Reasoning Learning Environment (SRLE) was discussed within the group and the basic principles of Cobb and McClain (2004) for the design of the sequences of tasks were adopted.

The first cycle in the design research took place in the subsequent meetings. The tasks' characteristics were informed by the principles of SRLE, and they were designed with the intention of providing meaningful contexts in which students would be challenged to formulate questions based on real data, analyse data with the *TinkerPlots* and to make data based informal inferences. Teachers generally worked in pairs, planning the lessons, collecting data and reflecting about them. One or both teachers in the pair implemented the sequence of tasks in one's class, and were supported by the other in the classroom management and data collection. After finishing the sequence, the teachers elaborated a short written report about each task and shared their experience in the group meetings. The materials produced and the joint reflection in the group informed the tasks' reformulation and the conditions for their implementation in the subsequent cycle by other teachers. In the last two meetings the teachers presented, in pairs, the reports of the teaching experiments, and discussed them with the group. Finally, they produced a written report that extended their oral presentation and presented the main ideas on the experiment.

Retrospectively looking at this group, we start conceiving it as an emergent inquiry community, where co-learning inquiry occurs as researchers and teachers are learning together through inquiry (Jaworski, 2008) about challenging tasks, the students' activity with technology and the opportunities for promoting their statistical reasoning. We share Jaworski's perspective that in this group "teachers and didacticians are both practitioners and researchers" (p. 312). The collaborative engagement of all members, even though they assumed different roles and identities, made possible the development of insider and outsider research (Goodchild, Fuglestad, & Jaworski, 2013). The projects' extended live period also contributed to emergent relationships among the participants, mainly among the pairs, and among university researchers and teachers.

METHODOLOGY

This study represents our first attempt to analyse teachers' participation and their professional development in the project. Among the eleven teachers in the group, we choose to start by studying Maira's participation as an exploratory case study (Cohen, Manion, & Morrison, 2007), since we were surprised by her reluctance in applying the tasks that were being designed in the project in her class. She was one of the two teachers who had previous experience with the software, and showed more enthusiasm in getting involved with the project from the beginning. Nevertheless, for several weeks we doubted she would apply the tasks in her classroom. Finally, Maira decided to apply two tasks with one 5th grade class that she considered very problematic, and this gave us the motive for trying to understand how her participation in the inquiry community evolved throughout the project, which is our research question.

All group meetings with the teachers were audio-recorded and the teachers' written reports from their teaching experiments were collected which allows us to understand the teachers' participation in the emergent community of practice. In a case study research, the identification of critical incidents or events is crucial for understanding the case (Cohen, Manion, & Morrison, 2007). Hence, after a careful reading of transcripts of all meetings, we looked for key moments in the project where Maira's discourse expressed more or less sense of belonging to the emergent community, and we used the notions of engagement, imagination and (critical) alignment (Jaworski, 2008; Wenger, 1998) to characterize her changing participation. According to the adopted methodology, the results are presented as a chronological narrative of events and combine description and analysis (Cohen, Manion, & Morrison, 2007).

RESULTS: THE CASE OF MAIRA

We identified four key moments related to Maira's participation in the project which define the four sections in the data analysis.

Discussing the theory

In the third meeting, the group discussed some theoretical principles concerning SRLE, based on one paper from Garfield and Ben-Zvi (2010). Maira connected the overall perspective about teaching of those authors with an episode from her classroom, which occurred a few days earlier. She expressed her difficulties in dealing with students' lack of interest in school, learning difficulties and disruptive behaviour in the classroom. Her decision to develop group work with these students produces evidence of her motivation to try new methods.

> ... this paper dates from 2010, but almost 20 years after my initial training course, we still debate the issue of the traditional teaching (...) Therefore there is a high resistance and I think it has to do with us ... I have been resisting but this week I promoted small group work [in that class]. I imagined it would be chaotic and this and that ... And again, I must admit: it didn't happen. The mistake is ours, because, perhaps, this was my best lesson since the beginning of the school year.

Conceiving her usual practice as different from the one reflected theoretically in the paper, Maira recognizes that there is a personal and general resistance to change, expressing *critical alignment* with it. The results gave her more confidence in assuming some risks with those students: "We have to belief and take a chance even in [the perspective of] a chaotic lesson (...) and then, maybe, we really have surprises".

In this meeting, Maira also attempts to give meaning to some theoretical ideas by trying to think about their connections with the statistical knowledge she considers possible to develop with 5th graders. Speaking about possible tasks for promoting students' statistical reasoning, the main goal of the project, she considers that:

> This kind of tasks that nurtures relationships is important, at least to develop their vocabulary and bring them close to this kind of argumentation, to the discourse that is used in statistical reasoning.

In the project's initial phase, Maira shows *critical alignment* to her practice, disposition to *engage* in the project, and motivation to *imagine* to do things differently. However, she is permanently thinking on the characteristics of the students and conjecturing about how those theoretical ideas (that the project seem to adhere to) might be adapted for her classroom (critical alignment with the project).

Thinking about the tasks

During the 5th meeting where one of the tasks for the 8th grade was designed, several discussions about the situations that could be adapted for the 5th grade occurred. In the next transcript, we observe that Maira shares her opinion about the tasks that she considers more suitable for her 5th grade class and the sequence in which they should be integrated.

> They [the students] need to have data that are very close to them, so they can recognize them ... Because... this is a completely different representation from the ones that they are acquainted with. They are used to see graphic, pie graphics, bar graphics, at least in the press, isn't it? (...) But if we want them to manipulate those data with a deep understanding, what each one represents, I think that is has to be something closer to them, as in the first task [discussed] where they insert the data.

In this moment, the teacher expresses engagement with the project, making suggestions that reflect what she knows the students' needs in this school level, and the kind of work they will do with the software.

Later on, Maira begins to question the opportunity for implementing the tasks in her classroom. In the 7th meeting, she speaks again about the students' misbehaviour, lack of interest in school, resistance to learning and lack of autonomy. She rehearsed many strategies but none seems to succeed:

> And also already tried to have a small text explaining what they have to do, [and asked them] "Let us read!" and "Let us do!", "No"! It's no use. (...) I really have to be by their side, even if it is only to read and to follow. So it will be an experience, perhaps... very difficult. (...) I have not been able to solve this.

It seems that in face of the circumstances, Maira starts questioning the possibility of developing the tasks as foreseen. It is still possible to identify her will to be engaged with the project, making part of the joint enterprise, but she also reveals a strong disbelief in obtaining any positive outcome from the implementation of the designed tasks with her students.

Discussing the lessons

Close to the end of the project, when teachers were presenting the results from their teaching experiments, Maira reacts to one comment of one of the two university researchers that she interprets as one of surprise for the lack of involvement of the students from a different teacher in the first lesson they used the software *TinkerPlots*.

> I know it is hard to hear when we have software which we, as teachers, find fantastic, that "with this the students became more restless". But we have to think in a different way ... It is obvious that the software has great potential, and it would be great that this was a magic formula that when entering the classroom the kids would be attached and involved ... but that is not going to happen... However, I think we should not lose this enthusiasm!

It seems that Maira is interpreting the project expectations in a simplistic way: we presume to have excellent results with these tasks. Maybe she has assumed some disappointment in university researchers concerning the effort made in producing the materials for the classroom and having the students not so involved has they expected. It appears that she is also manifesting her frustration by what she says next: "We have to consider that we have many obstacles".

Later in the same meeting, Maira reflects on the nature of tasks that have been designed and the difficulties her students had in dealing with their openness. She considers that such situation makes her reflect on the lack of opportunities their students previously had for working with such kind of proposals:

> [Students say]: "I don't know how to do! What shall I do?". And they were a bit lost. And that is also due to the fact, maybe, that her teacher ... didn't work certain kind of situations that can be explored and not the traditional "Calculate ... Determine", isn't it? Hence, there is a lot to do, really. I consider that this make us [think].

In this meeting, Maira expresses her lack of engagement with what she considers to be one motivation in the project, as she interprets the expectation of the university researchers. However, the difficulties faced by her students in understanding the questions in the tasks made Maira question her own practice and therefore exhibiting also critical alignment with it.

Inquiry on the students' activity

In the subsequent meeting, when presenting the results of her teaching experiment, Maira shows a small video from one moment of the collective discussion in the class, to share with the group of teachers something she valued positively:

> ... I thought this is interesting because, if you notice, they are really discussing this issue. And this issue of being probable or not is not so obvious (...) Carlos seems to be surprised "How come you did not understand?!" (...) But to her [another student] it was still confusing. And I think that the example given by Bianca was excellent and it was also very good that the other girl questioned that it was not the same thing, because in her representation a big difference was visible.

The interactions among the students are highly valued by Maira, as she identifies that they are discussing important statistical ideas, especially because this kind of social practice never occurred before in this class. In her final report about this experiment, Maira argues that the students' interactions that took place had "a noteworthy impact on improving the students' argumentative discourse", as well, allowed her to have "a clearer understanding of the informal inferences [that] emerged in groups and nurture a discussion of statistical ideas".

However, Maira also questions her role in those moments, considering that she could have explored more deeply some statistical concepts:

> ... when he spoke about "more than 50%" ... I could have taken that situation but then I left it because, in the meanwhile, the other girl was trying to speak ... It might seem ease but that [discussion] was a bit confusing, and sometimes we cannot grasp all.

Another important remark concerning Maira and her colleague's reflection about the teaching experiments was their proposal of a new category to analyse students' statistical reasoning, because they felt they had interesting results which did not fit the framework it was provided by the university researchers (by Makar & Rubin, 2009). Maira explained that the category appeared as they observed that students made generalisations that rely on the context: "which are statements that they do that come from their personal experiences and beliefs". In the final report, she refers explicitly the activity of her class and identifies different difficulties and limitations concerning students' statistical reasoning.

DISCUSSION

In this teacher's case it is possible to see elements connected with belonging to an emergent inquiry community. Maira engaged with the project in noteworthy ways since she expressed her ideas in the meetings, made an effort to understand the theoretical ideas conveyed, and was highly involved in reasoning about task design. Imagination was also present in her participation since she started to conceive the possibility of developing new approaches with the class even if this represented a big challenge for her. In an initial moment she appeared to believe that students could accomplish the project's expectations regarding the work with the designed tasks but as the project unfolds many doubts arise. We observe that Maira experienced a tension between the adherence to the joint enterprise of the group, with critical alignment to her established practice, and the non-participation in the implementation of the tasks, assuming that those can only be applied in certain favourable circumstances.

Close to the end of the project, through her inquiry about the taught lessons with the tasks designed in the group, she came to express her critical alignment: there were difficulties and the results were not in line with what she had imagined according to the project's goal – developing students' statistical reasoning. From her point of view, the conditions for developing the lessons according to the adopted model were not adequate. It was her further inquiry on data from those lesson supported by some theoretical ideas that helped her to notice some positive results and to evaluate what happened in some instances of the teaching experiment in a different way. This made her question her attitude and role in the classroom, and the mathematical situations she usually proposes in the class. The critical alignment with her practice, presents "possibilities to develop and change normal states" (Jaworski, 2008, p. 314).

The analysis of Maira's participation in the project shows that some characteristics of an inquiry community were present in the practices of the group, being inquiry an important aspect to consider. The developmental research project created many opportunities for discussing ideas among the group, and stimulated the teachers to negotiate changes to practice. Therefore, inquiry emerges in the participation of this teacher as an attitude that informs her practice and that it is not reduced to a technique applied in some situations (Goodchild, 2014). The results concerning just one teacher reveal the project's potentiality for promoting teacher development through a cycle of collectively planning demanding tasks, experiment them in the classroom and reflecting collaboratively about the outcomes using one analytical framework.

However, when we analyse the tensions Maira experienced concerning her engagement with the project, it appears that her active involvement in the mutual process of negotiation of meaning about the joint enterprise and its purposes was absent in some instances. This requires further research to understand how this was experienced by the other participants, and the implications of this participation to their professional development.

ACKNOWLEDGENT

This work is supported by national funds through FCT – Fundação para a Ciência e Tecnologia, Project *Developing statistical literacy: Student learning and teacher education* (PTDC/CPE-CED/117933/2010). We thank Ana Isabel Mota, research assistant in the project, for her extensive work in data collection for this study.

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A model of theory-practice relations in mathematics teacher education

Kaj Østergaard

Roskilde University, Roskilde, Denmark
VIA University College, Horsens, Denmark, kost@via.dk

The paper presents and discusses an ATD based (Chevallard, 2012) model of theory-practice relations in mathematics teacher education. The notions of didactic transposition and praxeology are combined and concretized in order to form a comprehensive model for analysing the theory-practice problematique. It is illustrated how the model can be used both as a descriptive tool to analyse interactions between and interviews with student teachers and teachers and as a normative tool to design and redesign learning environments in teacher education, in this case a lesson study context.

Keywords: Anthropological theory of the didactic, teacher education, lesson study.

THE THEORY-PRACTIC PROBLEMATIQUE

Establishing coherence between theory and practice is one of the main challenges in mathematics teacher education (e.g., Bergsten, Grevholm, & Favilli, 2009). In Denmark more than four out of ten student teachers experience a lack of coherence between the teaching of general educational science and didactics taking place at the university college and the practice of teaching in schools (Jensen et al., 2008). Throughout the last decades teacher education has become increasingly academic – which can be seen as positive – but concurrently, the practices at schools have become much more challenging due to increasing social and ethnic segregation, which affect schools particularly in disadvantaged neighborhoods. Therefore, many student teachers tend to focus on practical teaching tools rather than academic theories. This development causes a risk of a widening of the gap between theory and practice in teacher education.

The theory-practice divide can be regarded from (1) theory to practice or (2) from practice to theory. Ad (1) the questions are: How can theoretical knowledge be utilized to analyze and develop teaching practice in schools and how do we create a shared frame of reference from teaching practice to interpret the theory? Subject matter knowledge, pedagogical knowledge and pedagogical content knowledge are taught separately at university colleges but are in reality inextricably entwined with each other. The challenge is how to create interplay between the academic theories of mathematics and pedagogy and teaching practice in teacher education. It is crucial to create this interplay in order to legitimize the theoretical education and to place school knowledge in a wider context.

Ad (2) the teaching practice must be made visible and handled as the main object of discussion and theorization in the teacher education. This is necessary in order to ensure that student teachers' learning in and from teaching practice is connected to the theoretical education and brings about a critical view on the theories and research from a practical point of view.

These complex theory-practice relations in teacher education calls for a model, which can be used to describe and analyze the interplay between mathematical and didactical knowledge; teaching practice and learning in both teacher education and mathematics teaching in school. In particular, it is important that such a model can help differentiating between the different kinds of theory-practice relations in teacher education.

The aim of the research project behind this paper is to answer the following two research questions:

1) What different kinds of theory-practice problems appear in mathematics teacher education – according to the student teachers?

2) How can these theory-practice problems be conceptualized and analyzed within a model based on *The Anthropological Theory of the Didactic* (ATD)?

In this paper, the focus is on how the model can be used as a tool for analyzing empirical data from a lesson study project with teachers and student teachers. However, at first, the model evolved will be presented and discussed. The paper is round off with discussion of the benefit of the model in analyzing theory-practice relations in mathematics teacher education and on how such analyses can inform the design and use of lesson studies in teacher education.

A MODEL OF MATHEMATICAL TEACHER EDUCATION

ATD (Chevallard, 2012) provides an epistemological framework for mathematical knowledge. In ATD mathematical knowledge, regarded as a human activity including teaching and learning mathematics, is modelled by mathematical and didactical *praxeologies* (Winsløw & Madsen, 2008). Praxeologies consist of a practice block (*praxis*) regarding the questions *what to do* and *how to do it* (know-how) and a theory block (*logos*) regarding *why to do it* (know why). In addition to this, ATD models the *didactical transposition* of mathematical knowledge from *scholar mathematics* mainly evolving at universities to *knowledge actually taught* and *learnt* in schools (Bosch & Gascón, 2006, p. 56). The didactical transposition is divided into two steps. The first step is the *external didactical transposition* from *scholar mathematics* to *knowledge meant to be taught* – the mathematical knowledge as it is described, for example, in curriculum and textbooks. This step is often performed by people outside the school. The second step is the *internal didactical transposition* from *knowledge meant to be taught* to *knowledge actually taught* – this step is every day work for teachers.

The two concepts, praxeology and didactic transposition, both bring central theory-practice relations into focus – the first one inside an institutional frame (e.g., the school) and the second in a broader context between institutions. Together they provide a comprehensive picture of teacher education in mathematics, which can be usable to point out and analyse problems and constraints as the theory-practice problematique.

In the model below the two concepts are combined to form a model for analyzing the theory-practice problematique in teacher education (Figure 1). In my research the model is intended to be a tool for both descriptive and normative analyses. At first, the model is used descriptively to analyze different kinds of empirical data from two lesson study projects in connection with teacher education. On this basis, the model will be used normative to propose new ways to organize teaching practice, preparatory education and the theoretical education at the university college to improve the coherence between theory and practice in teacher education.

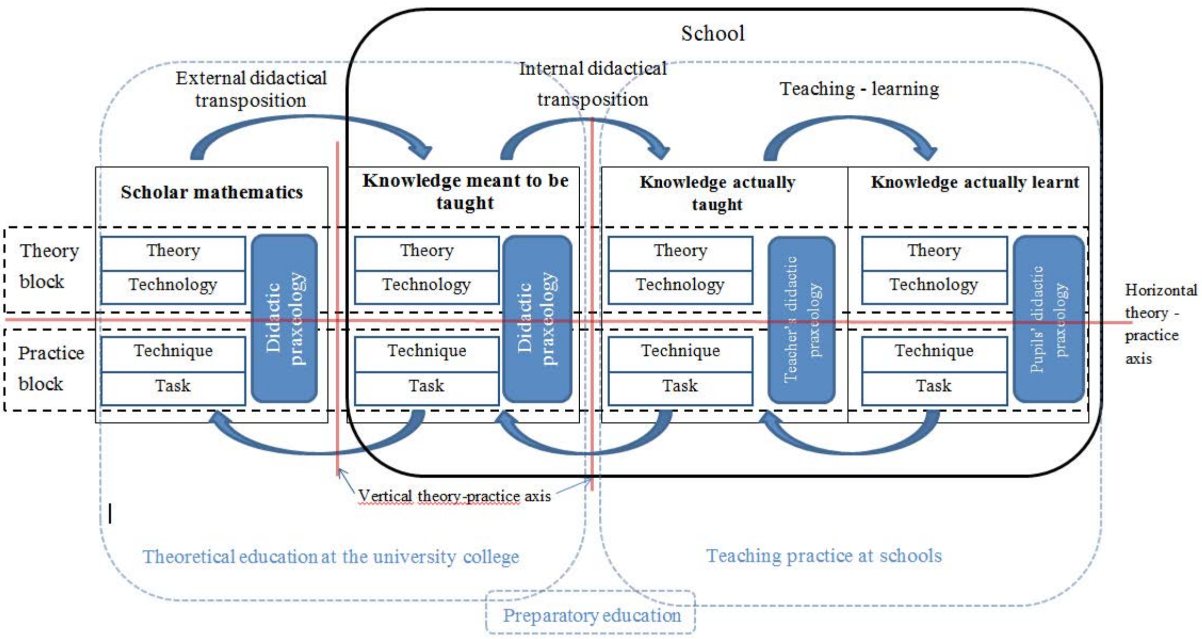

Figure 1: Teacher education model

The model consists of four columns containing the four kinds of knowledge in the didactic transposition. Each kind of knowledge is described by a mathematical praxeology with theory, technology, technique and task (see Winsløw & Madsen, 2008, for further details) depict with white boxes and a didactic praxeology, also with a theory and practice block, depict as blue boxes in Figure 1. By collocating the model and teacher education practice three different, pivotal theory-practice problems can be located – occurring in different forms. These are emphasized by red axes – two vertical and one horizontal axis.

The horizontal axis is dividing the practice blocks and the theory blocks. This axis stresses the divide between practical, procedural mathematics with emphasis on techniques to carry out tasks and theoretically *doing mathematics* by combining techniques and concepts, arguing, proving etc. The transcendence of this barrier is a crucial point for mathematical education – the higher level of abstraction in the theoretical block is a necessity but also a very difficult barrier to almost all pupils. Consequently, this axis is a significant problem area for teacher education both with regard to student teachers learning scholar mathematics and pupils learning mathematics at school and the relation between practice and theory block is an appropriate model in both cases.

The two vertical theory-practice axes are dividing, respectively, the scholar mathematics and knowledge meant to be taught and knowledge meant to be taught and knowledge actually taught. The divide in the first axis is treated at the university college. Comparison of scholar mathematics and knowledge meant to be taught is again highly relevant in teacher education to analyze what and why specific content is or is not selected for curriculum. It is pivotal for student teachers to be critical to this selection and to question the decisions in curriculum or textbooks. The arrows at the base of the model pointing "back", for example, from knowledge meant to be taught to scholar mathematics stresses that knowledge meant to be taught or actually learned can be taken as a starting point for analyzing the mathematical knowledge on the previous levels in the system. The latter of the vertical axes is dividing the theoretical education taking place at the university college and teaching practice at schools. To combine these two, university colleges often organize preparatory education as a special forum, depict as a small box in the bottom of the model.

The *internal didactical transposition* from knowledge meant to be taught to knowledge actually taught is everyday work for teachers and thus obvious content in mathematical teacher education.

The two columns to the right are a little different compared to the other kinds of knowledge. The relation between knowledge actually taught and knowledge actually learnt cannot offhand be described as a theory-practice problem because both are a part of the teaching practice at schools – the knowledge *actually* taught and learnt. Of course, teaching and learning can be described and analyzed by theoretical tools but the interplay at schools is a practice matter. As the transposition takes place inside school it is a part of the internal transposition but knowledge actually taught and knowledge actually learnt are closer connected and appears in a more direct interrelationship than the other kinds of knowledge. Student teachers are supposed to react to pupils' communication and learning, for example, during a dialogue in the classroom and adapt the teaching to the individual pupil or the specific class. Knowledge actually taught and knowledge actually learnt can be theoretically analyzed separately but are intertwined in practice. Therefore, the two kinds of knowledge are not separated in the model, but have a common borderline regarded as the interplay between the pupil's knowledge and the knowledge presented by the teacher in the form of the teaching environment presented.

THE LESSON STUDY PROJECT

The next section is an analysis of a group of two teachers and three student teachers' learning outcome from a lesson study project on the basis of the ATD-model. The lesson study was conducted in autumn 2013 in two classes grade 6 and 7 and the title was *"Similar – what does it mean?"* It was a part of a bigger lesson study project with the title *Trigonometry and inquiry based learning* involving 29 student teachers and 17 teachers conducted by a colleague and me. The empirical data from the lesson study consists of a lesson plan, video recordings of the two completions of the lesson, two 45 minutes interviews with one of the teachers and one of the student teachers and an article written by the student teachers. After the lesson study project I formulated an interview guide and accomplished the following data analysis on the basis of the ATD-model with a special focus on the three theory-practice axes.

Lesson study

Lesson Study is a Japanese form of action based development of teaching and teachers' and student teachers' teacher knowledge. In Japan, Lesson Study is an integrated part of both teacher education and teaching development in elementary school (Stiegler & Hiebert, 1999).

The ingenious and yet simple idea of Lesson Study is that the participants – student teachers and/or teachers – consider substantial didactical questions through mutual preparation, completion, analysis and reflections on one single lesson. Participants' observations and subsequent reflections on the pupils' learning and from this the didactical theme are crucial elements in the format. In the concrete project the student teachers studied trigonometry and inquiry based education at the University College before the project to be well-prepared to cooperate with the teachers. Together with the teachers, they prepared the lesson and formulate focus points for the observations to ensure that the observations are targeted at the didactical theme. A central element in Lesson study is the written *Lesson Plan* encompassing i.a. deliberations on mathematical, didactical and pedagogical aims of the lesson and hypothesis on pupils' strategies to solve the problems they are faced with. After a minimum of four hours of preparation one of the student teachers taught the lesson while the rest of the participants observed on the basis of the focus points. The lesson was evaluated immediately after the completion on the basis of the observers and the teacher's observations after a carefully worked-out plan. The evaluation resulted in suggestions to change the lesson and improvement of the lesson plan. Afterwards, the lesson was taught by a new student teacher in a new class immediately followed by an evaluation meeting where the second completion and the entire lesson study process was evaluated. The lesson plan was edited and the gained experience was described and discussed. At last, the student teachers wrote an article for *Matematik* – the Danish journal of mathematics teacher. It must be emphasized that the learning outcome from the lesson study not only – and not even mainly – relates to the lesson in question. On the contrary, lesson study is suitable to work with pedagogical and didactical problems on a generally level. The concrete and empirical basis opens up new possibilities to confront didactical and pedagogical principles with teaching practice at schools (for further details see, e.g., Lewis, 2002).

Data analysis: The lesson plan

The lesson plan is divided into three sections: First, some practical information concerning the participants, who taught the lessons, the name of the school, dates for completion of the lesson and the classes involved. The second part encompasses the tittle and aims of the lesson, competencies involved and working method. The last section is a detailed plan of the lesson containing mathematical focus point and learning goals of the lesson, a timetable, key question, teaching resources and useful tips for the teacher.

The lesson starts with a 10 minutes introduction to *geometric similarity* on the basis of every day examples of similar and not similar objects like a golf ball and a football, different sizes of Toblerone packaging (chocolate), enlarging/reducing in a photocopier and a pony and an Arab horse (not similar). After the introduction, the pupils receive a right-angled triangle cut of cardboard and the teacher asks the key question: *"You shall pretend that you are a photocopier and draw an enlarged and a reduced copy of the triangle".* This is the main mathematical task *t* of the *knowledge actually taught*. When the pupils have drawn the two triangles they must contact the teacher. The teacher then asks them two questions: *"How did you construct the triangles?"* and *"How can you convince me that the two triangles are similar to the one cut of cardboard?"* The two questions encompass the transcending of the horizontal theory-practice axis from the practice block to the theory block in knowledge meant to be taught. The teacher's didactic praxeology in connection to this mathematical praxeology is therefore a key issue of the lesson.

The crucial mathematical praxeology to be developed in the lesson study is based on the type of task T: *Given a right-angled triangle, how can you reduce/enlarge the size without changing the form?* A possible, predictable – and desirable – technique τ is to copy two or three angles from the cardboard triangle for instance by putting it on top of the paper and draw the angles and then reduce/enlarge the length of the sides. The technology θ to be realized by the pupils is firstly, that equiangular triangles are similar and secondary; the ratios between the lengths of equivalent sides are constant. Theory Θ – in this case the mathematical definition of similarity – is framing and justifying technology.

The lesson plan also points out some pivotal didactic praxeologies. One substantial didactical praxeology is concerning the inquiry based education mentioned in the category *mathematical working methods: "working in pairs – inquiry based education*" (IBE). IBE is concerned with the teaching-learning relation in the model – a theoretical idea about how pupils learn and from this how to teach. The example shows the delicate interplay between the pupil's and the teacher's didactic praxeology. The appertaining type of task in the teacher's didactic praxeology is how to set up a learning environment that makes the pupils investigate the mathematical task. The task is not explicitly mentioned in the lesson plan but two different techniques to solve the task appear in the following quotes: *"The pupils work inquiring with concrete materials and get the opportunity to reason on their own"* and *"Tips for the teacher: Be careful not to unveil the points".* So, the two main didactical techniques are to use concrete materials and to give the pupils opportunity to work out their own solutions (in pairs) without a standard procedure presented by the teacher.

The example shows how the model captures underlying mathematical and didactical considerations and the relations between these. In this case, the model is primarily used descriptively to analyse the lesson plan but it can as well be used normatively for instance to improve the design of the lesson plan template in the example about the ratios between the lengths of the sides by stressing the connections between mathematical and didactical praxeologies or type of task, technique and technology.

Video recordings of the lessons

The video recordings show that the student teachers to a great extent conduct the lesson as it is described in the lesson plan. They have experience with lesson study and know that this is important to focus the attention on the teaching instead of the teacher. During the section of the lesson where the pupils work with the problem in pairs they stick to the manuscript of the lesson, for example, *"Be careful not to unveil the points"*, and pose the planned question. For instance, in the following situation in grade 6:

Pupil 1: This one is double size
ST: How can you convince me, that it is the same triangle? Can you argue that they are similar?
Pupil 1: It has the same shape – and it has three sides
Pupil 2: And it is right-angled
ST: Yes. But so is this triangle (the teacher shows a triangle from another group). And your triangles are not similar to this one?
Pupil 1: No
ST: No, but they both have a right angle and three sides. Try to find out what the similar triangles have in common but these have not. Think about it... (The student teacher leaves)

The student teacher's first question is almost exactly quotation from the lesson plan. This question is difficult to answer to the pupils. Nevertheless, Pupil 1 refers to "same shape" as a colloquialism but unfortunately, the teacher do not respond to the suggestion and so the pupil do not get the opportunity to create a link to the mathematical concept – equal angles. This is the task of the didactic praxeology – to extend their understanding of the everyday word similar to a more exact mathematical interpretation. The example (and others alike) shows that the question does not encourage the pupils to investigate mathematical properties about the similar triangles and thereby get an opportunity to become acquainted with the theory block of the mathematical praxeology. The technique to solve the didactical task seems to fail. Maybe as a consequence of this, the student teacher improvises and reformulates the question: *"Try to find out what the similar triangles have in common but these have not."* This question is not mentioned in the lesson plan but it leads the pupils to examine mathematical properties because the question is posed in mathematics. An obvious answer to the question is that similar triangles have angles in common but ratios of the length of sides are not in the same way immediate obvious for pupils at this age. A new didactical task is therefore how the teacher can pose questions to lead the pupils to examine the ratios of the length of sides without "unveiling the point"? Analyzing the situation by means of the model could for example lead to a question like *"What will happen if you multiply the length of the three sides with the same number – 2 for example?"* The example shows that a problem concerning the didactic praxeology requires an analysis in details of the appertaining mathematical praxeology.

The video recordings show that the student teachers are very determined to follow the lesson plan as it is planned by the participants. The comprehensive preparation of the lesson and the very close connection to the theoretical education gives the student teachers an opportunity to try out their theoretical knowledge – both didactical and mathematical – in practice. Because they stick very carefully to the lesson plan there is a close connection between knowledge meant to be taught and knowledge actually taught and between the mathematical and the didactical praxeology – this is a crucial challenge in teacher education. Obviously, this challenge should be taken up in teaching practice but student teachers often find this very isolated from the theoretical education at the university college. In teaching practice the student teachers are "forced to act" – they have to teach a fixed number of lessons each week. Therefore, they often experience teaching practice as complex and stressful and fall into short-lived performance without coherence to their learning outcome from the theoretical education.

Interviews

Dialogue and working relationship between teachers and student teachers are – off course – important learning resources about school practice for student teachers. The interviews show that both teachers and student teachers experience significant differences between the dialogue involving teachers and student teachers in the lesson study compared to the usual teaching practice situation:

> Teacher (about teaching practice): Usual, when you have student teachers, it is vulnerable. Very often, you tell them what they did wrong or what they shall be aware of next time in the class instead of sticking to the point, the lesson, the content. (...)
>
> Teacher (about lesson study): Focus is on the lesson and not on the student teachers. We are not supposed to supervise them. We discuss what is working and what is not working about the lesson. We share a common responsibility to make the lesson work. We don't evaluate the student teachers but the lesson.
>
> Student teacher: In teaching practice, the teacher watches you when you teach, whereas in lesson study we are equal. We should all participate in the preparation of the lesson and we could all contribute to the lesson.

In teaching practice the student teacher usually prepare the teaching and teach a single lesson, while 1–3 of his or her fellow students and the teacher observe the lesson. Afterwards, the teacher supervises the student teacher in very close connection to the student teacher's presentations and interactions with the pupils in the lesson. The student teacher then tries to *"correct the mistakes"* before the next performance – some descriptions by student teachers and teachers indicates an inappropriate "trial and error" method. The strong focus on the student teacher's performance emphasizes the practice block of the teacher's didactic praxeology and – to a lesser extend – the teaching-learning relation in the ATD-model. According to both teachers and student teachers, it is unusual to discuss didactical and mathematical theory, curriculum and other topics in connection with the teaching on a general level – the two columns to the left in the model are almost absent in the dialogue. The interviews show that the teacher's didactical praxeology in knowledge actually taught is most often disconnected from both the appertaining mathematical praxeology and the mathematical and didactical praxeologies in the other columns. This is evident in many of the dialogs between teachers and student teachers in connection with teaching practice. Such practice off course implies a risk of widening the gap between student teacher's experience of theory and practice during their teacher education.

The interviews show two main differences between the dialogue between teachers and student teachers in connection with usual teaching practice and lesson study. Firstly, the dialogue in lesson study take place both before and after the teaching and especially the very long time spent preparing the lesson was emphasized as fruitful. The common preparing together with the lesson plan template makes the participants discuss and consider knowledge meant to be taught, the internal didactical transposition and the interplay between mathematical and didactical praxeologies. Secondly – as stressed by the teacher in the quote above – focus is on *teaching* and not the *teacher* in lesson study. This implicates for instance that knowledge actually learnt to a much higher degree is included in the dialogue in connection with lesson study than it is in the dialogue in connection with teaching practice and thus, the interplay between knowledge actually learnt and knowledge actually taught can be examined, discussed and related to knowledge meant to be taught.

There is a very clear consensus between teachers and student teachers that the dialogue in connection with lesson study to a much higher degree than the dialogue in connection with teaching practice includes a broader range of pivotal problems in teaching and learning mathematics. As a consequence, theory-practice axis in the model are treated and transcended more often.

CONCLUSION

The ATD model points out three different theory-practice problems in mathematics teacher education. It is crucial to put focus on all three axes and give student teachers opportunities to establish coherence between theory and practice in connection to the three axes.

Through different examples from a lesson study it is shown, that the model can be a fruitful tool to analyse teaching and learning contexts. Firstly, the model can be used as a descriptive tool to analyse and criticize planned teaching (lesson plan), actually completed teaching and the participants' experiences of the teaching with a special focus on theory-practice problems. Secondly, the model can be used as a normative, prescriptive tool for making adjustments to or changes in didactical designs for teacher education.

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Uncovering facets of interpreting in diagnostic strategies pre-service teachers use in one-on-one interviews with first-graders

Simone Reinhold

Leipzig University, Faculty of Education, Leipzig, Germany, simone.reinhold@uni-leipzig.de

The research presented in this paper focuses on prospective elementary teachers' proceeding in one-on-one diagnostic mathematics interviews. It goes beyond measuring the accuracy of teachers' judgments of students' achievements and analyses qualitative facets of diagnostic competence. Participants of mathematics methods courses were asked to prepare and conduct diagnostic interviews with children in grade one and reflect on their diagnostic proceeding afterwards. Findings of the research affiliated to this university teaching project lead to a model of strategic elements in PTs diagnostic proceeding and suggest types of diagnostic strategies. These may be realized or deliberately used to foster a sensitive qualitative diagnostic attitude.

Keywords: Prospective teachers, teacher education, analyses in one-on-one interviews, children in grade one, diagnostic strategies.

INTRODUCTION

Shulman (1986) or Ball and colleagues (2008) suggest distinct domains of teacher knowledge and point out *pedagogical content knowledge* (PCK) to be an integral element of teacher knowledge. PCK includes knowledge about *common* mathematical conceptions or misconceptions that are frequently encountered in the classroom. An interesting option to gain this kind of knowledge arises from teacher education settings where teachers examine *individual* cases: Analyzing a student's error to understand the underlying misconception refers to *knowledge of content and students* (KCS), which is regarded as a sub-domain of PCK by Ball and colleagues (2008, p. 403). Thus, the capability of "eliciting and interpreting individual students' thinking" can be found among the set of "high-leverage practices" (University of Michigan, 2015; Cummings Hlas & Hlas, 2012).

In this sense, identifying unique facets of the specific individual situation may contribute to a better understanding of widespread (mis)conceptions and provide an improvement of KCS (e.g., Peter-Koop & Wollring, 2001; Hunting, 1997). Dealing with individual cases may thereby foster the development of a teacher's diagnostic attitude and improve his or her teaching practices: If a teacher has detailed information on a student's individual mathematical concepts at his or her disposal, he or she gets the chance to design appropriate learning opportunities for this student. In this sense, diagnostic competence is an important element of adaptive teaching competence (Wang, 1992).

Recent studies concerning teachers' diagnostic competences mainly focus on measuring the accuracy of teachers' judgments (e.g., regarding a rank order within classes; cf. Südkamp et al., 2012). In these studies, diagnostic competence is "operationalized as the correlation between a teacher's predicted scores for his or her students and those students' actual scores" (Helmke & Schrader, 1987, p. 94). Individual mathematical learning processes which teachers try to capture during phases of concrete diagnostic activities are scarcely touched upon this understanding of the concept of diagnostic competence. But, focusing on high-leverage practices and on approaches of informal formative assessment (cf. Ginsburg, 2009), *how* do teachers arrive at a diagnosis of a student's conception via oral questioning or observation? As differences in accuracy might be due to teachers' different ways of diagnosing and analyzing, *how* do they get to an appropriate interpretation of a child`s utterances or how can they be helped to achieve appropriate diagnoses?

THEORETICAL FRAMEWORK

Diagnostic interviews in research, in the classroom and in teacher education

One-on-one diagnostic mathematics interviews stem back to the clinical method of interviewing developed by Jean Piaget. For educational research, one-on-one interviews provide a powerful method to gain insight into children's mathematical conceptions. Following a qualitative research paradigm, these conceptions can be interpreted from the children's utterances and activities they show while working on a problem. (cf. Hunting, 1997; Ginsburg, 2009).

To cope with the challenges of every-day classroom situations, teachers need a sensitive, constructivist views of their students' individual mathematical thinking and their progress in developing mathematical concepts. Thus, diagnostic interviews not only serve as a research method, but have also reached the classroom and may appear as little talks between teacher and student during a phase of individual working. In addition, research-based frameworks (e.g., concerning learning trajectories) resulted in the design of standardized task-based interviews to assess the range and depth of children's thinking in the context of mathematics learning in school. In these task-based interviews, in-service teachers actively explore facets of children's approaches to mathematics tasks. Prepared interview tools and empirically based growth points for the analysis guide teachers through these one-on-one diagnostic interviews and provide them with weighty arguments for their diagnoses. This may not only foster children's mathematical learning but also serve teachers' professional development (e.g., ENRP task-based assessment interview/CMIT/EMBI; cf. Clarke, 2013; Bobis et al., 2005; Peter-Koop et al., 2007).

High-quality programs for prospective teachers (PTs) engage them in concrete tasks which also include tasks of assessment or observation and focus on students' learning processes (e.g., Borko et al., 2010). Thus, studying students' mathematical conceptions in one-on-one interviews (which the PTs *themselves* prepare, conduct and analyze) offers substantial learning opportunities (cf. Prediger, 2010; Sleep & Boerst, 2012). Being involved in research projects that include interview assessments may also support the development of a sensitive diagnostic attitude (cf. Jungwirth et al., 2001; Peter-Koop & Wollring, 2001).

A process-oriented approach to diagnostic competence

Expertise in the area of diagnosing children's mathematical conceptions must not be restricted to teachers' accuracy in measuring children's achievements. It should additionally include rather vague aspects like diagnostic sensitivity, curiosity, an interest in children's emerging understanding and learning or the aptitude to gather and interpret relevant data in non-standardized settings (e.g., Prediger, 2010). Aiming at a framework to analyze processes and facets of diagnosing, it seems helpful to take a model into account which points out phases of the diagnostic process. In this sense, acting within a diagnostic situation in a one-on-one interview which intends to enlighten students' (mathematical) thinking can be regarded as an integral element of a circular process consisting of three dimensions, each including several components (Klug, 2011; Klug et al., 2013): Before trying to sum up information for a substantial diagnosis, it is crucial that the teacher sets the aim of the diagnosis in a preparatory *pre-actional phase*. This includes that the teacher should intentionally aim to watch the individual student's learning processes and therefore choose appropriate tasks and methods. The following *actional phase* includes data collection and data interpretation. Finally, the *post-actional phase* implies taking the necessary action from the data collection and interpretation in the actional phase (e.g., giving feedback, planning actions to foster). Activities in this phase also serve to prepare a repeated run through the diagnostic macro-process.

Cognitive elements in the micro-processes of the actional phase of diagnosing

Researchers in mathematics education have partially specified the challenges that in-service or prospective teachers face within such diagnostic macro-processes: Obviously, teachers actively do "construct knowledge by observation, experience, transfer and interrelation." (Bräuning & Nührenbörger, 2009, p. 945). Furthermore, there is a strong interest in the field of noticing and interpreting which can be analyzed when PTs face students' mathematical solutions. In this field, Ribeiro and colleagues (2013) investigate prospective teachers' *interpretative knowledge* which they regard to be part of SCK (*specialized content knowledge*, specified as sub-domain of content knowledge by Ball and colleagues (2008)). In these studies, the concept of interpretative knowledge is related to the ability of noticing and the authors point out that many PTs "find difficulties in interpreting children's solutions different from their own solution". Crespo (2000) and Kuhlemann (2013) offer similar results.

Focusing on *micro-processes within the actional phase in a one-on-one mathematics interview*; *collecting* data, *interpreting* and drawing *conclusions* have a major impact on the diagnosis which is derived from an interview and are likely based on different kinds of knowledge (e.g., KCS or SCK, see Figure 1). Here, proceeding in a one-on-one diagnostic interview is vitally influenced by cognitive processes and a person's (verbal) articulation (e.g., ways of questioning, confirming). Intentional decisions (e.g., switching between tasks) may reveal facets of the ongoing internal considerations.

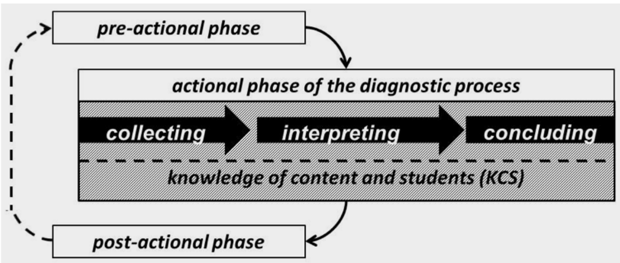

Figure 1: Differentiating the micro-process in the actional phase of diagnosing

Moyer and Milewicz (2002) identified general questioning categories (check-listing/instructing/probing and follow-up questions) used by PTs while collecting data in one-on-on diagnostic interviews. As there is no direct access to students' conceptions in these interviews, they "must be reconstructed by interpreting their utterances" (Prediger, 2010, p. 76) as "the interviewer attempts to construct a model of the student's mathematical knowledge" (Hunting, 1997, p. 149). Consequently, it is also important to reach a substantial perception of the diagnostic situation while interpreting. According to Barth and Henninger (2012, p. 51), this "includes the ability to structure the situation cognitively, the ability to change the focus of attention and the willingness and ability to adopt other perspectives" which leads to the generation and testing of hypothesis. Moreover, there is a demand "to know which information or knowledge sources play the most important role during the process of diagnosing students' learning prerequisites" (Barth & Henninger, 2012, p. 50). Yet, the implications of "gathering information, acting systematically" (Klug et al., 2013, p. 39) within the *actional phase* are not entirely clear for one-on-one interviews in mathematics education, so far.

RESEARCH QUESTIONS

The project *diagnose:pro* which is setting the frame for the study presented here emphasizes the need to sensitize prospective elementary mathematics teachers (PTs) to varieties, ranges and depth of young children's mathematical thinking. Therefore, graduate students (Master of Education) prepare, conduct and analyze one-on-one interviews about arithmetic problems with first-graders. These activities were part of a specific teacher education project at the University of Braunschweig (2011–2014) which is, due to space limitations, not presented in detail here. One part of the affiliated research project focuses on cognitive diagnostic strategies PTs use in their reflection and during the analysis of those interviews. To reach an empirically grounded theoretical framework for a qualitative view of PTs' cognitive activities in one-on-one interviews with children, the main purpose is to detect traits of these diagnostic strategies:

- What cognitive elements characterize the PTs' diagnostic strategies when diagnosing individual arithmetic approaches in one-on-one mathematics interviews with children at the beginning of grade one?
- Which types of (flexibly used) diagnostic strategies can be reconstructed from interviews they or others have been conducting?
- What kind of (pedagogical content) knowledge is included during the diagnostic proceeding?

METHODS

Making use of various approaches, data collection has been ongoing since 2011 and started with explorative studies via video-vignettes which led to written (mostly open) diagnostic comments of 31 PTs on diagnostic scenes. As analyzing these "diagnostic products" was not sufficient to answer the posed research questions, the following data-collection was shifted to video- and audiotaped peer-talks about mathematic diagnostic interviews: Here, students of two university courses (Master of Education, 28 participants in 2012) were asked to discuss about diagnostic scenes in video-vignettes. Finally, seven PTs (who had conducted a diagnostic mathematics interview with a first-grade child themselves) agreed to take part in retrospective interviews which complemented data collection in 2013. These interviews resembled methods used by Moyer and Mielewicz (2002). All PTs attended a mathematics methods course in the last year of their university studies which provided the opportunity to conduct individual diagnostic interviews with up to six first-graders per PT in cooperation with an elementary school. First drafts of these interviews were prepared at the beginning of the course where the PTs could refer to previous theoretical work on concepts of arithmetic learning trajectories and the method of task-based mathematics interviews (e.g., EMBI; Peter-Koop et al., 2007).

With only general advice at the beginning of the retrospective interviews, the PTs were asked to "analyze the interview" while watching the video-recording of an interview they had conducted. The PT was requested to stop the video at any scene in order to comment on the diagnosis he or she would derive from this specific situation or related observations. If comments were rather short or pure in detail, the PT was asked to explain what knowledge, information or evidence warranted his or her hypothesis. In addition to this concrete task (diagnosis of the child's conception or knowledge), the PT reflected on his or her proceeding in a more general way: Referring to the preliminary design of the interview, the PT was asked to comment on the choice of tasks selected, the wording of questions, on their own gestures or on deviations from the sketch. All re-interviews' analyses are based on Grounded Theory methodology and methods including open, axial and selective coding (cf. Corbin & Strauss, 1990). The interpretation, coding and contrasting comparison of the data are supported by ATLAS.ti which enables the research team to directly code video-data.

FIRST RESULTS

Analyses of the study's data support the notion that cognitive elements of PTs' approaches to diagnosis in one-on-one interviews often resemble basic processes in qualitative data analysis. This includes acts like *collecting*, *interpreting* and *concluding* within diagnostic micro-processes (see Figure 1). Furthermore, the findings contribute to the identification of sub-categories of collecting, interpreting or concluding and to interrelations among these sub-categories (see Figure 3). Excerpts from re-interviews with Ann and Sue, Master students in their last year of studies, exemplify facets of *interpreting* within the diagnostic micro-process of the actional phase.

Facets of interpreting: Comparing and contrasting

In her interview with six-year old Tom, Ann offers empty boxes for ten eggs and some chestnuts. The boxes of ten are partitioned in four fields (see Figure 2) since Ann intends to find out how children use these structures for counting. She assumes the children to use abbreviated enumeration, i.e, counting strategies including subitizing parts of an amount (cf. Besuden, 2003). Ann stops the video and comments on a scene where she has just put five chestnuts into the box (forming a row). Tom is asked to add further chestnuts in order to get a result of eight and fills two, then one more into the box. Answering Ann, he remarks "Because I left two free, one more'd be nine, then ten."

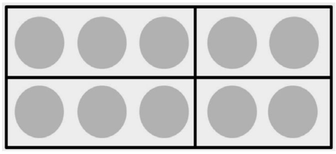

Figure 2: Structured box

Ann (07:08): And there I noticed that he, eh, always took ten as a starting point for the higher numbers, well, for eight and a moment ago for nine. He remembers, okay there are ten in the package, and then he always counts backwards.

In her comment, Ann compares and refers to Tom's previous work ("a moment ago"). *Comparing* details to a child's previous utterances or actions, to that of others or to the PTs own concept may also occur in terms of *contrasting* different scenarios:

Ann (08:30): Here, he saw, okay, there are four in one box and there are another four in the second box, well, four plus four equals eight, but he didn't do it that way in the next task. There he'd count single ones, it was done quite differently.

Facets of interpreting in a diagnostic micro-process: Coding

Sue uses the same kind of tasks in her interview with six-year old Ben. She wants him to find out how many chestnuts have to be added to four chestnuts (which are presented in the "square" on the right side of the box) to get a result of seven. Ben replies by first adding two (forming a "rectangle"), then one more to reach seven (Ben: "These are six, then seven."). Sue codes these activities by creating and applying the new term "auxiliary calculation":

> Sue (05:40): Responding to my enquiry, how he'd done this, now, how many he'd add, actually, I only wanted to hear three, well, he would seize on his, let's say "auxiliary calculation", six plus one equals seven.

PTs are similarly coding observed phenomena as they try to grasp unfamiliar, but obviously central aspects. Codes are often referred to later in the interviews (e.g., Sue's reference to the code "auxiliary calculation", 22:30) and also include substitutions for established terms (e.g., "shortcut" instead of "subitizing").

Facets of interpreting in a diagnostic micro-process: References to knowledge of content and students (KCS)

To describe the children's performances in the re-interview, PTs also try to make use of standardized terms that refer to previously acquired KCS and seize on theoretical concepts that were studied in the methods course before conducting the interviews:

> Sue (04:50): Well, at the beginning, Ben definitely used counting strategies. He saw those four and went on counting from that summand. He noticed, okay, if I add two then I'll get six, thus, he didn't go like "five...six", but he said, okay, two, that's six.

Although details of the counting strategy "counting on by steps of two" are not reflected here, referring to mathematical KCS tends to be an important element of PTs' diagnostic strategies: PTs do use information from their teacher preparation courses. They retain general knowledge of children's development of mathematical conceptions (e.g., "understanding of quantities"), but then remain unfocused in supporting their interpretation with this knowledge:

> Ann (15:17): But, Tom doesn't have, eh, a complete understanding of quantities at his disposal, partly he did, partly he didn't. It's when a child notices that a number is now, eh, bigger than the number before, or that one can draw conclusions from one equation to the next, that is connected to the first one.

Types of diagnostic strategies

Following Grounded Theory methodology, distinct types of diagnostic strategies with a stress on different elements of diagnostic proceeding (i.e., on the exemplified (sub-) categories) are detected. As indicated by the arrows in Figure 3, PTs' diagnostic strategies are far from a linear process and may be driven by general dimensions of diagnostic strategies (e.g., topographic or symptomatic search; Cegara & Hoc, 2006).

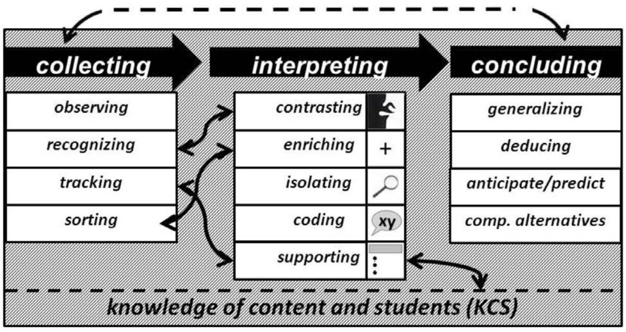

Figure 3: Sub-categories of collecting, interpreting and concluding

Following the strategy *descriptive collector*, the PT focuses on collecting and describing the actions, neglects both interpreting and concluding, and searches rather typographically. A *concluding collector* strategy is characterized by skipping elements of interpretation as collecting leads directly to conclusions which resembles findings of Crespo (2000). Symptomatic searches occur when elements of interpreting prevail in a *branched interpretation*. Here, interpreting, collecting and concluding are intertwined and frequently linked to KCS.

DISCUSSION

The findings of the study provide evidence of sub-categories of collecting, interpreting and concluding within micro-processes of the actional phase of diagnosing. They point at KCS within these processes and hint at a variety of strategy types. Thus, results enrich, for example, the idea of "interpreting" in the actional phase of diagnosing suggested by Prediger (2010) or Barth and Henninger (2012).

Bearing in mind that the findings are restricted to a particular type of tasks (arithmetic issues) and that they refer to a rather small number of participants (n=28 in peer-talks; n=7 individual interviews), the study outlines new topics in the field of teachers' professional development: It raises the hypothesis that reflecting on facets of interpreting in one-on-one interviews enhances PTs diagnostic sensitivity. This may increase their knowledge of assessing children's mathematical abilities and contribute to the consideration and implementation of "high-leverage practices": An awareness of "strategic diagnostic tools" might help to master diagnostic challenges in the classroom. Thus, further activities of the project *diagnose:pro* will explore how the findings (elements of diagnostic strategies/types of strategies) can be taken up in university courses and contribute to appropriate diagnoses of children's concepts in one-on-one interviews. Further steps also include using the developed model to qualitatively evaluate changes of PTs' interpretations over the duration of university courses and to analyze what leads to changes in PTs' diagnostic strategies.

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Teacher empowerment and Socioepistemology: An alternative for the professional development of teachers

Daniela Reyes-Gasperini, Ricardo Cantoral and Gisela Montiel

Cinvestav, Mathematics Education Department, Ciudad de México, Mexico, dreyes@cinvestav.mx

Teacher empowerment is an alternative proposal from Socioepistemology that postulates it as a tool for the professional development of teachers. The concept of empowerment is accompanied by the "problematization of knowledge" in both senses: mathematical knowledge and school mathematical knowledge. We assume that teachers will be better able to transform their educational reality, since they will have taken possession of the teaching knowledge. This new relationship to knowledge is not based more on mnemonics, but on what we consider to be the essence and "raison d'être" of knowledge that will allow the teacher to develop various strategies considering his group of students. In this paper, we will discuss "proportionality" for its high cultural value and its transversality in education.

Keywords: Socioepistemological Theory, teacher empowerment, problematization of knowledge, proportionality.

POSITIONING THEORY

While "the best-selling question" of the 90's was *how* to teach using various teaching strategies in order to make the understanding of certain mathematical knowledge more accessible to students at different educational levels, the Socioepistemological Theory posed a somewhat different question: *What* is it that we are teaching? *What* is it that our students are learning? That is to say, lets study and discuss the *nature of mathematical knowledge*, and thence, "reflect on" the *school mathematical knowledge*. Studying its nature does not implies just making an epistemological study, but getting a systemic perspective of the epistemological, didactical, social and cognitive dimensions of mathematical knowledge, it means, looking at them as a whole.

In terms of teaching practice, while the classical currents analyzed the tasks that teachers use in the classroom, the teacher-student interactions, the competition brought into play to solve math problems, the teacher's knowledge on how students think, know or learn a specific mathematical content, among many others, Socioepistemology wondered what and how is the professor's relationship to knowledge in a specified didactic relationship? It is for this reason that our line of research considers a necessary articulation between two theoretical elements: the functionality of the mathematical knowledge of proportionality (transversal notion in the educational system with high practical value in everyday life) and the theoretical construct of teacher empowerment (Reyes-Gasperini & Cantoral, 2014). On the basis of such an articulation we wove a conceptual framework in order to show that the *teacher empowerment, from a socioepistomological vision,* is a little known alternative to study the professional development of teachers *problematizing* mathematical knowledge.

Socioespistemological theory studies the social construction of mathematical knowledge. The education problem is not that of the constitution of abstract objects, but *their shared significance by its culturally situated use*. It is assumed that since this knowledge is socially constituted in non-school settings, its diffusion to and from the educational system forces it to a number of changes that directly affect their structure and functioning, so that also affects relationships established between students and their teacher. The socioepistemological research promotes a *decentration of the object*, that is, to pay attention to the *practices* from which it emerges and not just on the mathematical object per se. Socioepistemology delimits the role of historical, cultural and institutional setting in human activity, so the problem that motivates the research can be student's difficulties in learning a particular concept; however, studying it seeks to contribute to an alternative vision that includes the associated social practices and, to that extent, provide a social and cultural look of mathematical knowledge (Cantoral, 2013, Cantoral, Reyes-Gasperini, & Montiel, 2014).

SOME LINKS TO TEACHER EMPOWERMENT WITH MATHEMATICAL KNOWLEDGE

Empowerment is a *social phenomenon* typically studied in various disciplines and approaches, from social (Martínez Guzmán Dreyer & Silva, 2007), feminist (Camacho, 2003), from the Psychology Community (Montero, 2006), or from an educational point of view (Howe & Stubbs, 1998, 2003; Stolk, de Jong, Pilot, & Bulte, 2011). While each of the disciplines has a particular focus on the phenomenon, they all concur in their main characteristics that we have synthesized in the following way: empowerment is understood as a process of the individual in collective work (interaction is required in collective work), which parts from the reflection to be consolidated in action, which is produced by the individual without the possibility of being granted (collaborative work will be necessary but not sufficient to promote empowerment) and, above all things, *transforms the reality of the individual and his context.*

In particular the projects that aim to promote teacher empowerment (Howe & Stubbs, 1998, 2003; Stolk, de Jong, Bulte, & Pilot, 2011) provide teachers with tools to design new situations emphasizing contextualization, either by knowledge of new research related to the topic, as well as by the sample of situations that provide a context to what they already know. All with the aim of obtaining an attitude of leadership, confidence and improvement in their practices towards education, emphasizing the fact that they may acquire the power to take the reins of their own growth. While we may coincide with the results that are expected to be achieved, we believe that this type of analysis is reduced to only a pedagogical interpretation.

Our proposal, given the socioepistemological character that is added to this phenomenon, incorporates the notions of *problematization of mathematical knowledge (PMK) and problematization of school mathematical knowledge (PSMK)* keys to boost teacher empowerment. The action of *problematizing* the school mathematical knowledge is done with the knowledge that teachers use in the educational system. Now, why do we differentiate *PMK* from *PSMK*? The *PMK* refers to the fact of "making a problem out of knowledge", an object of training analysis, locating and analyzing its use and its *raison d'être*, namely refers to the study of the nature of said mathematical knowledge, for example related to the proportionality, on the basis of questions like: What problem did the *notion of proportion* come to resolve that could not be resolved without them? Are the problems more difficult when the magnitudes are heterogeneous than when they are homogeneous? Why are problems on the fourth missing value worked on if the comparison is not represented there? Where do the proportions appear in the civilization? What characterizes the relation of proportionality? Among many others. The socioepistemological study based on teaching, epistemological, social and cognitive dimensions of knowledge can make up a *unit of socioepistemic analysis* (*USeA, UASE in spanish*) that causes a singular symbiosis between, and from, the four dimensions, in order to generate a theoretical framework that challenges mathematical knowledge, and subsequently, school mathematical knowledge.

In contrast, when we work with the *PSMK*, we draw on the knowledge that is fundamental to the educational system. Based on the *USeA* an activity guide is designed which confronts the typical educational activities in order to put the teacher in a learning situation and thus generate spaces for the *PSMK* to be performed (Reyes-Gasperini, Cantoral, & Montiel, 2014). We understand the *PSMK* as the action part of the introspection, the gaze of the learner and uses available in their everyday life.

It is necessary to mention at this point that the socioepistemological theory rests on four fundamental principles (Cantoral, 2013): the regulatory principle of *social practice*, the principle of *contextualized rationality*, the principle of *epistemological relativism* and the principle of *progressive resignification* or *appropriation*. These four principles underlie the *PSMK*, well this problematization will allow the teacher to consider that social practices are the foundation of the construction of knowledge (regulation of social practices), and that the context will determine the type of rationality with which an individual or group – as a member of a culture – builds knowledge whilst he/she can express it and put it to use (rationality contextualized). Once this knowledge is put to use, that is to say, it is consolidated as knowledge, its validity will be relative to the individual or the group, as it emerged from their construction and their arguments, which gives that knowledge *epistemological relativism* Thus, because of the evolution of the life of the individual or group and its interaction with various contexts, such enriching knowledge of new meanings will be redefined constructed to this moment (progressive redefinition).

Therefore, the links between empowerment and mathematical knowledge are given by the articulation of the typically social phenomenon with the socioepistemologic character that underlies its main action: the *PSMK*.

PROBLEMATIZATION OF MATHEMATICAL KNOWLEDGE (PMK) AND PROBLEMATIZATION OF SCHOOL MATHEMATICAL KNOWLEDGE (PSMK): THE CASE OF PROPORTIONALITY

In our current research, we take the proportionality, as a mathematical notion, to work the *PMK* and the subsequent *PSMK* with teachers. In order to give some guidelines for what leads to the construction of the USeA, we part from an idea rooted in the educational system and society in general, however, we should make it plain that here there will only be presented some examples of what the USeA is in its entirety. If we asked in a generic way, "What is proportionality?" most would answer that it is a form of variation: "When *more is more* or *less is less*", or, "when using the rule of three". In these responses, we find two aspects to consider, first, the memory of a "recipe" to identify a problem of proportionality; on the other, an algorithm with which to solve said problem. Some typical examples given in this respect are: the price in buying tortillas in kilos, the distance traveled in a certain time, among others; but they all have the same prototype. The use of colloquial language allows the fluidity of a set mathematical thought, but will inevitably redefined subsequently, for example, at a formal level, it should be reflected in written form at a level of symbolic object and consider "the notion of constants proportionality" as the product or the reason for the magnitudes. Piaget's theory (1958, quoted in Noelting, 1980) considers proportionality as the hallmark in the development of formal operations, therefore, we ask, have students (or teachers in our case), developed this type of reasoning? The idea of spending an additive relationship to a multiplicative relationship seems to be the fundamental idea that has been pursued in studies concerning proportionality.

In problems of proportionality, usually, it is asked, "How many hours it take to travel 25 miles?", for us it is important to reflect on the notion of speed, considering it as relation between distance and time, rather than solely the notion of "missing value". Perhaps this is an obstacle that so far may not have been considered as an issue that cuts across the colloquial and has to do with a germinal idea of what the notion of reason is. To do this we ask, what is the nature of proportionality? Is it a continuous or a discrete nature?

Given this situation, we question whether it is possible that the question "how many" can generate a centered answer among students in the quantification rather than the relationship. Well, if we asked what is or what are the relationships that can be established between the magnitudes, perhaps we would be giving the students a relationship beyond the concentration in number and quantity to think about. That is, if from a psychologist point of view the research reports that proportional reasoning is related to the development of formal operations and is complex, if not impossible, to achieve such reasoning, it would seem that we should think that the way that said knowledge is addressed is alien to the reality of the student and teacher in our case.

We conducted a formal analysis of the theory of proportions addressed by Elements of Euclid in his Book V, we work on par with the idea of incommensurability. Hence we affirm: if there is no such thing as a common measure, how can these quantities be measured? The problem of measuring, was replaced by Euclid as the problem of comparing. This is the fundamental question that gave rise to the theory of proportions between magnitudes. So, is the condition caused by the inability to measure what has led to the need to compare? Just as the inability to advance time which has led us to predict (Cantoral, 2013).

In this respect it is stated that this theory emerges to address two specific problems of the time. On the one hand, before the conflict that had suffered the Pythagorean theory regarding the impossibility of assigning a number to the ratio of two quantities, theory of proportion was redesigned in such a way that "you could talk about reasons and proportions, without specifying whether or not they were considered commensurable magnitudes" (Guacaneme, 2012, p. 104), where the greatest merit of the theory described in Book V is the possibility of comparing incommensurable magnitudes (Corry, 1994, quoted in Guacaneme, 2012). On the other hand, the elements are intended to present the mathematical theories under a deductive axiomatic scheme. Now, if the theory of proportionality arises as the possibility of comparing incommensurable magnitudes, it is logical to think that if most of the problems encountered in the literature that are to do with the missing fourth, these kind of problems does not always require a proportionate reasoning (Lamon, 1999), for there will be nothing to compare because the amounts are given and you have to operate on them arithmetically, applying the rule of three most of the time. In addition, they can announce themselves with the structure of the missing fourth, without there being a proportional relationship between the magnitudes, however, the students will resolve it since the characteristic they believe to be enough to apply the simple rule of three is that both magnitudes increase (while the issue is to characterize the type of growth). So far, we conjecture that it is necessary to return to emphasize the relationship between the magnitudes by *comparing* them.

During the process of research on proportionality, we have studied the famous "Cauchy functional" that gave light to analyze the difference between the additive and multiplicative thinking in depth, creating a new look at the nature of the proportionality. There are four functional Cauchy equations (Roa, 2010), in a later study we related each of the four functional equations at school-level with four functions that are of significant importance: Exponential function: $f(x + y) = f(x) \cdot f(y)$; $x, y \geq 0$; Logarithmic function: $f(x \cdot y) = f(x) + f(y)$; $x, y \geq 0$; Power function: $f(x \cdot y) = f(x) \cdot f(y)$; $x, y \geq 0$; Proportional function: $f(x + y) = f(x) + f(y)$; $x, y \geq 0$.

The Cauchy's functional served as sustenance to give evidence, in an analytical and graphical way, for the differences between a proportional linear function and a non-proportional linear function, since that function is only true in the first case. This allows us to analyse the proportional function from a particular property and not just from the classical ownership "double receives double" or from "the rule of three".

In our case, we study in depth the related with the proportional function, which, based on the USeA of Reyes-Gasperini (2011), it was stated with the models of proportional thinking reported in the literature. Let's start now by thought patterns, there is a qualitative thought that is the first to appear in individuals (Inhelder & Piaget, 1972) and is exemplified by the idea of the chorus "more is more; less is less".Godino and Batanero (2002) conducted a study based on (Noelting, 1980) where they report the following types of reasoning used by students to decide between two jugs of juice which is the one with the "stronger" taste. Their arguments are based on the comparison of the number of glasses of water and juice placed. The question posed is: "My mother has prepared two jugs of lemonade. In jug A she has mixed two glasses of water and one glass of lemon juice. In jug B she has mixed three glasses of water and one glass of lemon juice. In which of the two jugs is the lemon flavor is more intense?"

Even if the amounts are in play, the answer is not a quantity, but a relationship between them: which is more intense? As the authors say "the additive strategy would be to compare the difference between the glasses of water and the lemon juice in each jar" (Godino & Batanero, 2002, p. 439), but they ensure that this strategy will not be sufficient to address problems of greater complexity. Regarding the above, Carretero

$$¿f(x) + f(y) = f(x+y)?$$

$$\begin{gathered} f(x) = mx + b, b \neq 0; f(y) = my + b, b \neq 0 \\ (mx + b) + (my + b) = m(x + y) + b \\ mx + my + b + b = m(x + y) + b \\ m(x + y) + 2b \neq m(x + y) + b \\ \text{So, } f(x) + f(y) \neq f(x + y) \end{gathered}$$

$$\begin{gathered} f(x) = mx; f(y) = my \\ mx + my = m(x + y) \\ m(x + y) = m(x + y) \\ \text{So, } f(x) + f(y) = f(x + y) \end{gathered}$$

Figure 1

(1989) distinguishes two types of structures. On the one hand, those having a given relationship between homogeneous magnitudes (also called extensive by other authors) to those called scalar multiplicative models; and on the other hand, those having a relationship between heterogeneous magnitudes (also called intensive), to those called functional multiplicative models. Subsequently Lamon (1993) also makes a distinction as strategies for students to find the missing value in a proportion. He calls them inter models (corresponding to the multiplicative model scalar) and intra models (corresponding to the functional multiplicative model). The work with the different types of multiplicative structures around the acquisition of the notion of proportionality allowed Carretero to conclude that "the division is evidently a more difficult operation than multiplication, despite the underlying multiplicative structure" (Carter, 1989, p. 95). Thus, we conclude that the additive model precedes the scalar multiplicative model, which is less complex than the functional multiplicative model, however, they are all thoughts that underlie the idea of proportionality.

Moreover, G. Vergnaud works on the theory of conceptual fields considering them a set of situations that can be "analyzed as a combination of tasks of which are important to know their own nature and difficulty" (Vergnaud, 1990, p. 140). Regarding proportionality, he compares the conceptual fields of additive structures (those that require an addition, subtraction, or a combination of the two) and the multiplicative structures (those that require multiplication, division or a combination of the two). This allows him to generate a classification and an analysis of cognitive tasks and in procedures that are potentially at stake in each. This allows her to generate a classification and an analysis of cognitive tasks and in procedures that are potentially at stake in each.

He concludes by stating that "it is not superfluous, on the contrary, to emphasize that the analysis of the multiplicative structures is profoundly different from the additive structures." (Vergnaud, 1990, p. 144). This is to say, we can ensure that there will be tasks that demand a multiplicative structure, and others, an additive structure.

Therefore, not all problems deserve to postulate a proportional reasoning in terms of Lamon (1993), but as Vergnaud (1990) says, there are problems that can be solved by additive structures or pre-proportional reasoning, for example: "If one coconut costs 35, how much do 10 coconuts cost?".

This example is tackled by Carraher, Carraher and Schiemann (1991), where they see how a child solves "on the street a sales situation": <u>Client:</u> How much does one coconut cost?; <u>M</u>: Thirty-five; <u>C</u>: I want ten coconuts; How much is it for ten coconuts ?; <u>M</u>: (Pause) Three are 105 plus three is 210. (Pause) we are four short. It is ... (pause) ... it seems to be 350.

An immediate question, at this level of analysis is: Has the child developed proportional thinking? Our answer is yes, because the situation does not require a multiplicative structure, but reaches an additive structure (additive model composed seen above), and behold, our assumptions about the mathematical knowledge of proportionality: proportional reasoning viewed as the relationship between two magnitudes that remain constant, should be assessed whether developed or not, whenever the situation warrants a comparison, that is, an analysis of relationship type between the magnitudes, and not the discovery of a missing value. Hence the need to draw up a learning situation that involves a sequence of activities where different thoughts are progressively and systematically put into play.

Proportionality arises to address the inability to measure incommensurable magnitudes; therefore, as it has already been shown as evident, the current school significance induces us to look at a math problem with a different rationality for offering its epistemological nature. This may explain the academic failure of students to proportionality.

Given the fundamental idea that the inability to measure generates the need to compare, let's see what happens with a purely mathematical problem to which the scales for working have been removed with the idea of the type of relationship between the magnitudes more than in the quantification of the values.

Activity: To the right of the graph of the function f is presented. Does this represent a direct or inverse proportional function? Justify your answer.

Most teachers with whom we have worked on this activity (both secondary school teachers and students) argue that it is inversely proportional because "a plus x, minus y" (qualitative thought). This was the trigger

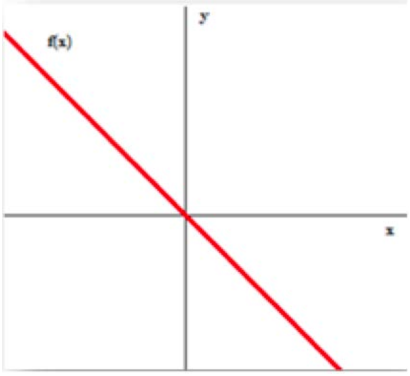

Figure 2

to replant the way we work proportionality in the education field, because it generates germinal mathematical errors supported by "rote simple recipes".

The work with teachers leads us to characterize "what it is proportional" (the proportional, as we use to say in our language of practices) as a relationship between two magnitudes whose ratio remains constant. First, is analyzed the constant rate of change, that characterizes all linear relationship () and then analyzes the constant ratio, that is maintained between its variables, which characterize directly proportional linear relations (). Thus, was worked on the relationship between the magnitudes as from different properties of proportional relationships. So rote recipes make sense and meaning.

Figure 3

To illustrate, we will show an activity we did with teachers, in order to address the idea that not every relationship which have the simultaneous increase or decrease characterizes a proportional relationship:

"Considers that the first figure is the original. Which of them could be considered an extension or reduction of it?"

After the teachers' discussions, where the notion of scale was at stake, we address them to reflect that it is not enough to consider the presence of an increase, we must emphasize the way this increase is done.

DEBATE AND FINAL CONDITIONS

The mathematical treatment of a transversal mathematical subject in all of mathematics education shows that you not only need to work with teachers on pedagogical issues of general teaching process, or only the contents as they are addressed in school. But to this type of study we add the need to problematize mathematical knowledge to then work with teachers posing questions of school mathematical knowledge and thus contribute to the professional development of teachers through the *change of relation to mathematical knowledge*, and not solely based in mnemonic rules or formulas with little meaning, but based on what we call "the reason for this mathematical knowledge and its frameworks that allow their use."

What we propose to be discussed in in the Group of Mathematics Teacher Education and Professional Development is how to generate, within the professional development of the teacher, areas in which the knowledge of the teacher is not classified, but through what the teacher has in its repertoire (background), deepen and challenge the school mathematical knowledge and change accordingly their relationship to knowledge. Thus we assume that teachers will be better able to transform their educational reality, since they will have taken possession of knowledge that teaches. This new way to relationship to knowledge no longer based on mnemonics, but on what we consider the essence of its purpose and allow the teacher to develop various strategies by the group of students he/she may work with. In short, we are studying the process of teacher empowerment, which we postulate as a tool that contributes to teacher development.

The line of research on teacher empowerment provided by socioepistemology brings a fresh, different look on dominant versions in the literature of professional development of the teacher in the field of school mathematics.

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Instructional coherence as perceived by prospective mathematics teachers: A case study in Chilean universities

Francisco Rojas and Eugenio Chandía

Pontificia Universidad Católica de Chile, Faculty of Education, Santiago, Chile, frojass@uc.cl

One of the notions that future Chilean teachers hold about their educators is the lack of coherence between the latter's instructional practices and the ways in which they are expected to teach mathematics in school. Upon this basis, we sought to characterize the Instructional Coherence of teacher educators, and particularly the way in which it is perceived by students. By applying a questionnaire to prospective teachers from two Chilean universities, focused on their teaching models, the relevance of replicating certain instructional practices in the school classroom, and the types of modelling observed in their educators, we were able to establish that coherence is perceived when prospective teachers notice, in the practices of their teacher educators, the characteristics of the teaching model that they themselves profess.

Keywords: Mathematics teacher educators, instructional coherence, modelling, pre-service teacher education.

CONTEXT

In Chile, international evaluations of both Chilean educational policy (OECD, 2009) and mathematics teacher quality (TEDS-M; Ávalos & Matus, 2010) have shown that the pedagogical and disciplinary education of the country's newly graduated and practicing teachers is not good enough to result in good performance. Even though major efforts have been made to improve Pre-Service Teacher Education, little light has been shed on the education processes of teachers in university classrooms (the so-called "black box"). In addition to this, and despite the relevance of teacher educators in this process, fewer studies have been conducted in Chile about these participants than about prospective teachers (Cisternas, 2011). Conducting more in-depth research on this issue could help to understand a factor which we consider essential in pre-service teacher education: the instructional practices of teacher educators and the perception of prospective teachers about the coherence between their educators' discourse and such practices.

THEORETICAL FRAMEWORK

Rojas and Deulofeu (2013) have observed that the teacher educator's instructional practices (mathematical-didactic activities designed and its classrooms management) are strongly related with the construction of the teaching models of prospective teachers. This educational process should include at least two aspects to be constructed: on the one hand, that the body of teacher educators offer future teachers opportunities to learn mathematics in the way their students are expected to learn (didactic model transference) (Chapman, 2008; Deulofeu, Figueiras, & Pujol, 2011), thus generating processes that model teaching practices; on the other hand, that the teacher educator introduce activities which constitute opportunities to learn to teach mathematics, in the sense of planning one's teaching, analyzing classroom management through classroom episodes, and working upon the basis of the mathematical production of secondary school students, which should establish a strong theoretical-practical relationship (Boyd et al., 2009; Gellert, 2005). The general purpose of these activities, in terms of design and implementation, is to allow prospective teachers to construct the knowledge necessary to teach high school mathematics. A major part of these activities depends on how the teacher educator manages them, that is, how he/she uses them to display their underlying didactic-mathematical approaches (Zaslavsky, 2007).

Instructional coherence

However, it is not enough to study the teacher educator's practices. If we consider Chilean students' demands for greater coherence in their educators (MINEDUC, 2005), it is necessary to advance a notion of coherence that is functional within the educator's job. In the literature, coherence is defined as the degree to which the main goals associated to teaching and learning are shared by everyone involved in the education of teachers, and also considers the degree to which learning opportunities are organized, both conceptually and logistically, to achieve these goals (Tatto, 1996). Within this concept, two types of coherence can be identified: conceptual (between the professional perspectives of those who work with teachers) and structural (associated with the design of learning opportunities) (Hammerness, 2006). Beyond capturing the notion of consistency, these definitions stress the idea that coherence requires alignment between ideas and learning opportunities (Grossman, Hammerness, McDonald, & Ronfeldt, 2008). However, none of these conceptions emphasizes the instructional coherence of the teacher educator, understood as the degree of alignment between his/her instructional practice and the *didactic transfer models* that he/she promotes in it, including the theoretical models that support them.

The alignment between the theoretical and didactic models of the teacher educator and his/her instructional practices in the university classroom may provide information about how teacher educators can become models for future teachers. Therefore, a teacher educator will display Instructional Coherence when his/her practices model the didactic-mathematical actions that he/she expects prospective teachers to acquire.

Modelling

The teacher educator is always an example for teachers; thus, when considering the widespread idea that teachers teach in the way they were taught, the modelling role that the educator acquires becomes relevant. In this regard, Lunenberg, Korthagen and Swennen (2007) state that the ways in which the educator models certain views of learning can be more important than the content itself. For the authors, these "ways" are grouped under the notion of modelling, understood as "the practice of intentionally displaying certain teaching behaviour with the aim of promoting student teachers' professional learning" (p. 589). So, considering these ideas, we could study Instructional Coherence in order to understand the ways in which a teacher educator makes explicit the message that he/she wants to convey to his/her students.

The authors define four types of modelling based on their literature reviews and their own research. These forms of modelling are grouped into implicit and explicit, with the latter having several degrees of complexity.

Implicit modelling: Even though the educators recognize that they must be good examples of the conceptions of teaching that they attempt to transmit, students often do not see these conceptions in practice. In fact, many educators do not manage to make their teaching models explicit, and their students' preconceptions about teaching and learning do not change significantly.

Explicit modelling: teacher educators should make explicit which choices they make while teaching, and why. Some techniques to achieve this goal could be journal writing, "thinking aloud", or co-teaching. Although these ways of making educators' didactic decisions explicit may be useful, they are not naturally observed in their actions.

Explicit modelling and facilitating translation into student teachers' own practices: even if educators explicit their decisions, students should be able to transfer them to their own classroom practices. This requires reflection and an analysis of the educator's instructional practices, combined with an attempt to define what they mean in teaching terms. From this starting point, students will be able to make their own decisions.

Connecting exemplary behavior with theory: it is clear that the theory-practice connection is essential in teacher education. For this reason, it is necessary to go beyond making pedagogical decisions explicit and giving students the chance to analyze them; students should connect practice with theoretical structures that allow them to explain these decisions and characterize them to inform their decision-making.

In order to identify which of these characteristics of their educators' instructional practices were present in the lessons observed, the questionnaire included referred to the modelling that students observed in their educators when conducting certain actions aimed at developing their knowledge for teaching.

Questionnaire dimensions

Both for educators and students, mathematics teaching involves the consideration of several theories that make it possible to construct a didactic model (Steiner, 1990) which has stable foundations and which can be implemented flexibly (Godino, 1991). Therefore, to study said models at any educational level, it is necessary to break down the practice of the participants involved according to the fundamental characteristics of the model that they profess.

In order to do this, and on instrumental terms, our questionnaire was created considering a set of theoretical approaches which are currently observed in Mathematics Education research. In this regard, Furinghetti, Matos and Menghini (2013) distinguish certain dimensions that make it possible to study theoretical teaching models. The first dimension, which emerged from early 19th century mathematics, concerns the promotion of mathematical thinking. This dimension also involves the promotion of statistical thinking, which is distinguished from mathematical thinking in that the argumentation of the former is based on data (Ben Zvi & Garfield, 2004). The second dimension is associated with the psychological-cognitive theories of the teaching and learning of mathematics advanced by Piaget, Vygostky, Dehaene–Gingerenzer, Bruner, Ausubel, and Van Hiele, among others. The third dimension defined here groups cultural and social approaches together. The theories it concerns are those of Freudental, Kilpatrick, Polya, and Shoenfeld; social epistemology, socio-criticism, the Theory of Didactic Situations, and the Theory of Didactic Transposition, given their sociocultural nature. In addition, considering the latest results of Lesson Study, we added a fourth dimension which is associated with the hermeneutic processes that characterize the Japanese teaching model.

The following are the characterizations that we have constructed for each dimension in order to illustrate how indicators present them.

RESEARCH QUESTION

By singling out instructional coherence as a key element in teacher education, due to its role in the construction of the teaching-learning models of future teachers, our study is guided by a fundamental question: Which characteristics of teacher educators' practices make students regard them as coherent? More specifically, this article focuses on *student perceptions about the coherence displayed by their educators* which can be useful as issues for reflection.

METHODOLOGY

Given that the purpose of this research is to assess students' perceptions about the instructional coherence of their educators, we used techniques and instruments capable of measuring this qualitative variable. Thus, we employed Likert qualitative measurement scales, because they are capable of generating a discrete ordered continuum of the students' perception level.

Sample

In order to look into students' perceptions about the instructional coherence of their professors, students

Dimension	Theoretical Characterization
Mathematical Thinking	This dimension is expected to capture characteristics of the mathematics teaching model which revolve around mathematical work, considering key aspects of mathematical or statistical thinking.
Psychological-Cognitive	This dimension is expected to capture characteristics of the mathematics teaching model that focus on cognitive-structural aspects of mathematics learners. It should also identify aspects that shed light on the reasons behind learners' behaviors and actions concerning mathematics.
Socio-Cultural	This dimension is expected to capture characteristics of the mathematics teaching model associated with the relationships established by social and cultural groups when learn or create mathematical meaning.
Hermeneutic	This dimension is expected to capture characteristics of the mathematics teaching model which are aimed at creating expertise and command of mathematical knowledge via the thorough use of processes and strategies.

Table 1

and educators belonging to education programs for secondary school mathematics teachers at Chilean universities were invited to participate. In order for a university to be selected, the following requirements had to be met: (a) having a teacher education program accredited for 4 or more years[1], (b) having had a minimum admission score of 550 points in the mathematics part of the test for the last 4 years[2], (c) having 14 or more Mathematics classes, and (d) 3 or more didactics/method classes in the program curriculum[3]. These values were established after analyzing the data for the 36 secondary education programs in Chile, since it was necessary to set a minimum quality level according to the parameters used in the national context. Of all the teacher education programs studied, only two met these criteria.

Specifically, the classes chosen were those belonging to the didactic or methodological area, since they are where students' mathematics teaching knowledge is strengthened, regardless of their formal name in each teacher education program. Finally, a total of 42 students, 11 from one university and 31 from the other, answered the questionnaires.

Questionnaire structure

The coherence variable is complex to study; therefore, in order to collect information about it from the students' perspective, an instrument comprising three parts (A, B, and C) was used. In part A, the students were asked about their academic trajectory in the teacher education program they are part of (number and type of courses taken) and their perception about their preparation for teaching the syllabus contents at different educational levels. In part B, they were asked about their beliefs concerning the main characteristics of mathematical activity in schools through a Likert scale (Likert I) that presented several strategies or methodologies which they would regard as necessary for their pupils to generate mathematical knowledge. In part C, they were asked about the educational process that they had experienced in their programs via two Likert scales. The first of them (Likert II) presented instructional practices, which can be observed in courses of a didactic or methodological nature. Students were asked which of these practices they believed were useful to replicate in schools, in order to identify which instructional practices were making an impact on the construction of their teaching-learning models. The second scale (Likert III) aimed to identify the type of instructional modelling used by their professors. In order to achieve this, the same indicators present in the previous scale were presented, but identified as actions performed by the educator. For each of them, students were asked to classify the indicator according to the modelling types described above.

Analytic process

In order to evaluate perceived coherence, this study analyzes parts B and C of the instrument. Likert I (Part B), was coded binarily, assigning 1 to "Yes" and 0 to "No", and Likert II (Part C), was again coded binarily, assigning 1 to the option "Useful to replicate" and 0 to the option "Not useful to replicate". The second scale of part C (Likert III) was coded binarily for each instructional model. That is, the implicit model was first identified with number 1, while 0 represented the rest; then, number 1 was used for the explicit model and 0 for the rest, and so on. In this way, 4 dichotomous scales were obtained, which made it possible to compare and group the indicators for each of the models. Binary scales were used because, to perform hierarchical and event tree analyses on the indicators for determining concentration in categorical variables when N is small, it is necessary that data be binary or be arranged into a Likert scale, that no normality be observed, and that no relations be present among them.

To analyze the binarily-coded scales, a hierarchical cluster analysis was performed using the Jaccard index, which makes it possible to determine the homogeneity between two indicators. These indicators reflect each of the characteristics of the mathematics teaching models within the frameworks established and described above. Index I is defined as $I = x/(x + y - z)$, with "x" reflecting the number of prospective teachers who chose indicator X, "y" reflecting the number of

1 The accreditation of programs certifies their quality according to their declared purposes and the criteria established by each academic and professional community (see www.cnachile.cl).

2 The University Selection Test (Prueba de Selección Universitaria, PSU) is a standardized measurement with a mean of 500 points and a standard deviation of 110. The selection process for students who wish to become teachers requires that they obtain at least 500 points in the PSU.

3 Teacher education processes are heterogeneous in terms of the number and types of classes that they offer their students. The programs available are concurrent and consecutive, with 8 to 24 mathematics courses and 1 to 7 methods (didactics) courses.

prospective teachers who chose indicator Y, and "*z*" reflecting the number of prospective teachers who chose both X and Y.

RESULTS

In order to characterize the students' perception of their professors' instructional coherence, three key elements reported by the above Likert scales were considered: the students' teaching models (Likert I), the usefulness of replicating the instructional practices of their educators in their own teaching (Likert II), and the type of modelling under which they observe said practices (Likert III).

Students' teaching models (Likert I)

With respect to the predominant mathematics teaching and learning models among prospective teachers, when asked "For a student to generate mathematical knowledge, it is necessary to", the hierarchical cluster analysis revealed two clusters with Jaccard homogeneity indexes over 70%.

The first cluster grouped the next indicators: 3. *Consider the mental structures of students in terms of the concepts' abstraction level*, 4. *Consider the discussion between students to generate the concept, 5. Show examples and counterexamples*, 6. *Create a representation of the concept in the student*, 8. *Consider that students have a certain knowledge and that they will use it to understand concepts*, 9. *Consider the socio-cultural aspects of students in connection with the activity*, 14. *Transform pure mathematical knowledge into knowledge that can be taught*, 16. *Present situations which are significant to the student*.

This set of indicators shows that students attach great importance to their pupils' previous knowledge, both cognitive and sociocultural. This reveals that the socio-cultural dimension, as well as the psychological-cognitive dimension, are among the elements that characterize the model that prospective teachers use to teach mathematics.

The second cluster concentrated the following indicators: 9. *To formulate a problem for knowledge to emerge in response to it* and 13. *To face the student with a problematic situation*. Both of them are associated with problem-solving as a strategy to generate knowledge. Said indicators are included in the socio-cultural dimension of the theories about how mathematics should be taught; specifically, they are linked with the Theory of Didactic Situations and Polya's notion that mathematics should be learned by simulating the activity of a mathematician.

Usefulness of replicating their educators' instructional practices (Likert II)

Concerning the instruction provided by university professors, the actions which are part of or characterize their teaching model are those which students believe would be useful to replicate in schools. Specifically, the following clusters displayed a Jaccard homogeneity coefficient over 75%.

Cluster 1: 1. *The way in which mathematical problems were solved*, 3. *The way in which students were made to participate*, 4. *The way in which discussions were generated about the mathematical learning activities conducted*, 10. *The way in which students were made to reason*, 11. *The way in which complex didactic and/or mathematical activities were approached (breaking something down into smaller elements, giving examples and counterexamples, using analogies, etc.)*.

Cluster 2: 17. *The way in which students' comprehension was verified*, 18. *The way in which mathematics education theory was used*.

Both clusters are part of the social and cultural dimension of how mathematics is taught, which intersects with the elements that characterize the students' teaching model. It is noteworthy that, even though the indicators ask students to reflect on the aspects in which the educator's work–his/her way of conducting activities in the classroom, on the one hand, and his/her "mathematical-pedagogical" work, on the other–it is precisely the "form" of mathematical-pedagogical action that is relevant when generating an impact on the knowledge of prospective teachers.

Modelling type observed in the educators (Likert III)

Finally, regarding the forms of modelling observed by the students, only transfer modelling displayed an association with the indicators in the clusters that characterize the students' mathematics teaching-learning models. Although the Jaccard coefficient was lower than those of previous classifications, it never dropped below 45% in any of the clusters of the transfer model.

Cluster 1: 14. *To identify students' mathematical errors*, 15. *To tackle students' mathematical errors*.

Cluster 2: 10. *To make students reason*, 3. *To make students participate*

Cluster 3: 4. *To generate discussions about mathematical learning activities*, 17. *To verify students' comprehension*, 19. *To motivate or involve students in classroom tasks or activities*.

As can be observed, the three clusters again belong both to the social-cultural and the psychological-cognitive dimensions. In addition, it is interesting to note that the aspects or actions associated with evaluation, such as providing feedback to students about their work, were observed to have 50% of homogeneity, but in the implicit model of the educator.

CONCLUSIONS

When comparing the dimensions *students' teaching model*, *usefulness of replicating educators' instructional practices*, and *type of modelling observed*, three strong associations can be observed. First, comparing the students' teaching models with the usefulness of replicating certain instructional practices (Likert I & II) reveals that those deemed useful correspond to the same dimensions that characterize their own teaching models, and are associated with aspects of the socio-cultural and psychological-cognitive theories of mathematical education. Second, when comparing teaching models with the modelling types observed in instructional practices (Likert I & III), the students manifest the same traits (indicators) that characterize their teaching model only for a specific modelling type: the transfer model. This implies that students observe characteristics of their own teaching model when the educator connects instructional practices with the reality of schools. Third, comparing the usefulness of replicating certain instructional practices with the modelling type displayed by the teacher educator (Likert II & III) reveals that those deemed useful are precisely those that belong to the transfer model. These associations indicate that, when the educator manages his/her class in such a way that allows him/her to relate didactic-mathematical actions with the school classroom, the student regards such actions as relevant because they are the ones which belong to his/her teaching model. In terms of perceived instructional coherence, students regard their professors as coherent when their practice reflects their teaching model.

We believe that the results provide empirical evidence of a phenomenon that we knew, but which we had not characterized. In this regard, it is worrying for our national teacher education system, as well as for other systems, to know that students only consider their teaching to be effective in terms of coherence when instructional practices match their didactic models. This prompts a question: to which degree has the education received affected students' conceptions of teaching and learning? In the case of the education of secondary school teachers, one of the main characteristics of education programs is their strong disciplinary focus, which contrasts with a brief period of pedagogical instruction. So, how has disciplinary instruction helped to change the traditional teaching patterns of school mathematics? This and other questions lead us to consider the need to know the initial didactic models of students, that is, how they see the teaching and learning of mathematics when they enter university, and how their views change as they progress in their teacher education program.

ACKNOWLEDGEMENT

This article is part of the research project "Estudio de las Prácticas Instruccionales del Formador de Profesores de Matemática de Educación Media: Construcción de Indicadores de Coherencia" [Study of the Instructional Practices of Secondary School Mathematics Teacher Educators: Construction of Coherence Indicators] (VRI INICIO 23/2013), funded by Pontificia Universidad Católica de Chile's Research Department.

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Word problems of a given structure in the perspective of teacher training

Marie Tichá[1] and Alena Hošpesová[2]

1 Academy of Sciences, Institute of Mathematics, Prague, Czech Republic, ticha@math.cas.cz
2 University of South Bohemia in České Budějovice, České Budějovice, Czech Republic

The paper reports on the results of a research study focusing on how to train prospective teachers for guiding their pupils through the process of solving word problems. The teaching experiment involved prospective teachers' work with graphical representations called branched chains to solve and pose word problems. The paper shows the difficulties the students were facing, presents ideas from discussions during joint reflection, tries to describe the changes in the students' skills and attitudes. It also points at the differences between prospective primary and secondary mathematics teachers.

Keywords: Inquiry based mathematics education, prospective teacher training, subject matter knowledge, knowledge base for teaching.

INTRODUCTION

This contribution is a follow up to a presentation on CERME8 (Hošpesová & Tichá, 2013), which focused mainly on the quality of prospective teachers' content knowledge and their ability to apply their knowledge of mathematics in the solving process of problems leading to inquiry based mathematics education (Artique et al., 2011). This became the starting point of this paper in which we discuss one of the possibilities of developing pedagogical content knowledge (PCK) and prospective teachers' inquiry based approaches to word problems.

As has been pointed out repeatedly, students learn mathematics by problem solving. While solving a problem, they learn to instruct themselves, to make decisions which methods of solving to choose. It comes natural when solving problems to explore, discover and justify. The whole solving process can be regarded as *the solver's dialogue with the problem* (How to begin? How to carry on at a particular point?) The solver modifies his/her decision on the basis of information from the problem and confrontation with the goal (with the question to be answered in the problem). He/she alternately asks questions or poses problems and solves these problems. This implies that problem posing and problem solving are intertwined.

A mathematical (word) problem should have certain properties one of which is structure. When posing a problem we should always bear this in mid. That is why we started to include posing of problems of a given structure into teacher training. We are convinced that this activity helps (prospective teachers) (a) get rid of stereotypical nature of problems they pose (he/she grows aware of a structure of a problem and the much needed variety), (b) get insight into the structure of a problem when solving it and (c) overcome the tendency to focus only on the sequence of calculations that lead to the result.

What we find crucially important is the need to develop prospective teachers' skills and ability to create (mental) representations, to model phenomena with mathematical tools, to formulate questions and pose problems. In the past years we have shown the diagnostic, *educational and motivational benefit* of integrating activities connected to problem posing in teacher training (Tichá & Hošpesová, 2013). We have discovered that problems posed by the trainees are often stereotypical in context, quality of the environment, representation and structure. The trainees did not find it hard to diversify the context and environment. However, it is very difficult to start considering deliberate, conscious work with the structure of the problem; both in problem solving and particularly in problem posing. That is why we have shifted our attention to this area.

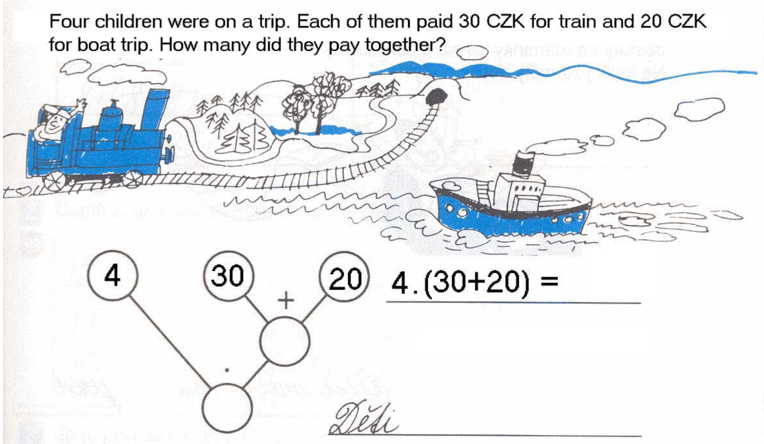

Figure 1

In the here referred study we would like to discuss how prospective teachers work with the structure of a word problem. With this aim we selected one part of Czech primary school curriculum, solving two-step word problems. In teacher training, we used specific graphical representations called branched chains (example in Figure 1 from the textbook for primary school level written by Kittler and Kuřina, 1994). Kittler and Kuřina understand branched chains as the model of the problem translated into the language of chains or as a record of the solution of the problem. We use these chains as the means of visualization, as graphical representation either of the problem structure or the problem solving procedure.

Similar graphical representations were used by Nesher and Hershkovitz (e.g., 1994). They refer to them as schemas for representation of two-step word problems.

Let us remark here that the concept of a schema is very frequent in mathematics education research and is used in various interpretations. These interpretations are often based on approaches and theories of cognitive psychologists such as Skemp (1971), Fischbein (1999), etc. A schema is habitually characterized as a mental structure with two main functions: it integrates the existing knowledge, and it is a mental tool for acquisition of new knowledge; a schema describes both the mental and physical actions involved in understanding and knowing. Also other characteristics are given: schema as internal representation of information, as a model of activity (action scheme), as a set of interrelated knowledge in a specific area, as representational structures that represent knowledge in the form of networks of connected concepts, as action structures, goal directed activity.

An analysis of the use of the concept of schema within different theoretical context of mathematics education is given for example by Davis and Tall (2002).

Hejný (2012) justifies that a schema, which in his opinion represents understanding, is the necessary condition for creation of a knowledge structure. Our approach to work with branched chains is analogical. We understand this work as the graphical means of visualization and a tool for grasping the structure of problems (data and relations between them) both when solving and posing problems. This means that we use these chains for deepening understanding of the problem solving process and the structure and/or construction of the problem.

TEACHING EXPERIMENT, ITS PARTICIPANTS AND METHODOLOGY

We tried to answer the following questions: What problems do prospective teachers pose when their structure is given by a branched chain? Are prospective teachers aware of the structure of the assigned problems? Whether and how do the students consider the structure of word problems?

It should be noted that in Czech schools it is a very common practice to solve word problems using various graphical representations. However, our experiment showed that most of the participants probably had no prior experience with branched chains.

Participants

We worked with two groups of participants – prospective teachers. The first group consisted of 36 prospective primary school teachers. Their undergraduate studies include courses in pedagogy, didactics and psychology, and also basics and didactics of all subjects taught at primary school level (mathematics, Czech language, arts, sciences, etc.). The other group consisted of 15 prospective secondary school mathematics teachers.

Teaching experiment

The participants solved a set of tasks in the seminar on didactics of mathematics. In the first stage the students (a) created branched chains to the assigned problems, (b) posed word problems of a specific structure represented by a branched chain, and (c) decided which branched chain represents a given word problem. In the second stage the students carried out joint reflection of posed problems.

Data analysis

The survey was planned and conducted as a qualitative study. We analyzed all students' written production as well as their verbal comments in the subsequent joint reflection. At the beginning we used open coding. After that we were looking for relationships between solutions and the students' subsequent discussions. What we could observe in our analysis was the gradual acceptance of branched chains and deepening of understanding. This could be observed both in the posed problems and in the students' contributions in the joint reflection. Our decision was to conceive this paper as an analytical story describing how students come to adopt branched chains and outline some of their misconceptions.

RESULTS AND DISCUSSION

We started by discussing branched chains (Figure 3) and asked prospective teachers how they understand and would call these "pictures". Their responses included: *"means of visualization, graphical representation, graphical record (of gradual addition), addition snake, alternative form of recording addition, a sequence of operations using brackets, analogy to addition "pyramids", family tree, tournament tree, mind map, Pascal triangle".*

Posing a problem of a given structure

In the beginning, participants were solving two tasks:

T1. Create a simple word problem to match Figure 2.

T2. Replace one piece of data in the simple problem you have just posed by a simple problem (e.g., as

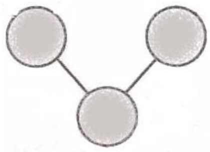

Figure 2

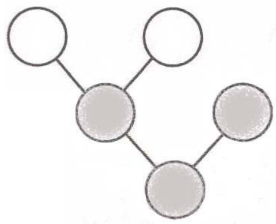

Figure 3

shown in Figure 3), thus you will pose a two-step word problem.

Most students were able to fulfil these tasks without difficulty and posed the required problems, e.g.:

> Problem 1 (Figure 2): Every school day, Matyáš gets pocket money from his mum and dad. He always gets 20 CZK from his mum and 30 CZK from his dad. How much money does Matyáš get every day?

> Problem 2 (Figure 3): Matyáš gets pocket money from his mum and dad. He always gets 20 CZK from his mum and three ten-crown coins from his dad. How much money does Matyáš get? (The author implicitly assumes commutativity.)

We could observe that in the beginnings some students (especially prospective primary school teachers) understood branched chains clearly as a representation of the solving procedure. Many students when solving T2 posed a problem which was set in the same context and which was a sequel to the story in T1, but they posed two separate simple one-step problems. As an example we present problems 3 and 4. Problem 4 is self-contained; it does not develop Problem 3. However, the follow-up discussion showed that the author had expected this pair of problems to be understood as connected, that the assignment of Problem 4 would be understood as a sequel to Problem 3.

> Problem 3 (Figure 2): Mum was baking biscuits. There were 6 lines of biscuits on a baking sheet. There were 3 biscuits in each line. How many biscuits were there on the baking sheet?

> Problem 4 (Figure 3): Mum divided baked biscuits among her two children equally. How many biscuits did each child get?

For some students it was hard to grasp "the language of chains". In the follow-up discussion the author realized that Problem 4 did not correspond to the figure from task T2 (Figure 3). That is why she visualized Problem 4 by a new branched chain (Figure 4); the words "divide between two children" was replaced by the sign for division ":" used in Czech schools. In fact she tried to make a kind of illustration of the problem; but Figure 4 made by her does not correspond to graphical representation using the language of chains, neither in the solving procedure, nor in the structure.

Another prospective primary school teacher posed two consequent simple problems 5 and 6. She also continued the story. When representing problem 6, she used representation which contains problem 5 and is a mere record of gradual addition.

> Problem 5 (Figure 5): We saw 3 giraffes and 6 sheep in the ZOO. How many animals did we see?
>
> Problem 6 (Figure 6): In the afternoon we continued our trip and we saw 1 hippo, 1 bear and 4 zebras. How many animals did we see during the whole day?

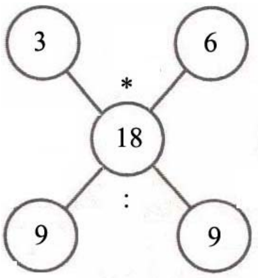

Figure 4

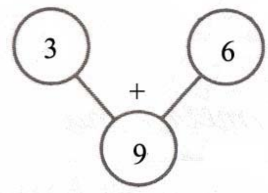

Figure 5

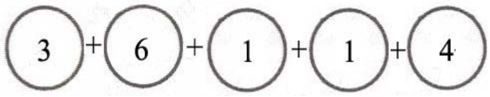

Figure 6

There were also inaccurate, ambiguous or even misleading formulations in various answers that could be interpreted in more ways in the posed problems. This became clear in the joint reflection on the posed problems which involved also solution of posed problems both by their creators and other participants.

Expanding the chain

Following the previous two tasks T1 and T2, the students were asked to work on modification of chains and structure of problems and solved the following task:

T3. Find other possibilities of "developing" the initial simple scheme (Figure 2) than shown in Figure 3.

In process of solving T3, the students proposed the expansion of a chain given for example in Figure 7 and returned to Figure 3 and reminded it. Doing that they formed for example a triplet of consequent problems 7–9:

> Problem 7 (Figure 2): I will buy 4 cakes at the price of 7 CZK each. How much will I pay?
>
> Problem 8 (Figure 7): I will buy 4 cakes at the price of 7 CZK each and bread for 19 CZK. How much will I pay altogether?
>
> Problem 9 (Figure 3): I will buy 1 poppy seed cake and 3 jam cakes. Each cake costs 7 CZK. How much will I pay?

This triplet of problems was then discussed by the students. After the discussion, one of the students expanded the original chain (Figure 2) into Figure 8. However, when posing Problem 10 she was not sure whether the chain corresponded to the assignment of the task T3 and to the text.

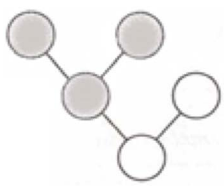

Figure 7

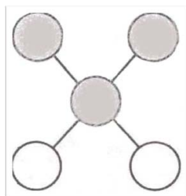

Figure 8

> Problem 10 (Figure 8): I will buy 4 cakes worth 7 CZK, i.e. (4 x 7) = 28. I will pay using a twenty-crown coin and add 8 CZK using two-crown coins, i.e. (20 + 8). Is this OK?

Other students proposed to visualize Problem 10 with the help of Figure 12. That is they realize that Problem 10 was not a problem with two operations.

Linkage of the problem to the chain

T4. Decide whether Jirka's (Figure 9) or Hana's (Figure 10) schemes "fit" the following assignment: *We were giving out notebooks. 15 were left in one package, 9 in another. I want to make packets of three from the remaining notebooks. How many packets will I make?* Justify your answer.

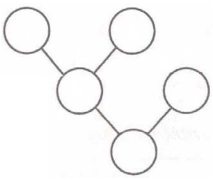

Figure 9

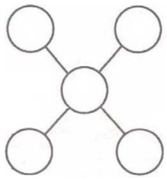

Figure 10

Within the frame of the joint reflection, one of the students suggested that the assignment of task T4 should be expanded and proposed the three chains in Figure 11 (i.e., Jirka's, Hana's, and Magda's).

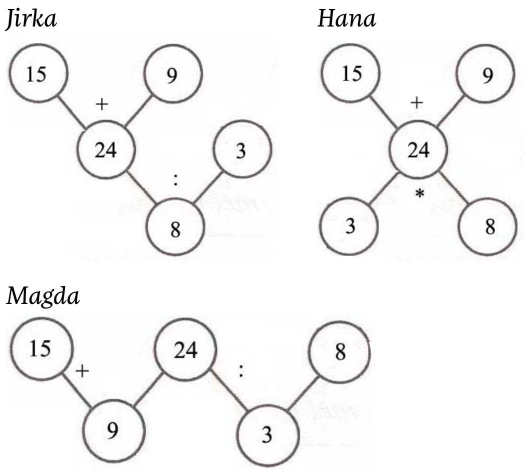

Figure 11

A very passionate discussion broke out in connection to the chain "Magda". One of the students proposed the chain should be "straightened". This idea launched a discussion on the meaning of numbers and symbols of operations in this chain. It was very useful and beneficial when students compared the chain "Magda" to a graphically seemingly very similar chain in Figure 14 (Problem 12).

Some other findings

We could also come across problems with more operations (multiple-step problems) that did not correspond to the assignments, which the students came to realize only in the joint reflection – for example, Problem 11 that was meant to represent chain in Figure 12.

> Problem 11 (Figure 12): I will buy 4 cakes at the price of 7 CZK. I have one twenty-crown coin and several two-crown coins. How can I pay?

Another student posed Problem 12 and created two chains to visualize it (Figures 13 and 14). This triggered a discussion on whether this is visualization of the solving procedure or the structure of the problem.

> Problem 12 (Figures 13 and 14): Honzik found 10 glass marbles on the pathway. He knew his brother was collecting glass marbles and that he would get 2 crowns for each. However there was a hole in his pocket and he lost 3 marbles on the way home. How much did he earn?

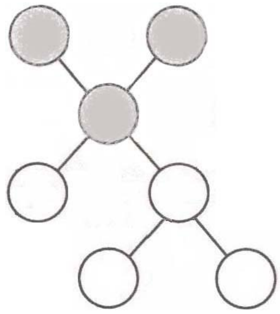

Figure 12

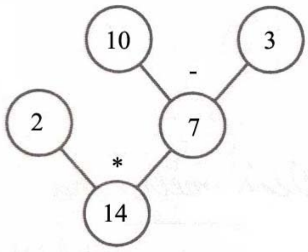

Figure 13

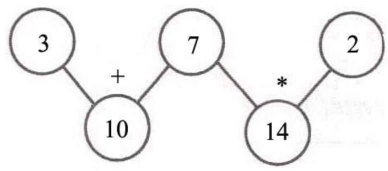

Figure 14

It is clear that students' opinions on the function, role of branched chains vary. However, most of the students agreed that a branched chain could help them solve problems.

We could also see that the effort to visualize required strenuous mental and intellectual effort. For some the will to use a chain at all costs may result in confusion. The chain stops being a benefit and becomes an obstacle. The student who had posed problem 13 was not able to decide whether it corresponded to the chains in Figures 15 or 16 and said: *"I am totally lost in this task."* and *"I think I first posed one problem and then a chain. When creating the problem I was already thinking about the model. Of course I could proceed the other way."*

> Problem 13 (Figures 15 and 16): The best basketball players were to be selected from three classes. Five were chosen from 2.A, they went to team A. 6 pupils were selected from 2.B, they were sent to team B. Eight pupils from 2.C were divided equally into both teams. How many players were in both teams?

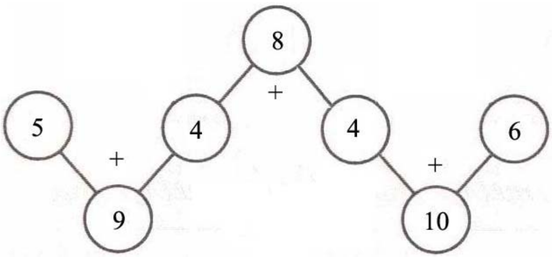

Figure 15

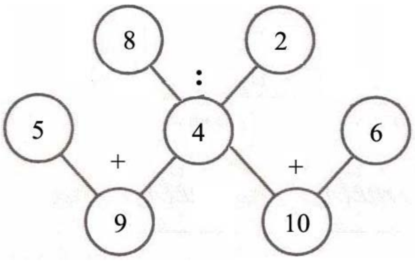

Figure 16

SUMMARY AND DISCUSSION

As we showed above some participants grasp branched chains as a representation of a problem structure, others as a representation of its solving procedure. This corresponds to the frequent view that a schema includes both a category of knowledge and the process of obtaining that knowledge. Both these perspectives could be come across. However, the conducted survey showed differences between the respondents: prospective primary school teachers usually understood branched chains as a record of the solving procedure (they did not mention the structure, the construction of the problem). In contrast, prospective secondary mathematics teachers were more likely to perceive a branched chain as a graphical representation of the structure of a problem. They stated that work with branched chains helped them "penetrate into the heart of the problem", grasp the problem. The following students' opinions could be come across:

> "… this is the first time I've seen visualization of word problems of this type … these schemas can help children who find it difficult to solve multiple-step problems to grasp the assignment."

> "I used to make an illustration of a problem only if everything else had failed. Now it's the other way round and I will teach my pupils to do the same …"

> "I can now get to know the structure of each problem. Thanks to this I can create also other types of problems than those they are usually assigned."

Both groups agreed that it was more difficult to pose a problem to match a model than to create a chain illustrating a problem. This regardless of whether the chain visualizes the problem solving procedure or its structure. This is documented by the following respondent's opinion: *"It's much harder for me to proceed from the chain to the problem … when posing a problem one must think very differently and this leads to better comprehension and in consequence to easier solution of problems."* However, we also came across a contrary opinion: *"I believe a branched chain helps us pose problems. It was easier for me first to construct the chain and then to come up with a story that fits it."* But some students refused the use of branched chains in teaching mathematics entirely.

Let us note here that joint reflection was a very important part of work in the seminar. Without this stage the students would have never grown aware of the fact that their problems did not meet the criteria from the assignment.

This makes us ask the following open questions: Why do some students understand branched chains as representation of the structure of a problem, what is its cause? Will this have impact on their understanding, on their competence to guide pupils while solving problems?

The relationship of this study to inquiry based mathematics education

We perceive work with branched chains for representation of word problems as (a) another substantial and stimulating activity leading to cultivation of PCK of prospective teachers and as (b) the field for application of inquiry based learning in teacher training (the concept is used in accordance with Artique et al. 2011). We are convinced that this approach is important in teacher training not only because of the fact that trainees get hands-on experience with inquiry based education but also because it develops creation of PCK. It is not insignificant that it brings other quality to the acquired knowledge, in this case focus on deliberate work with the structure of problems, which helps prospective teachers realize possible modifications of a problem and their effect on the cognitive difficulty of their solution, the potential of different problems. This should result in more deliberate and focused integration of word problems into their teaching.

When we study the issues related to problem solving, we build on the concepts of inquiry and discovery in mathematics education, i.e. on activities that are characteristic of inquiry based learning. The effort to improve mathematics education especially in the 1960s' resulted in bringing the issue of genetic teaching. This concept came out of the characteristics formulated by Bruner (1966) (who speaks of learning by discovery; to educate somebody means to teach him/her to get actively involved in the process of gaining, structuring and storing knowledge), Wittmann (1974) (genetic teaching is based on natural cognitive process in building and using mathematics), Freudenthal (1973) (genetic principle is characterized by guided rediscovery as a step in the learning process). These fruitful, powerful ideas are also as a rule present in the definition (indicated in the list) of the basic characteristics of inquiry based mathematics education and are reborn in a new form.

Benefit of posing problems of a given structure in teacher training

Some positive changes could be observed when the students were working with branched chains. We are well aware of the fact that the question how this change can be verified remains unanswered so far. Using a metaphor, similarly to force which is measured from its effects, the benefit of joint reflection and work with structure can be seen in changes, in a shift in certain aspects, as an indicator of these activities to development and refinement of PCK. We could observe the following changes:

— Change of climate (intentional work with structure when posing problems motivates students, improves their attitude to teaching mathematics and supports their self-confidence).

— Change of character of problems (from simple, easily solvable "textbook-like" problems that were often uninteresting, stereotypical and sometimes erroneously formulated, to problems that were challenging to pupils, were not common, whose assignment was varied (there were graphs, charts, tables ...), that allowed different solving procedures, whose solution often required reasoning.

— Deeper insight into a selected area, domain, deeper understanding of concepts and procedures (solving strategies and methods).

In our ongoing research we expand our database and we classify the posed problems and branches according to pinpointed characteristic properties.

ACKNOWLEDGEMENT

This research was supported by the Czech Science Foundation, project No. 14-01417S and RVO 67985840.

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Mathematical opportunities: Noticing and acting

Zelha Tunç-Pekkan[1] *and Hülya Kılıç*[2]

1 MEF University, İstanbul Turkey, tuncz@mef.edu.tr

2 Yeditepe University, İstanbul, Turkey

The aim of this study was to investigate how three pre-service teachers (PSTs) listen to students, notice Mathematical Opportunities (MO) and scaffold ideas based on MOs. There were 12 videos of three PSTs' interactions with a pair of 6th grade students respectively while studying fractions. We analysed videotapes and identified different number of MOs for each PST. The findings revealed that with the help of this research and teaching environment, all PSTs listen to the students to understand their mathematical thinking initially (meaning catching MOs) and try to follow-up on them in action in differing levels of sophistication. While most of the investigated MOs resulted in a mathematical solution, PSTs need to further develop appropriate scaffolding practices.

Keywords: Pre-service teacher education, mathematical opportunities, fractions, scaffolding, listening to children.

INTRODUCTION

Reform oriented teaching supported the idea that "all students should have the opportunity and the support necessary to learn significant mathematics with depth and understanding" (NCTM, 2000, p. 50). Reform-oriented teaching and its natural necessity, student-centred education, require teachers to attend to students' thinking both in planning and in action. Many researchers investigated how this mechanism work in terms of how teachers listen to, pay attention to students' mathematics and act accordingly (Franke, Webb, Chan, Ing, Freund, & Battey, 2009; Sherin, 2004). Giving importance to students' mathematics and using this construct in pre-service teacher education programs as component for quality teacher-education have been also argued in different research venues (D'Ambrosio & Campos, 1992; McDonough, Clarke, & Clarke, 2002; Philipp et al., 2007). While the research in pre- and in-service education of teachers helps us illuminate the issue of how to use students' thinking for practice, there is more to investigate for how this opportunity pre-service teachers become to notice children's mathematical thinking and act on such situations.

In this paper, we will report an intervention study that focused on three pre-service teachers and their interactions with three pairs of 6th grade students. The purpose of the study was to understand how pre-service teachers' (PSTs) notice children's mathematical solutions and steer their interactions using those notices i.e., how they use those constructs for the purpose of "scaffolding activities." In relation to this concern, we defined *Mathematical Opportunities (MOs)* occurred in the interactions and used those opportunities as basis for our web of analysis. Therefore, we had following research question:

To what extend do the pre-service teachers notice MOs and scaffold students' mathematical thinking during interactions (with the students)?

THEORETICAL FRAMEWORK

Mathematical opportunities

Our definition and use of mathematical opportunities for pre-service teachers might show parallelism with what Leatham et al. noted for teachers. It is that teachers should recognize such opportunities initiated from the students to build on the students' mathematical thinking (Leatham, Peterson, Stockero, & van Zeist, 2015). They called those opportunities as *Mathematically Significant Pedagogical Opportunities to Build on Student Thinking* (MOSTs) and described them as being observable student actions that enable teachers to make inferences about students' mathematical thinking, as being appropriate and important *mathematical point* to be focused on and as having potential to help students to understand the essence of the *mathematical point*. We build further on MOs for the purpose of understanding how they might provide opportunities for PSTs and what the affordances of those opportunities in action for the students.

Instruction based on students' thinking

A teacher-initiated or student-initiated conversation between the student and the teacher is valuable in terms of having a potential for creating teaching-learning opportunities. Van Es and Sherin (2002) stated that in reform-oriented teaching, teacher's noticing and interpreting students' actions are key components when adapting instruction in the moment. While many experienced teachers can learn how to notice and interpret over the years of practice, it is important that these concepts should be integrated in the teacher education programs as basis for reform-oriented teaching centered on students' thinking. From their research, van Es and Sherin (2002) identified the components of "noticing" as follows: "(a) identifying what is important and noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom interactions." (p. 573). For the purpose of providing learning opportunities for PSTs about "noticing," instead of the whole classroom interactions, we provided a setting where PSTs observed the two individual students solved mathematical problems. Then PSTs continued to observe when two students discussed about their solution of the particular problem and lastly we let them to interact with the students to understand, and extend students' thinking with their questioning and scaffolding activities. In this sense we think that providing such micro-classroom environment is important for PSTs to focus on "noticing" of a pair of students' mathematical thinking and provide in-action instruction around their observations of students' thinking.

Questioning and scaffolding

With the demands of reform-oriented teaching, the nature of the interactions between the students and the teachers has to change from traditional *show* and *tell* to more advanced interactions. Teacher's use of language, teacher's intentions, the use of representational and instructional tools, how an interaction is started, continued and ended are important for providing learning opportunities for the students. Anghileri (2006) discusses three levels of "scaffolding practices" (for further explanations, see Anghileri, 2006). The first level is basic and it is related to the environmental and physical materials in the classroom and their effect on orienting students for preparing learning. She defines Level 2 scaffolding as "explaining, reviewing and restructuring." Anghileri (2006) states that usually showing and telling had been an accepted explaining in traditional teacher initiated actions. However, she states that there are alternatives to showing and telling, such as reviewing and restructuring. Probing and prompting questioning types are mainly used in reviewing students' mathematical ideas. While probing questions are "to gain insight into students' thinking, prompting their autonomy and underpinning the mathematical understanding that is generated" (p. 42), prompting questions might "lock the teacher in the center stage" (p. 43) and might put the student into guessing mode of teacher's intentions behind the questions. Level 3 scaffolding is identified as the highest level of scaffolding such that we rarely observe this in classroom discourse, it is the scaffolding that helps students' extend their thinking and "specifically focused on making connections and generating conceptual discourse" (Anghileri, 2006, p. 47). While Anghileri's framework helped us to how to focus on scaffolding actions, we needed to develop a new framework, which helped us to explain PSTs' teaching practices in this study (see the Analysis part).

METHODS

Research setting and participants

Data collected for this analysis occurred in an after school program with the partnership of a university in Istanbul, Turkey and a local low SES middle school 6^{th} grade classroom. Participants included three senior PSTs specializing in teaching middle and high school mathematics (Anna, Betty, Carol; all names are pseudonyms) and three pairs of 6^{th} grade students (Pair A, Pair B, and Pair C; worked with Anna, Betty and Carol, respectively). PSTs and the students participated in this study voluntarily. All of PSTs had some informal teaching experiences such as tutoring mathematics, teaching in an after school programs voluntarily, however their teaching experiences were not homogenized.

Mathematics workshops

Mathematics workshops originally started with the idea to teach 20 local 6^{th} grade students difficult mathematics concepts using manipulative as an after school program. There were two university professors (authors) who did team-teaching in the workshops, five to seven PSTs who voluntarily came to workshops and helped the 6th graders whenever needed and one research assistant videotaped the whole class interactions. After each session, the professors and PSTs came together and discussed their observations related to what happened in the workshop and their observations related to students' understanding. About 11th week in the workshop program, we realized that PSTs were doing "show and tell" instead of paying attention to students' mathematical thinking and they were very directive in their questions. For the following eight weeks, we recruited three PSTs from the group, assigned them to particular three pairs of students, and focused on one mathematical topic, *fractions*. Sixth grade students were chosen based on the observations and performances they showed during the first 11 weeks. There were two mid-achieving pairs (Pairs A and B) and one-high achieving pair (Pair C).

In this paper, we will focus on smaller part of the data-4 weeks. During this 4-week period, each session was about 45 minutes and organized as follows: The first author introduced the activity and 6th graders individually worked on them. Later, 6th grade students worked in pairs and had chances to discuss their solutions with their partners. During this time period, PSTs only observed and video recorded students' individual work and their work in pairs. PSTs did not talk to students during this time period. This was on purpose since we wanted to have PSTs focus on students' activities while freeing themselves from the urge of teaching. This enabled them to observe the students' mathematical thinking process, how students communicated and negotiated their thinking to their partners. Later, PSTs were allowed to talk, interact and ask questions for 10–15 minutes. Finally, time permitting we had whole classroom discussions with 6th graders that the first author led.

ANALYSIS

We analyzed four sessions of three PSTs own-recorded videotapes. Data analysis consisted of analytic induction (Bogdan & Biklen, 2003). We reviewed all the videotapes one by one and identified MOs. We define MOs as 6th grade students initiated solutions and these solutions are usually interesting ways of mathematical thinking related to the fractions topic. For example, 6th grade students were asked to draw a number line and locate unit fractions such as 1/2, 1/3, 1/4, 1/5 on it. Betty's students treated the unit fractions as if they were whole numbers and they located them accord-

Identifier	Description	Coding
Opportunity	The mathematical situation that 6th grade students provide and overall how PSTs approached the situation	**Y:** Yes **N:** No
Opportunity in action	The interactions such that what kinds of questions PSTs asked to assess and advance students' thinking, what kinds of mathematical or mathematical pedagogical knowledge they used in their interactions, and how they proceeded and closed the conversations	**N/A:** If the PST missed the opportunity **Level 1:** If PST has surface level questioning. Conversation is mostly described as individual questions-answers. There is no big mathematical idea communicated in the conversation. Teacher mostly asks for explanation but does not take it to the further; she does not do anything with the explanation. **Level 2:** If PST asks probing and prompting questions. There is progressive conversation, which might include students' contribution, but it is mostly teacher-dominated conversation *with her mathematical goals in mind*. "Reviewing" and "show and tell" could be indicators of this level. In addition, PST might attempt to give examples and use materials to help students achieve her mathematical goal in mind but the students may/may not understand or make the teacher's intended connections. **Level 3:** If we saw evidences of PST's guiding students. Purpose of questioning is guiding towards a legitimate mathematical idea. The students also positively perceive questioning. Questioning might advance the students thinking. Students might have some ownership of the ideas developed during conversation. PST addresses misconception (if there were any) by providing mathematically valid examples to help the students to understand the teacher's mathematical goal.

Table 1: Coding scheme for analysing MOs

ingly with equal distances. This is an example of a MO that we investigated in detail. We made coding based on a grounded framework we developed ourselves (see Table 1) but it also shows some similarities to the focal points of van Es and Sherin's (2002) "learning to notice" and Anghileri's (2006) "questioning and scaffolding" frameworks.

We will introduce two examples from the common opportunities and discuss how the opportunity was used by two PSTs. In the third opportunity (see Table 2) the students were asked to find a fraction between 1/2 and 1/3, make a number line and place that fraction between 1/2 and 1/3.

Anna and Pair A

Anna's students' answer to the given problem is given in Figure 1. The students thought 1/2 and 1/3 as integers of 2 and 3 respectively. Therefore, a number which was bigger than 2 and smaller than 3 was 2.5 (In Turkey, comma is used to show decimals).

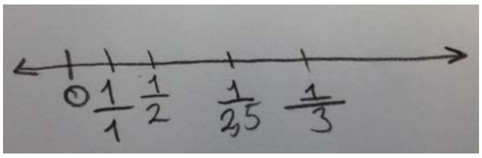

Figure 1: Anna's students' work

From their drawing we inferred that students probably thought that, except "0", the numerators of the fractions would be "1" and denominators would be "in order" starting from "1." Because they knew that 2.5 is between 2 and 3 then ½.5 would be the answer. Anna had the idea that fractions should be represented in a/b form where "a" and "b" are natural numbers but not decimals. In the following vignettes, Anna first wanted to discuss this point:

Vignette 1

Anna: How did you find two and a half? What kind of number is two and a half?
Pair A1: It has comma
Anna: Decimal
Pair A2: It is with comma, that is, it is not whole.
Anna: What kind of number system are we interested in? That is, how do we call the numbers [pointed to 1/2 and 1/3] in this system?
Pair A2: Proper fraction [it was supposed to be unit fraction].
Anna: Proper [Unit] fraction, right? Even it is improper, it is fraction. We show fraction like this [pointed to 1/2]. So, why did you write a decimal here [pointed to ½.5]?
Pair A2: We couldn't find anything else.
Anna: OK. How can you represent two and a half in other way?
Pair A2: 2/4
Pair A2: The half of 5. That is 1/2 of 5.
Anna: What do you do with 5, that is, how can you represent this as a fraction?
Pair A2: 1/5...
Anna: Your thinking is good but you are missing something. What did you do here? [Pointed to the previous number line] What did you do to show 1/4? What did you do 1?
Pair A2: We divided into 4 pieces.
Anna: What did you do with 5 and got 2.5?
Pair A2: 5, again we divided it into two.
Anna: OK. What kind of expression is that?
Pair A2: 1/2.
Anna: You are saying that 5 but then how did you divide it into 2.5? Where that division operation comes from?
Pair A2: Division operation...
Anna: Don't get confused by division. Here [pointed to previous drawing for 1/4] you divided into 4 because you have 1/4 at hand and to find it you divide one by four. Then, I am just talking about this 2.5. Because, why you wrote 2.5 here, it's interesting. Very interesting. I cannot think of it.

Then Anna drew a number line placing 1, 2 and 3 and then placing 2.5 between 2 and 3. The conversation continued with her questioning but because of the limited space we will not include it. Anna realized that students' thinking would lead to a correct answer if the problem statement asked a number between 2 and 3; an answer of 2.5 as a decimal would be acceptable. Her questioning shows us that she realizes students are not necessarily thinking in the same way that Anna thinking about conception of fractions. Anna's interaction with the students evolved around many topics. She first questioned students whether their answer was a fraction or decimal, then she wanted to have the students show 2.5 as a fraction (where she differed from the original answer of ½.5).

In our framework, PST notiaced the MO (coded as Y for Opportunity). Once she received answer of "one half of 5" from the students, she moved asking questions about how to represent it (students said 1/5 that Annaa did not investigate). She then asked them to relate to earlier example of 1/4 "where 1 was divided into four pieces" (as students verbalized). Anna then focused on centralizing the conversation on "division operation" and how students might have used "division" to get 2.5 from 5. For a while, the focus of the conversation and leading questions were unclear. Eventually, Anna asked how they thought ½.5 as in the same way they would give meaning to 1/4 (where students said it was one piece out of four pieces). Students did not give a definite answer to the question but they indicated that it seemed "illogical" after talking to Anna. The conversation ended without a satisfying situation neither for the students nor for Anna.

When we analysed the conversation, we observed that Anna controlled the conversation. Nature of her scaffolding and questioning changed throughout the conversation: she used some probing questions (e.g., how can you represent this as a fraction? So, why did you write a decimal here?) But interestingly the conversation did not lead to any productive ways of thinking on the students' part. The probing questions sometimes did not help the students and the teacher did not know how to use it to steer the conversation to help students gain some understanding. We coded these interactions as Level 2, for Opportunity in Action. We did coding for all three PSTs and on five common MOs (see Table 2 for the results).

Betty and Pair B

Betty's students' answer to the given problem is given in Figure 2. They thought a number between 1/2 and 1/3 could be found if the numbers are rewritten as 2/2 and 3/3. A number in between would be less than 3/3 and more than 2/2, so it would be 2/3.

Betty started the conversation by summarizing what the problem was asking. She wanted to make sure that students were also viewing the situation as she was. Then, students' answer was not an answer she expected. She asked "how" they found the answer. We coded whether teacher noticed this Opportunity as Yes. The conversation continued with teacher's questions focused on understanding how students came up with the answer of 2/3. Students introduced "wholes" and used number line as a conveying representation of their ideas. When Betty did not agree with students' answer of 2/3, she did not say this directly but asked them to locate 1/2 and 1/3 on the same number line. This was a good move in terms of reorienting students' thinking to what was asked in the original question situation. But she moved to using fraction strips, i.e., a tangible linear manipulative with colored parts and unit fraction symbols written on them. Betty first asked a general question, such as how to use the fraction strips to transfer that knowledge into the number line. Then the focus of the conversation moved to comparing 1/2 and 1/3 with the colored fraction strips. Even though with questioning, students were able to say, "1/2" is bigger than "1/3" they had hard time to understand what fraction might be in between those two numbers when fraction strips were used. Eventually, since the students were so immersed in the context, i.e., fraction strips, they were not able to look at all the other possibilities that included proper fractions such as 5/12 or 2/5. They were thinking that it should be a unit fraction that was overly written on the fraction strips. Betty indicated that she was surprised: "Now you say there is no fraction. But you were saying there was 2/3 before..." Betty's purpose of questioning evolved depending on what kinds of answers she received from the students. We coded Opportunity in Action as Level 2 since PST asked probing and prompting questions. There was a progressive conversation, which might include students' contribution, but it was mostly teacher-dominated conversation with her mathematical goals in mind.

FINDINGS

In the initial round of coding, two researchers (authors) checked all the videotapes that three PSTs recorded and identified different number of opportunities for each PST. We identified 9 opportunities for Anna and pair A (mid-level achieving), 10 opportunities for Betty and Pair B (mid-level achieving), and 15 opportunities for Carol and Pair C (high-level achiev-

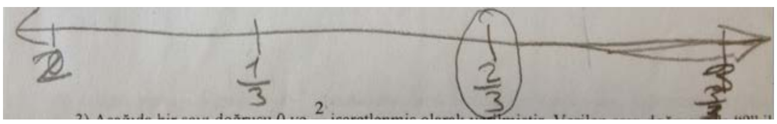

Figure 2: Betty's students' work

Pre-service Teacher	Description of the Opportunities	Opportunity	Opportunity in Action
Anna	Opportunity 1: Showing one fourth of an equilateral triangle Opportunity 2: Ordering unit fractions on a number line Opportunity 3: Placing a fraction in between two unit fractions (e.g., 1/2 and 1/3) Opportunity 4: Sharing an unknown amount, fraction multiplication and comparison Opportunity 5: Fraction division with manipulative and transferring to paper	Y: 5	Level 2: 5
Betty		Y: 5	Level 2: 4 Level 3: 1
Carol		Y: 3 N: 2	N/A: 1 Level 1: 2 Level 2: 2
Total		**Y: 13** **N: 2**	**N/A: 1** **Level 1: 2** **Level 2: 11** **Level 3: 1**

Table 2: Summary of the frequencies of the coding items

ing). This different number depended on the nature of the interactions PSTs and the pair of 6th graders had about the specific fraction problem. In the second round of coding, we overviewed all of the occasions and we identified 5 opportunities common for all the three PSTs (See Table 2). Based on the coding scheme we coded the PSTs interactions with students separately and achieved 0.88 consistency initially. Then we discussed the different coding and reached full consistency in coding.

With this experience, none of the three PSTs did "show and tell". They developed better questioning skills (Sleep & Boerst, 2012). Although, this was the case, as stated in other studies (e.g., Morris, Hiebert, & Spitzer, 2009; van Dooren, Verschaffel, & Onghena, 2002), one PST's lack of content and pedagogical content knowledge hindered to catch all MOs and address them effectively (see Carol's case, in Table 2). For Anna, while her interactions in defined MOs indicated as Level 2, her communication skills in Turkish deterred her from finding ways to use her content knowledge. This situation was directly related to her developing pedagogical knowledge for teaching fractions. Even though, PSTs listen to the children to figure out students' mathematical thinking, they need to further develop appropriate scaffolding activities. For instance, working with high-level achiever students who can argue their thinking more firmly might benefit PSTs growth more (see Carol's case). In addition, PSTs' own preparation, thinking ahead the necessary mathematical connections and planning towards some mathematical goal can improve their scaffolding actions (see Betty's case). Eventually, this might result in advancement of the conversations and 6th grade students' mathematical gains. In the working group, we will provide further evidences related to these claims and seek contributions from the group members.

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Facebook and mathematics teachers' professional development: Informing our community

Jorryt van Bommel[1] and Yvonne Liljekvist[1,2]

1 Karlstad University, Karlstad, Sweden

2 Uppsala University, Uppsala, Sweden

Nowadays, communication and cooperation on social network sites, such as Facebook, have become common. These kinds of sites are also used within teachers' professional development, both in formal and informal ways, as they create and form new opportunities to communicate and cooperate. In this paper, our aim is to discuss how mathematics teachers' informal participation in social network sites can inform the mathematics education research community.

Keywords: Professional development, social network sites, Facebook, mathematics teacher.

INTRODUCTION

It is well known that the present evolution of social media and social network sites transforms how people communicate, interact, and work together. Teachers use these different forums, such as: web sites, personal blogs, twitter or Facebook as resources in terms of networking, give and take advice and lesson plans, etc., for mutual benefit in their professional development (Manca & Ranieri, 2014). This is an informal professional development initiated and formed by the teachers themselves (Bissessar, 2014; Liljekvist, 2014).

Each forum serves different purposes: websites are often one-way communication and serve to inform others and share ideas[1]. Such ideas can be lesson plans; hence websites where teachers upload their own planning documents to be shared with other teachers occur (e.g., www.lektion.se). There are websites initiated by a municipality (e.g., www.pedagogvarmland.se), where active teachers, for instance, comment on pedagogical debates, or notify on-going developmental work. Furthermore there are teachers' personal websites where they gather materials of interest, such as lesson plans, but even research articles or links to other interesting web sites. (e.g., mattefroken.wordpress.com). Another form is the blog; some teachers have started blogs merely focused on their personal reflection in relation to their profession. 'Blogging' can be incorporated within a teaching position; for instance, one of the head teachers in a municipality can be responsible to share pedagogical ideas, etc. through a blog (e.g., http://pedagogstockholmblogg.se/vara-bloggar/).

All of these media are mostly a one-way communication – a monolog. However, other media offer the possibility for (instant) dialogue. The micro blog Twitter is one such way, where comments on events and actualities are given and shared with followers. Another dialogic resource is social community sites, such as Facebook. Teachers in Sweden have started to use this resource to a great extent in the past year. Facebook offers the opportunity to comment and to share, like twitter and other websites, but it also offers the 'members' to ask questions and to get response from other members. Thus the members themselves activate pedagogical discussions on mathematics teaching and learning. This differs from how more monologue Internet resources work. In the next section a brief overview of the situation in Sweden will be given – regarding the use of Facebook by teachers.

1 The examples given in this paper are taken from a Swedish context. However, the phenomenon of Social network sites is not limited to merely Sweden, therefore the examples can be generalized.

Teachers' professional development

In a review of research on teachers learning from teachers, White, Jaworski, Agudelo-Valderrama and Goya (2013) find that there is a complexity of settings in which teachers learn. The complexity is "influenced by both global and local forces, such as the recent pressure on teachers to meet different demands imposed on them [...] directly by politicians and national laws" (p. 421). There is interdependence between the institutional context and the teachers themselves as learners. Thus: even if practicing teachers need to change their teaching because of external reforms, it can be difficult to integrate reform practices due to institutional and social expectations (Nickerson, 2008).

White and colleagues (2013) show that the relationships between the support given (e.g., by expert/ experienced teachers, teacher educators, research/ researchers) and the supportee (e.g., the individual teacher, a group of teachers) can be of different kinds, such as teacher educators as guides, teachers and researchers working together, or teachers working together to design their own developmental activities. White and colleagues point out the similarities and differences between the knowledge that teachers and teacher educators/researchers respectively bring to the learning interface. They state: "neither group had all the knowledge that was needed for the development of teaching, but working together they could become a unified, powerful developmental force" (p. 422). Mutual respect and collaboration that allow the input of critical elements of knowledge, by teacher educators, colleagues etc., are found to be valuable to developmental practice and hence deepening their pedagogical content knowledge. In a review of teacher education Leder (2008) put forward the core factors of 'community building' and 'networking' as a means for in-service teachers' out-of-class meetings. She continues:

> Providing time out of class for meetings and involving academics to supplement the expertise within the group, for example, to add a stronger research dimension to the network's activities, have also been found to be facilitative, if not critical. (p. 361)

Nevertheless, White and colleagues (2013) stress that teachers' knowledge is pre-eminent in the in-school situations, and researchers and teacher educators have "much to learn about issues that influence what can happen in schools, and what is needed to put research-based knowledge into practice" (p. 422). As social network sites can be seen as such out-of-class meetings, a systematic study of social network sites will give the opportunity to look at the factors and issues of importance.

Social network sites

Bechmann and Lomborg (2012) describe three characteristics that define social network sites: 1) the communication is de-institutionalised, as each user has the ability to contribute, filter and share content, 2) the user is thus seen as both producer and participant, and 3) the shifting roles and communicative practice of the users can be described as interactive and networked.

Social network sites are widely used in Sweden; for instance, 50% of the inhabitants actively use Facebook every day (Findahl, 2013), although boyd[2] and Ellison in 2007 conclude that "...[social network sites] are primarily organised around people, not interests... structured as personal (or 'egocentric') networks with the individual in centre" (2007, p. 219). However, the rapid evolution of the Internet allows us now to see groups formed around a theme, and it is noteworthy that special groups for the issues of teaching and learning have been created on Facebook (see e.g., Bissessar, 2014; Ranieri, Manca, & Fini, 2012; Rutherford, 2010). Facebook offers the opportunity to comment and to share, as with Twitter and other websites, but it also offers the 'members' to ask questions and to get responses from others (henceforth: *posts and comments*). Thus the teachers themselves can activate pedagogical discussions of teaching and learning. This differs from how more monologue Internet resources work, such as blogs or web sites. Rutherford (2010) concludes that "Facebook provides teachers with an opportunity to engage in informal professional development that is participant driven, practical, collaborative" (p. 60).

Social network sites can be viewed as emerging communities of practice (Goodyear, Casey, & Kirk, 2014; Gunawardena et al., 2009). For example, some of the Facebook groups are specialised for mathematics teachers (e.g., "Mathematics course 2b for upper secondary school"; "Mathematics for lower primary school"). Other groups are gathered around more gen-

2 Note: danah boyd spells her name with lowercase letters.

eral themes in education (e.g., "The big five"; "Ipads in school"). Narrow thematic themes can have very few members, such as "Mathematics for course 2b in upper secondary school" with only about 60 members. Not surprisingly, more generic themes attract more teachers; hence the group "Mathematics for lower primary school" has 4 500 members. Generic themes that are interesting to all Swedish teachers can consist of groups of up to 20 000 members. Findings from various educational settings reveal how social network sites used as professional resources are not an isolated phenomenon and that the impact of social network sites on professional development varies (see e.g., Bissessar, 2014; Borbra & Llinares, 2012; Liljekvist, 2014; Manca & Ranieri, 2014; Pepin, Gueudet, & Trouche, 2013; Rutherford, 2010). However, Leder (2008) problematizes the tension between recognising results from particular studies and the wish to generalise the findings to broader settings. She calls for studies on whether, and when, interventions from professional development programs remain more permanently. She reflects on how the possibilities and the accessibilities of ICT are "opening new pathways for professional development with their own – often yet not fully realized – fresh strengths and challenges" (p. 368).

Some studies have been made of Facebook groups and teachers' professional development. One study investigated five Italian Facebook groups and focused on the motivation, activity level and outcome (Ranieri et al., 2012). This study gives valuable information about and insight into teachers' concerns and teachers' behaviour in the groups. Bissessar (2014) sees in her study that the teachers address issues on curriculum, and didactical and pedagogical concerns, but we need more studies to examine the extent to which critical discussions occur.

Aim of the paper

In this paper, we want to introduce a discussion in order to problematize some aspects when conducting a study in social network sites (in this paper: Facebook), and discuss how these kinds of studies can inform the TWG community on, for instance, how it influence teachers' practice. Webster-Wright (2009) points out in her review of research informing professional development practice that

> There is a need for more research beyond the 'development of professionals' that investigate the 'experience of PL' [professional learning] as constructed and embedded in authentic professional practice. (pp. 712–713)

She calls for research understanding more about the experience of professional learning – to support it more effectively; rather than just developing professional development programs. In teachers' social network sites, such as Facebook groups, we might find such an environment. In the following pages, we will outline some possible directions for our coming research on social network sites. We propose, with departure in our pilot study and literature review, three possible foci for further research into this new phenomenon: 1) mapping the arena, 2) inquiry into the collective knowledge created, and 3) consider the social network sites as extended working place learning. Our aim is to converge the discussion on the question: In what way can mathematics teachers' use of, for instance, Facebook inform the mathematics education research community?

SUGGESTION 1: MAPPING THE ARENA.

Some attempts have been made to map the arena of Facebook groups and professional development. One such study investigated five Italian Facebook groups and focussed on the motivation, activity level and outcome (Ranieri, Manca, & Fini, 2012). Such a map might be of interest concerning the Swedish Facebook groups as well giving valuable information and insights in teachers concerns and teachers behaviour within the groups. Two aspects are suggested to map: facts and professional development issues.

Facts

Concerning the facts: Statistics can be obtained on the fluctuation of the number of members in the groups. A relevant question to look at would be when people become a member, if that is related to specific timeslots in the year, or special events (e.g., yearly events, or more specific as launching new curricular goals). Statistics can also reveal when people are most active, what time of day for instance. This could shed a light on in what way Facebook is experienced as a formal or informal way of professionalization. One could argue for the idea that teachers, who are active on Facebook during working hours, look upon this phenomenon as a formal way of professionalization. Teachers who only are active during after-working-hours might look upon these Facebook groups as informal ways of professionalization.

A questionnaire posted on the site would give some first indications on the motives for joining the Facebook group, but also expectations of what teachers expect to gain from participation in the specific Facebook group. Members in this group could be contacted to conduct interviews in order to gain deeper insights in their motives and expectations of membership within the group. A strategic choice of members should be made: for instance new members, active members in responding to posts, or active members in posting a status, etc.

Questions to be asked will be of the kind: 'why did you chose to join the group', 'when do you decide to post a question within this group', 'what kind of questions do you post', 'what kind of questions do you respond to?', 'what kind of topics do you discuss' etc.

Professional development "issues"

A pilot study revealed that different types of questions appear in the groups at different time during the year. For instance, at the beginning of the school year a common post would be to ask for help with a good starting exercise for the beginning of the semester. Just before the summer holiday, a lot of teachers posted questions about textbooks – often explained by the motive that they were about to change textbooks and needed advice on qualities in different items. Prior to public holidays, questions were asked about suitable exercises for 'Easter', or 'Christmas', etc. Hence, the teachers' pedagogical considerations, as they differ over the year, can be studied more in detail. This informs teacher educators and professional development programs in terms of when to address specific topics or support teachers.

Furthermore, our pilot study revealed that different groups address different types of questions. For instance, one of the Facebook groups has a focus on discussing relevant and interesting research papers. All posts in that group concern either the choice/argumentation for a specific paper, or the content of the chosen paper. This indicates teachers' diverse conditions (e.g., depending on school level, topic, etc.) and what kind of support specific groups of teachers need.

As the description of TWG18 says, research has focussed on "topics like reflection, collaboration, or teachers' professional growth. In particular, models and programmes of professional development, as well as their respective contents, methods, and impacts were described and analysed" (Call TWG18, CERME9). With Facebook being another kind of arena: teachers' social network sites, to 'map' this new phenomenon inform the community of professional development of mathematics teachers on new reflections, collaboration and professional growth.

Mapping the arena is of importance as the discussion at CERME9 revealed that Swedish Facebook groups might differ from Facebook groups in other countries concerning the norms within a group. As we described a positive open non-anonymous atmosphere in the groups, other participants of TW18 at CERME9 described similar groups in other countries as hostile, with anonymous members, and a focus on complains.

Nevertheless, however interesting mapping the arena might be, more information is 'hidden' in the Facebook groups that could inform the community of professional development of mathematics teachers. Besides mapping the arena, two more suggestions are of interest. We don't suggest that further research consist of merely one of our suggestions, nor that is the only possible ways to conduct research. Mapping the arena probably is a prerequisite for the following two suggestions, starting with an inquiry of the collective knowledge in Facebook groups.

SUGGESTION 2: INQUIRY INTO COLLECTIVE KNOWLEDGE

Some studies have looked at single posts within specific Facebook groups. Rutherford (2010), for instance, has looked at one Canadian Facebook group, and categorized each posts in this group according to Shulmans' categorization of teachers knowledge. To continue on such categorization, we suggest going further than the individual posts, and hence look upon each Facebook group as a whole to make an inquiry in the collective knowledge. Facebook can be looked upon one of the emerging communities of practice (Goodyear, Casey, & Kirk, 2014; Gunawardena et al., 2009).

To do so, one could categorize all posts in a similar way as Rutherford did, but with the difference not to distinct each post, but to look upon the results as an indicator for the collective knowledge of the community of teachers in the specific Facebook group. Previous studies have looked at discussions where they have taken out single posts, meaning: if a question was posted and nobody responded, that question was not taken into consideration for data analysis (Rutherford, 2010). However, once teachers can formulate their questions, they have reflected upon their own teaching knowledge. Therefore, we argue, that both the questions posted and the reactions from the group show us the collective knowledge, expressed through the members' communication– including single posts. Again it is of importance to point out that the norms in the groups influence such collective knowledge and this might be different in different cultural settings as expressed by the TW18 at CERME9.

In the previous example, Shulman's categorisation was used to analyse the data. Different frameworks have described the knowledge needed for teaching mathematics (cf. Ball, Phelps, & Thames, 2008; Huckstep, Rowland, & Thwaites, 2003; Niss, 2004; Shulman, 1987). All of these have their own specifics (See Kaarstein, 2014, for an extended comparison of three of such frameworks) and hence, one of these could serve as a framework for analysis for posts within each Facebook group. Yet another option could be to analyse the mathematical content of the posts and focus on the mathematical topics addressed, including possible references to competencies (Niss, 2003) or proficiencies (National Research Counsil, 2001). The Swedish curriculum has changed recently and the pupils' possibilities to develop five competencies are the overarching principles in the current curriculum (Swedish National Agency for Education, 2011). Within Facebook groups for mathematics teachers at primary school, questions concerning the five competencies arise frequently. There is also one generic Facebook group 'The big five', aiming at teachers of all subjects.

This second suggestion, we believe, would inform the community of professional development of mathematics teachers via an explanation of the collective knowledge made in the groups, concerning the mathematical content as well as the knowledge needed for teaching mathematics.

SUGGESTION 3: EXTENDED WORKPLACE LEARNING

In the previous suggestions the attention has been on activity and on what kind of knowledge created in the Facebook groups of mathematics teachers. However, it can also be worthwhile to investigate how this knowledge is constituted. Social media and social network sites are, as we all know, used in professional development programs (for instance, 'Mathematical Boost', matematiklyftet.skolverket.se). In this paper, we want to discuss social network sides as an informal part of mathematics teachers' professional development; we suggest approaching social network sites as an arena for extended workplace learning. Professional development programs are culturally bounded and the Mathematical Boost is one such program that locally can influence in what way teachers look upon their extended workplace. Comments from TW18 at CERME9 implied that a detailed description of such programs is necessary in order to be able to clarify cause and effect.

Borko (2004) points out that teachers' discussions on work-related issues tend to be on a surface level, such as discussing ideas or materials; and that it takes support to foster critical discussions on teaching. She states that teachers need to "collectively explore ways of improving their teaching and support one another" (Borko, 2004, p. 7) in order to develop their teaching. Thus communication norms enabling a critical dialogue need to be established and maintained. In a study of Facebook groups it would be possible to study such communication practices, and examine to what extent these kinds of communication norms occur. Moreover, as we know the development of teacher communities is difficult and time-consuming (see e.g., Grossman, Wineburg, & Woolworth, 2001) are Facebook groups a way for teachers to foster such discussions – despite its instantaneous format and more or less loose gathered groups?

For instance, Bissessar (2014) sees in her study that the teachers address issues on curriculum, didactical and pedagogical concerns, but we need more studies to examine to what extent critical discussions occur. That is, does the informal arena of social network site nurture workplace learning? Can we see communication patterns changing due to changes in, for instance, curriculum, or due to impact of formal professional development programs? One way of looking at this informal arena would be via inquiry as a developmental tool in a community of practice, as described in Goodchild (2014).

Webster-Wright (2009) thinks it is a lack in research designs in professional development programs. She states that it is necessary for us to learn from teachers' authentic learning situations: "To gain further insights to enhance support for professionals as they learn, there is a need to understand more about how professionals continue learning through their working lives" (Webster-Wright, 2009, p. 404). We believe that studying; for instance, Facebook groups can help our community understand more, since it is a situated digital, hence extended workplace for the teachers.

ETHICAL CONSIDERATIONS

Although it is not the main focus of the discussion of TWG18, we need to address some aspects of ethical concern when conducting research on social network sites. Normally in a classroom the observer might influence the practice, and now a new question arise: in what way does the presence of researching members in groups influence the group?

This is not merely a methodological question; the blurred distinction between the private and the public in social network sites must also be considered (Bryman, 2008).

SUMMARY

In this paper, we have introduced a discussion continued at CERME9 in the Thematic Working Group 18 in order to problematize some aspects when conducting a study in social network sites. We proposed three possible foci for further research into this new phenomenon: 1) mapping the arena, 2) inquiry into the collective knowledge created, and 3) consider the social network sites as extended working place learning. Our aim was to converge the discussion on the question: In what way can mathematics teachers' use of, for instance, Facebook inform the mathematics education research community concerning mathematics teachers' professional development? Regarding issues of teaching and learning mathematics in social network sites, the uniqueness of the Swedish Facebook groups should be taken into account where an open, positive climate describes the conversations in these groups.

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How future teachers improve epistemic quality of their own mathematical practices

Yuly Vanegas[1], Vicenç Font[2] and Joaquin Giménez[2]

1 Autonoma Barcelona University, Faculty of Education, Barcelona, Spain, yulymarsela.vanegas@uab.cat

2 Barcelona University, Faculty of Education, Barcelona, Spain, vfont@ub.edu, quimgimenez@ub.edu

In this paper, we present how future mathematic teachers improve their self-reflection in didactical analysis competency. We draw on data collected from two groups of prospective teachers, using qualitative methods. We use a prospective teacher as an example to discuss how the training on the use of didactical suitability criteria and reflective tools lead them to learn about the experience, and explain in a deep way how they will change their own practices in the future.

Keywords: Mathematical quality, future teachers, professional development, didactical analysis.

INTRODUCTION

In this paper, we analyze how a specific pre-service mathematics teachers' training program produces changes in terms of future secondary school teachers' competence of didactical analysis. In particular, aiming at the growing and building mathematical knowledge for teaching (Zaslavsky & Sullivan, 2011) by using theoretical tools for self-analysis. Our general intention in such a program is to lead future teachers to develop the professional ability of reflecting about changing their own planning by using didactical analysis theoretical tools. In this paper, we focus on explaining how future teachers value the need of analyzing mathematical quality in their final reflections when they tell us how and why they will improve their school practice for the next future. The research relates to the dilemmas of task redesign that provides a framework that can be used for analysis of quality and suitability of tasks (Giménez, Font, & Vanegas, 2013). Our aim in this presentation is *to describe how a specific set of professional tasks, promotes the emergence of using theoretical tools from future teacher's reflections, when valuing epistemic/mathematical suitability of an instructive process in a multidimensional way.*

In our study we call 'professional task' those task s that we propose to future teachers in order to encourage them doing didactic analysis and developing their didactical analysis competences. We want to focus on some immediate effects of the task redesign done by the future teacher. In previous researches (Vanegas et al., 2014), it was found it when analyzing prospective teachers' thoughts emerging from their feedback [work assignments] with the researchers; and also emerging from our analysis of some impacts of the program itself. Such above mentioned development, it is stated when future teachers incorporate and use tools for the description, explanation and process valuation of mathematical school teacher/learning practices. Thus, our main hypothesis is that future teachers can reflect and value their own practices in a deep way, and to have ideas to structure a possible redesign that improves the quality of their own mathematics and teaching for the future.

THEORETICAL FRAMEWORK

In our research, we assume a learning through teaching perspective between the design of classroom tasks, the pedagogies associated with the effective implementation of tasks and the learning of mathematics (Leikin, 2006) and task design problematics. The role of the Mathematic teacher in such a framework is to select, modify, design, redesign sequences, implement and evaluate mathematical practices. The analysis and description of the mathematical activity is conducted using the theoretical constructs proposed by the 'Ontosemiotic' approach (OSA). According to this perspective (Godino, Batanero, & Font, 2007), the mathematical activity plays a central role and it is modelled in terms of systems of operative and discursive practices. From these practices the different types of related mathematical objects emerge building cognitive or epistemic configurations among them. Problem situations promote and contextualize the activity; languages (symbols, notations, and graphics) represent the other entities and serve as tools for action; arguments justify the procedures and propositions that relate the concepts. Lastly, the objects that appear in mathematical practices and those which emerge from these practices might be considered from the five facets of dual dimensions (Godino, Batanero, & Font, 2007). Both the dualities and objects can be analyzed from a process-product perspective, a kind of analysis that leads us to the processes shown.

In fact, there are two possible perspectives to analyze the quality of a mathematical practice. On the one hand, international experiences focusing on rigor and mathematical richness (Hill et al., 2008). They introduced a set of categories to measure the mathematical quality of an instructive process (Hill, 2010): (1) well done format, (2) work done in a connected way to mathematics, (3) richness of the mathematics itself, (e) work done with the students (4) errors and imprecise language used, (5) students' participation. It's also explained that for mathematical richness, we can see some indicators as: mathematics explanations, multiple procedures or solving methods, mathematical generalization development, use of a mathematical language, and general mathematical aspects of richness as representations, among others. On the other hand, by considering OSA we interpret that it is necessary to see the theoretical tools not as a priori solution, but an emerging process from the practice itself. We consider that it relates to a way of constructing quality criteria by the future teachers during the professional training process. Hypothetical trajectories should reveal not only the acquisition of theoretical tools for analysis, but also to see how personal didactical principles are assumed by future teachers in an open and personal way. The notion of epistemic quality proposed by OSA is centered on representativeness of mathematics taught by the future teachers in a lifelong learning perspective according a holistic meaning of mathematical objects to be learnt, understood as a pair (mathematical practices, primary objects and processes activated in such practices). The determination of such a global meaning or holistic meaning requires an epistemological/historical study about the origin and evolution of the mathematical object. We must take also into account the diversity of "using contexts" in which it's possible to play each of the possible configurations of these primary objects. There is a consensus on considering what epistemic suitability mean: Representative and articulated sample of diverse types of problems (contextualized, according different levels of difficulties, etc.); use of different modes of expression (verbal, graphic, symbolic...), also with translations among representations. In such a way we could consider an adequate level of mathematical language; definitions and procedures should be correctly stated and adapted to the level of the students; presentation of basic statements of the "topic" in order to, establishing relations and meaningful connections among definitions, properties, problems and so on. The complexity of perspectives about mathematical objects from OSA perspective, introduces the idea that a mathematical object is not simple but a complex system. The idea of representativeness as quality criteria is related to coherence and connectness aspects.

In a previous research, it was observed that epistemic suitability criteria and epistemic configurations, could be useful tool to organize self- reflection, but those tools were not used by future teachers in a pre-implementation as planning phase (Giménez et al., 2012). Now, our research aim is to confirm if the future teachers could emerge the suitability criteria from their final analysis. We also want to see how they use it, by describing a future teacher case study.

METHODOLOGY

According to the proposed aim for this study: a) a set of professional tasks had been designed; b) these tasks are implemented in the training program; c) the written productions of future teachers are considered as essential data and d) a student production is analyzed as a case study.

We assume a learning by teaching approach based on an inquiry and reflective practicing in which it was observed the tensions presented for task designers and teacher adaptation whether they are designing tasks for themselves or for others. Framework in which we design and implement diverse teacher training cycles as teaching experiments for developing didactical analysis competences among others. In particular, we discuss some effects of a teacher training cycle named "Epistemic Analysis" (as a part of a more general cycle of didactic analysis). The development of this cycle includes four main professional tasks: (a) First naïf analysis of a teaching episode about proportional reasoning, and refined analysis using suitability criteria (b) Looking for errors, ambiguities, and construction of different meanings by analyzing three episodes about perpendicular bisector concept; (c) Analysis of connectness and representativeness; (d) The need of improving richness of processes, by analyzing episodes of introducing integral concept. After that, the future teachers designed and implemented a lesson into their period of internship. They did a first reflection and valuation of the suitability from their didactic implemented unit. Finally, we ask for an improvement and re-planning of their lessons designs (for future implementation), within the Master's Final Project (MFP).

As a general schematic framework, we see in Figure 1 the developmental process.

During the cycle we present theoretical tools (suitability criteria, according Godino, Batanero, & Font, 2007) to conduct evaluative analysis to answer "what could we improve from our self-analysis?" We understand that the study of descriptive and explanatory analysis for a didactical situation is necessary to justify the evaluations. As the aim is to give tools for reflection, we use methodology based on case study design based research (Gravemeijer, 1998): the use of a real environment, to generate new and efficacy learning environments, collaborative enquiry perspective among trainer-researcher and future teachers, searching simultaneously construction of theories and practice innovation (Cobb, di Sessa, Lehrer, & Schauble, 2003).

The study was conducted by two groups of 32 future Secondary School teachers of Mathematics from the Interuniversity Program in Catalonia. For the results, we analyze the final work of a single student in terms of his use of tools. The data were registers on virtual platform, audiovisual comments about classroom episodes, final Master's work valuing mathematical quality about their implemented own practice. Now, we explain each of the analysis through the four main activities introduced.

(a) Task 1. Noticing mathematical objects and processes in a math task

A first task had been introduced to show how constructs emerge from a school practice as cognitive and semiotic conflicts, epistemic obstacles, types of norms, interactive, patterns of models of management, etc. It was selected an initial task in which students confront a short case study about proportional reasoning, using transcripts of a classroom situation. Such an initial *tasks* introduce the students for reading and analyzing the classroom example, by using their previous knowledge and beliefs of didactic analysis. During the first task, future teachers did naive comments about a proportion class talking about the comparison of two quarters with different density of population. It's easy for them to identify mathemat-

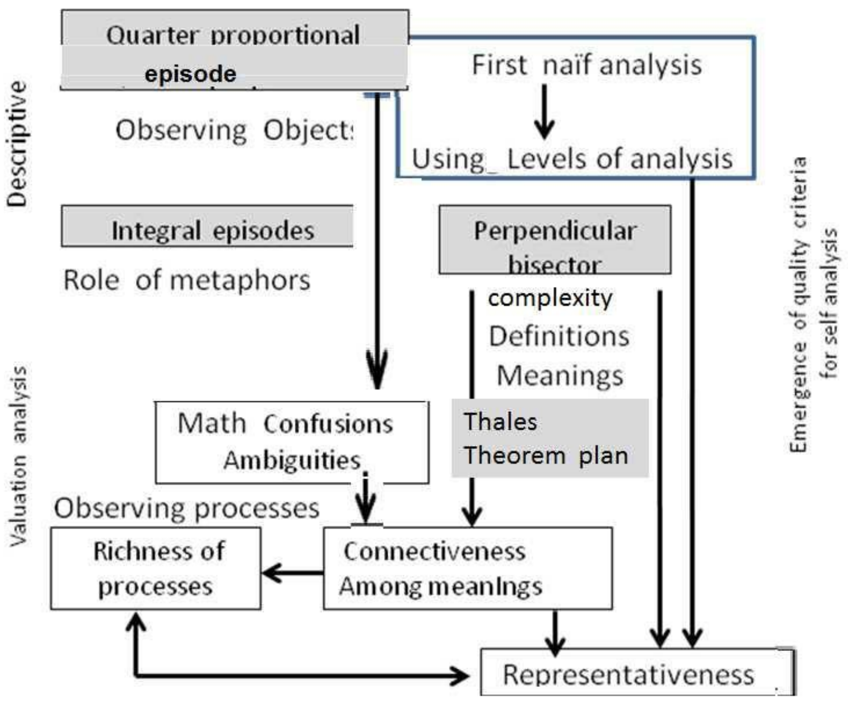

Figure 1: Scheme of different constructs involved in epistemic cycle

ical objects but it's difficult for them to recognize all the processes involved in the task.

Refining the analysis by using theoretical tools

It was analyzed a class about equations by applying structured suitability criteria. The students start by analyzing mathematical practices, by observing objects and processes. Then the trainer develops an example in which it was revised suitability construct. After that the future teachers reflect, improve and refined their analysis by using the notion of epistemic suitability. In such professional task, it's still difficult for the future teachers to identify some semiotic conflicts. Later, in the different subjects of the Master, the students realize other analysis of practices, objects activated in the above mentioned professional practices (problem, definition, proposition, representation and argument) and mathematical processes. Observing such initial analysis realized by the future teachers, we expect to improve some difficulties to distinguish between concepts and definitions. It's also found that during second analysis, the future teachers recognize their duplicities between definitions, propositions and procedures, and also their duplicities between propositions and thesis of arguments. It's still difficult for them to have a good description of practices because they overlap the configuration of objects and the description of processes. It's also difficult to see and to catalogue mathematical processes. The reflective process in this training period, try to add historical values of mathematical objects, relations among concepts, and to analyze how the use of resources can change the construction of different meanings. Therefore, it is expected to see mathematical structure behind the mathematical practice.

(b) Task 2. Analyzing the role of mathematical errors and ambiguities

Initially, the first proportion professional task was enough to see that if you did or introduced misconceptions or errors you will promote semiotic conflicts. It's common sense of quality the idea of coherence when using mathematical errors. But another task was introduced to review such idea about perpendicular bisector.

Analyzing the role of definitions and meanings

In such epistemic analysis, we notice that it's needed an adequate contextualization, in order to delete abuses, to focus on mathematical purposes, and to promote transference of knowledge. As a prototyping professional task to promote such analysis, it was proposed the observation of three short ways of introducing perpendicular bisector with 12–13 years old students, by observing three different teachers. The main idea to present a discussion about the different practices, objects and mathematics processes and to introduce a reflection associated to how each of these episodes contribute to introduce different kind of epistemic configurations and objects associated to three different definitions. First practice introduces perpendicular by the middle point of the segment. Second teacher use a definition as right-lane formed by the points being equal distance from the end points of the segment. Third teacher uses the idea of border of two regions determined by the closest neighbour principle) It was observed that both first and second teachers did classical proposals and management about the content and the classroom. The third teacher proposal is innovative not only because of the management but mathematically as a way of changing the regular use of mathematical content as a change of configuration of practices, objects and mathematical processes by using a non-routine task. Future teachers observed interpretation processes, communication of didactical and mathematical meanings, etc. Furthermore it appears a reflection about distinguishing complex processes from simple processes and also a general reflection about the idea of processes itself.

(c) Task 3. Analysis of connectness and representativeness of knowledge

To characterize epistemic quality, we analyze how representative and articulated set of activities/problems (contextualized, different levels of difficulty, different modes of representations and translations, etc.) had been proposed. It as a way to see the need of clarification of questioning and exemplifying, because in such situations we can see how different meanings appear, how we establish relations and connections among definitions, properties, and problems.

A prototypical example of this task analysis is a case based analysis upon a previous future teacher reflection that: first planned a sequence with 7th grade (13–14 years old students) for Thales theorem. She analyzed her own practice about Thales Theorem after the school practice, but she didn't notice a good mathematical connection among different activities. The trainer introduces epistemic configurations to see that some connections had been forgotten. The aim of this professional task is to recognize a deep level of analysis from such previous prospective teacher's practices (Choppin, 2011). Thus, the future teachers learn from this analysis, the need of connecting several epistemic configurations.

(e) Task 4. Improving richness of processes

It was decided to use the integral concept to start such a general analysis. The episodes presented come from a school experience in which a substitute teacher uses her common knowledge with a regular textbook to introduce integrals for 18 years old students. The trainer explains the idea of richness of processes, and talk with future teachers about the complexity of the integral concept by noticing seven meanings as geometric, result of a change process, inverse of a derivative, limit approach, generalized content, algebraic, and numerical method (Contreras, Ordóñez, & Wilhelmi, 2010). After that, it was questioned which processes they think it's possible to promote by using some problems. We considered if the situations activate some of the following processes: contextualization, algorithmisation, communication, argumentation and problem solving. In such analysis it's also discussed the relevance of some specific questions. After such task analysis, there was a consensus about the low level of accuracy in terms of promoting processes associated to different meanings.

RESULTS AND DISCUSSION

In order to reflect about how the future teachers introduce and interesting self-reflection, we exemplify some important unexpected acquisitions as a special case study to understand possible influences of the above professional tasks upon their epistemic valuation. The future teacher (N from now), was considered interesting for being analyzed because he is not mathematician, but economist, and he uses very carefully the theoretical framework above presented for his own reflection. Some teachers tell us that they didn't introduce mathematical errors. It is important to say that in previous years, they were not worried about it.

> I did an error introducing the notion of compatible undetermined system, by using an apparent good example of prices, but telling the students that there are infinite solutions. It was difficult to see, but I promoted an error, because the contextual situation is related to a finite set of answers. The children conclude that there are many solutions, but limited because of non existence of negative money. Nevertheless, I didn't say anything about the limited use of coins in Spanish currency. (N)

The same teacher explains explicitly how he will change a statement in a problem, because of the reflection.

The future teachers accept that they did ambiguities relating language problems that they identify as creating possible semiotic conflicts.

> I promote some statements, not enough clear as we see in the task … Finally, my tutor tell me about the need for clarifying why we used the expressions "It passes subtracting" "We delete denominators … and others". (N)

According formalism introduced, the future teacher assumes that he started with not necessary sentences. The teacher T relates such ambiguity to a theoretical article. For instance, he said about "*the need of searching analogies found because of an incorrect use of contextual framework*". He read a text from Reed to reflect about the use of two important variables influencing the decisions of the teacher.

> "The context understood as a set of traits perceive d in a certain problematic of real world involving objects, and facts"… But, the laws, principles, relations among quantities, and equations, constitute the structure of a problem". It is interesting that the future teacher explain some conclusions from this discussion: "the need to describe the similarities and differences among structures and surfaces of the source problems and aiming problems, because it influences the decisions about the equations presented to solve the aiming problems. It is also important to identify that familiarity can help the transference processes, but it also could be an obstacle to see the similarities and structural differences among problems". (N)

About richness of processes

The future teacher expressed the need to incorporate problem solving from the perspective of Polya, which had not been considered in his planning. He also explains the need to articulate the role of letters and unknowns.

> It would be necessary to change the status of quantities designed by letters We must identify the global traits of mathematical competency, not only because of my experience, but the studies conducted in which it's found the serious difficulties to produce right equations. If it's correct, is because of the use of Cartesian methods in a flexible way, producing a diversity of equalities for the same problem... It's also important to identify different patterns when we build equalities. (N)

We also found in his work, much unexpected reflections about the need of modelling processes in an inquiry perspective. The future teacher N tells about "*the emphasis on algorithms instead of modelling that gives opportunities to observe similar structures in the same model...*" Quoting Chevallard "*presented three important steps for modelling in algebraic situations: To introduce letters to define variables of the system, that gives opportunities for generalization and increase mathematical power. To establish relations among variables and to work mathematically to establish new relations...*". The future teacher N, not only recognize the students' difficulties but also indicate that "*some researches explain the possibility to validate the model and learning from such a perspective ... de signing courses using graphic calculators*".

About representativeness

The future teacher tells us that it's needed not only a look for meanings, but to see a historical, epistemological and curricular perspective. Then proposed a set of ideas about the Arabic way of solving problems to be introduced next time. In this case, he just offer a reflection about "*considering algebra as part of cultural legacy*".

About connectiveness

N explains the need for applications to other disciplines.

> As an economist, I can say the use of systems of equations to find equilibrium points, as intersection of different conditions, interpreted by curves of offer and demand... and planning problems of dead points... programming problems...We also use algebra as a process to solve engineering problems...chemistry problems, restoring digital images. About intramathematical connections, algebraic systems of equations are referent knowledge for optimazation problems... We assume that some of these knowledge must be introduced and adapt according the age of the students.

CONCLUSIONS

The professional tasks proposed, promote that the future teachers did positive reflections about their own practices. In each set of tasks and analysis we could identify the emergency of the different aspects characterising the epistemic suitability. Some of these are in agreement to which were presented by Wilson, Cooney and Stinson (2005).

The different type of analysis done during the implemented Program, not only gave opportunities for establishing categories and structure for the reflection, but also permitted to explicit how to improve the didactical sequence as we have seen in the case study. We consider that such attitude for changing from reflective analysis is important for professional development.

ACKNOWLEDGEMENT

The work presented was realized in the framework of the Project EDU2012-32644; financial help of ARCE 2014 and financial help given from Generalitat de Catalunya to the research team GREAV 2014 SGR 485

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TWG18

Posters

Professional development of primary teachers during a lesson study in mathematics

Valérie Batteau

HEP Lausanne, Lausanne, Switzerland
University of Geneva, Geneva, Switzerland, valerie.batteau@hepl.ch

We study the evolution of primary teachers' practices in mathematics by analyzing the effects of a professional development training called lesson study (LS) in Lausanne, Switzerland. We answer the following question: what practices will change and what will resist change?

Keywords: Teachers' practices, lesson study, professional development.

LS is a field of research and professional development in Asia, in US and in Northern Europe (Lewis & Hurd, 2011; Yoshida & Jackson, 2011). LS is a collective and reflexive process that involves a group of teachers and coaches. LS has four steps to a cycle (Lewis & Hurd, 2011): the group studies a mathematic subject (step 1), prepares a lesson plan (step 2), one of the teachers conducts the lesson while others observe (step 3), and finally, the group analyses and may revise the lesson (step 4), with the option of teaching it again.

Teachers' practices are analysed using the theoretical framework: the double approach (Robert & Rogalski, 2002, 2005) based on a French didactical approach and an ergonomic approach (Leplat, 1997) based on activity theory. In the ergonomic approach, the main goal is to distinguish prescribed work (the prescribed task, or what the teacher must do, planned together during step 2) and real work (the conducted task, or what the teacher does in reality, enacted during step 3). To appropriate the prescribed task, the teacher should modify it. We study the teacher's activity as a process of modifications between tasks (Leplat, 1997; Mangiante, 2007). The prescribed task includes the mathematic task, the mathematical knowledge, the lesson plan and the planning material. Leplat (1997) adds two tasks: the represented task (how the teacher represents the prescribed task and what he thinks the group attends of him) and the redefined task (the teacher redefines his task according to the prescribed task and his own professional goals).

In this framework, teachers' practices are seen as a complex, coherent and stable system. Regularities are observed in teachers' practices during three important moments of teacher's activity (process of devolution, regulation and institutionalization (Brousseau, 1997)) and correspond to teachers' strategies and choices.

MEHODOLOGY AND DATA ANALYSIS

The prescribed task is analysed *a priori*, which means we study the mathematic knowledge at play in the task, the possible resolutions and the didactical variables.

From the conducted task, we analyse the proceedings and the students' offered activities. We set up indicators to describe the teachers' practices in order to categorise their practices in i-genre. We define the represented task and the redefined task from research data (particularly with collective sessions enacted during step 2 and step 4).

We transcribe all video data (lessons and collective sessions) and we analyse with indicators (in Nvivo) video data, written documents and students' productions during the LS cycle.

SAMPLE

The group consists of seven primary teachers ranging from experienced, voluntary and generalist teachers, and two coaches. The LS process occurs over two years with two collective sessions occurring per month.

FORMAT CHOSEN

We present a figure of the LS cycle with first analysis, indicators and results applied to teacher's practices by visual material. Next to visual material there are short sections: (1) Context of the research and theorical framework, (2) Research questions, (3) Methodology and (4) References.

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A professional development program in formative assessment for mathematics teachers – Which changes did the teachers do and why?

Erika Boström and Torulf Palm

Umeå University, Umeå Mathematics Education Research Centre, Umeå, Sweden, erika.bostrom@umu.se, torulf.palm@umu.se

This study is part of a larger research project about a comprehensive professional development program (PDP) in formative assessment for mathematics teachers. The aim of the study is to investigate in which ways the teachers' classroom practice change, with respect to formative assessment, after participating in the PDP, and which some of the reasons may be for these changes. Fourteen randomly chosen grade-seven mathematics teachers participated in the PDP. The teachers were interviewed and their classroom practices observed, before and after the PDP. They also answered two questionnaires about the PDP. Preliminary results show that all teachers were motivated to change and did change their practice, but to varying degrees.

Keywords: Professional development, formative assessment, mathematics education.

BACKGROUND

Several studies have demonstrated that the use of formative assessment in classroom practice is one of the most educationally effective ways of increasing student achievement (e.g., Black & Wiliam, 1998; Hattie, 2009). But Wiliam (2010) also highlights that little is known about how to effectively help teachers implement a formative classroom practice, and that designing ways of supporting teachers to developing their formative assessment practice is an important issue. This study is part of a larger project about a comprehensive professional development program (PDP) in formative assessment for mathematics teachers. The overall aim in the project is to contribute to the understanding of factors that are significant in the support of teachers' implementation of a formative assessment practice. In this specific study the aim is to investigate in which ways the teachers' classroom practice change, with respect to formative assessment, after participating in the PDP, and which some of the reasons may be for these changes. In the project, we use a conceptualization, suggested by Wiliam and Thompson (2008), of formative assessment as practice based on an adherence to the "big idea" of using evidence about student learning to adjust instruction to better meet student learning needs, and a competent use of the following five key strategies: (1) clarifying, sharing and understanding learning intentions and criteria for success, (2) engineering effective classroom discussions, questions, and tasks that elicit evidence of learning, (3) providing feedback that moves learners forward, (4) activating students as instructional resources for one and another, (5) activating students as the owners of their own learning.

METHOD

Fourteen randomly chosen grade-seven mathematics teachers participated in the PDP. The PDP was process-oriented, focused on the "big idea" and the five key strategies of formative assessment and had duration of 24 full days distributed over 4.5 months. The teachers were interviewed and their classroom practices were observed, before and after the PDP. They also answered two questionnaires about the PDP. The analysis of the teachers' changes in classroom practice was carried out using the framework of formative assessment described above.

PRELIMINARY RESULTS AND CONCLUSIONS

After the PDP all teachers changed their classroom practice but to varying degrees. The most common and frequent change was that the teachers more often elicited evidence of student learning with the purpose of adjusting their instruction. Another common change was that they used more effective activities to engage and create thinking among all students during whole-class sessions. They did so by using an "all-response system" and a system for random distribution of questions when asking questions in class. These changes were mainly connected to Key strategy 2. Small changes connected to the other key strategies were also made. After the PDP the teachers were motivated to change their classroom practice since they believed in the idea of formative assessment, had tried activities and experienced positive effects on students and also felt that they had sufficient knowledge to develop their formative assessment practice. Some reasons to why the teachers did not change their classroom practice more was that the teachers experienced that they needed more time and that old habits are hard to change. This study may contribute with knowledge that can be used when designing professional development programs in formative assessment in the future.

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Analysing student teachers' lesson plans: Mathematical and didactical organisations

Louise Meier Carlsen

The University of Copenhagen, Department of Science Education, Copenhagen, Denmark
The University of the Faroe Islands, Department of Education, Tórshavn, Faroe Islands, LouiseC@setur.fo

I investigate 17 mathematics student teachers' productions, in view of examining the synergy and interaction between their mathematical and didactical knowledge. The concrete data material consists in lesson plans elaborated for the final exam of a unit on "numbers, arithmetic and algebra". The anthropological theory of the didactic is used as a framework to analyse these components of practical and theoretical knowledge.

Keywords: Didactic divide, lesson study, mathematical and didactical organisations.

INTRODUCTION

Mathematics teacher education is often described as consisting of several components such as mathematical contents, didactics and pedagogy. Bergsten and Grevholm (2004, p. 125) introduced the term *didactic divide* to indicate a lack of connection between such components. In this paper, I wish to investigate the degrees of didactic divide (or coherence!) by analysing certain productions of Faroese mathematics student teachers for an exam in a module that seeks to integrate the teaching of mathematical contents and didactics (in the context of algebra and arithmetic). This is my first research project as part of my work towards the Ph.D.

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

In the anthropological theory of the didactic, human practice and knowledge is modelled in terms of *praxeologies*, which consist of a *practice block* and a *knowledge block*. The practice block consists of *types of tasks* and corresponding *techniques*. The knowledge block contains *technology* (discourse on techniques) and *theory* (unifying discourse on technology). For more detail see Barbé and colleagues (2005). In our study we consider *mathematical* and *didactical* praxeologies (defined by tasks being of mathematical and didactical nature) and their mutual connections. We can then formulate the research question of the study as follows: *What links and gaps can we observe in the mathematical and didactical praxeologies of students as evidenced in their plans for a lesson on school algebra, produced in the setting of a teacher education course?* Naturally, a methodological problem is closely linked to this question: *How can we, concretely, extract the components of mathematical and didactical praxeologies from a lesson plan?*

CONTEXT AND STUDY

The mathematics teacher education programme at the University of the Faroe Islands can be described as mainly consecutive (cf. Winsløw et al., 2007) in the sense that general pedagogy and other generic courses make up the first two years of the programme, while the last two years comprise units with a main focus on contents and a secondary, integrated focus on didactics. One of these units in the third year is the source of our data for investigating the above research question. Halfway through the course, the format of *lesson plan* is introduced (in connection to the wider idea of *lesson study*) as a way to work with a school mathematical theme.

The task (a didactical one) is "to plan a lesson on a specific theme" (e.g., "how to add two fractions?"). The theme in turn can be described as a mathematical praxeology, in this case focused on techniques for adding fractions, descriptions and justifications of the techniques (technology), and mathematical links to similar practices (including theoretical notions such as commutativity, inverses etc.). The lesson plan will potentially articulate both elements of this mathematical praxeology and also present choices of didactic techniques, and as these are explicit (written) there will be elements of didactic technology and theory to analyse. The students elaborate (individually) one such lesson plan which forms the basis of the final oral exam in the course. Their presentation of the plan may take up to 20 minutes followed by questions from the lecturer. While the main data will be the lesson plans themselves, transcripts from the exam will also be used to complement the information regarding students' mathematical and didactic theory blocks.

My analysis focuses on the students' choice of didactic techniques, the justification of these techniques in relation to the mathematical praxeology, the preciseness of their didactic technology, their reference to explicit didactic theory and the connection between didactic and mathematical theory blocks (for instance, didactic theory on the teaching of fractions could be more or less connected to the theoretical components of the mathematical praxeologies proposed by the lesson).

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Questioning as formative assessment and its quality measurement

Veronika Hubeňáková and Dušan Šveda

Pavol Jozef Šafárik University in Košice, Faculty of Science, Košice, Slovakia, veronika.hubenakova@student.upjs.sk, dusan.sveda@upjs.sk

This paper presents the tool for measuring the quality of questioning in mathematics classes. Its construction is based on video analysis and interviews with teachers. Actually, we identified five different categories within the constructs of questioning. These rubrics have potential to be used in research and in teacher education

Keywords: Questioning, formative assessment, quality measurement tool.

BACKGROUND

Education in Slovakia has recently overcome a lot of changes. They were forced by poor results of Slovak students in PISA measurement. Unfortunately, this reform movement seems to miss its goal. The last mathematics PISA measurement revealed significantly worse Slovak students´ results in mathematics when compared to previous measurements (NÚCEM, 2012). Influential reasons of this failure are unchanged teachers´ practice and beliefs while the content and the aims of mathematics education were modified.

THEORETICAL FRAMEWORK

We consider questioning to be the one of the most important teachers´ skills. As pinpointed by Aizikovitsh-Udi, Clarke and Star (2013), the questioning is not only an issue of good questions, further, there is a strong importance of the way the questions are asked, timing and number of times each question is asked. Webb (2004) claims that classroom discussions present an ideal opportunity to explore students´ understandings and to inform instructional decisions. In good questioning, a teacher asks questions – to gain information about students´ knowledge, misconceptions, etc. – provides feedback and/or asks the next question based on elicited evidence. Therefore, the questioning fulfils the definition of the formative assessment introduced by Black and Wiliam (1998).

METHOD

At the beginning, we have reviewed scientific literature on the topic of formative assessment and prepared theoretical rubrics to describe different quality levels of the questioning (and three other constructs, connected to formative assessment).

The next step was to videotape four high school teachers with positive beliefs about inquiry based teaching. We found out the importance of this in our previous research (Hubeňáková & Šveda, 2013). Each teacher could say which topics are good for her to be videotaped, thus there are four different topics recorded, one for each teacher. Now, we are at the beginning of their qualitative analyses. The teaching passages of each class are fully transcribed. We use the software NVivo10 for their precise analyses. The next step will be the discussion with the teachers to choose the rubrics that are useful for them.

RESULTS AND IMPLICATIONS

We have found five different categories within the construct of questioning up to now: if and when are procedural versus conceptual questions asked; who is asked procedural versus conceptual questions; who is questioned; who answers questions asked to whole classroom; teachers reply on wrong answer. Elaborated rubrics are available upon request via email.

The implications of this work are in the research and in the teacher professional development, as well. We will be able to quantify the differences in teachers´

questioning practice and this enable us to observe its influence on students´ results and beliefs about mathematics. Hopefully, such information provides us with a good way how to enhance teachers´ questioning.

ACKNOWLEDGEMENT

This work was supported by the Slovak Research and Development Agency under the contract No. APVV-0715-12 and by grant VEGA 1/1331/12.

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Motivating student teachers to engage with their own mathematics teaching and learning

Suela Kacerja and Beate Lode

Bergen University College, Bergen, Norway, suela.kacerja@hib.no

The focus of our study is on student teachers' engagement in and with their own learning during a teacher training for primary school program. Different ways of working at the university college are combined with students' practicum at primary school. The aim is to motivate students to actively participate in, and be critical towards, their preparation in a meta-level in discussing their experiences.

Keywords: Teacher education, learning loop, primary school, practicum.

THE LEARNING LOOP AND CRITICAL MATHEMATICS EDUCATION

The study involves second year students during their mandatory mathematics course of 30 ECT in a teacher training for primary school program (1st to 7th grade). Different methods of working are integrated in the course where we attempt to engage students at different stages in their learning loop (Figure 1).

One part of the course is based on common lectures for all student teachers, where the lectures are held either by lecturers from the university college or primary school teachers. The largest part of the course is group-based lectures on mathematical concepts and didactical aspects of teaching/learning these concepts. An important part of the teacher education programme is students' practicum at different schools where they collaborate closely with practicing teachers and are followed by university college lecturers. During the first semester students have a group assignment in which they analyse data collected during their practicum based on didactical

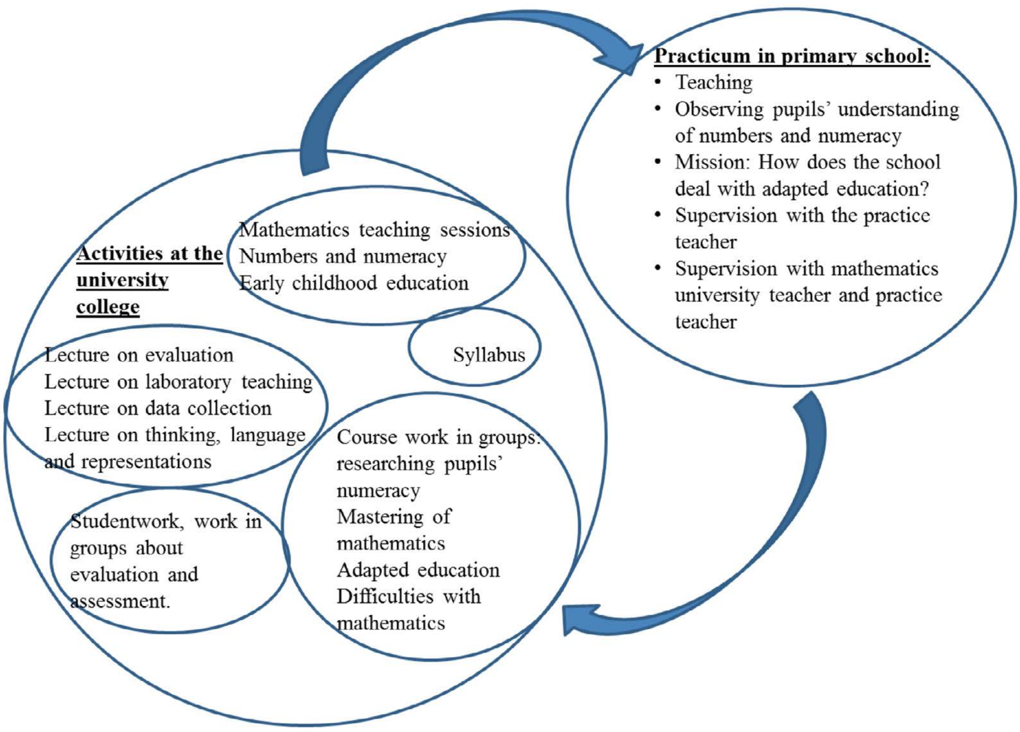

Figure 1: Student teachers' learning loop

concepts and theories included in the mathematics' course curriculum. The focus of the group work is pupils' learning of different mathematical concepts. During the semester(s) students also have different seminars on relevant topics in which they themselves prepare presentations of articles in groups for the rest of the class.

We have developed an initial learning loop inspired by an idea from Johnsen-Høines (2010). In our study we make clear the structure of the learning loop for the students and try to integrate the theoretical preparation of students with their practicum. The aim for this is to motivate students to be involved in their learning and their pupils' learning of mathematics by helping them in creating a meta-perspective, thus a whole picture, of their preparation at the university college. Our research perspective is one of critical mathematics education (Skovsmose & Greer, 2012) and it is concerned with students being critical to their own learning and teaching practice.

Skovsmose and Borba (2004) talk about current, imagined and arranged situations as "part of the research perspective in which classroom change plays a crucial role" (p. 214). We want to engage students in formulating imagined situations of teaching mathematics in the classroom, to teach those and then critically discuss the differences between the actual arranged situations and the imagined situations and reasons for such differences. This will hopefully help students to always find room and ways for improving their teaching in relation to pupils' numeracy, and to develop a critical attitude toward mathematics learning and teaching in general. We plan to use these critical discussions in teaching sessions at the university college where students summarize their practicum period.

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Using multimedia in mathematics teaching – New challenges for teachers' competencies

Julia Ollesch, Markus Vogel and Tobias Dörfler

University of Education Heidelberg, Heidelberg, Germany, olleschj@ph-heidelberg.de

The application of multimedia-based representations provides new opportunities for teaching mathematics. As the students are offered new options and new challenges, teachers are also faced with modified requirements respective to their teaching competencies. This project focuses on the development of an instrument to assess and foster multimedia teaching competencies.

Keywords: Computer, multimedia, video-vignettes, teachers' competencies.

Using multimedia-based representations in mathematics teaching could support the students to gain a different access to the underlying structure of the mathematical problem which probably leads to a deeper understanding of the topic (Ainsworth, 1999).

However, too many representations can provoke a cognitive overload (Chandler & Sweller, 1991). Furthermore, teachers have to be careful not to generate misconceptions through the representations (e.g., Hadjidemetriou & Williams, 2002).

While several studies investigated the effect of multimedia-based representations on learning outcomes, little is known of how to measure the competencies teachers need in order to take advantage of the potential of multimedia-based representations.

To measure these competencies, video-vignettes, which show various situations during mathematics teaching using multimedia-based representations, have been developed. These representations have been integrated in a classroom situation with interactions between students, teachers and the computer. The vignettes are constructed with a closed-ended question type, where each statement has to be answered on a scale from one to six according to its appropriateness for the situation. This type of assessment is thought to be effective to measure teachers' competencies (Blomberg, Stürmer, & Seidel, 2011).

35 vignettes have been developed, with 6 to 14 statements each, validated by a multistage expert reviewing process. The first step consisted of qualitative interviews with nine experts to assure relevance and clarity of the situations. In the second step, the vignettes were implemented in an online-tool, which was reviewed by 104 experts to get insights into the appropriateness and difficulty of the assessment. This is the data-base for the choice of the vignettes. For the current study only 10 vignettes are planned to be used. They are chosen from the data-base by sequentially applying different criteria, comparable to the stepwise selection procedure of Witner and Tepner (2011). The content of the individual vignettes was analysed separately after the second step of the expert reviewing process and only vignettes comprising interactions between students and computer were chosen, so that the number of vignettes could be reduced to 26 (criterion 1). A second analytical criterion does not refer to a whole vignette, but to each statement. The more of a consensus between the experts concerning a statement, the better it would be ranked. To measure this consensus, we will have a closer look at the quantity of experts that chose each of the possibilities from one to six. We are going to determine the absolute value of the differences between these quantities varying from the modal value and furthermore from the median, calculate the sum of the differences of each statement and standardise this value, so that the statements can be ranked. Furthermore, the experts' ratings concerning relevance (criterion 2) and clarity (criterion 3) will be analysed (as well as any available comments of the experts). Relevance and clarity of the vignettes were also ranked by the experts on a scale from one to six.

According to these four different criteria, the vignettes will receive a score corresponding to the criteria, so that ten vignettes with four statements each will be chosen for the pilot study in spring 2015.

In a next step, an experts' norm will be generated. A selection of vignettes, based on psychometric properties, will be used in the final assessment that will be validated in summer 2015.

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Implications from elementary mathematics teachers' lesson study experience

Deniz Ozen[1] and Nilufer Kose[2]

1 Adnan Menderes University, Faculty of Education, Aydın, Turkey, denizozen2009@gmail.com

2 Anadolu University, Faculty of Education, Eskişehir, Turkey

The purpose of this study is to reveal the experiences of lesson study being a model implemented by secondary school mathematics teachers participating in the Professional development project for the first time. Five secondary school mathematics teachers comprise the participants of the study. In the end of the study, acquirements and problems experienced by the teachers while implementing the model were determined. As a result, it can be said that lesson study is a model to be utilized by the teachers for professional development in the perspective of Turkey.

Key words: Teacher professional development, lesson study, experience.

INTRODUCTION

When we examine studies focused on professional development of teachers, the model of lesson study draws attention to the mathematics education research (Murata, 2011). Lesson study comprises of planning the lesson by the group as the teachers come together in a way that the student will learn the lesson efficiently, implementing and discussing (Lesson Study Research Group, 2002). This model is mentioned as the professional development process that Japanese teachers participate to evaluate the practices systematically to be more efficient in their professions. While this model improves teaching of teachers, it is also a professional development model contributing to benefiting from experience and knowledge of each other. In addition to this, it is emphasized that Murata (2011) lesson study model places the teachers in the center of efficiency of professional development. This study is quite significant in terms of revealing contributions of their professional development and experiences of mathematics teachers implementing the lesson study model seen in few studies since 2011 in Turkey.

METHODOLOGY

Participants of the study comprise of five mathematics teachers working at four different schools and having different years of service. Seminar was held with these teachers for a month before the sessions of lesson study started within the scope of the study. The model of lesson study was described at these seminars and also discussions on methods and techniques they could use during the process were held. Then the sessions of lesson study were carried out with teachers for 3 months at geometry classes. Data obtained from these seminars were analyzed by qualitative data analysis methods. In this poster, findings of the study will be presented as diagram under the titles of contributions of teachers to their professional development, problems they encountered and advantages of lesson study.

FINAL DISCUSSION

It was observed in this study that teachers benefited from the ways of thinking of each other in planning, implementing and discussing the implementation. Moreover, it was observed that teachers expressed that lesson study enabled them to see different methods and points of views. Teachers emphasized that since different opinions were presented during lesson study, a rich discussion environment was created and it enabled them to prepare an efficient lesson plan. In addition to this, they mentioned about the fact that it created team spirit and improved skill of making decision together. Also, since school conditions of teachers participating in the study were different and lesson study lasted longer, it was seen that they indicated that participation of teachers working at the same school would be more efficient for implementation of this model. As a result of this study, it can be expressed that the model of lesson study is a model to be employed for the professional development of secondary school mathematics teachers.

ACKNOWLEDGEMENT

This study was supported by Anadolu University Scientific Research Projects Commission under the grant no:1308E325.

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Lesson study as a professional development framework to support an exploratory approach

Marisa Quaresma, João Pedro da Ponte, Joana Mata-Pereira and Mónica Baptista

Universidade de Lisboa, Instituto de Educação, Lisbon, Portugal, mq@campus.pt

Our aim is to present the views of a group of basic education teachers involved in a lesson study as they discuss the features of tasks, students' difficulties, and how to conduct exploratory work about rational numbers in the mathematics classroom. Data collection was made through audio/video recording of the lesson study sessions, teachers' written reflections, individual and group interviews, and a researcher's journal. The results show that the teachers developed a vocabulary to speak about mathematics tasks, and developed an appreciation for students' reasoning processes such as generalizing and justifying as well as for classroom discussions as important moments for argumentation and negotiation of meanings.

Key words: Professional development, lesson study, exploratory approach.

LESSON STUDY AND EXPLORATORY APPROACH

Lesson study is a professional development process focused on professional practice with origin in Japan, which, in recent years, has been widely used in many countries. An important feature of lesson studies is their reflexive and collaborative nature (Fernandez, Cannon, & Chokshi, 2003; Perry & Lewis, 2009). Teachers identify an important issue and work together, analysing students' difficulties, discussing curriculum alternatives, and preparing what they expect to be an "exemplary" lesson. Afterwards, they verify to what extent this lesson achieves the desired objectives and what difficulties arise. Therefore, a lesson study is a process very close to a small investigation developed by the participating teachers on their own professional practice, informed by curriculum guidelines and by research results on the given issue. Our aim is to present the views of a group of grade 5–6 teachers as they discuss the features of tasks, students' expected difficulties and how to conduct exploratory work in the mathematics classroom. We focus especially on the professional learning of the teachers regarding the nature of the tasks, students' reasoning processes, and classroom communication.

A central aspect of lesson studies is that they focus on students' learning and not on teachers' work. Indeed, lesson studies aim to examine students' learning and to observe the way they learn. By participating in lesson studies, teachers may learn about mathematics, curriculum guidelines, students' processes and difficulties, and classroom dynamics. Lesson studies provide opportunities for teachers to reflect on the possibilities of an exploratory approach to mathematics teaching which is receiving an increasing support in curriculum orientations for mathematics education (e.g., NCTM, 2000). In this approach students are called to deal with tasks for which they do not have an immediate solution method (closed or open problems), constructing their own methods using their previous knowledge. Exploratory work creates opportunities for students to build or deepen their understanding of concepts, procedures, representations and mathematical ideas. Students are thus called to play an active role in interpreting the questions, representing of the information given and in designing and carrying out solving strategies which they must justify to their colleagues and to the teacher. However, conducting exploratory mathematics teaching is a serious challenge for teachers, demanding specific knowledge, competency and disposition.

RESEARCH METHODOLOGY

The methodology is qualitative, using participant observation. The participants are a group of grade 5–6 teachers of a school in Lisbon involved in a lesson study focusing on rational numbers. Data collection was made through audio/video recording of the sessions, teachers' written reflections, individual and group interviews, and a research journal made by a member of the research team.

RESULTS, IMPLICATIONS AND CONTENT OF THE POSTER

By participating in this lesson study, focusing on mathematical tasks and on students' reasoning, the teachers had many opportunities to get involved in doing mathematics through exploratory approach. They had many occasions to discuss the features of tasks that make them simple exercises or more engaging problems or explorations as well as features of reasoning processes such as justification and generalization (Lannin, Ellis, & Elliot, 2011) with rational numbers. Anticipating possible difficulties of students and looking at what they actually do in the classroom were key features of lesson study (Perry & Lewis, 2009) that were effective in leading the teachers to reflect and consider changes in their practice to conduct a more exploratory teaching in their classrooms. This led to significant learning regarding the nature of the tasks, students' reasoning processes, and classroom communication.

Using photos, schemes, and text, the poster presents the aims of the study, our framework on lesson study as a professional development setting, samples of the teachers' work on lesson study sessions, as well as implications of this kind of work for mathematics teacher professional development.

ACKNOWLEDGEMENT

This study is supported by FCT – Fundação para a Ciência e Tecnologia through Project PEST (PEst- OE/CED/UI4107/2014) and also through a FCT grant to Marisa Quaresma (SFRH/BD/97702/2013).

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Pre-service mathematics teachers' view of mathematics in the light of mathematical tasks

Johanna Ruge and Reinhard Hochmuth

Leibniz University Hanover, Hanover, Germany, ruge@idmp.uni-hannover.de

Mathematical tasks play a crucial role in mathematics education in the school context, in higher education and therefore also in the professional development of pre-service mathematics teachers. Pre-service teachers' views of mathematics are reconstructed in relation to experiences with mathematical tasks in those contexts.

Keywords: Pre-service mathematics teachers, mathematical tasks, view of mathematics.

INTRODUCTION

The findings presented in this paper are part of an ongoing qualitative study addressing learning experiences of pre-service mathematics teachers at university. Ten semi-structured interviews with pre-service mathematics teachers are analyzed using a combination of grounded methods (Strauss & Corbin, 1990) and objective hermeneutics (Wernet, 2009). The topic of mathematical tasks emerged to be worthwhile taking a closer look at in relation to the institutional dimension. Interestingly the topic of mathematical tasks is addressed in the interviews in relation to insecurities concerning the view of mathematics.

In her analysis of word problem tasks, Jean Lave (1993) has pointed out that the institutional dimension is an important aspect leading to a deeper understanding of meanings ascribed to mathematical tasks. She described that mathematical tasks fulfil a variety of different roles in the school context, which can be contradictory (e.g., "realistic" word problems being motivating to all students vs. the complication of tasks due to backward translation into mathematical language). Taking a closer look at teacher education, at least three institutional dimensions have to be considered; the pre-service teachers' experiences (I) at school level, (II) in subject matter courses at university and (III) in pedagogical content knowledge courses at university. These three dimensions do not stand alone, but are all embedded in the societal discourse.

One aspect of pre-service mathematics teacher's professional development is reflection on mathematical tasks. Taking a closer look at the influence of mathematical tasks on views of mathematics in relation to the different institutional contexts, the following question arises:

What meanings do pre-service mathematics teachers ascribe to mathematical tasks that they are confronted with throughout their university education?

The findings are discussed through the cases of Anna and Georg; both addressed modelling tasks in the interview as being a new aspect of mathematics for them.

THE ROLE OF MATHEMATICAL TASKS IN TEACHER EDUCATION

Bearing in mind the central role of mathematical tasks in mathematics education it is reasonable that they are an important topic for pre-service teachers. Experiences with mathematical tasks in school have already shaped students' views of mathematics (Hannula et al., 2005) and their perception of the teacher's role before they start their studies to become teachers. According to their view of mathematics, different meanings are ascribed to mathematical tasks in university courses: Anna's experiences with mathematical tasks in school led her to the stable view that doing mathematics will lead to definite solutions. In university courses she relies on sample solutions for learning mathematics. She perceives mathematical tasks provided by her subject matter courses to be guidelines of what she is supposed to learn. She plans on providing step-by-step instructions to her future pupils. Georg sees mathematics as a logic game. To him, mathematical tasks in university courses provide opportunities to learn a way of thinking.

Throughout their studies, pre-service mathematics teachers' views of mathematics are challenged by the different nature of mathematical tasks compared to those in the school context. For Anna and Georg, modelling tasks seem contradictory to their view of mathematics. Anna recognizes that modelling tasks differ from her view of mathematics and has difficulties integrating this new type of task into her mathematical practices and perceives teaching modelling in school as a burden. Georg recognizes, due to modelling tasks, the application facet of mathematics but even so he cannot imagine presenting the abstractness and application of mathematics in a coherent manner in a school context. At the time of the interview he favours promoting the application facet in school.

Different aspects of mathematics that seem contradictory to pre-service teachers need to be explicitly addressed and reflected on in university courses in order to help future teachers broaden their view of mathematics and integrate those aspects into their practice.

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Mathematics teachers' meaning making – Problematizing the process of learning in and from daily practice

Helén Sterner

Linnaeus University, Kalmar, Sweden, helen.sterner@lnu.se

Mathematics teachers' development and the understanding of what constitutes learning are an on going topic and highlighting the complexity in the processes of learning in and from practice. This study builds on the idea that mathematics teachers' professional development needs to be based on their classroom practice (Goodchild, 2008; Kazemi & Franke, 2004). Teacher participating in a working group, a learning community, and reflect on their own teaching and students learning. Working collaborative the mathematics teacher developed understanding of mathematical communication and mathematical reasoning in their teaching algebra.

Keywords: Developmental research project, mathematical communication and mathematical reasoning.

AIM AND RESEARCH QUESTIONS

Results from a pilot study showed teachers' difficulties with describing the concepts mathematical communication and mathematical reasoning as well as with using these concepts in their teaching. Based on the results from a pilot study and by means of using developmental research (Gravemeijer, 1994) the main study was designed as a collaborative work between mathematics teacher and a researcher. Also based on the results from the pilot study the reflection group decided to focus on; how could teachers develop classroom communication to stimulate students' mathematical reasoning? The main purpose of this study is to gain understanding of a process, of what and how teachers learn when participating in a reflection group.

THE THEORETICAL PERSPECTIVE

The theoretical perspective used in this study is *communities of practice* (Wenger, 1998), where learning is considered to be a function of participation and participants are constantly involved *negotiation of meaning*. From this theoretical perspective *reification* and *participation* are used in the process of *negotiation of meaning* (Wenger, 1998).

THE METHODOLOGY

This is a study of single case, consisting of five mathematics teachers from different elementary schools (grade 1–6). The five teachers and a researcher meet monthly in a workgroup, *the reflection group*. The project has used the model of *the developmental research cycle* (Goodchild, 2008, p. 208) as a methodology. This model is a cyclic process for both the professionals' mathematics teachers and the researcher. The model gives opportunities to analysing developmental activities with special emphases on the relation theories and practice (Goodchild, 2008). My interpretation and using *the developmental research cycle* (Goodchild, 2008, p. 208) is mutual between the researcher and the mathematics teachers, empirical data was generated from discussions and activities in the reflection group. The preliminary and on going results of the analyses fed back in the reflection group to provide development for practice (Goodchild, Fuglestad, & Jaworski, 2013). In the reflection group the teachers prepare mathematic tasks collaboratively, to be used in different classrooms. The participants use reflections in three different levels. These reflections are both individual and shared in the reflection group. The teachers make individual reflection before, during and after implemented lesson. When the teachers meet in the reflection group they discuss and reflect together both on own reflections, on the student work generated and on the videotapes from their teaching. The third level of reflection the researcher present as questions from previous meeting. These reflections are used as a tool to problematize the on going education in mathematics.

PRELIMINARY RESULTS

The study indicates how teachers in this learning community came to understand the concepts communication and reasoning in mathematics and shifted their ways of talking about students´ mathematical reasoning. I identify four shifts that teachers made in there learning about students' mathematical reasoning. *Understand, identify*, *interpreting and applies* the mathematical reasoning in their education.

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First-year teacher students' mathematical beliefs

Benedikt Weygandt[1] and Reinhard Oldenburg[2]

1 Goethe University Frankfurt, Frankfurt am Main, Germany, weygandt@math.uni-frankfurt.de

2 University of Augsburg, Augsburg, Germany, reinhard.oldenburg@math.uni-augsburg.de

This poster focuses on the beliefs accompanying first-year teacher students with respect to aspects of mathematics as a product of human creativity that might be designed in various ways.

Keywords: Teacher education, mathematical beliefs, genetic teaching.

INTRODUCTION

Transition issues are a frequent problem in academic mathematics education, particularly appearing in teacher students' education. Currently, there are several projects intervening in this field in German-speaking countries (e.g., Beutelspacher, Danckwerts, Nickel, Spies, & Wickel, 2012). We created a lecture, "Mathematical Genesis", held parallel to the corresponding calculus course in which the transition gap seems to be at maximum. The lecture deals with the development of mathematics in and by the society of mathematicians. First-year students learn about the importance of creativity and experience the process of developing mathematics. This study assesses effects of the lecture on their beliefs (cf. Weygandt & Oldenburg, 2014).

THEORETICAL FRAMEWORK, METHOD AND RESULTS

This research is based on Törner's theory and conclusions about German math teachers' beliefs, in which he stated the "key role [of beliefs] in the teaching and learning process" (Törner, 1996). Moreover, Törner, Grigutsch and Raatz (1998) identified four aspects of mathematics: "formalism-", "scheme-", "process-" and "application-aspect". Only the "process aspect" corresponds to the genesis of mathematics. Our initial hypothesis stated that this should be analysed in more detail as more aspects of mathematics that might be especially important in the beginning of academic mathematics education appear to exist. We adopted Törner's survey on prospective teachers' mathematical beliefs and added 60 items focussing on lecture-related subjects. During the course of the study we surveyed 178 first-year (mixed math and math teacher) students concerning their acceptance of each statement. Exploratory factor analysis of the 37 test items (introduced by Törner et al., 1998) verified those four factors with good reliability. An analogously conducted factor analysis of the newly designed 60 items suggested the exploration of five further factors. The extracted factors are homogeneous concerning their content. Thus we propose to add the following aspects to those postulated by Törner (1996): (a) "output efficiency", (b) "structure of mathematics", (c) "creativity" and (d) "universality". The aspect (e) "latitude" can probably be included as well. In order to get an idea of these new factors, Table 1 illustrates three aspects through corresponding example items.

The current research concentrates on an inter-aspect correlation matrix and compares the pre- and post-

aspect	example item(s)
(b)	Learning maths, it's a waste of energy to take a non-productive approach.
(d)	Mathematical objects are comparable to natural principles, i.e., they may be discovered, but are unchangeable. / Any extra-terrestrial intelligence would reach same mathematical conclusions.
(e)	If one dislikes the consequences of a definition, one may modify the definition accordingly.

Table 1: Example items for some of the five assumed aspects

aspect shift \ group	**attending MG (n=15)**	**not att. MG (n=22)**
scheme-orientation decreases	d=-.70 (p=.02)	d=-.45 (p=.04)
structure of mathematics increases		d=.60 (p=.01)
universality decreases	d=-.61 (p=0.03)	
latitude might increase	*d=.49 (p=.07)*	
application might decrease	*d=-.43 (p=.08)*	
output efficiency might decrease		*d=-.35 (p=.09)*

Table 2: Effect sizes of differences in mean values

test results of teacher students having or having not attended the lecture on "Mathematical Genesis" with the corresponding effect sizes in Table 2.

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TWG19

Mathematics teacher and classroom practices

Introduction to the papers of TWG19: Mathematics teacher and classroom practices

Despina Potari[1], Lourdes Figueiras[2], Reidar Mosvold[3], Charalambos Sakonidis[4] and Jeppe Skott[5]

1 National and Kapodistrian University of Athens, Athens, Greece, dpotari@math.uoa.gr
2 Nelson Mandela International School, Berlin, Germany
3 University of Stavanger, Stavanger, Norway
4 Democritus University of Thrace, Xanthi, Greece
5 Linnaeus University, Kalmar, Sweden

INTRODUCTION

This Thematic Working Group (TWG) together with TWG18 and TWG20 addresses questions related to mathematics teachers, teaching, and their development. TWG19 focuses particularly on mathematics teaching, including important micro and macro factors that frame it. Classroom research has been the focus for many years in mathematics education, but new theoretical and methodological directions have been reported in this group aiming to study on the one hand the overall complexity of teaching and on the other the particular aspects that characterize it.

TWG19 initially received 64 proposals (56 papers and 8 posters), and being the largest group in CERME9, it was split in two new groups (TWG19 and TWG20). Finally, 31 papers and 6 posters remained in the group. The reviewing process led to revisions in most of the papers and eventually 27 papers and 6 posters were presented in the conference and are included in the proceedings. The papers and posters were classified in three thematic groups: instructional practices (14 papers and 4 posters); classroom interaction (8 papers and 1 poster); and tasks and teaching resources (5 papers and 1 poster). The papers and posters in the first thematic group mostly concern general teaching practices, whereas there are some which focus on teaching practices related to the teaching and assessment of specific mathematical issues. In the second thematic group, the topics addressed are patterns of interaction, teacher's questioning, and classroom discourse. In the third thematic group task and textbook analysis is the main focus. Below, we discuss the papers and posters with reference to these thematic areas.

INSTRUCTIONAL PRACTICES

Most papers and posters in this area report on small-scale, qualitative studies of the interactions in one or a few school classrooms aiming to support practices for effective teaching, but also to gain a better description of teaching practices by looking for patterns of argumentation in classroom interaction (Zalska & Tumova). However, there are also examples of more quantitative approaches (Felmer et al.; Gunnarsdottir & Pálsdottir) and of researchers who do not report on their own observations of classrooms, but on teachers' interpretations of classroom events (Reid et al.) or task design (Opheim). Between them, the papers deal with all levels of schooling, from primary (Allard; Caseiri et al.; Gade; Velez & Ponte; Taylan), over middle (da Ponte & Quaresma; Zalska & Tumova) to secondary (Dias & Santos; Mata-Pereira & da Ponte; Reid et al.) and vocational (Opheim). One paper addresses instructional designs of novice teacher educators and the practices that unfold in their course for in-service teacher education (Kalogeria & Psycharis), and another one does not deal directly with interpretations of classroom events, but is based on interviews with teachers on how imposed assessment practices may significantly transform teaching practices in ordinary schools (Pratt). Also one of the posters deals with external evaluation and the teachers' attitude towards it (Signorini). Two posters explore the effects of formative assessment on teaching practice (Anderson & Boström; Vingsle).

Theoretical and methodological issues

Pratt's paper highlights the significance of taking contexts beyond the individual classroom and school into consideration. His theoretical framework, based primarily on Bourdieu, allows him to do so. *Context*, in one or other understanding of the term, is dealt with also in other studies (Reid et al.; Opheim; Taylan), and in fact the notion of context is mentioned in 10 of the 14 papers. However, it is not always clear how the term is used and how its significance is taken into account in the theoretical framework and the methodology. This seems to be an issue that needs attention in research on mathematics classroom practices.

The theoretical or conceptual frameworks that are used vary. Gade draws on CHAT, Reid and colleagues on Maturana (among others), whereas Allard as well as Kalogeria and Psycharis refer to the French school. Others base their studies on more local theories of classroom communication, problem solving, or the teaching and learning of particular mathematical topics. Clearly, there are strong limitations to what theoretical frameworks can be presented in a conference paper. Nonetheless, the interest in *context* combined with the limited attention paid to theoretical frameworks in most of the papers may indicate that it is a challenge to develop and use approaches that allow for a combined analysis of micro-, meso-, and macro issues of significance for instruction.

The approach in most of the papers is a carefully conducted qualitative analysis of unfolding classroom events, sometimes combined with stimulated recall interviews and textual analyses of tasks and student work. Other studies are based exclusively on interviews. In both cases, the research participant(s) are often purposefully selected teachers, who are either co-researchers engaged in teacher-researcher collaboration (Gade; Dias & Santos), experienced teachers whose participation in the study is based on their reputation and/or their participation in long-term teacher development initiatives (Allard; Taylan), or teachers who are engaged in focused collaboration with their colleagues on issues of interest to the study in question (Velez & da Ponte). In these studies, the research participants qualify as critical cases (Flyvbjerg, 2006), this means, as particularly useful cases that allow for conclusions of the type: *if reform intentions do not materialise with these teachers, they never will.* Although some of the studies invite this type of conclusion (Allard), most of them do not. As mentioned above, most of the papers report fairly positive findings, and the general conclusions seem more related to ways in which interventions or the work of competent teachers may inform teacher education or induction programmes. This is so for instance with a series of studies from Portugal by Caseiro and colleagues; Mata-Pereira and colleagues; da Ponte & Quaresma; and Velez & da Ponte and with the Turkish study by Taylan.

Reid and colleagues do not make use of critical cases, but adopt a second order perspective on classroom practice: they analyse groups of teachers' collective interpretations of video recordings of "typical" and "exemplary" lessons from their own classrooms. The intention is to understand the criteria that guide the teachers' observations. It is interesting that research participants focussed on different aspects of typical and exemplary lessons, when they had access to the videos and when they did not. Reid and colleagues argue that this indicates the research design needs to allow for different ways of accessing teachers' views of what is valued in mathematics education.

Substantive issues

There seems to be a shared vision of quality teaching underlying most of these papers. Some papers make explicit reference to *the reform*, but, even in the majority of papers that do not there appears to be a set of shared understandings inspired by current reform efforts about good classroom practice. In particular, students are expected to become involved in independent and creative activities much beyond their repetition or imitation of ready-made concepts and procedures as presented by the teacher or the textbook. In turn, this requires teachers, for instance, to organise problem solving activities (Felmer et al.); to orchestrate whole-class discussion and promote other forms of classroom communication (Gade; Ponte & Quaresma; Mata-Pareira, Ponte & Quaresma); to facilitate the development of students' ability to assess their own mathematical progress (Dias & Santos); and to base instruction on interpretations of the students' mathematical thinking, including their unexpected questions, comments, and suggestions (Taylan).

The papers mentioned report on relatively successful examples of instruction, when seen from the perspective of the researchers. Some studies are part of or follow up on intervention programmes and in these studies research participants generally appear to cope well with the challenges of teaching according to the reform (e.g., Felmer et al.). The most significant of these challenges appears to be the contingencies that arise as students are to make their own mathematical contributions to classroom practice. The specific contents of students' inputs are not necessarily part of the teacher's agenda, as their contributions may take the form of a surprising question or an unforeseen conjecture (Ponte & Quaresma). In this sense, the reform inserts an element of planned unpredictability in instruction that teachers need to capitalise on in the moment (Skott, 2004). One major challenge for teachers is to cope with the inherent unpredictability.

Despite the positive descriptions of the interactions in most of the papers, some point to problems in relation to the reform. Gunnarsdóttir and Pálsdóttir report on a study from Iceland on dominant ways of organising instruction in grades 1 to 10. There are examples of group work and whole class discussions, but students appear to spend most of their time working individually on textbook tasks. Allard's paper points to problems that arise because of difficulties involved in linking instruction to students' thinking. Her study on a highly qualified elementary teacher in France shows a tension between devolution and institutionalisation, that is, between handing over initiative to the students to support their independent activity and ensuring that what is learned is decontextualized and in line with the discipline of mathematics. One particular problem when taking students' thinking into account is that of the time needed for task completion. Opheim, working with vocational schools in Norway, suggests that the diversity of the student population poses particular problems related to timing for teachers' selection and use of tasks.

A somewhat different approach is adopted by Pratt. Working in the UK, he reports on part of a study of how a dominant assessment culture influences the position of teachers and the relationships among them. In turn, this is likely to significantly influence instruction.

CLASSROOM INTERACTION

The papers in this area consider various aspects of the interactions taking place in mathematics learning and teaching settings and the ways these interactions shape participants' learning. All but one of the papers concern small-scale, qualitative studies often situated or initiated in the context of a larger project. Only the study by Seker and Ader adopts a quantitative approach, focusing on teachers' written answers to items allowing for tensions between students' perceptions and teachers' intentions to be explored. The remaining research reports deal with aspects of interactions taking place in various mathematics education contexts: teachers' discussions in mentoring sessions (Mosvold); teacher practices when exploiting contingent opportunities (Kleve & Solem) or when aligning values to smoothen learning (Seah & Andersson); whole-class discussions (Drageset; Larsson); and students' grasp of teaching changes initiated by an intervention (Evans & Swan) or by a national reform (Wester, Wernberg & Meaney). Most studies are based on observations, the exceptions being the ones by Larsson and by Wester, Wernberg & Meaney, which are predominantly based on interviews. The interest in the impact of the emergent interactions on students' mathematics learning and directly or indirectly on teachers' learning about teaching mathematics is central to all the studies presented here, whose focus extends from the upper primary to the secondary school level.

Theoretical and methodological issues

The design and the implementation of the research reported in the papers are framed by the conceptual focus chosen, the theoretical perspective adopted and the methodology employed. The studies deal predominately with classroom practices related to promoting interaction and thus facilitating particular aspects of mathematics learning, mainly through teachers' management of teaching incidents (e.g., Seah & Andersson), classroom communication (e.g., Larsson), tasks (e.g., Evans & Swan), tools (e.g., Seker & Ader) and resources (e.g., Wester, Wernberg & Meaney). The theoretical frameworks employed in the studies concentrate on mathematics classroom interactions related to advancing students' learning or to developing teachers' knowledge for teaching. The former might refer to management (e.g., Drageset; Evans & Swan) or socio-cultural (e.g., Larsson; Seah & Andersson) issues of the interactions involved, whereas the latter frameworks adopt either a practice oriented (Kleve & Solem) or a discursive perspective (Mosvold). Finally, the methodological choices are qualitative in character (with the exception those of Seker and Ader), include mostly selection and analysis of classroom events, sometimes combined with interviews (e.g., Larsson) and involve teachers who agree or are willing to experiment with new ideas (e.g., Wester, Wernberg, & Meaney). The results reported are generally positive, illustrate how and shed some light on why interactions can promote classroom participants' learning, but may also limit their learning opportunities.

Substantive issues

The research presented in the papers under consideration is based on a shared view of learning as a process of pursuing challenging tasks and activities via student-student and/or teacher-student interactions. Opportunities for such tasks and activities to be introduced are sought in instances of the moment-to-moment decision making or in carefully planned and implemented teaching interventions, aiming in all cases at particular, well defined learning outcomes. In the former category, we find contingent moments of teaching seen as unplanned opportunities of supporting students' mathematical learning (Kleve & Solem); unexplained students' answers exploited as occasions for effective intervention (Drageset); tensions between teachers' intentions and students' interpretation taken as instances for improving participatory learning (Wester, Wernberg & Meaney); teachers' value alignment strategies related to the quality of the students' learning experiences (Seah & Andersson). As for the tasks and activities that are planned, some are based on and explore the power of theoretical models of learning, cognitively (Seker & Ader; Evans & Swan; Klein) or a socio-culturally oriented (Larsson; Mosvold).

The classroom practices supported by the tasks and activities utilized in the studies described above are characterised by significant to moderate teacher-student and only occasionally noticeable student-student interaction. For example, unexplained answers appear to attract teachers' attention (Drageset) and the same is true for contingent teaching moments (Kleve & Solem), leading to a range of teacher-student interaction strategies, which are generally beneficial for students' learning. Similarly, well-designed and established social and socio-mathematical norms contribute to pupils interacting effectively with one another, giving rise to an inquiry classroom culture (Larsson). However, overall, students find it hard to cope with and build on interactions with their peers in the classroom (e.g., Evans & Swan).

Overall, the studies reported here reveal that mathematics classroom interactions constitute critical teaching incidents that function in complex ways and offer opportunities for students' and teachers' individual and collective learning. However, the studies also indicate that there may be difficulties involved that we are just about to begin identify and understand.

TASKS AND TEACHING RESOURCES

Teaching can be regarded as plausible conceptions of teachers' professional practice, and most of the papers in TWG19 deal with such conceptions. Among the different approaches to conceptualize the work of teaching, one considers teaching as a didactic encounter between teacher and students about a certain mathematical content. In such an encounter, the teacher presents the content by using mathematical tasks or resources. Two of the papers discussed here, mainly concentrate on the content of textbooks and textbook tasks (Burke; Wijayanti), whereas three papers focus more on teachers' use of tasks in their work of teaching mathematics (Ayalon & Hershkowitz; Kwon; Matic & Gracin). In addition to these papers, a poster focused on mathematics teacher guides (Ahl & Koljonen).

Theoretical and methodological issues

The textbook is a central resource for mathematics teachers and students alike. Appraising, adapting and administering mathematical textbook tasks constitute professional challenges for the mathematics teacher. Textbooks are not only a source of mathematical tasks, but they also include demonstrations of techniques. When analysing Indonesian mathematics textbooks, Wijayanti focuses on tasks and techniques presented in the textbooks and uses Chevallard's Anthropological Theory of the Didactic (ATD) in her analysis. Burke focuses on the strategies deployed in textbooks and the pedagogical tasks embedded in them. As a theoretical framework, he applies Dowling's Social Activity Method (SAM). From such a theoretical perspective, he aims at describing possibilities for engagement between author and audience in a pedagogical relation.

Most textbooks also have a teachers' guide, and Ahl and Koljonen analysed the teachers' guides to the two most commonly used mathematics textbooks in Sweden. They apply content analysis in their analysis of these teacher guides.

Whereas the above-mentioned studies focus on the texts themselves – the textbook, the tasks, or the teachers' guide – the other three papers have a particular focus on teachers' use of textbooks and mathematical tasks in the work of teaching. In their study, Ayalon and Hershkowitz survey 17 Israeli secondary school teachers. They attend to the teachers' rationale for selecting particular mathematical tasks from the textbook in order to facilitate argumentative activity in the classroom – considering argumentation as a social process. Whereas these researchers investigate teachers' different explanations for the choice of tasks, Matic and Gracin analysed two Croatian primary teachers' use of textbook tasks. When applying the socio-didactical tetrahedron in their analysis, the connection between student, mathematics and textbook appeared important. In her attempt to conceptualize the teachers' work in supporting students' development of mathematical explanation, Kwon focused on the encounter between an expert teacher and five different cohorts of students around some mathematical tasks. Kwon's analysis revolved around core tasks of teaching, and this corresponds somehow to Burke's focus on pedagogical tasks.

Altogether, the papers and poster presented in this group vary considerably in their use of theoretical framework, instruments, and methods of data collection and analysis. This variation can be seen as a challenge for the further development of the field.

Substantive issues

From these studies, some emerging issues can be observed. First, there is the connection between teaching and students. Investigations of this connection differ among the studies, but the teachers' work cannot be dissociated from the students' learning activities. Second, the connection between mathematical tasks and the decomposition of teachers' practice into core tasks of teaching emerged as a central issue (e.g., Kwon). Such tasks of teaching are arguably important to study when attempting to conceptualize teachers' professional practice. Finally, an issue emerges in the various theoretical conceptions of teaching. The papers in this group adopt different theoretical frameworks, and different interpretations of core concepts like tasks and teaching are applied. Some papers appear to describe tasks merely as mathematical problems from textbooks (e.g., Ayalon & Hershkowitz), whereas others also discussed pedagogical tasks (Burke), or tasks of teaching (Kwon). Future studies in this TWG need to be more specific about their use of such core concepts and develop shared language necessary for establishing solid conceptions of the professional work of mathematics teachers.

GENERAL ISSUES – MOVING FORWARD

Research on mathematics teaching has been developed in mathematics education during the last few decades. Different research questions as well as theoretical and methodological tools have been formulated throughout the years. The papers and the discussions in this TWG show that we have taken some steps forward. For example, we have developed questions and frameworks that allow us to: look closer into critical moments of classroom interaction; consider contextual, epistemological and social issues that frame mathematics teaching; build general models that describe mathematics teaching; understand better in what ways the resources and tools that teachers use in the classroom transform mathematics teaching and learning. We have also started to consider mathematics teaching in its complexity and move away from dichotomies of what is "effective" or not and adopt deeper explanations of teacher's decisions and actions. We work with teachers in more collaborative contexts and care more about what our research says for them.

However, we still need to look more critically at our research concerning mathematics teaching. For example, many papers study the teaching of an expert teacher. Do we work under the assumption that there is an ideal teacher and/or an ideal teaching? We possibly need other interesting cases of mathematics teachers in different school contexts that will allow us to understand better the dynamic character of teaching. We also see that many studies use multiple research methods. Is there the underlying assumption that different methods provide differential access to the same phenomenon, for example, teachers' meaning making? It may be that different methods shed light to teachers' views and meaning making in decidedly different contexts that frame teacher actions very differently. In the discussions in the group, it appeared that because mathematics teacher education and teacher resources were discussed in other groups, we missed the dialectical relationship between mathematics teaching, resources and teacher education. Do we really have theoretical and methodological tools that allow us to study this dialectical relationship?

We close this introduction by addressing a number of challenging questions that were discussed in the group: How can we link students' activity to teacher's activity? How can we include the significance of context in our research on mathematics teaching? Can we listen to teachers' voices? How complementary are the different theoretical frameworks we use for studying mathematics teachers and teaching? Is there a coherent meaning underlying the different constructs we use?

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TWG19

Research papers

Analysis of teachers' practices: The case of fraction teaching at the end of primary school in France

Cécile Allard

University of Paris VII, LDAR, Laboratoire André Revuz, Paris, France, cecile.allardb@free.fr

My study deals with Institutionalization Process in French primary school. Institutionalization Process (IP) is defined in the Theory of Didactical Situations (TSD) as a process enabling to decontextualize and depersonalize knowledge. I focused both on what decontextualizing and depersonalizing imply in teachers' actions and on knowledge exposure, proposed by teachers. I particularly focused on fraction teaching at the end of primary school. My methodology allowed me to collect everything that is said diffused about fractions.

Keywords: Institutionalization, practices, fractions, decontextualization, depersonalization.

INTRODUCTION

In TSD (Brousseau, 1998) also explains that the aim of TSD was the production of mathematical situations because there was a lack of such situations in teaching, so questions about institutionalization (knowledge exposure) arose much later. More recent works question this process and show in what manner it is important for teachers to take over didactical memory of the class (Brousseau & Centeno, 1991). Butlen (2004), Butlen, Peltier-Barbier and Pezard (2004) show that a tension exists between devolution and institutionalization for the benefit of devolution. Coulange (2012) obtains similar results to those of Butlen and colleagues (2004) and of Margolinas and colleagues (2002). Coulange talks about deletion of the formulation and knowledge exposure for the benefit of practices that puts ahead a "permanent shoring". Observations and results of these authors lead us to question how expert school teachers (Tochon, 1993) deal with knowledge exposure. What are their constraints (Robert, 2001), which regularity and variability (Robert & Rogalski, 2002) will it be possible to determine when they institutionalize (for their meaning as for our)?

QUESTIONS AND OBJECT OF STUDY

In TSD, institutionalizing is defined as the action of depersonalizing and decontextualizing knowledge that arise following a situation of action. Pupils face a situation they need to solve by mobilizing knowledge and skills. The results of this situation lead them to build new knowledge. Brousseau (1998) thus shows one of the first institutionalization paradoxes: how to convince pupils that they have just learned something new despite the fact they were not able to solve the given problem. The teacher's main role here is to demonstrate and focus on this new knowledge. For this he has, on the one hand, to promote discussions between pupils in order for them to realize the great variety of their strategies involved to solve problems. On the other hand, they need teacher's support to agree due to their lack of a common language. In addition, the teacher has to show and name the new knowledge. These moments when the teacher names and shows the knowledge engaged will be called knowledge exposure.

Several recent French works (e.g., Butlen, Pezard, & Masselot, 2011; Coulange, 2012; Margolinas & Lappara, 2008) have shown difficulties met by junior faculty to deal with these knowledge exposures. Butlen and colleagues (ibid) show that teachers know how to devolve situations but meet strong difficulties to institutionalize knowledge. They then talk about tension between devolution and institutionalization. These difficulties to say and show involved knowledge at school, partly explain educational inequalities in France (some parents can help their children, some others not) (Rochex & Crinon, 2011).

A first study (Allard, 2009) has shown that written knowledge exposures, were few in number at the end of primary school (around twenty short texts for 36 class weeks for an average of 5 hours mathematics lesson per week). This small number of written traces let us assume the hypothesis that knowledge exposures are made orally most of the time. For my PhD I have tried to check this hypothesis.

These knowledge exposures can take place at the end of a lesson, at the beginning of another (reminder-phase), during the lesson, or during the correction of exercises. These knowledge exposures are quite fuzzy and I had to design a specific methodology to study them. We will describe it in a next part.

Although recent works are generally focused on junior faculty, we have decided to deal with skilled teachers. These teachers passed a specific exam (CAFIPEMF) to validate their expertise. These expert teachers are called PEMF (for "Professeur des Ecoles Maitre Formateur"). The exam consists in three parts. The first part consists in presenting two lessons in two different disciplines in front of a five-people jury (inspectors, PEMFs, academic advisors, university professors). The second part takes place in the classroom of a pre-service teacher whose practice has to be analyzed by the candidate to CAFIPEMF. The final event consists in making a short presentation about his/her professional dissertation. These PEMFS are teachers that receive and train teacher trainees. In this paper, I will approach the following research question: How do these skilled teachers deal with knowledge exposure?

METHODOLOGY

My methodology is comparative and qualitative. Comparisons are possible between several PEMFS but also, over several years for the same PEMF. Depending on comparisons, we had to use different methodologies. I followed five PEMFS over a year, including two for three years. In order to make comparisons easier, I set down some particular variables such as the use of a common manual and keeping to one mathematical problem: fractions. To make comparisons easier I filmed the teachers during all their lessons on fractions. In this paper, I will focus on the comparison of teaching practices of the same teacher for two years. I will therefore be able to talk about regularities and variabilities of these practices.

For this teacher, teaching fractions represents seven sessions of 50 minutes each for the entire year. I have transcribed all the lessons I filmed. These videos enabled me to retrieve a large variety of data. I mainly focus on the words expressed by the teacher depending on pupils' activities. I will particularly focus on the fidelity of the field of mathematics during three moments in the classroom: explanations given to pupils, before and after a research phase, and during knowledge exposure. These oral knowledge exposures are part of teacher-pupil dialogues. Through the analysis of these fragments of sentences, I will be able to put them together to recompose a text of knowledge as was shown to pupils. These knowledge exposures are studied according to their decontextualization level (Pezard & Butlen, 2003). I also kept the exercise books in which exercises were given and done and also the other exercise books that contained mathematical texts to be learned at home by heart. This data collection enables me to determine what kind of tasks pupils have done and what they have to learn at home. In France, it is forbidden to give written homework. Despite this prohibition that has been in effect since 1956, parents in certain social environments claim homework for their children. The only type of authorized homework, however, is reading and learning mathematical rules by heart.

These different data enable us to fill in the various practices components of PEMFs and to determine variabilities and regularities (Robert & Rogalski, 2002). I thus adopt the methodology described by Robert and Rogalski (ibid). These authors break down practice into 5 components (personal, social, institutional, cognitive and mediative) then recompose them. The personal component is only accessible by interviewing the teacher. It gives information on the relationship that the teacher has with mathematical knowledge, and choices made to help him/her carry out in comfortable manner his/her classroom teaching. The social component gives information on the teacher's working place, his/her colleagues and the social environment of his/her pupils (both in disadvantaged and advantaged areas). The official curriculum, the mandatory hourly amount of math lessons, the use (or lack of use) of certain handbooks and the relationship with inspection and inspectors, all give information on the institutional component. The cognitive component corresponds to teacher choices about content, tasks, organization and forecasts on how to manage. The mediative component is particularly relevant since it deals with improvisations, speeches, pupils' participation, devolution of instruction and knowledge exposures.

Thus, in order to describe practices and inform mediative and cognitive components, I focused on the choice of handbooks, the mathematical problems, on exercises books and on the teacher's speech. I then had the necessary's data to know how things are said and what is transmitted to pupils.

CASE STUDY

I chose to study the particular case of teaching fractions because it is a new notion that is introduced at the end of primary school. In French primary school, fractions are only studied under the subconstruct, called part-whole (Kieren, 1983). Introducing fractions is backed up by materials such as paper strips and circular areas.

I will name "Solene" the teacher I followed from September 2011 to June 2014. I previously followed this teacher from 2008 to 2011 but with a different focus. During these five years, I have collected enough data to describe her teaching practice carefully. 2011 was the first year when she had to deal with a double level class (CM1 and CM2: from 9 to 10 year old pupils). That year she proposed only three sessions on fractions. These sessions consisted in coding shapes. Shapes were subdivided into several equal parts. Fractions can be described by a numerator (colored area, part of a whole) over a denominator (all the parts, the whole). Solene proposed little written knowledge exposure. In 2012, she did not teach fractions to her group of CM2, but her colleague carried out this teaching.

We are now going to describe the five components found in Solene's particular case.

Personal and social components

Solene have a good personal relationship with mathematics. She declares that she loves mathematics and considers that she has a solid knowledge in this field. She has a Master degree in the field of Developmental Psychology. She is confident enough in her choices and her professional skills to use the resource designed by the ERMEL team (a group of PEMFs, didacticians and mathematics teachers). She has been using this resource since 2008 even though this handbook is considered to be difficult.

Solene has been a teacher for 15 years, 10 of which she served as a replacement teacher. In 2008, she took the responsibility of a class in a rural school and she still teaches there today. She passed the specific exam to become a PEMF (CAFIPEMF), in 2013. She collaborates with a colleague (also a PEMF). The school team is stable. Pupils who attend the school have a good general scholastic level (above average for national evaluations), and none of the parents are unemployed.

I can conclude that Solene has a good relationship with mathematics and works with a good team in a pleasant, working environment.

Institutional component

To describe this component, it is necessary to study the official curriculum proposed by education department in 2008. In this curriculum, teaching of fractions is spread over the last two years of primary school. The proposed progression is dedicated to decimal building "from fractions to decimal fractions". It relies, on Perrin-Glorian and Douady's (1986) work without explicitly naming them. The progression is built over several steps: learn, name and write fractions of the unit, know how to break up fractions into the sum of a whole number and a fraction of unit.

After this introductory work is carried out, teaching of fractions will begin. Decimal writing or "with dot" are introduced as a different way of writing, after the one of decimal fractions, to name decimals.

Arditi (2011, p. 95), in his PhD on the use of a written resource by didacticians, explains that, "the study of fractions is only a prerequisite to install decimals". Note that the French program adds an indication on using fractions in a particular context which is the coding of size measures. The official instructions have not changed over the three years of my study even though, in France, they are often updated. The last versions occurred in 1995, 2002, 2008 and will change again in 2015.

Solene uses the ERMEL handbook, which is recommended during teacher training. This resource is appreciated by the inspectors. In France, inspectors come to observe teachers in their classrooms. They check that the curriculum is followed conforming to official instructions and, finally, they evaluate teachers. This evaluation is important for career development.

Cognitive component

Solène spent seven sessions devoted to the teaching of fractions (including evaluation) and then continued with decimal fractions. Solene introduced fractions using paper strips. Using these strips, pupils draw and measure segments whose lengths are expressed under the form of fractions smaller than a unit or as the sum of a whole and a unit fraction. To understand what is proposed to pupils, it is important to describe the activity presented by the teacher. I noted that the cognitive path was identical in 2011 and 2013, because same activities and same instructions were given to pupils. The global project remained stable.

The first situation proposed to pupils by ERMEL aims to:

— Set in mind the first meaning of simple fractions: 1/2, 1/4, 3/4, 1/8, 3/8.

— Know and use the relationships between fractions, express them in multiplicative and additive writings.

Ermel offers two activities. In the first activity, the pupils are asked to fold a paper strip in order to cut out three quarters of the strip. Then an additional instruction is given: from a strip measuring three-quarters of a unit strip, pupils have to restore the unit strip. Below is a brief description of this activity.

The unit strip is named A. Pupils have two strips of the same width (B and C strips).

— B strip longer than the A strip

— C strip is three quarters the length of A strip.

The "A-strip" is shown on the board. The "C-strip" has two folds to visualize each quarter. Pupils are now given the task on the strip B to duplicate four times a quarter, two times a half or even duplicate the "C-strip", and then add one quarter. "C-strip": can be read in two ways¼ + ¼ + ¼ or ½ + ¼.

Solene has proposed this same activity for three years, and her instructions were always the same. Despite the similar teaching pattern, variations have also been noted. The first session is organized in the same way, and pupils carry out mental arithmetic on doubles and halves. Solene then goes on to the second step of the session by presenting the paper strips. Even though questions and global project stay identical, significant differences arise according to the duration of the sessions.

The first session lasted 48 minutes in 2011 and 90 minutes in 2013. In each year, the session is divided into three main parts (Table 1).

The duration of these three parts is well balanced, but, in 2013, the entire duration of the session was doubled. For all the other sessions I observed, the time for each session was also doubled. A notable difference can be explained by taking into account what the pupils have to say. In 2013, Solene clearly understood this situation and was ready to listen to her pupils.

Mediative component

To describe this component, I will rely on the transcription of what the teacher says about mathematics. I will then propose possible explanations of why the duration is doubled for the same session.

In 2011, Solene proposed ten calculations on halves and doubles. Solene spent a very short time on observing pupils procedures. She only listened to pupils who found the correct answer, and she concluded by giving the following rule: "cutting a number or figure is two times less, cutting a number or a figure by four is four times less".

The second part of the session is dedicated to the manipulation and folding of strips and to understand

	2011	2013
Part 1: thoughtful calculation : calculate doubles and halves	21 minutes	30 minutes
Part 2: fold a strip and cut three quarters. (see instructions for session A in Table 2)	10 minutes	30 minutes
Part 3: make a one unit strip from the three quarters of this unit strip. (see instructions for session A in Table 2)	17 minutes	30 minutes

Table 1

what a fraction of the unit (context part/whole) can represent.

In 2011, it was the first time the teacher introduced this concept, using the support of paper strips. The teacher exposed knowledge at two moments. The first moment occured just after bringing the procedures together. For example, she said that "four times one quarter is equal to a unity" without writing the number one.

> 46 Teacher: So initially, you have a half plus a half equals two halves (this written on the board as ½+ ½= 2/2), you have cut each half in half and four quarters (¼+¼+¼+¼) is equal to a unit. Is that okay?

At the time the aim of the lesson was given, Solene showed what she wanted to teach but keeping within pupils' vocabulary. She oriented her explanations towards resolution strategies. Knowledge exposures were targeted to the exposure of resolution strategies.

> 82 Teacher: I have a question ... the two strategies that have been used to do the work, is to add a quarter. Second strategy ... to use a half, to reproduce a half to obtain my strip. That's it! Perfect.

In 2013, the global scenario (Butlen, 2007) was the same with a view to the written preparations of the teacher. It is at the micro level that differences arise. There was no conclusion at the end of mental arithmetic. The teacher asked pupils to expose their strategies "a quarter is half of a half", and explained that it was more difficult to find the half of 24 that of 38. In 2013, the teacher let pupils elaborate more about their procedures. I can note that half is defined as the action of "cutting by two" and "it's two times less", whereas a quarter is defined as "the half of a half". Same expressions are used with reference to the school equipment: "cut in two". Two pupils did not manage to carry out these calculations correctly.

I can only hypothesize that pupils' knowledge is acquired by action. When pupils expose their strategies, they used their current vocabulary language instead of mathematical terms. They refer to actions as "cutting by two". The teacher also uses pupils' vocabulary, which may not help the pupils learn and understand mathematical terms.

The teacher questions numerous pupils and spends a long time explaining their strategies again without proposing other reformulations. The teacher uses pupils' vocabulary and does not propose other more "mathematical" vocabulary.

Most of the pupils manage to carry out the task, and it seems satisfying enough to go ahead with the rest of the session. The following session consists in folding sheets of paper to represent what one half is.

I therefore think that this phase of mental arithmetic was a means for the teacher to recall the knowledge of her pupils on what a quarter and a half are. The teacher reminds pupils of knowledge in a numerical context. I note that taking ½ of a number (operator point of view) is not the same point of view as the fraction which is as considered a part of the whole. Whether or not pupils are able to take this into consideration without the help of the teacher is hard to tell.

During this second step, Solene finally said and wrote that four quarters are "four times a quarter, but also three times a quarter plus one quarter". A pupil proposed "three quarters plus two quarters minus one quarter". This indicates that pupils are capable of producing equalities on fractions by the use of strips. They therefore produced more varied different equalities than in 2011, but all references to the unit disappeared.

At the end of the session, the teacher asked what they have learned today. She did not do this in 2011. Two pupils were capable of answering. They said that they had "learned quarters and halves" or "strips". The teacher concluded her lesson and reminded her pupils of the successive interventions. She proposed what seemed to be a synthesis of what was said: "These are fractions; we learned to fold strips to get quarters, three quarters in a strip plus several fractions enable to reconstruct a whole strip, OK?" The synthesis underscores actions on school equipment and vocabulary used.

These knowledge exposures are not prepared in advance. The knowledge exposure moment, is planned, but the content isn't. The knowledge exposure is produced in action after a discussion with the class. They re-use certain pupils' terms, excluding others. Fraction is defined by reference to an action using school equipment. Pupils are responsible for understanding that the whole strip is a reference to the unit.

632	Teacher:	the halves ... we learned the strips? We have not learned strips. We learned to do what with these strips? Three halves ... no, we did not learn to do three halves, we did quarters. How do you call this? (Shows a fraction) What is called?
635	Pupil:	fractions
636	Teacher:	These are fractions, we learned to fold the strips to get quarters, three quarters in a band plus several fractions allow us to rebuild a whole strip. Ok?

During the following lesson, Solene distributed a text to be learned by heart at home. This text was the same in 2011 and in 2013. Whereas she spent 20 minutes to exploit this text in 2011, it took her more than an hour in 2013. Pupils were asked in 2013 to comment on every line of the text that was distributed. Once again, I noted this difference in duration and the attention given towards what the pupils had to say.

CONCLUSION

For two years, I followed 5 teachers, but in this paper, I have only developed one case. The knowledge exposure does not develop mathematical exposure but develops methodological knowledge much more. These institutionalizations (knowledge exposure) remain strongly contextualized. The context does not remain entirely the same from one year to another and depends on interactions with pupils.

The study of the five components leads us to point out regularities and variabilities. For this teacher concerned, the teaching project is consistent concerning official programs and didacticians' recommendations (context of lengths). Apparently, the teaching project remains the same, although differences appear. I question these differences. The variabilities I noticed appear at the level of the mediative and cognitive components and more precisely in the content of knowledge exposure moments. What the teacher says orally seems randomly improvised and depends on discussions with the pupils. The oral content of the lesson leans on the pupils' vocabulary and moves away from mathematical vocabulary.

The game of questions/answers, difficulties met to name mathematical objects, to express the target taught, lead me to introduce the notion of negotiated institutionalization.

Expecting the pupils to be able to say what they have learned at the end of a session seems to be quite ambitious. Taking into account what the pupils declare they understood, by using imprecise vocabulary enables them to formulate and share their newly acquired knowledge between their peers. It can be questioned, however, if it is reasonable to propose incorrect and unclear mathematical definitions. This leads me to ask myself new questions oriented towards the practices of training teachers and more precisely on their professional gestures. What mathematical and didactical knowledge should a teacher acquire in order to achieve the transition from contextualized institutionalization to de-contextualized institutionalization?

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Teachers' attention to task's potential for encouraging classroom argumentative activity

Michal Ayalon and Rina Hershkowitz

The Weizmann Institute of Science, Israel, michal.ayalon@weizmann.ac.il

This study investigated secondary school mathematics teachers' attention to task's potential for argumentative activity in the classroom. Analysis of the teachers' choices of tasks and their explanations revealed categories that fall into two dimensions: (1) Attention to the mathematics in which the argumentative activity is embedded, focusing on three aspects: the mathematics inherent in the task; the mathematics related to the teaching sequence that the task is a part of; and meta-level principles of mathematics. (2) Attention to socio-cultural aspects related to the argumentative activity. Four attention-profiles of teachers were identified: Teachers who attended to both dimensions; teachers who were attentive only to the mathematics aspects inherent in the task; teachers who were attentive only to the socio-cultural dimension; and teachers who were attentive to neither of these dimensions.

Keywords: Argumentative activity, teachers, attention, task.

THEORETICAL BACKGROUND AND GOALS

The approach taken in this study is socio-cultural; therefore, we considered approaches to argumentation as a social process in which two or more individuals engage in a dialogue, and where arguments are constructed and critiqued (e.g., Ayalon & Even, 2015; Conner, Singletary, Smith, Wagner, & Francisco, 2014; Krummheuer, 1995). In recent years, there has been a growing appreciation of the importance of incorporating argumentation into mathematics classrooms: first and foremost because mathematics is intrinsically connected to argumentation. The principal facets of argumentative activity – justifying claims, generating conjectures and their justifications, as well as evaluating arguments – are all essential components of doing and communicating mathematics. In addition, accumulating research suggests that participation in argumentative activities – which encourage students to explore, confront, and evaluate alternative positions, voice support or objections, and justify different ideas and hypotheses – promotes meaningful understanding and deep thinking (Schwarz, Hershkowitz, & Phrusak, 2010). Moreover, involvement in argumentation in the classroom may provide students with a feeling of sharing responsibility with the teacher for the learning that occurs in the lesson (Forman, Larreamendy-Joerns, Stein, & Brown, 1998).

Research suggests that the teacher plays a crucial role in establishing norms and standards for mathematical argumentation in the classroom (e.g., Forman et al., 1998; Yackel, 2002). These roles include prompting students to establish claims and justifications, encouraging students to critically consider different arguments, explicating the argumentative basis of students' claims, and supplying argumentative support that was either omitted or left implicit, presenting to students what constitutes acceptable mathematical arguments, and modelling specific ways of constructing and confronting arguments. These various roles express the rich and complicated dimensions of establishing learning environment in which mathematics and argumentation are integrated (Yackel, 2002).

Research also suggests that argumentative activities are not often implemented in class, and that teachers need adequate training and scaffolding if they are to improve classroom argumentation (Conner et al., 2014; Hiebert et al., 2003). A possible first step towards offering such assistance may be to investigate teachers' approaches to and sensitivity towards argumentation as a practice in the mathematics classroom. Our study goes to this direction: It investigates teachers' attention to the argumentative potential of mathematical tasks.

The concept of attention is currently used in studies concerning mathematics teaching (e.g., Paparistodemou, Potari, & Pitta-Pantazi, 2014; Sherin & van Es, 2005). Common to these studies is the assumption that teachers' ability to notice is a key feature of teaching expertise, and that such an ability can be improved. This study shares the above assumption, and we focus on teachers' attention to task's potential of utilizing argumentative activity, i.e., what teachers take into consideration when they choose tasks from their textbook with the aim of promoting argumentative activity in their mathematics classroom. The specific research question was: What do teachers attend to when they choose tasks that, in their view, encourage argumentative activity in their mathematics classroom? Unlike other studies, our focus on teacher attention was not part of teachers' training; rather we intended to learn on teacher's attention towards argumentation in teaching, within a situation close to their day-to-day individual work of planning their teaching.

METHODOLOGY

Participants

A group of 17 secondary school mathematics teachers in Israel, at the beginning of a year-long in-service-course, participated in the study. Their teaching experience varied from 1 to 30 years; each holding a B.Ed. in teaching mathematics or B.Sc. in mathematics. The teachers were not explicitly exposed to the issue of argumentative activity before the research was conducted. All the teachers used the same 7th grade mathematics textbook in their classrooms.

The research tool

We developed a questionnaire that focused on the teachers' practice of selecting mathematical tasks. The teachers were individually asked: (1) to choose three tasks from a unit of the 7th grade textbook (Ozrusso-Hagiag, Bouhadana, Friedlander, Robinson, & Taizi, 2012) that they view as encouraging argumentative activity, and (2) to explain and justify their choices. All the teachers taught this unit in their classes while the data were being collected.

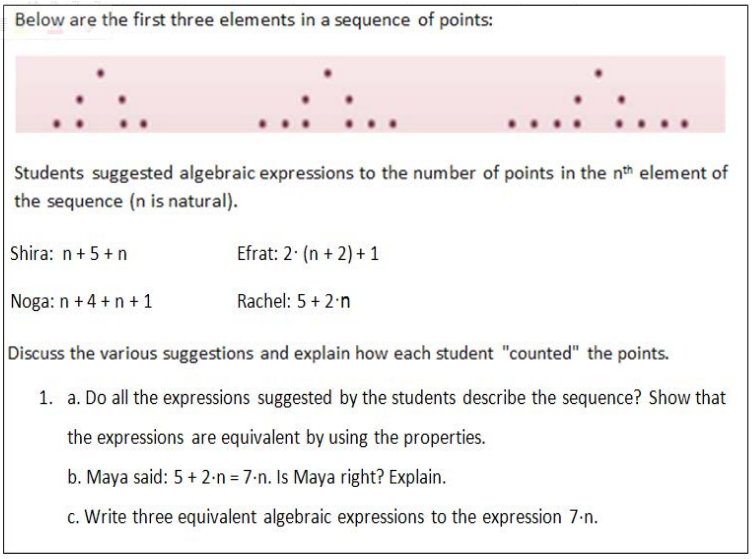
The textbook and the unit

This is the fourth unit of the 7th grade textbook, a part of the *Integrated Mathematics* junior-high school curriculum. This curriculum is specifically designed as a series of trajectories of tasks, one of the main characteristics of which is to engage students in conjecturing and justifying. The preceding three units include some work on algebraic generalization of patterns and some drilling on the properties of real numbers. The unit comprises five lessons; each includes tasks for direct teaching used as part of the whole class discussion and exercise tasks (total of 72 tasks). The main mathematical emphases are: using algebra as a tool for generalization; acquaintance with equivalent algebraic expressions; and purposeful use of simplification and substitution.

Data analysis

The 17 teachers produced 52 responses (16 teachers chose three tasks as they were asked and one teacher chose four tasks); each response consisted of the choice of a task that will encourage argumentative activity in the classroom and an explanation for their choice. Overall, 21 different tasks from the unit were chosen by the 17 teachers. As a first step, we identified the tasks' distribution within the unit, including the lesson the task was taken from (1–5), its function in the lesson (direct teaching, exercise), and the number of teachers' responses per task. We then took two directions when we analyzed the data: Top-down direction and bottom-up direction. The reason why we used top-down analysis was to learn how researchers, as presenters of the scientific educational authority, conceptualize the chosen tasks as encouraging argumentation. Our intention was to use this conceptualization as a backdrop against which to highlight and characterize the teachers' attentiveness, which emerged from the bottom-up analysis.

Top-down analysis of the 21 chosen tasks' affordances for argumentation, by adapting a widely used analytical framework suggested by Stylianides (2009), and was adapted by Bieda, Ji, Drwencke, and Picard (2014). This framework is commonly used to examine the opportunities provided in mathematics textbooks to engage in what Stylianides called reasoning-and-proving (RP), which means "the overarching activity that encompasses... identifying patterns, making conjectures, providing non-proof arguments, and providing proofs" (Stylianides, 2009, p. 259). Following this framework, we coded each chosen task according to the (1) Purpose of the RP Problem, e.g., making claims, making justifications; (2) Intended Outcome of the RP Problem, i.e., the type of justification expected: proof-type argument (demonstration or generic example), or non-proof-type argument (empirical or rational). For example, the task presented in Figure 1 was taken from the direct teaching part of lesson 1. Question a


Below are the first three elements in a sequence of points:

Students suggested algebraic expressions to the number of points in the n^{th} element of the sequence (n is natural).

Shira: n + 5 + n   Efrat: 2· (n + 2) + 1

Noga: n + 4 + n + 1   Rachel: 5 + 2·n

Discuss the various suggestions and explain how each student "counted" the points.

1. a. Do all the expressions suggested by the students describe the sequence? Show that the expressions are equivalent by using the properties.

   b. Maya said: 5 + 2·n = 7·n. Is Maya right? Explain.

   c. Write three equivalent algebraic expressions to the expression 7·n.

Figure 1: A task chosen from the direct teaching section of lesson 1

in the task engages students in generalizing visual patterns algebraically and introduces them to equivalent of algebraic expressions. Several teachers chose question b, which addresses a common mistake. Using Stylianides' framework, and assisted by the textbook's teacher guide (Integrated Mathematics, 2012), we coded the Purpose of the RP Problem as making a claim (Maya is not right) and a justification (e.g., multiplication precedes addition). The Intended Outcome of the RP Problem was coded as a proof-type argument of demonstration (e.g., relying on the properties of real numbers). We analysed in the same way all the 72 tasks within the five lessons unit.

Bottom-up analysis of the 52 teachers' responses, through which the categories of attention to the task's potential for argumentative activity were generated and consolidated, and then used to identify teachers' attention profiles, as will be elaborated below.

In both phases of analysis, the data were coded independently by each researcher, followed by a comparison of the codes. All disagreements were resolved by discussion and a consensus was reached.

FINDINGS

The distribution of the 21 chosen tasks across lessons

The distribution of the 21 chosen tasks was found to be rather homogeneous across the five lessons, and also across their function in the lesson (direct teaching or exercise). Out of 52 teachers' responses, the number of responses per each of the 21 chosen tasks was between 1 and 7.

Top-down analysis of the tasks' affordances for argumentation

Using this framework, we analyzed the 21 tasks chosen by the teachers, which revealed that the purpose of 19 of them was to make claims and justifications and the purpose of the other two tasks was to make claims; this is where 32 tasks of the 72 within the whole unit asked for justification, while the other 40 asked for claims only. In 16 out of the 19 tasks the intended outcome was a proof-type argument of demonstration, which is "at the top of the hierarchy" (Stylianides, 2009, p. 280), In the other three tasks, the intended outcome was an empirical argument, where from the 32 tasks within the whole unit, aimed at justification, 24 were of proof-type and 8 of empirical argument. Thus, we can conclude that the teachers chose tasks that by and large match Styliandies' RP spirit.

Bottom-up analysis: Dimensions of teachers' attention to the task's potential for argumentative activity

Bottom-up analysis of the 52 teachers' responses led to identifying categories that fall into two dimensions of attention:

— (D1) Attention to the mathematics in which the argumentative activity is embedded (23 out of 52 responses).

— (D2) Attention to socio-cultural aspects related to the argumentative activity, such as student-teacher interactions and the nature of the class discussion (20 out of 52 responses).

Note that each response may refer to one of the dimensions, both dimensions, or neither dimension.

Upon further analysis, we found that teachers focused on three aspects of dimension D1: (D1a) the mathematics inherent in the chosen task itself (23 out of 23 responses; this means that each response that was attentive to D1 was attentive to D1a); (D1b) the mathematics related to the teaching sequence that the task is a part of (9 out of 23 responses); and (D1c) global/meta-level principles of mathematics that transcend the particular task (8 out of 23 responses).

When looking at each teacher's responses separately we found that each teacher was consistent regarding the nature of his/her attentiveness across his/her 3–4 responses – i.e., the responses associated with the 3–4 tasks that each teacher chose fell in the same categorization in terms of D1 a-c and D2.

Overall, four different profiles of teachers' attention were found, as shown in Figure 2. It is worth to note that there was no correlation between a certain profile and a certain chosen task: different tasks were chosen by teachers of the same profile, and the same task was chosen by teachers from different profiles.

Next, we describe the various profiles in more detail and with examples.

Profile 1: Teachers whose all responses referred to the two dimensions of attention: the mathematics (D1) regarding its three categories (D1a–c) and the socio-cultural aspects (D2) (three teachers).

Example: A profile 1 teacher chose the task presented in Figure 3, which was taken from the exercise section of lesson 2. In the previous lessons students became acquainted with methods for justifying the equivalence of algebraic expressions.

She explained her choice (our coding appears in brackets after each utterance):

> Here different algebraic expressions may be suggested, some correct and perhaps some incorrect. It is possible to write all of them on the board and ask the students to say what they think about each expression (D2). The answers should be based on mathematical justifications: Use of the properties of real numbers to prove that the expressions are equivalent and use of substitution for a counter example or use of the properties to show that the expressions are not equivalent (D1a+D1c). If there is a disagreement about a certain expression, they (the students) will have to convince each other until they reach an agreement (D2). It is possible

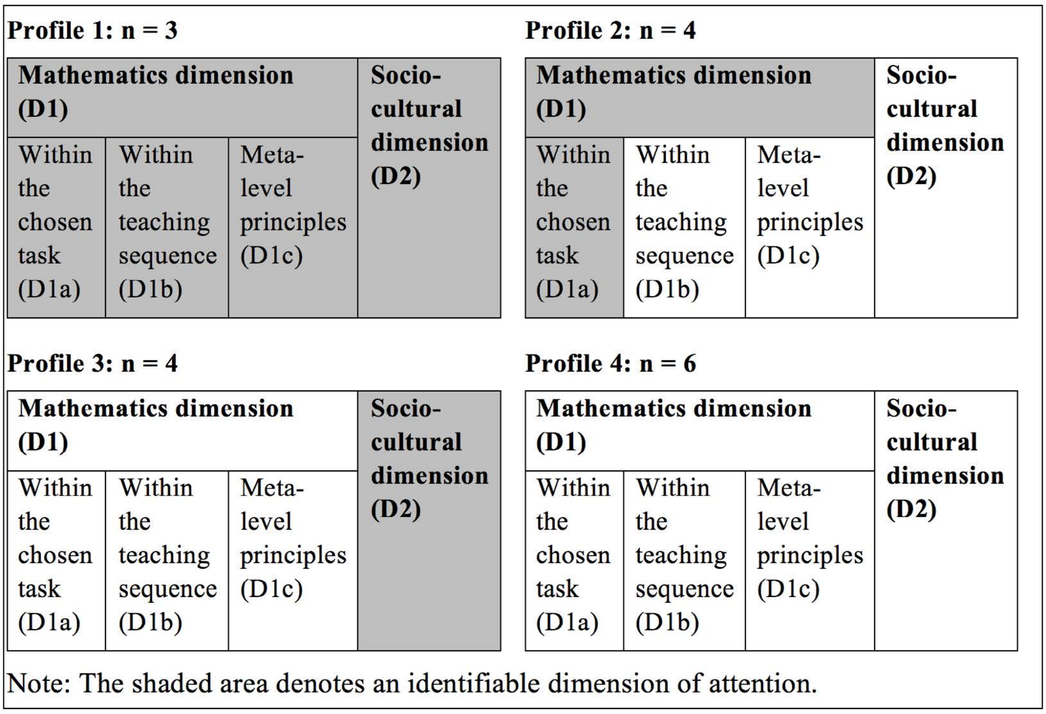


Figure 2: Profiles of teachers' attention

Write five algebraic expressions that are equivalent to the expression $1 + 6(x + 3)$

Figure 3: A task chosen from the exercise section of lesson 2

> that first students will suggest numerical examples as a proof for equivalence, but they will have to convince me and their peers that it is correct (D1a, D2). If no objection is raised in class, I might suggest a counter example in order to show them that an example is not necessarily sufficient to prove equivalence (D1c, D2). I want them to move to a general algebraic justification (D1c), which is one of the goals of this study-unit (D1b). If a student will justify equivalence by substituting a number, it is an opportunity to talk about the idea that one example is not enough for proving equivalence. Here is an opportunity to talk about how, in general, it is acceptable to justify in mathematics (D1c).

In her response, the teacher considers the mathematical justifications related to the task itself, the role of the task in the unit's learning trajectory, and the general principles of proof in mathematics, and interweaves them all in the socio-cultural process of shaping collective argumentation in class.

Profile 2: Teachers whose all responses referred to the mathematics within the task (D1a) and not to the other elements in D1. In addition, they did not refer to socio-cultural features (four teachers).

Example: A profile 2 teacher chose question b in the task presented in Figure 1, taken from the direct teaching section of lesson 1. This question was aiming at addressing a common mistake. In explaining her choice, the teacher considered only the mathematical claims and justifications related to the task:

> The task involves argumentative activity in which the argument is that Maya is not right because multiplication precedes addition, or you substitute a number on both sides of Maya's equation and receive different values, meaning there is no equality here (D1a).

Profile 3: Teachers whose all responses referred to the socio-cultural dimension (D2) and not to the D1 aspects (four teachers).

Example: A profile 3 teacher also chose question b in the task presented in Figure 1:

> In this activity students are requested to explain whether Maya is correct or not. Each student holds a different point of view. I will ask for more and more arguments. Students will describe their opinion and justify it and will have to convince their friends and me, or change their opinion and together reach the right answer (D2).

In her response, the teacher considers the socio-cultural aspects of organizing collective argumentation including raising arguments, convincing peers, and reaching a consensus. However, the response lacks any mathematics.

Profile 4: Teachers whose all responses were not considered to relate to any of the categories of attention (six teachers).

Example: A profile 4 teacher chose the same task and wrote:

> Is Maya right? Explain. Engaging in this activity requires students to raise arguments.

DISCUSSION AND CONCLUDING REMARKS

We consider as a major achievement of this research the possibility of getting close to a group of mathematics teachers and revealing some of their views concerning the argumentative potential of mathematical tasks in a textbook's unit for the teaching-learning processes in their classrooms. Our methodology was naturally integrated into the teachers' work, allowing us to reveal teachers' actual choices of "argumentative tasks" and their genuine attentiveness to different dimensions of what they consider the task's argumentative potential.

In examining the teachers' choices of tasks through the lens of established research "tools" (Bieda et al., 2014; Stylianides, 2009), we found that most of the chosen tasks are of the proof-type argument (RP) of demonstration, which is "at the top of the hierarchy" (Stylianides, 2009, p. 280), whereas more than a half of the unit-tasks did not ask for argument of any kind. This suggests that teachers were attentive to tasks that afford argumentation. Moreover, viewed collectively, the teachers attended to important rich dimensions of argumentative activity in their explanations. These dimensions – the mathematics in which the argumentative activity is embedded and the socio-cultural aspects related to the argumentative activity – reflect the complex process of establishing argumentation in the mathematics classroom and the simultaneous tasks teachers need to manage in order to facilitate it (e.g., Yackel, 2002). However, the attentiveness of individual teachers in the group was quite diverse; whereas some teachers were attentive to both the mathematics and social dimensions, for other teachers, such attentiveness was partial or nearly absent; they attended only to the mathematics embedded in the task, or only to the social situation, or did not exhibit attentiveness in either of the dimensions.

Our finding of consistency of the dimensions of attention found within each teacher's explanations for their choices and the fact that a same task was chosen by teachers of various profiles, suggest that the effect of the teacher's approach to argumentation was greater than the effect of a particular task. It is worth to note that no correlation was found between a teacher's profile and teaching experience, neither between a teacher's profile and education.

Restricted to the rather small sample, the findings of this study raise several intriguing questions and issues for further research. One is related to possible connections between what teachers attend to when choosing tasks and their actual teaching in the classrooms: Does a teacher who attends to the social aspects of argumentation but does not attend to the mathematics aspects in which the argumentative activity is embedded find it difficult to interweave the pedagogical practices successfully with the mathematical ideas (e.g., in presenting to students what constitutes acceptable mathematical arguments, in supplying argumentative support that was omitted or left implicit)? Does a teacher who focuses on the mathematics aspects but does not attend to the social aspects of argumentation make less room or find it difficult to support collective argumentation in the classroom?

Another question is related to the finding that the teachers who attended to the mathematics aspects within the task only, did not attend to the social aspects at all, whereas teachers, who attended to the mathematics aspects within the task, within the teaching sequence, and to global principles, attended to social aspects as well. Can this point at certain connections between attention to the mathematics in a broader sense and attention to social aspects? And if yes, what are the implications concerning the education and support planned for teaching focusing on argumentation? Further study will address such questions.

Paparistodemou and colleagues (2014) showed that teachers can learn to enrich their attention by reflecting on their teaching. We plan to devise ways to support teachers' development of attention, taking into consideration two main issues identified in this study: One is the diversity of the teacher population concerning attentiveness to argumentation in the learning-teaching process. The other is that this diversity ranged between attentiveness to mathematical aspects and socio-cultural aspects, and no attentiveness at all (at least not explicit). The fact that some of the participating teachers considered both dimensions as an integral part of enhancing argumentation processes in the classroom is encouraging; apparently teachers are at least partially open to adopting these "habits of teaching".

Still, another issue that emerges from our findings is related to design. We saw that the same task was chosen by teachers of different profiles; i.e., teachers saw differently "through" the task. This raises the question of how we can make a task more transparent in its argumentative potential so as to be "seen" by varied population of teachers.

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Being told or finding out or not: A sociological analysis of pedagogic tasks

Jeremy Burke

King's College London, Department of Education and Policy Studies, London, UK, jeremy.burke@kcl.ac.uk

Pedagogic tasks can be considered as a resource for engaging an audience and manipulating their apprenticeship. Different strategies can be deployed according to the pedagogic aim and what constitutes competence in the practice being pedagogised. The position taken here is that message acquirers will assume a degree of authority in the practice and the degree to which that is achieved is dependent on the possibilities made available in the task(s) set. A sociological analysis recognises that the engagement between author and audience in a pedagogic relation will lead to a distribution of message across positioned acquirer voices; the argument presented here seeks to map those possibilities in an explicit way. Texts on trigonometry are taken to exemplify this.

Keywords: Social Activity Method (SAM), pedagogic strategy, pedagogic tasks, textbooks, trigonometry.

There has been concern expressed about the use of textbooks in school mathematics teaching from a number of perspectives including a preponderance of procedures and calculation rather than conceptual development (Dole & Shield, 2008) allied with an authority position assumed in the author voice (Herbel-Eisenmann & Wagner, 2007). Whilst procedures are frequently used in mathematics practices there are other features which are available to make up a broader competence. My argument is based on the premise that in a pedagogic relation acquirers of the practice may have a range of subjectivities made available: a high level would be indicative of attaining some recognition as an adept in the practice and a low level suggests a degree of dependency (e.g., Dowling, 2009, p. 244). My concern here is the pedagogic strategies deployed in mathematics teaching may lead to the availability or restriction of access to some independent competence in the practice. My analysis is sociological insofar that I am concerned with social strategic action within pedagogic relations.

I wish to draw on Dowling's (2013) Social Activity Method (SAM) in which he outlines the practice being pedagogised as the yet to be acquired esoteric domain. The esoteric domain is recognisable through strongly institutionalised (I+) signification. In school mathematics texts I+ expression will include a specialised use of words, algebraic notation, technical diagrams or graphs. The expression will link to I+ mathematics content. The esoteric domain practice is realised through an assemblage of strategies that includes procedures and instrumentation, but also interpretive action including "discursive definitions, principles, theorems [...and] visual exemplars" (Dowling, 2013, p. 332). For further recent work in SAM see Dudley-Smith (2015), Olley (2015) and Burke, Jablonka and Olley (2014).

DISCURSIVE SATURATION

In Dowling's (1998) study of the very widely used School Mathematics Project (SMP) scheme in the UK, which was provided in sets of books, colour coded according to a 'level' of ability attributed to the readers. Here he found that there were different textual strategies played out in the Yellow (highest) and Green (lowest) series. Of particular note was a difference in the degree to which the texts provided access to the esoteric domain. A key distinction was between strategies which make the principles of the practice explicit within language (high discursive saturation) and those which are tacit in this respect (low discursive saturation) (Dowling, 2013, p. 322).

Textbooks, as extensively used resources, provide a good indicator of the specificity of classroom practice, and gives an indicator of the intensity of discursive saturation provided in a pedagogic message. For this reason, I have looked at a current, widely used, textbook series focused on the General Certificate of Secondary Education (GCSE) examination set for school students aged 16 years across England. The GCSE examinations are set and administered by examination boards one of which is Edexcel. The books I am referring to are from the Heineman/Edexcel texts, 1996 and the Pearson/Edexcel texts, 2010. The earlier series were divided into three tiers Higher, Intermediate and Foundation and the later series into Higher and Foundation, which, like the SMP books, construct a body of potential readers as having different 'ability'. I have taken, here, an example from the introduction to trigonometry from the Higher tier Books where there appears to have been a shift between the Heinemann/Edexcel texts (Pledger & Kent, 1996) and the Pearson/Edexcel texts (Pledger & Cummings, 2010) in terms of the discursive saturation of the texts.

Trigonometry

In the first chapter on trigonometry the 1996 book provides a detailed exposition based on a unit circle, generating three functions sin x, cos x and tan x. The limits of the range of the functions are introduced and then their application to right angled triangles. The hypotenuse of the triangle in the unit circle becomes analogous to the scale factor of enlargement which in turn explains the ratio sin x = opposite/ hypotenuse and so on. The text here is DS+, making relatively explicit the form and structure of trigonometric functions. I notice that the strategy in this book is also integrative of the esoteric assemblage (in providing the non-discursive visual resource to work in relation to the well specified theorems and procedures). There is, then, more *interpretative* work to be done by the ideal reader compared to the *proceduralisation* exemplified in the next paragraph

In the more recent Pearson/Edexcel textbook each chapter is written to a template where the topic is introduced largely referring to esoteric domain content. The presentations are focused on stepwise procedures for answering word problems, with call-out text boxes pointing where to do calculations, where to 'remember' certain steps or conditions and so on. The book provides an introduction to trigonometry thus:

> Key Points
>
> — The hypotenuse (hyp) of a right-angled triangle is the longest side of the triangle and is opposite the right angle. The other two sides are named adjacent and opposite. The side opposite an angle is called the opposite side (opp)
>
> The next side to this is called the adjacent side (adj)
>
> — Here is a right-angled triangle with its hypotenuse of length 1.
>
> The length of the opposite side (opp) in this triangle is known accurately and is called the *sine* of 70° and is written sin 70°.
>
> Its value can be found on any scientific calculator. Not all calculators are the same but the key sequence to find sin 70° applies to many calculators.
>
> (Pledger & Cummings, 2010, p. 383)

This text is considerably more limited than in the earlier book. There is no detail about what trigonometry is or where it comes from, as in the earlier text. It provides abbreviations and reference to calculator generated values, perhaps a literal 'black box'. A little later the book offers SOHCAHTOA to help memorise the three specified trigonometric ratios. The exercise that follows this introduction directs the use of a calculator to find the value of the trigonometric function given an angle, for example,

> Use a calculator to find the value of
>
> a) sin 20°
>
> b) b. sin 72.6°
>
> c) c. cos 60° [...]
>
> (ibid, p. 385)

This text provides more *procedural* work utilising both discursive (e.g., the procedural definitions of trigonometric ratios) and non-discursive (e.g., a calculator) resources, than the 1996 text.

In both of these books there is a pattern of introducing new content with an explanatory text of some form. Stacey and Vincent (2009) looked at modes of reasoning deployed in these introductions but they differentiated between, "a set of instructions that explain, for example, how to set up a stem-and-leaf plot ... [and] a deeper sense of explanation and connection involving derivation, justification and/or proof of a new mathematical result." This is consonant with my findings, but I shall argue that I have provided a finer analytical purchase on the strategies deployed.

OSTENSIVE AND RESERVED PEDAGOGIC MODES

One way of approaching a lesson plan is to consider what task will be set with which the audience can engage, and what prior introduction is needed in order to facilitate the engagement with the task. This presents a task as a two stage process of presentation and activity. In these texts, and those referred to by Stacey and Vincent, the content is *introduced* before the task activity. The task activity is anaphoric in relation to the pedagogic presentation: that is the presentation may make sense only through the task that follows it. The pedagogic mode, in these texts, is to point to the *how* or *why* attributes of the topic, which is redolent of Wittgenstein's description of *ostensive* explanation, pointing at the content and providing the expression, Wittgenstein (2009, 17§28). Wittgenstein also considers some limitations of "ostensive explanation" and here an alternative is conceived as a cataphoric pedagogic mode, where the task activity refers to the detail yet to be explicitly given. I turn now to an example of this, drawn from a different text altogether.

Shadows

Ollerton (2002, p. i) argues for an apparent freedom that is afforded teachers by not using a textbook in order to wrest "control away from authors of schemes and texts". Instead he offers a range of "starting points and extension ideas". Typical of these is a task, 'Shadows', included in the Association of Teachers of Mathematics (ATM) *Points of Departure, vol. 4*, as follows:

> 63. SHADOWS
> A child is standing near a lamppost. What happens to the child's shadow
>
> — if the child walks directly towards the lamppost
>
> — if the child walks in the other direction [...]
>
> ATM (1989)

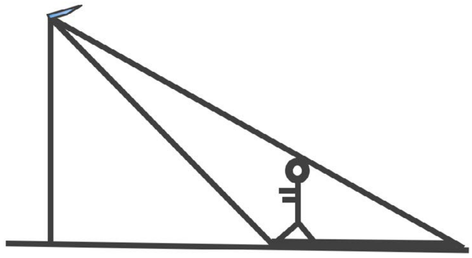

Figure 1

Here the introduction is minimal. A diagram is given showing how the shadow is formed, but other than that there is no prior guidance. In terms of a pedagogic task there has been little action on the part of the author. There is no explanation given as to what to do next. There is no procedure to follow. There is no answer at the back of the book. What then might be expected of a student engaging with such a task? It could be addressed through taking various examples, perhaps using a dynamic geometry package. Or it may lead to a recognition of 3 different triangles which are all similar and consequently have the same ratio of sides. In this case, the focus would be on a tangent ratio, and recognising that as the distance from the lamppost changes, so does the angle of elevation. However, that the similarity of the triangles maintains the ratios of the sides is fundamental in theorising trigonometry, in which case the task becomes DS+, that is the principles are made explicit.

This appears to be the obverse of the example from the GCSE textbooks. Instead of an ostensive explanation there is none given in advance of the *situation* presented. There may have been prior lessons on finding areas, or on strategies, such as Polya's heuristics, for dealing with "investigations", but these are not available from the text in this case.

The point to note is that there is still a pedagogic relation. The authorial voice sets a task, and there is little option but for the audience to engage in some way with it, and I rather suspect that Mike Ollerton's pupils, also, cannot opt out. This is because, as Bernstein (2000, p. 32) proposes, there are (at least) two discourses: the one relates to the disciplinary discourse (instructional) and the other a non-disciplinary discourse (regulatory). The latter will typically position the instructional author as the authority in respect of the disciplinary discourse also (Burke, 2011). The pedagogic relation is maintained even without an ostensive explanation. I consider this to be the case of a *reserved* pedagogy. The principles of evaluation of the practice are still retained by the author who will determine whether the performance of the audience is adequate. The ATM book does not provide any 'answers' and the opportunity is provided for definitive statements of coherence and arguments for solutions to questions developed in response to the situation. This would constitute a move towards a highly discursively saturated text (DS+), albeit one that is produced in the first instance through the engagement with the task.

However, whilst this has the appearance of the open text (Eco, 1984) it is not ultimately a *writerly* text (Barthes, 1974). The closure will come in the evaluative judgment of author, although this has been held in reserve for the duration. I will return to the discussion of the open task.

Tarsia

The three examples given outline tasks which are differentiated in terms of the discursive saturation of the texts produced, either as prior presentations or as performances based on engagement with the task. A fourth example will suffice to complete a schema. There is a widely used task activity referred to as Tarsia puzzles where students tessellate parts of a larger design according to the connection between two statements given on the sides of two pieces. An example of a puzzle requires pairing trigonometric identities, and here the blogger reports that pupils were keen to (Roy, 2013). Another example is given on the National Centre for Excellence in the Teaching of Mathematics (NCETM) website of a triangular puzzle constructed from 16 equilateral triangular pieces. A teacher says:

> I produced a triangular Tarsia puzzle for my class on the topic of number sequences. The questions consisted of number sequences with two missing terms [...] I gave out the puzzle to the students with no input at all and was specific in not providing any initial assistance. Although some students were disconcerted at first, they soon settled to the task and quickly solved the puzzle. (NCETM, 2010)

The task has no prior introduction from the teacher, so similar to Shadows, there is a reserved pedagogy. However the reservation appears to have lasted a relatively short period of time as the student "quickly solved the puzzle". The solution does not, as it stands, lead to making the principles of the 'solution' clear. In fact the teacher notes that students use a strategy of answering easier questions first suggesting a probabilistic response to the final questions. The task activity consists of a set of components to be assembled, but this is a DS- text.

PEDAGOGIC TASKS

I am now able to show these four tasks as pedagogic strategic engagements in a SAM type schema. The horizontal axis distinguishes DS+ and DS- discourse. The vertical axis distinguishes the pedagogic mode as Reserved or Ostensive.

The cells now give a typification of a strategic process of task engagement. With reference to the 1996 trigonometry text, there was a clear exposition of the principles followed by word problems including some of an analytical nature. The text was DS+ and the pedagogic mode was ostensive. This can be characterised by the anaphoric relation *exposition and problem*. The more recent GCSE textbooks had a rapid demonstration of how to find the value of a trigonometric function using a calculator and the application of SOHCAHTOA, followed by exercises using a calculator. This I am showing as *demonstration and drill*. The 'Shadows' task, on the other hand had a cataphoric task-pair relation. The task had no prior introduction other than the situation. However the production, following engagement can be considered in the form of a composition, which in mathematical terms would include an argument for both the strategy adopted and the coherence of any statement – 'discursive definitions and principles' (Dowling, 2013, p. 332) The task offers access to the esoteric domain

	Discourse	
Pedagogic mode	DS+	DS-
Reserved	situation/composition	components/assembly
Ostensive	exposition/problem	demonstration/drill

Figure 2: Pedagogic tasks

albeit through the action of engagement with it. I have termed this pair as *situation and composition*. The final cell is exemplified in the Tarsia task, with no introduction and little development of esoteric domain principles, comprising the presentation of *components* to be *assembled*.

The scheme does not fix action around pedagogic tasks. Empirically there will be activity by both the practice adept and the audience being potentially apprenticed to the practice. There can be moves between the task instantiation and the task activity. The way that the task is read will also depend on a number of factors.

READER THEORY AND CUEING

Weinberg and Weisner (2011) are concerned with how students use a textbook and develop an analysis which contrasts the intended, implied and empirical reader of the text. They note that whilst the text constructs an ideal reader the empirical reader may not conform with expectations. The pedagogic tasks outlined above may be responded to in a way which seeks to change the strategy initially promoted. The teacher in the Tarsia report observes that the "students were disconcerted". Weinberg and Weisner note that undergraduate mathematics students might look for a rule or procedure to guide them through their task activity. If the opening presentation had been in exposition mode, the students might respond as though it was a demonstration. Similarly the presentation might be in one format, but a response from the reader might lead to a change. In the Shadows example, the situation was given and a response could be to raise a question, "What should I do now?". Such a response is also strategic. Maintaining a reserved pedagogic mode the response to the cue to close down the task would be to ask a question in response. This could be as bald as, "What do you think you should do?" More productive questioning might be along the lines of Brown and Walter's (2004) 'What if?' and 'What if not'? If the response was to provide a procedure, 'compute the ratio of the height and base of the triangles' the move would be towards demonstration and drill.

Weinberg and Weisner also draw on the idea of Eco's open text where the invitation is for the reader to compose in response. This is the appearance of tasks such as Shadows, but the task is given in the context of a mathematics classroom. The text is already subject to its setting, and the reader expectation of (minimally) a mathematics classroom will also be a feature of a reader response. I wish to argue that the text is not open and that the relation between the author and reader is pedagogic – whether it is the author of the textbook or the empirical adept.

The form of responses to cues might be considered as further questioning to maintain a reserved DS+ strategy or simply a correction to maintain an ostensive DS- strategy. In the other two cases, then an explanation maintains an ostensive DS+ strategy and a hint a reserved DS- strategy. This is shown in Figure 3.

Herbel-Eisenman and Wagner (2007, p. 8) are also concerned about the positioning of students within mathematics textbooks and they provide a framework to examine "the way a textbook might influence a mathematics learner's experience of mathematics." The analysis is based on "the social positioning experienced by students" (ibid, p. 10).

They are careful to note that an analysis of a text does not give an unequivocal reading of the positioning effect on the reader. This is consonant with Dowling and Burke (2012) who pointed out that the various gendered representations in mathematics textbooks appeared not to have had an effect on girls out-performing boys in GCSE, at least up until 2010 when the format of the exam was changed. It seems likely that the pedagogic tasks deployed will have more of an effect on outcome, which is perhaps why the later Pearson/Edexcel texts provide, almost uniformly, tasks formed from demonstration and drill strategies which have been associated with a gradual rise in the number of GCSE passes at grade C and above over recent years. The 'effect' on learning mathematics might be different from an effect in terms of a wider social difference in other settings.

	Discourse	
Pedagogic mode	DS+	DS-
Reserved	situation/composition/question	components/assembly/hint
Ostensive	exposition/problem/explanation	demonstration/drill/correction

Figure 3: Pedagogic tasks and adept cuing

Stacey and Vincent (2009) in their study on reasoning in school mathematics textbooks concluded:

> ...the critical point for developing students' mathematical reasoning is whether they understand that some modes of reasoning are indeed part of the acceptable range of reasoning in mathematics, whilst others serve a local pedagogical purpose, such as helping them remember a rule, building connections between topics, making mathematics plausible. Textbooks could more often make these distinctions explicit, and in so doing, give students a stronger sense of mathematical justification. (Stacey & Vincent, 2009, p. 287)

Indeed textbooks could be more explicit about the strategies they are deploying but it is also perhaps as critical that mathematics teachers have some clarity about the resources they are using. The current Pearson/Edexcel textbook adopts, almost entirely, a pedagogic strategy of demonstration and drill, similar to the observations of Dole and Shield rather than distinctive, contingent strategies observed by Stacey and Vincent. The schemas I have given provide a coherent description of the process of interaction and pedagogic strategy in pedagogic tasks, demonstrated here through mathematics texts. They might also prove to be productive in classroom contexts to analyse the process of task presentation and student engagement in terms of whether they are being told, or they are finding out, or if there is a ubiquity of procedure.

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Elementary teacher practice in project work involving statistics

Ana Caseiro[1], João Pedro da Ponte[2] and Cecília Monteiro[3]

1 Instituto de Educação da Universidade de Lisboa and Escola Superior de Educação de Lisboa, Lisbon, Portugal, anac@eselx.ipl.pt
2 Instituto de Educação da Universidade de Lisboa, Lisbon, Portugal
3 Escola Superior de Educação de Lisboa, Lisbon, Portugal

This study aimed to analyze the practices of an elementary teacher in conducting project work involving statistics. The study follows a qualitative case study design with data gathered by means of video and audio recording of the teacher's class. The results showed that the teacher followed an exploratory approach with constant reflection during the teaching process. The results also showed a diversity of types of communication and teacher's interventions during the development of the students' projects and a high concern for students' involvement.

Keywords: Teacher practice, statistics, project work, communication.

INTRODUCTION

In Portugal, the 2007 Mathematics Curriculum of Basic Education emphasizes statistics, recognizing that this is a key area in modern society in which students must have a good preparation. To achieve this, students must carry out statistical investigations in school (Groth, 2006). Wild and Pfannkuch (1999) suggest that this is an important way of the learning of statistics.

Teachers' practices are one of the key factors that influence students' learning (Ponte & Serrazina, 2004). So, in order to support the quality of teaching and learning, it is important to develop research on this field (Franke, Kazemi, & Battey, 2007). As project work is about formulating questions and finding information to answer these questions in an engaged and participative way, our research question is to know how do teachers explore students' ideas and promote discussions during this process.

TEACHERS' PRACTICE IN STATISTICS

The concept of teachers' practice is a complex one. O'Donnell and Taylor (2007) emphasize the tasks that teachers use in their classes. Among the various kinds of tasks, project work can be used to foster students' learning as it favours their affective and cognitive involvement. Alarcão (1996) defines six phases in carrying out project work: (i) Choosing significant problems based on the desire to solve them or at least to study ways of solution; (ii) Establishing a plan and outlining ways of doing things, providing resources, planning possible interventions, dividing tasks and managing time; (iii) Contacting with reality, doing data collection through field work; (iv) Organizing and processing data, by comparing, analysing and reflecting on the data; (v) Preparing presentations to others to make known to others the results and the processes experienced; and (vi) Presenting the most significant aspects in a motivating way followed by evaluation of all the work carried out.

The stages of a statistical investigation are similar to those of project work. Problem solving using data is carried out through the investigative cycle (Burgess, 2007). Graham (1987) and Franklin and colleagues

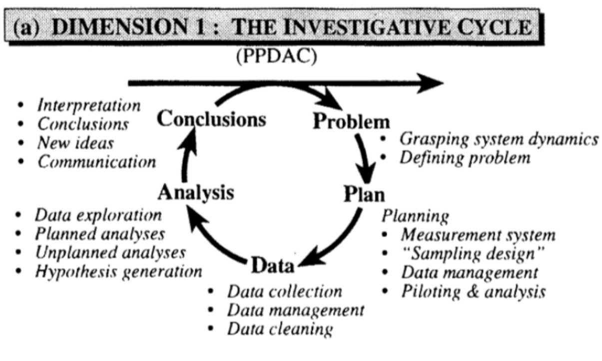

Figure 1: The statistical investigation cycle (Wild & Pfannkuch, 1999)

(2007) refer that a statistical investigation typically involves four steps: asking a question; collecting data; analysing data; and interpreting the results in an organized manner. Wild and Pfannkuch (1999) consider five basic phases of a statistical investigation: choosing the problem, defining a plan, gathering data, analyzing it, and drawing conclusions (Figure 1) and consider three additional key dimensions in statistical work: types of thinking, questioning cycle and provisions.

As Makar and Fielding-Wells (2011) refer, the formulation of the problem is important and should be related to the students' interests. The research question is the starting point of the work and is often overlooked by teachers who end up focusing on the other phases of the cycle. The problem to be investigated should take into account aspects such as the age level and the mathematical development of the students (Ponte, 2001). The second phase of the investigative cycle (plan) is also an important step. According to Shaughnessy (2007), little time is devoted to this phase. Most students are only taught "pre-statistical" topics, being presented with a ready-made problem formulation, plan, and production of data. The next two stages (data collection and analysis) allow students to become familiar with statistical concepts and representations. Finally, in the last phase of the cycle (conclusions) students should be able to verify if their initial questions were answered or whether is necessary to formulate and conduct a new investigation.

Another key aspect of teacher's practice refers to the nature of the classroom communication (Franke, Kazemi, & Battey, 2007). Brendefur and Frykholm (2000) propose a model with four types of communication: (i) uni-directional communication, where the teacher dominates classroom discourse while students listen, so they can reproduce, i.e., in this type of communication the main role belongs to the teacher; (ii) contributive communication, where there is a large number of interactions between actors (teacher and students and among students), although mostly short and where the teacher continues to have a major and dominant role; (iii) reflective communication, where the work carried out is subject to constant reflection, with the students engaging actively in discourse and having freedom to express themselves; and (iv) instructive communication where the teacher systematically draws on students' contributions to improve teaching and learning

Wood (1999) notes the importance of teachers leading students to justify their choices, encouraging everyone to participate in classroom discussions. Based on previous research, Ponte, Mata-Pereira and Quaresma (2013) developed a framework for analyzing the teacher's actions in conducting mathematical discussions, distinguishing four main types of actions: (i) inviting, in which the teacher promotes the initial involvement of students in the discussion; (ii) supporting/guiding, in which the teacher leads the continued participation of students through questions or other interventions; (iii) informing/suggesting, in which the teacher introduces information, providing arguments or validating student responses; and (iv) challenging, in which the teacher seeks that students make generalizations and justifications and question the responses of their colleagues.

METHODOLOGY

The methodology is qualitative and interpretative, following a case study design (Bogdan & Biklen, 1994). We study the case of a grade 4 teacher, Maria (fictitious name) who is 28 years old and has been teaching for 8 years in a private school in Lisbon. After getting her degree as a primary school teacher, Maria strived to deepen her mathematics preparation, making a master's degree in mathematics education and is now enrolled in a doctoral program in mathematics education. In addition, she regularly attends mathematics teachers' professional meetings. In this paper, we analyze her case, as the construction of the other cases is still in progress.

Data collection was carried out from April to June of 2013. Usually, she proposes statistics tasks with situations in which a graph is given and the students had to answer questions about it. The teacher showed some discomfort with this practice, stating that "I would like to do a work like that with other curriculum topics [such as environmental issues] including project work by the students".

The data presented in this paper was gathered through video and audio recording in seven classes taught by Maria. It is presented in the various stages of the investigative cycle of Wild and Pfannkuch (1999) and analyzed according to the types of communication proposed by Brendefur and Frykholm (2000) (uni-directional, contributive, reflective, and instructive communication) and teachers' actions suggested by Ponte, Mata-Pereira and Quaresma (2013) (inviting, supporting/guiding, informing/suggesting and challenging).

The decision to conduct project work with the students was taken on a collaborative group work that was composed by the first author of this paper (hereafter referred to as researcher), Maria and two other grade 3 and 4 teachers. In this session the researcher proposed to discuss the investigative cycle (Wild & Pfannkuch, 1999) and an article about different types of tasks, which included an explanation of project work. After this discussion the teachers decided to use this material to plan their lessons in the frame of project work. For teachers, a key idea was to engage students in their own learning, motivating them through interesting themes, and having them working in small groups. As the teachers had already worked tables and graphs with the students, it was decided that this work would verify the students' knowledge on these representations. Thus, the work was planned, discussed and adjusted together in the teacher collaborative group. The research presented here refers to data collected in Maria's classes in which the researcher assumed the role of participant observer, collaborating with the teacher and interacting with students.

The introduction of the task sought to involve the students in all decisions, especially in deciding how to format their project as a statistical investigation. The only condition imposed was to use quantitative data that could generate statistical representations. The first lesson was intended for proposing the project work, with the teacher seeking to motivate and engaging students. In the second lesson, in a plenary session, all suggestions for topics and study questions were discussed. The following three lessons were reserved for the collection, compilation and processing of data. The sixth lesson aimed to analyse the data and to prepare the presentation and the seventh lesson was designed to the presentation of results to the whole class.

MARIA'S PRACTICE

Introduction of the task and choice of the projects

Maria begun by informing the students that they would develop a project work involving data handling and explaining the different phases that they should follow. The teacher referred that all the work would be discussed with them. A student asks:

Ana: But we are supposed to do a project work? It takes so long.
Teacher: I do usually tell you what to study? No. Now what we want to do is to make a project work. It takes time, very well, so we have to organize ourselves. Do you know the planning sheet that we usually do? We will also do it to organize our work.
José: You can choose to be a 2 in 1.
Teacher: The theme is of your choice and our concern is that you have to present information in graphical representations.

Thus, the teacher introduced the work seeking to engage students in making decisions about the whole process. This introduction led the students to begin thinking about what they would like to study. The teacher guided the students' work from a perspective of reflection by all (*reflexive communication*), promoting their early involvement in the discourse (*inviting* actions), giving them freedom to express themselves. The challenging nature of this proposal made students to promptly begin participating in the discussion.

The students indicated some topics of interest. The class discussed whether their study made sense to carry out and was of general interest. Many students made suggestions:

Beatriz: We had the idea of making Chinese characters. Are more than ours.
Teacher: Well, are symbols, so it is so hard.
Mafalda: In which countries are different from the letters A, B, C, D, E.
António: It was also turning to ask what they think: think how many letters have alphabet X.
Teacher: But that's just an opinion, an answer.
Mafalda: They could say how many think they have every number of letters and then give the correct answer.
(...)
Teacher: The group of Rome: "What monuments? What are the Roman gods? What is the number of visitors who visited one (or

more) monument each year?" What they want to know with this project work?
Leonor: We want to know the number of visitors to monuments per year.
Ana: Why not make the Greek gods who are over?
José: Why do not the mean or the mode of the number of visitors?

This dialog shows that students were quite participative, making suggestions for the work of their colleagues. The teacher kept a low level of intervention thus enabling the discourse to be carried out mainly through student-student interactions (*reflexive communication*). All the students proposed themes and questioned their colleagues who received their opinions and questions in a constructive way. The teacher involved all the students in the discussion (through *inviting* actions) and promoted their participation, leading the discourse through *supporting/guiding* actions. In the end nine themes were raised by the students to study: "Monuments and Roman gods"; "Stars and planets"; "Curiosities about the world"; "Sharks"; "Rally Dakar"; "Butterflies"; "The strangest animals in the world"; "Trivia about writers"; and "Professions".

The plan

The teacher began by referring the importance to design a plan to serve as guide to carry out the subsequent work. With this goal Maria distributed a sheet of paper with a model in order to guide the students to organize the work reinforcing the idea that they should record the questions they wanted to study and begin to think about the representations they intended to build with the data collected to answer each of their questions. The teacher emphasized that they should think about what kind of graph they considered appropriate to use.

A student asked the teacher if they could put some information about the topics under study without doing graphical representations. At this stage, Maria showed a more directive attitude, not allowing a discussion of some aspects of the work she had already decided (as the work plan) or had to decide at the time of the class (such as the introduction of a part writing more information on the subjects). In spite of allowing students to be inquirers, she took a leading role intervening at all times of discussion in the large group (*informing/suggesting* action). In turn, when supporting the small groups in the development of their work plan, the teacher showed a less interventionist attitude and a more questioning role. She led some groups to think about the issues that their work would address and on the adequate representations concerning the data they would collect. Thus the kind of communication privileged by the teacher was *contributive,* and also, at times, *reflective*.

Data collection

To collect the data they need for the project, some groups decided to search for information in magazines, books and internet, while others decided to construct a questionnaire. The teacher supported the groups in this crucial phase of the investigation cycle, especially two groups that had never done this kind of work. One of these groups decided to study the occupations of parents of students in the class, based on a small questionnaire. The teacher provided some guidance questioning the students about the various aspects they could ask and suggested some elements which students could not be aware:

Teacher: And no matter whether you are male or female?
Afonso: But to know that we have to put that "is the father of Monica" or "is the mother of Duarte".
Teacher: But that is what is important or you just want to know the occupations of the parents of the class in general?
António: Oh, we could get to put a cross in the genre, and then write the name of the child.
Teacher: Oh, good. And they should not put the objective of the questionnaire?
(...)
Afonso: We want to know the unemployed.
Teacher: But knowing what about the unemployed?
Afonso: The number.
Teacher: And do you not ask parents what college degrees they have? And you are not interested to see if the parents are working in your area?

This excerpt shows the help that Maria gave to her students who had never built a questionnaire. When she felt that the students could reflect on their issues and could improve by themselves the wording of the questions of their questionnaires, she performed *supporting/guiding* actions. When she realized that the work was not moving forward she suggested what students should put in the questionnaire, with *informing/suggesting* actions. In this phase the teacher was both a guide and a reflective participant privileging *reflective communication.*

Data analysis

The students had to process and analyze the data (organizing it in tables in Excel and then making a graphical representation). The teacher circulated by the several groups in order to support their work and to assist in solving their questions. When checking that the question was not just specific of a group or referred to aspects already worked out and discussed in class, Maria conducted a whole class discussion:

Teacher: Now stop a little because now I want to discuss something with you. All of you can help that group studying butterflies... The group of butterflies discovered several species of butterflies and so now they want to make a bar chart and want to put here [in the x axis] the name of the butterfly and then here [y axis] heights. And my question is: this is a bar chart?

Gonçalo: It's not because it has no frequencies.

Teacher: And what is the frequency?

Gonçalo: It was for example if they where [x axis] had 2m to 5m. Imagine that these two [butterflies] one has 3m and another has 4m, then both were in that range and so it was a bar chart with frequency.

Teacher: I do not know if it is okay but what he is saying is that maybe it would be a good idea to arrange size ranges. (...) You must have what the Gonçalo said: frequencies, how often it happens, the number of times, the number of butterflies with 20cm (...).

In this example the teacher strived to involve the entire class in discussing an aspect of interest to several groups, the construction of a bar chart. She took the opportunity to explain students about the concept of bar graph, stressing the distinction between variable and frequency. The teacher assumed the central role of the classroom discourse by systematizing some knowledge of statistical representations, and the role of students was to listen to her explanations. At this stage of the investigative cycle the teacher tended to have a *uni-directional and contributive communication*, and her main actions where *informing/suggesting* and *supporting/guiding* students.

Conclusions and presentations of projects' results

Due to the end of the school year students ended up not having much time to draw conclusions from the data analysis that have made, and for preparing presentations. Maria suggested that all groups prepared a presentation with the data that they had already collected and analyzed so far. Thus some studies only presented a brief analysis of the data, and the teacher questioned some aspects to lead the students in reaching some conclusions. For example, the group that conducted a study on the Rally Dakar elaborated the graph (Figure 2) that displays information about the bikes used by the winners between 1979 and 2011. The students were asked by the teacher:

Teacher: So if we want to win you have to be with a ktm?

Miguel: The ktm is usually that almost all winners use. It is almost always the ktm wins.

Despite the short time available for this phase was still possible for the teacher to discuss some important aspects with their students, with a great student's involvement in the discussion, having the possibility to express themselves and to question the colleagues. At this stage of the investigative cycle, the role of the teacher became more questioning and challenging providing the formulation of conjectures by students.

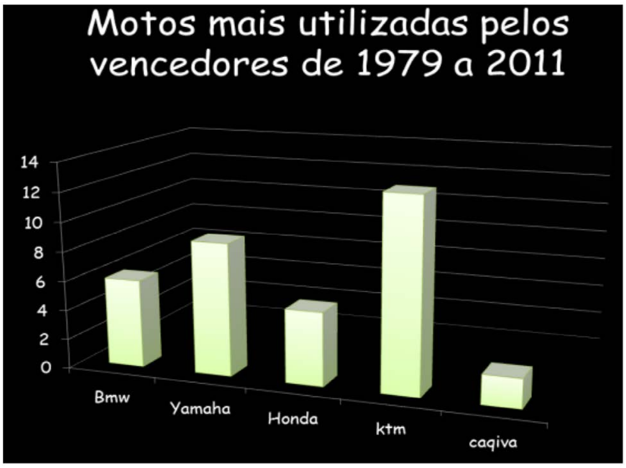

Figure 2: Graph constructed by the group, whose project was on the Rally Dakar

FINAL REMARKS

In the introduction of the task and in the phases of *problem*, *data* and *conclusions* of the investigation cycle (Wild & Pfannkuch, 1999) the teacher encouraged the students to get involved in the classroom discourse with complete freedom of expression. The communication developed along these phases may be seen as *reflective communication*. In the reflection phase of the *conclusions* the communication promoted by the teacher tended towards *instructive communication*. In the phase *plan*, Maria tended to promote *contributive communication*, notably in the whole class discussions in which she allowed small interactions between the students but made many validations and kept a major role in the decisions. This aspect may be due to the fact that she did not feel safe in carrying out project work involving statistics, as it was the first time she was doing it in her class. In turn, during small group discussions, the teacher enabled the students to discuss the issues and representations they wanted to do, which demonstrates *reflexive communication*. In the *analysis* phase, Maria tended to dominate the discourse explaining and transmitting knowledge, assuming a dominant role in the classroom discourse. Sometimes she put some closed questions and gave direct responses, which seems in line with *uni-directional communication* and *contributive communication*. This may be due to the fact that she felt the need to address some statistical aspects with the class that had already been worked before, as she just realized that such aspects had not been understood by students.

In terms of the actions of the teacher, they varied depending on the progress of the class and the development of the projects. In the introduction of the task these actions were mainly *inviting* students to participate, which also happened in the choice of the *problem*. At this stage, Maria also took *supporting/guiding* actions in the discussions. During the phases *plan*, *data* and *analysis*, the teacher tended to *informing/suggesting* actions, which during *data* and *analysis* are complemented by *supporting/guiding* actions. Finally, the phase of the *conclusions* the teacher tended to *challenging* actions, seeking to lead her students to go further

Finally, the choice of project work methodology shows that the teacher follows an exploratory teaching approach. In this approach the teacher does not have full control, to the extent that, for example, she cannot predict what topics and issues the students will select to study and how they will want collect the data. In summary, and responding to the research question, how do teachers explore students' ideas and promote discussions during the project work process, we see that the teacher enabled students to express themselves, using mostly *supporting/guiding* actions and following a *reflexive* communication.

ACKNOWLEDGEMENT

This work is financed by national funds through FCT – Fundação para a Ciência e Tecnologia, under the project Developing Statistical Literacy (contract PTDC/CPECED/117933/2010).

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An assessment practice that teacher José uses to promote self-assessment of mathematics learning

Paulo Dias[1] *and Leonor Santos*[2]

1 Secondary School of Moita, Moita, Portugal, paulo.dias.7@gmail.com

2 University of Lisbon, Institute of Education, Lisbon, Portugal, mlsantos@ie.ul.pt

In this paper, we present some results concerning an assessment practice, composed of oral interaction between a teacher and students that a mathematics teacher, José, in a context of a collaborative work, uses to develop the capacity of self-assessment of his secondary school students. The assessment practice developed by José includes two ways to promote self-assessment: self-assessment of the answer and self-assessment of performance. While the former is ensured by cognitive and motivational strategies, the latter is mostly based on meta-cognitive strategies.

Keywords: Assessment practices, self-assessment, mathematics learning, collaborative work.

INTRODUCTION

The tasks that simultaneously serve to teach, learn and assess students' performance have an effect on their school results, mainly because they are not associated with the practice of common assessment and aims to promote self-assessment skills for mathematics learning (Hodgen, 2007). Moreover, when teachers analyze students' answers and give them an appropriate feedback, they deepen the knowledge that students need to organize information and reply (Price, Handley, Millar, & O'Donovan, 2010). This practice helps the student to get a better knowledge structure, allowing an adequate self-questioning, contributing therefore to develop the capacity of the students to self-assess. It promotes mathematical efficiency and self-assessment of the mathematical knowledge, skills and capacities (Quinton & Smallbone, 2010).

But in Portugal, as in several other countries, this kind of practice is really still far from reality in mathematics classes (Santiago, Donaldson, Looney, & Nusche, 2012). We studied, in a collaborative work context, assessment practices done by secondary school's mathematics teachers, whose aim was to promote self-assessment of mathematics learning. In this text, we present only part of the study (Dias, 2013), focusing upon a secondary school mathematics teacher, José, and one assessment practice, the oral interaction between teacher and students (IT-S). In particular, we aim to answer to the following research question: What is the nature and the characteristics of IT-S assessment practice of a secondary school's mathematics teacher, developed in a work context of collaborative nature, which seek to promote self-assessment of learning?

THEORETICAL OVERVIEW

The study of teaching practice of mathematics teachers is relevant to understand their performance in the classroom (Ponte & Chapman, 2006). The understanding of the meaning given to the mathematics teachers' decisions contributes to deepen the knowledge of how the mathematics teachers work in the classroom. Regarding assessment practices in specific, it is not sufficient to assess whether students have mastered facts and algorithms or developed attitudes, skills and knowledge advocated in mathematics curriculum. It is necessary that assessment practices reflect teaching and learning processes. For this purpose, rich tasks are required (Smith & Smith, 2014) and the assessment questions have to be constructed so that, when analyzing the answers of the students, it is possible to get an idea of how students organize information (Price et al., 2010). To accept that the students have an essential part in the construction of their knowledge implies that the teacher must pay particular attention to feedback processes and self-assessment of learning (Santos, 2002).

The focus on self-assessment of learning is justified by the importance of student success in mathematics assignments and consequently in mathematics learning. Self-assessment develops the ability to assess a task and to implement the necessary corrections or adjustments. It is a group of actions that the student develops when regulates his or her own work (Zimmerman & Schunk, 2011). Self-assessment is the process in which students develop the strategies needed to achieve the desired objectives, creating conditions for a successful learning. But self-assessment of mathematics learning do not develops spontaneously (De Corte, Mason, Depaepe, & Verschaffel, 2011). It is the teacher who has the responsibility to promote it through several actions. This study emphasizes the oral interaction, conversation between students and teacher while performing a mathematical task (Henning, McKeny, Foley, & Balong, 2012). Because it is an intentional action from the teacher and occurs in the daily work of the classroom, may be considered an assessment practice for learning (Wiliam, 2007).

Interaction and communication in a mathematics classroom is definitely essential to improve student learning (Santos & Semana, 2012). Nevertheless, some aspects have to be respected. On one hand, teacher has to use relevant information about students reasoning and ways of learning in order to deal properly with the process of teaching and learning (Pinto & Santos, 2006). On other hand, some conditions have to be respected concerning the act of questioning, such as not correcting errors but giving clues, not confirming but asking in a way that leads the students to develop a convincing argument about their reasoning.

IT-S is characterized by questions, stimuli and directions given by the teacher during the implementation of a mathematical task. The impact of this practice may be conditioned by the opportunity of the intervention and the increased student confidence in building their mathematical knowledge (Schwarz, Dreyfus, & Hershkowitz, 2009). In this practice, the teacher should avoid correcting errors and adopt an attitude that contributes to students formulate questions independently. Students can be referred to their own productions or to the proposed tasks, or can be suggested to share and discuss, in pairs or in groups, their interpretations of the answers (Henning et al., 2012).

METHODOLOGY

Following a qualitative and interpretative methodology approach, through a case study design (Stake, 2009), the assessment practice of a secondary school's mathematics teacher, José, has been observed along two school years. José was chosen because of his recognized experience (31 years of teaching, both in schools of basic and secondary education), and his willingness to develop assessment practices that promote self-assessment.

The collaborative context (Jaworski, 2007), constituted by José, another secondary school's mathematics teacher and the researcher (first author), was created by invitation from the researcher, and has as main objective to develop and implement assessment practices to promote self-assessment. Episodes of classroom reported in almost all the texts read by the group were considered essential to trigger discussions and to define the assessment practices to study. After each lesson, there were moments of reflection between the three members of the group. The selection/creation of mathematics tasks was a great challenge for teachers when seeking to integrate assessment, teaching and learning (James, 2006; Pinto & Santos, 2006) and to promote self-assessment of mathematics learning (Black & Wiliam, 2006). Noteworthy the assessment practices planned by the group features challenging tasks (Smith & Smith, 2014), including the themes Trigonometry, Geometry and Functions.

Data collection was done in a 11th grade class (students aged 16–17) and included observation of 10 mathematics classrooms (A), that includes 5 tasks, respectively with trigonometry, geometry and mathematical functions, and collaborative work sessions (ST) from February 2009 to April 2010, with audio recording, a structured interview to José at the beginning of the study (E), and documental analysis, that collected documents used by the teacher and the mathematical work done by the students.

Data analysis was performed by content analysis considering three domains, planning, implementation and reflection (Clark & Peterson, 1986). Planning, before class was focused on the role of the teacher in interpreting, managing, planning and putting into practice his curricular choices. The implementation, during class, focused in the actions of the students during the development of the task and the role of the teacher in the oral interaction between teacher and student. Finally, the third moment, reflection, after class, helps teachers to make progress in their professional development and to build their own way of knowing. In this moment, the teacher makes explicit the strengths and weaknesses of his assessment practice.

In each of the distinct phases sought by José's actions showed his intention to promote the success in the task, either by teacher's questioning of the analysis, and the analysis of the self-regulation of mathematics learning by the student. Later, elements were organized as seen in Table 1.

Self-assessment of the answer	Commitment with mathematical tasks
	Stimulus to individual strategies
	Articulate student ideas
Self-assessment of performance	Self-regulate mathematics efficiency
	Self-assessment

Table 1

ORAL INTERACTION TEACHER AND STUDENTS (IT-S)

Planning

The assessment practice was defined and planned within the collaborative work. Initially, students' difficulties were identified. According to José, the most relevant ones were written and oral mathematics' communication as well as the understanding of mathematical content, although students master some techniques in solving exercises. In this context, the IT-S planning gave attention to the selection of the task and to the support given to students during the task, avoiding correcting the errors and to give too many guidelines for shaping. According to José, questions and stimuli are guidelines that can help students *to increase student confidence in improving their mathematical knowledge*:

> I understand the students' difficulties, but how can I help them? I might answer through interaction. Asking, stimulating, and giving clues to each one. [ST, 12nd]

Looking for the best approach in the classroom was essentially one of the texts discussed in the collaborative group (Santos, 2002). To avoid correcting errors was considered an important contribution *to help students to think for themselves* while solving tasks. Some of the questions suggested in the text were harnessed by José as an example of how it should be the role of the teacher in the classroom:

> "What have you done?", "Why did you take that option?", "Why did you think like that?" "Where did that idea come from?", "In which other situations does this process may be applied?", "If you want to convince someone that this is true, what would you say?" and I would wait for their reaction. [ST, 12nd]

In achieving the tasks, José prefers that students work in pairs, considering that this method may help students to learn:

> If students work in pairs, they can help each other, explore, understand and look for the solutions. [E, 1st]
>
> José recognizes the importance of questioning to help students to accomplish the task. This action neither includes the correction of errors, nor provides many guidelines. He seeks students to explore their work in pairs or in groups, encouraging discussion among students.

Implementation

While the pairs of students develop the task, José observes and interacts with them.

In a task of Trigonometry (Figure 1), José pointed out for the importance of the task assignment and called attention for it, trying *to commit the students to the task*:

José:	Read again. You should go on after understanding the picture \ figure and what it is said about it.
David:	But we just have to look at it and we understand.
José:	The information is important. It is essential to the solution. [A, 1st]

In this task, students had difficulty in selecting the information and José *stimulus to individual strategies*,

The figure illustrates a simple process to determine the radius of Earth. This method consists in measuring the angle α, horizon depression angle, from a high altitude point from which one can see the ocean.
The figure is not to scale. Consider that:

— B represents the observation point;
— C means the centre of the Earth;
— α is the amplitude, in degrees, of the depression angle, ($0^{\circ} < \alpha < 90^{\circ}$);
— h is the altitude of the place, in kilometres;
— the triangle [ABC] is rectangle in A;
— R , is the radius of Earth, in kilometres;
— $BC = R + h$

1. Show that:
$R = \frac{h \cos \alpha}{1 - \cos \alpha}$

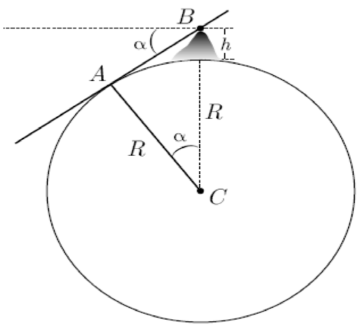

Figure 1

supporting this selection and helping them to find a solution strategy, encouraging them to continue (speaking 9), guiding them to the analysis of the figure and from the strategies designed for each pair of students (speaking 5):

1 David: Teacher, we already know, it is the trigonometrically reason: sine, cosine and tangent
2 José: Are you going to use the three?
3 Alexandre: No, no. we will see. the triangle needs the opposite cathetus AB
4 David: No way! AB is useless, what matters is h!
5 José: Where is h?
6 Alexandre: On the hypotenuse. but we also have R in it
7 David: We will do it with the cosine
8 Alexandre: We have the adjoining cathetus and the hypotenuse that goes for the cosine and then we find the value of h, can that be?
9 José: it can be, there are many ways, that´s yours choice. Go on! [A, 1st]

In the same task, other item (Determine the value of h, when $R = 1000$ meters and $\alpha = 60^{\circ}$ and when $\alpha = 45^{\circ}$. Present the results rounded to units. Compare the two previous results), he managed the same with two other students (Magda and Ruth). These students worked $\cos 60^{\circ} = \frac{1000}{1000 + h}$ and $\cos 45^{\circ} = \frac{1000}{1000 + h}$ equalities. From these equalities, they said, just watching the figure, that as h increases, the value of the angle has to increase too. José feels a little hesitant, because he was not expecting this task solution. He *articulated students' ideas*, not validating the work and asking students to articulate these ideas so that they may be able to present arguments of the result:

Rute: Is this true?
Magda: Adjoining cathetus upon the hypotenuse
José: Yes, it is right, but you did not use the equality $R = \frac{h \cos \alpha}{1 - \cos \alpha}$
Magda: But it is the adjoining cathetus upon the hypotenuse or not?
Rute: It is the cosine!!!
José: Yes, but how do you justify the comparison?
Magda: When h increases, the fraction value decreases and the obtained angle increases.
José: But you have to convince me that it is right! [A, 1st]

José wanted the students to identify the errors and report it orally. He believed that this explanation would help them to *self-regulate mathematics efficacy* and the ability of self-questioning about some important issues, such as how to work with the calculator:

> For instance, a student answered $R = \frac{2{,}35 \cos 1{,}5564^{\circ}}{1 - \cos 1{,}5564^{\circ}}$, so $R \approx 0{,}034$, and other student answered $R \approx 1{,}35$. I can identify that the first changed in the calculator "*Rad*" for "*Deg*". I would accept it as correct and alerted the student for the lapse. But I could not consider correct the second one. After some experiences, I could identify that the second student wrote on the calculator, 2,35 × cos 1,5564 ÷ 1 − cos 1,5564. In both cases, they are serious mistakes. [ST, 13nd]

In this assessment practice, José encourages the identification of mistakes. For example, he gave clues to the students so that they may *self-assess their performance*:

David: Are these the calculations?
José: How did you do the equations?
Alexandre: I think it was very fast, but it´s right.
José: Remember the change of signs...
David: Let´s see but this way leads us to what we wanted [A, 2nd]

In another task (Functions) after identifying a student's difficulty, José questioned him and, in some cases, the error was identified by the student *through self-assessment*:

David: It is 40 days.
José: Are you sure?
David: From 4 to 40 it is 30 plus 10.
José: But it is 4th January.
David: January
José: And the month of January has 30 days?
David: Okay, so t=41days [A, 3nd]

When students recognize difficulties to understand the mathematical situation, José do not hesitate to give them a clue to proceed, to pursue the approximation of their work to what the teacher expected them to do (*self-assessment*):

Alexandre: I am going to need help. I cannot understand.
José: Write first the formula of the area of the figure that you want to estimate, to calculate.
Alexandre: The area of the triangle?
José: Yes and now you can go on. [A, 5nd]

Reflection

José highlighted the improvements made by the students and the promotion of self-assessment as two aspects to value in his assessment practice. IT-S has raised José knowledge about the students, about the difficulties of writing and oral communication, poor connection between trigonometry and geometry content, the increase mode of understanding of mathematical content, and how to keep them motivated students.

In the first task, he realized through the help he gave the development of the student *capacity to select information*. This support, according to José, was essential to meet the first items of the task and *to motivate the students* to achieve it:

> I had to help them at the beginning, but I think that with that help they improved and accomplished the first questions. It also motivated them, it guided them to go through. [ST, 13nd].

José remembered the episode where he was confronted by the unexpected strategy followed by Magda and Rute, which he decided to not validate. He justified his decision by the need to verify the extent to which students arrived to the presented solution and *to understand the knowledge* that students have applied:

> They were confident that they were right, but when would they give up? (...) I left it unknown so they could think over it and that really amazed me, they knew what they were doing and rightly. [ST, 13nd]

In his opinion, students used strategies that work out things they learned previously. Strategies are not always predictable, but *to use these strategies with confidence* reveals effectiveness:

> Magda and Rute answered item solving it in a way I wasn´t expecting, but even so it was correct. It is like this, students use methods that I was not expecting, and we must validate them. These students knew what they were doing and were very confident about it. [ST, 13nd]

Difficulties concerning the understanding of information given in a short text are obstacles to self-assessment, which, according to José, make it harder to understand the question:

> When students read the assignment of a task and are not able to select the information, sometimes they get lost through the text. They do not even read the given information, which makes it much more difficult the approach of self-regulation strategy. They cannot use the knowledge they have, because they do not understand the worksheet. My questions sometimes do not make any sense to the students. [ST, 15nd]

One essential characteristic of this assessment practice is the existence of an understanding between teacher and students, so that they can understand the issues that the teacher poses. It is important that students understand what is asked in the tasks (on paper) or orally (inquiry in the classroom).

This reflection was important to José as it allowed him to identify different behaviors in the students: students participated more, discussed the tasks among themselves and produced work. In addition, José avoided answering directly to the questions of the students, chose to encourage them to overcome the difficulties without help, which resulted in students taking the necessary attitudes to comment on each other's ideas. This outcome has led José to reflect on his practice, making him particularly aware of the positive consequences of a reduction in the number of teacher inputs in the interaction with students.

FINAL CONSIDERATIONS

After observing and analyzing José's practices, we can infer two ways to promote self-assessment of learning mathematics: answer and performance. Self-assessment of the answer includes the action of the teacher, providing monitoring of the final work of the students, while the performance is concerned to the action to monitor processes and knowledge needed to find a solution. *Self-assessment of the answer* is provided by cognitive strategies and motivation for students who face questioning and oral feedback (Santos, 2002). Since planning, José prepares possible practices such as not correcting errors to help students to be autonomous in their cognitive strategies. He also pays attention to the need of students' confidence in doing mathematics, continuing in encouraging them during the development of the tasks. Once achieved, it leads students to increase the engagement with mathematical tasks, to develop strategies and to promote the integration of students' own ideas about learning mathematics.

Self-assessment of performance is assured by meta-cognitive strategies. It occurs to assess the efficiency of the mathematics performance of students and to encourage self-assessment of mathematical strategies to solve the tasks. José demands that their students deem complete answers, and present arguments to support their reasoning. He promotes self-assessment by not confirming the answers and guiding students to read the question again. To promote self-assessment, José also seeks to assess the depth of knowledge of mathematics and to identify errors when needed, leading students to think about the results (Quinton & Smallbone, 2010).

Although this assessment practice was new to José, he developed it over a whole school year, supported by a collaborative context, which led him to develop his self-confidence and ability in using it, recognizing positive aspects in the capacity of self-assessment of their students and, consequently, in the learning of mathematics. José assumed that his role is to support students' knowledge and learning processes, helping them to learn. Questioning, in IT-S, assumed the characteristics of oral feedback (Hodgen, 2007): to be focus on the task and not on the student; to be challenging, and to require achievable action. The IT-S practice shows progress in how students participate and engage in mathematical tasks, albeit the full understanding of the factors that increase the motivation and commitment of students with the mathematics tasks can and should be further explored.

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Teachers' response to unexplained answers

Ove Gunnar Drageset

UiT – The Arctic University of Norway, Tromsø, Norway, ove.drageset@uit.no

This paper studies students' unexplained answers and how teachers respond. The data is from observations of teaching in five different classrooms at Norwegian upper primary schools. Using frameworks and concepts usable to describe classroom discourse on a turn-by-turn basis, it is found that teachers more often attend to details of how and why when responding to unexplained answers than in general. This creates opportunities to learn for the students and opportunities for teachers to gather information usable for formative assessment. It is also observed that these teachers rarely attend to details of how and why when students answers incorrectly and by this limiting opportunities to learn.

Keywords: Communication, discourse, IRE, teachers' response.

INTRODUCTION

Several scholars have studied how teachers orchestrate classroom discourse in general. While the IRE pattern (Initiation – Response – Evaluation) (Cazden, 2001; Sinclair & Coulthard, 1975) only offers two concepts describing how teachers' intervenes, others have developed concepts such as extending, supporting and eliciting (Fraivillig, Murphy, & Fuson, 1999), advocating, reformulating and challenging (Alrø & Skovsmose, 2002), and simplification, requesting details and notice (Drageset, 2014c). Such concepts enable us with tools for describing communication in more detail, and also inspect how teachers responds to different types of student interventions.

In a recent study Drageset (2014a) has described five different types of student interventions (explanation, initiative, teacher-led responses, unexplained answers and partial answers) and described how teachers responds to these (Drageset, 2015). Unexplained answers might be one of the most interesting types of student interventions because it describes student comments where the reason for the answer is not given. The aim of this article is to go one step deeper into the data and re-visit the unexplained answers and study how teachers respond to these.

This aim resulted in the following research questions: How do teachers respond to students' unexplained answers? And what might this mean for students' opportunities to learn mathematics?

LITERATURE REVIEW

Conversation analyses developed from the hypothesis that ordinary talk is a structurally organized and ordered phenomenon (Hutchby & Wooffitt, 1998) where turns are the most fundamental feature (Sacks, Schegloff, & Jefferson, 1974). The default option is that people take turns of speaking one at a time (Sidnell, 2010). But even if the turns are sequentially organized, it is not possible to characterise a conversation as a series of individual actions, instead each turn is thoroughly dependent on previous turns and individual contributions cannot be understood in isolation from each other (Linell, 1998). This means that in a study of teacher and student turns (interventions, comments, responses, answers) it gives no meaning to study or describe turns isolated from the sequence. A description of the role of each turn is in fact a description of how it relates to prior turns and how it affects subsequent turns.

One example of such a description is the redirecting, progressing and focusing framework (Drageset, 2014c) where each single teacher turn were studied related to how teachers used student comments (turns) to work with mathematics. This developed into thirteen categories in three groups describing different ways in which teachers orchestrated the mathematical discourse in the classroom. The framework describes three types of redirecting actions (put aside, advising new strategy, and correcting questions) four types of progressing actions (demonstration, simplification, closed progress details, and open progress details) and six types of focusing actions (enlighten details, justification, apply to similar problems, request assessment, recap, and notice).

Using the same data, Drageset (2014a) developed five main types of student interventions; explanation, initiative, teacher-led responses, unexplained answers and partial answers. The most frequent type of student turns were teacher-led responses, and this is an illustration of how dependent a turn might be of prior turns as teacher-led responses are more or less given by the teacher through the prior turn (typically a question). Both (Sidnell, 2010) and (Linell, 1998) describes that usually only one or a few responses are preferred or more relevant than others and when the preferred or relevant response is given no explanation is needed. Teacher-led responses are a strong example of this. Unexplained answers are different from this as the reason of the answer is not given during the turn or becoming obvious from prior turns. The answer might be obviously correct or incorrect to the teacher and skilled students, but no information about student thinking or how the student arrived at the answer is given. The answer seems to come out of a black box. One such example could be when a teacher asks how much 1/4 added to 1/3 is, and a student only answers 7/12. It is obvious that students that do not immediately see that this answer is correct would benefit from an explanation about how the student was thinking to arrive at 7/12, and according to Franke, Kazemi, and Battey (2007), making details explicit is one of the most powerful moves a teacher can make. This means that unexplained answers create an opportunity for the teacher to focus on how to calculate or why an answer is correct or incorrect, either by telling it, asking the student to tell or challenge other students to explain.

Another example of a framework describing teacher and student comments on a turn-by-turn basis is the eight communicative features suggested by Alrø and Skovsmose (2002); getting in contact, locating, identifying, advocating, thinking aloud, reformulating, challenging, and evaluating. This framework does not differ between student and teacher turns. Advocating relates to justification and student explanations, challenging relates to redirecting actions, thinking aloud relates to enlighten details and student explanations, and evaluation relates to notice, recap and put aside. By relating it does not mean that they are identical, but that these concepts seem to describe related phenomenon.

While Alrø and Skovsmose (2002) and Drageset (2014b, 2014c) both describe frameworks intended to cover all different types of teacher and student turns in the observed classrooms, others describe teacher actions related to specific purposes. One such example is the Advancing Children's Thinking framework (ACT) (Fraivillig et al., 1999). The ACT frameworks were developed by intensive studies of one skilled teacher, describing three different teacher actions; elicit children's solution methods, supporting children's conceptual understanding, and extending children's mathematical thinking. While the eliciting and supporting components focus on the assessment and facilitation of mathematics with which the students are familiar, the extending component is focusing on further development of the students' thinking.

Another phenomenon is the tendency teachers have to reduce the complexity of tasks and rules. One way of reducing the complexity is by adding information, hinting or even changing the task in order to help the student find a (the) correct answer. Brousseau and Balacheff (1997) describes this as the Topaze effect, and the category of simplification (Drageset, 2014c) essentially describes the same. Anther way to reduce the complexity is described by Lithner (2008) as guided algorithmic reasoning where the teacher takes care of the process while the students answer basic questions. Closed progress details (Drageset, 2014c) is quite similar to guided algorithmic reasoning, describing how the teacher splits up a task into smaller steps, decides the method to be used and asks students basic question (typically calculations) with just one correct answer. Such reduction of complexity is seen as a hinder for students learn and understand mathematics (Lithner, 2008), probably because it reduces their opportunities to work and struggle with important mathematical ideas. And according to Kilpatrick, Swafford, and Findell (2001), opportunities to learn is considered the single most important predictor of student achievement. It might be obvious that when students are exposed to a topic they have a better chance to learn it than students that are not. But opportunities to learn is also about how students are exposed to topics, and teaching plays a major role in creating learning opportunities through emphasis on different goals, expectation for learning, time allocated, kinds of tasks, kinds of questions, kinds of responses accepted, and the nature of the discussions (Hiebert & Grouws, 2007).

The above frameworks relate to how learning can be fostered or hindered, and Wiliam (2007) offers five key strategies related to assessment for learning. One is to clarify and share learning intentions and criteria for success, a second is to engineer effective classroom discussions that elicit evidence or learning, a third is to provide feedback that moves the learners forward, a fourth is to activate students as instructive resources for one another, and a fifth is to activate students as owners of their own learning. The frameworks offered by Alrø and Skovsmose (2002), Drageset (2014b, 2014c) and Fraivillig and colleagues (1999) provides us with tools to describe mathematical discourse in the classroom in detail on a turn-by-turn basis. But there is still an open question if and how this can help us understand more about how teachers can engineer discussions in such a way that it elicits evidence for learning, how the feedback can more learners forward, and how to activate students as owners of their own learning. It is also an open question how such framework can help us describe how the opportunities to learn vary in quality between different situations and classrooms. The devil might lie in the details, and in a recent study Drageset (2014b) studied how students explained and teachers responded. This resulted in a description of three different types of student explanations; explaining how, explaining why and explaining concept. But even if the three types of student explanations were quite distinct, no major differences were found in how teachers responded to these. Looking for further detail, this article will look deeper into how teachers respond to students' unexplained answers and what this might mean for their opportunities to learn mathematics.

METHOD

This study is based on the same data that were used to develop the redirecting, progressing and focusing framework (Drageset, 2014c) and the five types of student comments (Drageset, 2014a). Based on a survey of 356 teachers, five teachers from upper primary (year five to seven, students aged 11 to 14) were selected for further study. These five teachers had a variation related to the survey constructs of mathematical knowledge for teaching and beliefs about teaching and learning. They all had several years experience as mathematics teachers and were educated as general teachers, which is the typical education for Norwegian teachers. All mathematics teaching for one week was filmed in each classroom (typically four lessons of 45 minutes). The camera followed the teacher, and a microphone attached to the teacher recorded all conversations in which the teacher was involved.

During the development of the frameworks describing teacher and student comments every turn were studied, describing its role in the conversation, grouping similar turns and developing categories gradually using a grounded theory approach. In the study reported in this article, the students' unexplained answers were re-visited, inspecting how the teachers responded to different types of unexplained answers in different ways. Unexplained answers are the ones where no information is given about how the student reasoned. This means that important details are hidden for the teacher and fellow students. An overview over different types of teacher responses to unexplained answers give a deeper insight into how these teachers use, or not use, the opportunities to make the hidden details explicit.

FINDINGS

By simply counting different types of teacher responses to unexplained answers and overall responses in the five classrooms, some interesting differences occur. As Table 1 illustrates, the five teachers tend to use progressing actions less frequently when responding to unexplained answer, and instead uses redirecting and focusing actions more often. A first impression is that the teachers uses the opportunity to focus on the answer more often when it is unexplained, but also more often tries to change the students approach by redirecting.

	Redirecting actions	Progressing actions	Focusing actions
Responding to unexplained answers	22%	36%	42%
Overall response to student comments	11%	55%	34%

Table 1: Responses to unexplained answers versus overall responses

	Redirecting actions	Progressing actions	Focusing actions
Responses to unexplained and *correct* answer	2%	17%	36%
Responses to unexplained and *incorrect* answer	20%	11%	3%
Responses to students *unable* to answer	0%	8%	3%

Table 2: Responses to unexplained answers separated for correct, incorrect and unable to answer

In order to understand what this means it is necessary to go one step deeper. The unexplained answers could be divided into three distinct groups or sub-categorise. One group is the correct answers that come without any explanations what was done, how the student was thinking or why this is thought to be correct. Another group is the incorrect answers, which vary from those close to correct to those where the student simply chooses a strategy than cannot work, and also these have in common that no information about the solution process or thinking is given. The third group of answers is those were the student is unable to answer or come up with a suggestion and where there is no information about why the student struggles. As Table 2 illustrates, these sub-categories gives us new information.

One striking difference is how redirecting and focusing actions follows different types of unexplained answers, which illustrates how a turn is thoroughly dependent of previous turns (Linell, 1998). It might not be surprising that redirecting actions mainly follows incorrect answers or strategies, as there is less need to redirect correct answers. Then it might be more interesting that focusing actions almost exclusively follows correct answers and rarely follows incorrect answers.

In general, teachers use redirecting actions to guide students towards other strategies, progressing actions to help students progress towards an answer, and focusing actions to make students work with, or to point out, mathematical ideas. So far we have observed that redirecting, progressing and focusing actions are used more or less often based on how the prior turn looks like. But which types of redirecting, progressing and focusing actions were used, and which types of such actions are used more or less frequently?

Just over half of the unexplained answers were correct. The main response to these was the focusing actions, and especially requesting students to enlighten details or requesting justification. When a teacher requests students to enlighten details the teacher typically asks how or what ('how did you find that answer?', 'what did you think when you solved this task?'). This is about making details explicit, which according to Franke and colleagues (2007) is one of the most powerful moves a teacher can make. In addition to this, such information is the basis on which a teacher can make formative assessment. A justification is typically requested by asking 'why is this correct'. This is different from requesting students to enlighten details as a relevant answer to why something is correct involves mathematical argumentation and not just a description of what is done to reach the answer. Also justifications are important for other students to understand or discuss. Together, both requesting students to enlighten details and to justify their answers are about making details explicit, which again might create opportunities to learn how to solve, think and reason. Requesting justifications are also vital for the teacher to get insight into a student's thinking and sense making which again is necessary for the teacher to be able to carry out formative assessment. It is particularly interesting to see that these five teachers' responds with requests for justification three times as often following a correct unexplained than they do following student comments in general.

About one third of the unexplained answers were incorrect, and as Table 2 illustrates the teacher responses changes strongly. The most typical response was to redirect students towards another strategy, and the main way this was done was by asking correcting questions. Typically, these were questions that involve a correction such as 'yes, but what if…'. It is hardly a surprise that teachers tries to guide students when they answers incorrectly. However, it is interesting to see that they are rarely exploring student thinking or reasoning when answers are incorrect. By only exploring thinking and reasoning when answers are correct some opportunities to learn are lost, for example since it is difficult to involve students in real discussions if only the correct are presented. Also, the teachers loose access to vital information about what students think and why they are not reaching a correct answer. This seems to tell us that these five teachers do not think such information is of any use to themselves or to other students.

The least frequent of the unexplained student answers were when the students were unable to give an answer. The teachers typically responded using the progressing actions of simplification and closed progress details and the focusing action of notice. Simplification is about making the task easier so that the student might be able to progress. This typically involves extra information and hints that reduces the complexity, and sometimes the teacher changes the entire task to create what Brousseau and Balacheff (1997) labels a Topaze effect. Closed progress details are rather equal to guided algorithmic reasoning (Lithner, 2008) and describe a situation where the teacher takes the responsibility of the process while the student contributes with answering basic tasks. Simplification and closed progress details describes two different ways to reduce complexity. It is not surprising that teachers try to help students that are unable to answer by reducing the complexity, and this might even be a good idea. The danger is when teachers consistently reduce complexity of tasks because then the students loose opportunities to learn the important mathematical ideas at their grade or level. The focusing action of notice is different, as it describes actions where teachers point out important information during task solving. This seems to be done in order to help students remember what we already know and can use, or help them to get back on track. It is also typical that notice is used to point out important arguments or things to remember.

In addition to the above, Table 2 also illustrates that a relatively large part of teacher responses to correct and incorrect answers are progressing actions. But looking into the different types of progressing actions does not give much information, it only reveals that it is used less responding to unexplained answers than in general.

CONCLUSION

The aim of this study was to look closer into students' unexplained answers and how teachers respond to them. Unexplained answers are defined to be those where information about the solution strategy or student thinking is not observable, neither during the student turn nor during prior turns. Using the redirecting, progressing and focusing framework (Drageset, 2014c) it was possible to see that teachers tended to more often use redirecting and focusing actions and less often use progressing actions than in general. By dividing the unexplained answers into three distinct sub-categories (correct, incorrect and unable) it became possible to observe that most redirecting actions came as a response to the unexplained answers that were incorrect and most focusing actions came as a response to the correct ones. By looking into the different types of redirecting, progressing and focusing actions it was found that as a response to correct answers teachers typically requested students to explain how and what (enlighten details) and why (justification). Also, it was found that responding to incorrect answers teachers typically guided the students by asking correcting questions, and when responding to students unable to answer the teachers typically reduced the complexity of the task (simplification and closed progress details) or pointed out important elements or earlier findings (notice).

Kilpatrick and colleagues (2001) states that opportunities to learn are the single most important predictor for student achievement. If so, one should look at which opportunities are given during discussion and not. By requesting details (enlighten details and justification) the teachers make these explicit for other students to reflect upon, discuss or ask, and for the teacher to understand the students thinking, reasoning and understanding. Making details explicit is important for student learning in general (Franke et al., 2007), and this is about creating opportunities to learn the important mathematics by attending to thinking, strategies and reasoning. But it is worth to emphasise that these teachers often requested such details when responding to an unexplained answer that was correct, and rarely did it when the answer was incorrect. This means that opportunities were lost, both for students to explore the reasons for the error, and for the teacher to gather information about students incorrect or incomplete thinking as a basic for formative assessment. Also since students need to struggle with mathematical ideas to learn (Hiebert & Grouws, 2007), something important might be lost if the students are only struggling with understanding what somebody else already understands and rarely have to struggle with something incorrect or incomplete and how to develop from there.

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Developing student questioning when problem solving: The role of sample student responses

Sheila Evans and Malcolm Swan

University of Nottingham, School of Education, Nottingham, UK, Sheila.Evans@Nottingham.ac.uk, Malcolm.Swan@Nottingham.ac.uk

This paper describes intervention resources designed to provide opportunities for student self-regulation, with a particular focus on setting subgoals when problem solving. Each task includes a range of pre-written inter-related "sample student responses" that expose students to multiple perspectives on an unstructured non-routine problem. After students attempt the problem they are given the responses to collaboratively complete, critique and compare. We explore students' capacity to adopt another person's (the sample student's) goals in order to complete a solution, and their capacity, through the use of comparison, to identify worthy criteria when critiquing the completed solutions. We then reflect on how we can make subsequent improvements to the resources.

Keywords: Problem solving, comparing solutions, peer assessment.

INTRODUCTION

Many studies cite planning and monitoring as key discriminators for problem solving success (e.g., Schoenfeld, 1992; Carlson & Bloom, 2005). During the initial planning phase, the subgoals students set determine the mathematical and self-regulating strategies used. Novice problem solvers often set vague, unstructured goals or their goals are flawed (Juwah et al., 2004). They often use naïve, inefficient strategies (e.g., trial and error), rather than considering the more powerful methods at their disposal. They pursue unfruitful or inefficient lines of enquiry relentlessly, without stopping to consider alternative strategies (Schoenfeld, 1992). Furthermore, they remain uncertain of the criteria to judge the quality of their work (Bell et al., 1997), other than checking the correctness of the answer. In contrast, expert problem solvers spend time setting hierarchical goals (Schunk & Zimmerman, 2012), carefully monitor their progress against these goals, and persist in the face of obstacles (Schunk & Zimmerman, 2012). They routinely use these goals to step back and ask themselves or their partner questions such as 'Where is this strategy going?', 'Should it be so complicated?', or 'Does this solution make sense?' (e.g., Schoenfeld, 1992). Answers to which may prompt a change of direction in order to improve, for example, their solution's appropriateness, elegance, efficiency or generalizability. Furthermore, using subgoals makes progress visible, and their realization may sustain motivation to persist (Schunk, 2006).

Empirical studies suggest that students might develop these self-regulating skills by critically reflecting on the work of others (e.g., Pintrich & Susho, 2002). In so doing, students' criteria for success are made visible for scrutiny (Black & Willam, 1998), differences surface, and opportunities arise for students to reflect on, and adapt their success criteria to accommodate new values. Through a series of case studies Juwah and colleagues (2004) found that providing students with opportunities for peer (and self) assessment encouraged the identification of goals implicit in solutions and judgments about how these goals related to their own solutions to a problem.

We have carefully designed resources intended to help students develop their self-regulating skills. In these resources, students are asked to interpret, complete, compare and critique pre-prepared, handwritten "sample student responses" to non-routine, unstructured problems. The responses are designed to simulate different ways students may solve a problem (Evans & Swan, 2014) and provide opportunities for students to use and reflect on the goals set by others. We begin by explaining the theory and method behind the design of these resources, then discuss how our intentions were interpreted in the classroom and detail the subsequent improvements.

THEORETICAL BACKGROUND

This study is rooted in a design research paradigm. Design research involves both the development of intervention resources and studying what happens when they are enacted in the classroom. Accordingly, the design process of iteratively designing, testing and revising a resource and the research process of conjecturing, collecting, analysing data and contributing to theory, occur simultaneously and in parallel. Thus the development of an intervention forms a symbiotic relationship with the development of the research. Within this flexible environment, both the intervention and the initial research questions or conjectures may be refined. This flexibility is particularly beneficial when the research base is thin and provides only limited guidance for the design of an intervention (McKenney & Reeves, 2012).

The design of the resources emerges from the findings from a large design research project (Swan & Burkhardt, 2014) but with some distinguishing features. Namely, the pre-written sample student responses are all incomplete. Thus the context has been mathematized; the students' task is to complete the mathematics and communicate results. This design structure provides students with an opportunity to ask themselves questions about each sample student's goals. Questions such as: 'What is this student doing?' and 'Why are they doing that? and 'What should they do next?' This awareness of goals set can positively influence their own performance when solving problems, promoting self-regulatory skills and productive goal-directed action, engendering persistence in the face of obstacles (Schunk & Zimmerman, 2012). After completing solutions, students attempt to *explicitly* compare and connect them. To prevent students from simply comparing handwriting or checking for mistakes, responses were short, accessible and error-free (Evans & Swan, 2014).

Comparing artifacts is routine practice in other disciplines. For example, English Language students may be asked to compare newspaper articles describing the same event. The literature suggest this practice, particularly if supported by a meaningful framework, focuses students attention on similarities and differences, and so facilitates the noticing of more features than if artifacts were viewed separately (e.g., Gamer, 1974; Chazan & Ball, 1999). Accordingly, in this study, students are asked to compare alternative approaches to non-routine unstructured mathematics problems. Thus encouraging students to ask themselves questions such as: 'What are the differences between these two responses?'; 'How do these differences benefit or constrain the solution?'; 'Why do *x* rather than *y*?' By encouraging students to not only to make sense of a solution but to make judgments about its quality, may shift their perspective from viewing solutions as a process, to viewing them as objects to be evaluated. This shift can promote deeper understanding of the mathematics (Sfard, 1991).

We know from the literature that transferring learning from one problem situation to another can be challenging as students often form highly concrete, context-specific, understandings of the solution (e.g., Gentener, 2003). This may be partially addressed by exposing students to multiple solutions, particularly if these solutions are compared rather than considered individually (e.g., Catrambone & Holyoak, 1989). By comparing students' focus on structural, often abstract, commonalities rather than idiosyncratic, situation-specific, surface features (Gentener, 2003). A study within mathematics education supports these findings. The study (Rittle-Johnson & Star, 2007) likewise focused on transferring methods studied in one context to another. Students learnt to solve equations by either comparing alternative methods or by reflecting on each method separately. The students in the 'compare' group made greater gains in procedural knowledge and flexibility to solve routine problems in multiple ways and comparable gains in conceptual knowledge. Although the studies on comparing solutions did not involve unstructured, non-routine problems, (we could locate no studies of this kind) we conjecture that comparing solutions to these types of problems could help to improve students' 'flexibility' when solving other problems. Thus increasing their capacity to monitor their progress against interim goals as their solution is slowly created.

METHOD

The resources for each lesson include a task and a detailed teacher guide. Based on materials from a larger US project (Swan & Burkhardt, 2014), the interventions represent the initial phase in a design research cycle of the UK study. Feedback from this phase will inform the refinement of resources, methods used for data collection and analysis for the UK study. The intervention lesson described here was the first in a series of four taught to 30 students in a top (advanced) set Year 9 class in a UK secondary school. The students had little experience of working with unstructured problem solving tasks or sample student responses, however, the teacher had taught many such tasks

Structure of the intervention lesson:

— Students worked on the task in a prior lesson. This provided the teacher with insight into the ways students were understanding and representing the problem.

— After the teacher briefly reintroduced the problem to the whole class, students worked first individually then in pairs, completing sample student responses.

— Because students were not used to comparing responses, the teacher briefly explained, using a non-mathematical example, the benefits of making comparisons.

— Students then glued the now completed responses to a poster and interpreted, completed and compared the solutions.

— In a whole-class discussion students reviewed what they had learned.

Figure 1 shows the problem used. Figure 2 shows the pre-written student responses. We carefully designed the responses to encourage students to make connections between approaches in order to create or strengthen networks of related ideas (Silver et al., 2005) and enable students to achieve 'a coherent, comprehensive, flexible and more abstract knowledge structure' (Seufert et al., 2007).

We summarised students' individual attempts to solve the problem. However, the prime source of data is the 15 student posters, each produced by 2 students. Throughout this paper, the word 'set' defines a group of assessment comments on 1 poster about 1 response. The word 'response' refers to the incomplete 'sample student' work, and 'solution' refers to a (real) student's attempt to complete a response. We used a grounded theory approach to assess the 45 sets of assessment comments made by the 15 pairs of students about the 3 responses. To interpret the comments we used 3 themes corresponding to the 3 tasks students under-

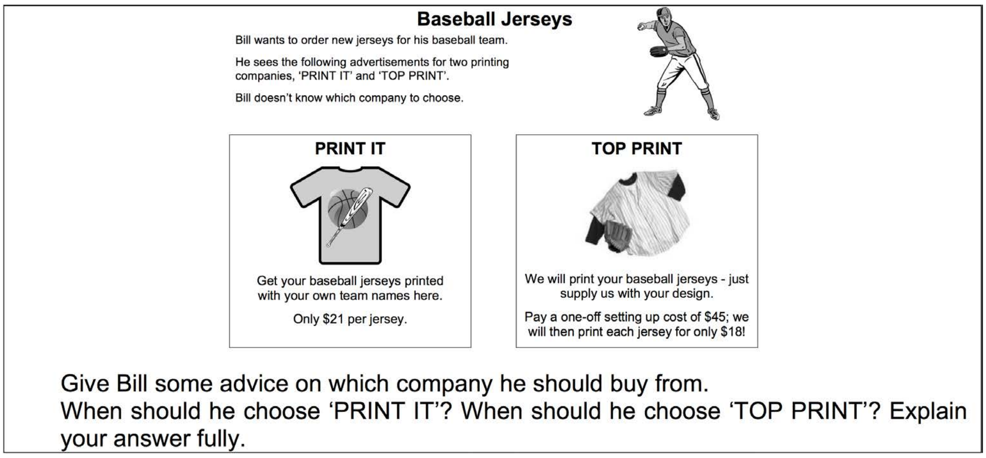


Figure 1: The problem

Dylan

Print it cost $21n
Top Print cost $45+18n
$21n = 45 + 18n$
$3n = 45$
$n = 15$
n = 16 Print it $336 Top Print $333

Cath

Number of jerseys	Cost at Print it	Cost at Top Print
2	$42	$81
4	$84	$117
8	$168	$189

Ezra

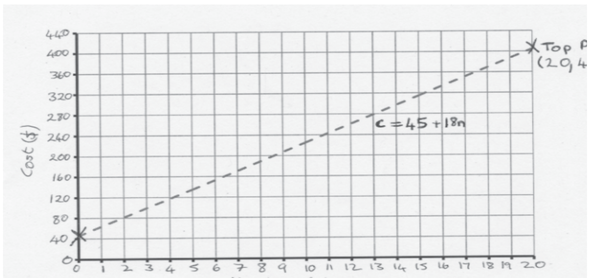


Figure 2: The three pieces of sample student work

took: how students completed; assessed and connected the sample student responses.

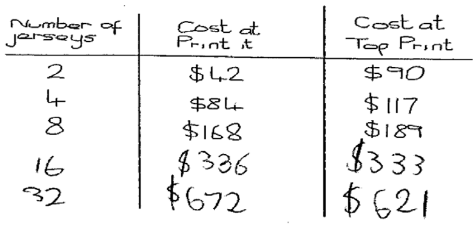
SECTION 4: RESULTS AND DISCUSSION

When initially attempting to solve the problem one student used a graphical method and another an algebraic method. The rest of the class used a form of 'trial and improvement'. This concurs with our earlier research (Evans & Swan, 2014) that suggests students often prefer this method rather than, for instance an algebraic strategy. Trial and improvement can forge a way into the problem but the information available within answers are often limited. For instance, trends may not be revealed. Furthermore, most students failed to effectively communicate their answer to 'Bill', thus overlooking the purpose of the problem. These results agree with the literature that suggests students often disconnect mathematical representation from the context of the problem and make little attempt to reconnect them (e.g., Friel, Curcio, & Bright, 2001). As trial and improvement was the commonly used strategy, students were to be exposed to two *new* approaches and a familiar one in the intervention lesson.

How students completed the responses

Table 1 summarises how pairs of students completed each response and attempted to advise 'Bill' (e.g., 'Top Print is cheaper after 15 jerseys').

Despite (or possibly because of) most students figuring out the correct answer on their own, some failed to complete the sample student responses. For instance, one pair of students added two more rows to Cath's table (Figure 3).



Number of Jerseys	Cost at Print it	Cost at Top Print
2	\$42	\$90
4	\$84	\$117
8	\$168	\$189
16	\$336	\$333
32	\$672	\$621

Figure 3: Response of two students

These students understood the context, but responded in a superficial way, by finding something procedural to do. They followed the pattern in the first column, and used the procedures for calculating costs correctly, but did not engage in solving the problem using Cath's method. Thus students did not attempt to understand and adopt Cath's goals. This may be due to the teacher not fully explaining the purpose of the activity at this stage: once complete the solutions were to be critiqued and connected.

Consonant with students' original attempts to solve the problem there was a notable lack of attempts to interpret their solution in terms of the context (21 out of a possible 45. Of these, only 3 pairs explicitly advise 'Bill', the remaining 18 simply explained what the solution showed). However, again not fully understanding the purpose of the activity may discourage students from providing all 3 answers (12 out of the 15 pairs of students did attempt to recontextualise at least one of their solutions). Students may assume they would simply be repeating themselves.

How students assessed the responses

32 of the 45 'sets' of assessment comments suggested students were making direct comparisons between the responses. For example, comments such as 'it is clear to see the pattern [Ezra's response] compared to Dylan's'. There were numerous other assessment comments that

		Did/did not attempt to advise Bill
Dylan (Algebraic)	4 pairs of students substituted $n = 15$ into the expressions $21n$ and $45 + 18n$. Most of the remaining pairs substituted a combination of $n = 14$, $n = 15$ and $n = 16$ into the expressions. Most did not explain their work.	6 / 9
Cath (Numerical)	7 pairs of students figured out the prices of the jerseys when $n = 14$, $n = 15$ and $n = 16$. Others figured out between one and four prices. 9 pairs of students figured out the cross-over point, $n = 15$. Usually the existing table was extended to accommodate these figures.	7 / 8
Ezra (Graphical)	All pairs of students successfully plotted a 2nd line on the graph. There was very little written work.	8 / 7

Table 1: Summary of how students completed the sample student work

implied students were making comparison. For example one pair made the two comments 'only talked about one particular value [Dylan's response] and 'showed the full range of results in a graph [Ezra's response]. Our evidence suggests that most students were not simply considering the attributes of individual approaches to the problem, but were using comparison to draw out the relative advantages and disadvantages of each. Most students (33 of the 45 sets) were able to make at least two comparative comments on each response and only 8 'sets' made totally positive or totally negative comments; indicating students were using a range of questions when assessing the work and did not feel compelled to declare one solution as the 'correct one'. This behaviour contradicts the commonly held assumption that mathematical solutions always consist of one right response amongst a hazardous field of wrong ones. It appears that as students compared solutions, similarities, differences, advantages and disadvantages were revealed, discouraging the emergence of a 'best' solution. We used 5 categories to investigate the nature of the assessment comments:

- *Assessments about clarity*. These comments referred to the personal challenges of understanding the response. For example, 'Easiest to understand'
- *Assessments about accessibility*. These comments referred to the personal challenges of using the method. For example, 'Cath's method may take a while to do'.
- *Assessment about fitness for purpose*. These comments referred to students' assessment of the legitimacy of the response given the context of the problem. (E.g., 'hard to find an exact price, big scale, so pretty much guess work'.)
- *Assessment about the incompleteness of the method*. These comments arose despite students being asked to complete each solution. For example, 'doesn't answer the question'.
- *Undefined assessment comments*. These were comments we were unable to categorise. For example, 'easiest'.

We then categorised all assessment comments into those expressing advantages and those expressing disadvantages. The results of the coding are given in Table 2.

A large proportion of assessment comments drew on a student's personal perspective (9 + 17 = 26) rather than on whether or not the solution was fit for purpose. For example, 'Quite complicated if you don't get it'. This was unsurprising. What students' notice in a solution and the questions they ask themselves about it are often influenced by past experience of mathematics classes. In a traditional concept-focused classroom a problem is often used by the teacher to introduce a new technique, then students practise and illustrate the technique using similar problems; what Burkhardt and colleagues (1988) calls 'exposition, examples, exercises.' It follows that students may assume, when critiquing solutions, their task is simply to decide if they understand it and if they could use the method to solve other problems. Accordingly, students may ask questions such as 'Do I understand this method?' or 'Do I have the maths needed to undertake this method?', or 'Would it take me a long time to solve a problem using this method?' These are legitimate questions, however they do not critique the mathematics used, nor the validity of the solution within the context. To do this, students need to ask further questions. Questions such as 'Is this method efficient, elegant, generalisable?' and 'Is this method suitable for the given context?' and 'Is the answer appropriately communicated?' We were encouraged to note 57 comments did answer questions such as these. For example, Figure 5.

Assessment about:	**Algebra (Dylan)**	**Table (Cath)**	**Graph (Ezra)**	**Total**
clarity	3 (1,2)	5 (4,1)	1 (1,0)	**9 (6,3)**
accessibility	7 (7, 0)	5 (0,5)	5 (1,4)	**17 (8,9)**
fitness for purpose	11 (1,10)	16 (9,7)	30 (17,13)	**57 (27,30)**
incompleteness of method	3 (0,3)	0	0	**3 (0,3)**
Undefined	4 (3,1)	7 (5,2)	1 (0,1)	**12 (5,3)**
Total	**28 (12,16)**	**35 (18,15)**	**37 (19,18)**	**98 (49,49)**

Table 2: The numbers in brackets refer to the (advantages, disadvantages)

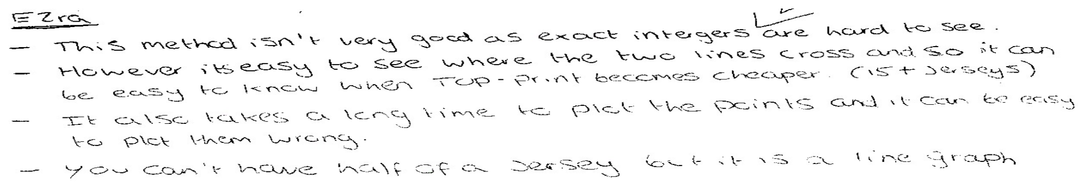

Ezra
- This method isn't very good as exact integers are hard to see.
- However its easy to see where the two lines cross and so it can be easy to know when Top-Print becomes cheaper. (15+ Jerseys)
- It also takes a long time to plot the points and it can be easy to plot them wrong.
- You can't have half of a Jersey but it is a line graph

Figure 5

Now we will analyse the nature of these assessments. Rather than critiquing the graphical solution from a personal perspective (6), students clearly preferred to focus on the suitability of the graph within the context (30). The positive comments include: 7 on how the graph allowed them to easily see a trend, 4 on the wide range of costs, and 3 non-specific comments about the appearance of the graph (e.g., 'easy visually to see'). There were 11 negative comments about the graph's lack of accuracy due to the scale (e.g., 'hard to find an exact price, big scale so pretty much guess work'). The tabular method drew 16 comments about the appropriateness of the solution, including positive comments about the accuracy of the costs (2), ease of comparing companies (3), and presentation of specific costs (2). Negative comments referred to the lack of a range of costs (2) and poor visual representation (2). Negative comments about the algebraic method referred to lack of values (4) and lack of detail about which company was cheaper (3). Thus we can detect themes across all three solutions: the accuracy of the work, the range of values used, and the ease of comparing costs. It is conceivable that these themes were instigated through the act of explicitly comparing solutions. For instance, an advantageous property noticed in one solution, may then be assessed in another.

How students connected the responses

Most comments about how solutions were linked were generic. For instance, 'The two solutions both used the same formula.' This concurs with research suggesting the need for instructional prompts that draw students' attention to how methods are linked (e.g., Chazan & Ball, 1999). However, time was another important factor contributing to the quality and number of comments on how responses could be connected. Students simply did not have enough time to complete the task.

CONCLUSION

To effectively complete the responses students needed to adopt and use each sample student's goals. To effectively critique each solution they needed to recognise the goal of the task (the comparison of the companies) and the role they were to adopt (advisers to 'Bill'). Students struggled with these activities. For instance, when completing responses students were able to follow algorithms, but sometimes failed to engage with their purpose, rendering insubstantial solutions. Furthermore many students failed to effectively communicate their results to the intended audience, 'Bill'. These findings highlight the difficulties students have recontextualising the mathematics both as they complete a solution and as they communicate the results. Possibly indicating some students were not asking questions such as 'Why am I figuring out this value?' or 'How does this result help Bill?' To ask these questions students need to engage in the 'student's' hierarchical goals.

The findings do support the perspective that comparing properties of solutions to problems does indeed draw out the affordances and limitations of each but without compelling students to decide which one is correct. Students often critiqued solutions from a personal viewpoint, focusing on whether they understood the method and whether they would be able to use it again. However, many students did also critique the suitability of each sample student response within the context of the problem. This was particularly the case with the graphical response.

Although we cannot make any generalisations beyond this classroom setting the findings from this study together with feedback from trials of other lessons will help shape a further iteration of all the resources. It is clear students need more support when undertaking these activities. For example, to help students complete the responses we will suggest teachers provide opportunities for students to reflect on the goals 'students' have set. Furthermore, if teachers ensure students understand the purpose of this activity, then they may be motivated to not only complete the mathematics, but also interpret it in the context of the problem. In so doing, differences may emerge in information gained from each completed solution.

This in turn may help students undertake the next activity, critiquing the solutions.

We endeavour, through the resources and teacher instruction to further raise awareness of what gets noticed when critiquing, and whether what is noticed and critiqued is of relevance to the context. This may help students move from simply noticing features of the solutions from a personal perspective to noticing them from the perspective of the context of the task. In this case, the 'sample student's' and 'Bill's' perspective. For example, it may require just a small shift in perspective to move from asking questions such as 'How long would this method take me to do?' to 'Is this an efficient and elegant method?' We also plan to design follow-up task in which 'Bills' goals are slightly altered. Students will need to think carefully about the criteria for success when planning a strategy and monitoring its progress as it unfolds on the paper or in discussion. We will continue to frame these student tasks within the activity of comparing solutions. We regard this as a successful design strategy.

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Problem solving teaching practices: Observer and teacher's view

Patricio Felmer, Josefa Perdomo-Díaz, Valentina Giaconi and Carmen G. Espinoza

University of Chile, Center for Mathematical Modeling and Center for Advanced Research in Education, Santiago, Chile, pfelmer@dim.uchile.cl

In this article, we report on an exploratory study on teaching practices related to problem solving of a group of 29 novel secondary mathematics teachers. For this purpose, two independent instruments were designed, the first one is based on lesson observations, and the second one is a questionnaire answered by teachers about their teaching practices while working on non-routine problem solving with their students. For each instrument, we perform a statistical analysis to define relevant dimensions regarding problem solving teaching practices and we compare these new dimensions. We find that results from the two instruments are coherent in the case of quantity of problem solving in lessons and quality of teaching practices. These results are encouraging for further studies in this direction.

Keywords: Problem solving, observed teaching practices, self-reported teaching practices.

THE IMPORTANCE AND CHALLENGE OF ANALYZING TEACHING PRACTICES

No doubt that the analysis of teaching practices is crucial for many aspects of education, for teacher, school or program evaluation, for more general purposes of defining and understanding good practices or for theoretical studies aiming to understand the teaching-learning process.

Methods used to collect teaching practices data have an ample variety, for example teachers/students reports, lesson observations and questionnaires, just like the type of analysis done with these data that could be qualitative, quantitative or a vast variety of mixed approaches. Having all of them advantages and disadvantages, depending on the type of collected data and the analysis method used, the research could show in-depth, small-scale results as in the case of qualitative analysis of videotaped lessons, or a more general view of teaching practices as in the case of quantitative analysis of teachers' surveys.

In particular, the analysis of teaching practices through questionnaires is of great value since it allows studies with a high number of teachers. PRIMAS project is an example of this analysis, in which teachers' self-reported practices in inquiry based learning in different countries are described (Engeln, Euler, & Maaß, 2013). On the other hand, teaching practices observations by a research team, without the teacher intervention (external analysis), has been widely used in mathematics education too. These analyses have been conducted with different purposes: to study a specific feature of teaching and compare different types of teaching practices in relation with this feature, for example the mathematical quality of instruction (Hill et al., 2008) or the presence of mathematical elements as definitions, properties or processes (Badillo, Figueras, Font, & Martínez, 2013), to identify teachers' participation patterns in the class (Skott, 2013) or to analyze attributes of teachers that could affect their teaching practices, for example their beliefs about mathematics and teaching (Stipek, Givvin, Salmon, & MacGyvers, 2001).

Nevertheless, how self-reported practices and externally observed practices are related is not an obvious issue. Important differences may come from distinct conceptions of "practices", that is, the researcher has something in mind which differs from what teacher have in mind or they may come from biased answers based in social desirability.

To sum up, there are many important reasons to analyse teaching practices and, at the same time, collecting data to do that is a critical point (Maaß & Artigue, 2013).

Our interest in methods for collecting teaching practices data is in the context of a R&D project related with mathematics teacher's professional development on problem solving practices. In this project, teachers design and implement problem solving lessons with their students, which are later discussed jointly between teachers and researchers. In the context of two previous projects we have designed two instruments to collect data from practices from a group of secondary mathematics teachers: a Lesson Observation Rubric (LOR) and a Questionnaire about Problem Solving (QPS). We are interested in the information that can be obtained from each instrument separately and we want to explore on the information that can be drawn using both instruments at the same time.

The research question of this study is: Which are the dimensions that characterize teaching practices in relation with problem solving when using each of the two instruments and how do they correlate? To answer this question is relevant from a data collection methodological point of view and also for deepening in the understanding of the teaching practices related with problem solving.

TEACHING PRACTICES RELATED WITH PROBLEM SOLVING

The framework for designing research instruments and analysing data to answer our research question includes a conception of problem and of teaching practices that promote students' work in a mathematical environment.

We agree with Schoenfeld (1985) considering that

> Being a problem is not a property inherent in a mathematical task. Rather it is a particular relationship between the individual and the task that makes the task a problem for that person [...] if one has already accessed to a solution schema for a mathematical task, that task is an exercise and not a problem. (p. 74)

Other characteristic that some authors give to a problem is that it needs more time to be solved (e.g., Kloosterman & Stage, 1992).

Our experience with in-service teachers shows that some teachers think that a problem is just a math exercise contextualized in the "reality", and that many teachers use the term "non-routine problem" to refer to what Schoenfeld call "problem". In order to face this common misconception among Chilean teachers, the QPS teachers answered starts with the following definition:

> We are going to consider that a non-routine problem is one where the person that solves it doesn't know a strategy or algorithm to solve it. Some of the characteristics that non-routine problem usually have are: They need more reflection and time than regular problems or exercises; they cannot be solved with a simple rule or just remembering and applying a known fact; usually they can be solved using different strategies; they can have one or more solutions; they are challenging for the person that is solving them and they don't have to be confused with problems with real context, to have or not real context doesn't determine a non-routine problem.

In relation with teaching practices the key pedagogical aspect of problem solving tradition is that students have to think mathematically. To accomplish this it is necessary to have student centred teaching practices and a climate that support for the autonomy. Swan (2006) uses the term "student-centred' to describe practices where

> [T]he teacher takes students' needs into account when deciding what to teach, treats students as individuals rather than a homogeneous body, is selective and flexible about what is covered and allows students to make decisions, compare different approaches and create their own methods. Mathematics is seen as a subject open for discussion. It is not necessary to cover everything on the syllabus and time may be taken to explore and discuss. (p. 63)

METHOD

Participating teachers

This study considers teaching practices from 29 novel secondary mathematics teachers (between 1 and 4 years of teaching) randomly selected from graduates of three of the Chilean reference universities for teacher formation (10 teachers from two of the universities and 9 from the other one).

Each teacher selected a class, in which he/she was teaching, to be observed during three consecutive lessons. The level of each class went from 7th to 12th and the topic treated during the observed lessons was varied, like numbers, algebra or geometry. Some months later (from 6 to 12 months) teachers answered a questionnaire about their teaching in relation with problem solving practices taking into consideration the same class that was observed.

Classroom observation

For each teacher, three consecutive lessons of the selected class were observed and coded (2 live lessons and 1 video recorded lesson), completing 270 minutes of observation. It was chosen to observe only three lessons since, according to methodological research findings; no significant information is obtained by observing more than 3 lessons (Hill et al., 2008). A lesson observation rubric (LOR) was designed to code lessons by sections of 7.5 minutes. The LOR is divided in three main parts described next, with 14 practice indicators in total.

Work dynamics: in this part, the rubric identifies if time is dedicated to mathematics or not and the general way teacher and students interact. It is a first approximation to know if the lesson focus is on the teacher or the students. There are 5 indicators: down time (DT), direct instruction (DI), individual work (IW), group work (GW) and class work (CW). Each 7.5 minutes the indicator which is most observed is marked.

Class management: in this part, more specific characteristics of teaching practices are collected using 7 indicators: teacher does not participate (TNP), teacher makes questions (TMQ), teacher returns responsibility (TRR), teacher does not answer (TNA), teacher gives solutions (TGS), non-routine intervention of a student (NRIS) and teacher promotes discussion (TPD). When one of these indicators was observed during a segment it was marked.

Type of task: in this part, the type of mathematical task was collected using 2 indicators: routine task (RT) and non-routine task (NRT), depending if student were performing routine or non-routine tasks. Just one of them was marked in each time segment except in the case both were present during a very similar amount of time.

For our analysis we use all variables described above as percentages. In the case of DT, DI, IW, GW, CW, RT and NRT they are the percentage of segments were the indicator was marked, for example if for a teacher 9 segments out of 36 were marked non routine task, his score in NRT was 25%. In the case all the other variables, they are percentages of the total number of variables marked in the three lessons, for example if TRR was marked 3 times in the 36 segments and all other variables were marked 15 times, TRR was 20%

Questionnaire (QPS)

The questionnaire answered by teachers (QPS) is designed to measure the quantity and quality of problem solving (PS) activities used by them. The QPS starts with the definition of non-routine problem indicated in the framework (section 2) and it requires teachers to answer considering that definition. The questionnaire was answered through an online form sent by e-mail.

Quantity of PS was measured asking about the frequency of use of PS in their lessons with one item with options from 1 (never) to 5 (almost every lesson). Quality of PS was measured with 19 items (see Appendix A) based in scales of autonomy supportive climate (Leroy, Bressoux, Sarrazin, & Trouilloud, 2007) and scales to measure the level of student-centred approach and teacher-centred approach (Swan, 2006). Also some items were created to complete our view of a good problem solving activity as indicated in the framework. The teachers were asked to answer each question with the following prompt "When I use non-routine problems". Each item in this part has 6 options in a Likert scale ranging from 1 (never) to 6 (always).

Statistical methods

To summarize the variables from each instrument we made a Principal Components Analysis. The purpose of this analysis is to create interpretable and reliable scales that allow us to simplify and organize the data, understanding the limitations of using this method with low number of individuals. We made the analysis over the correlations matrix and, in order to have more interpretable results, we applied a Varimax rotation. The number of factors was chosen with the analysis of the scree plot. We define new scales averaging the items that had the biggest loadings in each factor. If a factor had items with negative and positive loadings we multiplied the items with negative loadings by -1 after calculating averages and reliabilities.

In the case of QPS, the principal components analysis was made with a bigger sample (N=240) of secondary and primary mathematics teachers applying for participating in a Problem Solving Workshop. Nevertheless, when we look for correlations between variables of both research instruments we used the common sample of 29 novel secondary mathematics teachers described at the beginning of the section.

ANALYSIS

Dimensions characterizing teaching practices by external observation

The statistical method applied to data from the LOR retained 4 factors explaining 71.2% of the total variance. The first factor explains 27.9 % of variance and the items with highest loadings were GW, TNP, TMQ(-) and DI(-) (the items with (-) had negative high loadings). With these items we create the new scale or dimension *Classroom Management* with a Cronbach's alpha of 0.681. We interpreted this factor as a dimension that describes the class management because a high positive score in this factor means that the teacher used most group work (GW) without intervention (TNP) and a negative score means that the teacher used mostly direct instruction (DI) and make questions (TMQ).

The second factor explains 16.2% of the variance and the items with highest loadings were TGS(-), NRT, TRR and CW. With these items we create the new scale or dimension *Quality Practices* with a Cronbach's alpha of 0.62. This factor put together variables that we presented in our framework as related with a good problem solving activity. First the frequency of non-routine tasks (NRT) and return the responsibility (TRR) are key variables. To have the frequency of working with whole class (CW) also makes sense, because CW was distinguished from direct instructions (DI), and only was coded if the protagonist were the students and they were working in a mathematical task. Teacher return responsibility was coded when at the moment of been asked teacher answered with a question or guide that allows the student to build, develop or discover a mathematical idea. The frequency of teacher gives solutions (TGS) has a negative loading; this makes sense to us because a teacher that gives the answers kills any attempt to think. So, having a high score in this dimension means that a teacher has a high frequency (compared with the other teachers) of non-routine tasks, returns the responsibility to students, uses whole class discussions and usually does not give the answers or solutions to their students.

The interpretation of the other two factors was not clear enough so we decided not to use them in the subsequent analysis. Studies with larger samples would be necessary to clarify these factors.

Dimensions characterizing problem solving teaching practices by teachers' self-report

The statistical method applied to data from the QPS retained 4 factors explaining 67.9% of the total variance (see Table 1) The first factor is formed with items that indicate students' behaviour denoting student centred practices, for example student's discussions (8,16) and work independently (3). The second factor was formed by items that indicate teacher's behaviour describing student centred practices, for example the teacher promotes that students takes their time (12) and guides with questions (11). The third factor is interpreted as a teacher centred dimension because most of the items situated the teacher as the protagonist. The last factor is about class organization because a high value in the factor means that teachers group their students (1) and a low factor means that teachers set their students working individually (6). Since this dimension has a very low reliability we do not use it for the correlation analysis.

In the next two sections we will combine the information obtained with the analysis of data from both research instruments: LOR and QPS.

Factor	% of Variance	Items with highest loadings	New Dimension	Reliability Cronbach's alpha
1	31.8	8, 13, 16, 14, 19, 3, 4, 9, 7	*Student centred (students)*	0.919
2	16.2	12, 18, 5 11	*Student centred (teacher)*	0.841
3	12.9	17, 10, 2, 15	*Teacher centred*	0.768
4	7	6 (-), 1	*Class Organization*	0.291

Table 1: Description of factors and the new dimensions from the QPS

Relation between observed and self-reported quantity of problem solving

Regarding the quantity of problem solving, the QPS has one item asking explicitly for the frequency of non-routine problems, and the LOR has an indicator of non-routine task. The correlation between the average of observed non-routine task and self-reported quantity of non-routine PS is r=0.447, p=0.015, N=29. This shows that information reported by teachers is not so different from information registered in lessons observations, in relation with the quantity of non-routine tasks or problems used with students of the observed sample. Nevertheless there are some particular cases as teachers reporting they use PS most of the time and where non-routine task was not coded during the observed 3 lessons. Cases like this require a deeper analysis we expect to make in a future study.

Relation between observed and self-reported quality of problem solving

In the analysis of the relation between the observed and self-reported quality of problem solving teaching practices we have to recall that all 19 items of QPS were answered under the prompt "When I use non-routine problems". This lead us to consider with a different weight the answers to items, depending on the reported frequency of use of PS in classes: we multiply the answer of each item by 1, 2, 3, 4 or 5 according to frequency of use of PS reported by the teacher.

In the following table, we present correlations between the new reliable dimensions obtained through principal components analysis starting with data from LOR and from QPS.

Regarding the dimensions from the QPS, we highlight the significant high value between the Student centred (student) and Student centred (teacher), dimensions 1 and 2, which may be interpreted as coherence in the answers of the teachers, those teachers that have a student centred practices have students performing student centred activities. In the same direction the significant positive correlations between 3 with 1 and 3 with 2 suggests a sort of incoherence in teacher's answers. However, having weighted the items answered by teachers with a weight based on a question in the same QPS (quantity of RP) may have increased artificially the correlations between the variables within QPS.

The most important result in Table 2 is the significant correlation between the dimension 5 with the dimensions 1 and 2. In this case, we are correlating dimensions obtained through our statistical analysis for the two instruments, LOR and QPS. These three dimensions express quality in teaching practices related to problem solving measured with two different instruments. These findings are in the same direction as the positive correlation between the frequency of observed non-routine task and self-reported quantity of non-routine PS mentioned above, telling that the two instruments are coherent. Certainly the results of this analysis are not conclusive, but it encourages going deeper in the analysis and tuning of these two instruments with larger samples.

DISCUSSION

We would like to summarize some results found with each LOR and QPS instrument, highlighting those characteristics of teaching practices related with problem solving where both instruments coincide. This is to be done having in mind that this is an exploratory study and that the instruments were not designed for information triangulation purposes. Teaching practices are multidimensional in nature, nevertheless a principal components analysis of both instruments lead us to summarize the information obtained through a big quantity of indicators, (14) and items (19), in two main types of dimensions: those related with classroom management or classroom organization and those linked with teaching practices promoting students development of problem solving skills and autonomy, in the terms described in our

		1	2	3	4	5
QPS	1. Student centred (student)	_				
QPS	2. Student centred (teacher)	0.858**	_			
QPS	3. Teacher centred	0.549**	0.583**	_		
LOR	4. Classroom Management	0.143	-0.018	0.275	_	
LOR	5. Quality Practices	0.369*	0.543**	0.178	-0.075	_

Table 2: Correlations between the new QPS and LOR dimensions **p < 0.01 *p < 0.05

framework. From our point of view, this is an interesting result although, since both instruments are new, further research is needed to prove the stability of these dimensions.

Correlations indicated some coherence between the information obtained with both instruments. For example, teachers that reported to use more non-routine problem solving activities were the same for whom non-routine task was coded in the external observation of the lessons.

Correlations between dimensions in the case of observed teaching practices point to that there is no relationship between classroom management and quality practices.

Correlations between the three dimensions expressing quality in teaching practices related to problem solving measured with two different instruments were high and significant, showing that the two instruments are coherent in this crucial dimension of our analysis. This is the most important finding in our study that encourages to conduct further studies in this direction.

To sum up, both instruments became useful to identify dimensions related with teaching practices on problem solving and there was coherence between the information obtained using each of them, what it is interesting from a methodological point of view, especially having in mind that classroom observation is very expensive in relation to questionnaire answering.

Moreover, it will be interesting to investigate if obtained dimensions for problem solving teaching practices are stable or not, taking a bigger sample, and go deeply in some cases with a qualitative study that let us to understand better some of the results of this analysis.

ACKNOWLEDGEMENT

Funding from FONIDE F721209, CONICYT-PCHA/Doctorado Nacional/2013 21130684, PIA-CONICYT Basal Funds for Centers of Excellence Project BF0003 and Basal CMM project is gratefully acknowledged.

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APPENDIX A: QPS

We present an English translation of the items. The original version was applied in Spanish. The items with an (i) are considered to measure practices that we hypothesized are not related to good problem solving activities.

1) I organize my students working in groups

2) If my students take too much time in find the solution of a problem, I solve it in the board. (i)

3) My students solve problems independently

4) My students express their different strategies to solve their problems even though they are wrong

5) I walk around the tables watching my students work

6) I organize my students working individually (i)

7) Usually I get amazed with my students ideas

8) My students are able to discuss with each other different ways to solve non-routine problems

9) I use plenary discussions with all the class

10) My students depend too much on my help to move forward with the problems (i)

11) If a student is too frustrated with a problem, I try to guide him/her only with questions

12) I promote students to take time to solve non-routine problems

13) My students generate different solution strategies

14) My students make interesting questions

15) If a student is too frustrated with a problem, I show him/her how to solve it (i)

16) My students discuss their own mistakes

17) My students progress too slow in the problems resolution (i)

18) I ask questions all the time

19) My students explore new problems arising as a result of the problems we are working on

Teacher-researcher collaboration as Formative Intervention and Expansive learning activity

Sharada Gade

Umeå University, Umeå, Sweden; Sharada.Gade@umu.se

Extended teacher-researcher collaboration is reported in this paper, by drawing upon cultural historical activity theory (CHAT) perspectives. Teaching is herein conceived as dialectical practice in which teachers are both shaped by and shape instructional practices. Three instructional interventions conducted at a Grade four mathematics classroom in Sweden constitute and exemplify the construct of Formative Intervention. Teacher-researcher collaboration which paralleled such conduct next exemplifies the construct of Expansive learning activity. Such transformation and change sheds light on how mid-level taken for granted phenomena in schools can be worked with and around, besides contributing to the motivational sphere of students and teachers.

Keywords: CHAT perspectives, collective unit of analysis, transformative agency, Formative intervention, Expansive learning activity.

TEACHER-RESEARCHER COLLABORATION

In this paper, I draw upon cultural historical activity theory (CHAT) perspectives and exemplify the constructs of *Formative intervention* (Engeström, 2011) and *Expansive learning activity* (Engeström, 2001). In doing so I draw on my extended collaboration as university researcher with a school teacher Lotta, as Charlotta is known, at her Grade four mathematics classroom. Such collaboration relates to project funding obtained by Lotta from the Swedish school authorities (Skolverket Dnr 2009:406) towards promoting students' communication in mathematics and includes three specific interventions. First, action research in relation to students' use of the mathematical = sign (Gade, 2012), second a problem posing practice in relation to students' use of textbook vocabulary (Gade & Blomqvist, 2015) and third, Lotta's plenary conduct of exploratory talk in relation to everyday measures (Gade, 2014). The very nature of teacher-researcher collaboration changed during such conduct from my being participant observer in Lotta's classroom, to her becoming theoriser and co-author of scientific reporting of our collaborative research. I argue that while the conduct of classroom interventions exemplifies Formative intervention, my extended collaboration with Lotta exemplifies Expansive learning activity.

Prior arguments in three research domains steer arguments in this paper. In the first, mathematics education research which seeks linkage between theory and practice in a comprehensive manner (Arbaugh, Herbel-Eisenmann, Ramirez, Knuth, Kranendonk, & Quander, 2010). Highlighting for reflective rationality instead of instrumental rationality, it is also sought that researchers become stakeholders in teachers' instructional practices just as teachers become stakeholders in classroom research (Krainer, 2011). Second, arguments in action research recognise the theory-practice relation to be a practical issue, wherein personal knowledge teachers have and their instructional realities are often found denied and/or generalised in idealised theory formulated by researchers from outside (Elliott, 1991). Finally, it is recognised that most research at K-12 levels of schooling is not conducted by K-12 teachers, leaving out and silencing the voice of teachers and their world of teaching (Cochran-Smith & Donnell, 2006). In three interventions alluded to in this paper, Lotta and me respond to many an issue here outlined. Lotta's own role is conducting these included working to stride the theory-practice divide, become stakeholder in research and contribute to K-12 research as a Grade four mathematics teacher. In doing so Lotta took on two significant roles. First and as teacher she elevated the problem of students' faulty use of the = sign in everyday instruction to one worthy of resolution via action research. She also steered instruction to have students make explicit use of textbook vocabulary and conducted a plenary of talk for students to explore understanding of everyday measures. In each of these Lotta was willing partner in deploying CHAT constructs which I brought to bear as researcher. Second and in this process Lotta went on to became co-author and theoriser of research, lending voice to its conduct and reporting. In such manner of reflective rationality, Lotta and myself became stakeholders in each others' professional practices (Krainer, 2011). More recently I unpacked our extended collaboration with different theoretical lenses (Gade, 2015). Lotta, her class teacher and me also report our problem posing practice in a teacher's journal (Persson, Blomqvist, & Gade, submitted). It is against this backdrop that I ask, What manner of change can teacher-researcher collaboration, identified by constructs of Formative intervention and Expansive learning activity, bring about?

CHAT BASED UNDERPINNINGS

Cultural historical activity theory (CHAT) grounds discussion in this paper with two arguments. First that the capacity to teach and benefit from teaching is a fundamental attribute of human beings. Second that educational processes are conceived as being active at three level – the student, the teacher and the environment between them. In fact CHAT perspectives view teaching as a practice wherein a teacher and his or her environment are in a dialectic whole, in which teachers not only shape but are shaped by instructional practices (van Huizen, van Oers, & Wubbels, 2005). With dialectical exploration of public and personal meanings, CHAT disavows a transmission model of teaching and conceives teacher subject knowledge to develop within instructional practices (Ellis, 2007). Such a conception is at odds with objectified, individualistic and dualistic conceptions of teaching, and proposes an alternative wherein teachers could take actions based on negotiated outcomes of collaboration and participation in a community of learners (Shulman & Shulman, 2004). This latter contests the idea that teachers are heroic individuals given societal status in lieu of their ability to work autonomously facing all odds in everyday practice (Edwards, 2010).

Bronfenbrenner's (1977) *Transforming experiments*, wherein environments can be restructured to bring unrealised potentials of participants to life is useful in realising viable practitioner collaborations. Engeström, Sannino and Virkkunen (2014) term Transforming experiments as *Formative interventions*, elucidating three assumptions that underpin their conduct. First the principle of double stimulation, which highlights how human beings use not one but two stimuli to overcome the problem situation they find themselves in. With the first stimuli being the problem situation itself, the second stimuli is used to make a meaningless situation meaningful resulting in individuals regulating their own behaviour, for example, the ticking of a clock in a waiting area is used to kill time, besides leading to volition. Second the principle of ascending from the abstract to the concrete, whereby any phenomenon is studied holistically from as many perspectives as possible (Luria, 1979). In the CHAT tradition the term abstract means underdeveloped, lacking in connections and thin in content as against concrete phenomena which are mature, well connected and rich in content (Blunden, 2010). Third Engeström and colleagues' (2014) principle of transformative agency,

> Transformative agency differs from conventional notions of agency in that it stems from encounters with and examinations of disturbances, conflicts, and contradictions in the collective activity. Transformative agency develops the participants' joint activity by explicating and envisioning new possibilities. Transformative agency goes beyond the individual as it seeks possibilities for collective change efforts. (p 124)

The three assumptions outlined above which underpin Formative interventions, allow for non-linear, agentic conduct of classroom interventions. Engeström (2011) further outlines four distinguishing features of these. First the starting point of Formative interventions are not pre determined but found embedded in the life activity being studied. Second, in resolving any problematic the individuals involved gain agency in its conduct. Third, that any pedagogical idea utilised in a Formative intervention has potential of being utilised later on as well. Finally and in such conduct the role of the researcher is to conceptualise and support the growth of interventions, as these evolve over time. I highlight these very aspects in Lotta's conduct of the three instructional interventions, which taken together constituted a Formative intervention.

The nature of teacher-researcher collaboration as realised by me and Lotta not only paralleled our interventions but expanded qualitatively over time, from my initially being participant observer in her classroom to her becoming co-author and theoriser of research reported. It is such manner of transformation that Engeström (2001) terms *Expansive learning activity*. Arguing against reactive forms of learning based on dualistic conceptions of the mind, Engeström (1987) argues for learning as an expanding and historically evolving activity. In line with Engeström, Lotta and me overcame many a contradiction we faced in her classroom practice, resulting in our collaboration becoming a case of Expansive learning activity. By this is meant that it was possible to view the learning that transpired during our collaboration in three distinct ways (Engeström, 1999). First and instead of benign mastery of what was already learnt hitherto by us as practitioners, our learning involved partial destruction of the old in our intentionally intervening and conducting an action cycle to restore say her students' faulty use of the = sign. The realisation of such conscious reflection was also possible in the problem posing practice, as well as Lotta's plenary conduct of exploratory talk with respect to everyday measures. Second and instead of conceiving transformation and change in individualistic terms, in our interventions we conceived students' development in collective terms involving all students in Lotta's classroom. Finally and instead of vertical movement along hierarchical levels, it was possible to conceive students' learning and development as a horizontal movement across subject specific borders. In addition, Lotta and me participated in each otherσs professional practice as stakeholders. Our object was not to become the other but realise new activities at the margins of our existing practices. Our realisation of co-authorship and theorising with Lotta is illustrative of such horizontal, as against hierarchical aspects. Detailed in Gade (2015) and geared towards Lotta's project goals, our collaboration evolved into newer forms of activity which grounded in her classroom realities were also not envisaged beforehand. As articulated by Engeström (2001),

> The object of expansive learning activity is the entire activity system in which the learners are engaged. Expansive learning activity produces culturally new patterns of activity. Expansive learning at work produces new forms of work activity. (p. 139)

I now turn to outline the instructional interventions which together constituted our Formative intervention, outlining the development of teacher-researcher collaboration as Expansive learning activity in the section that follows.

FORMATIVE INTERVENTION

Before detailing Lotta's instructional interventions at her Grade four classroom, I mention our collaboration to benefit from my conduct of a pilot study with her prior batch of Grade six students (Gade, 2010). It was during summer vacation in between that Lotta took the initiative of applying for funding of a project she conceptualised in terms of communication and mathematics. Yet since such a topic is broad in spirit and scope, it was only in some topics of the curriculum that we designed and conducted instructional interventions. In line with Engeström (2011) the very starting point of the overarching Formative intervention I discuss in this paper, was not determined ahead of time but found embedded in Lotta's everyday instruction. This happened when Lotta came upon her students' faulty use of the mathematical = sign. In Lotta reporting this to me, we designed and conducted an action research cycle based on CHAT perspectives of self-directed activity (Bodrova & Leong, 2007) and explicit mediation (Wertsch, 2007). While we detail the background, rationale and conduct of action research in our reports (Gade, 2012; Blomqvist & Gade, 2013) it is reasonable to assume the incidence of faulty use to transpire in other mathematics classrooms as well. Yet Lotta's actions of highlighting the problem to me, in our collaboration, held with it an expectation that we address the problem by means of research. It was then that I drew upon CHAT constructs to conceive a relevant and implementable semiotic practice. Continued participation by Lotta's students' in four stages of this practice led not only to restoring students' appropriate use, but also to Lotta achieving satisfaction of such use as teacher. We realised two significant aspects in these actions. In the first we implemented a practice that was active at three levels – Lotta's students, Lotta as teacher and the environment between them. In the second, we enabled Lotta to utilise her teaching in a dialectic manner, wherein she not only shaped her students' learning and development but was herself shaped by the instructional intervention she led (van Huizen et al., 2005). In line with Engeström (2011) Lotta's agency was realised in her conduct of the action cycle, with that of her students as they worked in dyads and offered mathematical statements of equality. In this intervention we handed out numbers and signs on slips

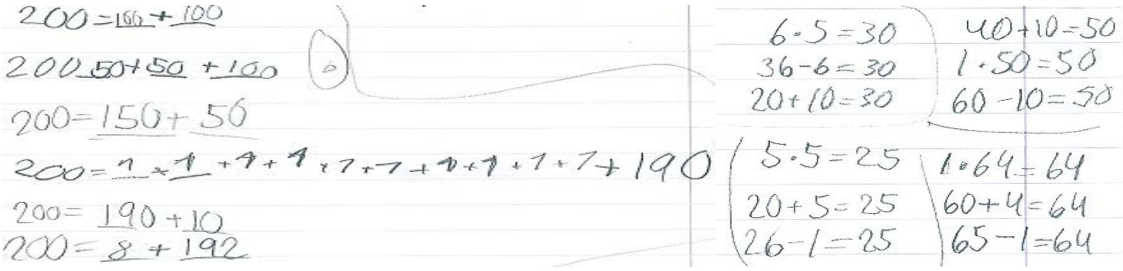

Table 1: Example of students' statements of equality in *lappar* based activity

of paper (*lappar* in Swedish) asking students to pick these at random and offer mathematically accurate statements. Two excerpts from such work detailed in Gade (2012) are given in Table 1.

We used *lappars* to hand out textbook vocabulary in the problem posing intervention as well (Gade & Blomqvist, 2015). As example while conducting this intervention Lotta herself partnered with a student Noel whose problem was as follows,

> 1000 people voted for president Noel but 600 voted for President Blomqvist. How many more voted for President Noel than President Blomqvist.

In the same session Ulla and Sara, two students who sat in desks behind Noel, posed the following problem almost in retort.

> There is a presidential election in the United States for the President. Noel was dismissed. President Ulla received 320 votes, President Sara 165 votes. How many people voted?

We next utilised *lappars* in Lotta's plenary, handing out improbable questions on everyday measures to provoke talk, for example: Can Eva and Anton measure the length of Sweden on foot?; Can Lars and Iris measure their age in decimetres? (Gade, 2014). An excerpt from Lotta's plenary conduct of the first question is given below:

Lotta:	Once more
Eva:	Can Eva and Anton measure Sweden's length with/on foot
Many:	Nooo ... @@
Lotta:	No, no, not now, Eva and Anton will attempt this question ... only them first
[??]:	It works (*Det går* in Swedish)
...	...
Liam:	You have to go straight ahead
Ulla:	What if you go into a building
[??]:	Then you go over the house
Lea:	Noel! Do you have anything good to say
Noel:	And you can go through the house ... and you can go inside the house and jump off the balcony ...
Leon:	And what if it is a high building
Nils:	If you have a map, you can take that, you can look how much a foot is and use the scale of the map ...

In realising students' agency in the three interventions we conducted, my own role as researcher saw me drawing upon relevant CHAT constructs, sharing these with Lotta and supporting her instructional practice. Where for the = sign intervention we drew on action research literature (Ladkin, 2004), in the problem posing intervention we drew upon van Oers' (2009) guidelines for Developmental education and for plenary conduct of Exploratory talk we drew on pioneering work of Barnes (2008). It was also the case that in each of the three interventions we conceived Lotta's teaching as a dialectical practice, in which Lotta shaped students' learning and development as well as reflect (Krainer, 2011) and be shaped by pedagogical actions she herself took (van Huizen et al., 2005). In was in this manner that Lotta was able to factor her practical knowledge (Elliott, 1991) besides develop rich nuanced understanding of pedagogical actions within her teaching (Ellis, 2007). Lotta went on to acknowledge our extended collaboration to be equivalent to her doing a professional development course in her classroom, which was rich preparation for her to foster a community of classroom learners in the future as well (Shulman & Shulman, 2004).

I finally discuss the four CHAT assumptions which Engeström and colleagues (2014) outline as underpinning any Formative intervention. First is the notion of double stimulation, in terms of which human beings resolve problem situations they find themselves in through a second stimulus, so as to gain volition. I argue that in each of the three interventions Lotta conducted, her students gained both agency and volition. I also argue such agency to be aided by their participation in instructional practices we had set up, in which the second stimulus was provided by the many *lappar* we utilised in a pedagogical sense and the CHAT constructs we utilised in a theoretical sense. The use of these together served the purpose of double stimulation. The understanding that Lotta and myself reached, was a result of observing and reflecting on each stage of the cyclical manner in which we deployed either intervention. Such an approach enabled us to view each of the three interventions from as many angles as possible and understand how various aspects were interrelated (Luria, 1979) besides how each intervention appeared as a pedagogical whole in concrete practice (Blunden, 2010). Finally and importantly it was in our conduct of interventions as classroom practice, that Lotta's students were able to jointly participate as well as contribute to collective transformational agency (Engeström et al. 2014). As example Lotta went far beyond treating the faulty use of the = sign by one of her students as a stand alone case and welcomed successive whole class interventions. Having shed light on the manner in which Lotta's instructional interventions exemplified the construct of Formative intervention (Engeström, 2011), I now turn to examine the manner in which teacher-researcher collaboration evolved correspondingly over time.

EXPANSIVE LEARNING ACTIVITY

As mentioned earlier on, my pilot study with Lotta's Grade six students prepared ground for teacher-researcher collaboration (Gade, 2010). It was during this study that either of us had opportunity to gauge each other as working professionals and take the many small steps which went on to eventually realise what Engeström (2001) terms as Expansive learning activity. In line with CHAT perspectives such a process was dialectical in spirit in that we both related and understood each other's actions. I was able to observe Lotta's teaching as participant observer and Lotta too was able to gauge how I interacted with her students, sharing a trick on one occasion and stoking their interest in mathematics on another. This led to her suggesting that I work with a few students who either needed special attention she did not have time for, or to those who were able to comply with her instructions before all others. From gauging each other and in terms of these actions, Lotta accepted me as a professional whom she could trust her students with. I argue that it was this trust that she took for granted when applying for project funding, whose aims were realised over time in the interventions which constituted our Formative intervention. In these it was possible to do away with older relations that students had with mathematical signs, textbook vocabulary or even talk and build relationships which we considered productive and mathematically rich. Our reporting of these aspects allowed for co-authorship and theorising by Lotta as K-12 teacher (Cochran-Smith & Donnell, 2006). Not only did we come up with and carry out new forms of shared activity (Gade, 2015) we also changed the very object of our collaboration from our jointly facing contradictions to conceiving alternatives and conducting interventions. As argued by Edwards (2010) and in the cumulative history of collaborative efforts, we brought our expertise as teacher and researcher to bear in our joint actions. In place of individual autonomy we lay emphasis on thoughtful practice and inclusive reflexivity. Our professional expertise was thus a negotiated one, which was not only born of practitioner struggle but also altered many times over. In line with Engeström and colleagues (2014) our agency was a transformative and collective one which realised systemic change. Such manner of change is captured once again by Engeström (2010, p 88) in 'Expansive learning is a process of material transformation of vital relations.'

CONCLUSION

I conclude by highlighting an outcome of significance about my collaboration with Lotta, which is that in our combined efforts we did not fail. Our actions, judgements, trust as well as relationships nurtured with one another, students and mathematics all contributed to the manner of outcomes I report in this paper. In line with a CHAT driven agenda, we worked towards as well as achieved transformation and change. Three observations follow. First and as recognised by Elliott (1991) there is reason to view the theory-practice relation as a practical issue for teachers. We saw the kind of efforts that Lotta and me took upon ourselves to bring CHAT based theoretical constructs to bear within everyday classroom instruction. Second and in line with Engeström (2001) such efforts necessitated expansive forms of learning, which were neither reactive nor predetermined but realised in our reflective actions which were grounded in Lotta's classroom. Such actions have potential besides of seeding as yet unforeseeable expansive learning in our trajectory ahead. Finally, such meta-level analysis sheds light on two historical trajectories, that of successive interventions conducted and teacher-researcher collaboration realised in parallel. Such insight in turn reveals how mid-level taken for granted phenomena in school, lying between rules and budgets on one hand and curricula and textbooks on the other can be worked with and around. Such actions and knowledge have potential besides to contribute to the motivational sphere of students and teachers within teaching-learning in everyday mathematics classrooms (Engeström, 2008).

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ENDNOTE

1. This research remains indebted to Charlotta Blomqvist, Cecilia Persson and their students.

Instructional practices in mathematics classrooms

Guðný Helga Gunnarsdóttir and Guðbjörg Pálsdóttir

University of Iceland, School of Education, Reykjavík, Iceland, gudnyh@hi.is

In the last decade, research on instructional practices has been carried out in mathematics classrooms in Iceland, mostly in the lower secondary school. In this study, the structure of 51 mathematics lessons in all compulsory school grades were analysed. The data came from a study called Teaching and Learning in Icelandic Schools, where 518 lessons from all school subjects were observed. To shed light on the structure of the mathematics lessons, diagrams were made which included the categories: non-mathematical work, teachers' public interaction with the whole class, individual seat work, assessment, group work, and playing of games. The analysis revealed that in most of the mathematics lessons the students were working individually in textbooks. There was little public interaction between the teacher and the class but the teacher went around the class and interacted with the students. There were a few examples of varied instructional practices that emphasized group work and discussions.

Keywords: Instructional practices, mathematics lessons, lesson diagrams, compulsory school.

INTRODUCTION

Findings in research on instructional practices in Icelandic mathematics classrooms have indicated that there is little variation in teaching approaches. In research by Savola (2010), where he compared lessons in Icelandic and Finnish schools, it was noticeable that in Icelandic classrooms the teacher often had no public interaction with the whole class and students were often working on their own pace in textbooks, while the teacher walked between the desks and interacted with students.

Karlsson (2009) also observed lessons in Iceland and Finland. According to Karlsson's results, Icelandic teachers were more likely to stay on the sideline and were not as central in the classroom as the Finnish teachers who also used more versatile teaching methods than the Icelandic ones. A recent study on the teaching of mathematics at lower secondary level in eight schools all around Iceland showed that in 56% of lessons students were working individually on workbooks. In 35% cases, there were some interactions between teachers and students around the topic followed by individual work on problems. In 6% of lessons, students were working in groups on tasks and in 3% they were playing some kind of games. During individual work, students often helped each other and the teacher encouraged them to do so both in public and also when walking between the desks (Þórðardóttir & Hermannsson, 2012). The study by Karlsson (2009) was the only one that contained some data from mathematics teaching at lower levels.

In Iceland, limited research has been done on mathematics teaching and learning and it has mostly been done by researchers who are stakeholders in many respects (teacher educators, authors of curriculum etc.). We therefore find it important to use available data, gathered by others, to add to our knowledge. In this paper, we use data from a recent study called *Teaching and Learning in Icelandic Schools* (Óskarsdóttir et al., 2014). In the study, 51 lessons in mathematics at all grade levels from 1–10 were observed. We gained access to the observation protocols and wanted to explore what they could tell us about what goes on in mathematics classrooms in Iceland, or more precisely: How are mathematics lessons structured? Of the observations, 32 were from grade levels 1–7, which have not been studied much previously. We therefore also wanted to find out if instructional practices at different grade levels were similar or if we could spot any differences.

We think that comparing teaching within a culture allows educators to examine their own teaching practices from different perspectives by widening known possibilities, in a similar way as comparing teaching between cultures does (Hiebert et al., 2003). It can reveal alternatives and stimulate choices being made within a country. It is also important to know what actual teaching looks like on average so that national discussions can focus on what most students experience. Using data from a general study on instructional practices also allows us to examine whether the practices in mathematics are any different from practices in general.

THEORETICAL BACKGROUND

Research on general instructional practices in Iceland

Various studies on general instructional practices in Icelandic compulsory schools have been carried out in recent years. According to Sigurgeirsson, Björnsdóttir, Óskarsdóttir and Jónsdóttir (2014), these studies all indicate that direct and teacher-centred methods are most widely used and to a much less extent methods where students take a more active role. Curriculum materials guide the teaching in academic subjects to a very large extent and the most common structure of lessons is a short introduction by the teacher followed by individual seatwork. There are though indications that instructional practices in lower grades are more varied than on lower secondary level.

According to a 2013 TALIS (Teaching and Learning International Survey) study (Ólafsson, 2014), Icelandic teachers on average seldom review content recently taught, they also more rarely review homework or try to relate new knowledge to daily life, than teachers in the other TALIS countries. There is a considerable difference here. For instance, 38% of Icelandic teachers state that they often or almost in every lesson review content recently taught, where the TALIS average is 78%.

The study *Teaching and Learning in Icelandic schools* (age levels 6–15) was conducted in 2009–2010 (Óskarsdóttir et al., 2014). The study was done in cooperation with many stakeholders from universities, school authorities, schools, teachers and parent organisations, and partners from an architectural firm and an information technology firm. The study focused on many aspects of teaching and learning, like the learning environment, student learning, teaching strategies and internal structures. A special focus was put on the development towards individualised and cooperative learning advocated by school authorities both on the local and national level. The framework for the research project was a model of school practices developed as an evaluation tool by educational practitioners in Reykjavík School District. Data were gathered by using multiple methods like observations, interviews, focus groups, questionnaires and action research in 20 schools out of 175 schools in the country. Three of the schools were chosen because they had been designed with the aim of changing the instructional practices from traditional to more open and individualized learning. Other schools were randomly chosen. In total, 518 lessons in all school subjects were observed, and 240 interviews were conducted with students, teachers, principals and other staff in schools (Óskarsdóttir et al., 2014).

The results showed that teaching strategies that can be labelled as “direct” are most commonly used and strategies like discussions, group work and project work which are recommended in national curriculum guidelines are rarely used (Sigurgeirsson, Björnsdóttir, Óskarsdóttir, & Jónsdóttir, 2014). It is also noticeable that there is a considerable difference between teaching strategies depending on school level. The teaching of the youngest students seemed to be more varied than the teaching of upper grades. Schools in the study were grouped into three categories, based on whether teachers mostly worked alone with their class, or classes if they were subject teachers (6 schools), were team teaching and responsible for a whole year group together or mixed age groups (9 schools), or a mixture of both (5 schools). The teaching strategies used in schools where team teaching was the norm, were more versatile. The results also indicated that the development towards more individualised learning in light of the frameworks used is not very advanced.

International research on instructional practices in mathematics

In recent years, several studies have been conducted in different parts of the world with the aim of identifying common features of mathematics teaching in countries scoring relatively high in studies like TIMSS or PISA or by teachers who are considered outstanding math teachers in their respective countries (Clarke, Mesiti, Jablonka, & Yoshinori, 2006; Hiebert et al., 2003). Even though it is impossible to generalize or identify a common lesson script on the basis of these studies, some important characteristics of effective mathematics teaching have been located.

The TIMSS 1999 video study brought to light that no single method of teaching 8th grade mathematics, was observed, in all the relatively high achieving countries taking part in the study (Hiebert et al., 2003). However, all eight-grade classrooms in all seven countries shared some general features. Mathematics was often taught through solving problems and 90% of lessons made use of a textbook or worksheet of some kind. Lessons were organized to include some public whole class work and some private individual or small group work. It was most common for students to work individually rather than in pairs or groups. The lessons included some review of previous content as well as some attention to new content and the teachers usually talked much more than the students. It was also observed that a variety of methods were employed rather than a single shared approach of teaching mathematics. Each country combined and emphasized instructional features in various ways, sometimes different from all other countries and sometimes partially the same.

Boaler's (2006) long-term study of mathematics teaching in three different schools in the US, showed that students from a school called Railside both enjoyed mathematics more and reached higher levels of mathematics than the students in the other two schools. According to Boaler, their success was a result of the unusual approach to mathematics at the school. The classes were heterogeneous, the students worked in groups on group-worthy problems that could be solved and represented in different ways. Moreover, the students spent a lot of time discussing mathematical ideas, learnt to help each other and were made responsible for teaching their peers. The lessons were 90 minutes long and the teachers worked closely together while preparing their teaching and shared the same ideas about teaching and learning.

Hiebert and Grouws (2007) identify, by reviewing research on the impact of classroom teaching on student learning, two main features of classroom mathematics teaching that facilitate students' conceptual development. These features are, firstly, an explicit attention to mathematical concepts and connections between ideas, facts and procedures, and secondly, that the students were given opportunities to engage in and struggle with important mathematics. According to Hiebert and Grouws (2007), these features seem to be general and operate across various contexts and teaching systems.

In their review of research on classroom practice in mathematics, Franke, Kazemi and Battey (2007) focus both on the teachers' role in mathematical work and students' experiences in the social context of the classroom. They point out that the nature of mathematical discourse in classrooms is central if teachers are to gain opportunities to learn from their practice. Students' individual work cannot alone provide such opportunities. In creating opportunities for discourse, teachers also need to attend to the social and socio-mathematical norms in the classrooms and develop relationships with their students where they take into account the students' cultural backgrounds. The IRE discourse pattern (teacher *initiated* question, student *response,* and teacher *evaluation)* is still prevalent in many mathematics classrooms, and this needs to change. The IRE discourse pattern falls within the exercise paradigm (Skovsmose, 2001) where the teacher presents some mathematical ideas and the students work with selected tasks from textbooks. The teacher presentation can vary in length. It can take up to a whole lesson and the students could also be working with exercises for the duration of the lesson. Justifications of the relevance of the exercises are not a part of the lesson and there is usually only one answer to the task.

Even though it is clear that classroom practice is complex and many cultural differences can be found when studying mathematics teaching across cultures (Givvin, Jacobs, Hollingsworth, & Hiebert, 2009; Franke, Kazemi, & Battey, 2007; Hiebert et al., 2003), there seems to be consensus regarding the idea that both teachers and students need to play an active role in the mathematics classroom. It is important that students both actively engage in mathematical discussions, making sense of mathematical concepts, and that teachers are able to learn from their students and develop their practice. Students also need to be engaged in solving challenging problems and given opportunities to share and present their ideas (Givvin, Jacobs, Hollingsworth, & Hiebert, 2009).

DATA AND DATA ANALYSIS

Our data consist of observation protocols from mathematics lessons made by researchers in the research project *Teaching and Learning in Icelandic Schools* described above (Óskarsdóttir et al., 2014). The researchers came from various disciplines within the University of Iceland and the University of Akureyri, mostly from general pedagogy. None of the researchers had specialized knowledge about mathematics teaching and learning. The observers made detailed notes in an observation protocol during the lessons. The focus of the observations was more on the progress of the lesson and the students' activity during the lesson than on the content and how that was dealt with. In some of the observation protocols, it was clear what the focus of the lesson was but in others there was no mention of the content of the lesson. This has its limitation but nevertheless we feel the observation protocols give us an idea of what is happening in the classroom and how the teaching is organized. We, the authors of this paper, have been actively engaged in teaching math teachers and making curriculum materials for a long time and are therefore well known to most math teachers in Iceland. We felt that by using this data we could gain some information about mathematics teaching in Iceland without collecting the data ourselves and thereby probably influencing the results.

As mentioned previously, Savola (2010) studied mathematics lessons in Iceland and Finland. He videotaped 20 lessons (two from each teacher) in each country. He made lesson diagrams for each lesson based on the coding of his data. His main coding categories were review, introducing new content, practice/applying, and other. The category 'other' included classroom management, mathematics management, homework, interruption, social talk and independent learning (IL). Three different types of IL were noted, between-desk instruction, where the teacher walks around the classroom and helps, and teacher or student presenting at the front, addressing only few students while the others are working individually.

Johansson (2006) studied videotaped lessons from three Swedish teachers, considered competent mathematics teachers in their community, at lower secondary level (4–5 consecutive lessons from each teacher). She tried to identify a common lesson script and in coding her data she used four main coverage codes: classroom interaction, content activity, organization of students and textbook influence. She also directed her attention to the teachers' activity and how often specific events like problem solving, assignment of homework, assessment, goal statements, summary of lessons etc. occurred within a lesson.

In our analysis, we started by reading carefully all the observation protocols. We then formed some categories on basis of the data with categories used by Savola (2010) and Johansson (2006) in mind. Our data is much more limited because it is only based on written notes by the observers and not video recordings and therefore does not allow a fine-grained analysis. Classrooms practices are complex and by analysing lesson structure we try to capture some important elements of both the form and the function of the lesson. However, it has its limitations and researchers should be careful not to draw too many conclusions on the basis of this kind of data, but it can shed some light on important aspects of classroom practices (Savola, 2010; Clarke et al., 2006).

Our main categories were: non-mathematical work (a), teacher's public interaction with the whole class including presentation of new material and checking and assignment of homework (b), individual seatwork (c), assessment (d), group work (e), playing of games (f). We made a diagram of each lesson using these codes and also described in few words what was happening in each part of the lesson, for instance, whether the students used textbooks or not. This made it possible to spot differences and similarities across the sample.

FINDINGS

We summarize our findings according to grade levels.

Grade levels 1–4

For grade levels 1–4, we have 19 observations. Most of the lessons (13) were 40 minutes but six had duration of 60–80 minutes. In almost all the lessons it took at least five minutes before the actual lesson could begin and in ten of them the mathematical work was finished five minutes before the actual lesson ended. In ten of the lessons there was some public interaction between the teacher and the students lasting 5–15 minutes. In most cases the teachers explained algorithms and procedures or discussed and showed students how to work with pages from the textbook using an overhead projector. When explaining algorithms and procedures, the teachers were not using textbooks but supplemented the textbooks with material from other resources. The biggest parts of the lessons students were working individually in textbooks or on worksheets provided by the teacher. Activities like working with attribute blocks, playing dice games, working with Tangram and unit cubes were inspired by textbooks. During individual seatwork, the teacher circulated and assisted students and in nine lessons an unqualified assistant or another teacher/specialist was present either working with specific students or assisting in general. When working individually in textbooks students sometimes worked with the same chapter or worksheets but in at least six lessons they worked at their own speed but with the same textbook or workbook. In three lessons the students worked in groups. Two of these lessons were organized as workstations where the groups worked on different activities. In one class, all the workstations focused on practicing multiplication but in the other, a first grade class, the activities were unrelated but varied. In the third lesson with group work, all students were working with attribute blocks, first making a picture together in a group and then they got some time for free play with the blocks.

Grade level 5–7

From grades 5–7 we have 13 observations. Most of the lessons were 40 minutes but three were 80 minutes long. In ten of the lessons it took about five minutes before the mathematical work started. Here, in six lessons, there was some public interaction between students and teachers and it centred on guiding the students through the textbook and reviewing homework. In one lesson the teacher was discussing properties of geometric forms with the students and then they were to make their own forms but it was not clear whether this was inspired by the textbook or not. In seven lessons the students were working individually on their textbooks almost the entire time of the lesson while the teacher was circulating and assisting them. In one lesson the students were working on a test for 30 minutes and then they handed in the test and were given an opportunity to work on it again the next day with the aim of looking things up at home and then improving their solutions. In one 80 minutes lesson, the students, after completing a self-assessment, worked in groups planning a table tennis tournament and after that they played the tournament and then came back to class and discussed and shared their solutions of the task.

Grades 8–10

From grades 8–10 (lower secondary level), we have 19 observations and eight of the lessons were 60–80 minutes long. Here it took less time for teachers and students to get to work with the mathematics. In most lessons they had started within 2–3 minutes. Eight of the lessons started with some public interaction between teacher and students, either by reviewing homework or presenting new material or problems. Even though in most lessons these interactions only took around ten minutes, there were examples of them taking from 20 and up to 60 minutes, and usually the teachers tried to engage the students by asking questions and encouraging discussions. In three classes the students were divided into two groups where a part of them worked individually while others were taught by the teacher or took a test. In five lessons the students worked individually on textbooks during the whole lesson, and in six more lessons, the students worked individually after a short introduction by the teacher. In many of the lessons, the students seemed to be working on the same topic even though there were examples of individual students working on material from higher grade levels, even upper secondary level, or students working with different material from the rest of the class because of learning difficulties. In three of the classes the students worked independently according to a plan made for the chapter or a month with a daily quota. In one lesson, students worked on an assessment. In one school two lessons were observed where students worked in mixed age groups on problem solving. Here the lesson started with some introduction of important concepts in relation to the problem at hand and then the students were divided into groups.

DISCUSSION

The findings from this study support previous findings on instructional practices in Iceland and add some new knowledge about mathematics teaching in compulsory schools as a whole. In many lessons there was a strong focus on individual seatwork where students worked mostly on textbooks. The teachers seemed to rely on textbooks and their contents for the whole class, aimed at helping or guiding students through the problems or exercises in the textbooks. Students were active in their work and the teachers moved around the classroom and helped them. There were examples of teachers creating opportunities for whole class discussions about topics or ways of working sometimes in the beginning and sometimes in the middle of lessons in connection with the individual seatwork. However, there were also many lessons with no public interactions and therefore the teachers were not creating enough opportunities for students to elaborate on and discuss with others mathematical concepts and connections. More whole group discussions should also help the teachers to develop and learn from their practice (Franke, Kazemi, & Battey, 2007). The socio-mathematical norms in the classrooms seemed to be that you learn through working on problems/exercises in textbooks, and the communication between teachers and students was centred on supporting the students in completing the work. This is what is often termed as traditional mathematics teaching or the exercise paradigm (Skovsmose, 2001). We do not know much about the nature of the tasks that the students were working on but from the observations it seemed like the teacher put an emphasis on guiding the students through their work with the problems. In several cases, the teachers in the lowest grades worked through the pages in the textbook by using overheads. With the older students, the teacher often discussed specific problems in textbooks. In six lessons, students worked in groups and there the teaching was more in line with what many researchers (Franke, Kazemi, & Battey, 2007; Hiebert & Grouws, 2007; Boaler, 2006) claim to be important features of effective mathematics teaching.

Our findings reveal that instructional practices in mathematics are similar to the general results on teaching practices in the study *Teaching and Learning in Icelandic Schools* (Sigurgeirsson, Björnsdóttir, Óskarsdóttir, & Jónsdóttir, 2014).

The lesson diagrams gave a good overview of the lessons and their structure and drew attention to similarities and differences both within grade levels and between grade levels. In the lower grades, the teachers used more time on non-mathematical work, which is not surprising. It could also be seen that teachers use considerable more time working with the whole class as the children get older.

The data consisted of observations from only one lesson in each class. Previous research has shown that lesson patterns can vary considerably from lesson to lesson with the same teacher (Clarke, Mesiti, Jablonka, & Yoshinori, 2006; Hiebert et al., 2003). In the data there was little information about the mathematical content of the lessons and what the teachers chose to emphasize in their interactions with the students. As mentioned earlier, most of the observers came from general pedagogy and they did not note what the focus of the lesson was. They sometimes referred to pages or specific problems and from that we could see whether the class was working on the same problems at the same time or not, but it provided limited information about the goals with the lessons.

From this study it is evident that Icelandic teachers use a considerable time walking between the desks interacting with students. Research has shown that there can be great variations in regard to what the teacher is actually doing when he is circulating in the class and that affects the quality of the teaching (O´Keefe, Xu, & Clarke, 2006). Therefore, it would be interesting and valuable to find out what characterizes the teachers interactions with students in Icelandic mathematics classrooms.

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Investigating two trainee teacher educators' transformations of the same resources in technology enhanced mathematics

Elissavet Kalogeria[1] and Giorgos Psycharis[2]

1 University of Athens, Educational Technology Lab, Athens, Greece, ekaloger@ppp.uoa.gr

2 University of Athens, Department of Mathematics, Athens, Greece

We address the documentational work of two trainee teacher educators in the context of their practicum in an in-service program concerning the use of digital tools in mathematics. Since they drew upon the same existing resources, we investigated the operational invariants – i.e., implicit aspects of their knowledge and epistemology underlying their designs – that influenced and differentiated their documentational work. The identified operational invariants were (a) their focal points during the observation of other teacher educators' classrooms, (b) the constrains and opportunities provided by the educational context, (c) their epistemologies regarding the role of technology in the teaching of mathematics as well as their conceptions of trainee teachers either 'as students' or 'of students'.

Keywords: Documentational approach, teacher educators, digital tools.

INTRODUCTION

In this paper, we study the didactical design and corresponding material developed by trainee teacher educators in teaching mathematics with the use of digital tools. The trainees drew upon existing resources as they started to teach in teacher education classrooms during their practicum and created their own documents. Our focus is on the factors that influence the development of the trainees' documents based on the same existing resources and the classroom implementation of their design.

The study took place in the context of an in-service program adopting reform-oriented perspectives to train teacher educators into the use of digital tools in the classroom of mathematics. The aim of the program was to provide the participants with methods, knowledge and experience in in-service teacher education and to educate them in the pedagogical uses of digital technologies for the teaching and learning of mathematics. One of the reform aspects of the approach for teacher education (see Kynigos & Kalogeria, 2012) concerned teacher educators' and teachers' active engagement in creating their own didactical design and material as coherent part of their professional development. Taking into account that teacher educators have very few resources to draw on directly (Zaslavsky, 2008), it was critical for the trainees to get used to developing their own material. In this course, the trainees were engaged in designing and generating resources in the form of microworlds and scenarios (i.e., structured activity plans addressing critical aspects of a pedagogically sound use of technology for the teaching and learning of mathematics). A structure [1] for addressing these aspects was developed by Educational Technology Lab (http://etl.ppp.uoa.gr), which participated in the design of the course and the corresponding material. The training program took place in specialised University Centres (UC) for 350 hours. The participants were experienced qualified mathematics teachers but the majority of them had no previous experience in the pedagogical use of digital tools. The plan was to employ the newly trained teacher educators in wide-scale 96h courses to groups of teachers in specific Centres for Teacher Education Support (CTES). The trainees in UC were given material by the trainers after each lesson and an official document containing theory and a set of twelve generic scenarios as a basis for organizing their subsequent teaching in CTES. During the course, the trainee educators gained significant experience with the pedagogical use of five categories of digital media: Computer Algebra Systems, Dynamic Geometry

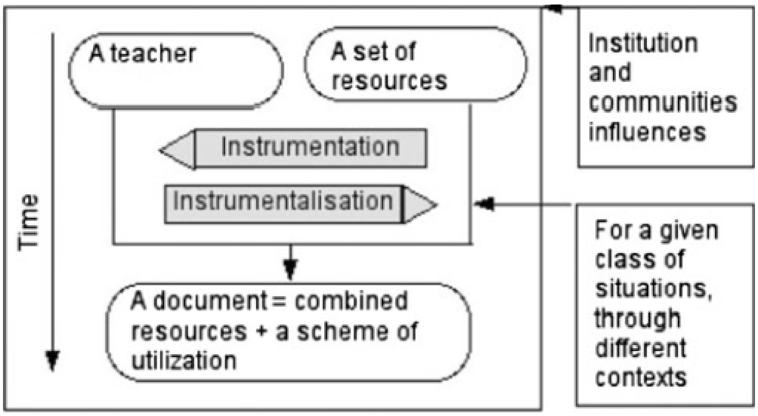


Figure 1: Documentational genesis

Systems, Programmable software, Simulations and Data Handling tools. By the end of the course the trainees had to have developed one scenario for each of these categories as well as scenarios for the practicum. Practicum was part of UC official structure provided shortly before the end of the course, so as to engage trainees in field activities and give them the experience of implementing their design in real classroom conditions and reflecting on it. Practicum took place in 30 hours and it was divided in two parts: teaching in school and observation – teaching in CTES. Here in focus is the second part consisted of (a) observation of other teacher educators' teaching in CTES, (b) design of a 3-hour lesson for teachers in CTES under the supervision of a mentor, (c) implementation in the classroom, (d) presentation of design and implementation in whole class special reflective sessions, (e) activity report by the trainees.

THEORETICAL FRAMEWORK

Over the last years a number of researchers have indicated that the study of resources in practice and context constitutes an important theme in mathematics teacher education and deserves a focus of attention (Adler, 2000). We adopt the documentational approach of didactics according to which the teacher's work is developed with and on resources in a dialectic process where design and enactment are intertwined (Gueudet & Trouche, 2009). An implication of this approach is that curriculum material is not conceived as a static body of resources that guides instruction but rather as a set of objects amenable to changes depending on the teacher's didactical design. Gueudet and Trouche (2009) use the term resources to describe a variety of artifacts such as a textbook, a piece of software, a student's sheet, discussions with colleagues etc. Through a class of professional situations and teachers' experience, the existing resources are transformed into documents according to the formula: *Document = Resources + Schemes of Utilization.*

A scheme of utilisation of a set of resources incorporates "*practice* (how to use selected resources for teaching a given subject) and *knowledge* (on mathematics, on mathematics teaching, on students, on technology)" (Gueudet & Trouche, 2011, p. 401). Practice entails observable parts of teachers' stable behaviour for a given class of situations (called usages). On the other hand, knowledge embodied in resources is implicit and can be inferred from the usages. This knowledge is intertwined with teachers' beliefs and it is difficult to distinguish one independently of the other (Thompson, 1992). The constituent elements of this kind of knowledge are the *operational invariants* built through different contexts of using the resources.

Creation of documents is considered as unfolding through a dual process of *instrumentation* (the resources act on the teachers and influence their activity) and *instrumentalization* (teachers act upon these resources as they appropriate and reshape them) (Gueudet & Trouche, 2009). This process is named documentational genesis (ibid, see Figure 1) and gives birth to a new entity: a document, which can be further transformed to a new document over time.

The importance of interrelating knowledge and epistemology, the difficulty to distinguish them, as well as their influence on the everyday teaching practice has been stressed by many researchers (Nespor, 1987; Thompson, 1992; Ernest, 1994). However, the relation between teachers' epistemologies and practices is complicated and not linear as it is strongly affected by constraints and opportunities afforded by social context (Ernest, 1994). For example, a teacher's conceptions regarding the teaching of mathematics may be rooted in the epistemological paradigm of absolutism while her/his own practices might be closer to the paradigm of fallibilism and vice versa. Thus, interconnections between teacher's practices and epistemologies formulate the following different roles in her/his own teaching (ibid): instructor (i.e., targeting skills' mastery and correct performance), explainer (i.e., targeting conceptual understanding with unified knowledge) and facilitator (i.e., targeting problem posing and solving). At the same time each role signifies a different stance towards the curriculum and the corresponding teaching material and leads to three patterns of curriculum use (ibid): the strict following of a text; the modification of the textbook approach, enriched with additional problems and activities; and the construction of the mathematics curriculum by the teachers themselves. As we will analyze later in the paper, the trainee teacher educators of our study – despite the fact that they were given the same initial resources – adopted different roles and corresponding didactical designs when they started teacher education themselves in the context of their practicum. We consider that these differences were determined by a combination of operational invariants that was unique for each one of them. Thus, in order to explain the differences between trainees' design and implementation, it was our choice to investigate the underlying operational invariants both in the design of documents and their usages in real educational contexts. Existing research on teachers' documentational work highlights the need to identify categories of operational invariants permitting refinement of the analysis of schemes of utilisation (Gueudet & Trouche, 2009).

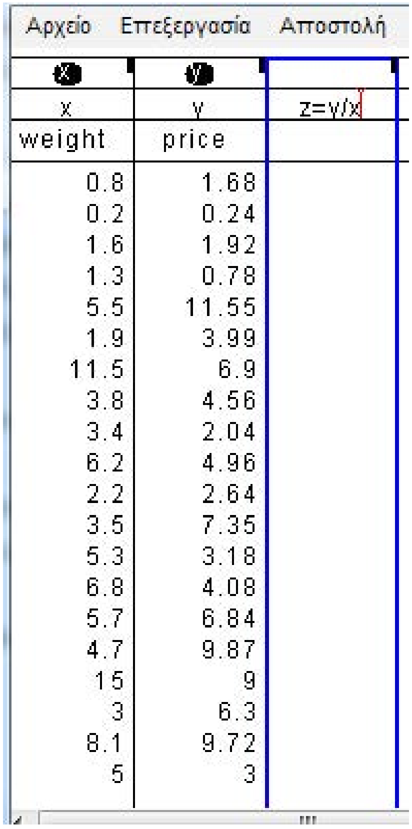
In this study, our aim is to investigate how the formula *Document = Resources + Schemes of Utilization* works when different teacher educators build the development of their documents upon the same existing resources. Thus, we investigate the underlying factors (i.e., operational invariants) that influence trainee teacher educators' design. We looked for these operational invariants within the space where the processes of instrumentation and instrumentalisation take place.

METHOD

The UC class of trainee teacher educators that we analyse here consisted of 16 qualified mathematics teachers (five of them had a doctoral degree, one was a PhD student and the rest held a Master degree). The data we analysed consisted of: (1) verbatim transcription of the discussions that took place during the 30 hours of the practicum in the UC classrooms, (2) the material constituting the trainees' designs for their lessons in CTES (scenarios, worksheets, ppts, etc.) and (3) their activity reports. The activity reports were templates in which trainees had to insert text describing aspects of their designs and their experience from its implementation. From the analysis of the 16 trainees' documentational work, in this paper, we present two cases with the aim to highlight differences in trainees' design of documents and implementation in teacher education classrooms. Particularly, we chose Ian and Jim as exemplary cases because their teaching in CTES was based on the same official scenario. This allowed us to view comparatively the documents they created and thus to address the underlying operational invariants. Our role as academic trainers and mentors in the practicum allowed us to capture the evolution of their documentational work in all phases of the practicum. In resonance with Gueudet and Trouche's (2011) principles regarding methodological aspects of research on documentational genesis we chose to (a) analyze Ian's and Jim's work in time periods in and out-of-class (reflexive investigation principle), (b) address their decisions taken in order to formulate their design through its use (design-in-use principle), and (c) consider their work embedded in and influenced by different collectives (e.g., teachers in CTES) (collective principle). We used data from different time periods: excerpts from their observations in CTES that took place before their design during the reflective sessions, the official material and its transformations by Ian and Jim including the arguments with which they documented their options and presented in the reflective sessions and finally Ian's and Jim's activity reports. In the analysis we adopted a data grounded approach (Strauss & Corbin, 1998). Initially, each one of us (the two authors) worked separately for coding trainees' work and identifying operational invariants. After reaching a common consensus, we jointly completed the analysis.

ANALYSIS

Ian's and Jim's lessons in CTES concerned the teaching of linear functions ($y = ax$) with Function Probe (FP) [2]. Their teaching was based on one of the twelve official scenarios provided for the course. The problem



x	y	z=y/x
weight	price	
0.8	1.68	
0.2	0.24	
1.6	1.92	
1.3	0.78	
5.5	11.55	
1.9	3.99	
11.5	6.9	
3.8	4.56	
3.4	2.04	
6.2	4.96	
2.2	2.64	
3.5	7.35	
5.3	3.18	
6.8	4.08	
5.7	6.84	
4.7	9.87	
15	9	
3	6.3	
8.1	9.72	
5	3	

Figure 2

included in this scenario was: "A salesclerk sales three products A, B, C of different prices. After every sale he records the quantity x (in kg) and the amount of money y received. When he had completed twenty sales he passed the values in two columns in the Table window of FP so as to check: (a) how many sales were made by each product and (b) if there had been mistaken sales. In how many ways can he conduct the check?". The indicative design of the official scenario suggested the following teaching sequence in three phases.

<u>Phase 1</u> *(Exploring regularities – relations between different groups of proportional amounts):* It is provided a ready-made table containing 20 pairs (weight, price) of 20 sales in two corresponding columns. Teacher can chose or discuss with the students the possibility to fill in one more column defined by the ratio $z = y/x$ (Figure 2). In this case, students are expected to notice that this ratio takes three particular values (0.6, 1.2, 2.1) except one. These values can be an indication for the existence of linear relations $y = ax$. Then the students send the points (x, y) to the Graph window (Figure 3) and they are expected to notice that the three values of the ratios correspond to three different groups of collinear points. Also they are suggested to construct the graph of $y = x$ and stretch it dynamically with the mouse so as to coincide to each of the existent groups of points (Figure 4). The corresponding formula for each transformation of the graph appears in the upper right corner of the Graph window. The students are expected to explore the problem either algebraically (i.e., by linking the corresponding ratios to linear functions) or geometrically (i.e., by dynamically manipulating the graph of $y = x$) and to connect the three ratios that encountered in the table to the three values of the coefficient a of $y = ax$ respectively.

<u>Phase 2</u> *(Testing the formulas of proportional amounts through different representations):* The students can be engaged in generating the exact prices of each product in three different ways: (a) In the Table window: they can construct one new column corresponding to the weight of each product (use of command 'Fill in' with values from 0 to 10 and step 0.1). Then, they are asked to construct three new columns so as to calculate the exact prices of each these products using the formulas $y = 0.6t$, $u = 1.2t$, $z = 2.1t$. This way the salesclerk will be able to know the exact price for each product until the weight of 10 Kg without calculating it. (b) In the

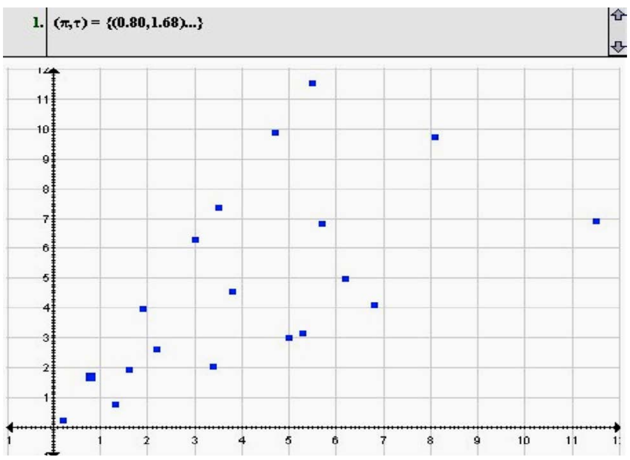


Figure 3: Sending points to the Graph

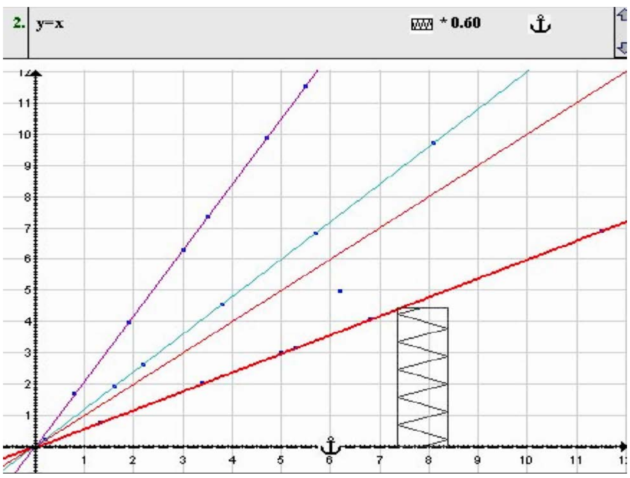


Figure 4: Stretching $y = x$

Calculator window: they can create three buttons (see [2]) for calculating all prices of the products through a functional relation. (c) From the Graph to the Table window: they can take points from the graph and send them to two columns in the Table standing for x and y respectively.

<u>Phase 3</u> *(y=ax+b as a transformation of y=ax):* $y = ax + b$ can be investigated through vertically stretching $y = ax$ at the Graph window and constructing new columns in the Table window defined by functional relations (e.g., $w = 0.6t + 0.5$).

Ian's documentational work

During his observation in CTES Ian, in both his activity report and the reflective sessions, concluded that in order to design his own lesson in CTES had to take into account the following aspects: (a) the different levels of teachers' familiarization with digital technologies, (b) difficulties with particular tools/functionalities of FP (e.g., creation of buttons in the Calculator), (c) difficulties in conceiving the links between the different representations of FP. Thus, he started his lesson in CTES by demonstrating particular functionalities of FP in a whole class session through the use of an interactive whiteboard. Then, he gave to the teachers a worksheet with the ready-made table of the official scenario. The worksheet covered the three phases of the official scenario and it was structured in the form of small steps-instructions insuring the correct use of FP tools for the requested activities. For instance, instructions concerning the first phase were the following: "Table window: Fill in the 3rd column with the ratio *y/x* and press enter", "Write down the resulting values and explain what they show", "Table window: Send points to the Graph", "Graph window: Graph → Graph choices → Check 'Show transformations' → Click on the icon '*y*=...' to create the graph of $y = x$". In the second phase, Ian provided a detailed account of the three ways by which teachers could have the prices of each product through the use of the Table, the Graph and the Calculator window respectively.

In his activity report Ian describes his choices in the design as shown in the following excerpt: "The development of my design follows the existing official material of CTES with a differentiation as regards the worksheet which is more instructional due to a better time control ... Though the worksheet should provide learners with freedom to pose and answer their own questions, I chose to make it more instructive – probably more than needed – so as to support teachers' familiarization with the available tools while solving the problem ... Another reason for choosing a more instructive perspective stemmed from the fact that the trainer of this particular CTES classroom had a similar approach in his own teaching. Thus, I targeted a more smooth transition from teachers' trainer to myself ...". During the reflective sessions in UC, Ian again justified his choices in the design of the worksheet after being asked by another trainee if he had been satisfied by his teaching in CTES and if he would change something in it: "I would prefer a less instructive design. But I was anxious about time. Through my choice, the worksheet was completed by all teachers in time. They liked the environment and seemed to have learned better the FP functionalities".

Teachers' and students' engagement in exploratory activities with digital tools was a central idea underlying the design of the UC course. Ian seemed to have shared this view, but he finally designed a rather instructive lesson in CTES. Thus, Ian seems to consider teachers mainly 'as students' and his design targets the development of two kinds of knowledge: (a) about technical aspects of software and (b) about the ways by which the available tools can be interrelated to subject knowledge. For instance, he provides detailed instructions on how to approach the targeted functional relation through the different FP windows (Table, Graph, Calculator).

The operational invariants underlying his design were related to the following factors: (a) his emphasis on constrains and opportunities afforded by the context (e.g., time restrictions, technological environment) which led him to follow instructive design and practices respectively, regardless of his (possibly different) epistemological conceptions for the teaching and learning of mathematics; (b) his observation in CTES concerning teachers' difficulties with FP and the previous teaching model adopted by the trainer in CTES; (c) his conception of trainees mainly 'as students' who need detailed instruction in order to overcome their difficulties. All these elements influenced the instrumentation/instrumentalisation interplay in his design leading to the integration of small-step instructions in the worksheets.

Jim's documentational work

Jim observed another trainer's lessons in CTES that preceded his own lesson and included teachers' initial familiarization with FP tools apart from the stretch tool. Thus, he started his lesson by demonstrating in the interactive whiteboard technical issues related to the use of FP with an emphasis on the use of the stretch tool. His worksheet included the following open tasks for the teachers (without any kind of instructions, even for aspects of the software): 1) The teachers were asked to work in groups of two and provide both an algebraic and a geometrical solution for the given problem. Construction of tables of values for the three products (in the Table window) or creation of buttons (in the Calculator window) is suggested. 2) Group discussions for the potential findings of a group of students and ways to approach them. 3) Whole-class discussion of the above findings. 4) Whole-class discussion for the added value of FP. 5) Design of indicative questions for the students by each group of teachers. 6) Presentation of groups' design in the classroom.

Describing his lesson in one of the reflective sessions in UC, Jim mentions: "I gave to the teachers a table with two columns corresponding to the weights and the prices for 20 sales of the three products. There was also a third empty column. I did not tell them to calculate a ratio or to send the two columns to the Graph. I was curious to see how many groups would be able to find them ... I asked them to solve the problem algebraically and geometrically. I recommended the creation of columns in the Table window for the values of the different products as well as the creation of three buttons in the Calculator ... I organized the class in groups of two following the model of their trainer ... All groups found at least one solution. I passed through the groups and encouraged them to find another one ... They discussed in their groups, they explored, I showed them how to stretch the graph of $y = x$. I was stressed by the possibility of not being able to use the FP tools. Thus, I had prepared another worksheet including more detailed instructions for the tool use. Finally, I didn't give it to them. If someone had faced problems with FP I would have given it to him. I believe that the level of instruction should be gradually decreased. When we first introduce new software we need to prepare a more instructive worksheet. But in the next lessons the worksheets should be as open as possible. The aim is to support teachers' thinking. You cannot tell them continuously 'press this' or 'press that'....".

Jim's design is closer to the ideas favored by the UC course. He took the risk to organize his lesson around an open worksheet and to confront potential teachers' difficulties with FP tools on the spot. In his design he initially considered teachers 'as students' and through a social process which evolved in six phases he came to view them as 'teachers of students'. Particularly, in phase one, he engaged teachers in exploring the problem (geometrically and/or algebraically), while in all subsequent phases, the teachers were asked to adopt the role of 'teachers of students' through a series of activities. At the instrumentation level, he took into account the model of teaching adopted by the official trainer in CTES as well as the fact that the main functionalities of FP had already been taught in previous lessons. Besides, his conception of the nature of worksheets regarding the degree of instruction led him – at the level of instrumentalisation – to incorporate open tasks. Facing the challenge to balance the correct use of technological tools and their integration in creating meaningful mathematical representations, Jim gives priority to the second. Jim is not interested in technological skills per se but how these would be incorporated in teachers' didactical design. Jim's choices reveal that he targets trainees' more complicated forms of knowledge as his design intertwines linear functions and its pedagogy with technology (approaching $y = ax$ algebraically or geometrically through ratios and dynamic manipulation of $y = x$). In terms of Ernest (1994), he adopts the role of facilitator who targets problem posing and solving and favors the development of meaningful material by the teachers themselves.

The operational invariants underlying the instrumentation/instrumentalisation processes were related to the following factors: (a) his epistemological conceptions of the ways mathematical knowledge should be approached though the use of technological tools; (b) his observation concerning the teaching agenda of the official trainer in CTES; (c) his conception of trainees initially 'as students' who work in groups to explore different solutions of the given problem with the use of tools and subsequently as 'teachers of students' who are facilitated by his mediation to exploit the tools and transform them into didactical instruments for mathematics teaching at the professional level.

CONCLUSION

Our analysis revealed the development of two kinds of documents: an instructive one and an open one. Taking into account the formula *Document = Resources+Schemes of Utilization*, we conclude that differences in the nature of the two documents can be explained through corresponding differences in the trainees' schemes of utilisation, i.e., identification of different operational invariants for each one of them. Both trainees intended to continue the CTES trainer's teaching model and to address the potential teachers' difficulties with FP. Ian's instructive approach seemed to have also been influenced by his concerns of differences in teachers' familiarisation with FP tools as well as by the time restrictions. While Ian aims to control all the above factors through an instructive document, Jim's design emphasises teamwork and appropriate teaching interventions by himself.

Thus, in the context of teacher education in technology enhanced mathematics, operational invariants that influence trainee teacher educators' documentational work in the context of their practicum seem to be closely connected: (a) to their experiences from the observation in CTES classrooms (focusing either in the trainee teachers' difficulties or the model adopted by the official trainers of the respective classrooms); (b) to the importance they attribute to the constrains and opportunities provided by the wider educational context; and (c) to their epistemologies regarding the role of technology in the teaching of mathematics and the ways they conceive trainee teachers, for example, emphasis on technology per se favouring the conception of teachers mainly 'as students' or on the integration of technology as a coherent part of teachers' designs favouring the transition of trainees from 'teachers as students' to 'teachers of students'. The first two operational invariants are directly linked to the teaching practice taking place in a real educational context and it is useful to be taken into account in every teacher education course. The third operational invariant reveals parts of the trainee teacher educators' beliefs that seem to influence the ways they design and use resources for teachers' meaningful integration of technology. Thus, the third seems to be a catalyst in favouring less instructive approaches by the trainee teacher educators and at the same time it indicates a domain for interventions in the design and further enrichment of the course. A potential suggestion for reinforcing the reform aspect of the course (or similar courses at the level of educating teacher educators) could be an earlier introduction of practicum. According to our findings, practicum provides a context dense of opportunities for challenging and questioning trainees' beliefs as they externalise them through the cycle 'design-implementation-reflection'. Thus, an early introduction of practicum in conjunction with appropriate mentoring could help trainees to negotiate and redefine their beliefs towards a pedagogically sound use of technology in their teaching.

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ENDNOTES

1. 1. Title, 2. Scenario's identity (author, subject area, topic), 3. Rationale (innovations, added value by the use of technology, students' learning problems addressed), 4. Context of implementation (grade, duration, location, prerequisite knowledge, social orchestration of the classroom, goals), 5. Phases of implementation (sequence of activities, roles of the

participants, anticipated teaching/learning processes), 6. Possible extension, 7. References.2

2. FP is a multi-representational software with three windows: Table, Graph and Calculator. Function graphs can be produced in a number of different ways, for example, inserting a formula for the function, "receiving" ordered pairs (x, y) from a table ("x" and "y" columns can be generated). Particular icons allow horizontal and vertical transformations of functions (translations, reflections and stretches) through direct actions on the graph. Stretching is carried out with the stretch tool that allows mouse-driven horizontal and vertical stretching. In the Calculator, the user can incorporate functional relations into buttons which reserve these relationships for future use.

A contingent opportunity taken investigating in-between fractions

Bodil Kleve and Ida Heiberg Solem

Oslo and Akershus University College of Applied Sciences, Oslo, Norway

In this paper, we discuss a contingent opportunity from a 7th grade mathematics lesson about decimal numbers, percentages and fractions in Norway. The question if adding numerators and denominators of two fractions was a way to find an in between fraction, occurred. The teacher's response to this question, which we saw as contingent opportunity, is our focus here. Although the teacher did not have the substantive mathematics knowledge needed to give an answer to the problem, many mathematics activities took place in the lesson.

Keywords: Contingent opportunity, teachers' mathematical knowledge, mathematising activities, argumentation and proofs.

INTRODUCTION AND RESEARCH QUESTIONS

When teaching mathematics, teachers are faced with many opportunities – both expected and unexpected. Awareness of opportunities and teachers' decision-making when they occur, is crucial with regard to students' learning outcome (Bishop, 1976) and thus important issues in mathematics teacher education. In the mathematics research literature, several possible responses to contingent moments are discussed (Kleve, 2010; Rowland, Huckstep, & Thwaites, 2005; Rowland, Turner, Thwaites, & Huckstep, 2009; Rowland & Zazkis, 2013). In responding to contingent opportunities, teachers may run a risk.

In a mathematics lesson about fractions, decimal numbers and percentages in a 7th grade in Norway, which we observed, the problem finding an in between fraction arouse, and data from this classroom event create the background for this paper.

Our research question is:

— How can a teacher's decision making to an unplanned opportunity open up for mathematics discussions, even if the teacher does not know the answer?

In order to answer this question, we focus in particular on the mathematizing activities that took place and how the teacher took advantage of the contingent opportunity when it arose. We also investigate how the teacher's knowledge appeared to influence the course of the lesson and thereby supported the students' learning opportunities.

A CONTINGENT MOMENT FROM THE RESEARCH LITERATURE

Earlier research have suggested ways in which teachers may respond when a student suggests how to find a fraction between two given fractions (Bishop, 1976; Rowland & Zazkis, 2013; Star & Stylianides, 2013; Stylianides & Stylianides, 2010). However, authentic data, which display teachers' responses to students' contributions to the task or from classroom discussions, have not been presented. In these studies, how students *may* respond to the task to find a fraction between fractions and how teachers *may* respond to a student's response, have created the background for discussions among prospective teachers.

Referring to the fractions example[1], Bishop (1976) asked the question: "How would you deal with that response?" (p. 41) and he emphasized teachers' decision making, to be at the heart of the teaching process.

1 Teacher: Give me a fraction which lies between ½ and ¾
Pupil: ⅔
Teacher: How do you know that ⅔ lies between ½ and ¾?
Pupil: Because 2 is between the 1 and the 3 and 3 is between the 2 and the 4

Rowland and Zazkis (2013) presented possible (tacit) conjectures made by the pupil who responded to Bishop's fraction example[2]. However, they "did not know how the lesson proceeded mathematically nor whether the teacher took the advantage of the opportunity" (p. 150).

Stylianides and Stylianides (2010) referred a similar scenario in order to exemplify Mathematics Knowledge for Teaching. In this scenario, a student, Mark, suggested adding the numerators and denominators of two fractions to find one in between, and he illustrated that on a number line. They focused on implementation of special tasks in teacher education and discussed how the task (a student's response to finding an in between fraction) could support the development of mathematical knowledge for teaching.

Star and Stylianides (2013) used the fractions task for prospective teachers in mathematics teacher education in order to discuss teachers' mathematical knowledge asking the question: "What can be considered as procedural/conceptual knowledge?" (p. 172).

TEACHERS' MATHEMATICAL KNOWLEDGE

In order to investigate teachers' mathematical knowledge, several frameworks have been developed (Ball, Thames, & Phelps, 2008; Fennema & Franke, 1992; Rowland et al., 2005). Both Ball et al. and Rowland et al. based their work on Shulman's categories of knowledge (Shulman, 1986, 1987). Stylianides and Stylianides (2010) related their work to Ball and Bass (2000; 2003). Using the abbreviation, MKfT, they emphasised mathematical knowledge *for* teaching:

> This specialised kind of mathematical knowledge is important for solving the barrage of mathematical problems of teaching that teachers face as they teach mathematics: offering mathematically accurate mathematical explanations that are understandable to students of different ages, evaluating the correctness of students' methods, identifying mathematical correspondences between different student solutions of a problem etc. (Stylianides & Stylianides, 2010, p. 161)

They suggested that the teacher's mathematical knowledge would influence the course of the lesson and students' learning opportunities. Linked to argumentation and proof they discuss both teacher's mathematical knowledge, which sometimes may deviate from conventional mathematical knowledge (misconception), and knowledge, which is consistent with conventional mathematical knowledge. Harel and Sowder (2007) carried out a study, finding that students' use of numerical examples, as a way of proving, was prominent. In everyday discourse "no rule without exceptions" and "the exception proves the rule" are accepted ways of "proving" something. Studies on teachers' mathematical knowledge and their conceptions about proofs in mathematics, show similar conceptions as revealed by Harel and Sowder (Martin & Harel, 1989; Stylianides & Stylianides, 2010).

Contingency is one of four dimensions in the Knowledge Quartet, KQ, which is a framework for Mathematical Knowledge *in* Teaching (Rowland, 2008; Rowland et al., 2005; Rowland et al., 2009). The KQ provides us with tools for analysing how teachers draw on different kinds of mathematical knowledge in order to support learners in the classroom situations. Contingency is informed by the three other dimensions in the KQ (Foundation, Transformation and Connection) and is about situations in mathematics classrooms, which are not planned for. Identifying "contingent moments" or contingent opportunities in order to analyse aspects of a teacher's mathematical knowledge has proven to be helpful (Kleve, 2010; Solem & Hovik, 2012). The teacher's choice whether to deviate from what s/he had planned and the teacher's readiness to respond to pupils' ideas, are important classroom events within this dimension. The Contingency dimension has also been expanded in a later research (Rowland, 2008; www.knowledgequartet.org; Rowland et al., 2009; Weston, Kleve, & Rowland, 2012). One such expansion is "Teacher's Insight" which is demonstrated when a teacher is realizing that children are constructing the mathematical ideas and something that sounds 'half baked' which means that they are in what Vygotsky (1978) termed Zone of Proximal Development (ZPD) where teacher can help with a scaffolding question or two (www.knowledgequartet.org). In their article, Rowland and Zazkis (2013) discussed several contingent opportu-

2 (C1): Whenever the numerators of three fractions are consecutive integers, and the denominators likewise, the second fraction will be between the two
(C2): Whenever the numerators of three fractions are in arithmetic progression, and the denominators likewise, the second fraction will be between the two

nities, which may occur in a mathematics classroom and whether they are taken or missed. One contingent opportunity that they discussed was similar to the one that actually happened in the lesson we analyse here.

METHODOLOGY

Our data are taken from an ongoing action research project which is about classroom conversations in mathematics focusing on what questions teachers ask (Solem & Ulleberg, 2013). Seven experienced teachers participated in the project. In this project, how questions could be used reflectively in planning a lesson, implementing the lesson and in order to analyse the lesson, were investigated (Ulleberg & Solem, in press). All teachers were observed in teaching a mathematics lesson in own class. We have chosen an episode from a lesson with a teacher, here called Kim. The reason for choosing this episode was not that it was representative for the research project, but that we found it especially interesting with regard to the contingent possibility, which occurred, and the teacher's response to students' input. Our data are field notes (including pictures), transcribed audio recordings from the lesson, and the teacher's prepared notes for the lesson. The lesson is from grade 7 (12–13 years old), and a whole class discussion had taken place. How the teacher drew on his mathematical and pedagogical knowledge in orchestrating the whole class discussion in this lesson is discussed in Kleve and Solem (2014). The teacher's plan for the lesson was to place fraction, percentages and decimal numbers on a number line.

ANALYSIS

As a challenge, the teacher wanted the students to find a fraction that was bigger than 3/5 but smaller than 4/5. According to the teacher's notes for the lesson, finding the fraction in between was supposed to be done by finding equivalent fractions with denominator 10. A student, Lea, suggested 7/10, because "I only doubled both and found what was in between". Then the following contributions from other students in class occurred:

Stud 1: Now it is the same again. 4 plus 3 makes 7.
Stud 2: It was the same last time we did it.
Stud 3: It happened then too, but you said it was a coincident.
Kim: But, may be, there is a pattern here?
Students: yes, yes..
Kim: Wait a minute....
Stud 4: But last time it wasn't ten, but now it is ten, isn't it?
Stud 5: We can add them both, getting ten

Kim then went back to how to get from 3/5 to 6/10 ensuring that all students were following the discussion. He asked Ada to explain to the others. Another student supported Ada saying:

Stud: You just multiply by 2 both in the upper and in the lower.
Kim: We double. Ok. Then back to the exciting thing, we talked about. Some of you are starting to see a pattern, which I am not sure I can see. Hanna?
Hanna: Yesterday we had the same, or we multiplied both with two. And then we saw that- that those numbers, I mean, the numerators, if we added them, it became the same as, yes, as the answer. And it happened now again.

Kim did not miss this contingent opportunity (Rowland & Zazkis, 2013), but responded by letting the students come up with suggestions, and after stud 3, he asked if there might be a pattern. Thus, he deviated from his agenda, which was to find a fraction between two fractions through expanding the two. He eagerly joined the students in searching for a pattern.

In their article, Stylianides and Stylianides (2010) discussed two possibilities from a similar scenario. One was that the teacher might consider the students' method to be correct because it works in different examples. The other possibility was that unless the method was shown to work for all possible cases, it could not be accepted as correct. Let us see what happened in our lesson:

Kim: Ok. But if it has happened twice, are we then sure it will happen next time?
Tiril: No, but there is a rule which is that if it happens once it is a coincident, twice, it is especially, three times, it is Ok.
Kim: Three times, then it must be like that? Does such rule exist?
Students: it is, ... no, no
Students: Can just be luck
Kim: OK, let us try

This suggests that Kim had knowledge about the distinction between empirical argumentation and proofs in mathematics. Although he expressed doubt about Tiril's rule (a method is correct if it works in three different examples), he invited his students to try a third time (the first was from yesterday, the second was 3/5 and 4/5), suggesting 1/4 and 2/4 and asked for a fraction in between. He challenged Januscha to explain:

Januscha: One plus two, I added the numerators, which made three, and the denominators, made eight. And to be sure, I doubled it too..
Kim: And then you got 3
Januscha: of 8
Kim: Did you all find the same? (Students confirmed)
Kim: OK, So we can see a pattern here now. But are we sure, does this apply to absolutely all fractions?
Students: That we don't know.
Ronja: I think, I don't know if it applies to fractions with different denominators
Kim: Ok?
Student: I think it applies to all fractions, because if 1/4 – if you multiply by two you get 2/8 and then you get 4/8 and that is in between (explaining that 3/8 is between 1/4 and 2/4).

Kim now suggested that they were seeing a pattern, but he emphasized uncertainty that it applied to all fractions. Thus, he exposed his students to hypothesize and to come up with conjectures. Ronja's question, if it applied to fractions with different denominators, was one conjecture, which opened up the possibility to find a counter example.

Kim did not follow up the students' hypotheses at this point, but he insisted further investigations with fractions with same denominators, and drew attention to the distance between the numerators. Proposing 3/5 and 5/5, he asked: "what if the distance between the numerators are bigger than one"? The students did not find this challenging at all:

Student: But if it is bigger distance between them, there is no challenge. What is it that is between 3/5 and 5/5? It's got to be 4/5!
Kim: Good. I agree that the challenge may disappear a little, but we have to check this out now. We are curious to know, aren't we?

Here, we suggest that the task had changed. Kim's point was no longer to find a fraction between two given fractions but to investigate if the "rule" -adding numerators and denominators to find an in between fraction- worked in different situations, also when the solution was obvious. Kim's proposal to have more than one between the numerators resulted in an "obvious example" (3/5 and 5/5) suggested by a student, who expressed that it was not challenging at all. Instead of being engaged in Kim's obvious example, the students contributed with further conjectures (which are partly contracting):

— I think it always goes when there is one even and one odd numerator

— When there are two even numbers or two odd numbers there is always a number between

— When there is one number of each, then it goes

— When there are two even or two odd, it will not work, but if there are one even and one odd, it works.

This demonstrates that the way Kim was teaching opened up for students' engagement and participation in mathematizing activities. Making conjectures and hypotheses are important ingredients in the learning of mathematics. The conjectures the students made, were general (about odd, even, and different numbers) and not linked to concrete numbers. Kim's response to these hypotheses were:

Kim: Ok- good. We are getting short of time, but this we *have to* check out. How can we check it all out?
Students: Do it many times
Kim: Shall we use "Tiril's method" and try: if it works the third time, it is ok, or what, how shall we do this?
Students: Why don't we try?
Kim: Ok, Then, firstly, I would like to try once having more than one in between. Let us do that first. Afterwards we can do it like you suggest with different denomi-

nators. So if we have…. We can take 3 of 8 and 5 of 8, then we have, the distance here is 2.

The students now agreed to go for Kim's suggestion first, for which he gave the following reason:

> Just so you do not believe…. that we can do it this way with the rest – that we just can add them together.

Now an eager discussion now took place. They had expanded the fractions to 6/16 and 10/16. We can hear students saying, "yes it goes", "it is four in between". Kim asked if they could choose between 7, 8 and 9, when a student said:

Student:	The middle is 8/16 and if we take 3+5 you get 8.
Kim:	Yes, good, it works.
Student:	It would have been the same if we just took different fractions. 3 of 8, and 4 of 8, or 4 of whatever. Just making different fractions.

This discussion ended up with the following statement from Kim:

> You know, now we are within an area in which I haven't checked either. Neither do I know for sure where this entire end up. I am not in the position to see all the patterns here either. Therefore, I find it very exciting. [] At least. Here we have lots of material we can work on in the future. But, you know what? We haven't done what we were supposed to do in this lesson (to place the fractions, percentages and decimal numbers on a number line)

This statement shows that Kim neither knew if it was possible to find an in-between fraction generally by adding the numerators and denominators, and consequently nor to prove it. Although he expressed that he did not know where it all would end up, he expressed excitement and was challenged to work further with the issue.

DISCUSSION

In this lesson a contingent opportunity occurred. Several students suggested that it was possible to find a fraction between two fractions by adding the numerators and denominators (Fareys mediant). The analysis above shows that the teacher deviated from the agenda and followed up the suggestions from the students. In grasping this contingent opportunity Kim took a risk. Although he neither knew if the conjectures were correct, nor had the advanced mathematical knowledge needed for proving the conjectures made, he deviated from the agenda and incorporated the students' suggestions in the further course of the lesson. Hence we see, that the mathematical knowledge he had, and also the mathematical knowledge missed, influenced the course of the lesson and thus the students' learning possibilities. Together with the students, he investigated their suggestions and claims. In asking "But are we sure, does it apply to absolutely all fractions?" Kim demonstrated mathematical knowledge about mathematical proving and signaled that it is not sufficient with numerical examples to generalize in mathematics. On the contrary, trying out different examples could have opened up for a counter example, which we suggest that he was looking for. This indicates that his substantive content knowledge incorporated what is a mathematical proof.

Since Kim did not have the mathematical knowledge needed to answer if adding the numerators and the denominators would make an in between fraction, he was not in a position to say "yes, it is correct", nor "it is not correct". The only tool Kim had was to try, and this he did in collaboration with the students. He encouraged his students to come up with several conjectures. The mathematical activities which took place incorporated hypothesizing if the rule, adding the numerators and the denominators to find an in between fraction, would work in different situations: fractions with different denominators, fractions where both numerators are odd, both numerators are even, one numerator is even and one is odd, different distances between the numerators.

The conjectures made:

- Adding numerators and denominators made an in between fraction (common denominators).
- Does it apply to all fractions?
- Does it apply to fractions with different denominators?

— No, but there is a rule which is that if it happens once it is a coincident, twice, it is especially, three times, it is Ok.

Formulating hypothesis, making conjectures and searching for patterns are important activities in learning mathematical thinking. The way Kim took advantage of students' input and questions, offered possibilities for the students to participate in these activities. Kim was sensitive to the students' mathematical knowledge (knowledge of content and students) and posed questions accordingly (knowledge of content and teaching). Posing appropriate questions as Kim did, is extremely demanding (Kleve and Solem, in press) and this episode displayed how he drew on his mathematical knowledge for teaching.

Although Kim did not have the necessary command of substantive knowledge to prove the rule (Fareys mediant), he did not close the door for mathematising activities, and a fruitful mathematical discussion to take place.

IMPLICATIONS FOR TEACHER EDUCATION AND RESEARCH

Research referred in the beginning of this paper, have suggested ways in which teachers *may* respond to a similar contingent opportunity, which actually happened in the lesson we observed and have analyzed here. Authentic records of teaching, whether they are videos or transcripts of data, are extremely powerful sites for learning: "Teachers and researchers are finding that analyses grounded in actual practice allow a kind of awareness and learning that has not previously been possible" (Boaler & Humphreys, 2005, p. 4).

Our teacher did not know the answer. However, these data demonstrate for student teachers that taking the risk and grasping the contingent opportunity may lead to fruitful mathematising activities. In mathematics teacher education, proving $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$ algebraically is possible. Another possibility is to show with functions and graphs (Stylianides & Stylianides, 2010). However, these proofs are not accessible for children in 7th grade. Therefore discussions how to illustrate Fareys mediant for these children may be relevant. For example, if the juice analogy referred in Rowland and Zazkis (2013), is an accepted argument for "the adding numerators and adding denominators procedure" can lead to good discussions among student teachers.

In further analysis of data from this teacher's teaching, we are interested in finding out what socio-mathematical norms were established in his class and what role the norms played for the mathematical activities that took place, such as use of language and classroom discussions, and hence students' learning possibilities.

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Supporting students' development of mathematical explanation: A case of explaining a definition of fraction

Minsung Kwon

University of Michigan, Ann Arbor, USA, mskwon@umich.edu

This paper aims to conceptualize the work of supporting students' development of mathematical explanation. To provide an empirical basis, I analyse instructional interactions managed by the same teacher for teaching the brown rectangle problem for different cohorts of students across five years. The four core tasks of teaching are (1) attending to the organic structure of the mathematical task; (2) mapping the scope of answers onto the targeted mathematical ideas; (3) hearing the mathematical needs embedded in students' explanations; and (4) distributing and building a mathematical talk collectively.

Keywords: Decomposition, the work of teaching, mathematical explanation, fraction.

INTRODUCTION

Teaching is often described as a complex activity because it involves managing multiple relationships simultaneously with students and with content over time (Cohen, Raudenbush, & Ball, 2003; Lampert, 2001). To make this complex work doable and learnable for teachers, especially for beginning teachers, several scholars (Boerst et al., 2011; Grossman et al., 2009; Sleep, 2012; Thames, 2009) have addressed the need to decompose the work of teaching into its constituent components. As a result, a core practice of teaching has begun to be decomposed into nested practices with varying grain sizes. Despite these initiatives, the call for "a specific technical language for describing the implicit grammar and for naming the parts" (Grossman et al., 2009, p. 2069) has not met an agreed-upon robust framework yet. For example, in decomposing the work of steering instruction toward the mathematical point, Sleep (2009) identifies seven core tasks of teaching and further decomposes each core task into strategies and problematic issues. The core tasks she identifies are not mutually exclusive but rather might be enacted simultaneously. Furthermore, in decomposing each core task into strategies, she does not associate it with particular teaching moves. On the other hand, in decomposing the work of leading a mathematical discussion, Boerst and colleagues (2011) start with the larger grain size of domains (e.g., leading a discussion) and then specify it into a smaller grain size of techniques (e.g., revoicing), while articulating intermediate practices (e.g., clarifying student thinking) that connect between domains and techniques. A brief review of literature on decomposition, despite focusing on a different domain of teaching practices, gives a particular prominence to the structure of decomposition, the level of decomposition, and the link to teaching moves or discourse moves.

Given the lack of agreed-on grammar for decomposing the work of teaching, this study aims to decompose one of the key teaching practices that are crucial for accomplishing the ambitious goal of developing mathematical power and mathematical proficiency for all students: the work of supporting students' development of mathematical explanation. The practice of giving, hearing, and evaluating explanation has been considered an important goal for learning because it resolves cognitive dissonance and facilitates cognitive development in the process of knowledge construction. More specifically, giving explanations can serve as opportunities for students to reflect on their own thinking and to reconstruct their existing knowledge, while hearing others' explanations provides opportunities for students to appropriate language that a teacher or more advanced students use, to recognize any cognitive dissonance that contradicts their own understanding, and to use others' explanation as a resource to extend their own knowledge.

Despite its crucial role for learning, there is a general consensus that most students do not have sufficient opportunities to develop their own explanations in U.S. mathematics classrooms (e.g., Stigler & Hiebert, 1999). One reason might be that many teachers believe that giving an explanation to students is more efficient and less complicated than eliciting an explanation from students. Even if this belief is being challenged as greater emphasis is being placed on eliciting an explanation from students, it is pedagogically demanding work for teachers. This is well captured by Cohen's (2011) metaphor in describing challenges in extending students' knowledge as he writes:

> Teachers and learners face the same gulfs of ignorance, but from different sides. Learners must somehow build bridges across the gulf, but these bridges are often fragile because the learners work from relative ignorance. The teacher's assignment is to help learners build those bridges, but they work from greater knowledge. ... Rather than helping learners construct and reconstruct bridges of their own, teachers present the finished results of their learning. That reduces the likelihood that teachers can cultivate a practice of teaching, for it can limit learners' understanding. (Cohen, 2011, p. 106)

This metaphor also applies to challenges in supporting students' development of mathematical explanation. On the one side, students do not have sufficient language to explain their mathematical ideas (Forman & Larreamendy-Joerns, 1998) and their explanations are distant from disciplinary explanation (Leinhardt, 2001). On the other side, teachers often present the compressed, polished, and finished form of mathematical explanations to students rather than helping students construct their own explanation. Considering this demanding but crucial work, this study examines what is entailed in supporting students' development of mathematical explanation, and particularly, the ways of using instructional resources to that end.

METHOD

The methods for studying teaching have adopted terms, concepts, and techniques from other disciplines (e.g., grounded-theory; ethnography), but have not further articulated how the selected method addresses issues that particularly matter for teaching. To make an explicit connection between the phenomenon being studied and the method being chosen, I briefly articulate the study design, which is situated in the instructional triangle (Cohen et al., 2003).

Teaching is often examined as a single case in which a teacher teaches a particular topic for a single group of students, but multiple cases of teaching are also examined. In such an examination, a variety of methods are employed. One way of examining multiple cases is to maximize the variation of components within the instructional triangle—a teacher, students, and contents. For example, to identify elements of expertise of teaching, Leinhardt (1985) contrasts performance of expert teachers with that of novice teachers. In another example, in order to identify the common model of instructional explanation and to specify the features of instructional explanation in each subject, Leinhardt (2001) analyses instructional explanations in history and in mathematics.

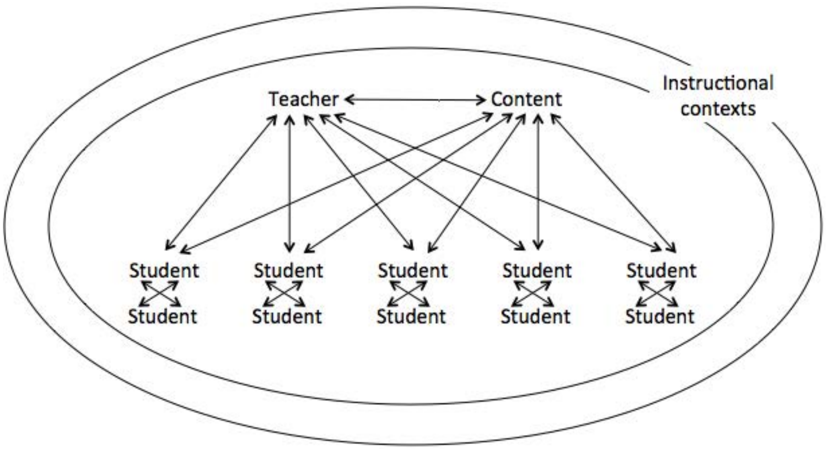

Figure 1: The study design

This paper approaches the problem in another way, minimizing much of the variation among components of the instructional triangle in a more controlled context in which only students vary (see Figure 1). As one of the greatest predicaments of teaching is its dependence on students (Cohen, 2011), analysing instructional interactions managed by the same teacher teaching the same mathematical task to different cohorts of students without substantial differences in students' mathematical abilities is crucial to identify the core tasks of teaching across the particulars of students and unfolding instructions. This method untangles the ways in which the same teacher adjusts the work of supporting students' development of mathematical explanation wherein each cohort of students brings different mathematical ideas, stances, issues, language, ambiguity, and difficulties in explaining the same mathematical task.

To provide an empirical basis, I analyse a longitudinal data set from the Elementary Mathematics Laboratory (EML), a two-week summer mathematics program for entering fifth graders taught by Professor Deborah Ball at the University of Michigan's School of Education, across five years (EML2007, EML2008, EML2009, EML2010, and EML2013). There are no prerequisites to participate in the EML, but it mainly focuses on students who are struggling with learning mathematics rather than students who are outperformed in mathematics. Considering the process of recruitment, there were no substantial differences in students' abilities in mathematics across five years. Each year, approximately 25–30 students, who are ethnically, racially, linguistically, and socioeconomically diverse, participate in a whole-group mathematics class every morning during the two-week program.

As part of a large-scale study which analyses instructional interactions managed by the same teacher for teaching four different mathematical tasks across multiple years, this paper mainly focuses on the brown rectangle problem (see Figure 2). The brown rectangle problem has been used with slight variations in the layout of the rectangle (i.e., where the shaded part is located; the rotation of the drawing), the colour of shaded parts, the inclusion of written problem statement on the poster, the presentation of two sub-problems (posting together vs. posting separately), and the wording of the problem statement (the big rectangle vs. the rectangle; shaded in vs. shaded

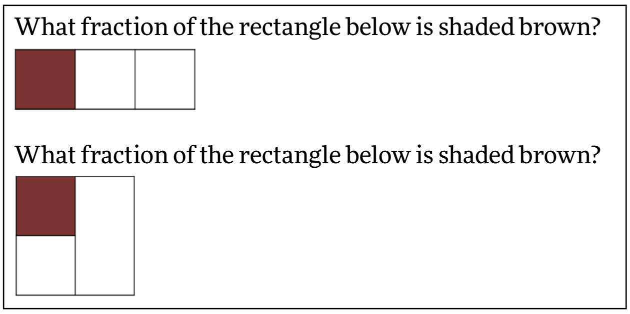

Figure 2: The brown rectangle problem

in brown), but the mathematical demand remains the same across years.

The analysis of each individual year provides detailed images of explanations that individual students produce, the process of constructing a mathematical explanation collectively by each cohort, and instructional supports that the teacher provides to develop a mathematical explanation for the brown rectangle problem. The cross-year analysis illustrates that instruction for teaching the same mathematical task unfolds somewhat differently, even by the same teacher. The similarities across multiple cases become strong candidates to be scaled up into the coherent structure of supporting students' development of mathematical explanation, whereas the differences across multiple cases offer analytical opportunities to examine whether or not the particular instructional feature plays a role in supporting students' development of mathematical explanation. In doing so, I do not treat such differences as discrepant or disconfirming evidence, but use the differences as the data to reveal the underlying structure of the work of teaching to support students' development of mathematical explanation. In addition, the differences observed across years do not necessarily represent the characteristics of the expert teacher's teaching practice. Looking at the multiple uses of the brown rectangle problem by the same teacher to different groups of students allows to elicit the demands entailed in the work of supporting students' development of mathematical explanation and framing the underlying structure that serves to meet such demands.

PROBLEMS OF STUDENTS' INITIAL EXPLANATIONS FOR THE BROWN RECTANGLE PROBLEM

In order to understand what is entailed in supporting students' development of mathematical explanation, it is critical to first diagnose problems that students have in explaining the brown rectangle problem. Some of these problems are more generic and apply to all mathematical tasks, but others are more unique to a particular mathematical task. If each mathematical task would require a specialized form of reasoning to develop a mathematical explanation, the problems that students struggle with would be different and the supports that a teacher needs to provide would be different accordingly. Identifying such problems, both generic and unique to the type of mathematical task, contributes to revealing how a mathematical task plays a role in the work of teaching and the use of instructional resources. The following list below is a more general characterization and a more comprehensive collection of problems that individual students have in offering an initial explanation for the brown rectangle problem across five years.

- Having difficulties providing, hearing, and constructing an explanation
- Not establishing the mathematical grammar to describe the objects to be explained
- Using inaccurate language in which its intended meaning is different from the accepted mathematical definition
- Using pre-defined mathematical terms
- Skipping the logical structure of naming a fraction or paying attention to the partial components of naming a fraction
- Losing the purpose and focus of what is being explained
- Not building correspondences between an answer, an explanation, and representations
- Heavily using demonstrative pronouns
- Grounding explanation on non-mathematical reasons or procedural knowledge
- Missing the key definitional ideas of naming a fraction

Despite sharing these similar problems in explaining the brown rectangle problem, the collective process of constructing mathematical explanation does not always remain the same across five years.

THE CONSTRUCTION OF COLLECTIVE RESOURCES: DIFFERENCES ACROSS COHORTS AND IMPLICAITONS FOR TEACHING

The process of recruiting and selecting EML students is quite similar from one year to another, so it is presumed that there are no substantial differences in students' mathematical abilities across five years. Despite the homogeneous features of the EML cohorts across five years, the cohort's mathematical ideas, stances, dispositions, and issues do not always remain the same. The observed differences are (1) the answers that the students collectively discuss in a public space; (2) the proportion of the students who produce correct answer to the students who produce incorrect answers; (3) the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal; (4) when the key idea of "equal" emerges; and (5) mathematical issues that matter the most for each cohort.

First, the answers that each cohort discussed collectively in a public space are not the same. For the first part of the brown rectangle problem, only one correct answer (1/3) was proposed in the EML 2007, the EML 2008, and the EML 2010, but three answers (1/3, 2/3, and 2/6) were proposed in the EML 2009 and two answers (1/3, 1/2) were proposed in the EML 2013. For the second part of the brown rectangle problem, four answers (1/4, 1/3, 1 and 1/3, and 1/2) were proposed in the EML 2007, two answers (1/4 and 1/3) were proposed in the EML 2008, one answer (1/4) was proposed in the EML 2009, five answers (1/4, 1/3, 1/6, 2/8, and 4/16) were proposed in the EML 2010, and three answers (not a fraction, 1/4, and 1 and 1/2) were proposed in the EML 2013. Even though the teacher made similar attempts to elicit multiple answers, different groups of students brought a different set of answers in a public space. Beyond attending to the number of answers elicited in a public space, an important task of teaching includes (1) not dismissing any proposals made in a public space but unpacking the reasoning behind the proposals; (2) introducing the key incorrect answers if they are not brought by students; (3) mapping the proposed answers to the targeted mathematical ideas; (4) deciding what needs an immediate agreement or disagreement and what needs to be preserved; and (5) customizing questions, probes, and prompts based on the dynamics of proposed answers.

Second, the proportion of the students who produced correct answer to the students who produced incorrect answer is not the same across cohorts. For the first part of the brown rectangle problem, nearly all of the students came up with the correct answer. On the other hand, for the second part of the brown rectangle problem, the incorrect answers were more prevalent than the correct answer in the EML 2007, the EML 2008, the EML 2010, and the EML 2013, but most of the students recorded the correct answer in the EML 2009. If the proportion of correct answer to incorrect answers might be related to the mathematical stance that students bring to the instruction, an important task of teaching includes (1) surveying the composition of students' mathematical ideas; (2) ensuring that a mathematical stance is not influenced by the idea held either by the majority of students or by advanced students; and (3) customizing questions, probes, and prompts based on the proportion of the students who produce correct answer to the students who produce incorrect answers.

Third, the intensity of counterarguments made against a competing proposal and the process of being convinced by a competing proposal was not the same. Except in the EML 2009, students proposed the key incorrect answer of 1/3 for the second part of the brown rectangle problem, but the degree of defending the incorrect answer and what made them being convinced by the correct answer was not the same. Some cohorts were more easily convinced by the idea that adding a line makes equal parts, but others were more resistant and hesitant to accept the idea because it contradicts their non-mathematical perception that adding a line changes the problem. The process of reconciling the competing proposals was not the same across five cohorts, but all of the cohorts ultimately arrived on the agreement that making equal parts is an important idea for naming a fraction and drawing a line provides an easy access to seeing the equal parts. As the intensity of counterargument and the resistance of accepting the competing proposal increased, the cohort constructed richer collective resources to convince others who had a competing proposal. It is not an easy task for a teacher to support students to have a strong stance on their mathematical ideas and to have them sustain their perseverance, but detecting such a moment, confronting competing ideas, and providing sufficient opportunities to defend one's proposal is an important task for supporting students' development of a mathematical explanation.

Fourth, the key idea of "equal" emerged at different stages of developing a mathematical explanation. It was early proffered by a student who proposed the answer of 1/4 in the EML 2007 as well as by a student who proposed the answer of "not a fraction" in the EML 2013, but emerged in the process of comparing between the equally partitioned rectangle and the unequally partitioned rectangle in the EML 2008, 2009, and 2010. Eliciting the targeted mathematical idea and developing the accurate mathematical language is key for developing a mathematical explanation, but an important task is not just accepting the targeted mathematical idea offered by a single individual student, but providing supports for students to use those collective resources.

Lastly, the mathematical issues that matter the most for each cohort are not always the same. For the second part of the brown rectangle problems, the EML 2007 cohort spent a significant amount of time to make sense of 1/2, the EML 2008 cohort discussed whether or not the line changes the problem, the EML 2010 cohort engaged in removing the existing line or adding an additional line to make unequal parts, and the EML 2013 cohort spent time making sense of 1 and 1/2. An important task of teaching is to adjust the instructional time according to the mathematical issues that each cohort struggles with the most.

In basic ways, the students and their mathematical proficiency were similar across years, but each cohort brought different mathematical ideas, stances, dispositions, and issues to explain the brown rectangle problem. Thus each cohort developed different collective resources that became available for use either by the teacher or by students. In comparing the mathematical ideas, stances, dispositions, and issue brought by different groups of students, I offer the following observations. First, the mathematical scope and terrain of collective resources that each cohort establishes varies to a certain degree, but all of the five cohorts develop the key ideas for naming a fraction. Second, there are variations in what collective resources are available for use to develop a mathematical explanation across cohorts, but the practice of constructing collective resources is quite the same. Third, some collective resources are for immediate or necessary use, but others remain in reservoir or are optional for use either by a teacher of by students. Fourth, the same mathematical issue is treated differently based on the established knowledge that each cohort constructs. Lastly, eliciting multiple answers has been considered an important pedagogical practice for fostering students' mathematical abilities and enriching mathematical discussion, but how the proposed answers could be used as resources for maximizing the development of mathematical explanation needs to be further examined.

THE CORE TASKS OF TEACHING

The four core tasks in supporting students' development of mathematical explanation for the brown rectangle problem are: (1) attending to the organic structure of the mathematical task; (2) mapping the scope of answers onto the targeted mathematical idea; (3) hearing the mathematical needs embedded in students' explanation; and (4) distributing and building a mathematical talk collectively.

The first core task is attending to the organic structure of the mathematical task. This core task includes (1) focusing on mathematical or non-mathematical attributes which impact the construction of an explanation (e.g., "big rectangle"; the affordance of sticky line; drawing the rectangle on the grids); (2) not attending to mathematical or non-mathematical attributes which substantially distract from the construction of an explanation; (3) recognizing how the design of the mathematical task creates or eliminates confusions and how the design of the mathematical task makes the key ideas implicit or explicit.

The second core task is mapping the scope of answers onto the targeted mathematical ideas. This includes (1) being aware of the scope of answers that students propose; (2) deciding which of the proposed answers needs an immediate acceptance or denial and which needs to be preserved; (3) not delving into the ideas that students do not have a shared access to; (4) not diverging into the ideas that seriously deviate from the targeted mathematical ideas; and (5) spending sufficient instructional time on scaling up the proposed answers to the targeted mathematical ideas.

The third core task is hearing the mathematical needs embedded in students' explanation. This core task includes (1) recognizing inaccurate or inconsistent language use that impedes building a mathematically acceptable form of common knowledge; (2) deciphering the vague, unclear, or implicit idea conveyed by students' explanations; (3) providing supports to build mathematical connections or correspondences instead of repeatedly asking general questions; and (4) recognizing the skip of or the deviation from the logical structure of building an explanation.

The last core task is distributing and building a mathematical task collectively. This core task includes (1) not exclusively relying on one students' contribution; (2) being attentive to the trajectory of constructing a mathematical explanation; (3) appropriately or sufficiently using a private space and a public space; and (4) making each other's contribution accessible in a public space.

DISCUSSION

Given that one of the greatest predicaments of teaching is its dependence of students, it is important to figure out how instruction might unfold with different groups of students. On the one hand, one might speculate that instruction would unfold in the same way by the same teacher teaching the same mathematical task because a teacher might make the same decisions based on his or her knowledge, skills, disposition, and instructional goals. On the other hand, one might suggest that instruction would unfold in a dramatically different way even by the same teacher teaching the same mathematical task because teaching entails being responsive to students. The question of how instruction unfolds with different groups of students might be answered based on one's personal sensibilities or perceptions built through years of their own teaching experiences, but it is not yet rigorously examined in the field how instruction managed by a teacher teaching the same mathematical task is likely to unfold differently with different groups of students; how collective resources are likely to be constructed differently with different groups of students; and what is the underlying structure of using collective resources with different groups of students. By analysing instructional interactions managed by the same teacher teaching the same mathematical task for different cohorts of students, this study contributes to identifying core tasks of teaching across the particulars of students and unfolding instructional dynamics.

The four core tasks of teaching are not just a mere collection of temporal stages, general pedagogical strategies, instructional routines, or discourse moves, but devised to structurally and attentively capture the essential elements of instructional interactions. Approaching through pedagogical strategies or discourse moves might be one way of examining what is entailed in supporting students' development of mathematical explanation, but it entails the risk of losing some key elements of instructional interactions. Instead, this study conceptualizes core tasks of teaching by taking into serious account the three-pronged arrows that a teacher has relationships with in the instructional triangle (students, content, and students-content) and by anchoring the core tasks of teaching into these relationships. These four core tasks are neither sequential nor mutually exclusive. Even though there exist differences in what bring to foreground and what leaves as background, all four core tasks of teaching attend to the coordination between students and mathematics.

ACKNOWLEDGEMENT

The data used in this paper were documented by the EML project at the University of Michigan. The author would like to thanks to the EML project giving me a permission to use the data. The opinions in this paper are those of the author, but not necessarily reflected by the teacher, Ms. Ball. This paper is a part of the author's doctoral dissertation.

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Exploring a framework for classroom culture: A case study of the interaction patterns in mathematical whole-class discussions

Maria Larsson

Mälardalen University, Västerås, Sweden, maria.larsson@mdh.se

Research is needed on frameworks that support teachers in the important and challenging work of orchestrating productive problem-solving whole-class discussions. The aim of this paper is to explore a framework for classroom culture with the overarching goal of supporting teachers in conducting class discussions focused on argumentation as well as connection making. Analyses of video-recorded whole-class discussions result in the articulation of some difficulties in clearly distinguishing between certain interaction patterns within different classroom cultures. The overall findings, however, suggest that the framework can be useful for characterizing interaction in terms of an inquiry/argument classroom culture.

Keywords: Interaction pattern, whole-class discussion, classroom culture, inquiry/argument, instructional practice.

INTRODUCTION

There is great consensus within the mathematics-education field that mathematical instruction needs to provide opportunities for students to participate in instructional practices that develop their mathematical competencies (NCTM, 2000; NRC, 2001). To understand mathematics, reflection and communication are key (Hiebert, Carpenter, & Fennema, 1997). Participating in whole-class discussions of multiple solutions to a challenging problem have great potential to allow students to reflect and communicate. However, interactions in various reform-oriented classrooms differ significantly, and it is important to relate these differences to students' thinking and learning. Supporting teachers in engaging students in interaction that promotes their mathematical thinking is central, and frameworks for teachers' actions can help. More research is needed on such supportive frameworks.

I have previously (Larsson, 2015; Larsson & Ryve, 2011, 2012) investigated ways that teachers can plan and conduct productive whole-class discussions of students' different solutions to challenging mathematical problems and have discussed ways that Stein, Engle, Smith, and Hughes's (2008) model can support teachers in this important and demanding work. In short, the model consists of five practices that build on each other: anticipating, monitoring, selecting, sequencing, and connecting student solutions. However, this model does not explicitly focus on how a teacher could productively interact with students in whole-class discussions.

Among frameworks that focus on interaction (e.g., Boaler & Brodie, 2004; Brodie, 2010), I find Wood, Williams, and McNeal's (2006) proposal especially interesting because they have found that an inquiry/argument classroom culture is closely associated with higher cognitive levels of student thinking. Moreover, Franke, Kazemi, and Battey (2007) see the promise of connecting Wood and colleagues' (2006) interaction patterns to Stein and colleagues' (2008) model, which is central to my research. Wood and colleagues (2006) distinguish between two types of reform-oriented classroom cultures: strategy-reporting and inquiry/argument. In the latter culture, there is a "major shift in participation from an emphasis on the child reporting her/his different strategies to the children as listeners taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas" (p. 235). The role of the listening students is hence crucial for distinguishing between the two types of reform classroom cultures. Wood and colleagues (2006) state that their most important finding is the differences between the two reform classroom cultures.

The overarching goal of my research is to help develop frameworks that support teachers in conducting productive whole-class discussions that focus on argumentation as well as connection making (see Larsson, 2015). In relation to this broad aim, this particular paper explores Wood and colleagues' (2006) framework for interaction patterns. More specifically, it aims to articulate the difficulties, if any, in distinguishing between interaction patterns—in particular, reform interaction patterns. I delineate the conceptual framework and the methodology that I use before presenting my results, illustrated by a fine-grained analysis of one particular whole- class discussion.

CONCEPTUAL FRAMEWORK

I use Wood and colleagues' (2006) conceptual framework for investigating specific interaction patterns in the whole-class discussion that I analyze. The purpose of their framework is to better understand which opportunities for learning arise in various classroom cultures. Wood and colleagues (2006) divide the interaction patterns into three categories: (i) patterns common to all instruction, (ii) patterns of conventional instruction, and (iii) patterns of reform instruction. The only pattern common to all instruction is *Collect answers*, in which the teacher collects answers to a problem with the purpose of making them public. I have summarized the interaction patterns that characterize conventional instruction and reform instruction in Tables 1 and 2.

METHODOLOGY

The data source for this paper is a collaboration with a very proficient teacher regarding mathematical problem solving discussions with over 15 years of teaching experience. I observed this teacher during eight days in one academic year without making interventions, with a particular focus on the teacher's orchestration of whole-class discussions of students' different solutions to challenging mathematical problems. The teacher strives to engage students in inquiry and argumentation in a collaborative spirit, making it interesting to analyze her whole-class discussions with Wood and colleagues' (2006) framework. Data consist of transcribed video-recorded lessons focusing on the teacher during whole-class discussions, audio-recorded pre- and postlesson teacher interviews for every lesson, and collected student solutions. To interpret the videotaped, transcribed whole-class discussions, I performed a fine-grained analysis of four whole-class discussions using Wood and colleagues' (2006) conceptual framework. All lines were coded in segments, each of which was categorized into one interaction pattern according to Wood and colleagues' (2006) descriptions (see my summary in "Conceptual Framework," above). One additional person coded one of the whole-class discussions. The categorizations were then compared, and we discussed them to resolve differences.

ANALYSIS AND RESULTS

As an illustration of how this particular teacher interacts with her students in a whole-class setting, I use excerpts from a discussion in sixth-grade about students' solutions to the problem "Houses of Cards" (Larsson, 2007). Students allowed for far-reaching generalizations, considering that they were only in sixth grade. This discussion has been chosen because it reflects the typical way this teacher interacts with her students in a whole-class setting.

Interaction pattern	Description	Purpose	Initiator
IRE (Initiate-Respond-Evaluate)	Teacher asks a test question, students' responses are confined to yes/no or right/wrong, and the teacher evaluates.	To check what students know.	Teacher
Give expected information	Similar to IRE, but students' answers can be more open.	To check what students know.	Teacher
Funnel	Teacher leads student(s) to the answer by a number of test questions.	To correct an incorrect student answer without telling the answer.	Teacher
Teacher explain	Teacher gives (often lengthy) explanations of key mathematical ideas/concepts.	To tell students what they are expected to learn and know.	Teacher
Hint to solution	Teacher gives a hint that takes away the challenge of the problem.	To ensure that students reach a correct answer quickly without struggle.	Teacher

Table 1: Conventional-instruction interaction patterns (summary of Wood et al., 2006)

Interaction pattern	Description	Purpose	Initiator
Explore methods	Students explain their solution strategy.	To give multiple solution strategies.	Teacher or student(s)
Inquiry	Teacher or student(s) ask questions because they do not understand.	To understand.	Teacher or student(s)
Argument	A student listener challenges an idea because (s)he disagrees, after which students participate, taking turns.	To reach to a resolution.	Student listener
Teacher elaborate	Teacher elaborates on a student's explanation because information is lacking.	To provide more information to the students.	Teacher
Proof by cubes	Teacher uses material either to find the correct answer or to gain understanding.	To get to the correct answer or provide insight.	Teacher
Proof of answer by student explanation	Teacher lets student(s) explain their correct solution.	To ensure that the class hears a correct solution.	Teacher
Focus	Teacher first provides a summary before asking a question that focuses students on what they need to resolve.	To orient students toward key aspects.	Teacher
Build consensus	Teacher tries to have the class agree on a key mathematical idea.	To establish common ground in the class.	Teacher
Check for consensus	Teacher checks with students to see whether they have questions or comments on a student idea.	To open up for questions and comments before moving on.	Teacher
Develop conceptual understanding	Teacher asks a question that addresses a specific idea or concept.	To facilitate students' conceptual understanding.	Teacher
Pupil self-nominate	A student voluntarily offers an idea or insight that goes beyond the topic and explains/justifies the idea.	To have students exercise their autonomy as participants.	Student

Table 2: Reform-instruction interaction patterns (summary of Wood et al., 2006)

Houses of Cards (Larsson, 2007)

Albin and Melvin are building houses of cards as the picture shows.

1) How many cards does a house of cards contain that has

   a) 3 floors?

   b) 4 floors?

   c) 5 floors?

   d) 12 floors?

   e) n floors?

Figure 1

2) A house of cards consists of 408 cards. How many floors does it have?

3) Make up a problem of your own and solve it.

Four different student solutions were discussed by the whole class for the general case of the problem. Table 3 displays the solutions in the order they were brought up for whole-class discussion.

Before the whole-class discussion, the students have worked on the problem individually and in pairs. First, Paula and Johanna's unusual strategy is explored. The two students help each other, trying to explain their strategy, whereupon the teacher asks whether anybody understands (checks for consensus) and then asks for the n^{th} figure if it is odd. Johanna explains again and inquires: "But I don't really know how to find a formula for it." The teacher asks: "Can someone else find out how they could write it? The n^{th} figure is $n \cdot 3\cdot$ something [shows in the table]. If we have 13, it's 7. If we have 11, it's 6. If we have 9, it's 5. If we have 7, it's 4." In several turns, the students and teacher collaborate, which is central to inquiry instruction (Wood et al., 2006), to find that "something" must be $(n + 1) / 2$, and Paula concludes, "That works. Then it's $n \cdot 3$ times $n + 1$ divided by 2." Then Axel gets to explain his strategy (see excerpt below). The interaction patterns

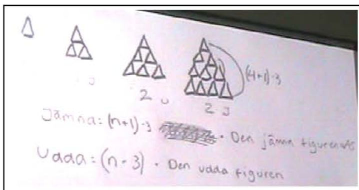
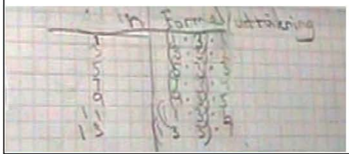
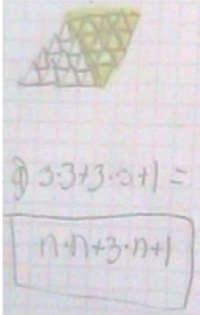
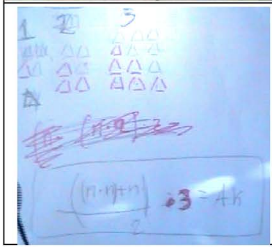
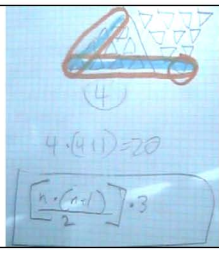
  1.Paula and Johanna after discussion: $(n \cdot 3) \cdot (n + 1) / 2$	 2. Axel after discussion: $n \cdot (n + 1) \cdot 3 / 2$
 3. Lena and Frida $((n \cdot n) + n) / 2 \cdot 3$	 4. Andreas $(n \cdot (n + 1) / 2) \cdot 3$

Table 3: Student solutions in class discussion of "Houses of Cards," in sequential order

from Wood and colleagues' (2006) framework that my analysis yielded are included as headings.

Explore methods

1 Axel: I thought a little like Paula, like she did on this one with—what was it?

2 Students: Flowers.

3 Axel: With flowers, exactly.

4 Students: Aha.

5 Axel: Then I thought that you have to add another card house—but sort of upside down, on top of. And then I was doing that for a very long time. Finally, I arrived at that it first gets $n \cdot 3$ here [points at the tilted pile with four triangles]. Yeah, I have to write that, I think [refers to his expression]. 'Cause this is Figure 4. $n \cdot 3$ [points at the tilted pile] and $n \cdot 3$ plus 3 [points at the bottom row]. And then you see here 1, 2, 3, 4, so it's 4—no it's 1, 2, 3, 4, 5—it's five of these stripes here. And then you've got to take times $n + 1$.

Check for consensus

6 Teacher: Does anybody understand what he's saying?

7 Students: No. Yes. Hmmm...

8 Axel: Should I explain again?

Build consensus

9 Teacher: We don't understand anything.

10 Frida: Or do you mean like this? Wait, wait. Eeh, I think I might know what you mean. First $n \cdot 3$. 'Cause that's Figure 4, right?

11 Axel: Yes, it's Figure 4.

12 Frida: Yes, Figure 4. Yes, then you see there on the edge that it's four triangles going alongside there, and that's the same thing as n.

13 Lena: He counts the cards separately, I think.

14 Axel: No, but check this out—

15 Frida: First he calculates times, and then he calculates it times 3.

16 Lena: Ahaa.

17 Frida: Do you understand? So it's $n \cdot 3 + n \cdot 3 + n \cdot 3 + n \cdot 3$. And then you divide [.] Yeah, but then it gets [.]

18 Johanna: No, it's simply to do $n \cdot (n + 1)$. No, I'm just kidding [she laughs]. Yes, but check this out [points at the board]—n, there on the edge, and then $n + 1$. And that times, so that's like a quadrangle, but it's nudged. And that divided by 2. So $n \cdot (n + 1) / 2$.

19 Axel: That you've got to multiply by 3 for each of these [.] It's 3 in each: $n \cdot (n + 1) \cdot 3 / 2$.

20 Johanna: Yeah. So $n \cdot (n + 1) \cdot 3 / 2$. [Teacher writes it on the board.]

21 Axel: It works.

Lena and Frida's solution and Andreas's solution are then explained and discussed.

Check for consensus

22 Andreas: Do you understand? [Refers to his explanation of his own solution.]

23 Students: Yes, I get it. Yeah.
24 Teacher: Do you understand? Yes?

Argument

25 Paula: They're a little different [refers to solutions 2 and 4], or I don't know, because there it's times 3, also divided by 2, but there it isn't times 3 divided by 2. But maybe it doesn't matter.
26 Lena: They take times 3 in the end.
27 Paula: Exactly. They take times 3 without having divided it.
28 Lena: It's the same thing.
29 Paula: But there it's times 3 divided by 2, and there it isn't.
30 Lena: Yes, here it's—
31 Paula: But maybe it doesn't matter.
32 Teacher: Exactly. If you think of [writes and talks] $3 \cdot (8 / 2)$, what's that, Paula? 12 [writes and talks]. What's $(3 \cdot 8) / 2$?
33 Paula: It's also 12.
34 Teacher: Yes, that's also 12. So it's the same thing.

Focus

35 Teacher: So now my question is—now we've got 1, 2, 3 different [points at the formulas], and Andreas's here—that's another one—that's four different. My question is: Are all different formulas? Are all different?
36 Lena: Actually, they're all the same thing. Everything makes the same thing, the same answer. That means that they're all equal; you just write it in different ways.
37 Teacher: Exactly, they're all correct; it's the same answer, and it's the same card house. You just write it in different ways. This means that you can use algebra, the mathematical language, to express the same thing in different ways.

After an initial exploration of Axel's method [1–5], when he tries to explain his strategy and also makes a connection to Paula's solution to a previous problem, the teacher checks for class consensus by asking, "Does anybody understand what he's saying?" [6], with the purpose of opening the floor for questions and comments. Since the students' answers vary [7], Axel asks whether he should explain again [8], whereupon the teacher states, "We don't understand anything" [9]. Now comes the really interesting part [10–21]: instead of leaving Axel alone to try to explain once again, classmates help him out in a collaborative manner that is characteristic of an inquiry/argument culture, according to Wood and colleagues (2006). Together the students try to make sense of the ideas and build a shared understanding; they build consensus.

Note that the teacher does not need to say anything during these turns while the students build consensus. The pupils build upon one another's statements. This must be seen as a result of the social and sociomathematical norms (Yackel & Cobb, 1996) that this teacher had established in her classroom during a long period of time. Since I have observed and interviewed the teacher in connection with many lessons, I know that she strives to foster collaborative discussion in which students help each other and "are listening and participating students at the same time" (Interview). This approach hence relates to the role of students as active listeners who try to understand, question, validate, and build on one another's contributions. In my view, this example amply illustrates the connection between classroom norms and the interaction patterns that develop (Wood et al., 2006). On her own initiative, Frida helps Axel out, saying, "Wait, wait. Eeh, I think I might know what you mean" [10]. She ensures that they are talking about the same figure number and continues to explain [12]. Lena also contributes by describing how she thinks Axel has reasoned [13], which helps her understand [16] after she gets additional input from Frida [15]. Frida actually addresses Lena directly with her question, "Do you understand?" [17]. When Frida continues to explain, she hesitates [17], whereupon Johanna steps into the discussion [18], making clear that "No, it's simply to do $n \cdot (n + 1)$." To Johanna's formula, $n \cdot (n + 1) / 2$, Axel adds [19] "That you've got to multiply by 3 for each of these [.] It's 3 in each: $n \cdot (n + 1) \cdot 3 / 2$." Johanna clearly shows her agreement [20]. Axel seems content in his concluding comment: "It works" [21]. A correct formula for Axel's strategy has now been produced through the class's collaboration. The students act as "listeners taking over the role of the teacher in questioning, clarifying, and validating mathematical ideas" (Wood et al., 2006, p. 235), a salient feature of an inquiry/argument classroom culture.

After Lena and Frida's and Andreas's solutions have been discussed in a whole-class setting, four algebraic formulas are displayed on the board for the number of cards. Andreas checks the class for consensus by asking, “Do you understand?” [22], and the teacher repeats his check [24], whereupon Paula argues that Axel’s and Andreas’s solutions are different. Paula and Lena take turns [25–31] in an Argument interaction pattern, trying to resolve whether the two formulas are equivalent (the teacher steps in to help [32–34]). The teacher summarizes and focuses the discussion on a key aspect by asking whether all four formulas are different [35], and Lena says that “Actually, they’re all the same thing.” [36]. The teacher concludes the discussion by restating this key mathematical idea [37].

DISCUSSION

I find Wood and colleagues' (2006) framework straightforward for distinguishing between conventional and reform interaction patterns. The categories cover the interactions well. However, I see some difficulties in making a clear distinction between certain reform interaction patterns. Wood and colleagues (2006) state regarding the pattern *Check for consensus*, “The teacher participates by checking with the students and listening to find out if they have any questions or comments about an idea, strategy, or concept that a student explained.” So far so good. They continue: “The student explaining may be asked further questions or to re-explain by the listening students. In some cases, listeners give another different strategy for solving the problem or offer further explanation. The outcome is public agreement on the validity of an idea or concept given by the student explaining” (p. 255).

I am concerned about the second part whose interpretation constituted the major difference in the interrater coding. In my interpretation, a *Check for consensus* consists solely of checking with the students to see whether they have questions or comments on a student’s idea. This aligns with Wood and colleagues' (2006) statement that “Checking for consensus initiated by the teacher appeared to be a final attempt to open the discussion so any child could make comments or ask questions before moving on in the discussion” (p. 235). The teacher’s check can then be followed by, for example, a student asking a question in order to understand (*Inquiry*) or challenging an idea (*Argument*) or by the teacher trying to establish common ground on key ideas (*Build consensus*) in the class (cf. the short *Check for consensus* [6–8] and [22–24], followed by *Build consensus* and *Argument*). However, the interrater coding made clear that another interpretation could be that the entire interaction is a *Check for consensus* since “the outcome is public agreement” (i.e., [6–21] constitutes one extended *Check for consensus*).

Making clear that a *Check for consensus* is solely a check and is followed by other interaction patterns would render the framework more straightforward. Further, my analysis suggests that in addition to the teacher (as stated by Wood et al., 2006), a student might also initiate such a check (cf. Andreas’s *Check for consensus* [22]). Just as it is difficult to determine when the pattern *Check for consensus* ends, it is not completely clear when the *Inquiry* pattern ends. My interpretation is that the *Inquiry* pattern consists solely of the act of asking and does not include the clarifications that follow. Again, outlining clear criteria not only for when an interaction pattern starts but also for when it ends would make Wood and colleagues' framework less ambiguous.

Solutions are purposefully selected by the teacher when using Stein and colleagues' (2008) five-practices model. Therefore, the pattern *Exploring methods* is hard to distinguish from *Proof of answer by student explanation* in terms of correct solutions; only their purposes differ (see Table 2). The purpose of deliberately selecting a correct solution for display can be both to provide multiple solution strategies and to ensure that the class hears a correct solution (cf. solutions 3 and 4 in Table 3).

The difficulties with Wood and colleagues' (2006) framework relate mainly to distinguishing between certain interaction patterns that are specific to different classroom cultures. Hence, the difficulties in interpretation affect only the relative distribution of the interaction patterns within a specific classroom culture, not whether the culture should be regarded as inquiry/argument or strategy-reporting.

Since Wood and colleagues (2006) have shown that an inquiry/argument classroom culture is closely related to higher cognitive levels of student thinking, I contend that it is desirable for teachers to strive to establish inquiry/argument interaction patterns. I see tremendous potential in using Stein and colleagues' (2008) model as a tool to guide teachers’ actions and support teachers’ development over time in their orchestration of whole-class discussions. My ongoing efforts intend to take into account argumentation as well as connection making in the Stein and colleagues

(2008) model. The exploration of Wood and colleagues' framework in this paper contributes to those efforts.

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Teacher and textbook: Reflection on the SDT-model

Ljerka Jukić Matić[1] and Dubravka Glasnović Gracin[2]

1 University of Osijek, Osijek, Croatia, ljukic@mathos.hr

2 University of Zagreb, Zagreb, Croatia

This study investigates in-depth how two teachers use the textbook in mathematics classrooms. Their use of the textbook was analysed in the light of a socio-didactical tetrahedron (SDT) which proved to be a powerful model to describe many aspects of textbook use that we encountered. Our results showed that the teacher acted as the mediator between the textbook and the students, but the lower part of the SDT helped us to identify the influence of social factors on textbook use which is an aspect that cannot be disregarded.

Keywords: Mediation, socio-didactical tetrahedron textbook, teacher, teaching.

INTRODUCTION

The issue of *how textbooks shape the teaching and learning of mathematics* has been the subject of many studies (Fan, Zhu, & Miao, 2013).Textbooks can be considered as artefacts which are dynamically used in classrooms, influencing the instruction (Johansson, 2006). Their content follows and largely reflects the requirements and intentions of the intended curriculum. Various studies have shown that teachers rely heavily on textbooks for lesson preparation, teaching new subject matter, practicing and giving homework assignments (e.g., Pepin & Haggarty, 2001; Johansson, 2006; Glasnović-Gracin, 2011).This paper investigates in-depth two mathematics teachers' practice in relation to the use of the textbook in the mathematics classroom.

THEORETICAL FRAMEWORK

Textbook-teacher-student

The teacher's role as a *mediator* between the textbook and the students has been considered by various authors (e.g., Luke, de Castell, & Luke, 1989; Love & Pimm, 1996). Pepin and Haggarty (2001) give the research systematization on the use of textbooks through six main domains, one of which relates to the teacher as the mediator of the text. They assert that it is the teacher who decides which textbook to use, when and how it is used, which parts to use and in what order, and when and to what extent the students will work with the text. Traditionally, mediating between the students and textbook has always been the teacher's function (Luke et al., 1989).

According to Remillard (2000), textbooks can alter the teaching strategies of teachers, but conversely it is the teachers who choose what to use and what not to use from the textbook. Sosniak and Stodolsky (1993) found that teachers have autonomy and that they like to mediate between the textbook content and their students. Also, teachers do not feel that the textbooks control their teaching. In addition, the authors believe that in order to understand the use of textbooks in classrooms, the thoughts, actions and working conditions of the particular teacher need to be taken into consideration. This brings us to the socio-didactical tetrahedron.

The socio-didactical tetrahedron

The use of textbooks in the classroom can be considered through the model of the didactical tetrahedron where the vertices are student-teacher-textbook-mathematics (Rezat, 2006). However, this model does not encompass societal and institutional aspects, which are also important factors in mathematics education. Therefore, Rezat and Sträßer (2012) proposed a more comprehensive model of didactical tetrahedron, called the socio-didactical tetrahedron (SDT). We imagine the basic didactical tetrahedron (Figure 1.) where the vertices are artefact-student-teacher-mathematics, as being put in such a position where the vertices student, teacher and mathematics lay on the bottom of the tetrahedron. These three vertices are extended by the social and cultural parameters, forming a comprehensive socio-didactical tetrahedron. The new bottom vertices in the SDT are: conventions and norms about being a student and about learning; conventions and norms about being a teacher and about teaching; and the public image of mathematics. These three vertices are connected to each other through the bottom edges. Since these social and cultural parameters lay in a complex relationship, other points on the bottom edges are highlighted, such as institution, noosphere, and peers and family (Rezat & Sträßer, 2012).These social and institutional parameters are often considered to be "hidden" or "less visible" because the persons involved are often not conscious of them. Therefore, we can see the SDT as an iceberg in the water: we often only consider the "visible" didactical tetrahedron involving just the teacher, student, mathematics and artefacts, and we forget (or we are not conscious of) the social and other parameters involved.

With all its highlighted points, the SDT model is a powerful enough tool to provide a structure for textbook use and to show the cultural interplay between educational and social context. For example, Rezat (2013) conducted a qualitative study on students' use of mathematics textbooks in Germany. The findings show that social impacts are important factors which influence students' use of textbooks. Some aspects of textbook use cannot be explained without taking into account the whole SDT.

Previous research and research focus

The study presented in this paper is a follow-up research on the large-scale study reported in Glasnović Gracin (2011). Using a questionnaire with multiple choice items, the previous study investigated the role of mathematics textbooks in lower secondary education in Croatia (grades 5 to 8). The survey involved nearly one thousand mathematics teachers, which is about half of the total number of mathematics teachers in grades 5 to 8 in Croatia. The results showed that teachers use textbooks for lesson preparation to a great extent. Participants said that new material is mainly presented by the teacher at the front of the class followed by students working individually on the textbook exercises. So, the teachers consider the textbook to be an important source of practice exercises for students. The survey also showed that most teachers select a textbook mainly according to the quality of its examples and problems. The coherence between the textbook methodology and the students' age was also an important factor which proved to be relevant for 64% of teachers.

In this new study, we wanted to examine in a more in-depth way how mathematics teachers use the textbook in their teaching practice. Since the previous study encompassed a quantitative method, here we wanted to use a qualitative approach. The aim was to investigate how teachers use textbooks in mathematics classrooms and to identify potential factors that influence such use. The findings are observed in

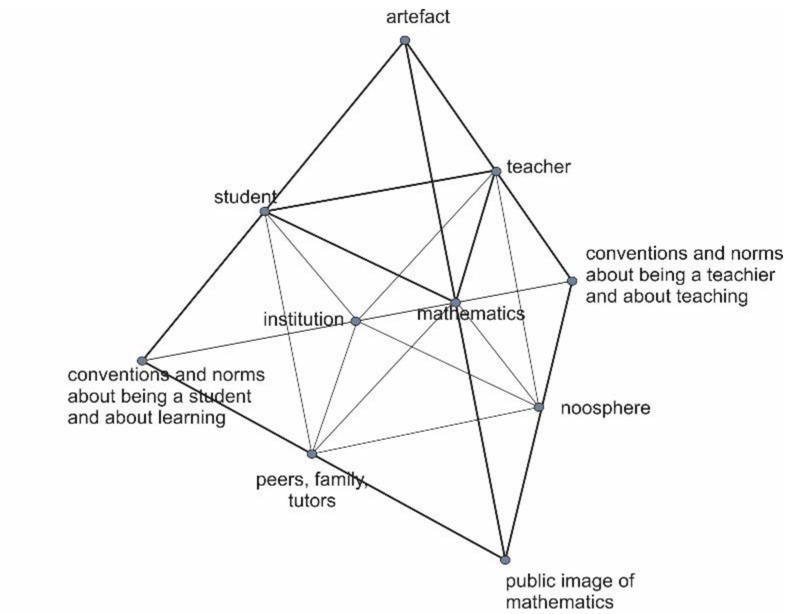

Figure 1: Socio-didactical tetrahedron

the light of the SDT, in order to better understand the teaching and learning of mathematics.

METHODOLOGY

Participants

The study involved two female mathematics teachers, Mrs. S and Mrs. D, from lower secondary education in Croatia (in our education system, this is a primary school, grades five to eight). The teachers chosen for this study are both experienced teachers: Mrs. S has 15 years of teaching experience and Mrs. D 20 years of teaching experience. Both participants have a degree in mathematics education and use the same textbook series. It is the most commonly used textbook series in Croatia, used by about 65% of the whole student population (data retrieved from the Ministry of Education, Science and Sport).The structure of this textbook can be described as the "exposition – examples – exercises" model. It has been found that this kind of model predominates in the structure of mathematics textbooks (Love & Pimm, 1996).

The Croatian education system

In the Croatian education system, primary school is compulsory and lasts for eight years. Secondary school is not compulsory, but the majority continues to go to school after finishing primary school. Secondary school lasts three or four years, depending on the type of school (i.e. vocational or grammar). One lesson unit lasts for 45 minutes in both primary and secondary school. Textbooks are compulsory and all textbooks used in schools are authorized by an official group of experts appointed by the Ministry of Education. As of 2010, teachers jointly select authorized textbooks for their school for the period of 4 years. Textbooks are usually purchased by parents, except for a short period from 2006 to 2008 when the Ministry of Education provided free textbooks for all primary school students in Croatia.

Method

In this study, we used qualitative strategies in the form of observations and interviews. During October 2013, we observed several lessons given by the participants. In that year, Mrs. S was teaching sixth grade, and Mrs. D grades five and eight.

For the purposes of observation, we designed three main observation categories. The categories were:

— *Minutes of textbook use* (How much time do the students and the teacher use the textbook?)

— *Impact of the textbook structure on instruction* (What is the influence of the textbook content and structure (title, language and symbols, order of worked examples, definitions and rules, didactic intentions) on instruction? Does the instruction follow the textbook page by page? What is taken from the textbook?)

— *Use of the textbook* (For what purposes is the textbook used during the lesson? How is new content introduced? Is the textbook used during the teaching of new mathematical content? What is a practice lesson like? Which sources are used for practicing and homework? Is the textbook used for practicing and reviewing? Did the teacher point out any specific figure, frame or picture from the textbook? For what purposes?)

Prior to each observation, we examined the structure of the mathematical content which was to be taught in that lesson. This helped us in making comments and answering the observation questions. After the classroom observations, we conducted a semi-structured interview with each of the participants. The interview questions come under three main categories, but the interviewer was able to ask additional questions based on the observed lessons or expand upon an interesting point arising during the interview:

— *Impact of the textbook structure on instruction* (Describe how you usually prepare for a mathematics lesson. Does the textbook, in your opinion, influence the structure of your instruction, e.g., using the same title as in the textbook for the lesson unit, using the same definitions, symbols, sequence, didactical approach, worked examples, figures? Explain.)

— *Use of the textbook* (Describe a typical lesson where new content is introduced. Describe a typical lesson with emphasis on practicing. Describe a typical revision lesson before a test. Describe how you select homework activities and from which sources. Do you significantly change your teaching style when you change the textbook?)

— *Strong and weak points of the textbook* (What do you find is lacking in textbooks? What do you like and what do you find helpful about textbooks?)

Data analysis was conducted using the constant comparative method (Corbin & Strauss, 1990). Data gathered from the observations and the interviews were coded thematically. This process allowed us to attend to, and report on, those aspects of textbook mediation that were common to both teachers as well as their differences.

RESULTS

Minutes of using the textbook

We observed four lessons of Mrs. S teaching grade six. The textbook was used intensively in all her observed lessons. Students had the textbook open on their desks all the time. In the first lesson, the textbook was used directly for 10 minutes when students worked on their own solving tasks from the textbook, and for the rest of the lesson the students and the teacher used it occasionally. In the other three lessons, the textbook was used for the entire 45 minutes.We observed three lessons given by Mrs. D. She used the textbook directly only in the first observed lesson for 15 minutes. In the other lessons, the textbook was not used by the students or the teacher at all.

Impact of the textbook on instruction

The textbook had significant impact on the lessons observed in Mrs. S's classroom. In the first lesson, the textbook was followed page by page for more than half of the lesson. The influence of the textbook was also visible in other parts of the lesson. The rules written on the blackboard were the same as the rules in the textbook. In the second lesson, to introduce and teach a new topic, the teacher used the motivational example from the textbook, as well as the worked examples and rules. The exercises for practicing were also from the textbook. In the third lesson, the textbook structure was heavily reflected in the lesson, since the textbook was followed page by page. The fourth lesson was preparation for the up-coming exam, where students solved problems from the textbook in the exact order as given in the textbook. In the interview, Mrs. S explained that she always uses the textbook in lesson preparation. The textbook structure influences her lessons and the textbook is the main source for practicing and for homework. Mrs. S explained that in this way she encourages the students to use the textbooks. She wants her students to use textbooks actively for two reasons. The first reason is "so that children can find at home what we did in class, especially if someone did not understand what I said in school". The other reason is so that students become independent and confident in using various resources, like books, encyclopedias or other curriculum materials. In this way she aims to prepare students for the active use of textbooks in upper secondary school, where the subject matter is more demanding than in the lower secondary grades.

Mrs. D did not follow the textbook page by page in any of the lessons observed. In the first lesson, rules were taken from the textbook, as well as some exercises for practicing. In another observed lesson, the textbook was not even open in front of the students; all the tasks were on a worksheet she had prepared, though some tasks were similar to those in the textbook. All of the lesson titles that she wrote on the board were the same as the titles in the textbooks. In the interview, Mrs. D said that the textbook was not her main resource for lesson preparation. Rather, she uses a variety of available materials such as the internet, a range of textbooks (old and new), professional journals, and ideas from colleagues.

The use of textbook

In the first of Mrs. S's observed lessons, the students used the textbook for learning new content. They were required to read the new unit from the textbook (Reciprocal fractions), become familiar with the new content, and work through the examples and exercises from the textbook. The teacher helped those students who had problems with understanding the textbook's content, thus she acted as mediator between the textbook and the student. The second observed lesson involved the division of fractions. The new content was presented through a teacher-led discussion with the class, where the teacher relied on the textbook, using the model of the chocolate bar (quadrilateral shape) as presented in the textbook. The discussion brought up some rules for fraction division, and the teacher also introduced the case $\frac{a}{b} : n$ which was not in the textbook, which some students remarked on. The practice tasks which followed were also from the textbook, as was the homework. The third and fourth lessons had a very similar structure. The teacher gave a long list of textbook exercises to be done, wrote the page number and exercise numbers on the board, and the students opened their textbooks and individually worked through the tasks. At the end of the lessons, the teacher assigned the homework, which was also from the textbook.

In the interview, Mrs. S said that she uses the textbook for teaching new mathematical content if she finds that the book covers the subject matter well. If it does, she does not devise her own exposition and examples. However, she says that she does not take all the definitions directly from the textbook:

> "I change a definition sometimes, not because my way is better, but so that students can see that the same thing can be said in different ways..."

At the beginning of the first lesson with grade eight, Mrs. D checked and went over the homework from the previous lesson (the homework was from the textbook).The teaching of the new content of powers was not related in any way to the textbook: the teacher used a story from the Rhind Mathematical Papyrus: "There are seven houses. In each house there are seven cats. Each cat kills seven mice. Each mouse has eaten seven grains of barley and each grain would have produced seven hekats. How many grains of barely will be saved?" The teacher pointed out the rules for the powers in the textbook and the exercises which students should do during the lesson. The teacher did not assign homework from the textbook, but gave a link to a web page with numerous tasks for practicing. The second and third observed lessons were in the fifth grade and in both lessons students worked in groups. In the second lesson, the students competed in a quiz. The tasks were prepared on a worksheet and were not the same as the textbook items. The third lesson was a revision of the topic of whole numbers. Here students participated in a game, which consisted of solving various tasks. The solution to each task was a piece of a puzzle that contributed to the solution of the overall puzzle. The tasks were similar to the items from the review section in the textbook. The game, however, did not come from the textbook.

In the interview, Mrs. D explained that she rarely introduces new procedures and concepts according to the textbook; she likes to use her own ideas. She uses the textbook for students to practice, but she uses other resources as well. For reviewing, she uses the textbook items, or composes similar ones herself. She considers that other textbooks do not greatly differ, having similar exercises.

Strong and weak points of the textbook

In the interview, Mrs. S described her attitude towards the official textbooks she was currently using in the classrooms. She was dissatisfied with the introductory sections of the lessons in the textbooks. However, she liked the selection of worked examples, that there are plenty of exercises from simpler to more complex ones, and that the exercises come right after the exposition.

> "I think it's right that when we learn something [a new concept], the exercises should follow on directly. Not be placed at the end of the chapter. This way parents can find them easily... Most of the parents aren't mathematicians and they don't always know what to give their children for practice, they might give them the wrong tasks."

Mrs. D said that she is satisfied with the textbook structure and exercises. But she also finds some textbook definitions and rules to be inappropriate for the students' age, so she rephrases them:

> "Definitions are sometimes unclear to students, so I translate them to be simple and clear".

She says that the main reason for choosing the current textbook series was financially and socially based:

> "Our school uses this textbook because... many of our students have brothers and sisters, we don't want them to have to buy new textbooks so we decided to continue to use this one and not to change it."

DISCUSSION AND CONCLUSIONS

Both participants consider their position as mediators between the textbook and students, in spite of the fact that they have different approaches to teaching mathematics and the use of textbooks. They both highly control what will be learned and practiced, how and when it will be learned. These intentions are related to the norms about being a teacher and about teaching because the teacher is supposed to prepare and shape the lesson actions. This is related to the SDT face *textbook-teacher-student* with the related social components placed on the bottom of the triangle: institution, conventions and norms about being a student and about being a teacher (see Figure 1).

Mrs. S uses the textbook as the central tool for teaching and encourages her students to use the mathematics textbook intensively. All the exercises done in the four observed lessons were taken from the textbook. Although she complained in the interview that she is not satisfied with the introductory sections of the textbook, in the observed lessons she used the material from the textbook. It may be that she relied on the textbook as a guarantee of stable quality (Pehkonnen, 2004) because the approved textbooks offer security and convenience for teaching (Love & Pimm, 1996). On the other hand, she claims that she encourages the use of the textbook because of the students. Learning how to use the textbook will help them when they attend the subsequent secondary school. Here we come to *institution* on the SDT. This also means that for her using textbooks does not mean just "learning mathematics from the textbook", but also "learning how to use a textbook, i.e. learning for life". Mrs. S mentioned another reason why she encourages the use of textbooks by her students – if the student was absent, or did not understand the subject matter in school, he/she is able to look at the textbook at home with the help of peers or parents. Here we come to *peers and family* on the other face of the SDT. The relationship *textbook-student-mathematics-peers and family* corresponds to the explanation given in the interview that the textbook helps parents to help the students at home. This is one of various reasons why Mrs. S follows the textbook so closely. This is also connected with the textbook being clearly organized – so that students and parents can catch up. The importance of the contents of the textbook being set out in a clear and simple way was mentioned by both participants.

The second participant Mrs. D prepares lessons using a variety of materials, the textbook being just one of them. She chose different and innovative ways both of presenting new concepts and practicing them. This is related to the norms and conventions of being a teacher and about teaching (Remillard, 2009). She likes the structure and the amount of exercises in the textbook, but she used it directly in only one observed lesson. The interview and the analysis of the observed lessons showed that she mostly used the same titles, symbols, definitions and rules as found in the textbook. For example, after a creative introduction to the idea of powers, her students opened the textbooks and copied the rule into their notebooks. The copying of rules and definitions into notebooks was also observed in Mrs. S's classes.

In the interviews, both participants said that they sometimes change the wording of the definitions and rules from the textbook to make them more appropriate for the students. This means that the teachers, during the lesson preparation, reflect on the didactics of mathematics in the textbook. This also corresponds to the norms about being a teacher, because the principle of compatibility of mathematical content with the students' age is one of the basic principles in mathematics education (Kurnik, 2009). The textbook content should be appropriate to the students' age, and at the same time it should be mathematically correct. The teacher should reflect on this interplay during the lesson preparation. This finding leads to the tetrahedron face with the vertices *teacher-textbook-mathematics*.

Mrs. D gave a reason of a social nature as to why the textbook is not frequently changed in their school. The same textbooks are used for a number of years, so that the younger generations can use the textbooks of their older siblings. This means that social and financial reasons are important factors in choosing whether to change the textbook or not. Such use of textbooks can be explained with the relationship within the SDT, *textbook-student-parents-public image of mathematics*. In Croatia, parents buy textbooks and this represents a great expense for them at the beginning of each school year. Also, from the public point of view, mathematics is concieved as a static discipline, with a long known set of concepts, principles, and skills (Cooney, 1985), and the learning of mathematics is concieved through practicing various exercises. Thus, from this persepective, there is no need to change textbook if the textbook contains a good amount of exercises.

All these findings lead to the conclusion that the reflection on the actions within the SDT model can help in understanding the teaching and learning of mathematics. The survey results obtained in Glasnović Gracin (2011) were not sufficient in comprehending the deeper reasons related to the teachers' use of mathematics textbooks. The qualitative approach presented in this paper confirmed and extended the survey results: Croatian teachers use textbooks for lesson preparation, exercises and homework. However, the findings of this qualitative study show in-depth how and why Croatian teachers use the textbooks in the classroom. One particular finding is related to the textbook characteristics. Teachers should have good mathematical knowledge for teaching, i.e. according to Ball and colleagues (2008), knowledge of content, students, curriculum and teaching, and to know the strong and weak points of the textbooks to be able to decide when to be the mediator between the textbook and students, and when to use other instructional resources for teaching beside the textbook. The study implies that teachers' use of the textbook is multilayered. This utilization is connected with the complex interaction between the STD components. Consequently, textbook use should not be examined separately from social and institutional influences.

Using the SDT model offers great potential and ideas for new studies. For example, the triangle *student-mathematics-textbook* is a very important face of the SDT. This aspect needs to be examined in more depth in order to better understand instruction and to improve instructional quality in the classroom.

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Interdiscursivity and developing mathematical discourse for teaching

Reidar Mosvold

University of Stavanger, Faculty of Arts and Education, Stavanger, Norway, reidar.mosvold@uis.no

This paper aims at further elaborating on a redefined theory of mathematical knowledge for teaching in terms of participation in discourse. This redefined theory of mathematical discourse for teaching is used to analyze data from mentoring sessions in initial teacher education. The results indicate that the mentor teacher, who is a more central participant in the mathematical discourse for teaching, is able to more dynamically switch between the different discourses related to mathematics teaching than the pre-service teachers—referred to as interdiscursivity.

Keywords: MKT, discourse, teacher education, mentoring dialogues.

INTRODUCTION

Learning is often defined as a process of acquiring knowledge; this metaphor of acquisition is popular in education research (Sfard, 1998). Following this metaphor, an aim of education is for students to learn something (i.e. acquire knowledge), and the teacher's responsibility is to ensure that students gain the required knowledge. In order to handle the tasks that are involved in this work of teaching mathematics, teachers need a particular kind of professional knowledge. Ball, Thames and Phelps (2008) have proposed a theory of mathematical knowledge for teaching (MKT) that has become widely used. Ball and colleagues define MKT as "the mathematical knowledge needed to carry out the work of teaching mathematics" (p. 395). Following Shulman's (1986) theory of teacher knowledge, they describe several subdomains of MKT. These subdomains of knowledge are seen as distinguishable and measurable. Measuring particular subdomains of MKT is, however, challenging. When analyzing the connection between teachers' responses to multiple-choice MKT items and additional written reflections, Fauskanger and Mosvold (2013) found that teachers use knowledge from different subdomains of MKT when responding to items that were designed to measure one particular subdomain only. Sticking with the metaphor of acquisition, a quandary emerges: How can teachers use knowledge from different subdomains of MKT when responding to an item, if these subdomains are distinct and items have been developed to measure distinct subdomains only?

Instead of adhering to the acquisition metaphor to knowledge and learning, I follow the participation metaphor (Sfard, 1998; Skott, 2013) in this paper when I look at mathematical knowledge for teaching from a discourse perspective. In doing this, I adopt Sfard's (2008) commognitive framework and follow Cooper's (2014) suggestion of redefining MKT into "mathematical discourse for teaching". Cooper argued that such a redefinition of the MKT framework is useful for interpreting data from professional development of teachers. My aim is to extend this redefined framework further and apply it in analysis of example data from initial teacher education. With this theoretical framework, I will argue, the quandary with the distinct subdomains of MKT becomes obsolete.

THEORETICAL FRAMEWORK

Before Shulman (1986) presented his ideas about teachers' professional knowledge, classroom research often had a focus on observable behavior of teachers and/or students. His main contribution was to focus researchers' attention on the role of teachers' knowledge. Shulman, who adhered to an acquisition metaphor, suggested that the knowledge required by teachers could be divided into several distinct categories. Among the most well-known are subject matter knowledge and pedagogical content knowledge, and these two categories are often presented as import-

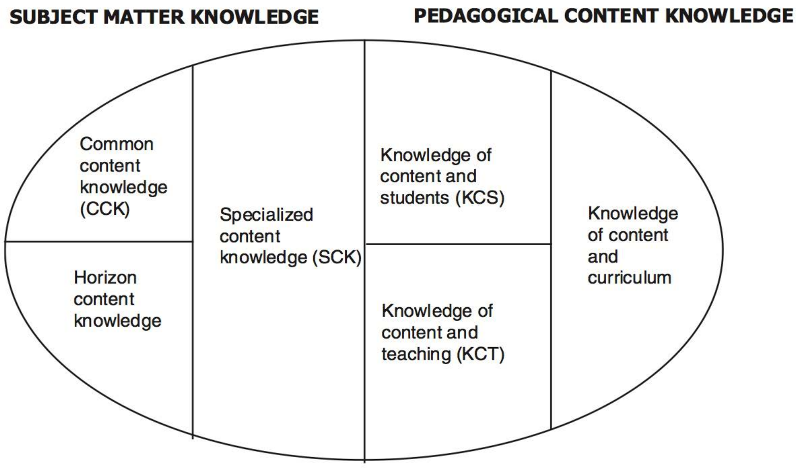

Figure 1: "The egg"—depicting MKT and its subdomains (Ball et al., 2008, p. 403)

ant aspects of teacher knowledge (e.g., Baumert et al., 2010). When Ball and colleagues (2008) presented their theory of MKT, they continued to scrutinize Shulman's initial categories. Subject matter knowledge and pedagogical content knowledge were divided into subdomains, and this model was depicted in a figure that has later been referred to with nicknames as "the egg" or "the oval" (Figure 1).

Sfard's (2008) theory represents a different view. A main idea is that cognition and communication are inseparable, and she combines the two terms into a new theoretical term: "commognition". She defines thinking as "an individualized version of (interpersonal) communication" (Sfard, 2008, p. 81). Knowledge is related to participation in discourse — not acquisition of an objectified entity — and learning is seen as a permanent change in discourse. This change can be either on an object level (where new words are introduced) or on a meta-level (where the rules of discourse change). For the researcher, then, the study of communication and participation in discourse(s) becomes pertinent.

Cooper (2014) applied Sfard's theory when he attempted to translate the MKT model into a discursive framework. He suggested redefining MKT as "mathematical discourse for teaching" (p. 338), and he substituted subject matter knowledge with a mathematical content discourse; pedagogical content knowledge was replaced with what he referred to as "pedagogical discourse for teaching" (ibid.). Building upon the ideas of Sfard (2008), he suggested that the following features could identify these discourses: 1) main *words* that appear in the discourse, 2) visual mediators that are commonly used in the discourse, 3) *routines* that are distinctive to the discourse, and 4) *endorsed narratives*. In this paper, I attempt to take Cooper's reinterpretation of MKT one step further and introduce the subdomains of MKT. I envisage this as a revised model consisting of several partly overlapping discourses (Figure 2).

Instead of following Cooper's approach and investigate words, mediators, routines and narratives in a discourse, I focus on participation in different discourses. I build upon the theories of Lave and Wenger (1991). They focused on how learners move from being peripheral participants to full or central participants in communities of practice. Instead of discussing communities of practice, however, I focus on communities of discourse (c.f., Sfard, 2008). I follow Sfard's (2008) definition of discourse as: "The different types of communication, and thus of commognition, that draw some individuals together while excluding some others" (p. 91). Instead of investigating how teachers increase their MKT, I attempt to study the process in which (pre-service) teachers move towards full participation in the mathematical discourse for teaching.

METHODS

In this paper, I do not investigate the complete process of becoming full participants in the mathematical discourse for teaching. Instead, I use data from mentoring sessions between three pre-service teachers and their mentor teacher as well as classroom data as an exemplification of one part of the process. The data material was collected as part of a larger project: Teachers as Students (TasS). An overall aim of this project was to investigate how pre-service teachers develop knowledge, skills and competence for teach-

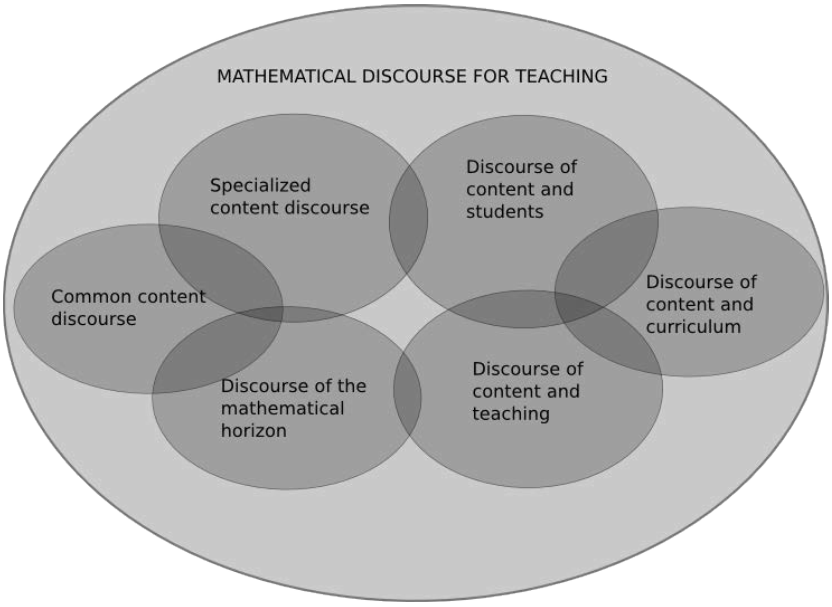

Figure 2: A tentative model of mathematical discourse for teaching

ing. In this paper, I use a discursive approach to investigate pre-service teachers' mathematical discourse for teaching.

In Norway, initial teacher education is organized as a four-year bachelor program. National curriculum guidelines require a total of 100 days of teaching practice, and teaching practice is supposed to be integrated in all subjects. The participants in this study were in the second year of their teacher education, and the data material was gathered in connection with a period of teaching practice. As example data, I use transcripts from two mentoring dialogues between a group of pre-service mathematics teachers and their mentor teacher before and after one lesson that was taught by the pre-service teachers. The group consisted of three pre-service teachers: Fiona, Rachel and Harry (all pseudonyms). The mentor teacher—who is a practicing teacher—is simply referred to as this in order to avoid confusion.

Recordings from the mentoring dialogues were transcribed verbatim, and these transcripts provided the foundation for my analyses. The analyses in this paper are meant to illustrate a possible use of an extended version of Cooper's (2014) redefinition of MKT into mathematical discourse for teaching as a framework, and I focus in particular on how the participants draw upon different discourses related to mathematics teaching in the mentoring dialogues.

RESULTS AND DISCUSSIONS

In the following, some excerpts from the data material will be presented as an illustration. A main emphasis will be on the mentoring dialogues before and after a mathematics lesson in 8th grade, but data from the actual lesson will be included in order to contextualize the mentoring discussions. The goal of the lesson was to enhance the students' understanding of the equal sign.

Pre-mentoring dialogue

When the three pre-service teachers met with their mentor teacher for a last mentoring session before teaching the lesson, their main focus was on discussing the lesson plan. They had discussed with the mentor teacher the day before, and they had agreed that they would have to adjust the level of the content they were going to present. In order to make it easier and more understandable for the students, they decided to start with a realistic problem. The context of the problem is that Fiona wants to go shopping, and she has to figure out how much money she could spend—given the various expenses that would be charged every month. With this problem as a starting point, Fiona explains how they are going to introduce the equal sign:

20. Fiona: And ask about how we are going to get that box (a), and what they did. Yeah, they subtract (b), and then you kind of take away 3500 from this side and that side, in order to balance [it]. And then we get to the

equal sign (c), and then kind of having scales and things like that, that kind of becomes a theme then, and work with the equal sign to really make them understand that it has to be balanced on both sides (d).

As we can see, Fiona has already included the unknown — the "box" — in this problem (a), but she is thinking of the equal sign as a main theme for the lesson (c). In this utterance, she navigates in a pedagogical content discourse. She describes how they will present the content for the students (d), and it can thus be interpreted as if she mainly draws upon a discourse of content and teaching. We can see that she anticipates a certain student response already (b), but her main focus is on the discourse of content and teaching. As a response to this, the mentor teacher goes into another discourse when she argues that they might be going too far too soon:

21. MT: Yes, what the equal sign really means (e). But isolating the box on one side, then you have really gotten far (f). Having something isolated on one side and move things, then you are really up there on the algorithm level right away (g). You might not have to talk about that at all, but I think they will see it when you say: "Okay, what's missing here?" They kind of see what is missing (h), and that is what the unknown is. It is something that is missing, and that you'll try to figure out. And then there are many ways of figuring it out. I mean, there are many ways of finding the unknown without rearranging and getting one box isolated, which is really the last part [in the process] perhaps. So try to hang in there as long as possible, only focusing on the understanding of what it is (i), and then one of the goals for the lesson is to understand the equal sign. Because then it is easier, and then you know what you want them to learn from the lesson. Being practical about it.

In the beginning of her utterance, the mentor teacher goes into a discourse of mathematical content. She starts by pointing at the true meaning of the equal sign (e), before shifting her focus to the mathematical horizon (f). Focusing on the mathematical implications of their choices in relation to the mathematical location of where the students are currently working, she claims that they have already moved to an algorithm level (g). I interpret this as an example of how the mentor teacher draws upon her experience from participating in a discourse on mathematical content in general and a discourse of the mathematical horizon in particular. The mentor teacher then draws upon her experience from the classroom and shifts into a discourse of content and students when she says that, "they kind of see what is missing" (h). Then she shifts again and moves into a discourse of content and teaching when she makes suggestions about how the pre-service teachers might present it to the students (i). The pre-service teachers still appear to be more peripheral participants in these discourses, and Fiona responds to the mentor teacher by drawing upon her experiences from the previous period of teaching practice instead (j):

22. Fiona: The equal sign is, in our last period of teaching practice (j), the equal sign and the understanding of an unknown was inseparable.

The mentor teacher follows up by confirming that the equal sign and the unknown constitute a sensible goal for the lesson. After this, they continue to discuss different aspects of the lesson plan until the mentoring session ends after 18 minutes.

Classroom discourse

In the lesson, the pre-service teachers start by presenting themselves, since this is the first time they are in that particular class in this period of teaching practice. After a round of presentations, Fiona starts teaching the lesson. "I have a problem that I want you to help me solve," she says. Then she explains that she wants to go shopping, but as a student in teacher education, there are certain expenses she needs to take into consideration. On the blackboard, she writes down the amount of money she gets from loans every month (6700 NOK), and then she writes down all the expenses below. The question is, "how much money is left for shopping?" After having given the students some time to think, one of the students presents 1200 NOK as an answer to the question. They spend some time discussing this before Fiona presents another similar problem: "It is the national day, and we have 200 NOK in our pocket. We want to buy ice cream, and an ice cream costs 20 NOK. How can we write down an expression that helps us calculate the number of ice creams we can buy?" The students come up with

Figure 3: Buying ice cream

different alternatives, and Fiona writes them down on the blackboard. She circles in the equal sign and asks if anyone knows the name of this (see Figure 3).

When a student responds, “is equal”, Fiona continues by asking what the equal sign actually means. A student responds: “what comes behind it is often the answer”. Fiona hesitates a bit before answering that it is partly correct, but that this is something they will come back to later in the week.

Post-mentoring dialogue

The day after having taught the lesson, the pre-service teachers meet with the mentor teacher for another mentoring session. The mentor teacher provides them with feedback, and they discuss different aspects of what they have observed in the lesson.

45. MT: And then you came to: What does the equal sign mean?
46. Fiona: Mhm
47. MT: “Behind it comes the answer”, said one of the students. And then another one said: “add together and insert [an] equal [sign] later”
48. Fiona: Yes, and then I thought...
49. MT: Andrew (name of a student) said it.
50. Fiona: (laughs)
51. MT: And you didn’t make much out of it really, and this was perhaps, you could have taken it further and shown: What does it become then? But you decided not to make much of it, and in this situation I think it was okay.
52. Fiona: Yes, I didn’t really≈
53. Harry: ≈I was thinking, when we were working with the cards, then he had the expression which is equal to, and the expression which is equal to, that’s what I meant.
54. Fiona: Yes, I was thinking≈
55. MT. ≈“If you have fifteen plus five equals twenty, and then minus three equals seventeen”, that’s how he said it.
56. Fiona: Mhm

The mentor teacher starts by referring to the question asked by Fiona in the lesson (45). The mentor teacher then follows up by presenting some of her observations from how students responded in the classroom (47). All of the pre-service teachers as well as the mentor teacher experienced the classroom discourse, but we see how the mentor teacher is the more central participant in the mathematical discourse for teaching (51) whereas Fiona and her fellow pre-service teachers are more peripheral participants. It can be argued that this relates to their different roles as mentor teacher and students in pre-service teacher education, but I argue that the differences can also be seen in how the participants navigate between different discourses. When evaluating Fiona’s response, the mentor teacher mainly draws upon a discourse of content and teaching, but then the focus shifts:

57. MT: Is this a correct use of the equal sign? (k) Because it is a very common thing to do for students, and it is a wrong use of the equal sign (l).

In this follow-up question (57), the mentor-teacher first draws upon a subject matter discourse (k), and then she combines this with a comment from a discourse of content and students (l): “it is a very common thing to do for students”. In the following, Harry tries to go back to the classroom discourse, but the mentor teacher interrupts him and continues:

58. Harry: He had≈
59. MT: ≈especially when it is multiplication and everything (m), then many [students] often add directly to the answer (n), and then the balance is really all wrong.

After having briefly drawn upon her experience from participating in a discourse of content and students (57), the mentor teacher is now back in a subject matter discourse (m), but she continues to draw upon the discourse of content and students (n) when saying that many students “often add directly to the answer”.

Fiona acknowledges the mentor teacher's statement with a "Mhm", before the mentor teacher continues:

60. Fiona: Mhm
61. MT: So you could have taken it further and shown: "Are both sides equal here?" and perhaps made them conscious about this already now, early on, to begin with (o). Because it is likely something many are going to have misconceptions about (p).

In this utterance (61), the mentor teacher draws upon both a discourse of content and teaching and a discourse of content and students. First, she reflects about how Fiona could have acted differently as a teacher (o)—which relates to a discourse of content and teaching—and then she adds a comment about this being something many students will probably have misconceptions about (p). The latter is part of a discourse of content and students. In this example, like in the other examples above, we clearly see that the mentor teacher does not stay within one of these discourses only, as if they were separate, but she continually switches between them—often within the same sentence. Throughout these examples, it the mentor teacher appears more dynamic in how she shifts between these different discourses—all related to a mathematical discourse for teaching—whereas the pre-service teachers are more caught in one discourse at a time.

CONCLUDING DISCUSSION

In the literature, researchers discuss supposedly distinct sub-domains of mathematical knowledge for teaching (e.g., Baumert et al., 2010). As a potential threat to the idea of the sub-domains of MKT being distinct, some studies indicate that teachers draw upon different aspects of knowledge when responding to MKT items (e.g., Fauskanger & Mosvold, 2013). When using a participation metaphor, however, it is unproblematic to consider that participants draw upon other discourses from which they have experience in participating. These discourses do not have to be completely separate, although they are different types of communication that include some participants and exclude others (Sfard, 2008).

In his attempt to bring mathematical knowledge for teaching under Sfard's (2008) discursive framework of commognition, Cooper (2014) distinguished between two components of what he referred to as mathematical discourse for teaching: a mathematical content discourse and a pedagogical content discourse. In doing this, he switched from an acquisition metaphor of learning and knowledge to a participation metaphor (cf. Sfard, 1998). Whereas Cooper used this as a framework for interpreting data from professional development, I have attempted to use it in the context of mentoring dialogues in initial teacher education. I suggest that the mathematical discourse for teaching is even more complex and compound than suggested by Cooper (2014), and indications of this can be found in the results presented.

Using a combination of Sfard's (2008) commognitive framework and the theory of learning as legitimate peripheral participation by Lave and Wenger (1991), I propose that the development of a mathematical discourse for teaching is related to the ability to dynamically draw upon different discourses for teaching mathematics. When discussing the planned and observed lesson on the equal sign, the mentor teacher uses a discursive move that can be referred to as interdiscursivity when she continually draws upon her experience from different parts of the mathematical discourse for teaching. The pre-service teachers appear to have less ability to use interdiscursivity, and this might be related to less experience from participating in the different parts of the mathematical discourse for teaching — or, put differently, that they are still peripheral participants in this discourse. There is a possibility, however, that this apparent difference can be explained by the different roles of the mentor teacher and the pre-service teachers in the mentoring dialogues. Cooper (2014) found that the learning situation in professional development is more symmetrical than what is often found in children's learning. In this study on pre-service teachers' learning in teaching practice, it can be argued that the situation is more asymmetrical. This issue of power relations could be investigated further in discussions among teachers with different levels of experience (but who do not have a mentor/student relationship). I have focused mainly on interdiscursivity in this paper, but this is only one of several issues that might emerge from analyses where the framework of mathematical discourse for teaching is applied. My discussion of this issue is meant to serve as an example, and I suggest that a redefinition of MKT in terms of participation in discourses should be further investigated.

With a discursive definition, a view of MKT — and knowledge in general — as some kind of object or hidden entity can be avoided. Discourse for teaching is not a latent or hidden trait, but something researchers can investigate and analyze more directly. I acknowledge that the framework of mathematical discourse for teaching is still in development, and further studies could for instance investigate a merging of this framework with Skott's (2013) framework of patterns of participation. Applying such a participatory and discursive approach to investigate MKT could then be seen as part of a larger initiative to develop a more coherent approach to understanding the work of teaching mathematics where acquisitionist terms are avoided.

ACKNOWLEDGEMENT

This study is part of the "Teachers as Students" (TasS) project, and it would not have been possible without the support from the Norwegian Research Council, grant no. 212276.

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Teachers' initiating change in practice due to variation of progression of didactical time

Linda G. Opheim

University of Agder, Kristiansand, Norway, linda.g.opheim@uia.no

This study aims at identifying situations where time can act as a condition for change in teacher practice. A design based research methodology was applied and teachers where offered help designing mathematics tasks they would want to use in the classroom. All the discussions from this collaboration were analysed using timescales as analytical categories as described by Assude (2005). The findings show that in a diverse class, the difference in learning pace between students can condition change in the teachers practice.

Keywords: Teacher practice, change, diversity, time.

INTRODUCTION

This research report emerges from a PhD-study on teachers' perception of what makes a 'good' task in vocational school mathematics lessons. I have worked closely with different teachers over a school year designing tasks they would want to use in the classroom in order to learn more about the teachers' preferences for mathematics tasks. One of the concepts emerging from my research is 'time', and this paper focuses on how time can be a condition, i.e., a prerequisite, for change.

Time is a recurrent issue for many teachers, and time pressure is a real day-to-day classroom experience which teachers have to live with (Assude, 2005; Jordfald, Nyen, & Seip, 2009; Leong & Chick, 2011). Time can serve as an obstacle when implementing new reforms (Keiser & Lambdin, 1996), integrating new technology (Assude, 2005) and probing understanding through whole-class discussions (Black, 2004). A teacher might also make didactical choices in the classroom based on time triggers (Leong & Chick, 2011), and avoid lesson preparations that are too time demanding (Goodchild, Fuglestad, & Jaworski, 2013). As a summary, lack of time is often seen as an obstacle for change in teachers' practices.

In my research I have noticed that time is an issue that surfaces often in the conversations with the teachers, but not always negatively. This triggered my interest to explore the data using time as an analytical category. The research question addressed in this paper is: *When can time be a condition for change in teacher practice?*

I will, in the following, present my theoretical framework of timescales before I give a brief summary of the political ideas behind the Norwegian educational system. With this as a background, I present my methodology followed by results before I round off with a discussion of the results.

THEORETICAL FRAMEWORK

In this paper, I will look at how time is viewed by the teacher, and how time can be seen as a condition for change by the teacher. In order to do this, I need to identify different categories of experienced time. Lemke (2000) argues that all human activity takes place on at least one timescale and in complex systems more than one timescales. For instance, an utterance of a single word can be viewed as a timescale even if it takes less than a second to articulate. Also one school lesson or a year curriculum can be viewed as timescales. Every process that stretches out over time can be viewed as a timescale. In my work I study how time can be a condition for change and, given that change is a process, it will be helpful to identify timescales for analyzing my material. With the help of timescales it is possible to identify at which point time is a condition for change.

Assude (2005) describes three theoretical timescales, and I will present and use the same timescales with a slight modification. The three timescales are didactic time, time capital and the pace of a course. The concept of *didactic time* is related to the work of Chevallard, and is defined as "related to scheduling the teaching of some knowledge" (Assude, 2005, p. 185). However, Assude (2005) also makes the claim that: "Didactic time is used as a gauge of the advancement of knowledge, and, in this sense, it's a framework, which regulates the activity of the teacher" (Assude, 2005, p. 185). This last quote is referring to learning (advancement of knowledge) and not only the scheduling of teaching. As I see it, the concept of didactic time might be used as referring to a teacher's scheduling of some knowledge for a whole class of students, or the concept might be related to the scheduling of teaching with respect to individual students' learning. I will in my analysis use both of these interpretations of didactic time.

The second concept which Assude (2005) identifies is *time capital*. This refers to the 'objective' time available for classroom work, for example, the lesson is 45 minutes long. The third concept Assude (2005) applies is *the pace* of a course or a part of it, which is viewed as how didactical time is advancing with respect to time capital. This is demonstrated by showing different linear graphs of how courses can be fast-paced, moderate-paced or slow-paced. This last concept might be viewed slightly different in many Norwegian classrooms, and I will give some information of the Norwegian educational system to demonstrate this.

THE NORWEGIAN EDUCATIONAL SYSTEM

Education for all and equality are important concepts in the Norwegian educational policy across political party lines, with a goal to reduce social inequality. (Markussen, Frøseth, & Sandberg, 2011) In the Norwegian educational system we also have "the comprehensive school" as an important political concept (Department of Education and Training, 2006–2007). We do not have special needs schools in Norway, and every pupil is entitled to be in a normal classroom. As a result, a typical class in Norway will have children with high and low achievers, different diagnoses and handicaps. The Education Act (1998) specifies: "Education shall be adapted to the abilities and aptitudes of the individual pupil, apprentice and training candidate" (§1–3). All pupils are entitled by law to get adapted teaching according to their abilities, and this is the teacher's responsibility.

The political concept of the comprehensive school has also influenced upper secondary school, and since 1994 every teenager has a statutory right of secondary schooling regardless of abilities and academic results. As a consequence, near to every teenager is now starting upper secondary school and many of the low achieving students apply for the vocational programs (Department of Education and Training, 2006–2007). The Education Act also applies to upper secondary school, and adapted teaching is a requirement for all the courses in the vocational programs. As a result, many teachers take it personally when their pupils fail subjects or quit secondary school. They have as a goal to give adapted teaching to all of their pupils, but the classes are diverse and the pace of a course can be viewed quite different from student to student. It is therefore difficult to make a general claim about the course being slow-paced, moderate-paced or fast-paced.

Norway is a country where accountability systems have never been approved for use in the education sector, even if there are some accountability devices in local quality-assurance systems (Christophersen, Elstad, & Turmo, 2010). Still, the Norwegian Prime Minister said in a speech in 2008 that "Teachers should have a clear responsibility for what students learn in school" (Christophersen et al., 2010, p. 2). It has not been made clear how a Norwegian teacher can be made accountable for students' learning and Christophersen and colleagues (2010) argue that this is not possible. The Prime Minister's speech might indicate a political shift when it comes to accountability in the future, but as of today the main focus in Norwegian schools are still on adapted teaching and not on the pupils' test results.

In this paper, I focus on two teachers working at an upper secondary vocational school in Norway, which is a school with diversity and where adapted teaching is mandatory (the Education Act, 1998). Both of the teachers reported in this paper, have read and approved of the paper before it was submitted for the conference. I will, in the following, show how an analysis of the advancement of different timescales previously identified can act as condition for change.

METHODOLOGY

In the PhD-project I was collaborating with four teachers; however only two of them are considered in this paper. The reason is that the initial analysis revealed that the concept of time was much more seldom mentioned by two of them, in addition these two are working at University level where adapted teaching is not required by law. The two teachers considered in this paper teach mathematics at a vocational upper secondary school. They both volunteered to be part of the research, and the aim of the PhD-project is a greater understanding of the choices teachers make when it comes to mathematical tasks and not to change the teaching. Findings from the TIMSS advanced 2008 study, show that the most dominating activity in Norwegian classrooms is by far solving mathematical tasks similar to those in the textbook, and this is also a predominant classroom activity in other countries (Mullis, Martin, Robitaille, & Foy, 2009, p. 162). This is one of the reasons I view mathematical tasks as an important issue to research.

To accomplish a greater understanding of the choices teachers make when it comes to mathematical tasks, a design based research methodology (van den Akker, Gravemeijer, McKenny, & Nieveen, 2006) was applied in the PhD study, and the teachers were offered help designing tasks they wanted to use in the mathematics classroom and which they felt would improve their students' learning. The collaboration lasted a school year, and all conversations between the researcher and teachers were audio recorded. A total of 20 hours and 49 minutes of audio recordings with the two teachers have been used in the analysis; the recordings consist of interviews, and discussions about the tasks, refining tasks and evaluation of implementing the tasks.

All of the recordings have been imported into NVivo and data reduced concurrently with data generation. Data reduction entails writing down what is happening and what is being said without transcribing word for word. The data reductions are detailed facilitating retrieval of relevant data material at a later point, but they are also the first phase of analyzing, serving as a guide, focus and help when collaborating with the teachers. The data has not been transcribed; rather all analyses have been done using the original audio recordings.

The data has been analyzed using techniques from grounded theory (Corbin & Strauss, 2008), and time emerged as a recurring issue. As a consequence, I went back to the data and isolated all the parts where time had been mentioned. This data have then been analyzed according to Assude's timescales as analytic categories. I have in the analysis considered how didactical time advances with respect to time capital, and how this can be a condition for change. For instance, could there be a point where didactical time is advancing so slowly that it conditions changes in the teacher's practice?

As before mentioned, my research question is *when can time be a condition for change in teacher practice?* Given my research design, I have unique data for analyzing reasons for wanting change from a teacher's perspective. The teachers are offered help to design new mathematical tasks to use in their classrooms without the researcher trying to influence them, and the reasons provided for the new tasks are therefore a good gauge of what the teacher sees as important to change. The research was not designed to bring about big changes, just adjustments to what was already being done. Teacher change is mostly viewed as a slow and difficult process (Sowder, 2007), however, some of the teachers in this project expressed clearly that they participated in the research because they wanted changes and they wanted to improve.

The teachers participating in the project want some changes, but it might differ how important this is to them. In order to view if the changes are something that really are important to them, I am also looking at the time and effort the teachers are putting into this project and how they view this investment. We know that time to prepare lessons is a valuable asset for the teachers, and they are often protective of how this time is used (Goodchild et al., 2013; Jordfald et al., 2009). So if the teachers are expressing that they want to invest time into my project, I view this as changes being rather important to the teachers.

RESULTS

I will start this section by presenting excerpts which are related to the teachers wanting changes, as explained above. One of the teachers expressed at the very beginning of our cooperation that: "I am doing this (participating in the research) because I feel I can benefit from it. I'm not doing this because I have a heart of gold". This is a clear statement from the teacher seeing this cooperation as beneficial for him, and he is looking for help to make changes. The other teacher is open and honest about how he is struggling with time management in his job at the moment, and expresses:

> I am not happy with how I've been doing my job now. I am underperforming. It's not satisfactory. I have been stressed, but I don't have any more to do than everybody else in this world. This is resulting in me not being able to 'be present'; I forget things and have an extremely bad memory at the moment.

Given that I have spent a total of about ten hours of discussions with this teacher in my project, I feel bad for costing him so much of his preparation time and I express this, but he responds:

> Yes, but... no! I've never had a thought of that. I've been thinking it's been beneficial for me all the time. Because then I get the help, and if I didn't have the pressure to make the index task, the lesson would have been extremely boring. So for me it's only been good.

With a teacher expressing so strongly that he has been lacking preparation time, but still views our cooperation as beneficial and as an asset, speaks volumes of the need for help to change. This teacher actually ended our last conversation with: "You should just make contact if you need it again, and this year has been very good for me and a help, so no burden for me."

To summarize this part, the teachers are expressing that this is a research project they see as beneficial to them, and they do not mind that it cost them time. They view it as an investment to get help to make changes in their practice. When analyzing what might condition this change, I have used the analytical categories *didactic time* (with respect to the whole class or to individual students), *time capital* and the *pace* of the course. I will, in the two following sections, present results from the analysis of how the pace is viewed both with respect to the whole class and individuals, and how time capital and didactic time is relevant in these examples.

Pace of the whole class

I argue above that it is difficult to view the general pace of a course in vocational programs in Norway, but sometimes the teachers make references to it. In this section, I will present teacher comments related to the pace of the whole class, i.e. didactic time with respect to time capital, and identify how didactic time and time capital are being viewed by the teacher.

In my recordings I found three categories of comments related to the pace of the class as a whole. These were:

- The curriculum in general: These comments are related both to being ahead of schedule or being on time. "I have found out that I'm almost a chapter ahead of the others. So I wonder if I need to slow down a bit."
- Time needed for open tasks: The teacher comments on the difficulty in knowing how much time the students will need for an open task.
- Structure or chaos in a specific lesson: For instance a teacher is commenting on the need for better structure in a task because the students are spending too much time on irrelevant stuff for learning mathematics

None of the teachers I talked to expressed any concerns about having enough time capital for the given curriculum, so it doesn't seem like they view this as a fast paced course, especially since one of the teachers even comments that he is ahead of time. When it comes to the pace for the whole class, the only concerns of the teachers were related to smaller sequences, i.e. a lesson or a task, and they are making suggestions for adjustments to improve themselves. I am therefore not considering these as big issues for the teachers in my research project. None of the teachers have mentioned the pace of the whole class as reasons for wanting changes in the mathematics tasks.

Pace of individual students

As I have expressed above, the teachers did not seem very concerned about the overall pace for the class as a whole group, however they did express concerns when it came to individuals. These concerns could be divided into two different types of comments: one related to students not doing anything at all in the lessons, and the other one related to students not understanding from the teaching. Here is one example of a teacher expressing concerns of students not doing anything in the lessons:

> I think the idea [of the task] is good, but it requires that the students actually engage in the task... And that is kind of the roadblock. When

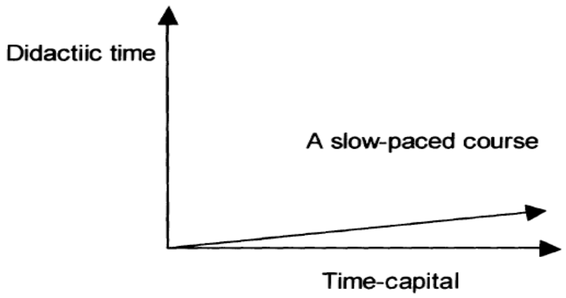

Figure 1: Diagram of a slow-paced course

> you asked me if we could cooperate, my idea was... What should I do to get the very lowest achievers of the students to have some kind of benefit? And what should I do to get them to do something?

These students are not only performing poorly, they are not doing anything at all during the lessons. The teacher talks about getting the students engaged as a roadblock and one of his main goals with this project is to get the low achievers to do something. Given that didactical time can be viewed as the advancement of knowledge, this teacher is here expressing that this is not happening at all for some of the students. This means that for these students, a graph of the pace of the course would look something like the diagram Assude (2005, p. 187) uses for a slow paced course (see Figure 1).

Students who are not doing anything at all during the lessons are the most extreme cases, but the teachers are also talking about students who are not benefiting from the teaching. This excerpt is from a conversation with a teacher where he is evaluating a ratio task we had designed and implemented:

> ...if I do it the "quicker" way which is: Hey, here is the formula, now do task one, two and three. And then there are those who manage to keep up with the explanations, they are done in five minutes. And then I use the rest of the time to explain to those who never get any of the things I explain (to the full class).

The teacher is here referring to a "quicker" way of teaching a formula, but where many of his students do not understand the explanation so he has to use the rest of the lesson trying to help them understand. This example shows how the teacher is struggling with students experiencing different levels of advancement of knowledge, or didactical time. Some of the students do not understand the mathematics from this type of teaching while others get it straight away.

The pace of individual students in a class is also what the teacher refers to when it comes to whether a new task is successful or not. Below is how a teacher is describing why he is happy with the task we designed:

> While here, here there is something everyone can do basically, and they have something tangible which they even can count if they need to. I even experienced that some of the low achievers seemed to have some Aha-moments, which they normally don't have. To sum up the benefits of the task: they get quickly started, everyone can manage something, they get active straight away.

Everyone being active and being able to manage something in the task, are success criteria according to this teacher. So didactic time is advancing at some level for all of the students, and these are the reasons the teachers are giving for the task being good.

DISCUSSION

Assude (2005) was, in her article, addressing the class as a whole when it came to the pace of a course, however I have shown that this is not necessarily sufficient for all classes. The teachers in this research project are mostly referring to individual students or a subgroup of students when they are talking about the pace of the mathematics course. The pace for the class as a whole does not seem to worry the teachers, but they are expressing concern when it comes to the pace of individuals or a subgroup of students.

I will argue that this difference in perspective is related to the diversity of a class. The more homogeneous a class is, the easier it becomes to talk about the general pace of the course. However, given the political ideal of the comprehensive school in Norway, there is great diversity in the classrooms of vocational schools. In addition to this, the teachers are obliged to give adapted teaching, which makes the differences in the pace of the course from student to student more obvious. In these classrooms there does not really exist *one* pace of a course, and the learning outcomes among the students might be very different.

This diversity of a class is, as explained by the teacher, a reason for their desire for changes in the mathematics tasks. Didactical time is not advancing for all of the students even if time is passing, so changes need to be made if they shall fulfill the requirements of the law when it comes to adapted teaching. If a student group is rather homogeneous and can keep up with the requirements of the curriculum, I assume the teacher will not worry much about didactical time. On the other hand, when a student group is very diverse, the teacher will be a lot more concerned about didactical time, given that it is so different within the same class.

If the student group is diverse, the teacher can choose to view this from two different perspectives. It is possible to have the perspective that the students will need to adjust to the teaching and the pace of the course, or you can have adapted teaching as a focus. The difference between these two perspectives comes from either a focus on the advancement of didactical time for each student as an individual or just an 'ideal' advancement of didactical time for the course. The teachers in this research projects were volunteers, and this might indicate a greater willingness to change, however the issues of diversity are present in the classrooms no matter the teachers' intensions.

The diversity of some of the Norwegian classrooms are extreme, however, there is some degree of diversity in every classroom. We might think that grouping students into ability groups in the form of setting, as they have been doing in England and Wales might give a more unanimous progress of didactical time with respect to time capital, but research shows that even in these classrooms, students feel the work they are given is either too hard or too easy (Boaler, Wiliam, & Brown, 2000). This shows that even if the problem might be more evident in the Norwegian comprehensive classrooms, there is evidence that this is also a universal classroom issue. Wilkinson and Penny (2013) argue that

> Within even the narrowest setting system, a set will contain students with considerable variations in attainment as well as learning style. It is therefore highly problematic to assume and treat setted students as intellectual homogenous. (Wilkinson & Penney, 2013, p. 10).

I have shown that some teachers feel the need to make changes, and I have argued that this is related to diverse classes where didactic time is not advancing for all of the students with respect to time capital, and this serves as a condition for change. Given that teacher change is mostly viewed as a slow and difficult process, identifying teachers who are so clearly expressing that they want change, can be of great value to further research.

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Teaching actions conducting mathematical whole class discussions

Joana Mata-Pereira, João Pedro da Ponte and Marisa Quaresma

University of Lisbon, Institute of Education, Lisbon, Portugal, joanamatapereira@campus.ul.pt

The aim of this paper is to understand teacher practice conducting whole class mathematical discussions. Teaching practice is seen as an activity based on a motive and carried out through a set of actions. The study is carried out in a grade 9 classroom of an experienced teacher. Data is gathered by classroom observations (video and audio recorded), and by discussions with the teacher. Data analysis is carried out based on a model about teachers' actions. The results show that a central challenging action embodies all discussion segments and that teacher's actions are deeply related to students' knowledge about the situation being discussed.

Keywords: Teachers' practice, teachers' actions, mathematical discussions.

INTRODUCTION

Nowadays, inquiry based or exploratory learning is regarded as a fruitful learning environment in school mathematics. It differs from other environments by the ways in which challenging tasks are used and whole class discussions are conducted. However, proposing such tasks and providing students' opportunities to present and discuss their reasoning poses teachers many challenges (Stein, Engle, Smith, & Hughes, 2008). Organizing and conducting whole class discussions is particularly important to students' learning, being an important feature of teachers' professional practice.

In an exploratory setting, working on a task typically involves the students working on a task individually or in small groups (Ponte, 2005). That is not the case of the task presented in this paper, as it is proposed, discussed and solved within a discussion, making this a rather special whole class discussion. With this study, by analysing this particular whole class discussion, we aim to understand teachers' practice during whole class discussions, focusing on the teachers' actions and on the mathematical processes involved.

TEACHER'S PRACTICE IN WHOLE CLASS DISCUSSIONS

Teachers' practice may be characterized as the activity developed by the teacher (Jaworski & Potari, 2009) that unfolds in actions established according to an action plan (Schoenfeld, 2000). In traditional classes the teacher control is very high and students' interventions are very limited. In contrast, exploratory classes have many inquiry and divergent moments. In such learning environment, conducting whole class mathematics discussions is not only challenging for the teacher, but essential to students' learning.

Features of teachers' practice during whole class discussions have been highlighted by several authors. Wood (1999) states the relevance of involving students in presenting their solutions and in discussing those of their colleagues. Potari and Jaworski (2002) refer that the level of challenge of teachers' questioning during whole class discussion is an important feature of teachers' practice. Stein, Engle, Smith and Hughes (2008) point out that an essential feature of teachers' practice is to shape students' incomplete or poorly phrased ideas into more precise and powerful mathematical ideas. They argue that a productive mathematics discussion (i) is supported by students' thinking, and (ii) provides important mathematical ideas. They also present a model to prepare and conduct mathematics discussions that include actions of anticipating likely students' responses, monitoring students' responses, selecting students to present their responses, sequencing students' responses, and making connections between students' responses and key mathematical ideas. More recently, Cengiz, Kline, and Grant (2011) identify as main teachers' actions in whole class discussions aiming to extend students

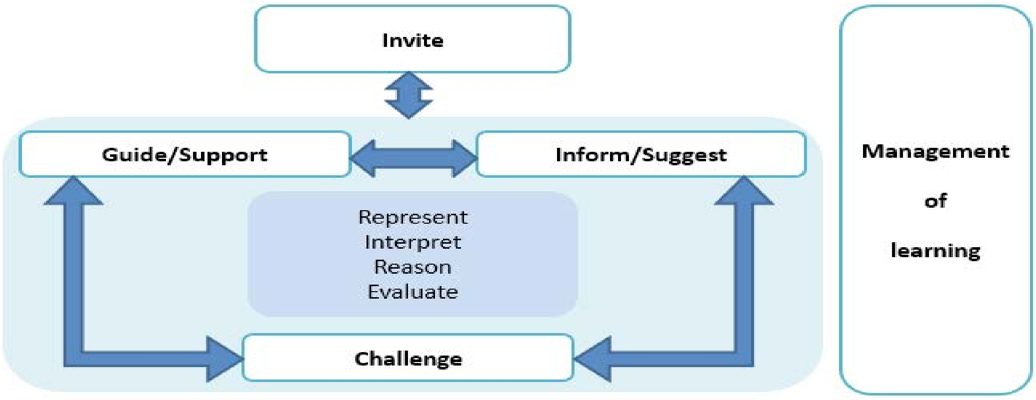

Figure 1: Framework to analyze teachers' actions (adapted from Ponte et al., 2013)

thinking: (i) eliciting actions that aims to elicit students' methods, (ii) supporting actions that aims to support students' conceptual understanding; and (iii) extending actions that aims to extend students' thinking. With a similar intent, Scherrer and Stein (2013) present a guide to analyze teachers actions (moves) during whole class discussion of cognitively demanding tasks that includes (i) begin a discussion, (ii) elaborate or deepen students' knowledge by furthering the discussion, (iii) elicit information, and (iv) other moves, like providing information and thinking aloud.

Teachers' actions during whole class discussions can be distinguished between actions directly related to mathematical topics and processes and actions that mainly relate to management of learning. Ponte, Mata-Pereira, and Quaresma (2013) identify four main categories related to mathematical aspects: (i) inviting actions – leading students to engage in the discussion, (ii) guiding/supporting actions – conducting students along the discussion in an implicit or explicit way in order to continue the discussion; (iii) informing/suggesting actions – introducing information, providing an argument or validating students' interventions; and (iv) challenging actions – leading students to add information, provide an argument or evaluate an argument or a solution. Guiding/supporting, informing/suggesting, and challenging actions, are the main support to develop whole class mathematical discussions, and involve key mathematical processes such as (i) representing – provide, revoice, use, change a representation (including procedures), (ii) interpreting – interpret a statement or idea, make connections, (iii) reasoning – raise a question about a claim or justification, generalize a procedure, a concept or a property, justify, provide an argument, and (iv) evaluating – make judgments about a method or solution, compare different methods. A model that relates teachers' actions in whole class mathematical discussions and these mathematical processes is presented on Figure 1.

RESEARCH METHODOLOGY

This study follows an interpretative and qualitative methodology. It is developed in a grade 9 class of a teacher with 13 years of experience. According to the teacher, the class has a very productive working environment and fruitful whole class discussions are possible because some of the students have a high performance and all of them are usually eager to participate. Data collection was conducted by the first author in the classroom and used direct observation (with audio and video recording of class) and document collection. Data analysis is based on the categories of the model presented in Figure 1, and is done with the support of NVivo software.

The situation analysed is a whole class discussion that aims to introduce how to solve incomplete 2nd degree equations generically represented as $ax^2 + bx = 0$. Before this situation, the students solved and discussed a problem that, implicitly, involves the zero product rule, but that was not explored with the aim of generalizing this rule to solve incomplete 2nd degree equations. Moreover, the teacher reminded how to solve incomplete 2nd degree equations as $ax^2 = 0$ and $ax^2 + c = 0$, that students had worked on in the previous lesson.

THE WHOLE CLASS DISCUSSION

We present some of the segments of the whole class discussion in order to analyse the teacher's actions to promote students' interpretation of the issues, reasoning and evaluation of solutions.

The first invitation

The teacher begins by writing down on the blackboard the equation $4x^2 - 12x = 0$ and, right after, she invites the students to solve this equation:

Teacher: . . . You have this equation, OK? An incomplete one, we are dismissing *c*. Who want to try to solve this equation with me? Just with what you already know. You already know how to solve 1st degree equations, how to solve some 2nd degree equations. I could tell you: Let's try to solve this equation. Irina, what would you do?
Irina: I would... Probably, I would get minus 12 to plus 12 in the other side [of the equal sign].
Teacher: So, I will write down [on the blackboard], $4x^2$ equals $12x$.
Irina: Exactly.
Teacher: It is true, this equation [$4x^2 - 12x = 0$] is equivalent to this one [$4x^2 = 12x$].

In this introduction, the teacher begins by inviting students to solve an equation. She also prompts students' answers by challenging them to make connections with their previous knowledge about equations. As Irina begins solving the equation, the teacher informs students by representing on the blackboard what she said. As this is the first time that students are solving this sort of equation, the teacher also suggests that Irina is going in the right direction by stating that both equations are equivalent. This suggesting action by the teacher leads Irina to keep going on her proposal to solve the equation:

Irina: Then, I would do the same procedure as usual. I would do squared *x* equals $12x$ divided by 4.
Teacher: [Writes down $x^2 = 12x/4$] Perfectly equivalent. Five stars. [Waits but Irina stands quiet.] Now the question is, when I am solving an equation, what is the aim?
Students: Find the value of the unknown.
Teacher: [Questioning Irina] Did you found the value of the unknown?

Once more, the teacher represents on the blackboard what Irina said and suggests that she is moving forward on her solving process. As Irina seems not to know how to continue, the teacher decides to highlight the aim of solving an equation, aiming to guide students on moving forward.

In this segment, the teacher's actions are marked by challenging the students on making connections between previous knowledge and this new situation. As students begin solving the equation, the teacher mostly uses guiding and suggesting actions to lead students to achieve the envisioned aim of using previous knowledge. Despite the aim of making connections, these guiding and suggesting actions are mostly representing actions, allowing students to stay focused on the solving process.

Overcoming misconceptions

As Irina does not know how to continue the solving process, the teacher selects Hugo from the students who want to participate:

Hugo: x^2 less x equals zero less 4 plus 12.
Teacher: Is that possible?
Students: No.
Teacher: Why? He would do this [writes down on the blackboard $x^2 - x = 12 - 4$]

Despite suggesting before that Irina was going in the right direction, the teacher does not suggest that Hugo is presenting an invalid proposal. Instead, she challenges the class to interpret Hugo's intervention and to further justify that interpretation. The teacher represents Hugo's proposal on the blackboard alongside with Irina's. Then, the teacher upturns the validation of Hugo's answer and, to do so, she suggests the students to establish connections with the monomial operations:

Teacher: Is this possible? This expression [$4x^2 - 12x = 0$] is equivalent to this one [Hugo's equation]? Remember that this is like a whole, a monomial, I cannot disconnect the coefficient from the literal part. I do not have something like this Hugo.

Towards this, Hugo presents another way to solve the equation, letting no time to his colleagues to justify why his first proposal was invalid:

Hugo: So... What if I add the x from 12 to x^2?
Teacher: So, first of all, can I add this two small monomials [x^2 and $12x$]?

Students: No!
Guilherme: You can.

Once more, the teacher challenges the students to interpret Hugo's statement, aiming to clarify the properties of operations with monomials. Most students seem to know that one cannot add two monomials with different degrees, but Guilherme states the opposite. So, the teacher promotes the analysis of this mistake, indicating that he can continue the solving process:

Teacher: So, do it.
Guilherme: $4x^2$ is the same has having $4x$ times $4x$.
Teacher: Is it?
Students: No.
Teacher: Guilherme asked if $4x^2$ is the same as having $4x$ times $4x$.
Students: No.
Teacher: Why not?

As Hugo, Guilherme proposes an invalid procedure, so, the teacher keeps challenging students to interpret his proposal and, to support students, she revoices Guilherme's statement. Again, the teacher challenges the students to justify their answer, but Guilherme realizes his mistake and corrects it right away:

Guilherme: But we could get, for example, $2x$ times $2x$, $4x^2$. And then, we would get $2x$ times $2x$ less $12x$ equals zero.
Teacher: OK, to that point it is OK. I will write down for you. $2x$ times $2x$, less $12x$ equals zero. It is also equivalent, OK?

With this new statement, the teacher suggests that Guilherme is doing a right procedure, arguing that this equation is equivalent to the previous ones. She also informs students about Guilherme's proposal by representing it on the blackboard. Then, despite being asked to keep going on his solving process, the student does not know what to do. So, in order to guide Guilherme, the teacher revisits the aim of solving an equation, which allows him to continue:

Guilherme: Not to stand again $2x$ times $2x$, after one would get... I do not know if I could divide by k. $2x$ less $12x$ equals zero divided by $2x$.
Teacher: This one here [$2x$] is multiplying the whole expression?
Students: No.
Teacher: When I divide, when I move to here dividing by $2x$, it means I divided the whole expression.

As the new proposal of Guilherme is not valid, the teacher suggests a reinterpretation of his statement and recalls the use of an equivalence principle of solving equations in order to show that he cannot do the procedure he is proposing.

In this segment, the students did several wrong procedures, highlighting that they do not have yet a mastery of monomial and polynomial properties. These wrong procedures led the teacher to challenge students to make connections with polynomial operations. As students have some knowledge about this topic, the teacher keeps using challenging actions instead of suggesting and guiding actions as she did in the previous segment. Nevertheless, when needed, the teacher uses suggesting and some guiding actions to move the discussion forward. All these actions are mostly around interpreting and reasoning about that interpretation.

An unexpected proposal

As before with Irina, Guilherme cannot move forward on solving the equation, so, the teacher invites another student to continue:

Teacher: Maria.
Maria: Teacher, getting back to that one [referring to $x^2 = 12x/4$].
Teacher: Getting back to this [pointing out the equation on the blackboard].
Maria: Is it possible to do x^2 divided by x equals 12 dived by 4?
Teacher: Oh, Maria did this... Have you seen what she has done?
Students: Yes.
Teacher: I am going to do what she has done, this x [on the left side of the equal sign]... She wrote this, I will continue [writes down $x^2/x = 12/4$].

As Maria begins where Irina stopped, the teacher guides students to focus their attention by revoicing and pointing out what Maria is referring to. As Maria presented her proposal, the teacher informs students by representing it on the blackboard. Before moving forward, a student states that Maria cannot separate 12 from x in $12x$, which leads to a segment of the discussion very similar to the segment *Overcoming misconceptions*. After clarifying some monomial properties, the teacher asks Maria to proceed with her way to solve the equation:

Teacher: And now what? Maria, I stopped here, what's next?
Student: x equals 3.
Teacher: In the other lesson we didn't get to this point. Now I am curious. Look here. If I would go this way, x^2 divided by x...
Students: It's x.
Teacher: Equals...
Students: Three!
Maria: So, it is possible!

Maria does not get to answer as there is a colleague that anticipates the response. The teacher asks again the question, suggesting students to develop the representation of the equation made by Maria. At this point, several students are following Maria's reasoning and completed solving the equation with no further support from the teacher. Thereby, Maria concludes that her way to solve the equation is valid. Nevertheless, the teacher explores the situation a little bit further by challenging the students to interpret the situation and then to justify their interpretation:

Teacher: Is it possible?
Students: It is.
Teacher: How can I see if the solution is correct?
Student: Replacing.

As a student readily answers how to justify that the value they found is a solution of that equation, the teacher suggests most of the computational procedures needed:

Teacher: So, let's do it slowly. Do it with me, slowly. Jorge, can you do it slowly? Here is 3 [referring to x]. x^2...
Lourenço: It's 9.
Teacher: 9 times 4...
Lourenço: 36.
Teacher: 36. 3 times 12...
Students: 36.
Teacher: 36 less 36...
Students: Is going to be zero.

This segment develops around Maria's proposal that emerge from the teacher's first challenge to solve the equation through making connections with previous knowledge. As students engage in the discussion about Maria's proposal, the teacher mostly suggests procedures that lead students to find a solution of the equation and, later, to justify that solution. In between, the teacher challenges students to justify that the value found is an equation's solution, which students easily do.

Comparing solutions

At this point, a student, by trial and error, figures out that zero could also be a solution, which led the teacher to introduce a discussion aiming the generalization of the zero product rule. This segment of the discussion had a first section where students factorized the proposed equation and a second section where the students figured out its solutions. Then, the teacher proposed some problems involving the zero product rule to students to solve in small groups. Previously to the discussion of those problems, the teacher recalls Maria's solving process:

Teacher: Earlier, I'll call it now "Maria's process". By Maria's process how many solutions have we found?
Student: That has one [solution].
Teacher: One solution. But when we did it by factorizing, we got two. My question is... We saw, we confirmed, that actually this equation has two solutions. The question is, what happened in this process, why zero does not emerge as a solution? This would be . . . x equals zero or x equals 3, two possibilities. Here [in Maria's process] only emerges x equals 3. Suddenly it looks like a solution has been hidden. Luís readily said that there was another despites this one . . . Why doesn't the solution show up here? [No student reacts.] Why did I found here x equals 3 and there I found out two solutions, zero and 3?

The teacher begins by saying what is going to be discussed, challenging students to evaluate the two processes of solving the equation. Mário says that solving using the zero product rule is the way to "obtain all possibilities", so the teacher guides the students to interpret Maria's process. But the students centre their attention on zero, referring to zero on the second member of the equation $4x^2 - 12x = 0$. At some point, Mário asks the following question:

Mário: Teacher, a question. Is zero the second solution in equation that... c equals zero [referring to c in $ax^2 + bx + c = 0$]

Teacher: That is a very good question. But my question... That is really a good question, it is very relevant. But my question here is why doesn't zero appear here? Before we go to that one. Why doesn't it appear here? And there it appeared.

Mário's intervention represents a very interesting generalization about zero being a solution of any 2nd degree equation like $ax^2 + bx = 0$. The teacher decides not to explore it but to move back to Maria's solving process, challenging students to get some justification on the lack of one of the solutions in Maria's process. A student gets back to the zero rule process' solution but the teacher guides students again to focus on Maria's process. Hélder intervenes:

Hélder: Because one cannot divide by zero.

Student: Yes you can!

Teacher: [Writes a huge 2/0] Question to the class...

Student: What a huge 2!

Teacher: It really is to be seen. Hélder said, here does not appear zero as a solution because Maria, when did this step, divides by x. And, if I place here a zero, what Hélder is saying is that it doesn't make sense to place here a zero. My question is, why? This equals to what?

Student: Zero.

João: Undetermined.

As Hélder's answer to the teacher prior challenge lead to a disagreement, the teacher challenges students to justify if Helder's statement leads to a justification or not. This discussion leads the students to conclude that it is not possible to divide by zero, as Madalena states ("Even so, no number can be divided by zero") and the teacher supports reinforcing ("No number can be divided by zero).

As the evaluation of the solving processes relies on the reason why Maria's process leads to just one solution, and that reason is justified, the teacher ends this whole class discussion.

In this last segment of the discussion, the teacher main action is challenging students to evaluate both solving processes. To promote students' activity towards evaluation aim, the teacher guides the students to focus on Maria's process and challenges them to justify the single solution it yields. In this segment there are few informing actions, and teacher's challenging actions lead to many students' reactions.

CONCLUSION

In this whole class discussion, we identify a variety of teachers' actions. We note that each segment is structured by a main action. As this class discussion aims to introduce new knowledge, one could expect that central actions would be mostly guiding, subtly or explicitly leading students to a solving procedure using the zero product rule. Nevertheless, attending this class characteristics, the teacher begins all four segments with a challenging action leading to quite unpredictable students' responses. This teachers' choice provided rich learning opportunities that would probably not occur if guiding would be the central action in each segment. We also note that, albeit marked by a main challenging action, all segments differ in their nature. Teachers' actions seem to depend on students' knowledge about the situation, as challenging actions are followed by other challenging actions when the students have the tools to embrace those actions and are followed by guiding or suggesting actions when the mobilization of students' knowledge needs further support.

Regarding mathematical processes, the teachers' central actions mostly rely on interpreting processes. Nevertheless, more demanding processes also occur as in *Comparing solutions,* where challenging evaluating actions assume great importance. As with the actions themselves, the mathematical processes addressed also depend on students' knowledge. New situations to students require a focus in representing and interpreting processes, while segments where students have some previous knowledge to rely on, reasoning and evaluating processes are more likely to occur, as in *Overcoming misconceptions* or *Comparing solutions*.

In summary, the model used to analyse teachers' actions (Figure 1), extending the previous models of Stein and colleagues (2008) and Cengiz, Kline and Grant (2001), contributed to better understand teachers' practice. Also, an interesting relationship between teachers' actions, mathematical processes and the situation itself emerged from the analysis with this model, which suggests the need of further research of teachers' practice during whole class discussions focusing on these features.

ACKNOWLEDGEMENT

This study is supported by FCT–Fundação para a Ciência e Tecnologia through grants to Joana Mata-Pereira (SFRH/BD/94928/2013) and Marisa Quaresma (SFRH/BD/97702/2013).

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Conducting mathematical discussions as a feature of teachers' professional practice

João Pedro da Ponte and Marisa Quaresma

Universidade de Lisboa, Instituto de Educação, Lisbon, Portugal, jpponte@ie.ulisboa.pt

We analyse teacher's actions during whole class discussions in exploratory classes (in which students are asked to design their own strategies) and their relation to students' learning. Data is collected through participant observation, with videotaping of lessons. The results show that the exploratory approach favours the emergence of disagreements among students and their formulation of generalizations and justifications provided that the teacher intertwines guiding and suggesting actions and makes appropriate challenging actions at key points. In such whole class discussions, the teacher has to make important decisions in relation to problematic situations raised by students' difficulties or unforeseen responses as well as by the need to figure out productive ways of continuing a discussion.

Keywords: Teacher practice, mathematical discussions, communication, reasoning.

INTRODUCTION

An exploratory approach to mathematics teaching seeks to propose students situations where they have to deal with tasks for which they do not have an immediate solution method or in which a new representation, concept or procedure may be useful. This approach creates opportunities for students to build or deepen their understanding of concepts, representations, procedures, and mathematical ideas. The students are called to play an active role in interpreting the questions proposed, in representing the information given and in designing and implementing solving strategies which they are called to present and justify to the whole class. This teaching approach is based in the fundamental distinction between task (the objective to be achieved) and activity (the work to be done to achieve this goal) (Christiansen & Walther, 1986). The work on exploratory classes develops usually in three phases (Ponte, 2005): (i) presenting and interpreting the task; (ii) carrying out the task individually, in pairs, or in small groups; and (iii) presenting and discussing results and doing a final synthesis.

In this study, we focus our attention in the work of the teacher in leading whole class discussions, in which students present and justify their solutions and question the solutions of their colleagues. We do not seek to establish a normative framework, saying what the teacher "must" do, but rather to analyze the phenomena that take place in the classroom, in order to understand the situations that occur and the actions that the teacher can do to promote students' learning. As students carry out exploratory work, the diversity of situations that may arise is very large and depends on the age level of the students, their mathematics ability, the culture of the classroom, and the mathematical topics under study. In addition, one must keep in mind the influence of other factors such as teachers' and students' concerns about assessments, school guidelines on curriculum management, textbooks and other resources available, physical conditions of the room, etc. In this way, our study has essentially an analytical stance, aiming to examine the diversity of actions that the teacher is called to undertake in whole class discussion moments and their relation to student learning.

THE DYNAMICS OF DISCUSSION MOMENTS IN THE CLASSROOM

Teachers' practices have an important influence on students' learning (Ponte & Chapman, 2006). An important aspect of such practices is the nature of the tasks that the teacher proposes to their students. If a task only requires students to select and apply a solution method that they already know, they have just to identify and carry out this method. By contrast, a task with challenging features (Ponte, 2005) or involving a high cognitive demand (Stein, Remillard, & Smith, 2007; Stein & Smith, 1998) may lead to a diversity of strategies that can be compared and evaluated, resulting in interesting classroom discussions.

Another aspect that frames teachers' practices is the nature of the classroom communication (Bishop & Goffree, 1986; Franke, Kazemi, & Battey, 2007). A fundamental aspect of communication are the questions posed by the teacher. Among these, inquiry questions that admit a range of legitimate responses are particularly useful. In addition, another important feature of classroom communication is the process of negotiation of mathematical meaning (Bishop & Goffree, 1986), leading students to make new connections among mathematics ideas, and helping the teacher to recognize their sometimes unforeseen points of view. Franke, Kazemi, and Battey (2007) stress the importance of processes that support students' language development, like revoicing. Whole class discussions provide opportunities for particular forms of communication, such as explanations and argument and are attracting a growing interest of mathematics education researchers (Bartolini-Bussi, 1996; Cengiz, Kline, & Grant; 2011; Fraivillig, Murphy, & Fuson, 1999; McCrone, 2005; Scherrer & Stein, 2013; Sherin, 2002; Stein, Engle, Smith, & Hughes, 2008; Wood, 1999).

The teacher role is to prepare the moment of discussion, taking into account the work carried out by the students and the class time available. In order to do this, Stein, Engle, Smith and Hughes (2008) highlight the importance of anticipating how students might think, to monitor their work, to gather relevant information, to select aspects to note during the discussion, to sequence the students' interventions and to establish connections among the different solutions during the discussion. A preparation made under these conditions is an important support for conducting a discussion. However, the actual development of a discussion involves other issues beyond the establishment of connections. Many of these issues cannot be fully predicted prior to the discussion, but create problems that the teacher must be prepared to face. As Sherin (2002) indicates, the teacher needs to be able to balance aspects relating to mathematics knowledge, which requires filtering ideas focusing students' attention in fundamental ideas, and aspects related to mathematical processes that require a frequent attention.

Seeking to identify situations of particularly productive discussions, both Potari and Jaworski (2002) and McCrone (2005) emphasize the value of challenging students mathematically. Wood (1999) underlines the potential of exploring disagreements among students, as teachers lead them to justify their positions and encourage other students to join the discussion. Fraivillig, Murphy and Fuson (1999) and, subsequently, Cengiz, Kline and Grant (2011) developed a framework of analysis for the teacher's actions in conducting mathematical discussions that distinguishes three main types of actions: (i) eliciting actions, to lead students to present their methods, (ii) supporting actions, to promote their conceptual understanding, and (iii) extending actions, to widen or deepen students' thinking. In another study, Scherrer and Stein (2013) developed an intervention to support teachers in analyzing whole class discussions based in four main coding categories of moves: (i) those that begin a discussion; (ii) those that further the discussion by elaborating or deepening students' knowledge; (iii) those that elicit information; and (iv) other moves.

With a similar intent, Ponte, Mata-Pereira and Quaresma (2013) developed a framework that assumes that the teacher performs actions directly related to the topics and the mathematical processes as well as actions that have to do with management of learning (Figure 1). Focusing their attention on actions related to the mathematical aspects, they point out that inviting actions are used to start a discussion and guiding actions allow leading students on solving a task through questions or observations that implicitly point the way forward. In informing/suggesting actions the teacher introduces information, presents arguments or validates students' answers. Finally, challenging actions seek to lead students to produce new mathematical knowledge. In informing/suggesting, guiding, and challenging actions it is possible to identify fundamental aspects of mathematical processes such as (i) representing (constructing, using, or transforming a representation), (ii) interpreting, including the establishment of connections, (iii) reasoning, including formulating a strategy to achieve a goal, producing a statement, generalizing procedure and justifying, and (iv) evaluating, making judgments about a concept, representation, or solution. A generalization may concern a definition, a statement or a procedure and a justification may be informal and related to the context of the situation or more formal as is the hallmark of mathematical work.

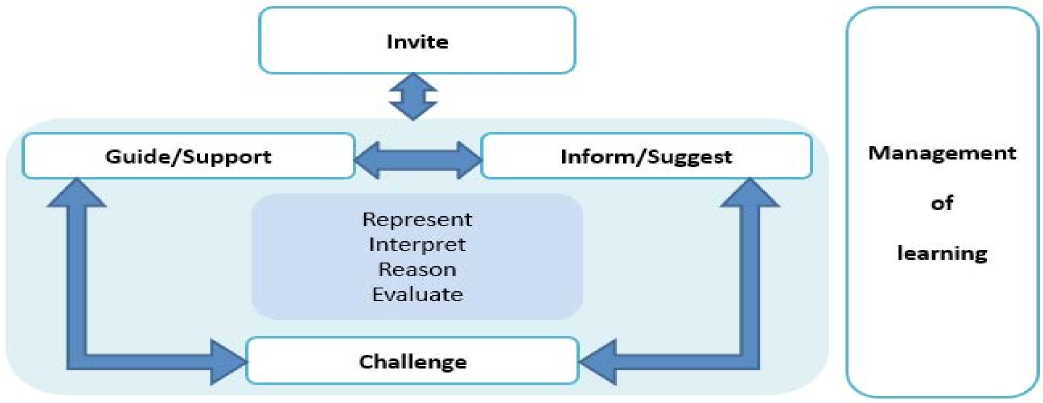

Figure 1: Framework to analyse teachers' actions (adapted from Ponte et al., 2013)

RESEARCH METHODOLOGY

This study follows a qualitative and interpretive approach (Denzin & Lincoln, 1989) using participant observation. Both authors assumed the role of teachers (striving to follow an exploratory approach) and researchers – as one conducted the class, the other acted as a participant observer. The grade 6 class, with 19 students, is in a rural elementary public school, in a deprived area. The students, usually, show little commitment to school activity and do not get themselves much involved in working in the mathematics class. The study involves five 90-minute lessons, in which students carried out several tasks presented in three worksheets. The first worksheet included diagnostic questions on comparing, ordering, adding and subtracting rational numbers, the second aimed to introduce the multiplication of a natural number by a fraction and the multiplication of two fractions, and the third was intended to develop the notion of operator in the context of problem solving. After the introduction of the task, the students began by working in pairs and the teacher monitored their work, helping them to move on, when necessary, but striving to not provide direct responses to the questions stated in the task. Finally, there was a whole class discussion, in a register of dialogical communication (Ponte, 2005).

The classes were recorded on video and the whole class discussions were integrally transcribed. Data analysis began by identifying the segments in the discussion of the solution of each task, coding the teacher's actions according to the categories shown in Figure 1. Then, we sought to establish relationships between these actions and specific events as regards interpretations, representations, and reasoning made by the students. For this paper, we selected two episodes that illustrate several aspects of these relationships.

DEALING WITH STUDENTS' DIFFICULTIES

In this episode we find two rather common situations in the classroom: (i) students with difficulty in understanding a written mathematical question and, (ii) students with difficulty in expressing their thinking. We show a first situation with several guiding and suggesting actions but where a challenging action proves to be critical and a second situation in which guiding and suggesting assume an identical role and there is a very low level of challenging. This takes place as students work on a task in a mathematical context (Figure 2) that asks to evaluate the validity of a statement involving two fractions. This task is a problem that requires the students to figure out that they either must find counterexamples or justify that the statement is always true.

Task 1. $\frac{2}{4}$ is larger than $\frac{1}{3}$, $\frac{4}{5}$ is larger than $\frac{3}{4}$. Do you think it is possible to make the following statement: "If we want to compare two fractions and we verify that one of them has the numerator and the denominator larger than the other fraction, we can immediately conclude that this is the larger fraction"? Justify your answer.

Figure 2: True or false? (task based on Lin & Tsai, 2012)

As students begun working individually, they immediately show difficulty in understanding what is asked in the question and the teacher realizes the need to promote a whole class interpretation of the statement and in helping students find a solution strategy:

Teacher: The two cases are true. (...) OK, this and this [$\frac{2}{4} < \frac{1}{3}$ and $\frac{4}{5} < \frac{3}{4}$] are true. May I always say that whenever the numerator and the denominator of a fraction are larger than the numerator and the denominator of the other fraction, then [the fraction] which has larger numerator and denominator is always larger than the second [fraction]? Does this always happen?

A student: No...

Teacher: How can you know if it always happens or not?

Daniel: Doing more fractions...

Teacher: Finding more examples... It may be a good suggestion from Daniel...

The teacher recalls the main aspects of the statement ("this and this are true") and then makes a more general statement ("when the numerator and the denominator of a fraction are larger than . . ."). The students realize that the statement is true in some cases but have difficulty in knowing what to do to know whether, in general, the statement is true or not. The teacher makes an inquiry question ("How can you know if it happens always or not?") and this leads Daniel to suggest a promising strategy. The teacher supports this idea and she revoices it formally in more appropriate terms.

In this first segment, as the students show difficulty in finding a strategy to answer the question, the first intervention of the teacher helps them to interpret the statement and is a guiding action, which is followed by an inquiry questiona challenging action. The final intervention, supporting the proposal of Daniel, is a suggesting action. The emphasis of the teacher's intervention is in interpreting (the task and its different elements) as a basis to support students' reasoning as the aim is knowing and justifying whether a given statement is mathematically valid or not.

In the sequence of this exchange, during the whole class discussion of the results, Guilherme, presents a counterexample for the statement, identifying fractions that satisfy the given conditions ($\frac{2}{4}$ and $\frac{3}{16}$) but which do not verify the inequality. To compare these fractions he converts them in percent. The teacher considers interesting that the whole class observe this solution, but some students do not understand it:

Teacher: So, Guilherme found... a way to transform fractions in percent and... He found that $\frac{2}{4}$ is 50%, right? And what did you find about $\frac{2}{4}$, $\frac{2}{4}$ was how much? Some of you discovered... Changed the fraction into a decimal...

Jaime: It was 0.5.

Teacher: Oh! It was 0.5. And so, 0.5 in percent is...

Guilherme: It is 50%.

Teacher: Oh! It is 50%. Oh, so that means that he arrived to the same conclusion as you did, but using a different representation . . . So Guilherme prefers to work with percent . . . So, he discovered that $\frac{2}{4}$ is 50% and $\frac{3}{16}$ is 18,75%... Why have you done this, Guilherme?

The teacher leads the students to compare the representations used by Guilherme (percent) and by the remaining students (decimal). With a final question, the teacher seeks to guide Guilherme, supporting him in explaining his solution.

In this second segment the teacher asks a student to present his solution, which she found to have a remarkable originality, but is faced with the problem that the student has great difficulty in explaining his reasoning. As many students are confused, the teacher's actions alternate between guiding and suggesting, with no challenging actions. There is much attention to representations and their transformations (converting between decimals and percent) but the focus of the teacher's interventions is in interpreting, revoicing the student's statements in a more understandable and correct way, in order to allow an interpretation and understanding of the other students in the class.

CHALLENGING STUDENTS

Next we show a situation that begins with a teacher challenge which is then followed by inviting, guiding and informing/suggesting actions, which lead the students to establish a first generalization connecting multiplication of an integer by a fraction with successive addition of fractions and a second generalization that highlights an understanding of equivalent fractions. It takes place when students work on the task shown in Figure 3 that asks for the value corresponding to seven repetitions of a certain magnitude in a contextualized situation. The students had not yet learned to multiply a whole number by a fraction. It was expected that they would solve the task through repeated addition, perhaps proposing by themselves a definition for the multiplication of a whole number by a fraction.

> **Task 2.** In the grade 6 class of the school Wide Horizon, the teacher made the following problem: "Every morning, Raquel drinks $\frac{1}{4}$ of liter of milk. How much milk does she drink in a week?" You must solve the problem yourself and justify your answer.

Figure 3: Task involving the multiplication of a natural number by a rational number

Two students solved the task using repeated addition of seven equal fractions. However, in their solution they wrongly indicate that $\frac{1}{4}+\frac{1}{4}$ is equal to $\frac{2}{8}$. During the whole class discussion, in a first segment, the teacher decides then to question how much is $\frac{1}{4}+\frac{1}{4}$. The students indicate several answers, some correct, such as $\frac{2}{4}$ and 0.50, and some incorrect such as $\frac{1}{8}$ and $\frac{2}{8}$. To guide the students in distinguishing among correct and incorrect answers, the teacher draws a pictorial representation (a rectangle divided in four equal parts) and asks again the students what will be the response.

Daniel, who had already presented a response to the question $\frac{1}{4}+\frac{1}{4}$ as $\frac{2}{4}$ and as a decimal (0.50), suggests a new answer, using the equivalent fraction $\frac{4}{8}$. The teacher notes that the student is thinking in fractions equivalent to , decides to validate his solution and asks for a justification (a challenging action):

Teacher: Exactly, $\frac{4}{8}$ would be also an answer. Why? Why is $\frac{4}{8}$ equal to $\frac{2}{4}$?...
Guilherme: Because it is 0.50.

Guilherme's justification is based on a change of representation. At this moment teacher decides to take the opportunity to recall equivalent fractions, emphasizing the relationship that exists among $\frac{2}{4}$, $\frac{4}{8}$ and $\frac{1}{2}$, and this leads to a new discussion segment.

Driven by the intervention of Guilherme and the suggestion of the teacher, Edgar suggests another equivalent fraction. The following dialog takes place:

Edgar: Oh! Teacher, I know another... 8 divided by 16 also does it!
Teacher: Also does it... $\frac{8}{16}$ also does it... Very good... Any other that also does it?

As two students (Juliana and Edgar), in the last class, had also made an interesting discovery related to this issue, the teacher encourages them to indicate it. Juliana corresponds to this invitation, stating a generalization:

Juliana: A number divided by its double will always yield its half."

The teacher challenges then the students to give more fractions equivalent to $\frac{1}{2}$, and they correspond in an enthusiastic way:

Teacher: Very well... $\frac{2}{4}$ is equal to $\frac{1}{2}$ that is equal to $\frac{8}{16}$... And I want another one!
Rui: So, now 16 divided by 32...
Teacher: $\frac{16}{32}$. And I want still another one...
Students: 32 by 64.
Teacher: Ah... Very good, $\frac{32}{64}$. Still another...?
Students: 64 and 128...

Other students join the discussion and suggest more fractions equivalent to $\frac{1}{2}$. The teacher supports this enthusiasm, revoices their suggestions using a correct fraction language and challenging them to find other fractions that satisfy the same condition.

In summary, at the beginning of this episode several students show that they do not recall the procedure to add two fractions with the same denominator. The teacher seeks to lead them to understand the rule to add two unit fractions, using for that purpose a pictorial representation. When all agreed that $\frac{1}{4}+\frac{1}{4}=\frac{2}{4}$, and assuming the opportunity provided by the fact that different correct responses were already provided, the teacher begun challenging the students to provide justifications regarding equivalent fractions and to find further equivalent fractions. In this episode, the teacher's most important actions are challenging, although one recognizes inviting, guiding and informing/suggesting actions as well. Starting from a simple procedural question, the teacher ends up leading an inquiry-oriented reasoning, with the establishment and use of a generalization to produce equivalent fractions.

CONCLUSION

This paper shows how teachers' actions may unfold during whole class discussions conducted within an exploratory approach. In the first episode the teacher seeks to support students in interpreting a written mathematical statement and leads those who solved a question correctly to explain it to their colleagues. The teacher provides some challenge but uses mainly guiding and suggesting actions, without indicating the solution to the students. Drawing on counter-examples, the teacher seeks to make the elements available to the whole class so that the students can figure out that the statement is false. In the second episode, after some work on pictorial representations to figure out a correct answer, the teacher challenges the students to present more answers, seeking the emergence of disagreements and, in response, the students produce a sequence of equivalent fractions. That is, in both episodes challenging is a critical action (Potari & Jaworski, 2002; McCrone, 2005) but needs to be underpinned by other types of actions. The way the teacher intertwines guiding and suggestion actions and makes appropriate challenging actions at key points is critical to foster students' involvement and to achieve the learning goals, notably, (i) when students present promising conjectures, (ii) when there is room for important justifications, and (iii) in situations that may prompt fruitful conjectures from the students. In addition, both episodes show how the teacher may promote the interconnection of representing and interpreting and create opportunities to foster students' reasoning, notably asking them for generalizations and justifications (Ponte, Mata-Pereira, & Quaresma, 2013).

These whole class discussion moments provide many opportunities for interpreting statements and using representations (Bishop & Goffree, 1986), for improving the students' language revoicing their claims (Franke, Kazemi, & Battey, 2007), for establishing disagreements (Wood, 1999), and formulating generalizations and justifications (Lannin, Ellis, & Elliot, 2011). However, whole class discussion moments also create many problems for the teacher, requiring the ability to deal with unforeseen situations and to notice opportunities for promoting students' learning (Scherrer & Stein, 2013). The discussion episodes presented in this paper include many moments in which the teacher needs to make decisions with respect to different situations, which are constituted as problems that she has to deal with in the course of the action. Some of these problems have to do with students' difficulties in understanding what they can do in a proposed task or in interpreting some aspect of a solution provided by another student. Other problems arise from unexpected responses from students, sometimes correct and other times incorrect. There are also problems which arise from the students' difficulty in explaining their reasoning. Finally, other problems arise from the need to manage, in a productive way, the range of students' responses and in keeping an appropriate pace for the classroom work. That is, besides planning the discussions and anticipating possible students' difficulties (Stein et al., 2008), the teacher must be ready to make important decisions in relation to problematic situations raised by students' difficulties in understanding the tasks, in figuring out strategies, in expressing themselves and by unforeseen students responses. In addition, at many points the teacher needs to figure out what is the most productive way of continuing a discussion. Such problems that conducting whole class discussions raise to teachers' practice creates an important agenda for research concerning mathematics teacher professional development.

ACKNOWLEDGEMENT

This study is supported by FCT – Fundação para a Ciência e Tecnologia through Project PEST (PEst-OE/CED/UI4107/2014) and also through a grant to Marisa Quaresma (SFRH/BD/97702/2013).

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Mathematics assessment, competition and professional capital

Nick Pratt

Plymouth University, Plymouth, UK

In this paper, I use interviews with two teachers, drawn from a small scale study of four schools, to illustrate the way in which mathematics assessment is used as a form of capital in teachers' wider professional lives and the effect that this is likely to have on teaching practices in mathematics. Assessment is high-stakes for English schools and outcomes are increasingly measured numerically in terms of annual 'pupil progress'. I use an analysis based on Bourdieu to illustrate how this affords competition between teachers for grades and potentially brings them into conflict with each other. I argue that the effect of this kind of professional capital is likely to be unhelpful in relation to what is known to be effective use of assessment in mathematics teaching.

Keywords: Assessment, Bourdieu, professional capitals, competition.

INTRODUCTION

The focus of this work is mathematics assessment from the perspective of teachers in English primary schools (5–11yrs) in relation to the significant part it plays in their teaching. This is set in the context of massive investment in mathematics education over the last 15 years, but with modest improvement and concern that English pupils are not keeping up with their international peers (e.g., Gove, 2013). Research (e.g., Boaler, 2005) is clear that one vital issue in effective learning of mathematics is that teachers, and pupils themselves, see the potential for learning as flexible and mutable; and that its opposite – perceiving it as objectified and immutable – leads to teaching practices which hold learners back. Pupils should be involved in understanding what they can and cannot *yet* do and involved in mathematics as a process of exploring interconnecting ideas. Despite this clarity, practices in English classrooms tend not to reflect it; a view of mathematical ability as a fixed characteristic of individuals, setting by such ability and teaching that is negatively differentiated so that weaker pupils (and the highest achieving) are not challenged mathematically, all prevail (Boaler, Wiliam, & Brown, 2000; Marks, 2014). This paper explores some of the reasons for this contradiction by trying to understand teachers' mathematics assessment practices from the perspective of professional status as a whole. In doing so it makes use of Alexander's definition of pedagogy as 'the act of teaching together with its attendant discourse' (Alexander, 2004, p. 11). Crucially, from this perspective, teaching practices can only be understood in relation to the politics of daily life within which teachers act; research which decontextualizes teaching and learning in this way is unlikely to affect practice in any significant manner.

THEORETICAL FRAMEWORK

Assessment judgments in the UK take place within a high-stakes environment of increasing competition at both school and individual level with quantified test outcomes being the key commodity for recognition of success and high status. Schools, and the individual teachers within them, are now judged on the basis of standardised numerical outcomes; ubiquitously referred to as 'data' by those who work with them. For anyone not associated with education in English schools it may be hard to appreciate just how much of a hold this assessment-driven culture has over the lives of those who work in them; to a large extent it dominates the way they think and talk about their work, with numerical assessment data being used as a proxy for the overall standard of education in a highly politicised landscape. The term 'progress' has been adopted by successive governments and policy makers to stand for the rise in pupils' attainment over time. Pupils' progress over stages of their educational life against subdivisions of National Curriculum levels has become the key measure by which schools are judged during inspection (Office for Standards in Education, 2012a) and schools can only be graded as 'outstanding' if the progress from Key Stage 1 (KS1 – 7 years old) to Key Stage 2 (KS2 – 11 years old) fits the approved pattern, regardless of the perceived quality of teaching during an inspection[1]. This means that the quality of teaching is understood to be validly and reliably represented by the measured progress. In turn this means that teaching *tends to get constructed backwards*; if the progress is not as much as it should be then teaching cannot be good, regardless of what is seen on the ground. Furthermore, the UK Coalition government has recently introduced new arrangements for performance-related pay for individual teachers (Hodgson, 2012) in which successful teaching gauged at an individual level is defined as all pupils making a specific number of points of progress across a year, regardless of circumstances. Assessment is therefore naturally very much part of teachers' everyday mathematics discourse and so, on the assumption that human activity is object-orientated (Vygotsky, 1978), this discourse and acts of teaching and learning mathematics will be mutually constituting in pedagogy.

To try to understand this relationship between assessment and teaching I draw on Bourdieu's notions of *field*, *habitus* and *capitals*. As Colley (2013) notes, though imperfect by Bourdieu's own admission the idea of the *game* as a metaphor is important in understanding how field and habitus interact with each other. The field (here the work of school assessment) is 'a space of conflict and competition constructed only through the human doings of human people' (p. 10) so that it 'is not only a set of external conditions which themselves have been devised or imposed ... [but] is also the agentic and partly subjective *playing* of the game through our *habitus*'. Field and habitus are not therefore separable, and certainly not representative of structure and agency respectively. Rather, they are mutually constitutive of each other, each produced through the other: habitus developed through repeated activity in the field; field developed through the playing out of activity through habitus. Thus, the question here is how the changing nature of mathematics assessment as a high-stakes, politicised element of schooling affects the ways in which teachers and pupils 'play out' schooling. In this respect, two other Bourdieusian ideas are relevant: *doxa* which refers to the dominant discourse of a field, separating what is thinkable from what is unthinkable; and *illusio*, a more conscious belief in the stakes (*enjeux*) of the game and the *belief that it is worth playing*.

METHODOLOGY

The project forms part of a wider study based in Plymouth University, UK, looking at the socially-constructed nature of teaching (see, e.g., Kelly, Hohmann, Pratt, & Dorf, 2013). It involves extended semi-structured interviews with primary teachers in four different schools (12 teachers in total) aimed at exploring their use and understanding of mathematics assessment (though inevitably, as non-subject specialists teaching the primary curriculum, teachers spoke about assessment in general too). In this sense it is a case study, exploring assessment as an aspect of the field of teaching and learning in the four case schools. Both teachers and schools were chosen purposively, the latter with different characteristics that might reasonably affect the way in which assessment takes place, including: inspection gradings; different arrangements of governance; and different proximities to competing schools. Data from all the interviews were analysed using a constructed, grounded approach following Charmaz (2006) and themes were developed. These themes are incorporated into the account that follows, but using examples from just two interviews, Tony and Sasha, to illustrate the claims, accepting that in such a short paper only a partial account of these themes is possible. Both interviewees are classroom teachers at the same urban school of 550+ pupils and have been teaching for 9 and 6 years respectively. Tony trained via a one-year postgraduate course and teaches Year 3 (7–8 yrs); Sasha trained on a four-year Bachelors course and teaches Year 5 (9–10 yrs). The school has had successful inspection outcomes over the last 15 years, based on attainment that is above the national average, but was recently downgraded from 'outstanding' to 'good' on account of progress not being consistent across the whole school. Such consistency, judged through assessment, is therefore a focus for all staff.

THE DISCOURSES OF SCHOOL ASSESSMENT

Despite the points about assessment used for institutional and professional monitoring made in the

1 At the time of writing the UK government is consulting on the removal of this system and its replacement by one in which pupils are judged against 'age-related expectations'.

introduction above, England has a long tradition of child-centred, progressive education and, more recently, assessment for learning in which the 'first principle of learning, is to start from where the learner is' (Hogden & Wiliam, 2006, p. 1). The inalienable nature of this as it relates to assessment is nicely exemplified by Tony who, right at the start of his interview is asked what 'role assessment plays in what you have to do to be a successful maths teacher'. He responds:

> Tony: Ok. For me the only [hesitating and thinking] thing assessment's for, obviously we're measured against it and the children are measured against it, but to be a successful teacher, for me, the only reason is to see what you need to teach the children and teach to their ability ... um ... and find those gaps and fill them and bring them on. ... what you want to do is only assess for what you are planning, what you are teaching.

But asked whether this is representative of how assessment is used in the school as a whole he declares:

> Tony: I think the trouble is that not everyone would agree with that. I think there would be some teachers who see assessment as a measure of progress, rather than assessment for learning. And I think all teachers would see those two things as part of assessment but where they put, where they see the emphasis might be slightly different.

In these two statements Tony describes the two contrasting discourses of assessment that were apparent in the data set, as well as the tension between old and new traditions. On the one hand assessment is formative, for teaching and learning; on the other it is a summative measure of progress. However, for both of these discourses there is also a question of who controls the information that assessment generates and therefore who makes use of it and what is expected from them. Children are certainly involved in this respect. Formative assessment is routinely used to set targets, usually in very systematic ways such that children know what they are meant to achieve next; and summative assessment outcomes are shared too so that my own son, for example, will routinely tell me that 'I am a level 5b'.

The language is interesting here as children are objectified 'as' their perceived ability level. Objectifying children in relation to 'their ability' has a considerable literature critiquing it (see, e.g., Boaler, 2008; Boaler et al., 2000; Cooper & Dunne, 2000; Muijs & Dunne, 2010), and the increasing emphasis on accountability seems to drive the need to become auditable, which in turn drives objectification of pupils as numbers on an assessment scale. Thus, in interviews we hear Tony describing the moderation of assessment, saying that the teachers in his year group 'looked at our level 4s'. Whilst this does not, of course, imply that teachers only see children in these terms, nor that there may not be benefits from doing so anyway, it does illustrate a creeping objectification of pupils in the discourse of schooling *as* their assessment level and runs counter to the research pointing to the value of a flexible view of learning potential. I argue that this is a necessary part of a system of high-stakes accountability since objectifying pupils as assessment levels enables them to be measured, and hence to create a narrative of 'progress' as also measurable. However, for this to be used to account for (and increasingly measure) teachers' work a second aspect of the narrative is necessary, which is the imposition of personal responsibility for this measure on teachers themselves. If one is to link pay to progress one must not only have a reliable measure of progress but must also believe that this is the result of the teacher's actions, so that these can be assessed and understood as the teacher-performance. So, for example, Sasha describes performance management meetings with the school's senior management team as 'a scary meeting', but says,

> Sasha: I've always quite looked forward to it. But again, I wonder if that's because I'm a successful teacher – it feels very strange saying that. If I'm a successful teacher, and if maybe I wasn't a successful teacher it would maybe be more worrying. And I know teachers in the past who've been hauled before heads or governors to kind of go 'your children have not made appropriate progress this year, why not?' ... the performance management cycle ensures that no-one's rubbish at their job.

In this light, aspects of the discourse that is created around school assessment include:

- The belief in a version of success defined in narrow, data-led terms
- The belief that progress is predictable and controllable across time
- The creation of official expectations of such progress
- Making individual teachers responsible for learning outcomes through their teaching

I argue that this discourse constitutes a *doxa* because as teachers spoke of their experiences in interviews these descriptions of assessment practice were generally 'granted [a] recognition that escapes questioning' (Bourdieu & Wacquant, 1992, p. 98). We see the attributes of control and predictability described here exemplified by Tony as he describes the typical shape of the graph of mathematics data for his class.

> Tony: Probably [the slope is] less gradual, very gradual, in the first term and then second term goes up a bit and then maybe curves over because you wouldn't go any higher than your SATs in May because you can't make any progress after that can you [laughs ironically]!

Note how the expectation on Tony and his own assessment practices constitute each other and control what is possible. Assessment does not just describe progress; it defines what is possible and what is not. Moreover, the doxa affords another shift, namely a move from assessment as a public good for the school, or more widely society as a whole, to also being a private good for the teacher. Teachers 'own' the data, in the sense that they both are made responsible for it and can make use of it as capital in their professional work. These are at work in the way Sasha describes mathematics assessment targets.

> Sasha: So the school target is something that everyone in the whole school is working towards. So whether that is a set amount of progress in a particular aspect of maths or just developing something, you kind of think about just that one thing. ... The class [assessment target] is progress of the children in your particular class or set and so again, obviously your assessment links up. And with the new pay and conditions thing, if you don't hit every one of your professional development targets you can't progress up the pay threshold...
>
> NP: Do you feel responsible for them [children's SAT scores]?
>
> Sasha: [immediately] Yes, really responsible, completely ... and some children you will be like, oh they were so close and then you feel a sense of disappointment for them and then you question whether or not you could have done more.

Moreover, this data is used and contested as capital by different parties.

> Tony: I mean like government need to be looking like they are doing a good job governing. It gets onto heads [headteachers] and they need to be monitoring that progress and they would say 'my school has made a lot of progress'. Deputies look at their key stage and say 'my key stage has done really well this year' [laughs]. And then teachers, 'my progress it is, it is' ... you're responsible for that data, so you have ownership over that data, but I suppose the stupid thing is ... is the children who are at the centre of it are the ones who are almost most removed from the talk of targets and data and .... Yes, it's ridiculous isn't it.

Again, Tony illustrates how the doxa of assessment works in opposition to the stated policy of assessment for learning with pupils at its heart.

ASSESSMENT: TEACHING AND MERITOCRACY

The culture of audit and accountability which promotes this doxa of assessment affords another idea; that teaching is meritocratic. The 'best' teachers are those whose children make the most progress, which, according to the doxa, can be reliably measured through assessment outcomes and attributed to individual teachers and their teaching. Thus, in describing herself as a successful teacher above, Sasha is not being immodest or arrogant. Rather, she is stating a fact since her outcomes are simply the right ones; though she seems aware of the possible social unacceptability of saying so ('it feels very strange saying that'). Habitus, the way actors are disposed towards the pedagogy of assessment, and field, the way they position themselves and are positioned by others in the professional setting, combine to normalize the practice of measuring outcomes and using these in particular ways. These outcomes can be understood as *les enjeux* (Bourdieu & Wacquant, 1992), the stakes in the meritocratic game of promoting one's professional status. That teachers adopt this position is not an example of either being forced to do so by structure or choosing freely to do so agentically. Instead it represents teachers' *illusio*, their sense of being caught up in the game and *committed to it in a belief that it is worth their investment*. Where doxa is a non-conscious participation in the dominant discourses of a field, illusio is a more aware sense of 'simply acting sensibly' (Colley, 2013, p. 11) in relation to the stakes of the game. By acting this way teachers implicitly say to themselves that they will be able to achieve things of value, and this may even outweigh acts of symbolic violence that they experience as a result.

Teachers therefore express their illusio in the manner in which they commit to the apparently meritocratic business of teaching and assessment because they believe it brings rewards, but also because the doxa within which pedagogy takes place appears to provide no alternative. It is important to understand that these work together. Just as it is not the case that teachers are unconscious of their actions, so neither do they exercise fully-conscious choice. The notions of doxa and illusio therefore try to represent the 'dance' of awareness that the teachers can move in and out of and, consciously or non-consciously, the way their actions are affected. For Sasha her illusio seems fairly straightforward and aligned with the doxa.

Sasha: and [my assessment] is all monitored really closely [in meetings with managers] in terms of percentages and progress in terms of points and any children who are flagged up as having regressed or not making as much progress as we'd have hoped, they're then flagged up and we put things in place, interventions and what-not. And similarly we do it to challenge [pupils] as well. Who do we need to challenge more?

In this instance at least, Sasha seems satisfied that assessment is valid, can legitimately be used to make decisions about teaching and that such decisions lead directly to outcomes such that teachers' work can be judged on a meritocratic basis.

In his interview however, Tony suggests a more arbitrary and political side to assessment, for example:

Tony: our stress is we should be recording children [with the grade] they are because then the [next] teacher could teach them effectively, and you feel like you are letting the children down if you don't record them accurately, but you know the stress of the school where it needs to look like a perfect gradual line. And we know that Ofsted would want that ...

Tony: Um, I don't think it [assessment] necessarily has an impact on progress on children. I think it helps people communicate the progress that people think children are making, but ... stress is necessary to some extent to perform to an optimum level, isn't it. But too much stress might tip people over, which... some people I know feel that way. And if the situation is where [pupils] are not moving, that's where the stress is bad, isn't it, but if they are succeeding then those meetings [about pupil outcomes] are usually quite rewarding, so it's usually quite ... it's never negative if you are jumping through those hoops.

Tony shows his awareness of, and challenge to, the doxa in his slight cynicism and way he seems to be questioning the effect of assessment for teaching and learning, but his comments still express his sense of illusio regarding the validity and use of assessment grades. Nonetheless, contrary to Sasha's view of assessment as supportive of learning, Tony describes his teaching practice as distorted by assessment in having to 'jump through hoops' with little 'impact on progress'.

ASSESSMENT: MERITOCRACY AND COMPETITION

Meritocracies, usually implicit in any neoliberal system, are essentially competitive. One rises up by being successful in whatever terms the dominant group in the system creates through the doxa; but as the stakes get higher obstacles to progress take on a greater significance. In a system where one's performance is judged mainly by being able to demonstrate a certain number of points of progress annually on an assessment scale, anything which makes this harder will be seen as a challenge; and since progress for one teacher starts where it left off for the previous teacher, end of year assessment has become a major focus and an issue of potential stress and anxiety for teachers.

> Sasha: … if you are in a position where actually you need that [grade] then it's very stressful because you then go 'but it's one child and it's one point and if I do it [alter the score] I'll get that' and you kind of go 'but professionally I can't do it, but I really need it …' it's that kind of whole, that real inner-turmoil about it.
>
> Tony: At the start of the year you look at data and you think how do I make progress with this lot and at the end of year you are stressing, I can't send them up like this they've done quite well, I'll put them down [laughs, but serious]. Every moment's stressful when you look at progress because you … I always found the stress of sending up children and how that would be perceived by [the next teacher] with the whole value-added thing, and I was always extremely careful that [it was] moderated *with* KS2. So sending up is incredibly stressful …and my stress at this moment is how my results which are relatively good are going to be perceived by the next teacher in year 4.

Despite the competition that assessment generates, Sasha's illusio suggests a stronger sense of investment in the practice of assessment. Perhaps because she is so successful within it and, regardless of the 'inner turmoil' described above, she notes that:

> Sasha: [The children] have no comprehension that you want to know how they've done and so you're stood over them [as they do tests] going 'yes, yes, yes, yes, nooo! Keep counting, keep counting' [laughs] and then you're like 'brilliant' or you're filling in their [marking] grids going 'amaaazing, look what they can do'. Very definite positives to it.

It seems that her habitus involves a strong commitment to these forms of high stakes assessment and despite her descriptions of stress she describes assessment as 'cathartic' because 'it confirms what I think about them' and 'it reassures you as a teacher'. This investment (illusio) requires a belief in the validity of the system of testing. For other teachers the illusio is just as strong, but is effected through other means, and where their pupils appear less successful the temptation to manage the system is clear. Indeed, more than just managing it, Tony claims that:

> Tony: I'd use the word manipulation! [laughs]. It is manipulated … for … for because of all the pressures. It's never manipulated up, it's certainly manipulated down. I've never manipulated up at all and that's, you know, I'm sure of that.

The best teachers were described above as those who made the 'most progress', but this is not quite the case. A better description is those who create the impression of having achieved the 'right progress'. Tony's manipulation is actually a *decrease* in the scores obtained at the end of the year and Sasha also reports 'moving their level back' so as not to be treated with suspicion by colleagues. However, whilst this seems altruistic, two further factors are important to appreciate. Firstly, Tony notes the robustness of the moderating system, saying that 'I don't know actually anyone who would send them up higher than you think because of the pressure of it being um [moderated]'. Similarly, Sasha claims that 'there is a tendency to perceive that a generous marker is bad'. End of year assessment appears well policed. However, progress is measured between two ends, and if you cannot move as far forward as you would like then you can simply start making progress from further back by challenging the judgements of colleagues.

> Tony: If you drop them [the previous teacher's transferred grades] four points [in your first assessment of the new year] then you are saying the other teacher's incompetent or a cheat and there is no other reason for doing that really. Some teachers, naturally … this thing really

> bugs me ... that some teachers don't trust other teachers and you see four points dip at the start of years. And they consistently do it.

Again, assessment is not simply the practice of recording mathematical achievement. It has the potential to distort teaching practices, requiring teachers to generate fictions to manage *their own* progress (not just that of pupils), and in doing so discover themselves in competition, even conflict, with each other.

IMPLICATIONS OF THE ASSESSMENT DOXA

As was noted at the start of this piece, research is clear about what we know to be effective mathematics teaching and the role of assessment in this; above all encouraging learners to treat mathematics as a network of related ideas and learning as flexible and mutable. Yet we also know that in England this is far from the case in practice and despite years of professional training and countless millions of pounds spent trying, pupils' experiences of mathematics are largely the same as they have always been; on average, these fail to extend those who are moving on with it and leave a long tail of those who are not (Office for Standards in Education, 2012b). Standing back and examining assessment from the perspective of teachers' wider professional lives offers some insights into this conundrum.

Teachers clearly work within a strong doxa surrounding assessment in mathematics and the evidence from the teachers here suggests that this doxa promotes a belief in a particular *epistemological* position: that mathematical knowledge is individual, not socially distributed; that it is largely propositional; and that learning is the business of its acquisition. This position then implies several implications for teaching: that acquisition should be predictable; that teaching should control and monitor this acquisition; that this takes place smoothly over time; and that teachers are responsible and accountable for it. No wonder then that whilst on the one hand teachers attempt to use assessment formatively to consider the learning of every child, on the other this is overlaid by practices that are designed to do professional work for themselves, as well as for the pupils. Whilst teachers still undertake assessment for formative purposes, ultimately the system requires them to objectify pupils' progress and to manipulate this into the 'right shape'. As we have seen, this is not a simple case of overinflating scores; it also involves downgrading them such that, potentially, 'the personal is used for the sake of the functional: students are included or excluded, valued or not, primarily on the basis of whether they contribute to the performance of the school' (Fielding & Moss, 2011, p. 52). Though this research has not observed these teachers' classroom practices directly it is not hard to speculate that, in general: teaching is likely to encourage pupils to see their mathematics learning as objectified and fixed rather than as constructed and mutable; and, rather than freeing children to explore the world of mathematics, teachers may well want to control mathematical thinking. Rather than simply being the business of evaluating pupils' mathematical achievements, assessment is a politicized process which alters classroom practice and through which *both* teachers and pupils are constructed as winners or losers.

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Québec anglophone teachers' pedagogies: Observations from an auto-ethnography

David A Reid[1], Annie Savard[2], Dominic Manuel[2] and Terry Wan Jung Lin[2]

1 Universität Bremen, Bremen, Germany, dreid@uni-bremen.de
2 McGill University, Montréal, Canada

This paper describes some characteristics of the pedagogy informing the teaching of anglophone teachers in Québec, on the basis of focus group interviews conducted as part of a Canada-wide comparative study. The paper also illustrates research methods embedded in an enactivist methodology that permit researchers to take advantage of the observer dependence of interpretations to gain insight into phenomena, like pedagogies, that are not directly observable. The dependence of results on methods used is illustrated in the case of the anglophone Québec focus group.

Keywords: Pedagogy, methodology, comparative research, teachers' beliefs.

INTRODUCTION

Large-scale international and national assessments have revealed a considerable range of student achievement in mathematics across Canada. When compared to international results, some Canadian provinces, notably Québec, rank among the top countries, while other provinces, especially in the Atlantic region, are significantly below the Canadian average. There are also some difference by the language of the school system. Students from the francophone (French speaking) system in Québec and from the anglophone (English speaking) system in Ontario achieved a higher average than their peers in the other language group in the same province (Brochu, Deussing, Houme, & Chuy 2013). A number of factors have been suggested to explain these differences including curriculum, gender, attitudes, beliefs, aspirations, time spent working outside school, parents' education, involvement and socio-economic status and school resources (see, e.g., Anderson et al., 2006; Beaton & O'Dwyer, 2002; Schmidt et al., 2001; Wilkins, Zembylas, & Travers, 2002). Teaching, which might be expected to have the most direct effect on student achievement, is considered less often. In a comparative research project (see http://www.acadiau.ca/~dreid/OT/) we seek to account for some of these disparities through a focus on pedagogy.

The nature of pedagogy

We make a distinction between *teaching* and *pedagogy*. Teaching refers to the observable practices of teachers and their interactions with learners. Pedagogy refers to what Tobin and colleagues (2009) call the "'implicit cultural practices' of teachers [...] practices that though not taught explicitly in schools of education or written down in textbooks reflect an implicit cultural logic" (p. 19). As Tobin et al. note, these implicit practices are related to teachers' "knowledge in practice" (Anderson-Levitt, 2002, p. 109) and "embodied knowledge" (Anderson-Levitt, 2002, p. 8). Such knowledge is related to Bruner's (1996) concept of folk pedagogy, the "taken-for-granted practices that emerge from embedded cultural beliefs about how children learn and how teachers should 'teach'" (p. 46). We see pedagogy as characteristic both of communities of teachers (grouped linguistically and regionally in our research) and of the individual teachers in those communities, being both a 'domain' and an 'orientation' in Maturana's (1988) sense. The two key features of pedagogy are that it is implicit and that it guides practice.

METHODOLOGY

The data analysed here comes from a larger project comparing regional pedagogies in middle school mathematics in four regions of Canada that show significant differences in student achievement. The regions chosen for comparison are Atlantic Canada, Québec, Ontario and Western Canada. In most regions two focus groups of teachers were formed, one whose language of instruction is English and one whose language of instruction is French. This was done as large scale assessments have revealed that there are differences of achievement along linguistic lines in some regions of Canada (Brochu, Deussing, Houme, & Chuy 2013) and this suggests there may also be differences in pedagogy along linguistic lines. Data from the anglophone focus group in Québec is analysed here.

Recalling Maturana's (1987) statement that, "everything said is said by an observer", we study teachers' pedagogies by examining teachers' observations of teaching. Our approach is similar to the multivocal ethnography approach described by Tobin (1999; Tobin, Hsueh, & Karasawa, 2009; Tobin, Wu, & Davidson, 1989) and we have adopted their terminology to describe the phases of research. Tobin and colleagues (1989) describe a layered process of documenting the implicit criteria of members of a community. This process involves working with members to construct a visual ethnography, an auto-ethnography and an ethno-ethnography. At each stage the teachers in the focus groups observe either their own or others' practices, first by creating a video record of their own practice, then by commenting on video recordings of classroom teaching within their region, and finally discussing video recordings of classroom teaching from other regions.

Visual ethnographies: Each teacher was asked to choose three lessons to be video recorded: a lesson that the teacher judged to be a "typical" lesson in her/his classroom; a lesson the teacher considered "exemplary"; and a lesson in which a topic related to fractions is introduced. Each teacher with a researcher collaboratively selected segments to be included in an edited video. An edited video of 20 minutes or less was produced by a research assistant for each lesson recorded by each teacher. These edited videos provide the visual ethnography of the teacher's teaching.

Auto-ethnographies: The teachers in each focus group viewed the edited videos from their classrooms and attempted to identify three that they feel show "representative" teaching in their region. The recordings of these focus group discussions form the first data set: as responses of regionally and linguistically internal observers they provide an auto-ethnography of mathematics teaching in each region. The three representative videos were used as stimuli for the other groups in the ethno-ethnography phase.

Ethno-ethnographies: Each focus group viewed and discussed videos from other regions, and in some case from other language groups. Encounters with other pedagogies offer the participants a way to reflect on their own familiar beliefs and practices, by comparison with others. The recordings of these focus group discussions form the second data set and constitute the ethno-ethnography of the pedagogy revealed in the videos.

The overall methodology for our research is enactivist (Reid, 1996). As noted above a key element of this perspective is that "everything said is said by an observer" (Maturana, 1987). This insight allows us to overcome a limitation of other studies of teaching practice, such as the TIMSS video studies (e.g., Hiebert et al., 2003). Pedagogy cannot be studied using approaches that involve external observers, as they have no access to what is implicit to the teachers themselves. However, by positioning the teachers as observers, one gains insight through what they observe and how they observe it into the implicit criteria that guide their observations.

In addition the research design includes self-observation by the researchers. In an enactivist approach, the process of analysis of data is an interrelationship, in which researchers find themselves learning new things within a context which is partially of their own creation. The changes which can be triggered in us, that is, what we can learn about the research context, are determined by our theories, beliefs and biases. What we learn is determined by what we know (Reid, 1996, pp. 205–206). In this paper, the analysis of the data was done by the first author, and so it is important to take into account his background as someone who himself was once an anglophone teacher in Québec, and whose teacher education occurred in Québec. However, he did not himself go to school in Québec, and so he is unlike the teachers in the focus group who experienced the Québec schools first as students and then as teachers. His teaching experience in Québec is also now two decades old, and things are no doubt different now. And his perspective has no doubt been modified by his more recent experiences doing school based research in other parts of Canada, and working with colleagues on school based research in England, France and Germany, as well as his main research focus on proof and reasoning.

ANALYSIS

The data analysed in this paper is drawn from the auto-ethnography of the anglophone teachers in Québec. The four teachers in this focus group all teach in the same school, at the grade 7–8 level (called "secondary cycle one" in Québec). They recorded their videos in their grade 8 classes. All the teachers have at least five years of experience teaching mathematics. Their school population is low income and low-middle class, with mostly homogeneous ethnicity. The rate of diagnosed learning difficulties in the school is high. One teacher left the group because of a stress leave, but gave permission for her videos to be used by the remaining three teachers.

The focus is the transcript of one focus group session, in which the teachers discuss first what exemplary and typical teaching is like, and then select the video about fractions they will share. The transcript can be divided into episodes based on breaks imposed by T, the interviewer. These are described in Table 1.

The transcript was analysed by coding it for the topic of the discussion. Teaching is complex, and so any discussion of teaching necessarily addresses some

Episode	Transcript line numbers	Description of episode
1	5–201	Responses to the question "What do you think a typical class in Québec in the English system looks like?"
2	203–266	Responses to the question "You're saying wow this is a really exceptional lesson. What would you be looking at?"
3	269–290	Following a pause, reactions to "It's interesting. I don't know if you guys are interested."
4	296–354	Discussion following T's responses to being asked by S, "What do you think T? About exceptional-"
5	373–405	Discussion following viewing of the first part of Video 1
6	409–419	Discussion following viewing of the second part of Video 1
7	423–444	Discussion following viewing of the third part of Video 1
8	449–494	Discussion following viewing of the first part of Video 2
9	499–611	Discussion following viewing of the second part of Video 2

Table 1: Episodes

Teaching related topics	**Gr**	Grouping (groups or pairs, think-pair-share)
	I	Interaction (student involvement, teacher prompting, brainstorming, student feedback)
	F	Format (chalk and talk, lecture, modelling, tell me what you need, student-centred approach)
	B	Basis (problem based, skill based, language based, reform, multiple solutions, examples)
	Pa	Pacing (working at the pace of the student, pressure to get through presentation)
	T	Technology & materials (Smartboard, Powerpoint presentation, notebook, worksheet)
	A	Assessment (student accountability, summative evaluation, monitoring, competition)
	Go	Goals of teaching
Learning related topics	**E**	Emotion (motivation, engagement, anxiety)
	H	How learning happens (memorising, repetition, activity, representations, creativity)
	W	What is learned (organisational skills, study skills, real world applications)
	Pr	Prior knowledge and experience
	Co	Specific mathematical concepts (equations, fractions)
Institution related topics	**Cu**	Curriculum (order of topics, goals)
	Ct	Communication between teachers
	Q	Teacher qualifications and perceptions (generalists, specialists, reputation)
	S	Stratification (difference, weaker groups, enriched kids, the strongest kids)

Table 2: Topics used in coding transcripts

aspects of teaching and neglects others. The topics addressed reflect a teacher's pedagogy, even before a particular position is taken. For example, referring to how students are grouped indicates that the topic of grouping is significant, whether preference is expressed for pairs, small groups, whole class construction or some pattern of combining groupings. The topics used in coding the transcripts are listed in Table 2.

These topics were generated from the data in an initial reading of the transcript, asking "What are the topics of this utterance?" for each speaker's turn. The transcript was then read a second time, and each utterance was coded with as many topics as fit. After this reading, several topics were only rarely used, and a third reading was done to check if additional occurrences of those topics had been missed.

Visualisations were then created to assist in in seeing patterns in the topics. For example, Figure 1 shows the topics discussed at the beginning of Episode 1, when the teachers were asked to describe a "typical class". The main focus is on Grouping (Gr), Format (F) and later Stratification (S). Interaction (I), the Goals of teaching (Go) and How learning occurs (H) also come up. The main focus is on topics related to teaching.

Figure 2 shows the topics discussed at the beginning of Episode 2, when the teachers were asked to discuss what an "exceptional lesson" looks like. Again, the main focus is on Grouping and Format as well as Interaction, topics related to teaching. As in Episode 1, Stratification also comes up later. Communication between teachers (Ct) is also mentioned.

These two sub-episodes display a pattern, of focussing on teaching related topics, and stratification, with little or no mention of learning related topics. This pattern was observed in five sub-episodes (1a, 2a, 4a, 7b, and 8c). Of these sub-episodes, four of the five occur in reaction to prompts to describe a typical lesson or an exceptional lesson. This suggests that teaching related topics are the first to come to mind when these teachers describe lessons.

Other topics related to teaching are discussed in sub-episode 1b (focussed on technology use) and sub-episodes 2b, 2c, 3b and 8b, focussed on the basis of teaching.

Episodes focussed on learning

Figure 3 shows the topics discussed in sub-Episode 1c. The focus shifts ways from teaching related topics, although technology, the basis of teaching, assessment and format of lessons are all mentioned. Instead the main focus is on an aspect of learning, specifically what is learned. The teachers are discussing the importance of learning good organisational skills in this sub-episode. "What is learned" is also the focus of discussion in sub-episode 4b, where the topic is learning about real world applications of mathematics and episode 9, the discussion of the second part of Video 2.

In sub-episode 1d, the main focus is on learning about integers, and the students' prior knowledge (see

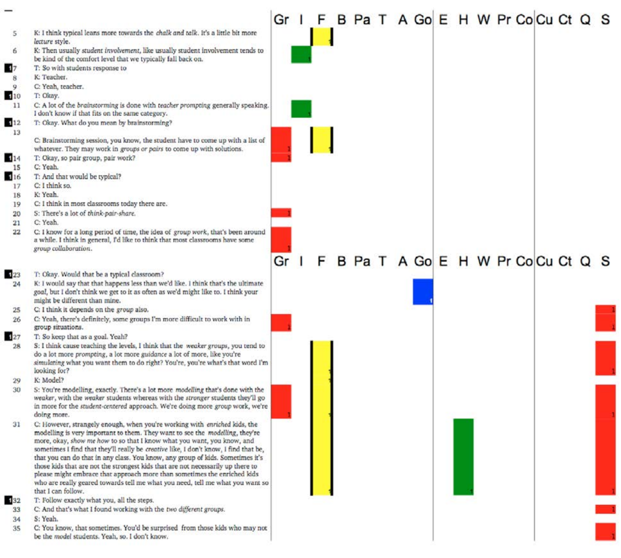

Figure 1: Topics discussed at the beginning of Episode 1

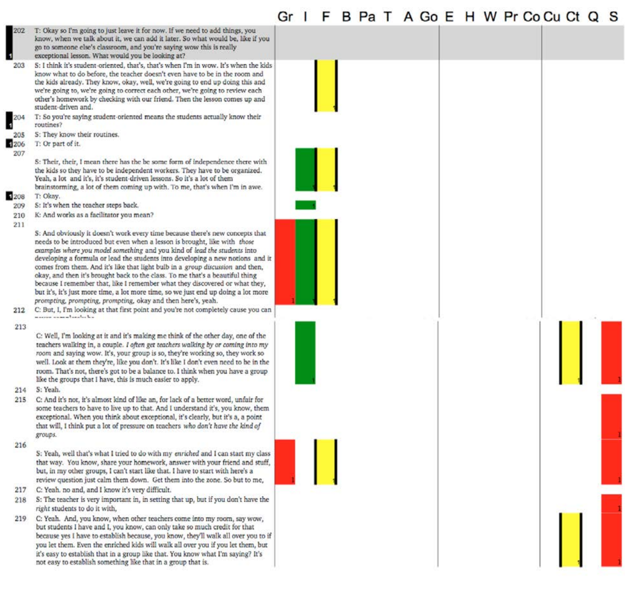

Figure 2: Topics discussed at the beginning of Episode 2

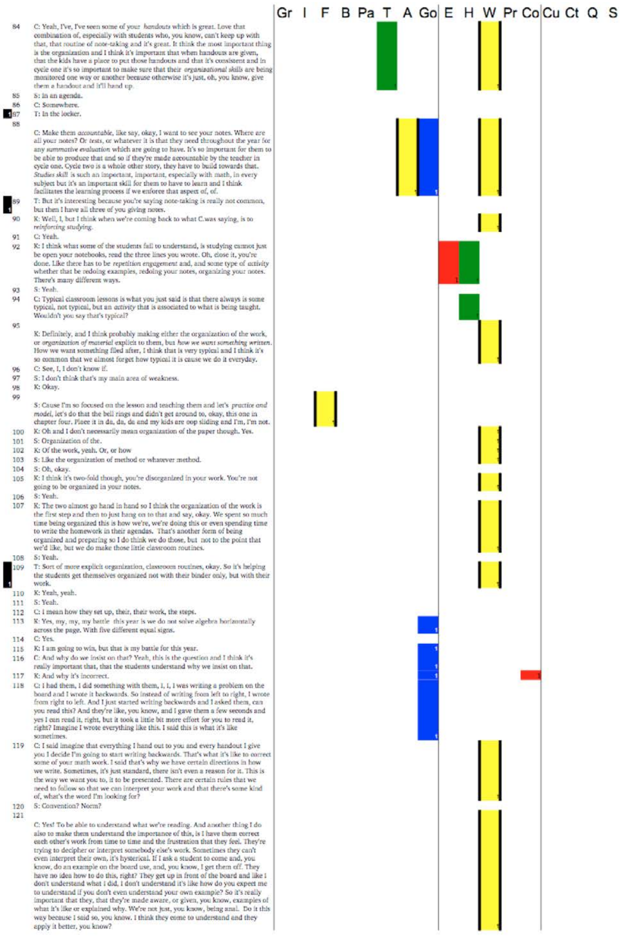

Figure 3: Sub-Episode 1c, focussed on what is learned

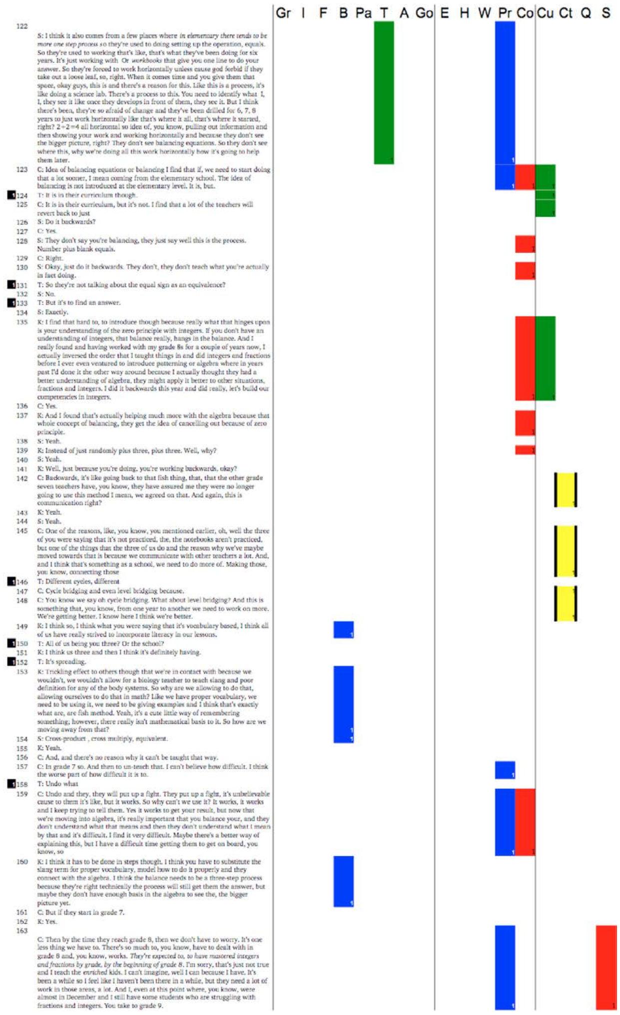

Figure 4: Discussion in sub-episode 1d, focussed on learning about integers and the students' prior knowledge, with a digression on communication between teachers

Figure 4). At one point the topic shifts to communication between teachers. The most extended discussion in which the focus is on specific concepts occurs in Episodes 5, 6 and 7, in reaction to watching Video 1. There the topics are the difference between negative numbers and subtraction in the context of algebraic expressions (in Episode 5), order of operations (in Episode 6) and equivalent fractions (in sub-episode 7a).

Other topics discussed related to learning include how learning occurs (sub-episode 8a) and students' prior knowledge (sub-episode 9b).

Other topics

In sub-episode 1e the topic of the intended curriculum came up very strongly. Topics related to learning, and assessment, also came up. Curriculum is also the topic of the discussion in sub-episode 4c. Again, topics related to learning (especially prior knowledge) and assessment also come up.

The discussion in episode 3 began with reflections on the experience of discussing teaching in the focus group, so the initial topics are emotions and communication between teachers. This shifts into reflections on the status of "Cycle 1" (grades 7–8) teachers compared to Cycle 2 (grades 9–11) teachers, who tend to be more specialised. The teachers returned to this topic at the end, in sub-episode 9c.

OBSERVATIONS

Table 3 shows an overview of the topics discussed. It makes visible a pattern in the teachers' discussions. In Episodes 1 and 2, when the teachers are first asked to describe typical lessons and exceptional lessons, they focus first on teaching related topics, especially format of lessons, student interaction, grouping, and the basis of teaching. The interviewer, T, also focusses on these topics in Episode 4 when the teachers ask her what she feels is exceptional. In Episodes 1 and

		1a	1b	1c	1d	1e	2a	2b	2c	3a	3b	4a	4b	4c	5	6	7a	7b	8a	8b	8c	9a	9b	9c	9d	9e
T	Gr	X					X					X														
	I	X					X					X									X					
	F	X	X				X					X						X			X					
	B							X	X		X							X		X						
	Pa												X								X					
	T		X															X								
	A					X			X									X								
	Go																									
L	E		X					X	X	X	X		X	X												
	H					X							X						X				X			
	W			X					X				X											X		
	Pr				X	X								X									X			
	Co				X	X									X	X	X		X	X						
In	Cu					X								X												
	Ct				X		X		X	X																
	Q									X															X	
	S	X	X				X																			

Table 3: Overview of topics discussed

4 however, the focus shifts as the discussion goes on, to topics related to learning: prior knowledge of students, specific mathematical concepts, what is learned and how it is learned.

In Episodes 5–7, after watching the videos, there is a striking inversion. The first topics the teachers discuss are related to learning, especially, in the case of Video 1, learning specific concepts. It is only briefly at the end of the teachers' discussion of Video 1 that they mention topics related to teaching (in sub-episode 7b). The discussion of Video 2 (Episodes 8–9) also begins with the topic of how learning occurs and learning a concept, but it then turns to topics related to teaching, before returning to topics related to learning after watching the second part of the video.

A further observation is that the topic of reasoning does not occur. This is noticeable primarily in that the observer in this case (the first author) has a strong interest in reasoning and so would be likely to notice any discussion of it by the teachers. The absence of this topic is an example of a finding that arises out of our methodological awareness that everything said is said by an observer.

CONCLUSIONS

These results are interesting in several ways. They reflect on the topics teachers themselves find most relevant when describing and reacting to teaching, and on the research methods used and the nature of teachers' pedagogies revealed by them.

The topics the teachers discussed overall are unlikely to be very surprising to researchers who are interested in teachers' pedagogies, beliefs and identities. However, it may be valuable to compare these specific results with results from elsewhere, and to consider sources of differences in both research methods and regional differences. We have made one such comparison, between this anglophone Québec group and the francophone Québec group and found agreement on the format of the typical lesson, the importance of mathematical vocabulary as the basis for teaching, the use of multiple representations (at least in exemplary lessons) and a belief that a high level of knowledge of the curriculum is important in planning exemplary lessons. However, there were also differences related to questioning, synthesis, and attention to student ability (see Manuel, Savard, & Reid, 2014, for more details).

From a methodological perspective it is thought-provoking and important that the topics the teachers discussed were different when asked to describe typical and exceptional lessons, and when reacting to videos of lessons. The teachers do not simply say different things in these two contexts, they focus on different *topics*. This means that a research design that relies on a single way of ascertaining teachers' views of teaching will miss some topics and overemphasise others. Within our larger research project, the other focus group sessions were run somewhat differently in the different regions, and in most cases teachers reacted to videos without having any prior discussions of typical and exemplary teaching. It will be interesting to compare the topics discussed in those focus groups with the topics discussed by this group.

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Teacher alignment of values in mathematics classrooms

Wee Tiong Seah[1] and Annica Andersson[2]

1 University of Melbourne, Melbourne, Australia, wt.seah@unimelb.edu.au

2 Stockholm University, Stockholm, Sweden

Teacher-student and student-student interactions in mathematics classrooms reflect what teachers and students value in mathematics teaching and learning respectively. These interactions shape and fine-tune developing values systems, which in turn affect the quality of the students' learning experiences. We propose that these teacher responses represent values alignment processes. We show three examples of teacher strategies, namely, redefining, reprioritising, and complementing. We will argue for the importance of teachers being able to develop strategies to facilitate values alignment in mathematics lessons.

Keywords: Values alignment, values, critical incidents, volition, teacher-student interaction.

INTRODUCTION

'When are we ever gonna use this maths?' (or its equivalent) must have been one of the most commonly posed questions by students all over the world. It seems that students find it important that there is utility in the knowledge or skill they are learning. They may be said to value *application* and/or *relevance* when learning mathematics.

It can also be assumed that the quality of students' mathematics learning will be affected by their teachers' responses to such a question, whether it is one in support of the student's valuing of *application/relevance*, one which implies a valuing of *understanding* ("this knowledge enhances your understanding of related mathematics content"), or one which reflects the valuing of *rationalism* (see Bishop, 1988).

Although it may generally be felt that teacher responses such as those above are guided by cognitive reasoning and/or affective dispositions, we argue here that importantly, these cognitive and affective functions are governed by the volitional variable of values. That is, what the teacher him/herself considers important – and indeed, values – concerning mathematics and mathematics pedagogies are embedded in the response offered to the students.

Accordingly, the teacher-student and student-student exchanges that take place in each and every mathematics lesson represent numerous negotiations of what the students and/or their teachers value, which may or may not result in agreements. Importantly, each of these represents an example of a critical incident (see Tripp, 1993) in that responses and outcomes affect the direction of subsequent classroom discourses and the extent to which planned lesson objectives are achieved. If successful negotiations about such critical incidents require that everyone concerned are satisfied with the outcomes of these negotiations, then the competing values would have become aligned in some way during the process.

This paper draws on empirical data to investigate what such values alignment incidents might look like in mathematics classrooms. We identify and categorise three values alignment strategies. To contextualise this discussion, we first review what is known about values as these relate to mathematics pedagogy and teacher-student / student-student interactions in mathematics classes.

THE ROLE OF VALUES AND VALUING IN MATHEMATICS PEDAGOGY

We consider values in the context of mathematics learning and teaching as a volitional construct. We have proposed elsewhere that:

> values are the convictions which an individual has internalised as being the things of importance and worth. What an individual values defines for her/him a window through which s/he views the world around her/him. Valuing provides the individual with the will and determination to maintain any course of action chosen in the learning and teaching of mathematics. They regulate the ways in which a learner's/teacher's cognitive skills and emotional dispositions are aligned to learning/teaching. (Seah & Andersson, 2015, p. 169)

The philosopher Ayn Rand wrote, "a being of volitional consciousness has no automatic course of behaviour. He [sic] needs a code of values to guide his actions" (1961, p. 97). Values guide decisions and actions (e.g., Park et al., 2011), and people and organisations defend or fight with passion for what they value. That is, values are expressions of will and convictions that provide us with a certain degree of 'stubbornness' to stay motivated and to persevere when we encounter barriers. This is not to say, however, that values are always expressed in the form of observable actions. Rather, the potential for action is the basis for valuing. Whether it is expressed in observable action or not depends on the context. This can be seen in Andersson's (2011) recent study, in which the upper secondary student participants indicated that they disliked – even hated – mathematics. However, Andersson's analysis show that their stories of mathematics learning experiences were connected to the context in which they were told. The students' stories and actions for learning mathematics changed as the contexts evolved.

As volitional variables, values have both cognitive and affective components as well. The cognitive components are visible through the choosing dimension of the valuing process (Raths, Harmin, & Simon, 1987). Valuing also has an affective dimension which can be seen in the way we often find ourselves embracing what we value in a passionate way, supported by associated emotions, attitudes and beliefs.

Considering any classroom interaction, then, what is conveyed (verbally or otherwise) and the subsequent responses would reflect what the teacher or student is valuing. These interactions represent negotiation attempts by all involved; students are aware of their capacities to adopt, resist or reject discursive positions. Given that there exists considerable within-class and within-school diversity of student cognitive and affective variables (Sullivan, 2015), the (mathematics) classroom discourses between teachers and their students – and amongst students – represent pedagogical spaces of contestation and conflicts. This is inevitable, and after all, the consistent domination of one particular (person's) goals and interests in any social gathering is very likely not desirable anyway, as discussed by Gutierrez (2007).

As a volitional variable, values (in mathematics education) not only motivate and guide decisions and actions; they also provide one with the will and determination to maintain courses of action in the face of competing actions and obstacles. In this way, values do more than what motivations do. In Kivinen's (2003) words,

> there is a distinguishing line between volition and motivation. Volition promotes the intent to learn and protects the commitment and concentration from competing action tendencies and other distractions. For example, a student may be motivated to read a book in the evening. He or she is more or less motivated to do so. The student takes the book and starts to read (motivation has done its work). Volitional processes (will) keep him or her reading, in spite of the fact that there is an interesting football match on TV. (pp. 26–27)

It is this sense of will and determination that is associated with valuing which would account for an individual bringing what s/he values to any interaction s/he is involved in. When such an interaction brings together different and potentially competing values that are embraced by teachers and students, this may involve resisting or rejecting decisions or actions representing the competing values. Thus, when a student asks when a particular mathematical concept or skill might be useful, the question reflects his/her valuing of, say, *application or relevance*. In the same way, the teacher's response will reflect particular values too, which demonstrate if these values are aligned with each other or not. In turn, anyone involved in the interaction can choose to pursue what s/he values with regards to the topic being discussed, although this will often be subjected to sociocultural norms and conventions such as power distance (Hofstede, 2011). We suggest that in order for a lesson to 'move forward' productively, teachers would have negotiated about the competing values such that there is achieved a certain level of alignment amongst the values concerned.

While values may indeed be stable when compared with variables such as interests and beliefs, we argue that the extent to which a value is embraced and prioritised is responsive to one's environment and is thus not fixed. In other words, opportunities for values teaching in mathematics education exist across all school years. Whereas values may be absorbed when one is young (Court, 1991), value priorities continue to be considered and evaluated throughout one's life in school and beyond. This may be seen in the valuing process conceptualised by Raths, Harmin and Simon (1987). Made up of three stages, that is, choosing, prizing, and acting, the first stage is related to choosing freely and amongst several alternatives, bearing in mind the consequences of adopting any one of these alternatives. We believe that it is this choosing activity that is periodically stimulated by phenomena that allows for one's value priorities to be monitored, assessed and fine-tuned.

VALUES ALIGNMENT IN THE MATHEMATICS CLASSROOM

Values alignment is a central feature of Senge's (2006) five disciplines of learning organisations (that is, personal mastery, building shared vision, team learning, mental models, and systems thinking). In particular, the discipline of building shared vision calls for aligned values in an organisation in order for the shared vision of the future to be co-created. The value of values alignment lies in the observation that

> all relationships – between one person and another, between the present and the future, between customer and product, a team and its goals, a leader and a vision – are claimed to be strengthened by aligned values. (Branson, 2008, p. 381)

Thus, for a teacher, being able to facilitate values alignment between what s/he values and what his/her students' value promises to strengthen the relationships, and is one of the keys to nourishing teaching and learning practices. Indeed, MacDonald and Shirley (2009) had proposed that the mindful teacher is one who, amongst other things, is proficient in establishing authentic alignment between his/her own values and professional practice, and is also successful in harmonising these values and policy. If some teachers are 'effective' in different classrooms whereas others perform well in particular classrooms only, this could be because the former have been successful in attaining values alignment in whatever classroom situation they find themselves in.

However, values alignment is not about ensuring that students' values are the same as their teachers'. It is thus different from values inculcation. Rather,

> building ... values alignment is about providing a cooperative and collaborative process whereby the members of the organisation can develop strategies, systems and capabilities that not only support those values that have previously been clarified as being essential for the ultimate success of the group as a whole but also are supported by the majority of the people within the group as acceptable guidelines for directing their behaviour. (Branson, 2008, p. 383)

That is, values alignment facilitates the co-existence of different values that are held by different people interacting together. In so doing, students can perceive that their knowledge, skills and dispositions are valued, and they can also feel inclusive in relation to their learning of mathematics.

IDENTIFYING AND INTERPRETING VALUES ALIGNMENT CRITICAL INCIDENTS

So, how do teachers and students negotiate the differences in what they value generally and in mathematics education in particular, so as to achieve values alignment in order to facilitate mathematics learning? In the absence of prior educational research on this topic, we referred to available data that were collected from a prior research involving values and valuing in the classroom (see Seah, 2005) as seen from the perspectives of classroom teachers of mathematics.

This prior research was conducted in secondary schools across Victoria, Australia. The methodology of the research then involved the identification of critical incidents (Tripp, 1993) in mathematics lessons when the teacher participants and their respective students were observed to be valuing different attributes of mathematics or of mathematics pedagogy. Lesson observations and teacher interviews were the methods employed. The research objective was to map out the range of strategies teachers employed to negotiate about differences in valuing between themselves and their respective students. The focus was on the various types of actions employed by the teachers, of which there were 7 (e.g., helplessness, amalgamation, appropriation). In the current study, however, the teacher responses to the episodes of values differences in the lesson transcripts were re-examined and re-analysed at a deeper level: instead of listing what the teachers did, here we have been attempting to interpret how the teachers' own valuing changed in the process of their negotiating about and aligning the value differences. The patterns observed amongst the 8 teacher participants suggested three such values alignment strategies, which we name redefining, reprioritising, and complementing. These are presented below.

Values alignment strategy: Redefining

> *Case 1.* Michael (a pseudonym), a mathematics teacher in a secondary school, noticed that his Year 10 students had been unwilling to work with concrete manipulatives such as geoboards and pattern blocks. "These are for young kiddies, sir!" they would say. Yet, Michael felt that learning is more effective when students are able to visualise the relevant concepts. Michael has since found a way round this issue, and his students are now exploring and understanding geometrical concepts using software programs such as dynamic geometry software, as well as online resources such as those hosted by the National Library of Virtual Manipulatives.

In this case, Michael's use of concrete manipulatives reflects his valuing of *visualisation*. However, this teaching approach was resisted by his students whose values were not aligned with the image of teenagers 'playing with blocks'. There was a potential here of a value conflict between Michael and his students, which could possibly result in the students being disengaged in his lessons. Michael resolved the potential value conflict through redefining what he and his students valued, coming to understand that in effect, his valuing of *visualisation* was underlied by a valuing of *exploration*. This was crucial, since the students' values were aligned with *exploration* as well; it was just that they did not want to feel like small kids playing with blocks and teddy bears. By redefining his valuing of *visualisation* with the use of digital learning technologies, Michael was able to plan and execute his lessons such that the dynamic geometry software and the online websites provided the students with opportunities to explore – and thus visualise – the relevant geometrical ideas and concepts in a form that was aligned with what the students value. Michael's valuing of *visualisation* had given him the will to resolve the value difference situation in ways which still allow for student visualising to take place, only that the means of actualising this valuing were now acceptable to the teenage students, who were understandably wanting to behave more adult-like and doing adult tasks. For his students, their positive response to the ICT use was an endorsement of their common valuing of *exploration*.

In this instance, values alignment was achieved through Michael's redefining what he values, such that its expression now is aligned with what his students value.

Values alignment strategy: Reprioritising

> *Case 2* Diane was an immigrant secondary school mathematics teacher from Canada, teaching in a small country town in Australia. When one of her students answered one of her questions about an algebraic equation by saying "just chuck in c, just chuck in the c", she responded that he was being too casual with his use of mathematical language. Diane's own mathematics learning experience in Canada had instilled in her a valuing of the *formality* in mathematics, a tradition that she felt needed to be upheld but which most students today would perceive as dry and boring. Thus she would have preferred her students to talk about "adding the constant, c".

Yet, Diane was deeply aware and concerned that she was teaching a class of mainly disengaged and underperforming students, and that meant that it would not be wise to get 'too caught up in those formal, scary things'. She was mindful that for these students, a valuing of *fun* would be a key motivator for them. In Diane's words,

> We see too many kids that's just, they come to the class and they are beaten already. Because they found it a difficult subject, and they don't enjoy it, they feel frustrated. It makes them, you know, they feel out of their depth, and that's just awful. If you're [i.e. the students] starting out that way, you know, I think we've got to really try. And if

> it's not [about] being so sticky about notations, then, you know, having a bit of fun with it then.

As such, she made a conscious effort to 'sacrifice 'plus c' for 'chuck in c' '.

Here, Diane realised that 'pushing' her students to share her valuing of *formality* and to use formal terminology would be counter-productive. This group of students needed first and foremost to be able to be interested enough in the subject, and to develop some confidence to acquire the skills and concepts required of them. The students' valuing of *fun* was a volitional force, which supported the cognitive and affective growth that they needed. Diane's understanding of this, and her subsequent reprioritisation of her valuing of *formality* and *fun*, resulted in values alignment between herself and her students. This reprioritisation of Diane's values is evident when she talked about the relative importance of notations/formality and fun/enjoyment, and how it would be her willing sacrifice to interchange the order of priority for the sake of facilitating her students' learning.

In this second case, values alignment was achieved when Diane in interaction reprioritised what she valued to achieve a common valuing with her students. Diane had not given up her valuing of *formality*. However, she also shared her students' valuing of *fun*, demonstrated through her provision of space for a different mathematical discourse in class. As a result of this reprioritisation between the two values within herself, Diane had achieved an alignment of what she and her students valued.

Values alignment strategy: Complementing

> *Case 3*. Amy taught Year 7 mathematics in a south-eastern suburb in Melbourne, in an area with a high concentration of Asian migrant professionals amongst its residents. Most of the students in her school were Asian, most of whose parents valued *competition* and *grades*. The school has also embraced the community's valuing of these two orientations. However, Amy had grown up valuing *co-operation* and (conceptual) *understanding*. For Amy, the need to confront the value differences between herself and the school/community was quite urgent, for she knew that she would not be able to teach mathematics effectively and professionally satisfyingly if she did not negotiate these differences soon enough. She talked to colleagues and some parents, and she referred to relevant literature. While she was not ready to give up what she had grown to value, she was also getting to understand how the students' and their parents' values were culturally powerful agents of engagement and motivation. At the same time, she felt that her students needed to learn to value *co-operation* too as a means of humanising competition, and that their developing meaningful understanding of (mathematical) concepts would further enhance their capacity to achieve even better grades in assessments. So in the last two years, Amy has developed mathematics lessons, which reflect the valuing of both *grades* and *understanding*, and more difficultly, *competition* and *co-operation*. Thus, her students strive to understand concepts while/before practising hard to attain proficiency. They are also able to work together and help one other, while enjoying pitting their mathematical skills against one another.

In this case, values alignment for Amy and her students was achieved through an acknowledgement of the different values, and a purposeful consideration of how they could co-exist and indeed, complement each other. Over two years, Amy developed pedagogical strategies that allowed for these pairs of potentially conflicting values to not just co-exist, but also to further support the inculcation of the other value in each pair. This complementarity reflects one of Hofstede's (2001) cultural value continua, *masculinity / femininity*. Here, the students' valuing of *masculinity* in the form of *grades* and *competition* has struck a balance through alignment with Amy's valuing of *femininity*, in the form of *understanding* and *co-operation*.

CONCLUDING IDEAS

In this paper, we have focussed on the day-to-day teacher-student and student-student interactions in mathematics classrooms, envisaging these as potentially critical incidents involving different and possibly conflicting values. We have drawn on empirical data to illustrate how teachers' facilitation of these critical incidents can actually be regarded as involving values alignment.

The examples listed above have been sourced from existing data given that there has not been any values alignment research study in our knowledge, an indication that values alignment in its various forms take place often enough during mathematics lessons. This highlights the importance for teachers' awareness of what they themselves value with regards to mathematics, to mathematics pedagogies, and to school education. With this self-knowledge, teachers are better empowered to respond to value differences/conflicts as critical incidents with effective values alignment approaches, thus maintaining a harmonious environment in the classroom. At the same time, such values alignment episodes also support students' cognitive and affective developments in a different way, that is, through the ways in which their own values systems evolve and mature.

Although it is not within the scope of discussion of this paper, it is also important to remind ourselves that not all values alignments attempt lead to productive learning/teaching. We observed in our data that despite the values being aligned and harmonious interactions being maintained, the aligned values might not support effective or productive mathematics learning/teaching. We also came across situations in which the values alignment attempts failed.

Although the values alignment strategies discussed above may well also take place in lessons other than mathematics ones, it is important that their deployment in mathematics lessons is highlighted to emphasise that the strategies can be as useful in mathematics pedagogy. The contexts within which Michael, Diane and Amy operated were uniquely mathematics educational. More importantly, the cases presented above relate values and valuing to mathematics pedagogies in very different ways. Indeed, the findings here which showcase the three values alignment strategies could well also address similar classroom tensions reported in Wester, Wernberg and Meaney (2015). The research reported here thus calls on the mathematics education research agenda to promote the need for a deeper knowledge of valuing as a volitional variable, and also of the values alignment process in the context of school mathematics pedagogy. Studies in this area represent cutting-edge, innovative mathematics education research; they promise to provide researchers and practitioners with a third, volitional approach to further understanding and improving mathematics learning in schools, complementing and strengthening existing cognitive and affective strategies.

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A study for profiling mathematics teachers regarding factors affecting promotion of students' metacognition

Vuslat Şeker[1] and Engin Ader[2]

1 Middle East Technical University, Faculty of Education, Ankara, Turkey, vuslat.seker@boun.edu.tr

2 Boğaziçi University, Faculty of Education, İstanbul, Turkey

The main objective of this study was to describe mathematics teachers' profiles on factors affecting their promotion of students' metacognition through developing profiling tools. In the light of this aim, four factors from the Framework for Analysing Mathematics Teaching for the Advancement of Metacognition -FAMTAM- (Ader, 2009) were used. The factors were (1) teachers' conceptualization of metacognition, (2) teachers' perceptions of students' features and needs, (3) distribution of mathematical authority in the classroom and (4) the external pressure perceived by teachers. The sample consisted of 314 middle and secondary school mathematics teachers. In this study, descriptive, correlational and causal comparative research designs were used. The main findings indicated that most mathematics teachers were conceptualized what metacognition is. They were aware of students' features and needs. They supported a learning environment where mathematical authority was exercised by students. They perceived high external pressure from various factors influencing their promotion of students' metacognition.

Keywords: Mathematics, metacognition, teaching of metacognition.

INTRODUCTION

Metacognition is briefly defined as the regulation of and knowledge about cognitive activities (Flavell, 1979). It has an important place in mathematics education (Lucangeli & Cornoldi, 1997). It appears in early stages of problem solving process with accurate representations and the planning of problem solving (Desoete & Veenman, 2006). Metacognitive activities improve students' mathematical learning (Jacobse & Harskamp, 2012). Therefore teachers must arrange learning environments where learners are encouraged to learn mathematics through exercising metacognition (Lombaerts, Engels, & Athanasou, 2007).

There are various teaching practices for the development of students' metacognition by taking their needs and the learning environment into account. The promotion of metacognition should be examined in a broader sense in order to contribute the development of students' metacognition. For example, why teachers promote or not promote metacognition in their classrooms should be considered in order to help teachers to create effective learning environments (Lombaerts, Engels, & van Braak, 2009).

Several factors affecting teachers' promotion of metacognition or self-regulation has emerged in recent studies (Dignath-van Ewijk & van der Werf, 2012; Lombaerts et al., 2009; Lombaerts, Engels, & Vanderfaellie, 2007). Mainly three different determinants namely, teacher characteristics, school context characteristics and pupil characteristics were described as factors affecting teaching practices of teachers on promotion of self-regulation (Lombaerts et al., 2009). Teacher beliefs, previous teaching experiences and educational experiences are given as teacher characteristics that affect the introduction of metacognition in teachers' teaching practices. Furthermore, curricular changes, timetables, number of students, textbooks, the relationship among teachers are certain examples for school context characteristics. Lastly, pupil characteristics affecting teaching practices on stimulation of metacognition or self-regulation are cognitive and metacognitive abilities of learners.

Ader (2009) developed a framework for analyzing mathematics teaching for the improvement of metacognition of students. The reason for developing such a framework came from "the lack of emphasis on teacher's role and teaching practices within the efforts to incorporate metacognition into mathematics classrooms" (Ader, 2013, p. 7). The framework consists of four factors; (1) teachers' conceptualization of metacognition, (2) teachers' perceptions of students' features and needs, (3) distribution of mathematical authority in the classroom and (4) external pressures perceived by teachers FAMTAM was believed to be a good source for "exploring the teachers' approaches to promotion of students' metacognition" (Ader, 2009, p. 282).

The conceptualization of metacognition (Flavell, 1979) is defined as a factor in FAMTAM. Conceptualization of such complex and multifaceted phenomenon is important to investigate since the complex phenomena can be interpreted and implemented in a different perspective because of the structure (Ader, 2009). Secondly, teachers' perceptions of students' features and needs is another component of FAMTAM since it is an indicator of how teachers act with respect to the features and needs of students for effective mathematics teaching (Jaworski, 1992). Therefore, the perceptions give encouragement for students to use metacognition in their learning progress. Thirdly, the distribution of mathematical authority is a factor described as the way teachers encourage learners to use mathematics since mathematics or discipline of mathematics can be taken as authority where members of mathematical communities are working on mathematics (Schoenfeld, 1992). Boaler (2002) identified the members of a community of a classroom that lack mathematical authority (i.e. mathematical problem solving) as not contributors in each other's mathematical learning, not doing mathematics, but as only receivers of mathematical knowledge. Hence, metacognition can be conceptualized as a way of practicing mathematical authority because metacognitive and problem solving processes are intertwined (Ader, 2009). Lastly, external pressures perceived by teachers were given as another factor in FAMTAM. External pressures stemmed not from classroom practices, but from policies of educational system and demand or expectations of educational institutions that make teachers feel pressure on their teaching practices, such as curriculum content, national exam, time constraint and so on (Ader, 2013; Lombaerts et al., 2009).

The purpose of the study is to describe mathematics teachers' profiles on factors affecting promotion of metacognition through developing profiling tools which were developed and validated based on FAMTAM. It is aimed to explore mathematics teachers' approaches to promotion of students' metacognition considering the four factors in FAMTAM. This study is significant in a way the developed scales describe a teacher profile with regard to the factors affecting metacognitive implications or promotion in their classrooms. As a result of the study, such profile identification may help researchers and policymakers to make sense of the effects of teachers' promotion of metacognition for future actions. When the profile of teachers based on pre-determined reasons or factors are determined, this can also help teachers to tackle some of the issues that influence their promotion of metacognition by eliminating negative conditions and supporting positive ones.

METHOD

Three research questions were investigated; (1) What is the teacher profile on the factors affecting mathematics teachers' promotion of metacognition? (2) Are there significant correlations between variables derived from FAMTAM as conceptualization of metacognition, perceptions of students' features and needs, distribution of mathematical authority and perceived external pressure affecting mathematics' promotion of metacognition? (3) Is there a difference in the four variables derived from FAMTAM according to teachers' demographic variables?

Sample

For the description of profiles of mathematics teachers on the factors affecting promotion of metacognition, the main study was conducted with 314 (161 from middle school and 153 from secondary school) mathematics teachers from Turkey. There were 175 female and 139 male participants. Of these, 34 participants were from private schools and 280 participants were from public schools. Moreover, 116 of 163 middle and 43 of 153 secondary school mathematics teachers were graduates of faculties of education. 4 middle school mathematics teachers did not state the faculty they graduated. Remaining teachers graduated from faculties of science and got a teaching certificate. The sample of main study was chosen conveniently. 199 of 314 mathematics teachers filled out web-version of the instruments. 105 of 314 mathematics teachers filled out hard-copy version of the instruments.

Instruments

Four profiling tools addressing the four factors in FAMTAM were developed and validated for reaching the purpose of the study. The factors affecting promotion of metacognition within FAMTAM were supported by related literature as discussed in previous section. Within the process of development of four profiling tools, two pilot studies were conducted. The first sample was used in order to develop the four scales. The second sample was used for assessing psychometric qualities of the instruments that were revised after the pilot study.

The Teachers' Conceptualization of Metacognition Scale consists of 9 items. The teachers are asked to state to what extent they agree with the importance of addressing the statements given in the items in their teaching (e.g., "students' planning of their thought"). The individual total score for this scale indicated the level of fit between teachers' conceptualization of metacognition and what has been mainly documented in literature (e.g., Flavell, 1979). The Cronbach's alpha level of this scale is .91 for reliability and two structures as metacognitive knowledge and metacognitive skills were observed through factor analysis so there was an evidence for construct validity.

The Teachers' Perceptions of Students' Features and Needs Scale consists of 6 items (e.g., "Teacher should help learner to improve metacognitive skills and knowledge by using various teaching methods (e.g., modeling, think aloud, direct teaching"). The items of this scale asked to what extent teachers agreed with the statements in the items. Cronbach's alpha level of this scale is .81 for reliability. Experts agreed on the items that within this scale they covered important issues regarding content provided through literature on teachers' perceptions of students' features and needs. Therefore, it was an indicator of content validity for this scale.

The Distribution of Mathematical Authority Scale consists of 10 items (e.g., "A learning environment should be constructed where teacher and students reason together"). The questions in this scale asked teachers to code the items according to the following criteria: "by considering the power of determining what is right or wrong should be on mathematics; such a classroom environment, state to what extent you agree or disagree the statements below". Cronbach's alpha level of this scale is .65. Four factors were observed in factor analysis and this was accepted as evidence for construct validity of this scale since each factor was derived from the definition of where mathematical authority resides in.

The External Pressure Perceived by Teachers Scale consists of 9 items (e.g., "change in curricular and teaching approach"). The questions in the items asked for teachers to state to what extent they agree or disagree with the listed factors about their effect on their teaching practices. Cronbach's alpha level of this scale is .73. Two dimensions obtained from the exploratory factor analysis supported the construct so there was evidence of construct validity for this scale.

STATISTICAL ANALYSIS

Since the aim of the study is to determine mathematics teachers' profile on factors affecting their promotion of metacognition through developing profiling tools, descriptive and inferential analysis were used.

For descriptive statistical analysis, means, standard deviations and possible range for four scales were calculated in order to describe the data set. Furthermore, the distribution of data for each scale was demonstrated through histograms. For correlational analysis, Pearson Product Moment correlation coefficient was carried out in order to seek answers for to what extent four variables were related to each other.

Group comparisons were done in order to observe how the profiling tools discriminated between scores of groups of teachers according to certain variables as gender, age, years of experience, education level, teaching level, school type. It was aimed to explore different aspects of the profiling tools. For this reason, firstly, normality test was also conducted. When the assumption of homogeneity of variances was satisfied, a one-way Analysis of Variance (ANOVA) was used. When the assumption of homogeneity of variances was violated, Brown-Forsythe F- ratio was used. In order to explore effect size, partial eta squared was calculated.

FAMTAM Variables	N	Possible Range	Mean	Standard Deviation
Conceptualization of Metacognition	314	9–45	38.53	7.06
Perceptions of Students' Features and Needs	314	6–30	25.89	3.23
Distribution of Mathematical Authority	314	25–50	38.83	5.55
Perceived External Pressure	314	9–45	34.07	6.36

Table 1: Descriptive statistics

RESULTS

The results showed that most mathematics teachers conceptualized metacognition in accordance with the commonly accepted conceptualizations in the literature. They were aware of students' features and needs. They supported a learning environment where mathematical authority was exercised by students. However, they perceived high external pressure from various factors influencing their promotion of students' metacognition.

Secondly, it was seen that there was a significant correlation between conceptualization of metacognition variable and distribution of mathematical authority variable. Although there was a significant relationship between these variables, the correlation coefficient was relatively low. Therefore, it was a weak positive relationship, $r=.16$, $p < .01$. It was same with the relationship between conceptualization of metacognition variable and perceived external pressure variable, $r = .20$, $p < .01$. Perceptions of students' features and needs variable was significantly, but weakly correlated with distribution of mathematical authority variable, $r=.21$, $p < .01$. Furthermore, same situation can be observed in the relationship between perceptions of students' features and needs variable and perceived external pressure variable, $r=.23$, $p < .01$. Only non-significant correlation was found between distribution of mathematical authority variable and perceived external pressure variable. The relationship between conceptualization of metacognition variable and perceptions of students' features and needs variable was found as significant, but there was moderate correlation between them, $r= .43$, $p < .01$. The result indicated that when the teachers had high conceptualization of metacognition that they got high scores in The Conceptualization of Metacognition Scale, then they better perceived students' features and needs based on metacognition.

Significant gender differences were observed on teachers' claims about their distribution of mathematical authority, $F(1,312)=8.86$, $p=.05$, partial , perceived external pressure, $F(1,312)=18.052$, $p=.00$, partial , and conceptualization of metacognition, $F(1, 263.96) = 8.24$, $p= .00$, partial, in favor of female teachers. Significant differences according to age, $F(3,310)=15.127$, $p=.00$, partial , and years of experience, $F(3, 282.90)=6.57$, $p=.00$, partial, were observed only on teachers' distribution of mathematical authority in favor of teachers with 20–29 age group and 1–5 years of experience respectively. There were also significant differences on distribution of mathematical authority, $F(1,312) = 8.06$, $p=.00$, partial, and perceived external pressure, $F(1,148.78) = 1.76$, $p =.01$, partial, according to teachers' educational background. Teachers with a master's degree supported the distribution of mathematical authority more and perceived less external pressure than teachers with a bachelor degree. In addition, significant teaching level differences were found on each factor in favor of middle school mathematics teachers. Lastly, perceived external pressure, $F(1,312)=5.75$, $p=.02$, partial, and teachers' conceptualization of metacognition, $F(1,59.22)=.01$, partial, also significantly differed on school types. Teachers working at a public school perceived higher external pressure and their conceptualization of metacognition's scores were lower than teachers working at a private school.

Most participants were found to conceptualize metacognition as a multiphase phenomenon which include those commonly accepted principles in the relevant literature. It shows that teachers' beliefs about the presence of metacognition in their teaching are positive. Although the positive results might be a result of social desirability, it is important to keep in mind that the teachers are aware of the importance of metacognition in mathematics classrooms. The awareness of teachers might lead them to introduce and promote metacognitive activities within their teaching practices (Lombaerts et al., 2009). It can be stated that better teachers' conceptualization of metacognition parallel to those in the literature reflecting the details and sophistication might influence teachers' promotion of metacognition positively when hampering factors were diminished or eliminated. As a remarkable and significant difference on teachers' conceptualization of metacognition was observed on teachers working at different school types which could be a result of the fact that secondary school mathematics teachers are mostly graduates of faculties of science (111 out of 153), not of faculties of education (42 out of 153).

Secondly, most participants perceived students' features and needs in terms of metacognition. They mostly stated they were aware of metacognitive characteristics of students and acting upon it. In order to promote metacognition and self-regulation, learning environments should be arranged accordingly (Lombaerts et al., 2007). Teachers who give priority to students' characteristics in terms of their developmental milestones, their way of knowledge construction, and participation of the learning process, are described as the ones taking "learning needs and experiences of students as starting point" (Vandevelde, Vandebussche, & Van Keer, 2012, p. 1563). Therefore the results showed that there is a positive approach to promote metacognition with respect to this issue. Furthermore, in perceptions of students' features and needs variable, only significant difference was found according to teaching level in favor of middle school level. The reason might be the nurturing, parent-like nature of primary education (Louis, Marks, & Kruse, 1996). As they stated, the concerns in high school shift to a more academic plane.

Thirdly, mathematical authority is rather a new concept that provides a student-centered learning environment for teachers and learners (Wilson & Llyod, 2000). As Amid and Fried (2005) pointed out when authority in classroom is discussed, most people including teachers imagine the teacher as "the head of a classroom" (p. 145). However, mathematical authority in the classroom might exist within a classroom when teacher authority as expert authority is eliminated (Amid & Fried, 2005). The results of the study showed the majority of participants stated a distribution of mathematical authority in which mathematics teachers guide learners to use mathematical concepts and procedures in order to reach conclusions through creating an environment where learners share their knowledge, discuss their mathematical thinking and form communities of practices (Schoenfeld, 1992). The results might be interpreted such that most participants might provide learners with a learning environment where mathematical activities, processes or problems can be interpreted and conceptualized through multiple viewpoints and where learners share and discuss their mathematical ideas through taking responsibility of their learning (Wilson & Lloyd, 2000). When group comparisons were investigated on distribution of mathematical authority variable, there are remarkable results found with respect to gender, age and education level. Younger and less-experienced teachers distributed mathematical authority well since the reason might be that they have more positive towards integrating educational innovations into teaching practices (Ghaith & Yaghi, 1997). Furthermore, teachers' educational background also contributed their adaptation to new perspectives in their teaching practices through learning more about reform changes. Therefore, as a further investigation, it is important to examine teachers' varying backgrounds in (e.g. ,their age, career stage, generational identity of teachers) with respect to authority within the classroom to have an idea about the effectiveness of educational changes (e.g., student-centered learning) (Hargreaves, 2005).

Lastly, external pressure is highly perceived by most teachers in the sample. Especially change in curriculum, timing, content, and students' attitudes towards mathematics, classroom size, parental expectations and achievement test were found to be important factors affecting teaching practices such as promotion of metacognition. School context characteristics consisting of classroom size, curriculum, parental expectations, expectations from principal and timing creates occupational stress or pressure on teachers that affect promotion of self-regulation or metacognition negatively (Lombaerts et al., 2009). When group comparisons were taken into account in terms of perceived external pressure, on the contrary to literature (Karaköse & Kocabaş, 2006), public school mathematics teachers perceived higher external pressure than private school teachers in this study. Furthermore, the study indicated middle and secondary school mathematics teachers also differed with respect to perceived external pressure in favor of middle school mathematics teachers. The reason for this result could be given as the characteristics of age-group taught and primary learning environment (Kokkinos, 2007).

For this study, four instruments as the profiling tools were developed and validated. The data were gathered from 314 middle and secondary school mathematics teachers. The results indicated that most mathematics teachers' current thoughts on three factors namely as teachers' conceptualization of metacognition, teachers' perceptions of students' features and needs, the distribution of mathematical authority were positive. However, most mathematics teachers felt external pressures on their teaching practices. The results indicated that although most teachers have positive approaches towards promotion of metacognition, external pressure they perceived might have a negative influence on promotion of metacognition as an educational innovation. The negative factors created pressure or stress on teachers so especially teachers hesitating integrating educational changes in their classrooms may have problems with adjusting their learning environment for example with respect to promotion of metacognition (Lombaerts et al., 2009). One limitation of this study is the use of self-report instruments as the single data source. Group comparisons for factors affecting promotion of metacognition are significant, but effect sizes of the group comparisons were relatively small. The small amount of explained variances is a limitation for explaining the factors affecting promotion of metacognition (Lombaerts et al., 2009). It would be informative to describe such data in-detail through using qualitative methods (Dignath-van Ewijk & Van der Werf, 2012). The researchers would investigate FAMTAM variables based on the items of the scales through qualitative methods to check whether such variables included in these scales and teachers' statements on FAMTAM match with teachers' existing practices. For the further studies, it would be better to describe the profiles of mathematics teachers with a larger sample size in order to improve the generalizability of findings derived from the profiling tools. All in all, the study showed that mathematics teachers have already had positive approaches towards promotion of metacognition with respect to conceptualization of metacognition variable, perceptions of students' features and needs variables and distribution of mathematical authority variable. However, perceived external pressure might have negative influence on their teaching practices. As a possible next major step in research on teachers' promotion of metacognition, the investigation to what extent the variables affect promotion of metacognition can be suggested.

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Characterizing a highly-accomplished teacher's instructional actions in response to students' mathematical thinking

Rukiye Didem Taylan

MEF University in İstanbul, Faculty of Education, İstanbul, Turkey, tayland@mail.mef.edu.tr

This paper is part of a larger study which investigates how a highly-accomplished teacher and two beginning teachers notice student thinking and respond to students' mathematical thinking as they teach concepts of multiplication and division in a third-grade classroom. The focus of this paper is on describing highly-accomplished teacher's instructional actions in response to student thinking which are different than that of the beginning teachers. The participant teachers' instructional actions were analysed utilizing a framework developed by Cengiz, Kline and Grant (2011). The results revealed that the highly-accomplished teacher challenged student thinking with counter arguments and introduced alternative representations more frequently, but complimented students less frequently than the beginning teachers.

Keywords: Instructional actions, responding to students' mathematical thinking.

INTRODUCTION

Creating instruction based on student understanding and thinking of mathematics is one of the essential practices underlying teaching for understanding (Ball, Lubienski, & Mewborn, 2001; Fennema, Franke, Carpenter, & Carey, 1993). Research shows that attending to mathematical thinking of students in professional development programs can help improve both teaching quality and student achievement (Carpenter, Fennema, Franke, Levi, & Empson, 1999; Kazemi & Franke, 2004). Although these findings establish the importance of mathematics teachers' understanding of student thinking and their pedagogical decisions based on this understanding, responding to student thinking in appropriate ways is a complex skill, which requires hearing and interpreting student thinking (Wallach & Even, 2005). Therefore, there is a need to understand this core practice of teaching and help teachers improve in their practices.

Due to the interactive and clinical nature of teaching, Grossman and McDonalds (2008) call for studies that investigate details of teachers' practices instead of their knowledge or beliefs. Grossman and McDonalds (2008) argue that research in teaching lacks "common pedagogies for helping novices learn to respond to student thinking in the moment," even though teachers' responses to students during interactive teaching is one of the major components of the teaching practice.

Berliner (2001) identified flexibility and adapting the lesson according to students' responses as one of the major skills that differentiate experts from novices. Among other studies, Borko and Livingston (1989) provided further evidence for this argument by observing and interviewing student teachers and their cooperating teachers before and after they taught lessons for a week. Researchers found that the cooperating teachers were much better than their student teachers at improvising the lesson according to unexpected student questions or comments.

There are relatively few studies that specifically focus on how teachers respond to student thinking during mathematics instruction (Even & Gottlib, 2011; Pierson, 2008). While many studies document different ways the way teachers respond to student thinking, these studies are either focused on classroom discourse (Even & Schwarz, 2003), teacher practices as they implement a specific curriculum (Fraivillig, Murphy, & Fuson, 1999), as part of an intervention program (Doerr, 2006), or in the context of a professional development program outside classrooms (Jacobs, Lamb, Philipp, & Schapelle, 2011).

Cengiz, Kline and Grant (2011) investigated how six elementary school teachers elicited, supported and extended students' mathematical thinking through classroom observations and interviews. The authors developed a framework building on the work by Fraivillig and colleagues (1999) that focused on instructional actions related to extending student thinking during whole class discussions. The authors conceptualized extending student thinking as "helping students move beyond their initial mathematical observations and further develop an understanding of a mathematical phenomenon" (p. 356). The most common instructional actions of extending student thinking were grouped under categories of encouraging mathematical reflection, going beyond initial solution methods, and encouraging mathematical reasoning. Using counter-speculation and introducing representations and contexts that are familiar to students were the least frequently observed instructional actions that supported or extended student thinking.

The purpose of this study is to describe characteristics of a highly-accomplished teacher's instructional actions, specifically in response to students' mathematical thinking. This kind of research may offer insights in teachers' learning as well as helpful learning experiences for novice teachers in improving their practices of building instruction on students' mathematical thinking. In this study, I prefer to use the term *highly-accomplished teacher* rather than expert teacher because the latter is not clearly defined and has many different connotations (Li & Kaiser, 2011). I draw upon Schoenfeld's (2011) definition of a highly-accomplished teacher as one who spends minimal time on classroom management issues and engages in diagnostic or responsive teaching most of the time.

METHOD

The case study methodology was used in the current study. Using this methodology allows the researcher to answer questions such as "how?" and "why?" while considering the influence of context on a phenomenon within which it is situated (Baxter & Jack, 2008). The purpose of the larger study (Taylan, 2013) is to characterize third-grade teachers' practices of noticing of student thinking and instructional actions in response to student thinking during teaching. Because the larger study explores noticing of children's mathematical thinking and teaching practices among three different third-grade teachers, one experienced and two in their first-year, it represents a multiple-case study which allows the researcher to explore differences and similarities within and among cases (Yin, 2003). The findings in this paper focus on the case of the highly-accomplished teacher.

The researcher observed and videotaped each teacher's mathematics classes for a week. Each third-grade teacher worked on topics of multiplication and division in the same school district during the time of data collection. The teachers wore portable cameras and selected moments of their instruction as they taught and reflected on the selected video clips during the interviews that followed each class. The interviews and teacher-selected video clips provided additional insights on the teachers' instructional decision making processes. The researcher's observation notes and transcribed video observations of each class together with teacher interviews and lesson plans allowed the researcher to have a robust understanding of teachers' instructional actions as they responded to student thinking in the context of each classroom.

Selection of case and background

Brooke was nominated for participation as a highly-accomplished teacher by the district mathematics coordinator and school principal, both of whom have observed her teaching before. Brooke has been teaching third grade for 6 years and she has 3 years experience of attending a contextualized and an intensive professional development program where she worked one-on-one with a prominent teacher educator and educational researcher.

Brooke taught at a school where many students came from low socio economical backgrounds. With regards to mathematics instruction in her classroom, Brooke aimed for her students to comfortably share their mathematical thinking and "think for themselves" (Brooke, the first interview) instead of seeking her approval. Brooke believed her students needed multiple types of opportunities in order to understand the content she planned to introduce.

The research project took place when Brooke's class was making a transition from learning multiplication to division. The activities that Brooke created and planned together with her colleagues involved different models of multiplication and division: students jumping on the number line taped on the floor to represent skip-counting situations, and cutting ribbons of equal length for wrapping gifts. Brooke described her teaching goals in the following way:

> One of our district objectives right now is having a variety of models for division and multiplication and make connections between multiplication and division and so we have done like discrete models and then we are supposed to introduce number line models. For some of my kids that have difficulties with number sense number lines are helpful, which we did in multiplication. Some of the things they used for multiplication might help them solve division problems more efficiently rather than having to repeatedly subtract and be inaccurate in their computation (Brooke, the first interview).

Brooke aimed to create meaningful experiences for her students so that they could make connections between the new topic of learning division and what they already knew in multiplication.

Data Analysis

Instructional actions examined in this study consisted of responses to student thinking (any spoken or written mathematics related ideas, justifications or generalizations) exhibited by teachers during instruction (Fogarty, Wang, & Creek, 1983). The nature of instructional actions as teachers responded to students' mathematical thinking was investigated by analyzing whole-class videos of classroom observations and field notes. Classroom talk pertaining to each lesson was transcribed, and instructional actions were analyzed only when they occurred as a response to student's mathematics related question, answer, comment or claim.

Guided by the frameworks of Fraivillig and colleagues (1999), Pierson (2008) in general and, Cengiz and colleagues (2011) in particular, teachers' instructional actions were analyzed in two steps. First, chunks were identified that indicated existence of instructional actions responding to a student idea or question. Second, these chunks of teacher responses were subdivided into more detailed segments.

By using the conceptual framework shown in Figure 1, teachers' instructional actions were categorized as supporting instructional actions, extending instructional actions and others. To ensure reliability, classification of instructional actions within one lesson was also checked by another educational researcher until reaching an agreement about the coding scheme.

RESULTS

The results of this study emerged mostly through the use of the conceptual framework (Figure 1) and observing instructional actions across teachers for different instructional actions. Apart from the analysis of individual instructional actions, observation of a phenomenon that was emphasized in previous research, flexibility of an expert (or a highly-accom-

Supporting actions	**Other**
Repeating student idea, claim, question	*Complimenting or evaluating*
Suggesting an interpretation	*Clarifying questions*
Introducing different representations	*Requesting basic information*
Reminding of the goal	*Redirecting to a peer*
Recording student thinking	*Providing hints*
Acknowledging student thinking	
Extending actions	
Inviting students to evaluate a claim	
Inviting students to provide reasoning and probing	
Challenging/providing counter arguments to student claims	
Pushing for alternative ways	

Figure 1: Instructional actions framework (Adapted from Cengiz et al., 2011)

Mrs. Paul has got this kid in her classroom named Sam. Here is the problem that she gave him: Mrs. Paul had 24 inches to wrap two presents. How long will each piece of ribbon be?

I want you to look at Sam's work and then I am gonna ask you whether you agree or disagree with Sam and to tell me why. Alright so here is his work. Sam wrote 24 + 24 =48 and 24 x 2=48.

If you agree with Sam's work you are going to go ahead and tell me why you agree. If you don't agree with Sam's work and his ideas, I want you to tell me why (Brooke, third lesson transcription).

Figure 2: A supplemental task after coming across a misconception

plished) teacher with regards to adapting instruction based on student thinking, deserved attention.

Although all three teachers provided evidence of noticing student thinking and employing instructional actions that supported student thinking, beginning teachers failed to introduce new tasks in modification to their lessons plans based on student strengths and weaknesses as they taught.

Brooke had the flexibility of changing the lesson plans by providing additional tasks she considered to be necessary. For instance, on the second day of observation some students used multiplication instead of division. On the third day, Brooke presented the following task given in Figure 2.

Most students who agreed with this misconception in the beginning of the lesson (more than half of the students in class) subsequently changed their thinking towards the end of the class as observed in student worksheets collected and teacher's assessment during interviews. Implementation of this task (Figure 2) could be considered as responding to student thinking based on the teacher's noticing.

In analysing classroom transcriptions of Brooke based on the analytical framework, several patterns emerged. First, Brooke exhibited instructional actions that had the potential to support and extend student thinking. In particular, *repeating a student idea*, *acknowledging student thinking* and *suggesting interpretation* were the most frequently used instructional actions that supported student thinking. With regards to instructional actions that had the potential of extending student thinking, the most frequently observed actions were *inviting students to provide reasoning and probing*, *inviting students to evaluate claims* and also *challenging/providing counter argument*s.

Brooke *introduced multiple representations of mathematical concepts* in her teaching and *challenged students* more frequently than the two beginning teachers did. On the other hand, Brooke did not *compliment student thinking* as frequently as the beginning teachers did in the study. Although most of the instructional actions were observed across both the highly-accomplished teacher's class and the beginning teachers' classes, some of the instructional actions almost solely occurred in Brooke's class. Therefore, it is important to provide details of how those instructional actions are enacted in order to understand the differences between the highly-accomplished and the two beginning teachers.

Challenging / providing counter arguments

Brooke challenged her students more frequently than did beginning teachers in this study, mostly by asking questions that helped students realize their own mistakes. Brooke specifically challenged her students in the first class, when she observed many misconceptions in her students' understanding of multiplication and division. Most of student misconceptions were revealed after Brooke introduced the task of finding the patterns between multiplication and division sentences. For example, the students worked on understanding the relationship between the two number sentences, such as 5 times 7 is 35, and 35 divided by 7 is 5. Several misconceptions came to surface during this discussion. For instance, some students thought that in division sentences, numbers are sequenced from the largest to the smallest number; such as in the example of 35, 7 and 5 (35 divided by 7 equals 5) while multiplication sentences go from the smallest to the largest numbers (such as 5 times 7 equals 35). Especially during the first lesson when Brooke observed that most students had this misconception, she challenged the students and provided counter explanations. The following excerpt from the first classroom observation transcription is a typical example of how Brooke challenged a student:

Student: I know that division starts with the big number and the first number in the multiplication sentence is the answer. And the multiplication goes from smaller to bigger.

Brooke: So is that pattern always true? What about this, what if I write 7 x 5 =35, is that number sentence true? Is it always gonna go from small to big? [*challenging, providing counter argument*].

As evident above, Brooke does not directly tell the student that his answer is problematic but instead she challenges his misconception by providing a counterexample in order for this student to arrive at this understanding himself. During the interviews Brooke noted that she avoided evaluative language such as "that is a wrong answer" because she wanted her students to think for themselves, independent of her approval or disapproval.

Introduction of multiple representations of mathematical concepts

Being aware of her students' weaknesses, Brooke used alternative representations to make the concepts more meaningful to them. Additionally, she believed that each student had a different way of learning and some models/representations made more sense than others to some students. Accordingly, she believed in using a variety of representations in her teaching, as evidenced in the following interview excerpt:

> The number line makes a lot of sense to some kids. It really helps them. For some kids it does not really make any sense. And some kids with the discrete model they are like "well, this is great." It makes complete sense, and for other kids it makes no sense. So sometimes that's why I like introducing a few ways to visualize it because I feel like different kids have different ways of thinking about it so it is nice to find something comfortable for them (Brooke, the first interview).

Further evidence of introducing different representations can be found in the following excerpt given below where Brooke helps two students by suggesting use of different representations when they have difficulties of cutting 45 inches of ribbon into 9-inch strips and finding the total number of strips.

Student: I know it is going to be 45 divided by 9 but I can't figure out the answer.

Brooke: Okay. What could you guys do, what tools could you use to figure that out? Could you use a meter stick or do you want to try number line that is erasable? [suggesting use of/introducing different representations] (Brooke, third lesson).

DISCUSSION

Analysis of the highly-accomplished teacher's instructional actions in responding to student thinking revealed distinct qualities. Although all teachers in the study worked towards building their instruction on student thinking via asking students to provide their reasoning, and restate and evaluate peer's thinking, the highly accomplished teacher's repertoire of instructional actions included challenging/providing counter arguments and introducing multiple representations, unlike novice teachers.

The highly-accomplished teacher's flexibility in building instruction based on her noticing of student thinking during teaching was a finding that was not surprising based on previous research findings (Berliner, 2001; Borko & Livingston, 1989). The task presented in Figure 2 was not included in Brook's weekly teaching plan. However, she developed and implemented this task based on what she believed students needed.

Some instructional actions were more common than others across all teachers. For instance, *inviting students to provide reasoning behind their answers and probing*, *repeating student answers*, and *inviting to restate peer's claims* were among the most common instructional actions that followed up student thinking. On the other hand, *challenging and providing counter arguments* to what students said was the least frequent instructional action for teachers. This result may not be surprising given the fact that it was observed rarely even among experienced teachers in Cengiz and colleagues' (2011) study. Lack of this particular instructional action, namely, *challenging and providing counter arguments* to what students said in beginning teachers' teaching was one of the most important distinctions compared to the highly-accomplished teacher's teaching. Challenging students' thinking with counter arguments is considered an important component of extending student thinking although it is not easy to come up with this type of arguments in the actual moment of teaching (Cengiz et al., 2011; Fraivillig et al., 1999). Being able to engage in this particular instructional action likely required Brooke to listen to students carefully and to be able to generate a counterexample in the moment that would confront their misconception by learning that it would not hold in every case. It may be the case that Brooke was able to challenge her students more frequently because her experience allowed her to develop certain schemata for providing responses when students had misconceptions (Borko & Livingston, 1989).

The instructional actions that require students to provide reasoning and evaluating peer's answers may not be enough to create discussions that would really benefit student learning if they do not take place during a well-planned math discussion. The selection and sequence of student ideas to be shared with the whole class are the key components to a quality instruction (Stein, Engle, Smith, & Hughes, 2008). The experienced teacher was confident in her choices as she selected particular students to share their thinking with the whole class and their sequence of appearance (Brooke, the third interview). This may be a difficult task for novice teachers.

Brooke *introduced different representations of the concepts* of multiplication and division frequently in her class. Brooke's experience in both teaching and professional development where she created curriculum materials and thought about appropriate models for student learning may have helped her to develop a larger repertoire of representations. According to Shulman (1986) being able to understand and present different representations of the same concept is an important component of teacher's pedagogical content knowledge.

CONCLUSION

Adapting instruction based on student thinking, challenging/providing counter arguments, and introducing multiple representations were characteristics of the highly-accomplished teacher's instructional actions in response to student thinking in this study. The findings have the potential to contribute to research in teachers' professional development, especially in creating schemata of instructional actions that novice teachers may learn from to become better teachers.

Providing illustrations of how experienced teachers employ a variety of instructional actions may prove valuable for novice teacher learning. Watching video cases of highly-accomplished teachers such as in this study, reading transcribed teaching or learning to provide hypothetical responses in the form of challenging students or providing multiple representations may be valuable tools for beginner teacher's learning of the profession of teaching.

There is a need for studies that investigate instructional actions of other experienced and highly-accomplished teachers as they teach different mathematical topics in different contexts. Considering future research, it may also be worthwhile to explore the relationship between specific instructional actions in response to student thinking and their impact on students' participation in mathematics discussions and achievement levels in mathematics.

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Promoting the understanding of graph representations by grade 3 students

Isabel Velez and João Pedro da Ponte

Universidade de Lisboa, Instituto de Educação, Lisbon, Portugal, velez@campus.ul.pt

This paper aims to understand how two grade 3 teachers promote students' learning of graph representations. The conceptual framework addresses representations and teachers' actions as students work on a task involving graph comprehension. Data collection includes observation and video recording of two classes. The results show that to promote the understanding of the graph representation teachers supported students in reading the data and in reading between the data, mostly by questioning.

Keywords: Teachers' practices, graph representations, graph comprehension.

INTRODUCTION

Representations are a key element on students' learning. Several authors have been studying the role of representations in mathematics learning (Bruner, 1999; Thomas, Mulligan, & Goldin, 2002). The NCTM (2000) indicates that representations help students to interpret, organize and understand the information given in a problem statement, to figure out how to reach the answer, and to monitor and evaluate their work. Therefore, it is important that teachers provide students with opportunities to learn and understand different types of representations (Bishop & Goffree, 1986).

Research regarding interpreting and using graphs has been receiving more attention. According to Friel, Curcio and Bright (2001) graphs are "used for data analysis function as discovery tools at the early stages of data analysis when the student is expected to make sense of data" (p. 132). They also refer that this side of the use of graphs in the classroom is related to the school curriculum. In 2012 Portuguese teachers were starting to use a new school curriculum (Ministério da Educação, 2007), that emphasized for the first time, the importance of working statistics from the early ages. In this paper, we aim to understand how two grade 3 teachers promote students' learning of graph representations.

REPRESENTATIONS AND TEACHERS' PRACTICE

A representation is a mental or physical construct that describes aspects of the structure of a concept, and its interrelationships with other concepts (Tripathi, 2008). Bishop and Goffree (1986) say that, to foster students' understanding of representations, the role of teachers is to facilitate "the interpretation of the many representations commonly used [and] encourage the connections to be exposed, so that pupils can share that knowledge" (pp. 335–336). Stylianou (2010) refers the importance of understanding more than one representation related to the same concept and states that teachers may select those which they regard as more appropriate to promote their students' understanding. When students explain their solutions this provides the teacher with the opportunity to understand students' answers and to know how they interpret a representation. As students work or present their representations, teachers may ask them to explain such representations, thus supporting the establishment of connections between representations and making conversions and treatments (Duval, 2006).

Statistical graphs are an important kind of representation. Goldin (2000) says that graphs may promote new learning, providing students opportunities to build their own knowledge and to participate in discussions. According to Friel, Curcio and Bright (2001) although all graphs have a similar framework, each type of graph has its own specifications and its own language. When a student reads a graph he or she must be able to describe, organize, represent, analyse, and interpret data, and relate it to its context. They state that "graph instruction within a context

Students' activity regarding representations	Teachers' actions
Designing/Choosing	Promoting free choice Hinting through questioning Providing explicit suggestions or examples
Using/Transforming	Challenge students through open questioning Asking to explain in a structured way Suggesting alternatives
Reflecting	Guiding or challenging to establish further connections Guiding or challenging to find conversions or treatments
	Promoting the evaluation of the work done Promoting systematizations

Table 1: Teachers' actions regarding students' representations

of data analysis may promote a high level of graph comprehension that includes flexible, fluid and generalizable understand of graphs and their use" (p. 133). The authors also refer that students' difficulties in reading and understanding graphs are associated (i) to establishing connections between data, (ii) to the graph with known situations, and (iii) to the task, students, and class characteristics. To map students' understanding of graph representation, Curcio (1987) proposed a framework with three levels of graph comprehension: (i) reading the data, responding to simple questions that just require direct readings; (ii) reading between the data, finding relationships in the data and making simple inferences; and (iii) reading beyond the data, answering questions based on extensions, inferences or predictions based on the interpretation of the data. Friel, Curcio and Bright (2001) suggest that, when students work on a task, teachers may question them, in order to promote the different levels of graph comprehension.

Teachers' practices strongly influence students' learning, and in particular the way they use mathematical representations in the classroom (Stylianou, 2010). Saxe (1999) states that practices may be regarded as recurrent and socially organized everyday life activities. An important aspect of teachers' practice is how they use tasks in the classroom (Ponte & Chapman, 2006). Often this is done through three moments: introduction of the task (which can be made by teacher or by actively involving the students), students' work (individually or in small groups) and whole class discussion and systematization of representations results (Ponte, 2005).

Regarding whole class discussions, Ponte, Mata-Pereira and Quaresma (2013), identify four types of teachers' actions (inviting, challenging, supporting or guiding and informing or suggesting). They give special attention to challenging actions by which teachers support students in discovering new information. Based on this perspective, we analyse the students' work on a task and we connect their activity and teachers' actions, focusing on how teachers promote the understanding of representations (Table 1).

Students' activity may involve designing a representation, using and transforming a representation or reflecting about used representations. Teachers' actions are closely related to students' activity. To support the students' in designing a representation teachers may (i) promote the students' free choice about an adequate representation, (ii) give students some hints about a representation, or (iii) give a suggestion or example that students should use. To promote the use/transformation of a given representation, the teacher may (i) challenge students using open questions to promote students' thinking about transforming their representations, (ii) ask students to explain their solutions in a structured way, or (iii) suggest students an alternative to their representations. In the reflection phase teachers can guide or challenge students to (i) establish connections between the used representations or others that could be used as well and (ii) do conversions and treatments of representations. The difference between guiding and challenging depends on the questions that teachers make (in a more open or structured way). Teachers can also involve students into (i) evaluating the work that has been done and (ii) systematizing information.

RESEARCH METHODOLOGY

This study is part of a qualitative research on the practices of a working group of four primary school teachers that in 2013 were in a school cluster in the surroundings of Lisbon, all teaching grade 3. The teachers of this working group were striving to promote problem solving with graph representations in their classes. The participants in this paper are two of those teachers, Rui and Catarina. We choose them because in their classes we observed episodes illustrative of a wide variety of interesting situations. Both teachers had less than 5 years of experience and already knew the students in their classes from the previous school year.

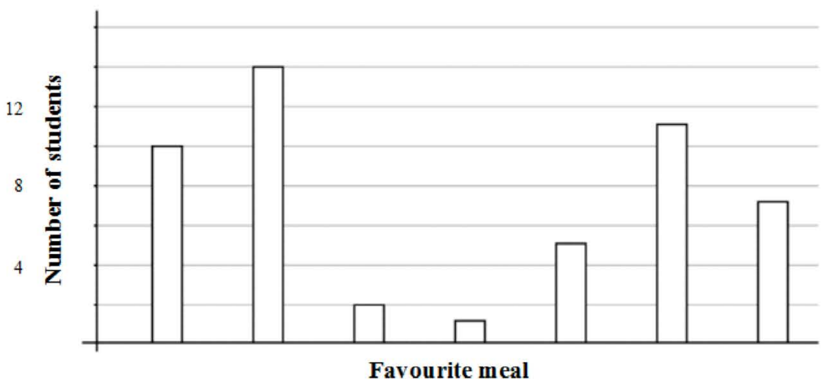
Data was gathered by video recording during class observations (whole class moments and individual teacher-student interactions). The first author collected the data and was a non-participant observer in the classes. Data was analysed trough content analysis regarding the different moments of classroom work on a task (Ponte, 2005), teachers' actions concerning the understanding of representations (Table 1), and the different levels of students' graph comprehension (Curcio, 1987). After transcribing all the audio and video data, we analysed teachers' and students' interactions and coded their actions with the categories of Table 1 and the levels of Curcio (1987). In this paper, we analyse the work carried out on a task on two different classes. The task was planned by the teachers' working group. At the beginning of the research, the teachers identified some topics that they wanted to teach (from January to June of 2013) and the first author suggested them some tasks. As the students had done some previous work with pictograms and graphs, the teachers chose a task involving a bar graph striving to address students' needs and difficulties in reading the data and reading between the data.

RESULTS

The task "The favourite meal" included a bar graph and a set of seven statements. Two questions (with sub questions) were made regarding the information provided in the graph and in the statements.

Interpreting statements a), d), f) and g) involve reading the data and interpreting statements b), c), e) involve reading between the data. As the information

Task – The Favourite meal



The schools' principal wants to know which students' favourite meal is. He received a report that included a graph and several conclusions:

a) The hamburger and fries were the most voted meal;
b) The number of students who chose the hamburger and fries were the double of those who chose roasted chicken;
c) The fish and chips had less 4 votes than hamburger and fries;
d) The spaghetti Bolognese was the second most voted meal;
e) The creamy codfish had 4 more votes than the baked fish;
f) 5 students voted on creamy codfish;
g) Some students voted in peas with eggs.

1) The principal received this little report with the graph but he was not pleased with it. He found the conclusions very confusing. After all, how many students voted? And how many voted for each meal?

2) Examine the graph that was done. What could be improved? What is well done?

Figure 1: Task presented to students (based in Martins & Ponte, 2010)

is already represented in a graph, the students do not have to design or choose representations – they are called upon to use, transform and reflect on given representations and on their own work.

Catarina's class

In the presentation of the task, Catarina begins by addressing the graph conventions (title, categories, vertical axis, horizontal axis) with her students, questioning them and providing some hints ("Why are graphs necessary?... What is the meaning of the numbers in the graph?..."). By doing this, she helps students in reading the data.

During the students' autonomous work, depending on their difficulties, Catarina walks around the room and asks students to explain their answers through open questioning or in a more structured way, seeking to understand their work. For example, regarding statement a), through open questioning, she challenges Jorge, a student who apparently does not have difficulty in solving the task, about his answer (he answers that the hamburger and fries matches to the second column). However, she is not sure how he read the data and continues questioning him about how many students chose this meal:

Jorge: Well... If this is twelve [points to the number twelve on the vertical axis], this should be thirteen...! Right?

That way Catarina could notice that although Jorge's written answer is correct, and his initial explanation seemed right, he does not have a proper understanding of the graph, assuming that the scale was of one element. The teacher continues to question him in a more structured way ("Let's look closer to the number of students...", "Where is the number zero?", "Let's complete the scale [in the vertical axis]"...). In this way, she leads the student to recall the graph conventions, focusing him in how to read the data, and guiding him to take a closer look at the graph scale on the vertical axis. Finally, Jorge discovers for himself his mistake.

In the whole class discussion, Catarina asks the students to present their solutions. She calls upon students according to their difficulties in reading the data and in reading between the data, especially reading the vertical axis scale and comparing graph bars that she had noticed in students' work. For example, Catarina had realized that António justified his answer to statement e) using just part of the information provided (if one column is greater than another one and if codfish has more votes than baked fish, codfish will be one column and baked fish another one). This prompts her ask him to present his solution to the class:

António: [as he points to the fifth column] [This is] the codfish...
Catarina: Why?
António: Because it has more than this one [he points to the third column].
Catarina: Hum... How many [votes]?
António: Two...
Catarina: But the codfish has plus four votes than the baked fish . . . How many votes does this has? [she points to the column chosen by the student]
António: Two...
Catarina: You have two votes...? So two plus four ... How much is it?
Antonio: Six...
Catarina: Do you have any [column] with six? . . .
António: No... Just five...!!

With this sequence of structured questions Catarina leads the student to read between the data, and António realizes that his answer was wrong. Afterwards, she continues to question him, focusing in how to read the data, and reviewing some elements of the graph representation that are necessary to solve the task, such as analysing the value of the remaining columns (reading the data) and comparing the with the scale of the graph.

In the discussion of the second question of the task, Catarina focuses in how to read the data, guiding students in reviewing the conventions of graphs. She uses open questioning, challenging students to find what they could improve ("What's missing?"). Some students say that "the graph is incomplete" and identify the two missing elements. Vanessa points to the missing categories on the horizontal axis ("It does not say the name of each meal (...) Here!") and another student refers to the missing title of the graph. At this point Catarina decides to challenge students, testing them about the size of the columns with a "bad example". On the black board, she erases some of the original columns and enlarges their width. As the students respond negatively to what she did, the teacher asks them: "Why are you reacting like that?! Why can't I do it like this?" The students' answers ("Because it would seem like a super meal", "It is occupying too much space!", "They must have the same width!") show that they know that all columns must have the same width. In the end the whole class discussion, Catarina systematizes with the students the information discussed and writes it on the board.

Rui's class

During the presentation of the task Rui provides some examples of how to read the data and questions students in a structured way. The teacher talks with them about graph conventions (role of horizontal and vertical axis) and states that it is important to understand what the axes represent ("This graph has . . . It has two axes... This axis here [points to the vertical axis] is the number of students that voted, right? And that [points to the horizontal axis], indicates the favourite meal").

During the students' autonomous work, most of them show great difficulty in understanding the graph representation and in reading the data. They have trouble in identifying the scale on the vertical axis and in relating the data on the graph to the sentences. As Rui realizes those difficulties, he decides to solve the task in a whole class discussion, asking students to explain their answers in a structured way. This decision motivates the class, prompting the students to reflect on the task and to participate in the discussion. Some of them start to read between the data. For example, on statement a), Bernardo justifies his answer saying that if the hamburger and fries is the most voted meal, then the corresponding bar would be the higher one ("The first [statement] indicates that the most voted meal was the hamburger and fries... So [the bar] that is the uppermost is this [meal]).

At this point, Rui invites all students to participate in the discussion and to present their solutions, and he challenges them through open questioning. For example, in statement b) he inquires a group of students that are talking about their solution:

Rui: Double of roasted chicken... What does it means? Explain me that!
Ulysses: The number of students who chose hamburger and fries was twice... Of those who chose roasted chicken... . . .
Rui: ... So what can I take from this? ... How am I going to do that?
Carolina: We have to go to the other [column of hamburger and fries]...
André: We have to see the numbers in this table [graph]!
Rui: We have to see the number in the graph, how?
André: Well... We have to see what number is [in each column]... We have to [compare columns and] see if [which] is twice of the roasted chicken . . .
Carlos: It's [the column] standing next to the hamburger with fries!
Rui: Come here to explain me how do you see that...

Carlos [as he goes to the black board]: I am telling you that... This [the second column] is the hamburger with fries and [the roasted chicken] is this [the first] one... Because...This [the first column] is eight and eight is the half of the double [the column that values fourteen]...

Rui begins by challenging students to read between the data and interpret the statement, namely the meaning of "double". Through open questioning ("Explain me that!", "How?", "What does it mean?...") he realizes that there were two strategies for dealing with statement b): comparing column size, such as Carolina suggests, or comparing column values, as André indicates. Prompted by Rui, André explains his answer.

Almost at the same time, Carlos insists that the first column is the right answer and Rui decides to question him in a structured way. For Carlos it is clear that he has to find the column which value is half of the hamburger and fries column ("half of the double") and he knows that half of fourteen is seven. However, as he could not find the column that corresponds to seven meals (because the number seven it is not identified on the vertical axis), he tries to compare column sizes and chooses the one that he thinks it is closer to seven. That way, he chooses the first column that he thinks it worth eight meals. Rui notices that Carlos is having difficulties in reading the data, and he promotes a comparison between the two strategies trying to make everyone understand why only one of them is reliable. Using Carlos as a representative of the users of the first strategy, Rui challenges him to explain how he can be sure that the value of the column that he chose is half of the column corresponding to the hamburger and fries. The student tries to read between the data, not acknowledging that he is not able to read the data. As the vertical axis does not have all the numbers identified, Carlos tries to compare the size of the columns, measuring them with his hands. Rui continues to challenge him through open questioning and asks Carlos and his classmates ("Do you think that column [first column] is half of the one of hamburger and fries?"). With this question, all the students begin talking at the same time, trying to pick the right column.

As Catarina did, Rui also realizes that the class is struggling to read the data (namely the vertical axis scale). He decides to explore the graph with his students making questions in a structured way. He points to the intersection between the two axes ("Which number is this one?") and that guides students to look at the scale of the vertical axis and to read the data. Leonardo, says "Oh! It's two by two!" and Miguel completes the answer of his colleague saying "It is two, four, six, eight, ten and twelve!" which makes the whole class understand the scale and therefore the graph representation. As a consequence, Rui decides that students can return to work autonomously. As the work goes on, the teacher analyses students' solutions and questions some of them through open questioning and structured questioning. Afterwards, during the whole class discussion of results, the students have no difficulties in explaining their answers.

In the second part of the task, Rui challenges the students through open questioning and they easily indicate what could be improved in the graph. For example, Bernardo states that the given scale (that he considers incomplete) made the graph difficult to read ("Because it had not all the numbers!") and André indicates that the graph should have a title ("The title!! It is missing the graph title!").

After the whole class discussion, Rui systematizes with the students the relevant information related with reading the data and reading between the data (title, scale, y-axis and x-axis, categories, size of columns) and he promotes the evaluation of the work done as well as what students have learned.

CONCLUSION

Teachers' actions as the students seek to use and transform the information provided on the graph and the statements mainly involved open questioning and questioning in a structured way to lead students to explain their answers. They also questioned students in a structured way to discuss the graph conventions and rules in order to lead them to understand the main characteristics of that representation.

As the students identified the relevant graph conventions they could read the data. This also allowed them to read between the data (Curcio, 1987). When the students were having trouble in reading between the data, the teachers decreased their questioning level in terms of graph comprehension (Friel, Curcio, & Bright, 2001), and sought to make sure that students could read the data by questioning them in a structured way about the graph conventions. When the students were able to read the data, the teachers increased the questioning level and challenged them through open questions. Rui and Catarina had to decide how and when to intervene, and how and when to increase or decrease their questioning level, helping students to understand the graph representation without jeopardizing the activity on the task, making it too easy or too hard.

During the discussion phase the teachers tended to guide and challenge students. Both of them systematized the most important information at the end of the discussion. As Bishop and Goffree (1986) suggest, the teachers chose students with right and wrong answers depending on what they wanted to explore. For example, Rui picked a situation when a student had a different solution from everyone else.

In conclusion, to help students to read the data teachers tended to question in a structured way and to help students to read between the data teachers tended to use open questioning. Both teachers used the whole group discussion in which students presented their solutions as an important learning moment in which they challenged and guided the students to explain their solutions, so that all of them could understand their colleagues' solutions. As the teachers recognized the students' answers, it was easier to help them to identify and correct their mistakes. The teachers used mostly open questioning and questioning in a structured way during the introduction, the students' autonomous work and whole class discussion, with very little resource to suggesting alternatives. The results that we present in this paper supports idea that the teachers' choice of tasks and handling classroom communication are crucial aspects of their practice (Ponte & Chapman, 2006), providing the required opportunities to develop students' mathematical learning.

ACKNOWLEDGEMENT

This study is supported by FCT – Fundação para a Ciência e Tecnologia through the Project Developing Statistical Literacy (PTDC/CPECED/117933/2010) and also a grant to Isabel Velez (SFRH/BD/97033/2013).

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Students' perceptions of Norms in a reformed classroom

Richard Wester[1], Anna Wernberg[1] and Tamsin Meaney[2]

1 Malmö University, Sweden, anna.wernberg@mah.se

2 Bergen University College, Norway

This paper explores tensions between the teacher's intention and the students' interpretation of a reformed classroom practice. Focus for this paper is particular on the social and socio-mathematical norms. The example presented in the paper is connected to the use of resources, such as manipulatives, to catch both explicit and implicit tensions between the students' perceptions of existing norms and teacher´s intention of supporting norms. These tensions could form and cause a barrier to students' opportunities to learn. Reform teaching is likely to fail if students do not share a similar understanding to that of their teacher in regards to their contribution to mathematics learning.

Keywords: Reformed teaching, tensions, social norms, socio-mathematical norms, students' perspective.

INTRODUCTION

School Inspection's quality review (Skolinspektionen, 2009) of Swedish mathematics teaching reveals that teaching in general is strongly connected to the use of textbooks. Students working individually in textbooks dominate Swedish mathematic teaching practice (Bergqvist et al., 2010; Kjellström, 2005). Alrø and Skovsmose (2002) term such teaching as the *exercise paradigm*. In the exercise paradigm, students are expected to learn how to master tasks by executing a series of procedures in order to gain the correct answer. According to Alrø and Skovsmose, an alternative to the exercise paradigm is *a landscape of investigation*. In that setting, students are encouraged to ask questions and to work together to investigate issues. The starting point is the students' previous understanding and that students are active seekers of learning.

In Sweden, a new syllabus (Skolverket, 2011) for mathematics education was implemented in 2011. It focuses on developing students' mathematical competences. The idea of mathematical competences in syllabus originates from the Adding Up report (Kilpatrick, 2001) and the KOM-project (Niss & Højgaard Jensen, 2002). Competencies provide another way to explain what it means to master mathematics. Both of these previous projects had the intention of changing teaching practice in schools (Boesen et al., 2014), by creating a broader view of what school mathematics means.

As a consequence of the introduction of the new curriculum and the reports such as the one by School Inspection, professional development in Sweden has concentrated on supporting mathematics teachers to adopt teaching practices which are less-textbook focused and require students to be more actively engaged. Although many teachers have been involved in extensive professional development of the kind advocated as best practice by mathematics education researchers (Rodgers et al., 2007), the impact on classrooms seems to be minimal. This follows a worldwide indication that changing mathematics teaching is challenging. For example, since the publications of NCTM's standards at the end of the 1980s, there has been a world-wide push to reform teacher-directed mathematics classrooms but this has met with limited success (Jacobs et al., 2006).

In trying to understand why this is the case, much research has focused on teacher change (see, for example, Clarke & Hollingsworth, 2002). There has been little research which investigates students' perspectives on reform teaching practices and whether their perspectives might contribute to the status quo remaining. One of the few studies is that of Graue and Smith's (1996), who investigated students and their parents' perceptions of reform mathematics classrooms. Graue and Smith showed that different students in the same class described the new teaching practices in various ways, almost as if they had experienced different things. There were often strong similarities between the students and their parents' stories. Graue and Smith concluded that when the students interpreted the teaching practices, they related them to their previous experiences of mathematics, some of which were shared with their parents. The students' stories also showed links between their perceptions of the reform classrooms and how they succeeded with the earlier practices. Those who considered that they were successful with the previous practices seemed averse to accepting the new teaching practices whereas those who had previously performed poorly talked more positively about the possibilities with the new practices.

THEORETICAL FRAMEWORK

There are many rules and routines that guide and frame the classroom practice (Jablonka, 2011). Without these, teaching would not be possible. Some rules are explicit, while others are unspoken and thus hidden. The hidden rules, you learn by participating in the practice. The rules are not fixed, but are changing continuously by the participants. To succeed as a student in school mathematics, it is not enough to know the mathematics thought. You must also cope with what it means to be a student in a mathematics classroom. One must be able to follow the rules of the classroom, both explicit and hidden.

To successfully reform a teaching practice means that there is a change in the prevailing teaching culture available in the mathematics classroom. Old understandings about teaching of mathematics are shattered and needs to be renegotiated in different ways. Cobb and Yackel (1996) describe the agreed rules that operate in a mathematics classroom as norms; social norms and socio-mathematical norms.

Social norms operate in all classrooms to regulate and frame the social interaction between teachers and their students. These norms are established, often implicitly, as agreements in the group, rather than by a single individual. However, individuals may have their own ideas about how the norms operate. Examples of social norms are expectations about explaining and justifying solutions, listening and trying to understand others' thoughts, etc.

In mathematics classrooms, there are ways of doing things that are specific to mathematics teaching and these Cobb and Yackel labelled socio-mathematical norms. Some examples of socio-mathematical norms are what is valued as an effective mathematical solution or what should be included in an acceptable mathematical solution. From participating in classroom practice the individual's perception are influenced by these socio-mathematical norms. The individual's perceptions will then affect the group's socio-mathematical norms.

In the examples scrutinised for this paper, Cobb and Yackel's (1996) framework provides as the analytical tool. Originally, this framework was used to analyse mathematics classroom interactions so that the taken-as-granted ways of behaving, *norms*, could be recognised and their role in determining what occurred in the classroom better understood (see, for example, Kazemi & Stipek, 2001). Cobb and Yackel found the framework had great potential for systematically structuring an examination of an otherwise complex and messy classroom. This framework therefore seemed relevant for exploring tensions between students and teachers.

DATA COLLECTION AND ANALYSIS

The data examined in this paper come from a larger study (Wester, 2015), investigating students' interpretation of their reformed classroom practice. The study was situated in a classroom where the teacher had made a major effort to change her teaching as suggested by the Swedish National Agency, defined earlier. Data from the students' perspective were gathered through semi-structured focus group interviews. These groups consisted of three or four students and were put together randomly. A round of interviews occurred during the spring semester of grade 8, the autumn semester of grade 9 and the spring semester of grade 9. The students were part of the first cohort to experience a new curriculum and a grading system which were implemented to support teacher change.

The extracts should be considered as illustrations, which exemplify how students from this classroom talked about their new teaching practice. Students' perceptions were compared with teacher's intensions. The different ways the issues were discussed were compared in order to identify tensions. These differences identified tensions operating. Three kinds of tensions occurred: *explicit tension*, *tension inside a norm* and *tension between different kinds of norms*.

Explicit tension

The explicit tensions are easy to get hold off. When discussing classroom practice, students express that they have a different opinion than the teacher about the practice in the mathematical classroom. That is an explicit tension.

The students described how they experienced the mathematics teaching had changed from previous school years. Working in textbooks was no longer seen as the obvious activity connected to a mathematics lesson. Instead, using manipulatives had become increasingly prevalent.

> Student 1: Yes, it's *very practical subject* at the moment, which I think is wrong. So those blocks and so, even drawing and so, which I think is wrong. It's really very practical. And then it's too little, there is very little with books. It's *not so much calculating in books* nowadays. And then it's very much like visualizing the numbers in front of you (2012-05-31). [*Ja, det är väldigt praktiskt ämne just nu som jag tycker är fel då. Så de med klotsar och så, även rita och så tycker jag är fel. Det är väldigt mycket praktiskt. Och sen är det för lite, det är väldigt lite med böcker. Det är inte så mycket man räknar i böcker nu för tiden. Och sen är det väldigt mycket vi ska se talet framför oss.*]

Students stated that teaching was no longer just about calculating in textbooks. The student gave examples of various practical activities. The first sentence in the quote shows explicitly that there is an explicit tension between the student's view about mathematical teaching and the new practice. The student draws on his previous experiences about mathematical teaching, working individually in textbooks (Kjellström, 2005; Skolinspektionen, 2009) to criticise the new practices. The student was not convinced about the value of a new teaching practice towards learning mathematics.

Potential tensions

The hidden tensions that are not expressed explicitly by the students are more difficult to detect. Just because students do not express tensions, do not impose they do not exist. We call these, *potential tensions*. Two kinds of potential tensions were found in the data; *potential tensions inside a norm* and *potential tensions between different kinds of norms*.

Potential tensions inside a norm

In next extracts, students discussed a task where they could use manipulatives. The task instructions were:

> Student 4: *You get to understand exactly how big a cubic decimetre is.* And you realize it can have different shapes. [*Man förstår hur stor en kubikdecimeter är. Och att den kan se ut på olika sätt.*]
>
> Student 1: It is *rather basic in the beginning*. You start with number one. Do one thing. [*Det är en rätt grundläggande början. Alltså att man börjar med ettan. Gör en sak.*]
>
> Student 2: It will take some time to cut and paste. [*Det tar ju tid också att klippa och klistra.*]
>
> Student 4: Yes. It takes a lot of time to put the whole thing together. Folding and taping. [*Ja. Det tar mycket tid att hålla ihopa den. Vika och massa tejp.*]
>
> Student 3: I think this is rather good, when you are supposed to make different shapes, to get *to know which sizes are possible* to have making a cubic decimetre. You have to understand how big or small a cubic decimetre really is (2012-12-19). [Sen är det rätt bra tycker jag när man ska börja göra olika former på dem att man lär sig *vilka mått man måste ha* för att få just en kubikdecimeter. Så man förstår hur stort eller litet en kubikdecimeter är.]

Students indicated that they knew what it was they were supposed to understand from participating in the activity; getting to visual the actual size of a shape which could hold a cubic decimetre. They also have to understand how the shape could vary. Seeing the different representations of a cubic decimetre is valued by the students, at least early in the group discussion. As the discussion continued, the interpretation of the task above changed. The given task was now discussed in relationship to its teaching context by the students.

> Student 1: That was what it was about in our *lesson before. How to do to make a correct*

calculation. How to get a cubic decimetre. All connected to this task. [*Det fick vi hela lektionen innan. Hur man skulle räkna ut det. Få fram en kubikcentimeter. Se vad det var till uppgiften.*]

Student 2: You have it all *in the textbook*. [*Och det stod ju i matteboken.*]

Student 4: Yes, but this is of course rather ... basic. [Ja men det här är ju väldigt ... grunden.]

Student 1: Actually. This task is connected to exactly what we were discussing in the lesson before. We had a whole lesson on how to calculate it. How to calculate volume of 1 cubic decimetre, and others. And the next lesson was about this task. [*Egentligen. Det här är en uppgift på det vi gick igenom på hela den lektionen. Så vi hade fått egentligen en hel lektion på hur vi skulle kunna räkna ut kubikdecimetern. Eller hur man skulle räkna ut kubikmetern, och massa sånt. Och sen efter det kom uppgiften.*]

Student 4: This task is typical at the beginning of a new chapter (2012-12-19). [*Den här uppgiften är vad man gör i början av ett område.*]

The students now connected the task to what they had done in the previous lesson, which had been about how to calculate volumes. The manipulative activity now shifted into a task about supporting to calculate volume of objects, even though calculations are not mentioned in the task instructions. Understanding the need to visualise the decimetre, discussed by the students in the first part, is now something you just need to do in order to do the calculation properly. The existing *socio-mathematical norm* expressed by the students is to be able to do the calculations (see Table 1). The students experience calculations are easier to do when they have produced representations of the different objects with manipulatives. They thus interpret the purpose of the manipulatives to support calculations and not the understanding of concept of volumes. The students´ beliefs about what school mathematics is about do not allow such an interpretation. In students´ beliefs and values, concepts have a secondary importance in relationship to mathematical procedures. Manipulatives are thus tools doing the mathematics (see Table 1).

The teacher's intentions with the specific task are not in the interviews with her. However, the teacher had expressed elsewhere how she considers manipulatives to support students´ understanding of mathematical concepts.

Teacher: I was teaching multiplication of fractions. The given task was 1/3 multiplied to 3/4. It is really handy if you know how to do it. 1/3 multiplied to 3/4 is just multiplication straight ahead and it will be 3/12. It is easy to rewrite as 1/4. That´s the way. Super easy. But I didn´t stop there. *Why does it works*? Then we used paper strips. (The teacher explained how the multiplication of fraction will be *represented* through the paper stripes). We *visualized*, made it *concrete* and explained. Yes it was not so smooth, if you compared it to the procedure (2013-02-01). [*Vi skulle köra multiplikation av bråk. Då ville jag att vi skulle titta på 1/3 gånger 3/4. Det är ju jättelätt att ställa upp ju. 1/3 gånger 3/4 är ju bara att gånga varandra. Det blir ju 3/12. Det kan man snabbt och enkelt göra om så blir det 1/4. Så gör man ju. Det var ju supersmidigt. Men så försökte jag liksom varför blir det så då? Och så tog vi pappersremsor. (Läraren förklarar hur man kan illustrera uppgiften med pappersremsor) Vi visualiserade, gjorde konkret det det var. Men det är klart lite krångligare. Det andra är bara att ta och gånger varandra.*]

In this quote, the teacher compared the learning of the procedure with explaining and understanding the concepts behind the multiplication of fractions. For the teacher the manipulatives is a tool helping students to gain mathematical understanding of concepts. Developing understanding requires more effort from the students than memorizing a procedure. As the teacher wanted to develop students' mathematical understandings, memorizing procedures was considered a contradiction to this. The *socio-mathematical norm* intended by the teacher is that mathematics is much more than calculating. School mathematics also aims to develop mathematical understanding and thinking (see Table 1).

The socio-mathematical norm suggested by the students and confirmed by the teacher, is that manipulatives are useful tools in mathematical teaching as they can represent and visualize mathematical concepts. But there is a tension in relationship to this norm. Students considered manipulatives useful as a primitive form for solving tasks, and thus less valuable than calculating. The teacher on the other hand, considered them to be a tool for gaining understandings. This is an example when both students and teacher talks about the same norm but give this norm different meaning. There is an example on a *potential tension inside a norm* (see Table 1; tension between students and teacher inside socio-mathematical norm).

Potential tension between different kinds of norms

Teacher's intentions of new socio-mathematical norms described above require a different teaching approach. The new socio-mathematical norms need supporting social norms containing teacher's role, students' role and general activity (Cobb & Yackel, 1996). The teacher wants students to be active seekers of learning through discovering and discussing while working with manipulatives in groups.

> Teacher: From there I put up different kinds of tasks that *used manipulatives*, we try to use the computers part because they actually have their own computers, *much discussion* and *group tasks* so they should learn to *communicate mathematics*, is also trying to get away so that math does not become a writing topic (2012-06-13). [*Och därifrån lägger jag upp olika sorters uppgifter med laborativa övningar, vi försöker använda datorerna en del eftersom de faktiskt har egna datorer, mycket diskussion och gruppuppgifter så dom ska lära sig att kommunicera matematik, försöker också komma ifrån så att matte inte blir ett skrivämne.*]

The students' own thinking challenged and developed through reasoning and discussion. The teacher wants *social norms* containing a student role, teacher's role and a role of activity adapted to a learning landscape (see Table 1; teacher's intensions of new social norms).

In the interviews, students easily express different ways of their new teaching practice. But they never talk about changes in their role as students or the teacher's new role. Instead they keep suggesting traditional roles to the teacher, which they believe make their learning more efficient.

> Student 3 It is better, I think, to work with exercises from the textbook instead of blocks and stuff like that. I think they are hard to learn from. Instead I *want somebody to show me how to do it. Then I have to practice on my own.* [*Det är bättre tycker jag att arbeta med uppgifter man får och inte hålla på med klossar och sådant. Jag lär mig heller ingenting utav det, utan jag vill ha någon som visar så här gör du så får jag träna själv.*]
>
> ...
>
> *Math should be something to calculate.* If you manage to do it in your head it is good. *It should not be necessary to lay it out.* (2012-06-06) [*Matte ska vara någonting du räknar ut. Tänker så blir det bra. Du ska inte behöva lägga ut det.*]

In these quotations, the student is talking about what mathematics is about (socio-mathematical norm). From their socio-mathematical norm the students suggest a supportive social norm; the role of the teacher is to show them how to calculate. Since they keep repeating the supportive social norm it could be seen as an indication of their awareness of the non established social norm.

In the same way have manipulatives has a role supposed to help students to calculate. Students believe using manipulatives is a good method if you do not know how to calculate. For instance, if you are on your way learning something new. But you are later supposed not to use manipulatives if you are able to calculate. This kind of value is an example on a socio-mathematical norm (see Table 1).

Students' suggestions of traditional social norms are heavily connected to their view of socio-mathematical norms. There is a *tension between students' view of*

	Students	*Teacher*
Socio-mathematical norms	School mathematics is about mastering calculations. Manipulatives are tools doing the math.	Math is more than calculating. School mathematics aims to develop mathematical understanding and thinking
Social norms		Creation of a learning landscape contains certain roles.

Table 1: Patterns of potential tensions

socio-mathematical norms and teachers intention of new social norms (see Table 1).

CONCLUSION

Changes in mathematics teaching from a traditional setting toward reform practices give rise to changes in social and socio-mathematical norms. However, it appeared hard for students to understand teacher's intentions of new classrooms norms. Hence a new curriculum and grading system, the change from mastering procedures to developing competences is not discerned by the students. They still believe that school mathematics is mainly about learning procedures. There is a tension between students' interpretation, and the teachers' intention of the reformed classroom practice. This teacher wanted her students to develop mathematical competences and not just learn the procedures. Nevertheless, for the students it is not yet a norm that conceptual understanding should be learned from teaching. This will affect what it is possible for students to learn from using different teaching resources. Students consider the use of manipulatives as interfering with their learning of mathematics. The resources are somehow able to do what the students are supposed to learn from activities. Thus, the students think it is better if they can do the mathematics without them. This is in contrast to the teacher, who is using these resources to support students developing mathematical thinking (Table 1, tension between socio-mathematical norm and social norm).

Students' interpretation of operating socio-mathematical norms interfere students' possibility to catch up teacher´s intention of new social norms. When students are not aware of new student roles and new teacher roles, they are hindered to participate in practise. Even if it looks like they are participating, tensions of this kind works as resistance. Without students are able to understand, there cannot be any agreement which provides new classroom's norms. This might be one explanation to limited success of reform teaching (Jacobs et al., 2006).

Understanding the inherent differences between viewing these resources as part of a social or a socio-mathematical norm, or conceptions of a socio-mathematical norm, provides an indication of why some students might resist reform teaching practices in their mathematics classroom and this will have an impact on the reality of their mathematics learning. This difference could be an explanation to why students do not learn from these resources what teachers aiming for.

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Relating arithmetical techniques of proportion to geometry: The case of Indonesian textbooks

Dyana Wijayanti

University of Copenhagen, Science Education Department, Copenhagen, Denmark, dyana.wijayanti@ind.ku.dk

The purpose of this study is to investigate how textbooks introduce and treat the theme of proportion in geometry (similarity) and arithmetic (ratio and proportion), and how these themes are linked to each other in the books. To pursue this aim, we use the anthropological theory of the didactic. Considering 6 common Indonesian textbooks in use, we describe how proportion is explained and appears in examples and exercises, using an explicit reference model of the mathematical organizations of both themes. We also identify how the proportion themes of the geometry and arithmetic domains are linked. Our results show that the explanation in two domains has different approach, but basically they are mathematically related.

Keywords: Proportion, arithmetic, geometry, textbooks, Indonesia.

INTRODUCTION

Trend of fragmentation of the school curriculum makes students visit a theme after the other, but somehow the rational for visiting them or the way they are related to each other does not become clear. Chevallard (2012) defined this trend as "visiting monument" in which students are encouraged to admire and enjoy without knowing about its *raisons d'être,* now or in the past. Thus, relation among themes is necessary, particularly for an important theme such as proportion. This theme appears in many different domains, such as arithmetic and geometry. When it goes across domain, the relationship is important because it is easier to relate a theme within a domain than across domains or across grades. Making those links could help students having the experience that mathematics is a connected body of knowledge not just a collection of tricks that teachers play out to satisfy the children.

The notion of "proportional reasoning" is often used to indicate what is needed to fully operate with a variety of mathematical objects and models such as scales, probabilities, percentage, rates, trigonometry, plane geometry, linearity, fractions, etc. The so-called "missing value problems" are a common way to introduce proportion problems to the students. For example, if it requires two hours of work to make 3 puppets, how many hours are needed to make 25 puppets? Here we can find three values in the task (2, 3, and 25) and one missing value to be found. In general, proportion is considered as a relationship between quantities. More abstractly, proportion is about two n-tuples of quantities related in the same way to each other (constant ratio), for instance the ingredients in two recipes of the same cake or the sides in two similar triangles.

The mathematical notion of proportion is antique and goes back as far as to Euclid (about 300 BC). In book V, definition VI it is said that magnitudes having the same ratio are called proportional (Fitzpatrick, 2008). In book VI, Euclid also defines the geometric notion of similarity. It is also said that similar figures are those which have their corresponding angles equal and the corresponding sides about the equal angles proportional. This definition is still used in the elementary curriculum, although it adopts a more modern formulation. In particular, according to Miyakawa and Winsløw (2009), it can be said that Euclid's notion of proportion is static (about a property of given quantities) rather than dynamic (in terms of a functional relationship between variables). The dynamic definition is common in the algebraic domain which has become dominant in scholarly mathematics over the past four centuries and has been conceived in terms of linear functions between real numbers.

In the Indonesian school, the proportion theme in arithmetic is introduced as ratio equality: given four numbers a, b, c and d, the equality $(a, b) \sim (c, d)$ indicates that *a* and *b* are in the same proportion as *c* and *d*. At the beginning of the seventh grade, students start to work with missing value problems. However, proportion also appears in the geometry and statistics domain and mostly in upper secondary school, in the introduction of functions.

The fact that proportion is found not only in arithmetic, but also in geometry, makes it interesting to investigate how textbooks introduce and treat the theme of proportion in geometry (similarity) and arithmetic (ratio & proportion), and especially how these themes are linked to each other. To formulate and investigate this phenomenon with more precision, we use the anthropological theory of the didactic (ATD) by (Chevallard, 1999, 2002) in particular the notion of *praxeology*. It is not our aim to address the issue of how proportion in arithmetic and geometry should be taught in lower secondary education (in Indonesia), or how it should be treated in textbooks. However, we expect that our research into the approaches chosen in textbooks will contribute to this discussion and give it more precision.

PROBLEM BACKGROUND

The research about how teachers use resources for teaching is increasing. Gueudet and Trouche (2012) point out that documentation work where teachers interact with resources, selects them and work on them is a central in teachers' professional activity. Considering textbooks as an example of resources, this study implies that the textbooks are not only seen by teacher as a text to follow, but also it can be used as a resources for teacher learning. However, Pepin (2012) argues that it is still less clear what kind of textbooks that can help teacher learning. In her study she used tool analysis for reflection and feedback to help teachers develop further understanding and enrichment of mathematical tasks. Pepin's study inspired me to use another tool to analyse task in order to enrich teachers' point of view of quality materials for teaching.

A textbooks analysis developed by González-Martín, Giraldo, and Souto (2013) who consider the case of the introduction of real numbers in Brazilian textbooks and found an unintegrated mathematical organization as knowledge to be taught. Furthermore, Hersant (2005) conducted a historical study of how the arithmetic of proportion appears in the French compulsory education from 1884 to 1988. Based on an elaborate reference model, she demonstrated how the teaching approach is changing over time. Finally, García (2005) proposed a reference epistemological model in terms of a sequence of *praxeologies* to study linear systems and linear functions, and he used it to identify proportional relations, both in arithmetic and in the study of functions, showing a quite poor coherence between the two as they appeared in school mathematics. Our proposal to look at proportions as they are treated in the arithmetical and geometrical domain has to be considered as an extension of Hersant's and Garcia's work relying on González-Martin and colleagues' methodology.

THEORETICAL APPROACH

According to ATD, knowledge is produced, communicated, learned and used in institutions, and depends on them. The relationship between knowledge in scholarly institutions and school institutions is described using the notion of *didactic transposition* (Chevallard & Bosch, 2014). Even though our study does not aim at analysing the full transposition problem, the notion of reference model enables us to analyse specific textbook transpositions of scholarly knowledge in a wider framework, without assuming this scholarly knowledge as a universal or fixed measure.

In ATD, mathematical activity is identified into two blocks. First, there is *practical block (praxis)* which is made of the type of task and technique to solve. Second, there is *knowledge block (logos)* formed by technology and theory to explain and justify the *praxis*. We refer to Barbé, Bosch, Espinoza, and Gascón (2005) for more details on these notions. The aim of this paper (and my thesis) is to develop a textbook analysis method based on ATD.

Our analysis of textbooks mainly considers *mathematical praxeology*, but the textbook itself is also a rich resource for developing *didactic praxeology*. Specifically, textbooks contain a wealth of explicit mathematical tasks, demonstration of techniques using specific technologies, and also *theoretical discourse* which explains, relates and justifies the technologies.

Reference model

Proportion appears as themes (collections of mathematical *praxeologies* that are unified by a common technology) with different theoretical frameworks (sectors) that appear in different school mathematical domains; we consider here mainly the domains of arithmetic and geometry, while related themes appear also in other domains, especially algebra and probability. We now present parts of our reference model to analyse the two themes separately. We mainly use a categorization of techniques which are sufficient to explain *praxis* (as the types of tasks are also evident) in this model.

Proportion in the arithmetic theme is defined as follows, for pairs of numbers: $(x_1, x_2) \sim (y_1, y_2)$ if $\frac{x_1}{x_2} = \frac{y_1}{y_2}$. More generally, $(x_1, ..., x_n) \sim (y_1, ..., y_n)$ if $(x_i, x_j) \sim (y_i, y_j)$ for all $i, j = 1, ..., n$. There are three main types of task that are categorized:

T_1^{Ar}: given $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ decide if $(x_1, ..., x_n) \sim (y_1, ..., y_n)$

T_2^{Ar}: given $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ compare $\frac{x_i}{x_j}$ for $i = 1, ..., n$

T_3^{Ar}: given $(x_1, ..., x_n)$, y_1 find $y_2, ..., y_n$ so that $(x_1, ..., x_n) \sim (y_1, ..., y_n)$

The actual tasks in textbooks are usually given through the description of a daily situation, such as using scales on a map, buying and selling goods. technique that is used for T_1^{Ar} is to calculate the ratios of related terms and conclude about the proportionality. In T_2^{Ar}, the technique consists in comparing ratio of corresponding numbers or magnitudes. For T_3^{Ar}, the detail technique can be categorised as proposed by Hersant (2005). We illustrate these types of tasks with typical textbooks task:

T_1^{Ar}: The price of 2 kg rice in shop 'A' is Rp. 5.000,- and the price of 5 kg rice in shop 'B' is Rp. 12.500,- . Do both shops have the same price of rice?

T_2^{Ar}: In Bu Ina's grocery, the price of a package containing 2 kg of sugar is Rp. 9.400,-and the price of a package containing 5 kg of sugar is Rp. 22.750,-. Which package is cheaper?

T_3^{Ar}: Two students can carry 15 books. How many books can 8 students carry?

In geometry, proportion is connected to the notion of "similarity". Two polygons of the same kind (triangles, quadrilateral, etc.) are defined as similar if the corresponding angles have the same measures and the ratio of the lengths of corresponding sides are equal. Common types of tasks in the textbooks treatment of similarity include the following which are closely related to T_1^{Ar} and T_3^{Ar} respectively:

T_1^{G}: Given two polygons with the same angles and also given the side lengths of two polygons, decide if the polygons are similar.

T_3^{G}: Given similar figures with corresponding sides $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ with $x_1, ..., x_n$ and y_1 known. Find the unknown sides $y_2, ..., y_n$.

There are some textbooks examples of the types of tasks T_1^{G} and T_3^{G}, together with corresponding techniques:

T_1^{G}: One rectangle is 12 cm in length and 8 cm in width. Another rectangle is 6 cm in length and 4 cm in width. Are they the similar?

T_3^{G}: Quadrilaterals *ABCD* and *LMNO* are similar. The side of *ABCD* are 6 cm, 10 cm, 12 cm and 14 cm in length, respectively. The shortest side of *LMNO* is 9 cm. Find the length of other side of *LMNO*.

To analyse the *praxeologies* proposed in the textbooks, we will mainly use the categorization of types of tasks explained above (the basic reference model).

CONTEXT AND METHODOLOGY

Indonesia is a big country with 252 million people inhabitant. It is a big market for publishing companies and makes among them are competing with each other. Furthermore, the fact that Indonesia still lacks of teachers and some of them have a weak educational background makes them dependent on the textbooks. Thus, it is the reason why the quality of textbooks should be controlled or become a priority for the ministry of education.

The proportion theme in arithmetic (Ar) is introduced in grade 7. In Indonesian, proportion is called "perbandingan" which literally means proportion. Proportion in geometry (G) is introduced in grade 9 as "kesebangunan dan kekongruenan" (similarity and congruence). We study two themes in three different textbooks for each of the two grade levels. We refer to the textbooks as shown in Table 1 and 2.

Code	textbook title	Authors, year of publication
Ar_1	Book for studying mathematics 1	Wagiyo, Surati, and Supradiarini (2008)
Ar_2	Contextual teaching and learning mathematics	Wintarti and colleagues (2008)
Ar_3	Mathematics 1: concept and application	Nuharini and Wahyuni (2008)

Table 1: Lower secondary textbooks grade 7, analysed in this paper

Code	textbook title	Authors, year of publication
G_1	Book for studying mathematics 3	Wagiyo, Mulyono, and Susanto (2008)
G_2	Contextual teaching and learning mathematics	Sulaiman and colleagues (2008)
G_3	Easy way to learn mathematics 3	Agus (2007)

Table 2: Lower secondary textbooks grade 9, analysed in this paper

To analyze the data, we studied how the above mentioned themes in the arithmetic and geometry domain were introduced in the textbooks (that is in the main text, rather than in the collection of exercises). We considered how proportion is introduced through examples (*praxis*), with explanations of techniques and use of theoretical justifications (*logos*). This way we identified the main elements of the mathematical organization to be developed by students in both themes, according to the textbooks. Furthermore, we were also interested in how much student autonomy was foreseen in solving tasks; for this, we considered the variety of tasks that are proposed for student work (in the exercise sections of the textbook) and also their degree of similarity to examples given in the main text. Obviously, the analysis of exercises as tasks is incomplete because there are no given techniques, unlike in the working examples. Furthermore, we identified how the proportion themes in arithmetic and geometry are linked, based on the characteristics of mathematical organization and the presence or absence of explicit cross references in the text.

RESULTS AND DISCUSSION

How proportion is introduced in the arithmetic theme

We first provide some informal remarks about the theoretical structure of the relevant chapters. The title of proportion theme was different in each textbook: application of algebra (Ar_1), ratio and proportion (Ar_2), proportion and daily life arithmetic (Ar_3).

There were three subchapters in the Ar_1: 1. linear equations with one variable; 2. problem solving in daily life arithmetic; 3. proportion. Between these three subchapters we find connecting sentences such as: “we often use algebra to solve economic activity' and 'in the daily activity a lot of things correspond to proportion” (Wagiyo, Surati, & Supradiarini, 2008, p. 108 & 115). Furthermore, the authors also relate proportion to fractions with a brief sentence. In Ar_3, there were two main subchapters: daily life arithmetic and proportion, and there was no explicit connection between them. However, the authors use fractions as a tool to introduce proportion: “in the last chapter you already learned that a fraction can be considered as a proportion of two numbers” (Nuharini & Wahyuni, 2008, p. 152). In contrast to Ar_1 and Ar_3, the authors in Ar_2 did not relate proportion with linear equations and daily life arithmetics. They directly discussed a new terminology called *'ratio'* to introduce proportion. Due to the focus of research, we will only consider the proportion subchapters.

To explain what proportion is, the authors tried to avoid formal explanations. For example:

$$\frac{\text{the price of one book}}{\text{the price of fives books}} = \frac{500}{2500} = \frac{1}{5}$$

> Comparison between the amount of the books and the prices give the same numbers, so that the amount of the book and the price are proportional (Wagiyo, Surati, & Supradiarini, 2008, p. 120).

The authors used daily life situations to introduce proportion, for example, selling and buying price. Furthermore, the technologies that cover proportion in these three textbooks were very similar. For further explanation, the following table shows the number of examples in three textbooks, given to the two figures of task T_1^{Ar} and T_3^{Ar}.

Type of task	Ar_1	Ar_2	Ar_3
T_1^{Ar}	2	1	1
T_3^{Ar}	5	4	7

Table 3: Number of examples in the main text

From the table above, we noticed that the textbooks have more T_3^{Ar} than T_1^{Ar}. Also, we found a considerable number of techniques related to T_3^{Ar}. For example: reduction of unit, multiplication by ratio, proportion, graph and etc. (for further study, we refer to Hersant, 2005, techniques).

Task type T_2^{Ar} appeared in the two textbooks (Ar_1 and Ar_2), but it was only as an exercise. In the exercise section, we also found a variation of T_3^{Ar} that is symbolised as $T_3^{Ar'}$. The example of this type of task is:

> $T_3^{Ar'}$: A contractor estimates a bridge to be completed within 108 day, if it is done by 42 workers. After working 45 days, the work is stopped for 9 days for some reasons. Determine how many workers, which must be added to finish on times (Nuharini & Wahyuni, 2008, p. 159).

To solve this type of task requires considerate student autonomy. They should master in mathematics modelling before they apply technique from T_3^{Ar}.

We can point out four important observations from the discussion above: 1. proportion is located in one subchapter with a connecting sentence to the previous subchapter and there is a large variation in the titles of chapter where proportion is explained; 2. the authors tend to use informal examples (daily activity case) to introduce the notions of proportion; 3. there are two types of task that are provided by textbooks in the main texts (T_1^{Ar} and T_3^{Ar}); 4. to work with exercises, students sometimes require to develop new variations of techniques demonstrated in the main task (T_3^{Ar}).

How proportion is introduced in the geometry theme

Proportion was introduced as similarity in the geometry domain. Similarity was always explained together with congruence. Due to the focus of research, we will only consider the similarity theme. We found connecting sentences in G1 and G3 which support students to relate arithmetic techniques to geometry techniques. G1 tries to connect similarity with proportion using scale , whereas this theme is also discussed in arithmetic domain. For further explanation, see the example bellow:

> We already studied about scale in the seventh grade. Pictures that are of same scale can be found by magnification or reduction. So that scaled picture has the same proportion as the real picture. We can say that the scaled picture and real picture are similar (Wagiyo, Mulyomo, & Susanto, 2008, p. 7)

However, G3 used plane geometry to introduce similarity and there was no connecting sentence in G2.:

> We already learned about triangles, rectangles, squares, trapezoids and kites. In this chapter, we will discuss about the similarity of those figures (Agus, 2007, p. 1).

To explain similarity, the authors of three textbooks used more formal definitions such as:

> Two figures are similar if the corresponding angles have the same measure and the ratio of the leghts of corresponding side is equal (Sulaiman et al., 2008, p. 10).

We counted the number of examples of the two types of task T_1^G and T_3^G:

Type of task	G_1	G_2	G_3
T_1^G	6	3	4
T_3^G	4	1	4

Table 4: Number of examples in the main text

From Table 4, we can see that the number of T_3^G was smaller than T_1^G, because in many examples T_3^G is explicitly given that the two figures are similar. Therefore, students only need to compute the unknown side. On the other hand, student need to consider the property of similar figures to solve T_1^G. Furthermore, there is also T_4^G whereas students need to develop their own technique:

T_4^G: A rectangular frame of photographs is 40 cm × 60 cm, and a rectangular photograph is 30 cm × 40 cm. Are the frame and the photograph similar? Suppose we modify the size of the frame so that the frame and the photograph are similar. What is the size? (Sulaiman et al., 2008, p. 8).

Students need to elaborate new techniques to solve task T_4^G because they face a new type of task which is different from the example.

From the above, we can conclude four things: 1. similarity is introduced together with congruence; 2. to explain similarity, the authors use formal definitions; 3. there are two types that appear as examples in the textbooks (T_1^G and T_3^G); 4. students sometimes require autonomy to develop new variations of techniques demonstrated in the main task T_4^G in the exercise.

How the arithmetic and geometry themes are connected to each other

Based on the discussion, we can see the relation between two themes, especially the use of arithmetic techniques as a part of geometry techniques. There are two types of tasks in arithmetic (T_1^{Ar} and T_3^{Ar}) which correspond to geometry type of task (T_1^G and T_3^G).

There is a relation between arithmetic and geometry themes, but we consider it to be relatively weak. From the data, there is only one textbook (G_1) in the 9th grade which refers explicitly to 7th grade proportion. The other textbooks discuss about plane geometry (G_3) and there is not even a connecting sentence in G_2.

CONCLUSION

We have identified the main types of tasks related to proportion in the domain arithmetic and geometry as they appear in the textbooks. The introduction of proportion in arithmetic theme is more informal than for the geometric theme. For example, in arithmetic the authors use daily life activity as a tool to explain what proportion is. While in geometry, the definition of similarity is based on formal definition. However, we also found that both themes have varied types of tasks, including tasks with a considerable variety as concerns the student autonomy required: from tasks that require techniques to be reproduced from an example to main tasks that require a technique to be developed independently by the students.

We found that textbooks establish two mathematically related types of tasks in each of two themes within different domains, and found that the explicit link between these types of tasks is relatively weak. Making this link is seen explicitely by students could help them experience mathematics as connecting body knowledge. Furthermore, we point out that methodology of this study can contribute as a new approach to analyse textbooks or a new approach to choose a textbook.

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TWG19

Posters

Investigating the voice of two Swedish mathematics teacher guides

Linda Ahl and Tuula Koljonen

Mälardalen University, School of Education, Culture and Communication, Västerås, Sweden, linda.ahl@mdh.se

This poster presents a study on the voice of two Swedish mathematics teacher guides. We have made a content analysis of teacher guides to the two most commonly used mathematics textbooks in Swedish lower secondary school Grades 7 to 9. Our theoretical stance is that curriculum resources can support teachers pedagogical design capacity. Our analysis shows that the voice of both guides speaks through the teacher with narratives and scripted instructions, instead of making visible the strengths and weaknesses with different teaching design considerations. The results could be used to further elaborate on how the teacher guides can be designed to support the pedagogical design capacity for a range of different teachers.

Keywords: Sweden, teacher guides, voice.

RESEARCH TOPIC

Mathematics curriculum resources are teachers' most important tools in mathematics education all over the world (Fan, Zhu, & Miao, 2013). Emerging research shows potential in curriculum resources to support teachers' pedagogical design capacity (PDC), i.e., support teachers to find productive ways to adapt curriculum resources to reach instructional goals (cf. Brown, 2009; Davis & Krajcik, 2005). This poster presents one piece of a larger study where the support for the teachers in their teacher guides' is investigated. Our research question is: *What characterizes the voice of two dominating Swedish mathematics teacher guides?*

THEORETICAL FRAMEWORK AND METHOD

Davis and Krajcik draw on the work of Ball and Cohen and present five high-level guidelines for how to design educative curriculum materials, i.e., with the intention to promote teacher as well as student learning. We used these guidelines to develop an analytical tool to analyze the content of mathematics teacher guides (Hemmi, Koljonen, Hoelgaard, Ahl, & Ryve 2013). In this poster, we focus on one of five categories in that framework, namely to make the developers pedagogical judgments visible to the reader. This category is manifested by the *Voice of* the teacher guide; how the authors communicate with the teacher. They can speak either *through* the teacher or *to* the teacher. The difference in addressing the teacher by through or to is nicely explained by Remillard (2012) in the following quote:

> Despite the invisibility of the authors, curriculum resources have a voice that is manifested through the way they communicate with the teacher. Most curriculum resources place primary emphasis on what the teacher should do. I think of this as talking *through* teachers. That is, the authors communicate their intent through the actions they suggest the teacher takes. Few resources speak *to* the teacher by communicating with teachers about the central ideas in the curriculum. (p. 112)

RESEARCH RESULTS AND IMPLICATIONS

Our analysis shows that the voice of the investigated guides mostly *speaks through* the teacher as narratives and scripted instructions, and only occasionally *to the teacher* about the strengths and weaknesses with different teaching designs that could facilitate teachers to make choices and to keep their autonomy. The results could be used to further elaborate on how the teacher guides can be designed to support the PDC for a range of different teachers.

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Formative assessment in Swedish mathematics classroom practice

Catarina Andersson and Erika Boström

Umeå University, Umeå Mathematics Education Research Centre, Umeå, Sweden, catarina.andersson@umu.se

Using formative assessment has woken interest in many countries because of the potential effect on student achievement. To investigate Swedish teachers' use of formative assessment in mathematics, this study used classroom observations and teacher interviews of 38 mathematics teachers. The teachers used formative assessment, but additional formative activities could support teachers to better take advantage of the potential in using formative assessment.

Keywords: Formative assessment, mathematics education.

BACKGROUND

Several studies have demonstrated that substantial learning gains are possible when teachers introduce effective formative assessment (e.g., Black & Wiliam, 1998; Hattie, 2009). Despite many reform initiatives concerning formative assessment taken in different parts of the world (Tierney, 2006) a more extensive use of formative assessment is still desirable (Cizek, 2010). To be able to estimate possible learning gains for Swedish students in mathematics, and need for additional content in teacher education and in-service training we need to know the characteristics of current Swedish mathematics teaching in relation to formative assessment.

Effective formative assessment can be conceptualized as practice based on an adherence to the fundamental idea of using evidence about student learning to adjust instruction to better meet student needs, and a competent use of the following five key strategies (Wiliam, 2010):

1) Clarifying, sharing and understanding learning intentions and criteria for success;

2) Engineering effective classroom discussions, questions, and tasks that elicit evidence of learning;

3) Providing feedback that moves learners forward;

4) Activating students as instructional resources for one and another;

5) Activating students as the owners of their own learning.

PROJECT DESCRIPTION

The study is a part of a larger research project on professional development in formative assessment in mathematics. Participants were 38 randomly selected primary and secondary school teachers. Using interviews and classroom observations, we investigated those teachers' classroom practices to answer the following research question: *How do mathematics teachers in the municipality use formative assessment in their classroom practices?* Using the framework of one fundamental idea and five key strategies in the analysis, we identified actions of the teachers that we called formative assessment activities.

RESULTS

The findings show that most teachers use 11 to 15 formative assessment activities, distributed over the five key strategies as well as the fundamental idea of formative assessment. There are only minor differences between the primary school and secondary school teachers in the extent and the ways the teachers use formative assessment. The overall picture of the mathematics teachers' current use of formative assessment shows both strengths and weaknesses. For example, the teachers adjust their instruction in several ways as a consequence of information about students' learning. However, the potential of the adjustments are not fully utilized since the teachers often collect such information rather seldom and not from the whole class. In addition, the teachers do not use questions or tasks consciously connected to specific learning intentions.

DISCUSSION

The study shows that the mathematics teachers use formative activities in their classrooms and that similarities predominates differences between the two groups of teachers. However, additional formative activities could support teachers to better take advantage of the potential in using formative assessment. For example, the teachers could use more goal-target questions and tasks that all students answer on mini whiteboards. The identified room for improvement implicates the need for formative assessment to be included in teacher education and in-service training for teachers.

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Teaching trigonometry based on the meaningful learning theory and the conceptual field theory

Marjúnia Edita Zimmer Klein

Universidade Federal do Rio Grande do Sul, Porto Alegre, Brazil, marjunia.klein@gmail.com

The present study sought to identify the previous knowledge a group of students possessed about trigonometry so that educational activities could be developed in order to teach, clarify and help students comprehend the related concepts. As an educator, I delved into Ausubel's and Vergnaud's theories in an attempt to help my students with conceptual and procedural difficulties to relate trigonometry to real life. Results show that identifying the students' previous knowledge and making the knowledge-in-action explicit results in a change of attitude both on the part of the student and on the part of the teacher.

Keywords: Mathematics education, high school, knowledge-in-action.

RESEARCH TOPIC

This applied study focused on the possibility of promoting learning in trigonometry by using a method based on the theory of meaningful learning and the theory of conceptual fields.

THEORETICAL FRAMEWORK AND METHOD

Ausubel and colleagues (1980) in his meaningful learning theory emphasizes the knowledge the learner possesses. According to it, teachers can identify the knowledge learners have in order to draw a map and outline a plan of how to teach them in an appropriate and significant way. Thus, meaningful learning is a process by which new information interacts with a structure of existing and specific knowledge, and as a result, there is a new piece of information, with new meaning. Vergnaud (2007) proposed a theory to provide a coherent structure and some basic principles to the study of the development of complex skills. Conceptual field is the set of situations, problems, relationships, contents, thoughts, and procedures that the learners use to give meaning to a given topic to comprehend the real world. In this realm, both theories contribute to comprehend how empirical knowledge evolves into scientific one as well as how pedagogical activities support the process. A group of students from the 2nd year high school of a private institution answered a questionnaire with 6 questions aimed at clarifying the students' prior conceptions of trigonometry. The answers helped developing educational activities about (1) the definition of the trigonometric ratios; (2) the contextualization of the trigonometric ratios for the construction of an instrument capable of measuring heights; (3) the definition of the number π; (4) the definition of the trigonometric functions in the trigonometric circumference; (5) the relationship between the congruous arches and the reduction to the first quadrant; and (6) the construction of plots for the sine, cosine and tangent functions. Right after the completion of each activity, there was a class discussion and the teacher intervened.

RESEARCH RESULTS AND IMPLICATIONS

Outcome analysis reveals that identifying the students' previous knowledge and making explicit the knowledge-in-action result in a change of attitude both on the part of the student and on the part of the teacher. The activities engaged the students, favored concentration as well as the clarification of doubts. They represented a possibility for the debate of ideas, an opportunity for reflection. Conceptual and procedural aspects of knowledge evolved from empirical knowledge into scientific knowledge.

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Attitudes of teachers towards the external evaluation system for the assessment of mathematical learning

Giulia Signorini

University of Pisa, Department of Mathematics, Pisa, Italy, signorini@mail.dm.unipi.it

The external evaluation systems for the assessment of mathematical learning are increasingly widespread in the educational context, nationally and internationally. They actually affect the teachers and their teaching practices with different effects. The poster shows a qualitative research in order to explore teachers' attitudes toward the external evaluation in Italy, with particular attention to their emotions.

Keywords: Teachers' practices, assessment, affective factors.

INTRODUCTION AND THEORETICAL PERSPECTIVE

Research in mathematics education has focused its attention on teacher education rather recently: mathematics teacher education is considered "an emerging field" (Adler et al., 2005). In this emerging but growing field there has been little space for study about the teachers' attitudes towards the external evaluation systems for the assessment of mathematical learning.

In the present age, the issue of evaluation system has become central in the discussions on education and external evaluations are increasingly present in many school systems. Therefore we believe that it is crucial to give voice on this important educational issue to one of the main actors of the educational process, the teachers, collecting their beliefs, emotions and attitudes. As a matter of fact, these affective reactions seem deeply to affect the emotions of teachers, also influencing their educational choices. In this perspective we believe that understanding teachers' attitudes towards the external evaluation of students' mathematical competencies can be the first step in order to exploit at the best from a didactical point of view the important feedback that an external evaluation can return to teachers and students.

In our theoretical framework, we assume a multidimensional definition of attitude where attitude is characterized by emotions and beliefs towards the object and their mutual interactions (Di Martino & Zan, 2010).

FOCUS OF THE POSTER

Our research focuses on the Italian teachers' attitudes towards the evaluation system for the assessment of mathematical learning promoted by INVALSI (National Institute for the Assessment of the Educational and Instructional System). In the Italian context, the impression is that many teachers suffer this kind of evaluations also because they feel the external evaluations as something "ready-made", far from "real school" and imposed from above without listening to teachers' opinions. This impression is largely perceived in many unofficial occasions but not scientifically analysed in Italy until now.

To understand the range, but above all the causes of the sometimes strongly negative reactions by the Italian teachers to this external evaluation, we have planned a research in two stages. A first stage based on the elaboration of an on-line questionnaire addressed to all teachers of every school level and a second stage that foresees the development of personalized interviews to the teachers who voluntarily share their e-mail for that in the online questionnaire.

The focus of the poster will be the first stage. We will introduce the methodological choice at the basis of the development of the questionnaire and discuss the first analysis of the data collected.

Concerning the methodological choices, they are in line with recent studies about teachers' attitude (Coppola et al., 2013): the survey was voluntary and anonymous, formed mostly by open questions in order to allow everyone to choose what he feels is the most important to say, with the words he feels as the best. Regarding the analysis of the data, the research is now in its first stage but we are developing what Demazière and Dubar (2000) call *analytical* approach: the text is analyzed in order to systematically produce sense starting from people's words. The final outcome of this process is the construction of a set of categories, properties, relationships: what Glaser and Strauss (1967) call *grounded theory*. A first result is surely the participation of Italian teachers beyond any expectations: in less than 2 months, about 2 thousands of teachers have filled our open questionnaire, confirming the teachers' desire to express their view about the current external evaluation of mathematical competencies. Among the first results of the data analysis, we have seen the differences between teachers' attitudes according to the school level, but also some different motives that origin a negative attitude toward the external evaluation.

CONTENT OF THE POSTER

The poster was divided in two sections: in the first we illustrated the argument, the conceptual framework and the methodological choices at the basis of the development of the used questionnaire; the second section was focused on the conducted analysis, providing the first results of the data analysis.

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Formative assessment: Teacher knowledge and skills to make it happen

Charlotta Vingsle

Umeå University, Umeå Mathematics Education Research Centre, Umeå, Sweden, charlotta.vingsle@umu.se

The study investigates a teacher's use of activities, knowledge and skills when conducting formative assessment during interaction in whole-class. This formative assessment practice includes eliciting information about student learning, interpreting the responses, and modifying teaching and learning activities based on elicited information. Results show that the teacher used activities that help students to engage in common learning activities and take co-responsibility for their learning. Furthermore, while orchestrating the activities the teacher used knowledge and skills that are complex, demanding and difficult.

Key words: Formative assessment, teacher knowledge, mathematics.

BACKGROUND

Several studies have demonstrated that substantial learning gains are possible when teachers use formative assessment in their classroom practice (Black & Wiliam, 1998; Hattie, 2009). At the heart of most definitions of formative assessment lies the idea of collecting evidence of students' thinking and learning, and based on this information modifying teaching to better meet students' learning needs. Such regulation of learning processes would require skills to elicit the thinking underlying students' oral and written responses, and the capacity to make suitable instructional decisions based on this thinking. Sufficient knowledge about the character and use of mathematics teachers' knowledge and skills when practicing formative assessment is lacking (Heritage, Kim, Vendlinski, & Herman, 2009). The aim of this study is to identify activities, and characterize the knowledge and skills, that a teacher of mathematics uses in her formative assessment practice during whole-class lessons. The definition of formative assessment by Wiliam (2010) was used as analytic tool to identify the formative activities. A framework based on Shulman (1986), and Ball, Thames and Phelps (2008) was used to characterize the knowledge and skills used by the teacher.

METHOD

The study is a case study of a teacher's formative assessment practice during mathematics lessons in year 5. This teacher is one of 23 teachers that participated in a professional development program in formative assessment during spring 2010. The students of these teachers significantly outperformed a control group at a distal post-test in the end of the school year compared with a pre-test in the beginning of the same school year (Andersson, Vingsle, & Palm, 2013). The lessons in mathematics was observed and audio-recorded for 2 months. In the study a number of examples of situations involving formative assessment have been analysed. The situations were chosen from interactive whole-class lessons from three different time cycles. The examples represent situations where the teacher elicits information of student thinking and learning in different ways.

RESULTS AND CONCLUSION

The main result of the study shows that the formative assessment practice is a very complex, demanding and difficult task for the teacher in several ways. For example, during short-cycle minute-by-minute formative assessment practice the teacher used knowledge and skills to elicit, interpret and use the elicited information to modify instruction to better meet student learning needs. In the minute-by-minute formative assessment practice the teacher handled new mathematics (to her), unpredictable situations and made decisions about teaching and learning situations in a matter of seconds. In addition, she also helped students' to engage in common learning activities and to take co-responsibility of their learning. Specialized content knowledge and Knowledge of students and teaching where the most useful characters of teacher knowledge used when the teacher searched to understand students' answer and to formulate questions based on incorrect answers.

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Distinct mathematics teaching practices: Patterns of argumentation

Jana Zalska and Veronika Tumova

Charles University in Prague, Faculty of Education, Prague, Czech Republic, jana.zalska@pedf.cuni.cz, veronika.tumova@pedf.cuni.cz

This analysis of several mathematics lessons in two teachers' classrooms looks for characteristic patterns of argumentation and justification, specifically at the presence and types of warrants required, provided, and accepted – in the social environment of the classrooms – for mathematical claims. A question is posed whether these could be useful as descriptors of distinct justification-related teaching practices.

Keywords: Argumentation, teaching practices, middle school, warrant types, participation.

THEORETICAL FRAMEWORK AND RELATED RESEARCH

The process of mathematical argumentation, explanation and justification in mathematics education has long been an important subject of study. Toulmin (1969) proposes a framework of analysis for arguments in general, and his model of argumentation has been successfully adopted, adapted and accepted as a useful tool in analysis of qualities of (students') mathematical argumentation itself as well as of the collective learning experiences in a classroom (e.g., Yackel, 2002; Rasmussen & Stephan, 2008; Krummheuer, 1995, 2007; Tabach, Hershkowitz, Rasmussen, & Dreyfus, 2014), helping to shed light on students' learning in a social context.

Yackel (2002) points out two different roles that teachers play in collective argumentation (in an inquiry-oriented environment): that of using it as a means to start introducing new concepts and that of carefully supplementing either data, warrants or backings for claims that would otherwise not be explicit to all students, and/or that help students make connections between mathematical concepts.

Krummheuer (2007) analyses argumentation in a mathematics classroom and adds the criteria of participation. He is then able to reconstruct "different grades of autonomy according to the interactional contribution of a student" (p. 60) as well as his status as a participant in the argumentation process.

Our own study also looks at the use of argumentation and justification in a mathematical classroom as a whole, i.e. a closed learning community, but through the lenses of two teachers' distinct beliefs and their respective teaching practices.

THE METHOD

Data were collected from two different classrooms: a cycle of teacher interview-lesson observation-interview was conducted for a sequence of 5 lessons in two lower-secondary classrooms with different teachers (the lessons were about percentage in one classroom and operations on fractions in another). The two teachers were selected as professing distinct beliefs about mathematics teaching and learning, one characteristically leaning towards an inquiry-based conception of the norms for mathematics teaching and learning (as described in Yackel, 2002).

Data collected from observations (transcripts of collective interaction) were analysed for episodes involving the justification of mathematical claims. Cases of argumentations were further characterized using Toulmin's model (especially for data, warrants, and backing) and analysis of participation on warrants and backing.

RESULTS AND IMPLICATIONS

While pupils' participation in providing warrants was proportionally similar in both classrooms, the differences that emerged concerned especially the types (forms) of warrants. Those varied more in the inquiry-oriented classroom, while argumentation in the other classroom was typically centred around procedures. There is evidence that backing for warrant forms and relevance (although not always correctness), is provided by the teacher in both classrooms. These results show that patterns of argumentation are partly useful as descriptors of a teacher's practice, however, more investigation is necessary in the connection between teacher's beliefs about the role of argumentation and the socio-mathematical norms of argumentation as well as the impact of mathematical tasks on argumentation patterns in a particular classroom.

ACKNOWLEDGEMENT

This work was supported by research grant GAČR P407/11/1740.

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TWG20

Mathematics teacher knowledge, beliefs, and identity

Introduction to the papers of TWG20: Mathematics teacher knowledge, beliefs, and identity: Some reflections on the current state of the art

Miguel Ribeiro[1], Fatma Aslan-Tutak[2], Charalambos Charalambous[3] and Julia Meinke[4]

1 Norwegian University of Science and Technology (NTNU), Trondheim, Norway, miguel.ribeiro@math.ntnu.no, cmribas78@gmail.com

2 Bogazici University, İstanbul, Turkey

3 University of Cyprus, Nicosia, Cyprus

4 Georg-August University of Göttingen, Göttingen, Germany

In this introductory chapter for the Thematic Working Group "Mathematics teacher knowledge, beliefs, and identity" (TWG20) at CERME9, we address the main issues discussed during our working sessions. We aim to provide a critical and broader view on the work being done, including the work undertaken at previous conferences, although not extensively. We also seek to take a new look at areas of potential improvement with regard to the focus of, discussion about, and problems inherent to research in the area of teacher knowledge, beliefs and identity, all with the goal of improving teacher education and practices. This paper brings to the forefront some critical features of this area of research and aims to contribute to the genesis of a new focus on our research and a new vision of our own roles as researchers and teacher educators.

Keywords: Teachers' knowledge, beliefs, identity, teacher practices.

INTRODUCTION

In considering possible theoretical and analytical perspectives with regard to teacher *knowledge, beliefs,* and *identity*, a long list emerges that includes different aspects and foci. Even so, having a TWG dedicated exclusively to these areas allows the simultaneous amplification and synthesis of such a list—both in depth and on a broader scale—in terms of theoretical and analytical approaches, as well as the goals of the research presented.

A broader and deeper view of the multitude of possible perspectives on teacher *knowledge, beliefs,* and *identity* is facilitated by an in-depth discussion of, and reflection on, different contexts, such as the variety of knowledge frameworks and conceptualizations among different teachers, evaluating versus understanding teacher knowledge, belief changes, and the existence of a mathematics teacher identity and its content. Focusing on the content and on the aims of such a multitude of approaches to these three areas, gives a deeper awareness of possible potentialities and misunderstandings within the area of mathematics education. The existence of a multiplicity of ways to perceive the same aspect is in itself a potential area for improving research (and education), and should also be a focus in order to better understand each of the three factors addressed in the TWG. The misunderstandings come, for example, from the different contexts and cultural heritage of the researchers themselves, which leads to the use of different interpretations of a notion (e.g., teacher knowledge) being linked with the aims of the research itself (e.g., understanding versus evaluating teacher knowledge). Examples are the discussions presented by Kuntze, Dreher, and Friesen (2015); Vasco, Climent, Escudero-Ávila and colleagues (2015); and Pizarro, Gorgorió, and Albarracín (2015) around teacher knowledge in different contexts and using different theoretical and analytical approaches. Overcoming such misunderstandings allows a broader view of the landscape, leading to a better understanding of where we are coming from and where we envisage going in the future. Obviously, diverse paths can be taken, and we consider this beneficial; but the different ways such approaches can contribute to the end goal should always be explicitly considered.

The creation of a unified understanding of the work being developed around the three focuses of the TWG requires evidence of how the multiplicity of theoretical and analytical approaches might contribute to the ultimate end point — the (student) teachers' learning. Knowing where we are heading — as well as where we are coming from —, this introduction will present and reflect upon the core aspects discussed in the TWG, expanding upon the content of the papers included in the proceedings and the already developed work. We note that although theoretical and analytical perspectives are perceived as intertwined, and thus incapable of being considered as two disjointed sets, we opted to deal with them in separate sections for operational purposes. In doing so, we advocate the need for more careful attention to the importance of attending to different perspectives and the roles they can play in research (process, findings, recommendations, and implications). Additionally, the large predominance of current research focusing on teacher knowledge also influenced the discussions we elaborate on here.

With this introductory chapter, the work developed in the TWG (focus upon problematic and core aspects of/for discussion when working in the area of teacher *knowledge, beliefs,* and *identity*) is discussed with some clarifications, and we call attention to the need for a further step forward in research in this area and its impact on education. We will also discuss the possible structure of future research.

SOME CORE THEORETICAL ASPECTS

When looking at the theme of our TWG, three areas of study are considered to deeply influence teacher practices: *knowledge*, *beliefs,* and *identity*. These can be considered as the core of a teacher's practices, with each area influencing and being influenced by the others (Ribeiro & Carrillo, 2011a). In doing so, our aim was to shed some light on the paths that still need to be devised to allow for a better understanding of teachers' *beliefs, identity,* and *knowledge*.

Beliefs

When teachers became a focus of research, one of the first attributes studied was their beliefs (e.g., Pajares, 1992), along with the inconsistencies that could be found between discourse and practice. Afterwards, the interest shifted to the ways teachers change their beliefs, leading in turn to a change in their teaching methods and awareness of their own practices—all influenced by their own experiences as students (e.g., Ebbelind, 2015; Sayers, 2013). Going a step further, another relevant point addressed was the development of teacher awareness about how and why they teach what they teach (e.g., Schueler, Roesken-Winter, Weißenrieder et al., 2015), making this transparent through the analysis of beliefs and thus making change possible.

Although a large amount of research has focused on beliefs and their manifestations in teacher actions, questions, or answers (e.g., Ribeiro & Carrillo, 2011b; Sayers, 2013), it is important to consider in conjunction with this the competencies and practices of teachers. Although beliefs have been a focus of research for a long time, there remains a lack of information leading to better and deeper understandings of the role, impact, and connection between teacher beliefs and the remaining core aspects of mathematics teacher practices (e.g., Potari, Berg, Charalambous et al., 2013) in order to improve student learning, understanding, and results. When addressing research being done on teacher beliefs (as well as their *knowledge* and *identity*), a core aspect of discussion in the TWG was the need to re-emphasize the human aspect of mathematics in the classroom—humanizing without de-mathematizing. Such foci would bring mathematics to the front, and is one of the core ways to bring more consistency (both internal and external) to the area, eradicating the question "Where is the mathematics in this mathematics education research?".

Identity

In researching identity, the particular aspects focused on within this multifaceted notion must be made explicit. In particular, the transformations involved in moving from student to teacher have to be taken into consideration. And, in an intertwined manner, the changes that occur over time in teacher practices, goals, and levels of awareness must also be examined, especially because some authors argue this leads to the development of a teacher identity (e.g., Adler, Ball, Krainer, Lin, & Novotná, 2005). Assuming the existence of such an identity, research on its development must consider the need to move from dealing with mathematics as a scientific discipline (as it is traditionally presented at university) to understanding it as primary and secondary school mathematics and the diversity of processes involved; or, in other words, teacher professional development as a learning process. In that process, the concept of teaching and its dynamic nature plays an essential role in the development of a mathematics teacher's identity (e.g., Rø, 2015). Thus, when attempting to contribute to the study of teacher identity development as a learning process, our own practices and knowledge as educators should also be examined (e.g., Mellone, Jakobsen, & Ribeiro, 2015; Superfine & Lin, 2014) in order to allow for placing the scientific discipline into a primary and secondary school context. Such inquiry can lead to an expansion of the notion of awareness as perceived by Mason (2002), taking into consideration the possible focus on differences and similarities between the identity of mathematics teachers and those of teachers of other disciplines. It is of special importance to consider the specificity of the content being taught and how that influences teacher identity, particularly when a teacher must teach several disciplines, as is often the case for elementary school teachers. This once more brings mathematics and its specificities in and for teaching to the front.

When thinking in terms of mathematics teacher identities, and considering research as a way of making us stop and think (Kilpatrick, 1981), some questions naturally emerged during discussion and reflection in the TWG. Some of these discussions and reflections are concerned whether identity is a self-concept, leading to a broader question about the concept of identity itself, particularly what identity is and what comprises it[1], as well as its dynamic versus static nature. Taking a broader view of our own practices as teacher educators and researchers (as if looking at our own work from a distance), the need is evident to address the importance and helpfulness of the concept of identity(ies), clarifying its content as well as its developmental process(es).

Knowledge

Research of teacher knowledge began, and in some cases remains, in the domain of identifying lack of knowledge—assuming a deficit. In order to overcome such a deficit perspective and, further, contribute to the creation of improved education (Potari et al., 2013), research should be done that works with teachers to develop and expand upon what they already know. Although the papers in the TWG mainly adopted this perspective, it is clearly one area where research, and a rethinking of the foci of such research, is still most needed. Thus, a step forward is still required when rethinking the foci to lead away from the deficit perspective and toward an understanding of what teachers know and how they know it (nature and type) (e.g., Fauskanger & Mosvold, 2015; Montes & Carrillo, 2015). This can then lead to the development of practices that enrich the levels of awareness and connections that contribute to improved education and, ultimately, practice. Bridging theory and practice is essential for such improvement, and core to building such bridges is defining the nature and goals of proposed tasks (e.g., Jakobsen, Ribeiro, & Mellone, 2014; Tirosh & Wood, 2008) for enhancing teacher acquisition of ideal knowledge that would allow them to foster fruitful mathematical understanding in their students.

A large number of teacher knowledge conceptualizations—visible in the different papers of this chapter—are grounded in Shulman's (1986) seminal work. The discussed conceptualizations also assume the Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK) domains, but with different interpretations of its content. Such interpretations associated with the devised focus of attention lead the research being done in different directions. Examples of such are the Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008); the Mathematics Teachers' Specialized Knowledge (MTSK) (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013), the Knowledge Quartet (KQ) (Rowland, Huckstep, & Thwaites, 2005) and the multi-layer model presented by Kuntze (2012). The use and development of such diverse conceptualizations can be perceived from one side as a richness of the field, and from another side as a constraint when discussing the core aspects of teacher knowledge. Such diversity contributes to enriching views on the content and factors that influence the development of teacher knowledge, contributing scaffolding for the necessary bridges to improving student results. But they can also be problematic from the viewpoint of the language used —the same wording with different meanings (Bardelle, 2010) — making it difficult to find a common ground when discussing the core aspects of teacher knowledge.

1 This same question can be asked concerning knowledge and many other concepts we work with and sometimes take for granted. Indeed, in a TWG focusing on teachers, this was one of the problems discussed (see part of the proceeding with TWG18 papers).

Although the conceptualizations of teacher knowledge have different points of views and foci, as well as different elements at their cores, what they all have in common is considering teacher knowledge as too complex to be examined on its own, as well as the impossibility of considering the different subdomains in an isolated manner. In that sense, the existences of subdomains are justified for analytical reasons — facilitating description and a deep understanding of teacher knowledge and practices. The subdomains are also considered to be a core element for devising and conceptualizing teacher education tasks that are focused on improving teacher knowledge and awareness (Jakobsen et al., 2014).

The existence of such a diversity of concepts of teacher knowledge, as well as the diversity of dimensions as core elements, is one of the reasons for the continued lack of consensus on what comprises ideal knowledge. In the TWG discussions, such lack of consensus led to reflection and research on a number of topics. These included the advantages and disadvantages of considering different subdomains in teacher knowledge conceptualizations (subdomains versus global knowledge); whether or not there is a need to measure teacher knowledge, and if so, how to best do so (effectiveness of instruments designed and the nature of the knowledge measured); the nature of the claims the researcher makes about teacher knowledge and the learning outcomes with such a foci of analysis (e.g., deficit, descriptive, understanding, measurement perspective); and the potentialities and constraints of each conceptualization in and for analyzing teacher practices.

SOME CORE ANALYTICAL AND CONTEXTUALIZED ASPECTS

In practice, the theoretical aspects focused on cannot and should not be taken separately from the analytical approaches considered. In that sense, in accordance with the different theoretical perspectives explored in this chapter's papers, a diversity of analytical approaches was used. Such diversity was linked not only to the different frameworks considered, but also to the specific aim of the particular research. In this section, rather than discussing the type and nature of the analytical approaches used in the papers (e.g., different nature of the case studies – what concerns the cases; different foci of analysis and instruments used; the nature of the research aim – evaluate or understand knowledge/practices, contribute for change, understand awareness), we will address and reflect upon some of the more crucial aspects that emerged when discussing research focused on teacher *knowledge*, *beliefs,* and *identity*, including sharing the processes and results.

One of the most pertinent topics is the content of analytical and contextualized aspects, commonly termed methods or methodology. On the one hand, this section should include the description, justification, and explanation of the methods used in the research (with or without an example of how they were used). On the other hand, it should also include other contextualized information that would allow the reader to better understand the hows and whys of the decisions made and the specific context in which these were made (minimizing misunderstandings). Although this has already been a topic of discussion in previous conferences (e.g., Potari et al., 2013), it still remains an underinvestigated area. In that sense, there are some essential, though often unmentioned, aspects needing to be addressed and taken into account when thinking and writing about analytical and contextualization options. Among these, we can consider aspects from (i) why a certain analytical process was chosen over any other, (ii) the particularities of the method used (e.g., taking into consideration the theoretical perspective used or developed), and (iii) the different analyses done (e.g., qualitative, quantitative, particular case study, or going beyond the particular case study).

With regard to (i), the reasons for selecting a certain analytical process must be made explicit, including not only justification for the process chosen, but also factors that led to the exclusion of other approaches. This allows for building common ground. Some of the aspects to take into account are the minimal requirements for writing about the analysis and results obtained so others effectively understand them and the degree of detail needed for describing the context, assumptions, and each theoretical perspective.

For (ii), the particularities of the specific analytical processes used and the implications of a descriptive versus analytical approach must be made clear. Such particularities and implications are linked with the potential for, or limitation of, work being done to impact education and steer toward a change in teacher *beliefs* and *knowledge*, allowing the development of teacher *identity*, and thus its impact in practice. See, for example, Ferreira and Ponte (2015) for teacher learning after a teaching experience or Zoitsakos, Zachariades, and Sakonidis (2015) for teacher difficulties in conceptualizing and suggesting mathematically correct instructional practices for an issue at hand. With regard to different subdomains in teacher knowledge, the subdimensions one might encounter and their impact (potentialities and limitations) on research, as well as for education and practice, must be taken into consideration. For example, what impact would they have in the elaboration of mathematics teacher identities? How can they be used as a resource to conceptualize tasks for teacher education? How can they refine the theoretical concept assumed? Keeping these questions in mind would contribute to an awareness of the need for developing further both theoretical and/or analytical methods used as well as ways to bridge research and practice, explicitly fulfilling the already identified gap concerning the need for clarification of the context and analytical procedures.

Concerning the different focus and processes of analysis (iii), and its impact on future research and knowledge generated, at least two perspectives must be kept in mind. One is how to feed back the research findings to inform the theoretical and analytical framework(s) used/developed. The other concerns a focus on what aspects can be brought into the discussion in order to go a step beyond merely describing the particulars of the situation(s) under study. When aiming at taking a step forward through the research to have a more effective impact on education, practice, and ultimately on student learning and understanding (e.g., Ebbelind, 2015), the previously mentioned two components need to be assumed in an intertwined manner. Considering the need for such an intertwined approach leads to reinforcing the connections and filling the gaps between such dimensions. Such reinforcement, and the associated step forward, seem to be crucial to clarifying the role, impact, and importance of case studies and large qualitative studies in improving education and promoting a deeper understanding of what seems to happen, and why, in the context of each study.

Thus, a need exists to clarify the choices made in research, keeping in mind the requirement of a balanced way of doing so, considering the broad range of research foci and analytical processes—all while also being aware of the different information required and the results obtained.

CONCERNS, POSSIBILITIES, AND FUTURE NEEDS

There is a diversity of theoretical and analytical approaches in the chapter's papers associated with different contexts and research aims. In this introductory paper, our aim was not merely to address the issues discussed during the working sessions. Rather, starting from these and thinking backward, our foremost concern was the possible horizons we could envisage moving toward at that time. With such an approach, we hope to lead the reader to think and reflect on the role of the diversity, depth, lenses, and aims associated with each of these studies. In doing so, we ultimately hope to contribute to thinking outside of the box in order to break the chains that constrain us, which ultimately restrict the effective potential of the work being developed.

During the TWG discussion of the papers and possible content and structure for this introductory chapter, a lot of questions for future research emerged. This provides evidence of the need for more focused attention to the what, how, and why of what we do, leading to envisioning potential impacts of the work with teachers and students to foster a more literate and numerate society globally. In that sense, addressing the different aspects and foci of needs (theoretical, analytical, and of context) is essential in order to allow a broader impact on the work of teaching (rather than merely remaining with particular successful cases). Such needs are of a diverse nature and associated with different aims, and therefore are associated with different target groups. When analyzing the papers and the questions raised from its discussion (e.g., the need for emphasizing teacher potential rather than deficiencies; bringing teachers and the mathematical content to the front; attending to teacher voices in the research being done — see, for example, Takker, 2015; Toor & Mgombelo, 2015) reflection on the researcher's role in the research and education process arises. There is thus the need for rethinking our own role—our own beliefs, knowledge, and identity — in the research being developed. Have they influenced the choices made and, necessarily, the path followed and results presented? We must consider ourselves and the work we develop not only as part of the possible solution to the problems addressed, but also as part of the challenges.

Possible points of focus for future research can be, for example: (a) having a profound understanding of a specific mathematical content (its meaning — core aspects of its content, ways to measure/develop it, analytical approaches and its implications); (b) connect the mathematical content knowledge (e.g., university mathematics) to the mathematical knowledge in primary and secondary school (changes and impact on teacher education — including their educators); (c) meaningful communication among communities with different professional identities (communicate findings to practitioners and policy makers to steer change); (d) development of theoretical and analytical tools to break some of the chains that constrain us (leave the known space of solutions and bring novelty to theoretical and methodological aspects, both as objects of and for research).

Having an open mind, developing an awareness (Mason, 2002) to pay attention to granular aspects, and making connections to better perceive and understand the big picture allows us to take a further step toward freeing research from the description and interpretations of particular cases (even when quantitative research is considered).

ACKNOWLEDGEMENT

This paper has been partially supported by the Portuguese Foundation for Science and Technology (FCT), project code (UID/SOC/04020/2013).

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TWG20

Research papers

Systemic Functional Linguistics as a methodological tool when researching Patterns of Participation

Andreas Ebbelind

Linnaeus University, Department of Mathematics Education, Kalmar, Sweden, andreas.ebbelind@lnu.se

This study highlights the role, if any, that teacher education programmes and experiences from other practices play in influencing generalist student teachers' tales of themselves as emergent primary mathematics teachers. The conceptual framework Patterns of Participation, PoP, is used when theorising and interpreting student teachers' becoming, and analysing the processual and dynamic character of immediate social interaction related to practice on a macro level. Therefore, this paper evaluates whether Systemic Functional Linguistics, SFL, can be a methodological tool used on the micro level. This paper shows that SFL structures the data in a way that makes interpretations through PoP possible.

Keywords: Methodology, Patterns of Participation, Systemic Functional Linguistics, teacher education, student teachers.

INTRODUCTION

Methodology can be understood as methods used for gathering information and the specific reasons for using such methods. Furthermore, methodology should concern aspects of how gathered information is transformed into data and why the transformed data are relevant in relation to a framework (Skott, 2014). The process of transforming gathered information into data is done in steps. Gathered information needs to be structured. Structured information is used when generating data. The generated data is then interpreted by using a conceptual framework. This means that there is a relationship between methodology and theory (Gee, 2014). This paper focuses on the first step of this transformation, the choice of methodological tool in relation to a conceptual framework, and on how this methodological tool is used for structuring information on a micro level in a study concerning student teachers at the primary level. Microanalysis allows access to finer details of gathered information (Jarowski & Potari, 2009), in this case highlighting the functionality of language.

In this study, concerning primary student teachers, the overall aim is to contribute insights about the role, if any, that teacher education programmes and experiences from other relevant past, present and future practices play in influencing generalist student teachers' tales of themselves as emergent primary mathematics teachers. Tales refer to stories teachers tell about themselves and can imply more creative or imaginative storytelling. Other relevant practices include those related to the teaching and learning of school mathematics, but they may also be non-mathematical and not institutionalised. Furthermore this study seeks to understand how these different practices and experiences are related, if at all, to each other.

This means that the methodological tool needs to disentangle elements of teacher education and relevant practices in relation to different entities, experiences related to these elements of teacher education and relevant practices, the tense in which these experiences are described, what role these experiences play in terms of degree of certainty and the relationship between these different practices. This interest draws on the highlighted concern that primary generalist teachers within their first years in the profession may not prioritise, or may have no opportunity to prioritise, the subject of mathematics itself (Palmer, 2013).

Emergent student teachers at the primary school level are typically generalists, expected to teach a range of different subjects in the future. Consequently, their level of education in each of the school subjects is modest, and their professional background is less linked to the teaching of specific subjects than to the profession as a whole. In Sweden, for instance, the combined course in mathematics and mathematics education for primary school student teachers is a 30 ECTS point course in the four-year teacher education programme (240 ECTS).

Recently, there has been extensive interest in student teachers' image of teaching mathematics, their knowledge gained during teacher education and how their image of themselves as teachers evolves in relation to practice. Standard procedures for assessing these have involved research about student teachers' beliefs, mathematical knowledge for teaching, and identity (Skott, 2013). The first two research fields have generally interpreted human behaviour as something enacted within a person and it is generally explained by acquisitionist research perspectives. The last research field has generally interpreted human behaviour as situated in practice and explains it using participatory research perspectives (Sfard, 2008).

However, in all three research perspectives attention to student teachers themselves as the unit of analysis has decreased over the years. The main focus is not on the student teacher. Instead, the focus is on, for example, the domain-specific beliefs, the knowledge needed for practice and the specific community of practice. In contrast, this study intends to focus on the student teacher as the unit of analysis and therefore uses the PoP framework. This is because PoP intends to describe the pre-reified processes that are said to precede the so-called construction of beliefs, knowledge and identity (Skott, 2013) and, above all, PoP places the student teacher at the centre of the analysis (Skott, 2014).

PATTERNS OF PARTICIPATION

When theorising and interpreting the role that teacher education programmes and experience from other relevant past, present and future practices play in influencing generalist student teachers' tales of themselves as emergent primary mathematics teachers, the conceptual framework Patterns of Participation, PoP (Skott, 2013) is used. PoP "seeks to understand how a [student] teacher's interpretations of and contributions to immediate social interaction relate dynamically to her prior engagement in a range of other social practices" (Skott, 2013, p. 549). Lerman (2013) highlights the need for frameworks that emphasise emergence and regards PoP as such a framework. He implies that Skott, through the PoP framework, tries to go beyond the demarcation of the acquisitionist and participatory perspectives.

PoP provides the researcher with thinking tools (Gee, 2014) to use when planning and designing a study, and also when interpreting the patterns in the immediate emerging social practice on a macro level. The conceptual framework is therefore used to go beyond the micro level in order to focus on the social practice and situation (Jarowski & Potari, 2009). PoP draws on two main theoretical sources, symbolic interactionism and social practice theory. Symbolic interactionism views humans as actors and reactors in situations, and it positions meaning as something that one engages in when experiencing things in the situation. Humans respond to the situation by interacting with others and with the self, and by taking the role of others (Prus, 1996). In social practice theory, student teachers' identity formation and learning results from shifting participation in relevant practices (Skott, 2013). Identity is viewed as imaginings of selves expressed in actions and created in immediate social interaction. Identities develop and are expressed in social practice (Holland, Skinner, Lachicotte, & Cain, 1998).

PoP focuses on immediate emerging social interaction. Immediate social interaction is concerned with the relationship between a social practice and a text produced within it. A text is considered any instance of language used as part of a situation. The situation is regarded as the "environment in which meanings are being exchanged" (Halliday & Hasan, 1989, p. 12). Individuals' interpretations of the situation emerge as they interact symbolically in past, present and future practices (Skott, 2013), intra-personally as well as inter-personally (Sfard, 2008), and this interaction contributes to the production of texts (Halliday & Hasan, 1989). This means that every situation is unique and the gathered information is situated in time and place.

Skott (2013) and Palmer (2013) use a qualitative approach inspired by grounded theory to disentangle prior engagement in immediate social interaction. Inspired by social semiotics, this study turns the disentangling of immediate social interaction in a different direction, that of linguistics (Ebbelind & Segerby, 2015). It does so for several reasons, for example, to focus the analysis on the functions of language while highlighting the uniqueness of the situation, and to incorporate a systemic way of structuring the gathered information on a micro level that specifically focuses on the text itself while highlighting the relations between different experiences in terms of language as functional.

To conclude, PoP has an interpretive stance that implies that human behaviour is different from objects that can be measured statistically. An interpretive stance also implies that there is no intention, in this study, to discuss cause and effect through different variables in relation to teacher education. However, questions of validity and trustworthiness need to be addressed. According to Gee (2014) and Skott (2014), validity in this kind of social research concerns the relation between theory and methodology, how they work together. This recognises that the macro level includes the micro level (Jarowski & Potari, 2009).

Therefore, the specific aim with this paper is to evaluate whether Halliday's Systemic Functional Linguistics, SFL (Halliday & Hasan, 1989) can be a proper methodological tool for disentangling the information gathered for this study. The reason for exploring SFL is that SFL is suggested and regarded as a powerful tool for visualising context in text (Herbel-Eisenmann & Otten, 2011; Morgan, 2006). To explore SFL, a short transcript will be analysed with this tool and then related to PoP in the discussion.

METHODOLOGY

The present study adopted a theory driven, multi-sited ethnographic approach. It involved three student teachers and followed them before, during and after different situations such as lectures, seminars, internships, study groups and examination work related to mathematics education.

The study was theory driven (Walford, 2009) because PoP (Skott, 2013) guided the choices made during the ongoing project. The study was multi-sited because the mode of construction was not a single site; instead, the mode of construction was a process, the emergent generalist teacher, that took place in multiple sites (Pierides, 2010). The ethnographic approach implies the use of multiple methods for gathering information, including field notes, interviews and the collection of other text material. The reason for using multiple methods is based on the assumption that different types of material contribute different types of information.

The gathered information needed to be structured in relation to the aim and conceptual framework. For this purpose SFL was used. Among research that has been conducted internationally in mathematics education studying the role of language, Morgan's (2006) use of SFL to visualise experience in school mathematical assessments is the most well known. Another example is Herbel-Eisenmann and Otten (2011), who set out to reveal mathematical meaning potentials construed in discourse and used SFL to visualise semantic structures of mathematical content. SFL views language as a resource that people use to accomplish specific purposes through the functions of language (Halliday & Hasan, 1989). The functions can be analysed at the clause level to understand how contexts are reflected in the linguistic choices that participants make in their text production.

Systemic Functional Linguistics

SFL states that every speech act is about something (ideational meta-function), is addressed to someone (interpersonal meta-function), and is presented and connected in a specific way (textual meta-function).

The ideational meta-function relates to the main verb and concerns how actions or experiences are articulated through the transitivity system. The transivity system illustrates how actors are connected to actions or objects (Halliday & Hasan, 1989). The main verb can either be a material process, in which there is an actor that does something; a mental process, where a senser is addressing a phenomenon; a verbal process, expressing something that has been said; or a relational processes, emphasising relations between objects.

The interpersonal meta-function relates to voice, tense and modality. Voice highlights the personal pronoun and entities visible in the text. Tense highlights whether the proposition is valid for the present, past or future. Modality relates to the degree of certainty in an utterance. This function answers the questions: What persons and entities are visible in the text and how are persons and entities connected to the text?

Finally, the context and language structures used to carry the meanings of the text are components of the textual meta-function. This function concerns the process of structuring the way information is conveyed. By looking at the theme and rheme one can view how clauses follow each other in thematic bindings. When several of these thematic bindings occur, a lexical chain is created. One can also view the use of conjunctions. Conjunctions show how different experiences are related to each other. For an extended description of the meta-functions see Ebbelind and Segerby (2015).

Background of the student and the text

The texts selected for analysis in this paper come from a semi-structured interview with a student teacher during her second internship, after two years and three months of study in the programme. The student teacher, who was in her early 20s at the time of the interview, began the teacher education course immediately after high school. During the first internship and the later 30-credit course in mathematics education, she did not talk much. When she did speak, her tone was low and her body language withdrawn. This was also observed during several interviews and observations. During the second internship the researcher was present, attending lessons and meetings. The observations made on these occasions marked a major shift in the student teacher's participation related to the mathematics classroom, from shy and quiet to straightforward and self-confident. The selected text from this interview, presented below, was then analysed through SFL.

The transcript

Researcher: So, in some way, you felt more secure. [Yes] Or if you were not secure, then you were at least much more determined. [Yes (content in intonation)] How do you experience this? Secure or determined, or the difference from last time.

Evie: I have got more training to stand in front of pupils and speak. More teaching hours on this internship than the other, that is probably the biggest difference.

Researcher: But have you spoken much about [the changes in] your voice? That you use your voice in a totally different way now than you did before. Is that something you have been speaking about or something that you have been trained in?

Evie: No, I think that a lot of this is from my soccer referee career and from different referee courses and refereeing at different levels. That comes naturally, how one uses one's voice for speaking to the players on the field in the same way one speaks to pupils in the classroom.

Researcher: But when did you begin this referee initiative?

Evie: I began refereeing at a senior level during the summer of 2010, but then the initiative began during the winter when I found out, found out that I was nominated to the next level in connection to the "elite camp for girls" this summer.

Researcher: Has anyone besides me pointed this out, about your voice?

Evie: No, no one from the course, but the inspectors when I referee (laughing), they point out that I speak well and distinctly when I speak with the players.

ANALYSIS

Ideational meta-function

The transcript is a sample of an interview between two people where the researcher guides the conversation. The student teacher makes linguistic choices when discussing her engagement in past, present and future practice. The researcher asks the student teacher about the perceived change through the mental processes *felt* and *were* and opens the interview and addresses this perceived change with the mental process *experience*. When describing the origin of this experience, the verbal process *speak* and the material processes *got* and *stand* are used. The researcher's sensing is answered with examples from the physical world. The researcher tries to identify the origin of the changed *voice*, or relationship, through the verbal process *spoken* and the material process *trained*.

The student teacher then uses the mental process *think*; however, *think* is used to describe the relation between *"soccer referee career"* and the researcher's utterance *"you use your voice in a totally different way"* and is therefore regarded as a relational process. Furthermore, the student teacher uses the material processes *referee* and *use* in relation to the first relational process *think*. The choice of saying *"one speaks to pupils"* is important in this transcript. Here, *speak* is a relational process between the management of a soccer field and a mathematics classroom. Speaking with soccer players and pupils learning mathematics can, through this choice of words, be interpreted as equal ways of communicating.

The researcher then uses the material process *begin* to locate the tense of the *"referee initiative"*, and the student teacher answers twice with the same word; however, *began* in the answer is considered a relational process. In the last interference by the researcher the verbal process *"point this out"* is used to see if someone else has registered the change. The student teacher emphasises that no one from the educational side has highlighted this change, but uses the verbal processes *point out* and *speak* in relation to inspectors from the Swedish Soccer Association.

Interpersonal meta-function

The researcher poses questions with prompts such as *"do you experience"*, *"have you spoken"* and *"when did you begin"*, and the student answers the questions. The student teacher's prior and present internships, the change as perceived by the researcher and soccer referee career appear in the interview as entities.

In the first section two questions and answers are concerned with this change that the researcher has recognised. In the third question, the researcher uses the mental process *experience*. When answering this question, the student teacher uses *I* and draws on her past and present experience through internship. In the second section the researcher uses the personal pronoun *you* five times. The first two *you* are related to the verbal process *spoken* and involve the internship supervisor, and the last three address the student teacher. In the answer, the student teacher uses *I* and relates to the fact that *one* needs to use one's voice both as an referee and as a teacher in the classroom. *They* is used to emphasise that the inspectors are the ones that have highlighted her way of behaving, *"I speak well and distinctly"*.

As already indicated, the modality is strong at the beginning when the student refers to having *"got more training"*. The tense is used to refer to the difference between the present and the past internship *"on this internship than the other"*. When the student teacher first mentions the "referee initiative", the validity becomes low through *think*; however the validity becomes stronger and ends with strong linguistic validity with *"they point out"*.

Textual meta-function

The researcher links the first and second questions together with the disjunctive conjunction *or* and the interrogative *if*. The comparative conjunction *then* is used to relate the two internship experiences to each other. The additive conjunction *and* is used to turn the low validity of the clause to provide the clause with stronger validity. The conjunction *but*, used in the last answer, is used to exclude the teacher education from the discussion and promote the inspectors. This is an adversative conjunction, which means that there is something contrary to what one might expect.

Looking at the theme and rheme in the transcript reveals that the feeling promoted by the researcher is followed up by the student teacher in relation to the different internship experiences (first and second internship). The researcher uses this rheme to formulate a question, theme, about whether the student teacher has articulated this difference. In the rheme the researcher asks if it might be something the student has been trained in (during internship). At this point the lexical chain is broken by the student teacher when adding the *"referee initiative"*.

DISCUSSION

This paper focused on methodology and the transformation of gathered information into data, more specifically on the use of Systemic Functional Linguistics as a methodological tool for structuring gathered information on a micro level, enabling interpretations using the conceptual framework Patterns of Participation on a macro level.

PoP seeks to understand how teachers' interpretation of and contribution to immediate social interaction relate dynamically to what Skott (2013) calls prior engagement in a range of other practices. This is in line with the aim of the study, which problematises the role, if any, that teacher education programmes and experience from other relevant past, present and future practices play in influencing generalist student teachers' tales of themselves as emergent primary mathematics teachers and how these different relevant practices and experiences are related, if at all, to each other.

As indicated in the first section, the methodological tool needed to: disentangle elements of teacher education and relevant practices in relation to different entities; identify the tense of these experiences; disentangle the role these experiences play in terms of degree of certainty; and visualise the relationship between these different practices and experiences.

Through the voice, interpersonal meta-function, persons, entities and social practices became visible in the text, for example, the referee inspectors *they*; the student teacher herself, *"I speak"*; the *"referee initiative"*; and the student teacher's participation in *"different referee courses"* and *internship* related to mathematics education.

The transivity system, ideational meta-function, indicated how actions or experience were related to persons, entities and social practices. This was done through the main process verb, for example, the verbal process *speak* when describing the origin of the experience indicated by the researcher, the mental process *think* when relating to the *"referee initiative"* and the material process *referee* relating to refereeing soccer games and how to act in classrooms.

The tense highlighted whether the proposition was valid for present, past or future time. An example of this is when differentiating between the present and the past internship *"on this internship than the other"*.

When disentangling the role that persons, entities and social practices play in terms of degree of certainty, modality was used. One can see, for example, that modality is strong in the utterance *"have got more training"* and low through the mental process *think*.

Finally, the methodological tool needed to visualise the relationship between these different practices and experiences, if any. This was done by looking at the relational process, ideational meta-function, through the use of conjunctions and theme/rheme. For example *speak* was used to describe the relation between the management of a soccer field and a mathematics classroom. Another example was the adversative conjunction *but* that was used to promote the referee inspectors instead of teacher education. Looking at the theme and rhyme in this transcript it can be seen that the lexical chain was broken by the student teacher when adding the *"referee initiative"*.

CONCLUSION

On a theoretical level Morgan (2006) highlights that SFL is a systemic way of visualising the relation between text and practice. Through the structuring of the information it is clear that the student teacher's *contribution* to an *interpretation* of immediate emerging social interaction can be linked in a fine-grained manner to the "*prior engagement in a range of other social practices"*, for example, internship (past and present) and the current referee initiative. It is also clear that the fine-grained analysis highlights small but important parts in relation to the aim of the study. However, PoP does not highlight relevant practices, but sets out to highlight the way the re-engagement is conveyed and connected. In this perspective, the textual meta-function models links between practices, experience and entities, but maybe more important, the relational process links do not just go forward in chronological order. The relational process also indicates links back in the patterns that are created in the emergent social immediate interaction. These different links combine the re-engagement in prior and present practices into patterns.

I hold that I am coordinating two different perspectives. Prediger, Bikner-Ahsbahs, and Arzarello (2008) make a distinction between coordinating and combining theories. They define coordinating as a term for bringing theories together that contain interpretations of notions that are compatible, whereas combining is when theories are only juxtaposed. In both PoP and SFL, immediate social interaction is central; more important, PoP set out to describe practices that are within the text produced in the immediate situation. SFL not only visualises the practices, it clearly shows how the different practices, in Skott's terminology, are linked together.

To conclude, SFL used as a methodological tool allows a text analysis on the micro level of the text, and PoP allows an analysis on the macro level related to practice. In this sense the theory and method seem to work together. The validity in Gee's terms is, from this perspective, strong.

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Why are Laura and Jane "not sure"?

Janne Fauskanger and Reidar Mosvold

University of Stavanger, Department of Education and Sports Science, Stavanger, Norway, janne.fauskanger@uis.no.

The mathematical knowledge for teaching (MKT) measures have been widely adopted by researchers. This paper reports on a case study of two mathematics teachers. The aim of the study was to investigate challenges regarding extensive use of "I'm not sure" as a suggested solution in the multiple-choice items. The connection between teachers' responses to multiple-choice MKT items, their written responses to corresponding open-ended questions (long responses) and group discussions are analyzed. The findings indicate that teachers' responses to multiple-choice items do not always correspond with the understanding revealed in responses to open-ended questions and in group discussions.

Keywords: MKT, measuring teachers' knowledge, multiple-choice items.

INTRODUCTION

This paper has a focus on measuring teachers' mathematical knowledge for teaching (MKT), and it builds upon the practice-based theory of MKT that was developed by researchers in the US (Ball, Thames, & Phelps, 2008). In connection with the development of this theory, sets of multiple-choice items were developed to measure MKT (e.g., Hill, Sleep, Lewis, & Ball, 2007). These items—which are often referred to as MKT items—were developed in the Learning Mathematics for Teaching project (LMT).

A multiple-choice item includes a correct answer—referred to as the key—and several wrong answers or distractors. In order to reduce the possibility of guessing, "I'm not sure" was included as a suggested solution in MKT items formatted yes/no (see Figure 1) in 2001 (Hill, 2007). In 2002, the researchers who developed these items were required to include "I'm not sure" on all number and operations content knowledge items that were piloted. To avoid changes in item parameters, "I'm not sure" was also included in the forms from 2004—which were later translated and adapted for use in the Norwegian context (Fauskanger, Jakobsen, Mosvold, & Bjuland, 2012). "I'm not sure" is always coded as incorrect, and teachers who are experienced with such tests would therefore immediately know that this suggested solution is a distractor. Some teachers would thus avoid this suggested solution, whereas other teachers might select it to avoid giving a wrong answer or to avoid guessing. In his often-cited handbook of development and validation of multiple-choice test items, Haladyna (2004) recommends that all distractors in multiple-choice items should be plausible, and, as a consequence, the suggested solution "I'm not sure" should be avoided. Many issues regarding the MKT items have been investigated already, but possible challenges related to the use of "I'm not sure" have not received much attention.

Previous studies conclude that teachers seem to have different reasons for selecting this alternative solution; some select "I'm not sure" based on uncertainty, whereas other teachers reveals instrumental and even relational understanding of the content of the items—but still select this alternative (Fauskanger & Mosvold, 2014). In this paper, we analyze data from a collective case study in an attempt to approach the following research question:

> What differences can be found between teachers' arguments for choosing the suggested solution "I'm not sure" in multiple-choice items?

This question is important in order to reveal challenges of extensive use of "I'm not sure" as a suggested solution in multiple-choice items developed to measure MKT. We focus in particular on two teachers' written long responses and discussions in group interviews regarding MKT items where they selected the alternative "I'm not sure". Cross case displays were constructed from teachers' long responses and their multiple-choice response supplemented with data from the interviews.

MEASURING TEACHER KNOWLEDGE

Teacher knowledge is imperative to high-quality teaching (e.g., Davis & Simmt, 2006), and it is thus relevant to gain insight into methods used to access and assess different aspects of teachers' knowledge (e.g., Hill et al., 2007). At the University of Michigan, researchers have developed a practice-based theory of mathematical knowledge for teaching (MKT). Hill, Rowan and Ball (2005, p. 373) define MKT as "the mathematical knowledge used to carry out the work of teaching mathematics". In the LMT project, considerable resources were invested in developing and validating sets of multiple-choice items in order to assess and access teachers' MKT (e.g., Schilling, Blunk, & Hill, 2007). Based on these efforts, Hill and colleagues (2004) suggest that the MKT measures can be used to measure growth in teachers' knowledge. The teacher knowledge that is measured by the MKT items further relates to the mathematical quality of instruction (Hill et al., 2008) and—to a certain extent—to student learning (Hill et al., 2005). Inspired by these promising results, researchers have adapted the measures for use both in and outside the US (Blömeke & Delaney, 2012).

Responses from multiple-choice items are expeditiously analyzed and they can be used at scale. Developing multiple-choice items that are intended for measuring something beyond procedural skills is, however, both tedious and demanding (e.g., Haladyna, 2004; Osterlind, 1997). When discussing measurement of teacher knowledge, researchers have argued that use of multiple-choice items might result in trivialization of the complexities of teaching and thus threaten validity (Beswick, Callingham, & Watson, 2012; Haertel, 2004). In his critical discussion of the MKT measures, Schoenfeld (2007) argued that these measures might test something other than they are supposed to. He further suggested that the multiple-choice format might complicate the content for the test takers, and confirming evidence of this was found in a Norwegian context (Fauskanger, Mosvold, Bjuland, & Jakobsen, 2011). An additional aspect of Schoenfeld's (2007) criticism was that the items actually measure a type of knowledge that is more procedural than intended, and more recent studies in Norway support this criticism (Fauskanger & Mosvold, 2015).

A standard multiple-choice item consists of two parts: a problem (often referred to as stem) and a list of suggested solutions. In the MKT items, the stem is typically situated in the context of the work of teaching mathematics. The list of suggested solutions contains a key and one or more distractors (often incorrect alternatives). Some MKT items differ from more standard multiple-choice items in that mathematically incorrect alternatives are not always included; the correct solution might then be: "all of the above" or "none of the above". The use of such items are often discouraged (Haladyna, 2004); if used, they should at least be used with caution (Osterlind, 1997). Another way in which MKT items differ from more standard

3. Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35 x 25 125 +75 875	35 x25 175 +700 875	35 x 25 25 150 100 +600 875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

	Method would work for all whole numbers	Method would NOT work for all whole numbers	I'm not sure
a) Method A	1	2	3
b) Method B	1	2	3
c) Method C	1	2	3

Figure 1: MKT testlet including "I'm not sure" as a suggested solution in all three items (Ball & Hill, 2008, p. 5)

multiple-choice items is the extended use of the suggested solution "I'm not sure" (Fauskanger & Mosvold, 2014)—first included in items that were formatted yes/no and later for all number and operations content knowledge items (Hill, 2007).

Different approaches have been made to distinguish between different categories of teachers' knowledge and understanding of the mathematical content; Skemp's (1976) distinction between instrumental and relational understanding is an archetype. Rote memorization of algorithms for two-digit multiplication (Figure 1) is an example of instrumental understanding, whereas relational understanding encompasses a deep, conceptual understanding. Skemp argued that students cannot develop relational understanding from instrumental teaching. Recent research has investigated this connection empirically and concludes that teachers' knowledge of facts and procedures have less positive effect on the quality of instruction and students' achievement relative to knowledge of concepts and connections (Tchoshanov, 2011). For this reason, the importance of exploring teacher knowledge accessed by measures as well as possible challenges regarding the use of multiple-choice items to investigate something as complex as teacher knowledge, becomes evident. It is relevant to carefully investigate these challenges—in particular the challenges revealed from extensive use of "I'm not sure" as a suggested solution.

METHODOLOGY

This study is part of a collective case study (Silverman, 2006), and two in-service teachers were chosen as cases to investigate the topic under investigation. When adopting a case-oriented approach, we considered the case as an entity and first looked for configurations and characteristics within the case before we searched for similarities and patterns across cases. The two teachers, who have been assigned the pseudonyms Laura and Jane, were participants in a professional development course. A total of 38 in-service teachers participated in the course, and 30 of these teachers agreed to submit multiple-choice responses to 28 MKT items (including the testlet in Figure 1). All of these items had a focus on number concepts and operations, and 18 of the items included "I'm not sure" as a suggested solution. In addition to submitting their multiple-choice responses to the items, the teachers also agreed to submit long responses related to each item and to discuss the items in groups afterwards. The questions prompting long responses were developed to tap into teachers' instrumental and relational understanding (Skemp, 1976) and varied across the 28 items. The group discussions were based on the same 28 items.

In a previous publication, we have reported on results from analysis of multiple-choice responses and long responses for all the teachers who selected "I'm not sure" as a response to a particular item (Fauskanger & Mosvold, 2014). Those analyses revealed that three groups of teachers could be distinguished between (see Table 1).

Laura and Jane both responded "I'm not sure", but their long responses indicated that Jane did so due to insecurity (group 1), whereas Laura indicated relational understanding in her long response (group 3). In our attempt to investigate different arguments for selecting "I'm not sure" as a response to MKT items, we therefore selected these two teachers as contrasting cases.

According to the official coding manuals from the LMT project, the suggested solution "I'm not sure" should be coded as incorrect. An underlying hypothesis would then be that teachers who select this response do not have the proper level of MKT to identify the key; they select "I'm not sure" to avoid guessing. In order to learn more about what challenges are revealed from extensive use of "I'm not sure" in multiple-choice items, we focus on what types of understanding could be found in teachers' long and oral responses when they select the answer "I'm not sure" on a multiple-choice item.

The unit of analysis is the individual teachers' multiple-choice responses, their long responses and their

Group	Group 1: not sure	Group 2: instrumental understanding	Group 3: relational understanding
Name of teacher	Erna, Frøya, *Jane*, Jan, Ada and Nina	Pia, Mons, Harald, Ola and Are	Sara, Inge, Ragna and *Laura*

Table 1: Teachers grouped according to their long responses

individual voices as revealed in group discussions. The group discussions were recorded and transcribed. We have applied an iterative strategy weaving back and forth between the empirical material and theories (Alvesson & Karreman, 2011).

We used Skemp's (1976) categories for our coding of the textual data. Excerpts from teachers' written and oral responses reflecting memorization of facts or rules, procedural computations or other aspects related to instrumental understanding were coded as instrumental, whereas excerpts reflecting understanding of concepts and connection between them, multiple solutions to non-routine problems or other aspects related to relational understanding were coded as relational. A third code, low/no MKT, was used to code excerpts where teachers' explicitly wrote or said that they did not know the content of the item(s) or excerpts revealing low level of MKT. In order to increase the reliability of the coding, the two authors coded the data independently and reconciled. In the few instances where there was a mismatch between our initial coding, we discussed and reached agreement.

THE CAse of Jane

Jane's long responses indicated insecurity related to the content in focus, and she responded "I'm not sure" to all three items in the testlet in Figure 1 (6a, 6b and 6c in our form). When asked how she would approach students who used methods like A, B and C, Jane wrote: "It is difficult to know when you do not understand the methods [the students have] used." This long response—along with the other long responses written by Jane—thus seems to support the hypothesis that the selection of "I'm not sure" implies lack of knowledge or insecurity. Coding her responses to the multiple-choice items in this testlet as incorrect thus seems reasonable. It can also be argued that the inclusion of "I'm not sure" has reduced the possibility of guessing with Jane, and that was the intention of including this suggested solution in the MKT items (Hill, 2007).

In the interview, Jane explains that she wants to teach an algorithm she is more comfortable with herself:

Jane: There is a point to explaining that your own way [of calculating it] is all right, in a way. I have used it for calculating for years, so it is natural to me. But, then again, we are different. Some like this and some like that. And it is similar to subtraction, when you borrow 10, if you say for instance 15 minus, or if you say three minus and then add the 5. We have shown both ways, and then it is up to them [the students] what they... Then we have said that it relates to what they, some like this and some like that, and≈

Ragna: ≈Open for all [to choose]≈

Inga: ≈And then, many [students] have the parents show it to them in a different way than the one they have been taught. Yes, that is difficult.

Interviewer: But do you experience that the parents have a common [algorithm]?

Jane: No. I haven't asked about that, but I know that many [parents] help their children setting it up, and with subtraction it is quite similar – with borrowing (...)

Interviewer: Anything else you want to say in relation to the items that were concerning different algorithms?

Jane: No, and we have been taught that there isn't only one standard algorithm (laughter)

As displayed in this excerpt, the reason why Jane selected "I'm not sure" seems to be that she is insecure about the mathematical content, and she wants to teach the students an algorithm with which she is familiar. The excerpt thus indicates that Jane's knowledge related to multi-digit multiplication is instrumental (Skemp, 1976). She reveals, however, that the professional development course has made her aware of the existence of multiple algorithms, and she seems to agree that different algorithms might be useful for different students.

THE CASE OF LAURA

Laura also selected "I'm not sure" as her response to several multiple-choice items. In contrast with Jane, however, Laura's long responses indicates that this was not due to low level of MKT or insecurity. On the contrary, Laura's long responses indicate deep conceptual knowledge (Skemp, 1976). In her long response related to a particular testlet — with a content focus related to place value and non-standard ways of decomposing three-digit numbers (items 1a-d in our form) — Laura argued that the stem could be interpreted in different ways and that the choice of key for each item would depend on this interpretation. The following is an excerpt from what Laura wrote: "Item a) is wrong by all means. Items b), c) and d) are wrong if it [the problem presented in the stem] is a closed problem, but they are correct if it is an open problem." By "closed problem" Laura seemed to have in mind the standard decomposition, and by "open problem" she meant "open" to non-standard ways of decomposing three-digit numbers. When highlighting testlet 1 as mirroring knowledge important for her as a teacher Laura wrote:

> To be able to do arithmetic one has to think flexibly when it comes to decomposing a number. 574 is not only 500 + 70 + 4. It could also be 400 + 170 + 4. 500 is 5 hundreds, 50 tens or 500 ones, etc. The students need to be familiar with this [non-standard ways of decomposing numbers] in order to be able to understand the four arithmetical operations [addition, subtraction, multiplication and division] and in order to develop flexible strategies for multi-digit arithmetic.

Laura is one of the teachers whose long responses—by relating the decomposition of numbers to understanding of "the four arithmetical operations" and "the development of flexible strategies"—indicate relational understanding (Skemp, 1976) of the content. In her long responses, Laura relates multiple decompositions to arithmetic, and multiple decompositions seem to be just as important for her as standard decompositions (cf., Jones et al., 1996). Her incorrect multiple-choice responses are thus inconsistent with her long response, and she responds "I'm not sure" despite of a high level of MKT. Laura seems to have responded "I'm not sure" due to the wording of the items included in this testlet. This brings forth issues related to item development and translation (cf., Fauskanger et al., 2012).

When analyzing Laura's utterances from the interviews, it also appears that she has a deep understanding—both in terms of mathematical content knowledge and pedagogical content knowledge. The interview data thus, seems to support the long responses from Laura.

Laura: I want to argue that these items are very much about understanding. (...) You have these [MKT items], which relate to understanding what the students actually do. And being able to identify it. And it is what they were able to divide into, I mean, normally we divide into all the hundreds, all the tens and all the ones. That is how to do it. And then they don't remember that they can change into ones and tens, and then they are stuck there. So this is very relevant.

(...)

Laura: It relates to what grade level it is in. So you could say, now I work in third grade, and we have approached the bigger numbers. The first point then is that they know that the numbers have different value if they are in the one, ten or hundred place. Then they know this, and this is the first thing they have to know. And then they know the place value system. But if they are not able to calculate, for instance 200 minus 4, because that doesn't work since there are no ones there [to subtract from in 200]. Then there is something they don't know after all, about knowing that they have different value. That is the first point. And then it is concerning the flexibility that this item implies. To see if they have [this]. This also has to become natural eventually. But I think it is important to know that there and there and there [points to the digits in the three-digit number] the values are different.

As displayed in these excerpts, Laura selected "I'm not sure" despite being secure about the mathematical content. The voice of Laura in the group discussion thus seems to support findings from her long responses and the reason why she responded "I'm not sure" in the multiple-choice items does not relate to her low level of MKT but rather her relational understanding (Skemp, 1976) of the content.

CONCLUDING DISCUSSION

In a previous study, we analyzed 15 teachers' long responses related to a set of multiple-choice items from the LMT project (Fauskanger & Mosvold, 2014). Based on those results, two teachers were chosen as contrasting cases in the present study. We analyzed the teachers' responses to multiple-choice MKT items, written long responses as well as discussion of the items in group interviews in order to learn more about the challenges related to extensive use of "I'm not sure" as a suggested solution in the MKT multiple-choice items.

The results from our analysis indicate that we need to be careful about how we interpret it when teachers select "I'm not sure" as a multiple-choice response. "I'm not sure" is always coded as incorrect, and our analysis of teachers' long responses as well as their interview discussions reveals that this alternative was indeed selected by some teachers who explicitly indicated that they could not identify the key due to their low level of local MKT (as Jane, see Table 1). This seems to be in line with the intention of introducing "I'm not sure" to the items, and the suggested solution reduces the possibility of guessing for these teachers (Hill, 2007). Other teachers, however, selected "I'm not sure" as a response to multiple-choice items although their long responses as well as their discussions in group interviews indicated that they had relational understanding (e.g., Laura). Teachers might draw on deep conceptual or relational knowledge (Laura), procedural or instrumental knowledge (Schoenfeld, 2007), or their lacking knowledge (Jane) when responding "I'm not sure". Our results thus indicate that the knowledge teachers utilize in long responses or discussions does not necessarily mirror the knowledge (that seems to be) used when selecting a certain multiple-choice response. The assumption that the multiple-choice response "I'm not sure" is correctly coded as incorrect should therefore be subject to further scrutiny, and the inclusion of this alternative response in MKT items should also be critically discussed.

Our analysis of teachers' long responses as well as their discussions in group interviews indicate that none of the 30 participating teachers were guessing—thus conforming to the intention of including "I'm not sure". This suggested solution is still problematic, however, since one cannot conclude whether the choice of this suggested solution is based on lack of knowledge as one extreme point or deep conceptual knowledge as the other extreme. One might advocate removal of the "I'm not sure" option from the items, but this would change item parameters and should not be done hastily. Instead, we call for cross-cultural studies that investigate teachers' motivation for selecting a suggested solution like "I'm not sure" when responding to MKT items. Recontextualizing the MKT items into scenarios for use in qualitative studies related aspects of teachers' MKT as done by Adler and Patahuddin (2012) might be a fruitful approach. These researchers argue that the carefully constructed MKT items provokes teachers' mathematical reasoning in relation to practice-based scenarios.

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Exploring pictorial representations in rational numbers: Struggles of a prospective teacher

Nadia Ferreira and João Pedro da Ponte

Universidade de Lisboa, Instituto de Educação, Lisbon, Portugal, nadiadferreira@gmail.com

We aim to identify the knowledge that a prospective teacher, Maria, uses in practice, focusing on her struggles and what she learned from her practical experience teaching rational number multiplication. Data was collected from lesson plans, observations, written reflections and semi-structured interviews. Maria developed her knowledge for teaching when anticipating solutions and errors and selecting representations. Reflecting on her practice, she realised that she was able to solve some tasks with symbolic procedures but could not represent them pictorially.

Keywords: Knowledge, practice, rational numbers, representations.

INTRODUCTION

The knowledge for teaching mathematics that prospective teachers need to develop and the way they develop it are controversial issues (Ball, Thames, & Phelps, 2008; Ponte & Chapman, 2015; Shulman, 1986). The practicum is a particularly important site to study such knowledge since prospective teachers are faced with circumstances that allow noticing important weaknesses and strengths.

To foster students' understanding of mathematics concepts and procedures, teachers are called to engage them in making connections among representations (NCTM, 2007). They need to support students' fluent use of symbols, grounded in informal representations (Ball et al., 2008; Ma, 1999; Ponte & Chapman, 2015). Rational numbers raise many difficulties for students and challenge teachers to promote conceptual learning (Lamon, 2006; Ma, 1999). Research has brought attention to prospective teachers' knowledge of rational numbers in different ways. For example, Isiksal and Cakiroglu (2011) studied prospective teachers' pedagogical content knowledge of fraction multiplication. Findings indicated that teachers have different perceptions of children's mistakes and employ different strategies including using multiple representations, using problem solving strategies, making clear explanations of questions, and focusing on meaning of concepts. However, no study has been found focusing on the use of informal and formal representations in teaching fraction multiplication and the struggles prospective teachers may experience in providing representations that enable students to develop their knowledge on this topic. The aim of this study is to identify the knowledge of a prospective teacher in the teaching and learning of rational number multiplication, with a focus on the use of informal and formal representations, analysing the knowledge mobilized in teaching practice, the struggles, and the knowledge built from the first practical experiences.

PROSPECTIVE TEACHERS' KWOWLEDGE

Teachers' knowledge includes mathematical and pedagogical content or didactical knowledge, both of which are of critical importance for teaching practice (Ball et al., 2008; Ponte & Chapman, 2015; Shulman, 1986). Mathematics knowledge involves conceptual and procedural aspects (Hiebert, 1988; Rittle-Johnson & Schneider, 2012). Conceptual knowledge is a network of concepts and procedural knowledge consists in rules or procedures for solving mathematical problems (Bartell, Webel, Bowen, & Dyson, 2012). Procedural knowledge may be part of conceptual knowledge. Procedures may be performed without understanding or may be performed knowing why and when, in which case we have mathematical conceptual knowledge. Conceptual mathematical knowledge of rational numbers involves knowing different representations and meanings and in order "to create one representation first we have to know what to represent" (Ma, 1999, p. 135). Conceptual knowledge allows us to connect topics (e.g., seeing multiplication as repeated addition).

Didactical knowledge concerns with how teachers teach (Ponte & Chapman, 2015). Teachers must anticipate students' common mistakes and misconceptions (e.g., generalizing addition procedures in multiplication), to anticipate students' solutions in specific tasks, and also know what students will find challenging, interesting or confusing. Teachers also have to be able to sequence tasks, to recognize the value of using certain representations, to pose questions, and to explore students' strategies. In addition, they need to understand the main ideas of current curriculum documents, identifying principles of teaching (e.g., NCTM, 2007).

Ponte, Quaresma and Branco (2012) characterize teacher's practice into two main aspects: the tasks proposed to students and the communication established in the classroom. In respect to tasks, teachers may choose to offer just simple exercises or also propose challenging exploratory tasks, problems and investigations in which students need to design and implement solution strategies based on their previous knowledge (Ponte, 2005). Classroom communication may be univocal or dialogic, depending on the roles assumed by the teacher and the students and the types of teachers' questions, including inquiry, focusing or confirmation questions (Ponte et al., 2012). Representations are an important feature of tasks, and may be categorized as pictorial (images), iconic (points, lines, circles), and notational (number line, arrows, vertical columns, symbols) (Thomas, Mulligan, & Goldin, 2002).

RESEARCH METHODOLOGY

This study takes a qualitative and interpretative approach (Erickson, 1986), using a case study design. The participant is Maria, a prospective elementary school teacher. She always wanted to be a teacher but is in a higher education program at a late stage of her life. In school she had mathematics up to grade 9. Maria reflects with ease, addressing her difficulties in an explicit way. She said that she had to study hard to know the content that she was going to teach and to figure out how to implement the didactical ideas that she had learned at university. During teaching program, she experienced exploratory learning and she wishes to provide such approach to her students. Maria already knew the grade 6 class with 28 students, in which her practicum took place. She interacted informally with her school mentor to discuss who would teach the different topics, deciding that they would give a total of six classes on rational numbers, three of each taught by each one of them, with Maria introducing the concepts and the school mentor providing practice.

Maria's lesson was observed and videotaped (Li). In addition, data was collected and analysed from initial (II) and final (FI) semi-structured interviews, and before (BIi) and after lesson (AIi) interviews. We also analysed the documents that she produced (lesson plans and reflections) and the field notes written by the first author during data collection. The interviews and videos were fully transcribed. The analysis is descriptive, seeking to characterize Maria's teaching. The transcribed conversations were first analysed according to four dimensions (conceptual/procedural mathematical knowledge and didactical knowledge about tasks and students). At a second moment the analysis was based on categories built from data. The intersection of the four dimensions enables us to highlight communication moves (Charmaz, 2006). We consider knowledge to be conceptual when there is evidence of understanding the reasons for using procedures and for knowing different representations and meanings of a situation. We consider knowledge as procedural when the teacher cannot relate informal and formal representations or when she cannot explain in the interviews why she did it. We also give attention to didactical knowledge in practice, focusing on knowledge about tasks, students and communication that takes place in the classroom. For example, teachers must design appropriate tasks, know what they will explore and relate the representations, anticipate student's solutions and plan how to orchestrate them. In addition, teachers should anticipate questions to help students understand the concepts in the context of productive classroom communication.

THE PRACTICE OF MARIA: LEARNINGS AND STRUGGLES

Despite her willingness to follow an exploratory approach and carry out considerable planning, Maria did not anticipate how to relate different representations. Comparing her agenda with her teaching practice, she realized that she did not clearly explore the concepts. That is, Maria is a case of a prospective teacher seeking to perform exploratory teaching but with trouble in preparing and carrying it out as intended.

Mapping the topic and anticipating practice

Maria taught three lessons on rational numbers. She wanted to introduce fraction multiplication with an emphasis on understanding: "I'm more interested that students understand the why of the result and the meaning of the result. What it represents..." (BI3). She made decisions about the tasks to propose, considering their nature and value, and chose a pictorial rectangular representation and different symbolic representations of rational numbers. She intended to promote discussions with inquiry questions. Next Maria had to negotiate the tasks with the school mentor who wanted her to use the textbook. She used the tasks of the textbook, but felt that her choices were limited. She also read articles and documents about rational number multiplication. She reviewed procedures and solved several tasks: "I saw everything always supported what was in the textbook" (BI1). She tried to include ideas that represent "good practice". In the end, she decided to explore first the multiplication of a natural number by a fraction and then the multiplication of fractions. However, she still had some unresolved questions and sought out the professors from her teacher education college: "It was only when the professor began making pictorial representation that it occurred to me! Only when I looked at this representation did I associate pictorial and symbolic representations" (BI3).

In her lesson plan, Maria anticipated students' solutions, errors and explanations to prepare her to help students overcome these errors. She thought, "If they ask this, what will I say? If I ask this, what might they say?" (BI). And she added:

> I consider students' possible solutions (...) in the multiplication of fractions, whether they follow the rule denominator times denominator and numerator times numerator (...) multiply denominators and maintain the numerators, [or generalize ideas from] addition, finding common denominator ... (BI).

In her view, pictorial representations might serve as a useful support for solving the tasks because "sometimes... I think we do the mathematics, we give the results, but what are we talking about? Which unit? What part? Part of what?" (BI3). As she explored the tasks, she encountered difficulties and said: "It's very difficult to imagine students' thinking, what will happen... Imagine them... It is a difficult exercise but very necessary for further practice" (BI2). In the lesson plan, she solved all tasks with fractions and pictorial representations except in the first task.

Maria used didactical knowledge when she considered the kind of tasks to propose, established a sequence of tasks, and anticipated students' potential solutions and common mistakes. However, when anticipating the solutions of the tasks she did not fully realize how challenging and powerful they could be. As a result of a learning experience with her university professor, she recognized the value of using pictorial representations; thus, it seems that she had developed didactical knowledge. Yet when she anticipated different symbolic solutions for the tasks and some pictorial representations, her procedural knowledge became evident. However, her conceptual knowledge would only show up in the practical experience.

Instructional practice

Maria began the lesson reviewing the homework. Then, she told a story to engage students in solving a problem: "With the candy that came in a box, Luís separated 6 bags of 2/5 kg each. Does the box weigh more or less than 3 kg?" This task combines discrete quantities (6 bags) and continuous quantities (weight of the bags). Dealing with these quantities requires suitable representations. Maria invited students to present their ideas, saying "How can we solve this problem? Who wants to help? Pedro!" The student proposed immediately "add 2/5 six times". She recorded Pedro's idea and asked the result. After 5 seconds she wrote the sum with the students' help "2/5+2/5+2/5+2/5+2/5+2/5=12/5". Immediately Pedro said "12/5 isn't more than 3 kg". Maria again recorded this on the board and posed inquiry and focusing questions to help others students to understand it:

Maria:	Pedro said that 12/5 is not more than 3 kg. Pedro, how did you think?
Pedro:	Because 5/5 = 1; 5/5 + 5/5= 10/5=2; 15/5=3.
Maria:	Exactly! And what do I have (pointing to 12/5)?
Pedro:	12/5.
Maria:	Then we know that we have at least 2 kg! Because 5/5 more 5/5 are 10/5! And there is 2/5 missing to one [more unit]. But how can we see that? Is there

another way of seeing this? And if we tried… (Turns to the board) How many bags do we have?

To illustrate Pedro's answer, she began the explanation with the support of a rectangular representation (Figure 1) and said:

> Let's suppose we have here a rectangle and each part of the rectangle is a bag. How much does each bag weigh? (…) 2/5 kg. And with the second bag? (…) 4/5. So we have… (Together with the students) 4/5, 6/5, 8/5 and the result is 12/5. (L3)

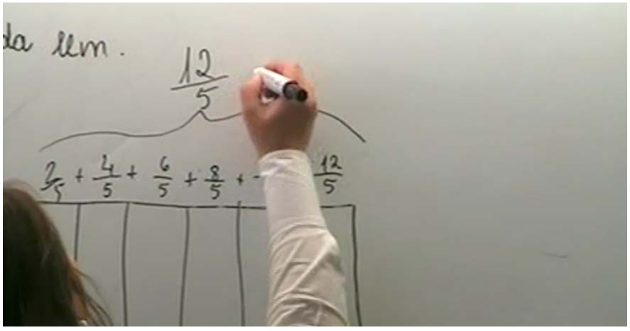

Figure 1: Representation of the first problem

This representation is rather confusing and students began complaining that they did not understand. Figure 1 illustrates that Maria was saying one thing and symbolically representing another. On a second trial, she divided each part in five parts unsuccessfully. Then she erased the figure and explained symbolically "we have to do 2/5+2/5+2/5+2/5+2/5+2/5=12/5". The teacher did not relate the pictorial and symbolic representations. She went on posing focusing and inquiry questions:

Francisco: Ok! And what if we do… Six times two and the five stays the same?
Maria: Ok! How can we do it?
Francisco: Six times two is twelve and the five stays the same.
Maria: Why don't you multiply the 5 too? And what if I do like this (6/1×2/5)? Now we have the numerator!
Pedro: Five times one is five…
Maria: And if I said that I multiply the (…) numerator and keep the denominator. Can I always do this? (L3)

The students rejected the rule and Maria, attempting to convince them, used another example with another denominator, stressed the rule comparing the procedure with the result, and moved to the synthesis and final answer:

> The problem said that Luís has six bags and each bag has 2/5 of a kilo. Then we do 2/5+2/5=4/5 and 4/5+2/5= 6/5 (…) 12/5. Then Francisco said it would be faster to do 6 times 2/5 which gives 12/5. And, finally, we see that we can multiply numerator and numerator and keep the denominator. (…) Pedro said in the beginning that the answer is less than 3 kilos because 12/5 isn't as much as 3 kilos… (L3)

In the sequence, Maria proposed a second task: "Rita cut a cake into 4 equal parts. But one quarter of the cake seemed to be a very big slice! After all, she only wants half a slice, or half of one quarter. Which part of the cake does Rita eat?"

Immediately a student explained that "1/2 represents half and one quarter is the final measure when we divide in 4 pieces. One half of a quarter is half of 1/4". Again, Maria asked how to represent this situation and drew a rectangle divided in four parts (Figure 2). She went on asking confirmation and focusing questions:

Maria: Is this what Rita eats? (Pointing to half of a quarter of the cake)
Francisco: It is half of a quarter! (…) She only wants half!? Since all the pieces were the same… I have to divide everything in the middle…
Maria: Oh! Interesting, Francisco! (…) What Francisco is saying is very important! But maybe it is better to divide in columns! It's the same reasoning! (L3)

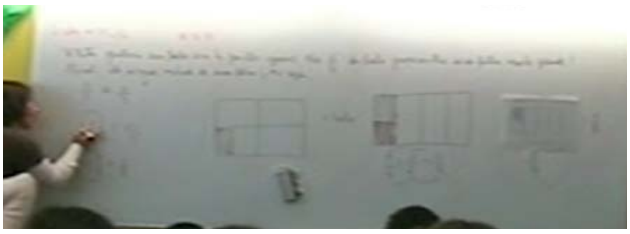

Figure 2: Different representations of the second problem

At this point, using the projector, Maria illustrated Francisco's idea. A student said:

> She eats one eighth. Rita divided the unit and only ate one part... The cake was divided into four parts and then divided in eighth parts... Halves that gives eighths... She ate one out of eight parts! (L3)

In her teaching practice Maria mobilized mathematical and didactical knowledge and showed weaknesses in both. When she explored the first problem, she used fractions and she did not consider the pictorial representation of multiplying a whole number by a rational number. She showed a procedural nature of mathematics knowledge and revealed some weaknesses in her conceptual knowledge when she was not able to pictorially represent the operation. In the second task she anticipated the pictorial representations and showed conceptual knowledge. However, she did not explore the connections between the pictorial and symbolic representations. Her discourse had changed. In the beginning, she had a dialogue with three students (one at a time) and tried to focus the attention of all students on the emerging ideas. She posed different types of questions, including inquiry, focusing and confirmation questions. In the second moment, she posed fewer focusing and confirmation questions.

Evaluating, reflecting critically, reviewing, and restructuring knowledge

In her reflection, Maria considered that this lesson had presented unexpected difficulties with the pictorial representation of the task and regarding the exploration of mathematical ideas related to the representations and their connections. This aspect caused her great anxiety at the end of the lesson:

> I shouldn't have used this kind of representation because they [the students] weren't used to it (...) Although I think that this representation helps explain why. The goal is for them to realize why in that case the result 12/5 appeared. It didn't fall from the sky! (AI3)

Maria recognized the value of using certain representations but considered giving up on the use of pictorial representation, which reveals some insecurity regarding her didactical ideas. She did not notice the mistakes that she had made on the board (shown in Figure 1) and merely felt that she had not handled the situation well and the students did not understand. After visualizing the lesson video she said: "I think this approach [using pictorial representations] is better for everyone, I am convinced. (...) The problem is not in the explanation of the problem! It's in me!" (AI3). She identified her mistakes in the first task: "What I wrote was not the same as what I was saying! And maybe that was the confusion! I caused the confusion because, in fact, what they are seeing is 2/5+4/5+..." (AI3).

During the interview, Maria tried to explain why she should not use this pictorial representation. She agreed that the reason was that she did not recognize the differences between quantities and that the result was an improper fraction. These difficulties raise questions about her conceptual mathematical knowledge.

Confronting the plan and the practice in the second task, Maria realized that she used different pictorial representations and different symbolic representations of the proposed problem (Figure 3). At the end of this lesson she evaluated her practice recognizing that: "I knew the procedure, I mastered the procedure! I memorized the procedure back and forth and not how to represent the concept [multiplication]!" (AI3).

Reflecting on her agenda and comparing it with her teaching practice, Maria concluded that the end result was positive but could have been better. She recognized that, although she had invested heavily in planning the lesson, she had not achieved her desired result. She noticed that in order to prepare these lessons she had mobilized her mathematics content knowledge that she learned in the teacher education program:

> I see rational numbers in a different way (...) Before I had fractions and decimals arranged in

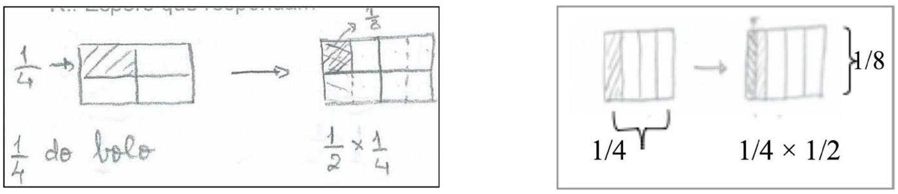

Figure 3: Representation of the second problem in the plan and in practice

different drawers (...) Now I see that they are different representations of the same number (FI).

When she studied the literature, she learned the relations between operations and said "when we multiply a whole number with a fraction we are simplifying repeated addition (like in 6×2/5, when I work with students starting from 2/5 +2/5 +...)" (FI). She did not explore the relationships between representations and she was able to see, during the interview, that the arithmetic expression that she wrote on the board did not represent the problem proposed.

Maria recalled general didactical ideas and these became meaningful for her. She realized the importance of planning and that her lesson plan was better after each practical experience. She realized that in order to be successful in her practice "the plans have to evolve (...). In the first plan [lesson before], I thought that it was perfect and then I went to practice and I realized that it could have been better..." (FI). She developed knowledge when anticipating solutions, errors and representations. Maria knew that a teacher must "lead students to understand the why of something" (FI) and that is why she knows that it is important to "unpack the pictorial representations" (FI). After the teaching practice, she continued to believe that good teaching leads to learning with understanding. She assumed that the teacher must support students in learning, by listening, asking good questions and helping them to build their knowledge.

CONCLUSION

Maria felt that she developed significant aspects of her knowledge about rational numbers during these lessons. Regarding her mathematics knowledge, she become more aware of the concepts related to rational number multiplication in fraction representation. Early in the lesson, her weakness in conceptual knowledge led her to take a procedural approach (Ma, 1999). However, she showed conceptual knowledge when she said that rational numbers may be represented by fractions and decimals and related addition and multiplication. However, she struggled to connect these with pictorial representations. As in other studies, we see issues related to connecting real-world situations and symbolic representations and connecting different representations of a concept (Ponte & Chapman, 2015). At the end of this teaching experience, Maria had learned about: how to represent improper fractions; the complexity of using both discrete and continuous quantities; connecting pictorial and symbolic representations; and the meanings of different expressions.

Concerning didactical knowledge, Maria was able to sequence tasks using ideas of several articles and recognized the value of using pictorial representations as tools to develop mathematical ideas. She mentioned that she had become more aware of anticipating students' questions, common mistakes, and solutions (Bartell et al., 2012). She also became more aware of when she needed to explain. In her instructional practice she posed questions and explored students' strategies, seeking to lead them to connect pictorial and symbolic representations. As in other studies, we see issues related to conceptual understanding, using multiple representations and curriculum materials and textbooks, planning, assessing students, and analysing mathematics teaching (Ponte & Chapman, 2015).

This study illustrates the struggles and learning that prospective teachers may experience when they strive to engage in an exploratory approach that requires strong mathematical and didactical knowledge (Ponte & Chapman, 2015). In order to propose challenging tasks and to use different representations, they need to develop a deep understanding of rational numbers. They must engage in analyzing students' strategies, offering student-focused responses, anticipating practice by solving the tasks and discussing common mistakes (Son & Crespo, 2009). This study shows how anticipating practice in a careful way is essential for prospective teachers.

These results show us that some aspects may only become explicit in practice. Teachers have perceptions of children's mistakes and different strategies that may be used including using multiple representations, using problem-solving strategies, providing clear explanations and focusing on the meaning of concepts (Isiksal & Cakiroglu, 2011). In this study we focused on the use of informal and formal representations in teaching fraction multiplication and on the struggles that a prospective teacher experienced in providing representations to help students to develop their knowledge about this topic. For example, Maria anticipated the solutions of the tasks in symbolic and pictorial representations but in class had trouble relating the two representations. In another words, practice was an appropriate context to see what she could do and the nature of her knowledge. These issues are important to know how teacher educators may prepare and support prospective teachers who are learning to teach for meaningful conceptual learning.

ACKNOWLEDGEMENT

This study is supported by national funds by FCT – Fundação para a Ciência e a Tecnologia (reference SFRH/ BD/99258/2013).

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“Responding to student ideas” as an indicator of a teacher’s mathematical knowledge in teaching

Oguz Koklu[1] and Fatma Aslan-Tutak[2]

1 University of Georgia, Athens, USA

2 Bogazici University, İstanbul, Turkey, fatma.tutak@boun.edu.tr

This paper describes and analyses two 8th grade mathematics instructions, delivered by two teachers on the slope and its algebraic representation. The focus of analysis is on teacher content knowledge which was analysed with Knowledge Quartet model (Rowland et al., 2005). The study examined whether teachers’ responses to students’ comments indicate the quality of teachers’ content knowledge. Findings indicated that teacher’ reaction to students’ comments would significantly reflect teachers’ depth and breadth of mathematics content knowledge.

Keywords: Teacher knowledge, knowledge quartet, contingency, slope, responding to students’ ideas.

TEACHER KNOWLEDGE AND TEACHING

Enhancing students’ mathematics thinking and learning is the utmost aim of overall mathematics education. One of the most important influences on students’ learning and mathematics thinking is the teachers and their professional knowledge (Even, 1993; Darling-Hammond & Ball, 1998). Teachers’ knowledge is crucial since teachers’ effectiveness is influenced by that knowledge (Gilbert & Gilbert, 2011; Wagner et al., 2007). In addition to its effect on students’ learning, teachers’ knowledge is influential in shaping their practices (Borko & Putnam, 1996; Even, 1993; Fennema & Franke, 1992; Rowland & Ruthven, 2011; Sherin, 2002; Shulman, 1986).

Shulman’s study (1986) opened a way to understand teachers’ complex and multi-dimensional character of knowledge. Shulman suggested seven knowledge categories. Among them, the first four were content-free and the last three ones were on content-specific knowledge (Rowland & Turner, 2007). Subject-matter knowledge (SMK), curricular knowledge (CK), and pedagogical content knowledge (PCK) constitute the content-specific knowledge that teachers have (Shulman, 1986). The work of Shulman (1986) suggested that content-specific knowledge becomes important in studying teachers’ knowledge.

Shulman’s conceptualization on teachers’ knowledge has been heavily used by mathematics education researcher community. Almost all study on mathematics teacher knowledge assumes Shulman’s categorization and conceptualization of teacher knowledge that is why we do not present a detailed account of Shulman’s ideas in this paper.

Researchers have studied mathematics teachers’ knowledge from several perspectives. While some researchers conducted research on teachers’ understanding of various concepts in mathematics (e.g., Ball, 1990), others have focused on investigating the relationship between SMK and PCK and teaching (e.g., Even, 1993; Rowland, Huckstep, & Thwaites, 2005; Hill, Rowan, & Ball, 2005).

Apart from the perspectives summarized above, discussions on the nature of teachers’ knowledge deserve an exclusive attention. The discussion on the nature of teachers’ knowledge yield a consensus that the knowledge employed in teaching is dynamic, more visible through practice, and should be studied in actual classroom setting (Fennema & Franke, 1992; Hodgen, 2011; Rowland & Ruthven, 2011; Wagner et al., 2007). Conceptualizing a mathematical knowledge for teaching would unlikely to be successful unless it carefully takes the classroom context of teachers’ professional work into account (Hodgen, 2011; Rowland & Ruthven, 2011; Sherin, 2002). The reason behind this failure may originate from the de-contextualization of teachers’ knowledge (Hodgen, 2011). Focusing the use of teacher knowledge in the practice of teaching may lead more sound results in investigating teachers’ knowledge (Hodgen, 2011; Rowland & Turner, 2007).

Considering our concerns on the nature of teachers’ professional knowledge, one framework which may provide studying teachers’ content knowledge during instruction is the Knowledge Quartet (KQ) (Rowland et al., 2005). The framework suggests four broad units in investigating teachers’ mathematical knowledge in teaching. These are (i) foundation, (ii) transformation, (iii) connection, and (iv) contingency.

The first unit of the KQ is rooted in the foundation of teachers’ knowledge, beliefs and understanding of mathematics and teaching. The remaining three units originate from a foundational underpinning. The second unit of the KQ, transformation, is knowledge-in-action which is visible throughout planning and teaching. The unit includes a teacher’s capacity in transforming the content knowledge into pedagogically powerful forms. Connection concerns the depth, breadth and coherence of relationships demonstrated in a lesson or between lessons. The fourth unit of the framework, contingency, encompasses to responding appropriately to the events and ideas which occur in classroom during instruction. It is about contingent action of teachers in the classroom (Rowland et al., 2005). Each of these KQ units is represented with codes. For example, there are three codes for contingency (responding to students’ ideas, use of opportunities, and deviation from agenda). There are eight-teen codes in total for four units (Rowland & Turner, 2007). Within its current form, the units of the framework are not directly linked to Shulman’s conceptualization of subject matter knowledge and pedagogical content knowledge. However, within the units, the framework aims to cover those knowledge types. The KQ has been grounded in classroom practice, and the findings have been open to enhancement and revision in the case of new research data. In this sense, the KQ can be effectively used to investigate the way content knowledge enacts during instruction. A detailed account of the framework may be reached from the researchers previous studies (Rowland et al., 2005).

THE TEACHING AND LEARNING SLOPE OF A LINE

In order to examine and evaluate teachers’ knowledge on a topic, it is important to address crucial mathematical ideas for the topic and literature on teaching and learning of it. Therefore, in this part of the paper, we will highlight mathematical importance of the topic ‘slope’ and some literature on teaching and learning of it. Slope is a deep and multi-faceted concept (Stump, 1999). Learning slope requires proper comprehension of important concepts such as ratio, rate of change, proportionality, covariation, and synthesis of different representations. Hence, understanding slope of a line in early grades is especially crucial for future learning. Comprehension of slope through algebraically, geometrical or by other representation as well as the connections between them requires conceptualizing an important set of concepts.

Slope is a fundamental but a conceptually complex concept for students in learning algebra (Lobato et al., 2003; Stump, 1999, 2001). Rasslan and Vinner (1995) investigated that majority of the nine graders did not realize that “the slope is an algebraic invariant of the line and therefore does not depend on the coordinate system in which the line is drawn” (p. 264). In addition, Saldanha and Thompson (1998) suggested that it may be challenging for students to understand “graphs as representing a continuum of states of covarying quantities” (p. 7).

Lobato and Bowers (2000) provided that students have various difficulties in learning quantitative complexity of slope. For example, majority of the participants in the study showed difficulty in understanding the role of change in rise and run on steepness of a line. In addition, students have difficulties in regarding slope as a ratio (Bell & Janvier, 1981; Leinhardt et al., 1990; Lobato et al., 2003).

The previous studies showed that there exist a number of misconceptions among learners on slope concept. Some of the incorrect ideas mentioned in the literature were: the quadrant where the line is located is related to slope value, changing slope alters y-intercept of the line, slope is the scale of the x-axis, and slope is the difference in y-axis (Lobato et al., 2003). Among them, slope-height confusion is very common. Being aware of those misconceptions is crucial in teaching slope.

THE STUDY

The aim of the study was to investigate teachers’ responses to students’ unanticipated questions as an indicative of their content knowledge. In other words, in this paper, findings about teachers’ knowledge of contingency (as defined in KQ) will be examined. Therefore, the research question is that *How does mathematics teachers’ responding to students’ ideas indicate their content knowledge in teaching slope of a line?*

The original study focused on two pre-service, two early novice (0–2 years of experience), and two more experienced novice (3–5 years of experience) middle school mathematics teachers` content knowledge on slope at 8th grade in Turkey. One of the researchers conducted individual pre-interviews, recorded 1–2 lesson hours of slope instruction, and finally conducted post-interviews. Pre and post interview data were used for triangulation but main data source was video-tapes of classroom instruction and researchers’ observation notes of the instruction. Instruction data were transcribed and open-coded. In data analysis, the researchers attempted to reach themes for the open coded data. Then, for each theme, one of the eighteen codes of the KQ has been assigned. Since assigned KQ codes belong to the units, we were able to categorize each significant episode of instruction in terms of the KQ framework units. Lastly, findings were compared between groups and similarities and differences were discussed. A detailed account of the framework, methodology, and findings may be reached from the thesis study (Koklu, 2012).

The aim of the current paper is to discuss teachers’ responses to students’ unanticipated questions to examine their knowledge on students` mathematical thinking. For the purpose, *contingency* episodes were used to investigate research question. Furthermore, data from two teachers (Akif and Merve, pseudonyms) was used because their instruction had clear contingency episodes; they encouraged students to share their opinions in the classroom. For each teacher, first, the instructions will be summarized briefly then one contingency episode will be provided. It should be noted that there were more contingency instances for teachers but because qualitative research findings report less raw data but more interpretation, we will give only one example for each teacher. Further examples and detailed information about the analysis can be found in Koklu (2012).

A PRE-SERVICE TEACHER’S CASE: ALGEBRAIC REPRESENTATION OF A VERTICAL LINE

Akif was a pre-service teacher in senior year who was practicing in 8th grade mathematics course. He described the aim of the lesson in the lesson plan as “*Students will be able to explain the relationship between slope and equation of a line.*” Akif segmented lesson into three phases. In the first phase, Akif introduced the slope concept. The second phase based on a task. The task included a scenario and a numerically represented tabular data and asked students to; (i) graph a line, (ii) write its equation in slope-intercept form, (iii) compute slope on graph, and (iv) investigate that slope and the coefficient of x are same in a slope-intercept form of line equation. The lesson followed by a computation on a series of exercises in which students were asked to match line equations and graphs of lines in coordinate plane. The line equations given for the exercise were as follows: $y = 8 - 4x$, $y = 2x - 2$, $y = 3x + 6$, and $y = -x + 2$. The teacher preferred first to find slope of lines on the equations. Then, he found the slope of lines on the graphs and related these slopes to the equations by using the relationship discussed.

The below episode was coded as *contingency* because of the teacher’s reaction to a student’s comment. The teacher applied the algebraic relationship between slope and the coefficient of x in slope-intercept form of line to compute slope. Then, a student raised an important idea.

Teacher: We said that the coefficient of *x* would be the slope, okay the coefficient of *x* since *y* stands separately [in one side of the line equation].

Student: If there would be a number instead of *y* what would be the answer?

Teacher: If there would be a number [he repeats student’s idea and waits a second] you mean no *y* [he meant the case in which no y appears in line equation].

Student: Yes.

Teacher: When we are write such an equation [he indicates the graph of $y = 2x$] we need to write in terms of *x* and *y*, think when we graph a linear equation you should remember that [he graphs a line which has a negative x-intercept and positive y-intercept] let’s x-intercept would be a and y-intercept would be b. When we write it, we were saying x over a plus y over b equals to 1 [he wrote $x/a + y/b = 1$].

Student: I do not know it.

Teacher: Okay, let’s skip it. Why there should be y here [thinks a second] if there was not a y, then how can we indicate the number of goods produced [he refers to the activity of the lesson] based on the number of employees.

Student: Right.

Teacher: One variable needs to depend on the other there is a relationship between the variables there is a combination of two hence I cannot assume that one [indicates the variable] is missing. Two of them [indicates the variables] have to exist at the same time. As a conclusion we write in this way.

This episode indicated that even though the teacher tried to respond student’s idea in multiple ways, he was *unable to suggest a correct explanation.* In other words, Akif was unable to recognize that a line equation does not need to include two variables. A vertical line, for example, is expressed by one variable. The data suggested that in the first explanation the teacher was unable to address student’s comment because his explanation already necessitates the existence of two variables. The teacher’s second explanation referred to his activity. According to his activity, a case such as the student proposed would not be possible. The explanation made by the teacher seemed to satisfy the student even though the explanation does not address the student’s question.

The data suggested that the teacher responded to student’s idea willingly even though his responses did not address the question. It seems that the teacher was not able to remember vertical lines. It was remarkable that the teacher did not resort to slope concept. The basic idea that the teacher should know is that slope is a concept which is defined on non-vertical lines. Mathematically, slope of a vertical line is undefined.

During data analysis we labelled the above case and the similar ones as our themes. Then, for each theme, codes of the contingency unit of the KQ have been assigned. Since assigned codes belong to the units of the KQ, we were able to categorize each significant episode of instruction in terms of the framework. To illustrate, the above episode suggested a student involvement to instruction hence we assumed that “responding to student ideas” code from the contingency unit of the framework would be an appropriate coding. Revisiting the episode several times suggested that the other two codes of the unit, “use of opportunities” and “deviation from agenda” might also be involved to the episode. As a result, this single episode suggested a very fruitful conclusion for our data analysis. Teacher’s inability to respond student idea refrained him using it as an opportunity and deviate from his plan. Even though, the data does not yet allow us to claim that the observed teacher lacks necessary knowledge on vertical lines and their representation, our finding in this part helped us to speculate the teacher’s current state of knowledge on vertical line, its connection to slope, and the necessary skills and professional knowledge on how to teach it in such a context. Overall, we claimed that the episode of a student asking a question and the way the teacher reacted enables us to say more about the teacher’s content knowledge on teaching slope.

A NOVICE TEACHER’S CASE SELECTING TWO POINTS ON A LINE IN ANY ORDER

Merve started instruction with the graph of y=2x and yield the slope by first forming a right triangle on the coordinate plane and second by creating a table in which x and corresponding y values are inserted from the points got from graph. She showed that the slope of the line is same as the coefficient of x in the equation. As a second phase, the teacher provided four lines that pass through origin in the same coordinate plane. The teacher formed a whole class discussion on the inclination of the lines and the *sign* of the slope. She used slopes of lines graphed ($m = 1$, $m = 4$, $m = -4$, $m = -1$) to help students compare lines. The teacher summarized the findings. Then instruction on finding slope from the graphs was given. For the next phase, Merve asked students to find the equation of a line which given graphically. Merve indicated that the slope value and the coefficient of x are same. Then, the teacher asked further exercise questions to apply the newly learned relationship on $y = 3x$, $y = -2x + 1$, $y = 12 - 3/4$, $2y = 12x + 6$, $5y = 10 - 7x$, $3x + 4y - 8 = 0$ and $8x + 2y = 9$. In these examples students reached the slopes by using the relationship. Lastly, Merve asked students to find slope of a line in which only two coordinate points are given, (1,2) and (3,6). The students had difficulties in reaching both the intercept points of the line and its equation. The teacher reminded that they knew how to write equation of a line from its intercept points. The last exercise question was about finding slope of a line which passes through (-2,4) and (-1,-5) and write the equation of the line by using the points and calculated slope.

To compute slope, the teacher used the tabular data (Table 1). Merve told that students should be careful in considering the first point. According to her, between two selected coordinate points, students should take the point as first if its x-value is smaller than the other point's. In brief, Merve chose (0,2) as the first and (3,0) as the second point in computing slope.

x	y
0	2
3	0

Table 2: Tabular data used for coordinate points of the line

A student objected to Merve by claiming that it is not necessary to fix a point as first or second. The below episode indicate the teacher's reaction to a student's comment.

Student: If we did not take 2 as the first value for y [in the formula]
Teacher: Every time from left [teacher indicated to the left side of the graphed line which has smaller apsis]
Teacher: Hey kids Ali again asked a good question he said that if I would write (3,0) at first what would be the conclusion he said that he found the inverse hey kids if you recognize I start writing with smaller x values from this side [pointing from (0,2) to right side] in writing the change I start with the smallest x values.
Student: The point where x is smaller.
An. student: From left to right.
Teacher: Got it. I start where x is smaller.
Student: If we had written it would turn out the same thing since we again would write y's first.
Teacher: Let's try [the teacher applies the slope formula again and reaches the same conclusion]
Student: Yes same
Teacher: Okay good
Student: I already did in this way [the student indicates that he used the last way]
Teacher: Okay good, yes we have reached the same conclusion.

The episode indicates that though the teacher did not suggest the rationale explicitly, she showed it procedurally on an exercise question. It was particularly interesting that the teacher did only resort to a procedure that may be regarded as a pure idea in teaching slope concept. To check whether the student's idea is correct she only applied the procedure. There is a fundamental knowledge that two arbitrary points on a line may possibly give its slope. Lines are not as vectors though there is a resemblance in shape. A vector is a directed quantity so it has initial and ending points. In contrast, lines do not start or finish anywhere. All of these arguments suggest that we are not restricted to behave a point say A, as the first point and B as the second. In either order, the slope will be equal. All of these important ideas may be showed through slope formula. This episode reminded Saldanha and Thompson's (1998) suggestion that it may be challenging for students to understand "graphs as representing a continuum of states of covarying quantities" (p. 7). There is some evidence here that teacher's overall knowledge of line and skills for teaching slope of a line has some slightly weak parts. However, she was also able to use students' ideas as teaching opportunity. Though this single episode does not indicate a big problem, it helped us to put more attention on the issue throughout the full video recorded data.

CONCLUSION

To conclude, responding to students' ideas, first of all, requires active and careful listening of students' content related expressions (Davis, 1997; NCTM, 2000). The study supports Sherin (2002) that students' behaviors such as elaboration of their ideas may be regarded as a chance for teachers to revise their content knowledge. Knowledge of students ideas is regarded as pedagogical content knowledge (Shulman, 1986, Ball et al., 2008) however teachers' responses to student ideas also indication of their subject matter knowledge too. Using Knowledge Quartet as the framework for analysis, we were able to focus on episodes of *contingency* which reflected teachers' knowledge on slope. Teachers may effectively use those episodes to increase their and students' knowledge (Franke et al., as cited in Sherin, 2002). The results are very consistent with the claims made by Rowland, Thwaites, and Jared (2011) that teachers' contingent action provide valuable information on the effect of teachers' content knowledge in teaching. Furthermore, as authors addressed *contingency* episodes are not isolated from other units rather includes them. The contingency episodes that were discussed in this paper not only show teachers' way of responding to students' ideas but also their knowledge of the subject too.

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Teachers' resources in analysing mathematical content and classroom situations: The case of using multiple tepresentations

Sebastian Kuntze, Anika Dreher and Marita Friesen

Ludwigsburg University of Education, Ludwigsburg, Germany, kuntze@ph-ludwigsburg.de

Using multiple representations for mathematical objects in the classroom is a key for fostering students' understanding. As teachers have to analyse both mathematical content and classroom situations in order to effectively accompany learning processes, they need corresponding professional knowledge and resources. As specific empirical research is scarce, this study aims at collecting first evidence about whether and how different samples of pre-service and in-service teachers analyse content against the idea of using multiple representations and whether these content-related resources interdepend with their awareness of the role of dealing with representations in classroom situations. The findings indicate expected differences in resources and interdependencies between CK and PCK.

Keywords: Teachers' analysing, PCK, CK, multiple representations.

INTRODUCTION

Teacher expertise is related to personal resources in various domains of content knowledge (CK) and pedagogical content knowledge (PCK, cf. Shulman, 1986). A reservoir of such CK and PCK is needed in particular when teachers have to analyse learning opportunities and subject matter on the one hand or specific interaction situations with learners in the mathematics classroom on the other hand. Making connections with professional knowledge in these two contexts belongs to the core of what is required from mathematics teachers in their profession. This holds in particular for being able to analyse content matter and classroom situations against criteria related to the use of multiple representations, as abstract mathematical objects can hardly be accessed without any representations (cf. Duval, 2006).

Consequently, there is a high interest in investigating whether secondary teachers from different school types are able to analyse (A) how mathematical content affords using multiple representations and (B) how problems for learners may emerge from a non-optimal use of multiple representations in classroom situations. To our knowledge this is the first study with a quantitative empirical scope on the potential interrelatedness of these domains of teachers' profession-related analysing.

In order to clarify our focus on teachers' analysing, we will in the following introduce into the theoretical background of this study, derive the research questions from these thoughts, inform about sample and methods, present results and discuss their implications for the theoretical and practical contexts.

THEORETICAL BACKGROUND

As mathematical objects are only accessible through representations (Duval, 2006), representations play an important role in the process of mathematical learning and its support in the classroom. Different representations can stand for a mathematical object (Goldin & Shteingold, 2001), so using multiple representations may support learners in learning more about the mathematical object behind these different representations (Lesh, Post, & Behr, 1987) and in distinguishing the mathematical object from its representations. As different representations mostly emphasise different aspects of the corresponding mathematical object, the use of multiple representations enables learners to build up a rich concept image (Tall, 1988; Ainsworth, 2006). Connecting different representations as well as changing between representations provide thus crucial learning opportunities (e.g., Ainsworth, 2006; Duval, 2006; Lesh, Post, & Behr, 1987; Ainsworth, 2006).

Using multiple representations for mathematical objects in the classroom is hence a key for fostering students' conceptual learning (see also Dreher & Kuntze, 2013). Accordingly, teachers need professional knowledge so as to foster their students' learning – and they have to be able to *use* this knowledge in analysis processes. Analysing mathematical content against the background of multiple representations (A) is probably as important as analysing related to ways of dealing with representations in the learning process (B). In the following, we will concentrate on the teachers' analysing and the underlying professional knowledge they refer to.

As a framework, this study uses a multi-layer model (Kuntze, 2012), which combines Shulman's (1986) domains with the spectrum between knowledge and prescriptive convictions/views (Pajares, 1992), which are considered as individual professional knowledge as well. As a third dimension, the degree to which professional knowledge is bound to content or classroom situations is used to distinguish different levels of globality (cf. Törner, 2002), resp. situatedness (vertically ordered layers in Figure 1). The cells of the model in Figure 1 should not be seen as completely separable – indeed, the extent to which components of professional knowledge are consistent across different cells may even be interpreted as an indicator of teacher expertise (cf. Doerr & Lerman, 2009). A more detailed description of this theoretical background is given in Kuntze (2012) and Dreher and Kuntze (2015a).

Professional knowledge related to using multiple representations is a resource (Schoenfeld, 2011) for teachers, which is the base on which they can draw when they analyse content matter related to learning opportunities or the interaction with students in classroom situations. By *analysing* we understand *an awareness-driven, knowledge-based process which connects the subject of analysis with relevant criterion knowledge and is marked by criteria-based explanation and argumentation.* The subject of analysis can be for instance an area of content knowledge (A) or a classroom situation (B). Both of these are in the focus of this study.

Analysing mathematical content with respect of criterial knowledge related to using multiple representations (A) means in particular that teachers can identify and connect different representations of mathematical objects, but also that they can think of examples of content in which the use of multiple representations plays an important role for example, for gaining in-depth conceptual insight or for simplifying problems (cf. Kuntze et al., 2011). Such analysis steps draw above all on content knowledge (CK). An example of a question which requires analysing content in such a way is given in Figure 2 (in the methods section): For answering this task, teachers have to be able to review examples of content matter according to criterial knowledge related to using multiple representations and hence to make connections between examples of content matter according to this criterial knowledge.

Analysing classroom situations against the background of dealing with representations (B) is more based on aspects of pedagogical content knowledge (PCK). In particular, the following processes play a role which are also connected to aspects of teacher noticing (cf. Friesen et al., in this proceedings book): 1) Identifying relevant situations concerning the use of representations, which marks the "starting point" of an analysis, 2) evaluating such situations in a critical way which is based on connecting relevant situations and arguments to corresponding elements of theory regarding the use of representations, and 3) presenting/articulating the results of the analysis. These processes should not be considered as ordered or completely

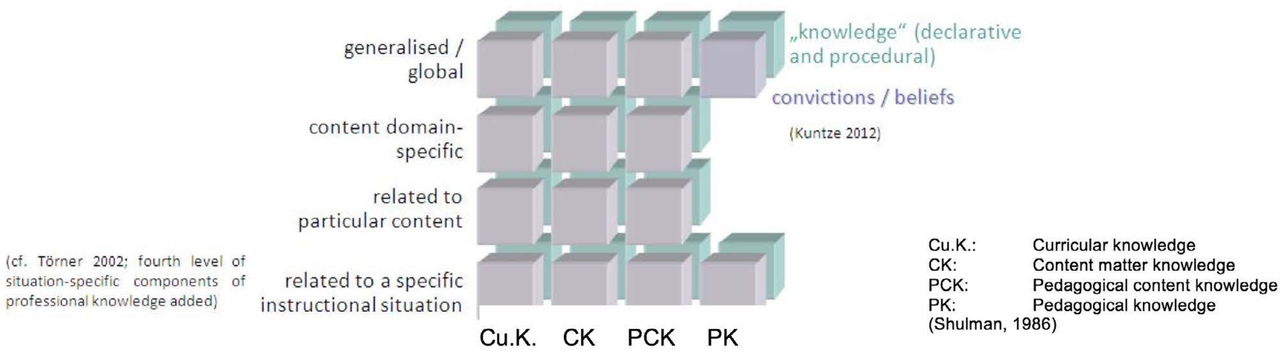


Figure 1: Multi-layer model for components of professional knowledge

separable, as there may be jumps or simultaneous processes.

In both cases, the analysis needs an initial awareness of criteria linked with using multiple representations. Moreover, making connections between this criterial knowledge on the one hand and specific content or characteristics of a specific classroom situation on the other hand is needed. Drawing on criterial knowledge affords explaining the subject of analysis and identifying arguments for the conclusions which are necessary for an appropriate answer. For this reason, potential interdependencies between teachers' personal resources in analysing merit attention.

Despite a clear relevance of the requirement of analysing the use of multiple representations in content matter and classroom situations for teaching mathematics, research focusing on teachers' analysing related to these aspects is scarce (cf. overviews in Dreher & Kuntze, 2015a, 2015b). More specifically, to our knowledge this is the first quantitative empirical study which focuses on interdependencies of teachers' analysing related to these domains of professional knowledge. For exploratory insight and against the background of differences (cf. Dreher & Kuntze, 2015a, 2015b), it is also necessary to explore whether there are differences between teachers from different school types and between in-service and pre-service teachers.

RESEARCH QUESTIONS

Against the background outlined in the previous section, there is a need of research exploring teachers' analyses of content matter (A) and of classroom situations (B) related to using multiple representations. Using multiple representations is thus an overarching concept (cf. Kuntze et al., 2011) which affords analysing and making connections between professional knowledge and the subjects of analysis, namely examples of content and classroom situations.

This study hence aims at answering the following research questions:

— To what extent do in-service teachers have professional knowledge necessary for analysing whether content matter can be connected with the overarching concept of using multiple representations?

— Are there differences between in-service teachers from different school types?

— How big are these differences in comparison with the knowledge and analyses of pre-service teachers?

— Does the quality of analysing content matter regarding the use of multiple representations interdepend with the in-service teachers' analysis of classroom situations based rather on PCK?

SAMPLE AND METHODS

The in-service teachers participating in this study belonged to two sub-samples: There were in total 58 German in-service teachers which consisted of:

— 34 teachers from academic-track secondary schools (20 female, 14 male; mean teaching experience 14.8 years, SD=13.4 years)

— 24 teachers from secondary schools for lower-attaining students (9 female, 15 male; mean teaching experience 10.7 years, SD=9.5 years)

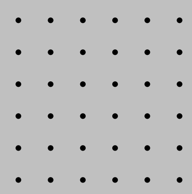
Moreover for answering the third research question, 117 German pre-service teachers were included (78 female, 35 male, 4 without data; mean age of 22.33 years, SD = 3.56 years) who had been studying on average for 2.19 semesters (SD = 1.12). Within this sub-sample of pre-service teachers, 61 were preparing for being teachers in primary schools, 32 in secondary schools for lower-attaining students (HSWRS), and 15 for working in schools for students with special needs (9 without data). An additional sample of 42 Austrian pre-service teachers preparing for teaching in academic-track schools were included for exploratory comparisons.

In order to tap the teachers' analyses of contents, a test was conceived with a focus on analysing contents and perceiving links across contents according to overarching concepts. The instrument concentrated on CK-related analyses connected with the overarching concepts *using multiple representations*, *dealing with infinity*, and *giving arguments/proving*. (cf. Kuntze et al., 2011). This study only focuses on the teachers' answers to two tasks related to using multiple representations, a sample task is presented in Figure 2. The answers were collected in an open format, the

On the right, there is a graphical representation of the definition of "square number". This representation affords an additional access compared to the symbolic definition ("if $q=n^2$ for a positive integer n, the q is called a square number").

Can you think of other mathematical concepts, for which a symbolic definition can complement with a non-symbolic representation in a similar way?

Please give as many examples as possible.



„A number that can be visualised like shown here in the form of a square is called a square number."

Figure 2: Sample task for analysing and connecting content according to the overarching concept of using multiple representations (Kuntze et al., 2011, p. 2722)

teachers were given as much time as they required to complete the test.

The answers were coded according to a top-down coding method (see Kuntze et al., 2011, for details and sample answers). Among others, the coding categories *existence of a codable answer*, *quality and transfer level of the examples*, and *embedding of examples provided* were used. An indicator score was generated in the following way: If the teachers gave at least one appropriate example, their answer was coded with 1 point, if at least one of the examples was from a different content area than the example given in the task, the full score of two points was assigned to the corresponding answer, as in this case we can assume that the teacher had shown awareness of criterial knowledge related to using multiple representations and analysed successfully at least one example of content matter.

In classroom situations, the quality of the use of multiple representations often depends on how carefully teachers link different representations and whether they accompany the students' understanding with this respect. For tapping the teachers analyses in this area, we used an instrument which is described in detail in (Dreher & Kuntze, 2015a). In this instrument, the teachers are given fictitious transcripts of classroom situations (related to the content area of fractions) in which the teacher changes the representation register unnecessarily and without explicitly providing help or explanation for linking the corresponding representations. In the situation in Figure 3, this is the case for the teachers' change to the pizza representation. The participating teachers were asked to evaluate how the given teacher's reaction supports the student's understanding in this situation and to give reasons for their evaluation. According to the design of the classroom situations, the participating teachers' analyses of these situations based on their PCK related to using multiple representations should hence lead to a critical evaluation of the way how multiple representations are used (Dreher & Kuntze, 2015a). A

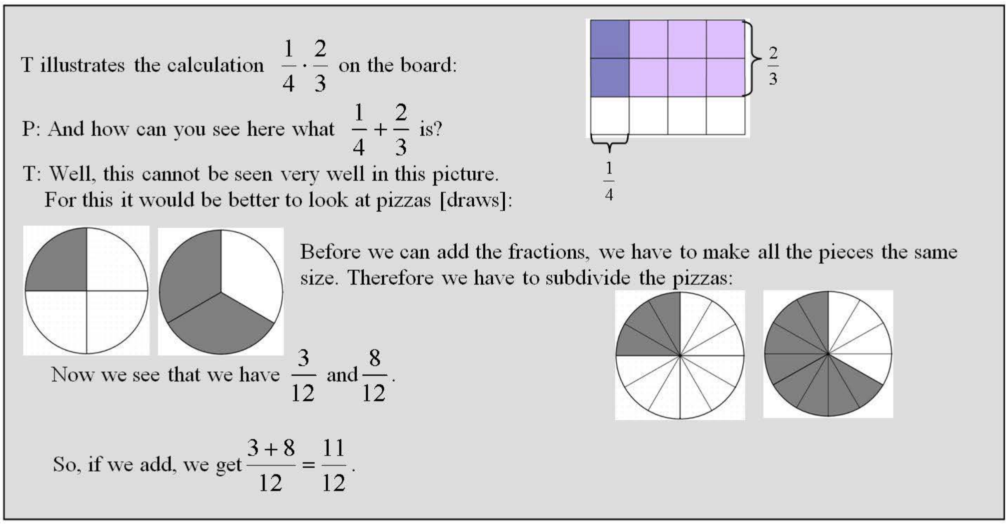


Figure 3: Classroom situation to be analysed according to the use of multiple representations (Dreher & Kuntze, 2015a)

corresponding top-down coding was consequently done with the teachers answers which were collected in an open format. Based on these codes, a score was calculated by counting successful answers. As there were four classroom situations, these scores ranged from 0 to 4 points (Dreher & Kuntze, 2015a).

For answering the fourth research question which focuses on interdependencies between the two analysis scores, correlations (Pearson) were calculated. As there was the possibility of controlling for specific CK related to changing between representations of fractions (for details about this additional instrument see Dreher & Kuntze, 2015a), this variable could be included in a regression model which affords describing the interdependencies in the focus of the fourth research question.

RESULTS

The first, second, and third research questions focus on the extent to which different samples of in-service and pre-service teachers analyse content matter against the background of the overarching concept of dealing with multiple representations. Figure 4 shows an overview of these results. The in-service teachers working at academic track secondary schools reached on average about two thirds of the full test score related to analysing and connecting content matter through the concept of using multiple representations. In contrast, the in-service teachers from HSWRS secondary schools were significantly less successful in these analyses (T-test; T=3.40; df=56; p=.001, d=0.90). However, each sub-sample of in-service teachers scored significantly better than the corresponding sub-sample of pre-service teachers (academic-track teachers: T=3.32; df=74; p=.001, d=0.76); HSWRS teachers: T=4.75; df=54; p<.001, d=1.23). The values of Cohen's d indicate that these are strong effects. The average score of the academic-track pre-service teachers was very similar to the score of the HSWRS in-service teachers.

As the data in Figure 4 is rather connected to the teachers' CK it may be of additional interest that we found in a corresponding study (Dreher & Kuntze, 2015a, 2015b) that there was a relatively similar pattern for teachers' analyses of classroom situations against the background of dealing with multiple representations, which is rather based on teachers' PCK. This points to the fourth research question, which concentrates on connections between CK-related and PCK-related variables.

CK, especially CK related to the particular content, is probably important for analysing classroom situations. For this reason specific CK (in this case related to the use of multiple representations with fractions) was also included in the study as an additional variable. Table 1 shows correlations, which suggest that for academic-track teachers the more overarching CK-related variable has played a more important role than the more specific CK variable (fractions) which is more close to the classroom situations the teachers had to analyse.

Deepening this insight, we included the variables in regression analyses (cf. Table 2). The results show that analysing contents according to the overarching concept of using multiple representations (B) interdepended more strongly with the quality of the analyses of classroom situations (A, i.e., the corresponding score as independent variable), than specific CK related to using multiple representations of fractions – and that in particular it plays a significant role even if the data is controlled for the fraction-related "local" CK component, as far as the in-service academic-track teachers are concerned.

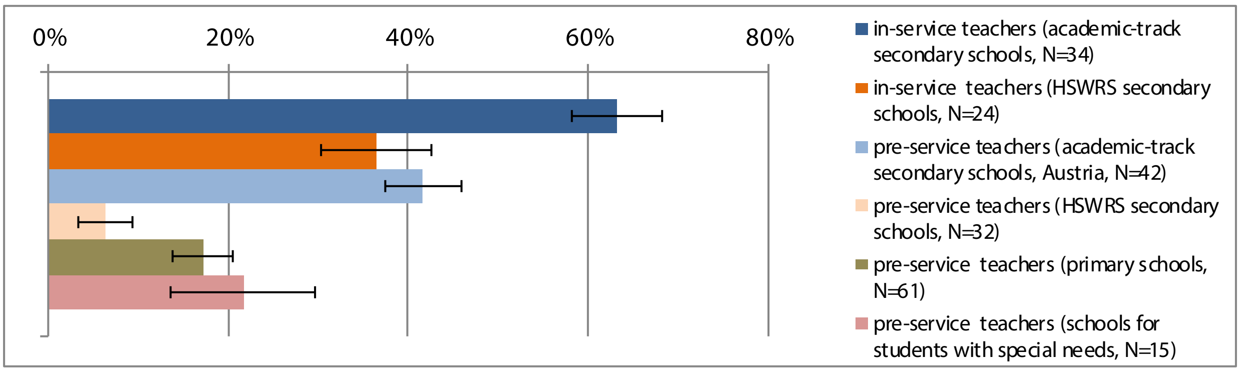


Figure 4: Analysing contents against the overarching concept of using multiple representations: Means and their standard errors

Correlations with score related to analysing classroom situations with respect to using multiple representations (B) (Dreher & Kuntze, 2015a)	Subsample of all in-service teachers (N=58)	academic-track in-service teachers (N=34)	HSWRS in-service teachers (N=24)
CK-related score „using representations in the content area of fractions" (Dreher & Kuntze, 2015a)	,128 ,333	-,031 ,863	,217 ,309
Score "analysing content matter with respect of using multiple representations" (A) (Kuntze et al., 2011)	,319* ,014	,365* ,034	-,021 ,922

Table 1: Correlations (Pearson) and significance values (2-sided)

		B	St. error	Beta	t	p
HSWRS in-service teachers	(Intercept)	,235	,077		3,046	,006
	CK-related score „using representations (fractions)"	-,206	,197	-,238	-1,050	,306
	Score "analysing content matter with respect of using multiple representations"	,028	,105	,061	,269	,791
academic-track in-service teachers	(Intercept)	,201	,132		1,523	,138
	CK-related score „using representations (fractions)"	-,146	,195	-,129	-,752	,458
	Score "analysing content matter with respect of using multiple representations"	,300	,130	,396	2,318	,027

Table 2: Regression analyses for subsamples of in-service teachers

DISCUSSION AND CONCLUSIONS

Before discussing the results in detail, it is important to recall that the study has clear limitations, as the findings hardly allow for generalisation. For instance, the samples are not representative for German teachers, and the research instruments should be extended in follow-up research. However, the findings afford answering the research questions and call for further investigation in related empirical studies.

The following key findings can be summarised:

— The results related to the first research question indicate that many of the academic-track secondary teachers were not able to connect the overarching concept *using multiple representations* to at least one example of mathematical content. Analysing content against this criterial knowledge appears hence to be very difficult even for in-service teachers from academic-track secondary schools.

— Teachers from HSWRS schools showed, however, even less elaborated analyses and a significantly lower professional knowledge related to connections with the overarching concept of using multiple representations. In line with findings from Dreher and Kuntze (2015b), the findings related to the second research question thus suggest that the culture associated with different school types or selection effects of teachers might play an important role and that there are school-type-specific professional development needs.

— As appropriate answers of the pre-service teachers were – as expected (cf. Kuntze et al., 2011) – even less frequent, the findings associated with the third research question point to a substantial need for teacher education.

— As far as the fourth research question is concerned, the results related to interdependencies may be seen as consistent with corresponding findings from the COACTIV study (e.g., Kunter et al., 2013), suggesting that there might be general interdependencies between CK and PCK. This study can add preliminary, but more specific insight by focusing on the teachers' analysing.

Follow-up questions arising from the findings open up several directions for further research: First of all, there is a clear relevance for the development of theory: What status does analysing related to overarching concepts (cf. Kuntze et al., 2014) have as a component of mathematics teacher expertise and to what extent do different contexts of profession-related analysing interdepend? But there are also more practice-related follow-up questions, such as: How can CK and PCK related to overarching concepts be supported best in professional development programs? When and how can growth in the corresponding domains of professional knowledge take place? Which role may be played by classroom experience or curricular knowledge? Finally, also aspects of the further development of the test instrument are in the focus (e.g., extensions to more overarching concepts, other test formats) and may contribute to a deepened understanding of the findings and teachers' profession-related analysing.

ACKNOWLEDGEMENT

The data gathering phase of this study has been supported in the framework of the project ABCmaths which was funded with support from the European Commission (503215-LLP-1-2009-1-DE-COMENIUS-CMP). This publication reflects the views only of the authors, and the Commission cannot be held responsible for any use which may be made of the information contained therein.

We further acknowledge the support by research funds of Ludwigsburg University of Education (project La viDa-M) during the data analysis phase.

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What does it mean as a teacher to "know infinity"? The case of convergence series

Miguel Montes and José Carrillo

University of Huelva, Huelva, Spain, miguel.montes@ddcc.uhu.es

This paper aims to explore the knowledge of infinity that three teachers deploy in making methodological suggestions for dealing with the convergence of a geometric series in a secondary level lesson. We will also illustrate the potential of the model of professional knowledge known as Mathematics Teacher's Specialised Knowledge (MTSK) as a tool for analysing the knowledge of infinity underpinning the three proposals. The mathematical knowledge revealed is considered, alongside suggestions for further exploration of the pedagogical knowledge of the three teachers regarding infinity.

Keywords: Infinity, professional knowledge, MTSK, series, secondary teacher knowledge.

INTRODUCTION

It is widely acknowledged that the nature of a mathematics teacher's knowledge of the subject is different to that of other people involved in mathematics. To this end, Ball, Thames and Phelps (2008) specifically identify the sub-domain *Specialised Content Knowledge* within their analytical model, MKT, in contrast to that of *Common Knowledge*, with the aim of underlining the kind of knowledge about certain topics required of a teacher, but not necessarily required of others who habitually draw on mathematical knowledge. An alternative model was recently presented, *Mathematics Teacher's Specialised Knowledge*, henceforth MTSK (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013), the premise of which is that the specialisation derives not from the unique nature of the teacher's knowledge in itself, but from the use to which this knowledge is put in teaching. It is from this perspective of specialisation, following Montes, Carrillo and Ribeiro (2014), that we consider the concept of infinity and its significance to the teacher's store of professional knowledge.

The concept of infinity is an interesting case because it touches many areas of the secondary curriculum, and is hence especially relevant to analysing teachers' mathematical knowledge. From our perspective, a teacher's knowledge of infinity is specialised by virtue of the fact that, as practitioners, they need to know how to deploy their knowledge in contexts of teaching and learning. This paper applies this perspective to the question of how this knowledge influences the methodological considerations involved in planning a lesson on the convergence of a series generated by successive powers of ½. In doing so, it hopes to illustrate the effectiveness of the MTSK model in revealing the specialised knowledge underlying teachers' decisions.

THEORETICAL FRAMEWORK

The MTSK model (Figure 1), which provides the theoretical background to this paper, consists of six sub-domains divided between Shulman's (1986) original dichotomy of *Subject Matter Knowledge* and *Pedagogical Content Knowledge*.

In terms of purely mathematical knowledge, it proposes three sub-domains. The first of these, *Knowledge of Topics*, brings together all topics involved in the teaching-learning binomial, including many aspects which are seldom made explicit to students, but can nevertheless be understood as fundamental to the teacher's knowledge. The second, *Knowledge of the Structure of Mathematics*, concerns the connections between areas, such as concepts which range across the curriculum (as in the case of the concept of infinity). Finally, *Knowledge of Practices in Mathematics* encompasses ways of doing and proceeding with mathematics. There are likewise three sub-domains within *Pedagogical Content Knowledge*, which aim to enhance the specifications proposed by Ball and colleagues (2008). The first of these, *Knowledge of*

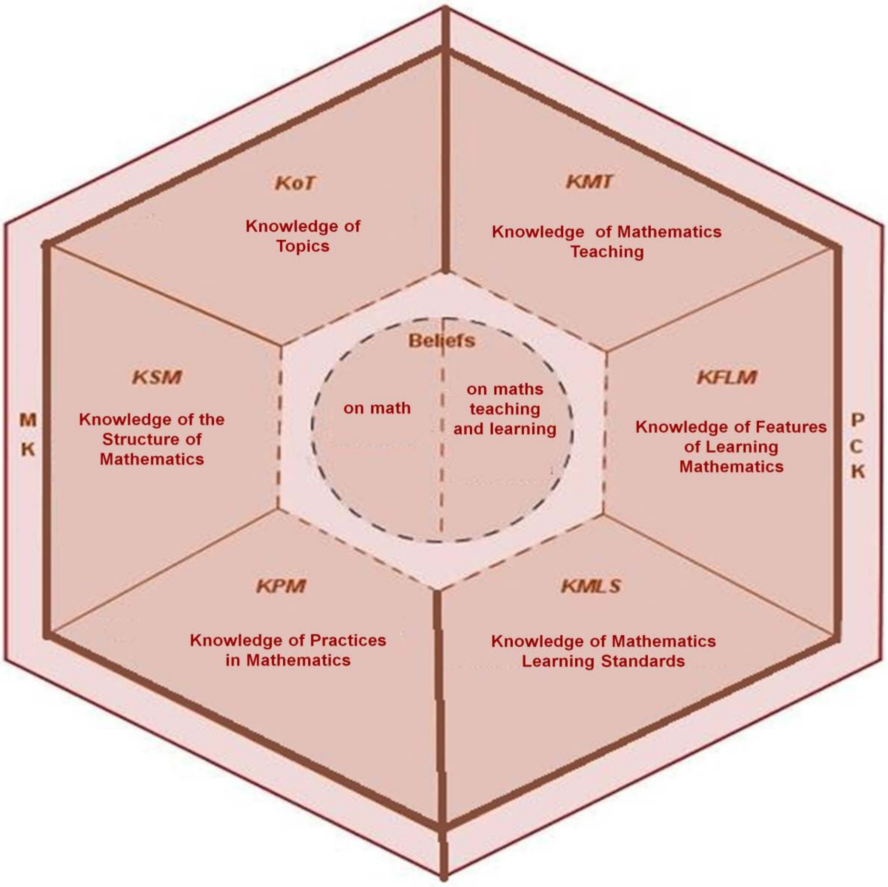

Figure 1: Mathematics Teacher's Specialised Knowledge

Mathematics Teaching, refers to knowledge about ways of presenting mathematical content in the classroom. The second, *Knowledge of the Features of Learning Mathematics*, concerns knowledge about how pupils learn, understand and interact with mathematical content. The last, *Knowledge of Mathematics Learning Standards*, includes knowledge relating to the curriculum and learner expectations in respect of year group, age and educational stage.

A professional understanding of infinity

The concept of infinity, as an item of human knowledge, has been widely studied over the centuries and at the present time is understood as a property of various sets (Moreno & Waldegg, 1991). With respect to research into infinity and learning, studies have considered the *intuition of infinity* (Fishbein, Tirosh, & Hess, 1979), the relation between conceptions of infinity and understanding the notion of limit (e.g., Sierpinska, 1987), how infinity is dealt with in class (e.g., Hannula, Pehkonen, Maijala, & Soro, 2006; Roa-Fuentes, 2013), and the development of students' understanding of situations closely associated with the concept (Arnon, Cottrill, Dubinksy, Oktaç, Roa-Fuentes, Trigueros, & Weller, 2014). Nevertheless, the concept of infinity has received scant attention as a component of professional knowledge, that is as something useful for education as opposed to a mathematical item to be studied in depth.

Efforts to research teachers' knowledge of infinity have thus tended towards regarding their subjects as "possessors of advanced knowledge" on account of their training background, including mathematics training, and their familiarity with the various concepts pertaining to school mathematics. But interest in teachers' knowledge also derives from their capacity to satisfy their students' curiosity and provide valid responses to their questions. As Hannula and colleagues (2006) state, "Most primary children are very interested in infinity, and they enjoy discussing the concept, if the teacher is only ready for it" (p. 1). Any such responses must be founded on mathematics; the teacher needs to be aware of what to say and why this is valid. In this respect, there have been numerous studies into the misconceptions about infinity of different groups of teachers, focusing on the epistemological basis for their thinking (e.g., Kattou, Michael, Kontoyianni, Christou, & Philippou, 2009), the nature of the misconceptions themselves, including taxonom-

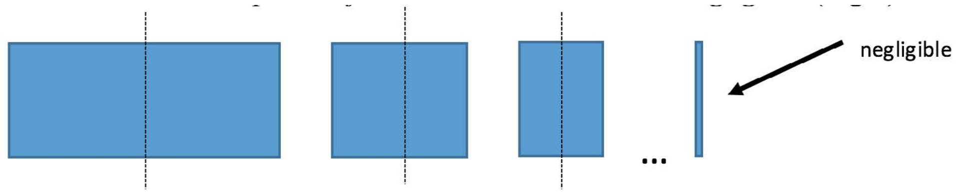

Figure 1: Graphical reconstruction of the explanation of Sandro

ical analysis (e.g., Yopp, Burroughs, & Lindaman, 2011), and the degree of cognitive development in relation to concepts associated with infinity (e.g., Weller, Arnon, & Dubinsky, 2009). Finally, some studies give special emphasis to how such misconceptions can be transmitted to students (Tall & Schwarzenberger, 1978). However, we would argue that the mere fact of being in possession of a valid response does not account for the full range of knowledge that a teacher mobilises when they construct an example, as shall be seen in the following section.

The potential of various models has recently been illustrated in relation to teachers' knowledge of infinity (Montes, Carrillo, & Ribeiro, 2014). In this paper, we consider the use of one such model, MTSK, in revealing the impact of the teachers' understanding of infinity on their planning for a lesson on the addition of geometric progressions.

METHOD

The data used in this paper are drawn from two separate but related case studies (Montes, 2011; Montes, in press), studying the knowledge of infinity of three teachers by means of an interview structured around the question, "How would you approach the addition of successive powers of ½ with a class of 16-year-olds?"

Here, we consider the different approaches adopted by the three teachers in response to this question. The first two, Sandro and Aaron, are mathematics graduates with 5 and 8 years' teaching experience behind them respectively, whilst the third, Manuel, also a mathematics graduate, was following a teacher training course at the time of the study. As noted above, all three were taking part in studies into their knowledge of infinity, and this meant that all of them took for granted that the interview about successive powers was concerned with the notion of convergence of an infinite series. Their suggestions for dealing with the concept in class are given below:

Sandro: *The usual thing is you say that they have a cake* [draws a rectangular cake], *and you eat half, and then half of the remaining bit, and then half again, and carry on like that a few times, and you see [...] You can also do it with a piece of paper, which you cut again and again, and each time you are left with a smaller piece, and they see that there comes a moment when the piece left over is so small as to be negligible* (Figure 1).

Aaron: *I'd take a piece of paper a metre wide as reference, and then one of half a metre, which I'd place on top. After that, another of a quarter of a metre, which I'd place on the right, and then one of an eighth, which I'd also lay on top. At some point, I'd get them to see that if you zoom in on the process, you get back to what was essentially the starting point, making a comparison between the piece left over and the original half-metre piece. I'd continue a few more times in the same way and get them to see that you can never get beyond that, however close you get* (Figure 2).

Manuel: *I'd get them to cut a sheet of paper in half, and then one of the halves in half again, and carry on like that five or six times until they see that you could continue with some 'super-sharp scissors' as many times as you like. Then I'd get them to write on each piece the fraction it represents, put all the pieces together*

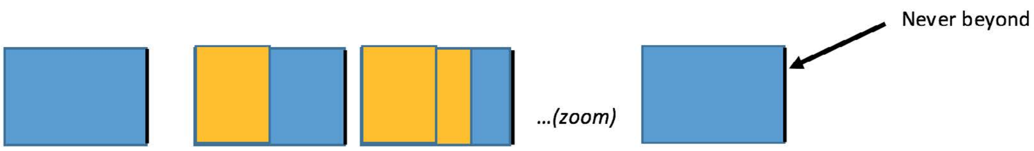

Figure 2: Graphical reconstruction of the explanation of Aaron

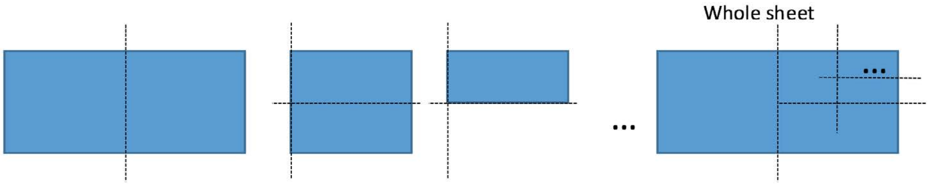

Figure 3: Graphical reconstruction of the explanation of Manuel

> *and imagine them all together, including the ones we could have cut but didn't, so that they became aware that all the pieces together comprise the whole sheet, which has an area of 1* (Figure 3).

ANALYSIS

This section takes the examples of three teachers talking about the same exercise and suggesting ways of dealing with it in class, and illustrates the potential these situations offer for exploring the knowledge displayed by the teachers beyond the strictly mathematical.

The three teachers take the same methodological approach in that they all recommend cutting paper in half, albeit that two – Sandro and Manuel – suggest the pupils themselves do the cutting, whilst in Aaron´s version it is the teacher who does this. In terms of the mathematical knowledge deployed, all three responses show understanding of the proposed scenario, predicated on convergence when adding a geometric series. However, the three scenarios reflect different understanding of the concept of infinity.

In the case of Sandro, his argument is based on the fact that after repeatedly cutting the sheet in half (in order to represent division by two) and putting the resultant pieces together, there comes a moment when the remaining piece "is so small as to be negligible." This alludes to a significant property of the remaining item in a series, underlying which is the idea of the infinitesimal. The knowledge revealed by the answer is a clear example of *Knowledge of Topics*, in which the topic in question is 'convergence of series'. Nevertheless, the concept underlying the solution and making it coherent, although not actually explicitly articulated, is that of the tendency to zero of the remainder in a series. This suggests a potential understanding of infinity (as the process is unfinished) which, given that the concept ranges across topics, is most appropriately located in the sub-domain *Knowledge of the Structure of Mathematics*.

Turning now to Aaron's answer, we can see that the mathematical foundation of his answer concerns the limiting effect of adding successive halves. Aaron diverts attention away from the total value of the addition towards the impossibility of this total ever reaching the total dimensions of the sheet. In mathematical terms, the element underlying this procedure is the concept of convergence of a sequence (of partial additions). Aaron makes the process a recurrent one so as to illustrate the fact that, however many times one cuts the paper in halves, there will always be one half lacking to complete the whole. The completion point constitutes a limit, which can be approached as closely as one wishes, but which can never actually be reached. This process view reflects a process-oriented and iterative perspective of infinity, again also associated with a potential understanding (Lakoff & Nuñez, 2000). Once again we are in the domain of *Knowledge of Topics*, in this case the topic being convergence of series. Aaron's familiarity with the definition of convergence means he puts the emphasis on the possibility of choosing any term and recognising that it never goes beyond the established limit, although it might be *very close*. This familiarity, rooted in his understanding of infinity, is likewise associated with *Knowledge of the Structure of Mathematics*.

Finally, Manuel's answer is suggestive of elements which are different from the previous answers, less in terms of the knowledge mobilised as the nature of this knowledge itself. It follows the same dynamic as the previous two in that it generates the $(½)^n$ series via the expedient of dividing a sheet of paper in two. During the procedure, Manuel introduces the idea of 'super-sharp scissors' to achieve the required degree of abstraction suggested by being able to cut the paper indefinitely, beyond the physical limitations. To the extent that this approach illustrates awareness of potential difficulties in the students' capacity for abstraction, the extract pertains to the sub-domain *Knowledge of Features in Learning Mathematics*. Afterwards, Manuel introduces an element which diverges significantly from the previous suggestions. Instead of centring attention on the process of repeatedly dividing in two, he pieces the sheet together again. This helps the students to make sense of the final result, graphically illustrating that the process of addition has a finite result equivalent to the surface area of the original sheet. The knowledge involved in this instance is that of *Knowledge of Topics*, which enables him to construct an example based on the meaning of the convergence of a series, and to take into consideration the implicit role of infinity in the expression 'all the pieces together', which provides an objective correlative for the notion of convergence of a series.

In each case, the proposed activity is underpinned by an understanding of the role of the concept of completion in demonstrating convergence, which is most pertinent to the sub-domain of *Knowledge of Practices in Mathematics*. Sandro makes use of the characteristic of series that if the remainder tends to zero, then the series is convergent. Aaron, for his part, draws on the fact that the annotation of partial additions implies convergence. On the other hand, Manuel bases his approach on the result of the addition, and as such it does not constitute a demonstration in the strictest sense of the word, although it is an interesting strategy to use with students.

Opportunities for further research

The analysis above is concerned chiefly with the purely mathematical considerations brought into play by the three teachers. Nevertheless, this is not the only kind of knowledge we believe they mobilise. The fact that the three suggestions are significantly different, in terms of both the mathematics involved and the manner of understanding infinity alerts us to the presence of *opportunities for further research* (Flores *et al* 2013), specifically into the responses which these three teachers might give to different questions about the appropriateness of their suggestions for dealing with this topic. With a view to developing teacher training materials such as vignettes for discussion, we think that further research into other aspects of these three teachers' knowledge is of interest as it will enhance the consistency of analysis. Some of the areas we feel would reward study are indicated below.

We think it would be interesting to ask the three teachers about the effectiveness of their approaches in terms of the students' potential understanding of the scenario. In this respect, although the first two approaches demand an understanding of the concept of limit if they are to be successful, they imply a potential understanding of infinity, something usually expected amongst students. By contrast, the third scenario implies considering the process as a whole, given that it demands an actual understanding of the infinite process, which is a more difficult concept to acquire. Exploring with teachers the use of the paper-cutting technique as a means of approaching the task also has the potential to reveal useful information about mathematics teachers' knowledge, going beyond the purely manipulative dimension of the technique towards understanding how it might help pupils to understand the concept in question. Finally, it would be interesting to go into greater depth regarding the suitability of the lesson plans in relation to the pupils' age, previous learning experiences of the topic, and syllabus expectations regarding the concept of infinity.

CONCLUSIONS

Infinity matters for teaching certain concepts, many associated with the area of mathematical analysis. For certain items on the syllabus, its importance to mathematical procedures at a conceptual level, such as the calculation of limits, is explicit. In the examples above, each teacher's consideration of infinity leads them to tackle the same question from different mathematical perspectives, each of which provides the epistemological underpinnings for the chosen approach, specifically the infinitesimal of the remainder, the iterative and recurrent elements of the limiting process, and the conceptualization of a sequence as a whole. We are aware that the examples covered in this paper offer nothing more than an initial consideration of the knowledge of infinity brought into play by teachers when they plan a lesson around the concept at an epistemological level. Nevertheless, we believe that they demonstrate the involvement of infinity in the teaching process, and explore a little further the big idea of *'dealing with infinity'* (Kuntze, Lerman et al., 2011). The examples also enable us to map the contours of the knowledge of infinity which the teachers bring to bear in their classes, which is extremely closely related to how they conceive of infinity, and complements studies of a purely cognitive nature (e.g., Weller et al., 2009).

Nevertheless, beyond purely mathematical considerations, we would argue that it is necessary to further explore those aspects of infinity which come under the auspices of *Pedagogical Content Knowledge*, as it is these that make the mathematical content relevant to the classroom. There are, too, the questions of the potential of different methodological approaches for different concepts, and the features of the pupils' learning of concepts involving infinity, as well as those of a curricular nature or associated with the performance expected at a particular age or level, all of which are also important for the teacher to be aware of.

In this paper, we have outlined some of the areas of pedagogical content knowledge associated with infinity which we believe merit further attention and which are of special interest to our work. The understanding and awareness of infinity which a teacher is required to acquire is necessarily one which takes a specialised perspective of mathematics teaching. The role of PCK is relevant from this perspective in terms of opportunities for further research beyond the purely mathematical aspects of the concept, such as understanding which features of the concept are involved in the process of learning. It is in this respect that the MTSK model is especially useful.

The different aspects arising from analysing teachers' professional practice with regard to infinity or related items should also be taken into account in the training of both primary and secondary teachers. For example, paradoxes could be used to nudge teachers to develop their understanding of key mathematical concepts, especially those associated with infinity, or could be a "vehicle for raising their pedagogical awareness of the development of mathematical knowledge" (Moshovitz-Hadar & Hadass, 1990).

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Primary teacher' approach to measurement estimation activities

Noemí Pizarro, Núria Gorgorió and Lluís Albarracín

Universitat Autònoma de Barcelona, Barcelona, Spain

Estimation appears as a relevant aspect to the new Curricular Guidelines of year 2012 of the Chilean Ministry of Education. In this paper, we analyze the mathematical knowledge for teaching of 112 Chilean teachers of Primary Education to deal with a classroom activity of discreet measurement estimation proposed in governmental curricular orientations. The qualitative-descriptive analysis developed in our study is supported by a reconstruction of the concept of measurement estimation. The results show that the knowledge of a significant part of the participant teachers presents some deficiencies, since they either propose to develop the activity without including perceptive abilities or they propose an action limited to a measurement exercise. Only 13.1% of the teachers state that when using the example given for estimation a perceptive-type exercise is required.

Keywords: Mathematical knowledge for teaching, measurement estimation, perception, referent.

INTRODUCTION

At the beginning of the 80's, the teaching of measurement gained emphasis in the curricula of some schools that answered to the guidelines of international organizations linked to the teaching of mathematics (Cockcroft Report, 1982; ICMI, 1986). These orientations were included by the Chilean Ministry of Education in the Curricular Guidelines of the year 2012, in which measurement has become the new curricular axis in primary education. In order to complement these guidelines, the Ministry itself has published some activity proposals in its study program.

The new Chilean curriculum considers estimation in more than twelve learning objectives in measurement and geometry. However, previously to this reform, measurement estimation hadn't had its own space in Chilean study programs. For this reason, probably, current teachers have never received training in this respect, neither in their school years nor in their further professional training. Therefore, we are facing a situation in which a mathematical concept bursts into the classrooms without any specific training being provided to the educators.

Considering the shortcomings identified by international mathematics studies regarding the teaching skills of teachers on the estimation of measurement (Chamorro, 1996, 1998; Joram et al., 2005; Forrester & Pike, 1998), estimation measurement is a concept that has not been investigated in depth, and in which difficulties have been detected in the learning of the students (Jones, Forrester, Gardner, Grant, Taylor, & Andre, 2012). At the same time, there are studies that claim that teachers work in estimating measurement is superficial and that they do not feel confident about working with students (Lang, 2001)

If to this we add that Chilean teachers have not had training in this respect, we consider it important to focus on how teachers interpret the estimation measurement activities proposed in government documents and official textbooks. Developing different learning objectives for the Chilean curriculum should be the main reference point for this.

As an orientation for our work we ask ourselves: How are measurement estimation activities treated by primary school teachers? In order to answer this question we take the Mathematical Knowledge for Teaching (MKT) proposed by Ball, Thames and Phelps (2008) as a theoretical referent and we carry out a qualitative-descriptive study in Santiago de Chile with 112 in-service Primary School teachers, who have taught or will be teaching measurement estimation to their pupils at a certain point in time.

Our aim is to investigate the possible treatment of measurement estimation activities in the classroom. Conversely, we intend to study which MKT will be needed by the teachers when introducing the new content in the mathematics curriculum. Our final aim is to establish guidelines that may back up future improvement actions of initial and continued training of teachers.

MATEMATHICAL KNOWLEDGE FOR TEACHING

The work of Ball and colleagues (2008), rests on Shulman (1986), emphasizes the use of knowledge in teaching and for teaching. They focused on the analysis of knowledge for teaching from the observation of the teacher's work in the classroom and outside, thus introducing the line of research of MKT. In the current study, we focus only on one component of MKT, Knowledge of Content and Teaching (KCT), which combines knowledge of mathematics and knowledge of its teaching. Examples of this type of knowledge are found in the ability to sequence content, propose starting, deepening and closure examples, and knowledge of the advantages and drawbacks of representations used to teach a specific idea.

MEASUREMENT ESTIMATION

When revising the literature on measurement estimation we often observe that different definitions of estimation include a wide range of tasks related to different fields of mathematical knowledge. These tasks may include mentally assigning a value to a magnitude measurement, an agile mental calculation of an arithmetic operation, or the calculation of the likely value of a statistical measurement, amongst others. Table 1 shows different definitions of estimation proposed by different authors.

Segovia and colleagues (1989) establish that measurement estimation differs from computational estimation due to methodological reasons. Hogan and Brezinski (2003) differentiate three types of estimation, according to the abilities they develop:

— Numerosity: refers to the ability to visually estimate the number of objects arranged on a plane within a limited amount of time.

— Measurement estimation: is based on the perceptive ability to estimate different magnitudes in common objects.

— Computational estimation: refers to the process by which an approximate value is rapidly determined for the result of a mathematical operation.

For Hogan and Brezinski (2003), computational estimation is a skill which evolves together with the rest of mathematical abilities. On the other hand, numerosity and measurement estimation are abilities that develop independently from that of estimating a calculation and are not grouped in the category of usual mathematical abilities. These authors explain that

	Definition	Relevant aspects
1	"A process to reach a measurement or measure without the help of measuring tools. It corresponds to a mental process that includes visual or manipulative aspects." (Bright, 1976, p. 89).	Mental process used to obtain a measurement. Visual perception or use of sense of touch.
2	"the ability to judge whether the result of a calculation which has been carried out or a measurement which has been taken seems to be reasonable and ability to make subjective judgements about a variety of measures" (Cockcroft Report, 1982, pp. 22–23).	Evaluation of the adequacy of results of calculations or measurements. Subjective evaluation of a measurement.
3	"assessment of the value of the result of a numerical calculation or of the measurement of a quantity, as a function of the individual circumstances of who forms it" (Segovia et al., 1989, p. 18).	Subjective evaluation of the results of calculations or measurements.
4	"the skill of making an educated guess as to the value of a distance, cost, size, etc. or arithmetic calculation" (Clayton, 1996, p. 87).	Conjecture on measure-ments and arithmetic ope-rations.
5	"Refers to a number that is a suitable approximation for an exact number given the particular context" (Van de Walle et al., 2010, p. 241).	Adequate approximation of a number.

Table 1: Definitions of estimation

in terms of the human brain, we have an accumulator that stores quantities in an approximate way. For instance, we know what a meter is, we can identify this concept in an object or in part of it and we can describe it using our body. This is possible due to the fact that we are aware of the adequate measurement of the length of a meter. This quantity "accumulator" may contribute to the development of people's visual perception by relating measurement estimation and numerosity with spatial abilities. We understand that this "acquisition" of the unit of measurement in use when performing estimations does not appear in the definitions shown above.

ESTIMATION IN EDUCATION

An important part of teaching measurement is focused on the development of their estimation. In the first place, this is due to the fact that estimation will further develop perceptive abilities (Inskeep, 1976; Cockcroft Report, 1982; Hogan & Brezinski, 2003). In addition, estimation comprises the foundation of the teaching of physical magnitudes such as length, surface area, volume or weight, amongst others (Hogan & Brezinski, 2003; Joram, Subrahmanyam, & Gelman, 1998; Joram, Gabriele, Bertheau, Gelman, & Subrahmanyam, 2005). When working on measurement estimation numerical abilities are also exercised (Boulton-Lewis, Wils, & Mutch, 1996; Joram et al., 1998; Sarama & Clements, 2009), in the sense of measurement (Clements, 1999) and in the numerical and spatial senses (Lang, 2001).

Despite the emphasis mathematics education attributes to measurement estimation, there are few studies that refer to the way estimation is developed in the classroom. Forrester and Piké (1998) observed that in the teachers' speech in the classroom there was a significant separation between measurement and estimation. Estimation was treated as a predictive hypothesis, in a vague and superfluous way, as a process failing to produce satisfactory answers to solve situations that only a measuring tool would be able to answer.

These authors observed that estimation was treated as a means of "thinking reasonably" that included riddles more often than value judgments based on referents, without the requirement of evidence of concept understanding. This outlook suggests mathematics is a synonym of rigor and exactitude. Therefore, an estimate should always be checked with a measuring instrument. Segovia and Castro (2009) explain that in most dictionaries the meaning of mathematics is characterized by exactitude and rigor. Following these criteria, estimation would be far away from "real mathematics", since the results it provides are not exact.

In our literature revision we have found few studies on teacher training related to measurement estimation. However, Joram and colleagues (2005) suggest the use of the estimation strategy of the reference point, as a seemingly appropriate didactic strategy.

GOAL OF THE STUDY

Since the Chilean Curricular Guidelines of the year 2012 include measurement estimation, it would be desirable to for Chilean teachers to have the adequate knowledge to use the activities proposed in the study programs provided by the Chilean Ministry of Education. In this context, the aim of our study is the following:

— To characterize the Knowledge of Content and Teaching (KCT) on measurement estimation to allow teachers to apply the activities proposed by the Ministry's guidelines in their classrooms.

THE STUDY

As a data collection tool, we have elaborated a questionnaire of ten open questions that were oriented by the different categories of MKT (Ball et al., 2008). The questionnaire was answered by 112 in-service Primary School teachers in Santiago de Chile. In this presentation we only analyze the answers to a question based on a task proposed in the guidelines for the curriculum of 7-year-old children (Figure 1).

The task proposed in the guidelines is an example of a measurement estimation activity for the teachers to develop in the classroom. The aim of this activity is estimate quantities up to one-hundred in concrete situations using a referent. With this question, we try to collect information about Knowledge Content and Teaching, because it can show how the teachers manage the content and activities in the classroom, and consider that the referent is a strength of the activity and that students should not be permitted to measure using the square as the unit of measurement.

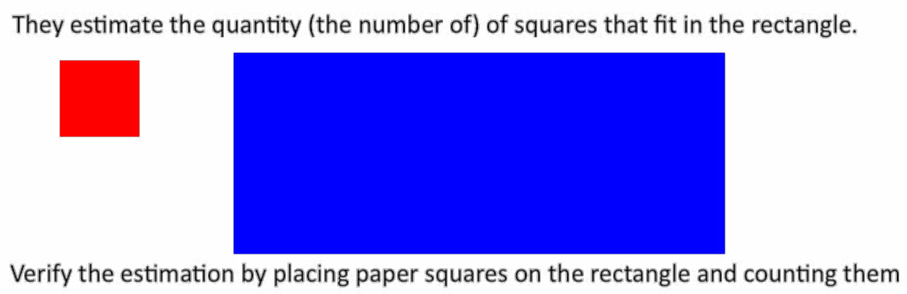

Figure 1: Estimation activity

In our questionnaire we ask teachers to explain how they would approach the activity shown in the classroom

DATA ANALYSIS

Since we are interested in identifying aspects of teachers' MKT around measurement estimation, the analysis of the data collected is qualitative. Therefore, based on the different views teachers have of activity 4, we have created several categories to characterise the knowledge of measurement estimation detected. When making a first reading of the answers, we prove the need to establish elements that determine if a task is actually about measurement estimation in order to back up our analysis. For this reason, we decided to elaborate a definition of the concept that included the fundamental elements identified during our literature review.

Our definition of measurement estimation is the following:

To perceptively assign a value, or an interval of values, together with a unit that corresponds to a discrete or continuous magnitude, by means of previous knowledge or by an indirect comparison to an auxiliary object.

This definition is based on three essential elements:

— Assigning a numerical value (V): in order to consider the task an estimation, the assignment of a numerical value is required; otherwise, the task only requires perceptive work. For instance, if when touching the water of a swimming pool we state that it is pleasant to swim in, the task is exclusively perceptive, given that we have not assigned any value to temperature and measurement units do not play a part in the activity (Bright, 1976; Informe Cockcroft, 1982; Segovia et al., 1989; Joram et al., 2005).

— Executing the task perceptively (P): in order to estimate it is necessary to use the senses, avoiding counting or the direct use of any measurement tool, standardized or not. When a measurement tool is used and a direct comparison is made, the action taking place is measuring, not estimating, even though the result may be expressed approximately (Bright, 1976; Hogan & Brezinski, 2003; Joram et al., 2005; Sarama & Clements, 2009).

— Relating perception to previous knowledge or to the mental image of an auxiliary object (R): In order to estimate referents are required, or to ideally possess a mental notion of a measurement unit; otherwise the activity would only require to guess or give a random result (Clements & McMillen, 1996; Hogan & Brezinski, 2003; Joram, 2003; Sarama & Clements, 2009).

Given that in the activity used in our question, the referent considered in the estimation has been given in the formulation, in our analysis we consider only how this referent has been used, the main analysis aspect of which is the perceptive component (P). Below we give an example of the analysis process and we introduce the different categories that organize the different ways teachers suggest approaching the activity in the classroom.

Use of the activity based on providing tools

Some teachers limit themselves to answering to what type of material will be used during the activity, without reporting particularly how these resources are planned to be used. For instance, one of the teachers answers: *"The students will be given both shapes (1 of each) made of paper card."*

We observe that the teacher's attention is focused on the material he will use in the activity instead of what is going to be done with it, which makes it impossible to guess how the teacher is going to develop the activity in the classroom.

Use of the activity as evaluation

Some of the answers given by the teachers suggest using the activity described as an evaluation exercise without a backup of mathematical processes associated to estimation. For instance, one of the teachers answers: *"First I would show the children this rectangle and after that I would hand in a piece of colored paper to each of them and would ask them to estimate how many squares can fit into the rectangle."*

We observe that the teacher is focused on the amount of squares that fit into the rectangle (V), but without offering any didactic instructions nor any other type of information that may allow us to understand how he plans to direct and develop the activity in the classroom. Therefore, we have no evidence that perceptive aspects may be included in the activity (absence of P).

"*Children: I invite you to estimate the area of the rectangle using the little square as the unit of measurement*". The teacher does the same as the teacher above, both give space for students to measure, estimate or guess.

Use of the activity as measurement

In the proposals of some of the teachers, we observe that they intend to go about the activity as a measurement task, with the use of non-conventional tools, explaining the way these are going to be used for the activity. This allows us to see that their proposal corresponds to a measuring activity. Answers in this category evidence a misunderstanding between the use of a referent to carry out an estimation and the use of a non-conventional measuring tool. Into this category we included answers such as: *"We need to know how many squares we need to cover the rectangle. After constructing the rectangle we should place squares of equal size on top of the former."*

We see that the teacher points out that the surface area of the rectangle should be covered by squares of the same size as the one given, suggesting a counting activity. Based on this idea, the referent then becomes a non-standardized unit of measurement. In this case, the students would reach an evaluation (V) that is not supported by a perceptive act (absence of P) but by a measurement.

Another example is this answer: *"Place the square on the rectangle to complete the rectangle with squares. The rectangle with the superimposed squares will be presented by the students"*

In this task, there is no perceptive work (absence of P) because the students must complete the rectangle with squares, perhaps drawn on it, to count how many squares fit. As with the teacher above, the square was used as a non-standardized unit of measurement.

Use of the activity as a measurement estimation

Some of the teachers regarded the evaluation of the amount of squares that fit into the rectangle as a perceptive activity, by using the square shape of the problem's formulation as a referent. We consider that answers of this type suggest using the activity as a measurement estimation exercise. Among them we find: *"Boys and girls, on the blackboard you can see a red square and a blue rectangle. Without standing up or leaving your seats, or as you prefer, please estimate how many times the red square fits into the blue rectangle, by placing the squares next to each other."*

We have identified that the intention of reaching an evaluation (V) is supported by a perceptive action (P), since the teacher does not allow a physical interaction between the students and the representations of the figures on the blackboard. In this case, the students would be obliged to carry out a visual perception exercise.

Another teacher answer: "*Without making any movement, think about how many squares you estimate will fit in the rectangle"*

We observe that when the teacher says "making any movement", s/he indicates that the student must not handle the square, thus steering the student away from measurement and supporting the perceptive action (P). In this way, the evaluation (V) rests on the perceptive action.

RESULTS

Table 2 shows the answer frequency for the different categories.

We see that only 16 of the 112 surveyed teachers have made a proposal that includes the development of per-

Category	Frequency	
Use of the activity based on the supply of tools.	4	(3.6%)
Use of the activity as an evaluation with a possible estimation or measurement.	38	(32.7%)
Use of the activity as a measurement.	25	(22.3%)
Use of the activity as a measurement estimation.	26	(13.1%)
Other answers	5	(4.5%)
No answer	23	(20.5%)

Table 2: Frequencies and categories of teachers' answers

ception and evaluation. Given that the referent used is found explicitly in the formulation, their proposal corresponds to an estimation task. These teachers show an adequate Knowledge of Content and Teaching for the development of the activity in the classroom.

On the other hand, 38 of the teachers make a proposal in which estimation is a synonym of evaluation of a magnitude, without justifying how the task is carried out. This makes us think that in the concept of estimation used by these teachers, perceptive processes are not relevant and estimation is not supported by mathematical processes. The development of the activity proposed is incomplete, presenting an incomplete Knowledge of Content and Teaching regarding measurement estimation.

A third group of 25 teachers considers that estimating a measurement is a synonym of measuring, using the referent as a unit of measurement. For these teachers, measurement estimation seems to be the same as measuring with a non-standard tool. In this last category, we include the answers to the questions that are exclusively based on providing the students with measurement tools without specifying their didactic use. This group is comprised of 4 teachers, who we may also consider to confuse estimation with measurement. Finally, the categories that include other answers and empty answers represent 28 of the 112 teachers, which represent a non-negligible percentage (25.0%) of the teachers surveyed.

General research results

This research is part of a wider study. We have used a variety of questions focused on different aspects of MKT estimation of measurement. In this way, we have found that there are gaps at different levels, e.g., teachers confused estimation of measurement with measurement itself, using non-standard units of measurement; teachers consider that the estimate of measurement is a superfluous task, like a guess; teachers do not observe the weaknesses of estimation measurement activities.

CONCLUSIONS

In the present study we have observed large differences in Knowledge of Content and Teaching on measurement estimation between Chilean teachers. Based on one of the activities proposed by the Chilean Ministry of Education to exemplify measurement estimation work in the classrooms, we note that a large part of the teachers do not possess an adequate specific knowledge to use these activities with the purpose for which they have been designed.

Specifically, only 13.2% approaches the activity with the required conditions to work on estimation in an effective manner. A significant part of the group does not acknowledge the importance of perception when estimating and do not mention perception explicitly. This allows the student to handle the activity both directly or indirectly, which would not be considered estimation. It should be noted that a large part of the teachers confuse measurement estimation and measurements based on the use of non-standardized measurement tools. This type of confusions can be related to the absence of estimation in the curriculum to this date. The surveyed teachers have not received any training that allows them to distinguish these two types of tasks.

In order to solve this situation of lack of knowledge, we consider that measurement estimation should be included in teacher-training curricula. It would be interesting for pre-service and in-service teachers training to include activities that required for them to carry out their own estimations and compare them to measurement processes It would also be useful for teachers to observe mathematical processes that distinguish the different types of estimation presented in our literature review, as a way of reinforcing their specific mathematical content. On the other hand, a revision of current curricular guidelines is paramount, in order to avoid creating larger ambiguities. If we consider it acceptable to verify the result obtained from early-stage estimation activities with the exact result, it is fundamental that teachers regard estimation as a task in itself, with its own validation procedures according to the context's requirements

Finally, we consider it crucial to define a mathematical concept precisely when introduced into an educational system. In addition, official programs and didactical guidelines accompanying textbooks should provide examples from appropriate activities, distinguishing them from those that are not. Otherwise, it is highly likely that the curricular changes the country wants to achieve may not be transferred to the classrooms effectively.

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Investigating mathematics teacher identity development: A theoretical consideration

Kirsti Rø

University of Agder, Kristiansand, Norway, kirsti.ro@uia.no

The paper discusses a theoretical framework for investigating the process of becoming a mathematics teacher, based on the need to better understand how mathematics and teaching combine in developing mathematics teacher identities. Identity is understood as a function of participation in and at the boundaries of various communities of practice, during university teacher education and at school. In order to reveal possible relationships between developing teacher identities and the discipline of mathematics, teachers' changing concept images of mathematics are embedded within the definition of identity.

Key words: Identity, identity development, transition, situated concept images.

INTRODUCTION

Becoming a secondary school mathematics teacher can be described as a role change, as one goes from being a university student who is learning and knowing mathematics for oneself, to becoming a mathematics teacher in school and being able to enable others to know it (Rowland, Huckstep, & Thwaites, 2005). It is about carving out a space for one's identity as a mathematics teacher (Alsup, 2006), as one is negotiating shifting conceptions of what mathematics teaching is or should be (Beauchamp & Thomas, 2009). In parallel with undergoing the transition from university teacher education to a professional debut in school, the character of the mathematics content is changing. It goes from being a scientific discipline represented by the discourse taking place between mathematicians and mathematics students at the university, to becoming a school subject as part of general education, preparation for everyday life and as a basis for higher education. Hence, the process of becoming a mathematics teacher is concerned with both; changes in practices and changes in the mathematics content.

The purpose of this paper is to understand better how mathematics and teaching combine in teachers' development of identities (Adler, Ball, Krainer, Lin, & Novotná, 2005). From research on mathematics teacher education, there is known a great deal about some of the specialty of mathematics teachers' knowledge (e.g., Ball, Lubienski, & Mewborn, 2001) and their beliefs (Phillip, 2007). However, there is a need to understand better the dynamics of mathematics teachers' learning, as they move across different practices in university and school and unite their experiences into a role as a professional teacher. According to Lerman (2000), this requires an extension of the unit of analysis from concerning the individual mathematics teacher, her mathematical knowledge and beliefs, to also including the social practices in which the teacher participates. In order to study how learning takes place within and across practices and institutions, I present a theoretical framework for investigating how prospective mathematics teachers make sense of undergoing the transition between university teacher education and school. Learning is here understood as identity development, when a person participates within and at the boundaries of communities of practice (Wenger, 1998). Further, I assume that the discipline of mathematics is a distinguishing characteristic of the learning context, meaning that the nature of practices of mathematics is fundamentally different from other disciplines. Consequently, developing an identity as a mathematics teacher is different from developing a teacher identity in other subjects.

In order to study identity development through the profession of a secondary school mathematics teacher, I offer for critique a situated understanding of teachers' concept images in mathematics as being part of their professional identity and development. The framework is used in a longitudinal interview study of a group of prospective secondary school mathematics teachers' accounts of their ongoing transition into the mathematics teacher profession. It is based on the assumption that the teachers' accounts provide a window into the sort of learning they can experience as they move from one setting into another (Jansen, Herbel-Eisenmann, & Smith III, 2012). Embedded in the learning process are the teachers' situated concept images in mathematics. Hence, the focus of the study is the teachers' meaning making when participating in different communities of practice, rather than identifying possible impact from communities on their concept images.

OUTLINING A FRAMEWORK FOR INVESTIGATING MATHEMATICS TEACHER IDENTITY

Identity in teaching has been explored from a range of theoretical approaches (Beauchamp & Thomas, 2009). It spans from categorising aspects of teacher identity in order to better understand and describe it and possible influences on teachers and their practice, to viewing identity as a function of participation in different communities (Wenger, 1998). The latter is in line with what Lerman (2000) has denoted as the *social turn* in mathematics education research, identified by the rise of theories viewing learning as participation in practices rather than as acquisition of new knowledge structures or beliefs. Following Wenger (1998), identity develops through "negotiated experiences of self" (p. 150), as a person interacts with others and regulate her participation according to the reactions of others to her. In other words, mathematics teacher identities exist both in teachers and in their relations with others (Ponte & Chapman, 2008). Consequently, mathematics teaching can be considered a complex personal and social set of embedded processes and practices that concern the whole person (Olsen, 2008).

Wenger's (1998) theorisation of identity in communities of practice enables me to study individual mathematics teachers through their social settings. Hence, the primary unit of analysis is neither the individual mathematics teacher, nor the learning communities in the transition from university teacher education to school, but instead the teacher-in-the-learning-community-in-the-teacher (Graven & Lerman, 2003; Lerman, 2000). The first part, teacher-in-the-learning-community, acknowledges that the object of study is more than individual cognition and affect, because learning is the development of modes of participating with others in society. The second part, learning-community-in-the-teacher, implies that participation develops identity in such a way that the practice becomes part of the individual. In other words, the focus is neither directed towards categories of mathematics teacher knowledge, nor is it directed towards measures of teachers' mathematical knowledge for teaching. Instead, I study mathematics teachers' developing sense-making of mathematics and mathematics teaching in light of their experience of participating within and at the boundaries of communities of practice in university and school.

In the following, I willpresent a framework that allows me to change the focus on the object of study, by placing respectively the collective and the individual in the foreground (Palmér, 2013b). With the collective in the foreground, I am interested in the mathematics teachers' community memberships in terms of modes of belonging: engagement, imagination and alignment (Wenger, 1998). Further, becoming a mathematics teacher is about reconciling various memberships across communities, and consequently, trying to establish continuity across community boundaries. By placing the individual in the foreground, I assume that when a teacher is stepping into a practice, the teacher is somehow changed. By orienting towards the practice, taking up new practices or marking a distance towards it, the teacher consequently develops new understandings of herself as a learner and doer of mathematics and mathematics teaching.

Wenger (1998) provides a general theorisation of learning and a superior framework for this paper, and I will in the following section elaborate on an understanding of mathematics teacher learning within and between communities of practice. In order to adapt the framework to *mathematics* teacher identities, I will argue for the necessity of combining Wenger's notion of identity with a situated understanding of Tall and Vinner's (1981) definition of concept image. This is built upon a sociocultural understanding of concept images (Bingolbali & Monaghan, 2008), and its compatibility with Wenger's concept of reification.

MATHEMATICS TEACHER LEARNING WITHIN AND BETWEEN COMMUNITIES OF PRACTICE

Based on the assumption that learning is located in the relationship between a person and the world in which the person participates, Wenger (1998, 2010) considers meaning making as a dual process. On the one hand, a person engages in different forms of *participation*, referring to the social experience of living in the world in terms of membership in social communities and active involvement in social enterprises. On the other hand, *reification* accounts for the way in which a person of a community builds her own meaning for her participation. For Wenger (1998), reification and participation are viewed as complementary concepts naming the process whereby individual and community experiences are shared and lead to the production of shared ideas and concepts. A mathematics teacher's identity can therefore be understood as the intersection of individual and social aspects of learning. Individual aspects represent who one is as a mathematics teacher due to one's knowledge and views about mathematics and mathematics teaching and learning. However, these individual aspects are continuously negotiated through the teacher's participation in different communities of practice.

For prospective mathematics teachers, learning can take place when they enter communities of practice during teacher training as peripheral participants and gradually develop as central participants (Lave & Wenger, 1991). A gradual change in participation, from the periphery towards the centre, can be described in terms of engagement, imagination and alignment (Wenger, 1998). Within the community, the prospective mathematics teacher *engages* with ideas about mathematics and mathematics teaching through engagement in communicative practice and develops the ideas through exercising *imagination*. Further, the prospective teacher *aligns* with conditions or characterisations of the practice. The term alignment has later been challenged by Jaworski (2006) in terms of *critical alignment*, a means of not just aligning with practice as established in the community, but of looking critically at that practice while aligning with it. The *boundary* of the community is constitutive of what counts as central participation and what does not. Other learning opportunities will therefore occur *at* the boundaries, as the mathematics teachers are exposed to foreign competences when they enter other communities in school and may try to create bridges across them and former communities (Wenger, 1998). I explore the transition between mathematics teacher education and the professional debut in school in terms of *boundary crossing*, as prospective mathematics teachers become participants in communities of practice in school and do the work of reconciliation of different forms of memberships in various communities. Hence, different forms of knowledge and understanding of mathematics and mathematics teaching may coexist. Developing a mathematics teacher identity during boundary crossing is then about continuously synthesizing what counts as legitimate knowledge within various communities. Simultaneously, the teacher is striving towards continuity across boundaries by maintaining a sense of self through time (Abreu, Bishop, & Presmeg, 2002).

Wenger (1998) discusses how participation and reification have the potential to create continuities across boundaries in terms of brokering and boundary objects. *Brokering*, as participation across communities, is provided by people who can introduce elements of one practice into another. A prospective mathematics teacher may act as a broker when moving between communities in university and school, and bringing ideas of mathematics and mathematics teaching stated in teacher education into communities of mathematics teachers in school. However, acting as a broker is challenging and uncomfortable, since it requires enough legitimacy to influence the development of practice within a community, and enough courage to address conflicting interests. Hence, entering school as a novice mathematics teacher can be described as an experience of confrontations between own expectations about teaching and expectations stated by teacher education, students in the classroom and colleagues. Another dimension of continuities across boundaries is constituted by *boundary objects*, being reified connections between communities (Wenger, 1998). An example is the guidance paper that student teachers are supposed to write and submit to a teacher educator at the university and their tutor in school prior to their lessons during teacher training. The paper may act as a boundary object when it bridges views and knowledge of mathematics teaching in university with experiences of classroom practice in school.

In Akkerman and Bakker's (2011) review of research on literature on boundary crossing, boundaries are defined as "sociocultural differences leading to discontinuities in action and interaction" (p. 133). The definition highlights that boundaries are not about sociocultural differences per se. Instead, boundaries are "real in their consequences" (p. 152). Thus, unlike describing sociocultural differences in university and school, I am interested in how the differences play out in and are being shaped by the process of developing mathematics teacher identities. By overcoming discontinuities, boundary crossing carries a potential to learn about practices and about one's own identity. Akkerman and Bakker (2011) present potential learning mechanisms that may occur when crossing boundaries, two of them concerning processes of making sense of practices in multiple contexts. *Identification* entails a renewed sense making of different practices and related identities, by encountering and reconstructing boundaries but not being able to overcome discontinuities. Learning in terms of *reflection* will result in an expanded set of perspectives and a new construction of identity, which in turn will have an impact on future practice. Here, reflection is the desirable learning outcome, when one is establishing continuity across boundaries by reconciling memberships across communities. For prospective mathematics teachers, crossing boundaries is both a valuable and a risky process. On the one hand, they are in a position to introduce elements of one practice into the other. On the other hand they face the risk of never fully belonging to or being acknowledged as a participant in any one practice.

MATHEMATICS TEACHER IDENTITY AND SITUATED CONCEPT IMAGES

As have been emphasised by other researchers (e.g., Palmér, 2013a), Wenger (1998) does not focus specifically on mathematics education and mathematics teaching, but on learning in its broadest sense. Yet others have discussed the possibility of Wenger's framework to take into account the full spectrum of locations of cognitive development, from in-the-brain to socially dependent (Van Zoest & Bohl, 2008). In order to comply with this request, Van Zoest and Bohl (2008) combine Wenger's (1998) theory of communities of practice with Shulman's (1987) heuristic of teacher knowledge, and an understanding of knowledge and beliefs as cognitive in nature. However, unlike viewing identity as dynamic and in continuous development, they describe it as something the teachers "carry with themselves as they move from context to context" (Van Zoest & Bohl, 2008, p. 338). Consequently, their framework is not consistent with my investigation of how teachers make sense of their ongoing transition, due to their present situation of participating in communities of practice.

Taking a different theoretical approach, Palmér (2013a, 2013b) connects Wenger's notion of identity with Skott's (2010) and Skott, Larsen and Østergaard's (2010; 2011) theory of patterns of participation. This is done in order to describe primary school mathematics teachers' professional identity development, "by including both the individual and the social parts of identity development" (Palmér, 2013a, p. 2851). Further, the theories are argued to be consistent within a situated learning perspective. However, in Palmér's (2013b) study of primary school mathematics teachers' way into their profession, the discipline of mathematics do not appear as a prominent part of the teachers' developing identities. Even so, I claim that prospective secondary school mathematics teachers need to some extent relate to and cope with mathematics both during university studies and in their profession. Hence, there is a need for developing an analytic tool which makes the mathematics within the teacher identity more visible. Inspired by Bingolbali and Monaghan (2008), I consider the notion of concept image (Tall & Vinner, 1981) as being a helpful construct when placing the individual mathematics teacher in the foreground (Palmér, 2013b). Tall and Vinner (1981) describe *concept image* as the total cognitive structure associated with a concept in an individual's mind, which includes mental pictures, associated properties and processes, strings of words and symbols. Unlike a *concept's definition*, the concept image is dynamic and develops differently among persons and through a multitude of experiences. Although concept image and concept definition are terms originating from cognitive theories of learning and a focus on individual student mathematical constructions, Bingolbali and Monaghan (2008) argue that the dynamic nature of concept image is consistent with a sociocultural perspective on learning:

> Indeed, the construct can be viewed in terms of Vygotsky's (1934/1986) *complexes*, a 'phase on the way to concept formation' (ibid, p. 112) and the original view of concept images as developing differentially over students through a multitude of experiences is essentially a contextual viewpoint. (p. 21)

Bingolbali and Monaghan (2008) base their argument on a study of undergraduate students' learning of the derivative, in which the context of learning is regarded as paramount. They found that students' developing concept images are closely related to teaching practices and departmental perspectives, respectively within mechanical engineering and mathematical sciences. I will take a step further by claiming that the dynamic nature of concept image is compatible with a situated perspective on learning. According to Wenger (1998), different types of memberships in various communities entail opportunities and limitations for developing practice, which consequently affects a mathematics teacher's knowing. Hence, I assume that situated concept images in mathematics are embedded in a mathematics teacher identity. It is about knowing mathematics for oneself and for teaching, which is related to one's views and emotions about mathematics and mathematics teaching. Further, concept images are dynamic in the sense that they are negotiated through interaction with other participants within various communities of practice.

A further argument for the compatibility of Wenger's notion of identity and a situated understanding of concept images can be found by the term reification. As part of the negotiation of meaning, reification is a way of giving form to our experience "by producing objects that congeal this experience into 'thingness'" (Wenger, 1998, p. 58). Simultaneously, reification shapes our experience, since forming a certain understanding about a topic brings a new focus for negotiating meaning within a community, leading to new ways of reasoning or acting. Reification of practice can appear as abstractions, symbols, concepts and tools, and becoming a participant in a community is then about growing into the practice in which one engages, including its reifications. Regarding the prospective mathematics teacher, she takes up new practices in mathematics and mathematics teaching by participating in different communities of mathematics students at university, mathematics student teachers during teacher training and teacher colleagues at school. These practices include knowledge of mathematics concepts, skills, and beliefs about mathematics and mathematics teaching, which I further assume provide a basis for developing situated concept images in mathematics.

Nevertheless, reification in terms of situated concept images does not mean that they are simply objects of knowledge and beliefs in mathematics that teachers have or gain, but instead they are both "a process and its product" (Wenger, 1998, p. 60). In other words, a situated understanding of concept image highlights its location in social practices, being continuously negotiated through participation. Further, situated concept images are only "the tip of an iceberg" (Wenger, 1998, p. 61), being indications of larger contexts of meaning realised in human practices. This is in line with Tall and Vinner's (1981) notion of evoked concept images, which is the part of a person's memory evoked in a given context, and which is not necessarily all that a person knows about a certain topic or area. Hence, there is a problem of gaining insight into mathematics teachers' concept images. However, based on a longitudinal interview study, including interview tasks in mathematics, I gain insight into an ongoing process of identity development on given times and over a prolonged period of time. I am therefore able to describe the dynamic nature of mathematics teacher identities and associated concept images, not as objects within teachers, but as objectifications of ongoing processes (Palmér, 2013b).

Taking a situated perspective on concept images, they do not count for comparing a teacher's knowledge and/or beliefs before and after undergoing the transition between university and school. Instead, the underlying question is under which conditions successful participation in university communities facilitates successful participation in school communities (Greeno, 1997). This re-establishment of participation across settings may lead to experiences of discontinuities, e.g., experiences of shifts in practices of mathematics teaching and in the mathematics content. By overcoming them, boundary crossing carries a learning potential of developing one's identity as a mathematics teacher and consequently one's concept images in mathematics.

FINAL REFLECTIONS

I have presented a framework for investigating teacher learning in the transition between university teacher education and the professional debut in school, using Wenger's (1998) theory of community of practice as a starting point. In addition to recognise the learning potential that resides within a community, the framework takes into account learning during boundary crossing, where reflection is the desirable learning outcome (Akkerman & Bakker, 2011). Developing an identity as a mathematics teacher is then about negotiating what counts as legitimate knowledge within various communities in university and school, comprising shifting conceptions of what mathematics teaching is or should be. Based on the assumption that the discipline of mathematics is a distinguishing characteristic of the learning context, I have argued for the necessity of combining Wenger's notion of identity with a situated understanding of mathematics concept images. They are related to the teachers' knowledge and views about mathematics and mathematics teaching, and continuously negotiated through their participation in various communities of practice.

I take the mathematics teacher's perspective and investigate identity development based on how prospective mathematics teachers make sense of their ongoing transition between university and school. This meaning making is situated within a context, based on the teachers' participation in various communities of practice. Hence, the way the teacher looks back and reflects on being a student teacher in mathematics and a novice mathematics teacher in school, and how she discusses mathematics and mathematics problems, may change and develop due to her present situation. In addition, the teachers' accounts provide a means for them to create continuity across time, in terms of maintaining a sense of self. I thus assume that possible changes or development in the mathematics teachers' accounts constitutes evidence for their identity development, in which the teachers' situated concept images of mathematics are embedded.

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Using Concept Cartoons to investigate future teachers' knowledge

Libuše Samková and Alena Hošpesová

University of South Bohemia in České Budějovice, Faculty of Education, České Budějovice, Czech Republic, lsamkova@pf.jcu.cz

In the study presented here, we address the issue of how to investigate future primary school teachers' responses to more or less expected children's answers and questions. We observe future primary teachers exposed to contingent situations mediated by an educational tool called Concept Cartoons, and analyse their responses with respect to mathematics subject knowledge. The Concept Cartoons presented in this article deal with addition of natural numbers.

Keywords: Concept Cartoons, teachers' knowledge, mathematics subject knowledge, knowledge quartet, contingency.

INTRODUCTION

The study presented here is a part of a three-year project focusing on opportunities to influence professional competences of future primary teachers through experienced *inquiry based mathematics education*. We realize a set of university mathematics courses for future primary teachers in which they can experience inquiry based education as students, and analyse its impact on their content knowledge. An integral part of the project consists of repeated diagnosis of mathematics subject knowledge of project participants. We generally observed inquiry based education from the perspective of the *knowledge quartet*, and realized that this kind of education is extremely rich in *contingent* (unpredictable) situations. As far as contingent situations are unpredictable, and it is difficult to simulate them systematically, we decided to imitate such situations by an educational tool called *Concept Cartoons*. We use this tool as one of the diagnostic tools in our project.

In this particular study we observe future primary teachers exposed to contingent situations mediated by Concept Cartoons, and analyse their responses. Our research question is: What aspects of future teachers' mathematics subject knowledge can we investigate using Concept Cartoons?

THEORETICAL BACKGROUND OF THE RESEARCH

Teachers and their knowledge

Describing and analysing teacher work is a very attractive field of recent international research in mathematics education. Starting with Shulman's widely accepted concepts of *subject matter content knowledge* (SMK) and *pedagogical content knowledge* (PCK) (Shulman, 1986), researchers try to analyse different kinds of teachers' knowledge, its content, relations, and obstacles in their formation.

An extensive research on mathematics subject knowledge of future primary teachers has been conducted by a group around Rowland. Their research resulted in the identification of aspects of the behaviour that seems to be significant as information about one's SMK or PCK in mathematics. They introduced 20 categories which were subsequently grouped into four broad dimensions: foundation, transformation, connection, and contingency – the so-called *knowledge quartet* (Rowland, Huckstep, & Thwaites, 2005; Rowland, Turner, & Thwaites, 2013, 2014). Knowledge quartet and its subsequent categories can now be used as a tool for conceptualizing the ways in which teachers' knowledge comes into play in the classroom.

As for the particular dimensions, *foundation* refers to teacher's theoretical background and beliefs, *transformation* concerns knowledge-in-action with central focus on representations (analogies, examples, explanations, etc.), *connection* refers to ways the teacher achieves coherence within and between lessons. The last dimension, *contingency*, involves aspects dealing with unpredictable (contingent) events in the classroom. It concerns teachers' responses to events that were not anticipated in the planning. The dimension of contingency consists of five subcategories: *responding to students' ideas* (RSI), *deviation from lesson agenda*, *teacher insight*, *use of opportunities*, and *responding to the (un)availability of tools and resources* (Rowland et al., 2014).

In our study, we shall investigate responses related to the RSI code:

> This code includes the ability to make cogent, reasoned, and well-informed responses to unanticipated ideas or suggestions from students. These teachers' responses are to students' contributions to the (mathematical) development of the lesson. These contributions are typically oral, but could be written. Our analysis of the data available to us identifies three sub-types of triggers in this category: 1) student's response to a question from the teacher; 2) student's spontaneous response to an activity or discussion; 3) student's incorrect answer – to a question or as a contribution to a discussion. ('Knowledge Quartet', 2012)

In addition to knowledge quartet we also observe future teachers' knowledge from procedural and conceptual perspectives. Hiebert and Lefevre (1986) characterized conceptual knowledge as a connected network of facts and propositions, and procedural knowledge as a structured set of mathematical symbols and conventions for their use, rules, algorithms and procedures. This construct was attacked by some researchers for inconsistency, and reconceptualised several times. We find the most suitable the reconceptualization given by Baroody, Feil, and Johnson (2007, p. 123):

> *Procedural knowledge* consists of mental actions or manipulations, including rules, strategies, and algorithms, for completing a task.
>
> *Conceptual knowledge* is knowledge about facts, generalizations, and principles.

Concept Cartoons

In our research we make use of an educational (and in this study diagnostic) tool called *Concept Cartoons* (CCs). CCs were developed in 1991 by Keogh and Naylor as a tool for learning and teaching science (Keogh & Naylor, 1993); lately they have been created also for other school subjects, for example, mathematics (Dabell, Keogh, & Naylor, 2008). Each CC is a cartoon-style picture showing a group of children in a bubble-dialogue based on an everyday situation, the children presenting different viewpoints on the situation (Figure 1). The alternatives displayed in bubbles may be based on real events, on classroom scenarios, on common conceptions and misconceptions, or might be prepared intentionally.

CCs are used mainly in the classroom to support teaching and learning by generating discussion, stimulating investigation, and promoting learners' involvement and motivation, i.e., as a tool oriented mainly on pupils (Naylor & Keogh, 2012). In the study presented here, we aspire to use CCs innovatively for investigating future primary teachers' content knowledge.

In view of the fact that each CC offers a situation not invented by the teacher, and children's various responses on this situation, a suitably chosen CC can provide the teacher an educational model of a contingent situation. In the context of the RSI code, each CC is an artificial reality that partially imitates triggers of the 1st type (it shows children's responses to a question, but the question was not asked directly by the teacher, and sometimes the question is not explicitly expressed in the cartoon), and with an appropriate choice of the content of bubbles it can partially imitate triggers of the 2nd type (children in the picture can response to other children's answers; this case would serve Peter's bubble in Figure 1 changed to "Kevin, you are not right, they scored more."), and entirely imitate triggers of the 3rd type (some answers in bubbles are incorrect).

As the RSI code admits not only oral but also written contributions from children (see the characteristics of RSI above), we place CCs on worksheets, and let the respondents react to them in written form.

DESIGN OF THE STUDY

Participants

Participants of the research are two groups of master students of primary teacher training from our Faculty of Education. This master's degree training lasts 5 years, and covers all the primary curriculum subjects. We involved 29 students of the 2nd year, and 35 students of the 3rd year. The 2nd year students

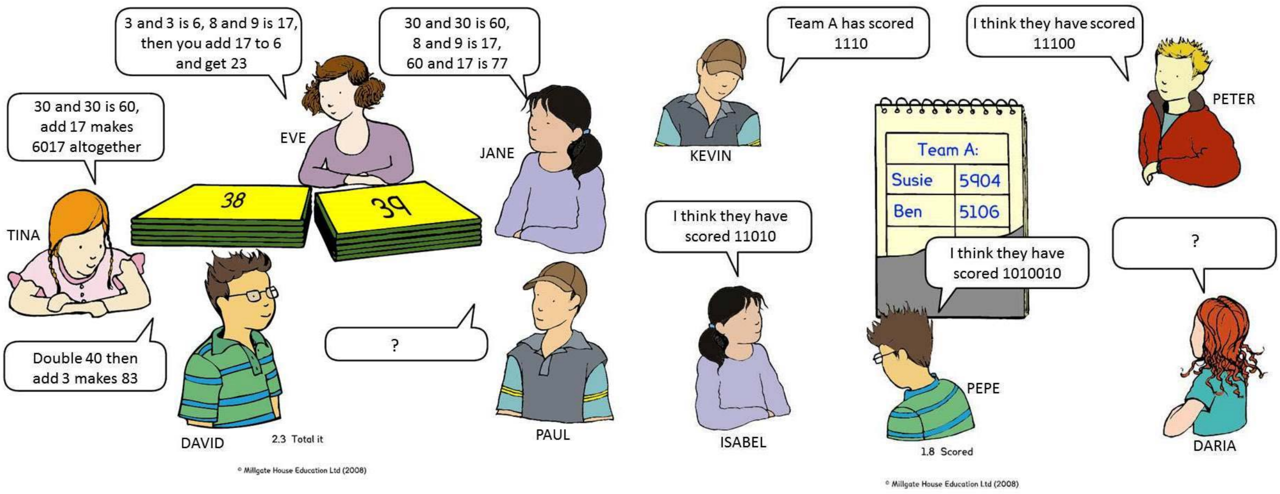

Figure 1: Two examples of Concept Cartoons; taken from (Dabell et al., 2008), slightly modified

have recently completed the "Natural numbers" part of Arithmetic courses, but have not attended any Didactics yet. The 3rd year students have recently completed the "Natural numbers" part of Didactics of mathematics courses.

Course of the study

In the data collection stage of the research we gave each respondent a worksheet consisting of four CCs on addition and subtraction of natural numbers (two of them are in Figure 1), with four common questions:

1) Which child do you strongly agree with?
2) Which child do you strongly disagree with?
3) Decide which ideas are right and which are wrong. Give reasons for your decision.
4) Try to discover the cause of the mistakes, and advise the children how to correct them.

Respondents were asked to fill in the worksheet individually. For all respondents it was the first occasion to work with CCs.

We processed this output qualitatively, using grounded theory methods (Strauss, 1987): we started with open coding, then grouped the codes according to similarities and internal relation into categories, and marked codes with plus or minus sign to denote positive or negative aspects (good or poor knowledge, correct or incorrect recognition, etc.). Then we several times reinspected all the output, looked for new fragments and new contexts, rearranged existing fragments and codes, and debugged the coding process as well as the process of sorting codes into categories.

As relevant for our study appeared the following code categories:

A) respondent's spontaneous response (the very first opinion) on ideas in bubbles

B) respondent's subsequent response on ideas in bubbles

   – recognition of right and wrong answers

   – recognition of procedures used by children, identification of the causes of mistakes

C) the way how the CC was composed, i.e. the nature of the CC

We distinguish between A and B categories, because in relation to contingency we consider as important an immediate response to the content of bubbles. We assume that such immediate response can be triggered by worksheet questions 1 and 2, while subsequent response is rather a matter of questions 3 and 4. More precisely, thought processes caused by questions 1 and 2 are different from those caused by questions 3 and 4 – respondents do not need to go into a deeper analysis while looking for a child with whom they strongly (dis)agree. Therefore, aspects related to A codes are mainly triggered by questions 1 and 2, aspects related to B codes are mainly triggered by questions 3 and 4.

In the next stage of the analysis we examined relationships between codes and between categories, from the perspective of the content of particular bubbles as well as from the perspective of particular respondents.

FINDINGS

The initial finding we made during the coding process was that participants' responses are substantially affected by the nature of the individual CC (i.e., by aspects hidden under the C codes). The effect is clearly seen when comparing the two CCs from Figure 1 – both of them show a situation that need to be transformed to mathematics, mistakes displayed in bubbles are standard, and each bubble contains no more than one type of mistake. But their other features differ, for example:

Figure 1a

- the arithmetic task is not explicitly stated, it needs to be revealed from the picture
- each bubble describes the procedure of calculation, and the result

Figure 1b

- the arithmetic task is outlined, numbers to sum are aligned below each other
- each bubble shows only the result of calculation

Thus, with the first CC respondents can comment results and procedures described in bubbles, and look for errors in procedures leading to incorrect results (and also in procedures leading to correct results). While with the second CC respondents can comment only results; it simplifies decision at a bubble with a correct result, but complicates decision at a bubble with an incorrect result: the procedures hidden behind the incorrect result are not described, and respondents have to make an attempt to discover them.

In the first CC, majority of the 2nd year respondents failed to decode the David's procedure, many of them even mentioned David as the child with whom they strongly disagree:

P28	Where did David take the 40?
P11	I strongly disagree with David. David – I do not understand how he came to the 40.

But the David's procedure is very inventive, with only a small mistake in the final. The bigger compliment goes to these respondents who praised David for his procedure, and advised him the right ending:

P31	David has a good tactic, but he mistook a sign, instead of adding there should be subtracting. David, you do not have those 2 x 40, you have less, so you must subtract.

On the contrary, the 3rd year respondents managed the David's issue without hesitation. Here the PCK acquired in Didactics courses comes into play – these respondents already attended Didactics on natural numbers where they learned how to utilize various counting procedures in the classroom.

Some 2nd year respondents had also difficulties with the Eve's procedure; they were not able to discover why she started with 3 + 3:

P22	Eve counted wrong from the start.
P20	Eve makes a sum of random.
P31	Eve does not watch orders, she is short of tens.
P27	Eve counted tens incorrectly; she took 3 + 3 instead of 30 + 30.
P12	Eve has a problem with counting of tens. Eve, we decompose 38 to 30 and 8, not to 3 and 8.
P33	Eve handled incorrectly the numbers; she did not take them as wholes, but separately. She did not realize that 39 is 30 and 9, and made 3 and 9 of it. Eve, you messed ones and tens, 39 must be expressed as 30 and 9.

To clarify the situation we should mention that counting tens separately during mental addition is not a standard procedure in our schools. On the other hand, the respondents recently attended Arithmetic courses on decimal numeral system, so that the excerpts above point to inflexible thinking of their authors.

In the second CC, respondents often succeeded in finding the procedure hidden behind the results, some of them suggested really credible rationale for why the mistake happened:

P14	Kevin completely eliminated the column with zeros.
P27	Kevin, if you count 0 + 0, you must enter the final 0 to the calculation.
P14	Peter probably thought that 0 is not a number, and added 1 to the next sum.
P37	Peter forgot to add 1 to the tens place, and added it later to the hundreds.

But there appeared also rationales that do not look likely:

P13	Peter swapped 0 and 1.

The tendency not to seek the procedure but only compare the bubble and the correct result, which is noticeable on the last line of the transcript above, appeared even stronger in some 3rd year students' outputs:

A03	Peter wrote the third '1' to the wrong place.
A10	Peter is coming to accuracy, but his result is 90 more. Peter, recount it again, your result is 90 more than the correct result. Pepe has 2 more zeros in his result. Pepe, your result is too high compared to the computed numbers. Recount again the example, and remove some numbers.

Among 2nd year respondents we found two who confused terms 'number' and 'digit' in their explanations. Further analysis of their responses showed that the problem might not be only terminological:

P24	Tina is right. If I add 17 to a digit 60, I get 6017. But if I count up, I would get 77.
P18	Tina and Jane are right, they followed the instructions exactly. David's bubble should be corrected: 2 x 40 = 80, add 3 makes 803.

These statements can be signals of deep misunderstanding of the concept of number. To be sure, we would need further data from these respondents.

During the analysis of completed worksheets it became apparent that in some responses to the first CC it is possible to distinguish between procedural and conceptual knowledge. This fact was especially evident for the following two respondents; both are 2nd year students, with average performance in Arithmetic. The first respondent answered the questions as follows:

P33	ad 2)	I strongly disagree with David.
	ad 3)	David is wrong. He confused it all.
	ad 4)	Instead of subtracting 3 from $40 \cdot 2 = 80$, he added it. He needs to have the whole counting explained again.

In the beginning, David is the only child in reply to a question with whom the respondent strongly disagree. Then we can see that the respondent knows the procedure that David used; she even says where the mistake is and what the correct version of this part of procedure is. Yet in the end she states that David needs to learn the whole procedure again. This respondent probably got her knowledge by rote learning. She does not understand the procedure as a sequence of individual steps, but as one indivisible whole. She knows the procedure very well; she is even able to compare her calculation with David's, and find the mistake. But she is not able to divide the procedure into individual steps, and repair just the wrong one. In her kind of understanding the only way how to repair the procedure is to learn it again as a whole. Summarized, in this task the respondent displayed no conceptual knowledge, and only superficial procedural knowledge.

With this respondent we can also clearly illustrate the difference between SMK and PCK: she knows how to count the example for herself (an indicator of SMK) but is not able to help the child (an indicator of a lack of PCK).

The second respondent wrote:

P37	ad 2)	I strongly disagree with David and Tina.
	ad 3)	Eve is wrong.
	ad 4)	Eve, you can calculate this way, but you have to write the second number under

the first, digits lined up in columns.
David, you must not round the numbers. If you round, you have to sum the numbers, and subtract their difference from the result.

This respondent has her knowledge too closely tied to the context in which it was learned (e.g., counting of numbers that are written below each other), so that the knowledge cannot generalize to other situations. As Hiebert and Lefevre (1986, p. 8) note, also this kind of knowledge used to be obtained by rote learning. Again, this respondent shows no conceptual knowledge, and superficial procedural knowledge. As in the previous case, the respondent displays SMK but no PCK.

CONCLUSIONS

In this study we introduced an educational tool called Concept Cartoons (CCs), and used it innovatively for investigating future primary teachers' mathematics content knowledge. For this purpose we prepared a set of CCs on an essential topic "addition and subtraction of natural numbers". These CCs imitate contingent situations, and we may observe future teachers' responses to more or less expected children's ideas.

As the results show, CCs are a very flexible tool, and we may investigate various aspects of knowledge with them. If the CC contains bubbles showing both procedures and results, then we might be able to distinguish clearly between procedural and conceptual knowledge, and between SMK and PCK (David & P33, Eve & David & P37). This kind of CCs also reveals when respondents have troubles to decode a simple procedure containing a mistake (David's and Eve's cases). The incorrect procedures presented by children might be ambiguous at first glance, and some respondents could display inappropriate spontaneous reactions – disagree strongly with an idea they do not understand (David & P11), or wrongfully blame the child to count randomly (Eve & P20). Moreover, the description of an incorrect procedure in the bubble can influence some respondents to incorporate the mistake into their own responses (David & P18, Tina & P24).

The CC containing bubbles with results only can serve as a useful supplement to the previous type. Respondents can display good transformation knowledge when looking for examples of procedures that could fit to incorrect results in bubbles (Kevin & P27, Peter & P14). On the contrary, some respondents can show a lack of PCK by just comparing the incorrect result in the bubble with the correct result, and giving the child advice without looking for the procedure hidden behind the mistake (Peter & P13, Peter & Pepe & A10).

The analysis of the data obtained during the research led us to the need to investigate deeper the question how different kinds of knowledge can be displayed through the mediation of CCs. This is the direction we will continue our research. We found out that further systematic triangulation from different perspectives (responses to CCs, Arithmetic tests, interviews) is necessary for the creation of a grounded theory.

We appreciate the advantage that CCs offer in comparison to other diagnostic tools such as videos or classroom scenarios: the possibility to prepare the content of the bubbles intentionally, on a chosen purpose. We expect that this feature shall allow us to investigate teachers' knowledge more deeply through presenting bubbles with alternatives that are able to reveal important aspects of teachers' knowledge but sometimes might remain unspoken in a real classroom.

ACKNOWLEDGEMENT

This research was supported by the Czech Science Foundation, project No. 14-01417S.

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Lessons we have (not) learned from past and current conceptualizations of mathematics teachers' knowledge

Thorsten Scheiner

University of Hamburg, Hamburg, Germany, Thorsten.Scheiner@uni-hamburg.de

This paper attempts to capture some of the breath of frameworks and models on mathematics teachers' knowledge in order to identify central lessons we have (not yet) learned from past and current approaches in theorizing and conceptualizing a knowledge base for teaching mathematics: there are accounts of the complex and multidimensional nature of teachers' knowledge but no accounts as to the reorganization of dimensions of teachers' knowledge in order to be more consistent with a constructivist view on learning and teaching; there are accounts of what teachers' knowledge is about but no accounts as to a structural description of teachers' knowledge. The paper highlights several unsettled issues of this research field and certain profitable directions for advancement.

Keywords: Teacher education/development, teacher knowledge.

MAPPING THE TERRAIN OF RESEARCH ON TEACHERS' KNOWLEDGE

With his influential construct of dimensions of teachers' knowledge in the 1980's, Lee S. Shulman (1986) at Stanford University has guided the research on teachers' knowledge in a new direction and, simultaneously, proposed an approach to educational reform that labelled teaching a *profession* (Shulman, 1987). Shulman (1986) promoted a *paradigm shift* in educational research by assuming the existence of a knowledge base that is special for the purposes of teaching. Since then, several interesting approaches, partly distinct and partly overlapping, have been developed that shape the current theoretical landscape in mathematics education research on teachers' knowledge. In the research literature, it is common to follow Shulman's (1987) conceptualization of a knowledge base for teaching including (1) content knowledge, (2) general pedagogical knowledge, (3) curriculum knowledge, (4) pedagogical content knowledge, (5) knowledge of learners, (6) knowledge of educational contexts, and (7) knowledge of educational ends, purposes, and values, and their philosophical and historical grounds. Several researchers have made attempts to identify features of mathematics teachers' knowledge that (may) matter in the work of teaching – in many ways, making Shulman's (1986, 1987) conceptualization of domains of teachers' knowledge, and, in particular, subject matter knowledge (SMK) and pedagogical content knowledge (PCK) specific to teaching mathematics.

The frameworks and models that shape the theoretical landscape in the conceptualization of and research on mathematics teachers' knowledge are within a broad spectrum of specificity, ranging along general, discipline-specific, domain-specific, and concept-specific frameworks and models. Various general frameworks contributed to the field, for instance, in (a) documenting teachers' resources (including knowledge), orientations (including beliefs), and goals as critically important determinants of what teachers do and why they do it (Schoenfeld, e.g., 2010), (b) highlighting that besides subject matter knowledge per se there is subject matter knowledge specific for teaching (Shulman, 1986, 1987), and (c) providing insights in teacher proficiency including the identification of various dimensions such as knowing students as thinkers and learners, reflecting on one's practice, among many others (Schoenfeld & Kilpatrick, 2008). Schoenfeld and Kilpatrick's (2008) contribution builds the bridge to discipline-specific frameworks since their work has been initially developed for identifying dimensions of mathematics teachers' proficiency but can and has been extended to a general (discipline-unspecific) framework.

A considerable number of research work is located in mathematics education research, providing both discipline-specific and domain-specific frameworks and models (e.g., Ball et al., 2008; Baumert et al., 2010; Blömeke et al., 2014; Fennema & Franke, 1992; Hill et al., 2008; Kilpatrick et al., 2006; Rowland et al., 2005; Tatto et al., 2008, 2012). These contributions, among others, are of interest in this paper since each contribution introduces and examines a particular approach in theorizing and conceptualizing the construct of mathematics teachers' knowledge. They are chosen because of their *complementary* power and their potential to study teachers' knowledge in a more comprehensive manner. Instead of reviewing each contribution in detail, the following section presents some central lessons we have (not yet) learned from these approaches.

Notice that, with few exceptions (e.g., Even, 1990), the mathematics education research community has almost neglected concept-specific frameworks. However, from the author's perspective, investigating teachers' knowledge at the level of specific concepts is an important issue that needs more attention in future research on teachers' knowledge.

THE RECENT DIVERSITY OF CONCEPTUALIZATIONS AS A RESOURCE FOR CURRENT AND FUTURE RESEARCH ATTEMPTS

The mathematics education research literature contains a broad range of approaches in theorizing and conceptualizing a knowledge base for teaching mathematics. The diversity of approaches is, of course, a reflection of the complexity of the research field that cannot be described, understood, or explained by only one theoretical framework. Different frameworks evolve for multiple purposes due to different needs in given contexts with different implications – some on a theoretical, methodological, and/or empirical level. The diversity of frameworks may provide a rich resource for future research attempts – the frameworks and models are important in their own right and may prove to be productive in some contexts.

The broad diversity of approaches starts with the versatile function of frameworks and models of teachers' knowledge: (a) as *tools* or (b) as *objects*. While most of the frameworks and models of teachers' knowledge are used as *tools* for guiding research practices, in particular for analysing data in empirical investigations, only a few function as an *object* of research – they are the aim of research practices. This distinction between 'tools for research' and 'objects of research' has already been made by Assude and colleagues (2008) with reference to theories in mathematics education. While the conceptualization by Ball and her colleagues (2008), for instance, can be understood as the result of an intensive 'job analysis', where 'conceptualizing a knowledge for teaching' was one of the goals, the conceptualizations by Blömeke and her colleagues (2014) and Tatto and her colleagues (2008, 2012) provide tools for empirical investigations in an international comparative, large-scale study. However, the distinction between frameworks as tools or as objects is rather inclusive (than exclusive) since the ways in which teachers' professional knowledge is understood and conceptualized impact on how teachers' knowledge is investigated, and vice versa.

THE KNOWLEDGE BASE FOR TEACHING MATHEMATICS IS COMPLEX AND MULTIDIMENSIONAL

The different approaches converge in an understanding that teachers' knowledge is complex and multidimensional. Although the discipline-specific models and frameworks mentioned above differ in detail, many of them converge in efforts to further *refine* the construct of subject matter knowledge (SMK) and pedagogical content knowledge (PCK). The following is an attempt to shed light on ways how Shulman's dimensions of SMK and PCK have been refined in the above mentioned contributions.

Subject Matter Knowledge (SMK)

The literature suggests that subject matter knowledge (SMK) can be further differentiated in terms of substantive and syntactic structures (Schwab, 1978), in terms of ways of understanding and ways of thinking (Harel, 2008), in terms of school mathematical knowledge and academic content knowledge (Bromme, 1994), among others. Each further distinction has shed light into important issues: Shulman (1986, 1987), for instance, emphasized Schwab's (1978) distinction between substantive and syntactic structures of a discipline. Substantive structures are the key principles, theories, and explanatory frameworks that guide inquiry in the discipline, while syntactic structures provide the procedures and mechanisms for the acquisition of knowledge, and include the canons of evidence and proof. As already noticed by Rowland and Turner (2008), the term 'syntactic' is mainly associated to the formal structure, thus, it seems that Schwab's (1978) choice of the word 'syntactic' is unfortunate since it does not capture the heart of the intended meaning that is, as argued by Rowland and Turner (2008), the heuristics of inquiry. However, Schwab's distinction has been an initial point to think about various dimensions of SMK.

In synthesis, it can be stated that several researchers have refocused on the centrality of SMK in teaching. However, crucial in the literature is the assumption that there is *unique* content knowledge for teaching mathematics and that having such knowledge is key to the enactment of rich mathematics. The notion of 'specialized content knowledge' introduced by Ball and her colleagues, described as pure content knowledge "that is tailored in particular for the specialized uses that come up in the work of teaching" (Hill et al., 2008, p. 436), is a key contribution in efforts to examine dimensions of mathematical knowledge considered as being crucial for the purposes of *teaching*. In contrast to the former refinements of SMK, the notion of 'specialized content knowledge' has the potential to go beyond just differentiating mathematical content knowledge in various (qualitatively different) sub-facets (such as to think about content knowledge in terms of procedural and conceptual knowledge, school mathematical and academic content knowledge, etc.). This 'specialized content knowledge' is not the kind that disciplinary experts would necessary possess. As a consequence, in contrast to Shulman (1986) treating 'subject matter knowledge *for* teaching' as equivalent to PCK, these considerations lead to the claim that there is pure mathematical knowledge specialized *for* teaching mathematics. Furthermore, it is argued that this kind of mathematical knowledge is not merely qualitatively but may be fundamentally different to SMK per se. This argument is rooted in the observation that SMK per se is primarily aimed at creating new knowledge, while SMK for teaching is essentially aimed at promoting students' mathematical thinking and learning. In this work, the former kind of knowledge is called *mathematical content knowledge per se* (MCK per se) and the latter kind of knowledge is called *mathematical content knowledge for teaching* (MCK for teaching).

Notice that 'mathematical content knowledge per se' is not equal to what Ball and her colleagues (e.g., 2008) described as 'common content knowledge' since it is not limited to the knowledge 'held or used by an average mathematically literate citizen' but may also include academic content knowledge, for instance. Mathematical content knowledge per se can be considered as not only including basic factual knowledge of mathematics but also the conceptual knowledge of structuring and organizing principles of mathematics as a discipline as described and operationalized in the TEDS-M framework (Tatto et al., 2008). Moreover, it can be described in terms of Kilpatrick, Blume, and Allen's (2006) *mathematical proficiency with content* including conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, productive disposition, and knowledge of structure and conventions, among others.

Pedagogical Content Knowledge (PCK)

Since Shulman's (1986) introduction of the construct of PCK, many researchers have added and further elaborated attributes and components of PCK. The above mentioned contributions provide various ways to refine the construct PCK, including, but not limited to, knowledge of cognitive requirements for learning, knowledge of students' conceptions, knowledge of epistemological obstacles of particular mathematical concepts, and knowledge of instructional strategies. Although even representing refinements of PCK, these subcategories are quite broad and often remain unspecified. Rowland and his colleagues' (e.g., 2005) work let to the identification of several subcategories that could be grouped into four units. Although their units are broad, the underlying subcategories provide some specificity. Another example is the work by Hill, Ball, and Schilling (2008) in making effort to conceptualize, develop, and test measures of teachers' knowledge of content and students (KCS). The same authors, although providing with KCS a subdivision of pedagogical content knowledge, state that even their subcategory is multidimensional (see Hill, Ball, & Schilling, 2008). Thus, various researchers assume that it is reasonable to further refine the various subcategories.

The subcategories of PCK identified in the above mentioned frameworks and models can be clustered into three dimensions, namely (1) an epistemological dimension, (2) a cognitive dimension, and (3) a didactical dimension. The *epistemological dimension* refers to knowledge about the epistemological foundations of mathematics and mathematics learning (see Bromme, 1994). For instance, Harel (e.g., 2008) calls for teachers' knowledge of epistemological issues involved in the learning of specific mathematical concepts, including knowledge of epistemological obstacles. The *cognitive dimension* refers to knowledge of students' cognitions (Fennema & Franke, 1992), in particular, knowledge of students' common conceptions (see Shulman, 1987), knowledge of students' cognitive difficulties involved in concept construction (Harel, 2008), and the interpretation of students' emerging thinking (Ball et al., 2008). In other words, it includes knowledge of how students think, learn, and acquire specific mathematical knowledge (Fennema & Franke, 1992). The *didactical dimension* refers to what Shulman (1986, p. 9) described as knowledge of "the most useful ways of representing and formulating the subject that make it comprehensible to others", including teachers' illustrations and alternative ways of representing concepts (and the awareness of the relative cognitive demands of different topics) (Rowland et al., 2005) and knowledge of the design of instruction (Ball et al., 2008).

In summary, it can be stated that the frameworks and models about teachers' knowledge mentioned above can be understood as elaborating rather than replacing Shulman's (1986; 1987) contribution within this field. The approaches taken and the conceptualizations of teachers' knowledge proposed are not inconsistent, nor are the identified dimensions of mathematics teachers' knowledge mutually exclusive. In contrast, the identified dimensions are *complementary* and provide, taken together, a more *refined* picture of conceptualizing the teachers' knowledge base.

The considerations proposed above demonstrate the multidimensional nature of mathematics teacher knowledge, in particular, the multidimensionality of SMK and PCK. Although the distinction between SMK and PCK is ambitious, several scholars take the view that the two categories, and, in particular, their corresponding subcategories, are useful tools in describing teachers' knowledge for research purposes and particularly in devising teachers' professional development programs.

MOVING AWAY FROM SHULMAN'S ORIGINAL CONCEPTUALIZATION

The above mentioned contributions present strong cases that progress can and has been made in the conceptualization of teachers' knowledge. As mentioned above, several scholars have particularly reformulated the concept of PCK, by refining sub-dimensions or identifying dimensions of teachers' knowledge and adding them to the construct of PCK. Thus, it can be seen that researchers have assimilated the notion of PCK and redefined it according to their beliefs or to findings from empirical studies. Although the mentioned studies represent reformulations of the concept of PCK, Shulman's conceptualization of PCK was still the theoretical starting point for these studies. In this process of further refinement and extension, however, researchers' understanding and interpretation of PCK have moved away from Shulman's original conceptualization. For instance, the concept of PCK has almost lost its most important characteristic, namely its topic specificity (Hashweh, 2005). PCK, according to Shulman's definition, is not only specifically related to topics within certain disciplines, but also research on PCK typically does not result in a description of 'expert teaching' as if there would be one optimal way to teach certain subject matter (see, Shulman, 1987). From the author's perspective, recent research on mathematics teachers' knowledge tend to ignore the complex nature of PCK as a form of teachers' professional knowledge that is highly topic, person, and situation specific (for overviews see, e.g., Abell, 2007; Van Driel & Berry, 2010).

A NARROW FOCUS ON THE DISCIPLINE

Many in the field of teacher education today take Shulman's conceptualization of the knowledge base for teaching for granted – accepting the view of pedagogical content knowledge (PCK) as an adaption of subject matter knowledge for the teaching enterprise, a process Shulman (1987) called *transformation*. However, with restricting PCK to the capacity to transform the subject matter of the discipline to subject matter of the school subject, Shulman places the subject matter content at the centre of conceptualizing the knowledge base for teaching. As a consequence, past and recent research on mathematics teachers' knowledge limited their focus on teachers' *unpacking of mathematics content* in ways accessible to their students. In doing so, the attention is focused entirely on the discipline. However, in being more consistent with a constructivist view of learning, the emphasis needs to be shifted from knowledge of the discipline to knowledge about how students' knowing and learning actually progresses. Thus, a reconceptualization of the knowledge base for teaching mathematics is needed toward a theory of teaching grounded in research on students' learning.

FROM REFINEMENT TO REORGANIZATION: TURNING THE REFINEMENTS ON THEIR HEADS

We have learned a great deal of the necessity for refining Shulman's initial work toward more specific descriptions of the knowledge base for teaching mathematics. Whereas it was important to initially identify and define various sub-dimensions of SMK and PCK and making progress in obtaining empirical evidence to support each piece of the puzzle, interpreting them in light of a model of cognition and learning certain subject matter may allow for the integration of the various pieces into one framework for mathematics teachers' knowledge. Thus, the time has come to move from further refining to reorganizing sub-dimensions of teachers' knowledge. As indicated above, the various refinements of PCK seem to converge in three domains, namely (1) *knowledge of students' understandings* (KSU), (2) *knowledge of learning mathematics* (KLM), and (3) *knowledge of teaching mathematics* (KTM). KSU refers to a cognitive perspective, KLM to an epistemological perspective, and KTM to a didactical perspective on this issue. In this work, knowledge of students' understanding (KSU), knowledge of learning mathematics (KLM), and knowledge of teaching mathematics (KTM), together with mathematical content knowledge per se (MCK per se) and mathematical content knowledge for teaching (MCK for teaching) *build* the knowledge bases that constitute the particular kind of knowledge that is considered as specialized for the purposes of teaching mathematics. In doing so, past and current approaches in research on mathematics teachers' knowledge are turned on their heads in the sense of taking the identified (and refined) knowledge dimensions as *building blocks* for the construct of 'knowledge for teaching mathematics'.

GOING BEYOND WHAT TEACHERS' KNOWLEDGE IS ABOUT: A WINDOW TO A STRUCTURAL DESCRIPTION OF TEACHERS' KNOWLEDGE

While the subcategories of mathematics teachers' knowledge identified in the above mentioned contributions are crucial pieces of the puzzle, we have not learned how these pieces fit together. In the past, the primarily focus was on what knowledge is held by teachers, and how that knowledge is used in practice. It seems that, with few exceptions, the literature has limited its focus on the content teachers do or should possess. However, a key theoretical concern arising in the realm of theorizing and conceptualizing mathematics teachers' knowledge is the question on how the knowledge is structured and organized. To put it in other words, what is missing in the current landscape of the conceptualization of mathematics teachers' knowledge are attempts to go beyond what the teachers' knowledge is about to include a *structural description* of teachers' professional knowledge. Drawing on the 'knowledge in pieces' framework developed by diSessa (e.g., 1993), Scheiner (2014) proposes to consider teachers' professional knowledge as a complex system of 'knowledge atoms'.

> 'Knowledge for teaching mathematics' is considered as the repertoire of 'knowledge atoms' that have been transformed along (1) knowledge of students' mathematical understandings (KSU), (2) knowledge of learning mathematics (KLM), and (3) knowledge of teaching mathematics (KTM), taking (4) mathematical content knowledge per se (MCK per se) and (5) mathematical content knowledge for teaching (MCK for teaching) as the cornerstones. (Scheiner, 2014, in press)

With this perspective, several angles for theoretical reflection on the *nature* and *form* of teachers' knowledge are presented, including those concerning the degree of integration, size, specificity, and source of teachers' knowledge. The notion of 'transformation', for instance, indicates that the constituent knowledge bases are inextricably combined into a new form of knowledge that is more powerful than the sum of its parts (concerning *degree of integration*), while the notion of 'knowledge atom' indicates that knowledge is of a microstructure, highly context-sensitive, and concept-specific and has to be considered as of a fine-grained size (concerning *size* and *specificity*). Notice that in contrast to Shulman and his proponents' work taking content knowledge and pedagogical knowledge as the constituent knowledge bases for teaching, it is KSU, KLM, and KTM, together with MCK per se and MCK for teaching that build the constituent knowledge bases for teaching mathematics (concerning *source*). A more detailed elaboration of first attempts towards a structural description of teachers' knowledge can be found in Scheiner (in press).

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Characteristics of out-of-field teaching: Teacher beliefs and competencies

Sven Schueler[1], Bettina Roesken-Winter[1], Jochen Weißenrieder[1], Anselm Lambert[2] and Matthias Römer[2]

1 Humboldt-Universität zu Berlin, Berlin, Germany, sven.schueler@hu-berlin.de

2 Universität des Saarlandes, Saarbrücken, Germany

In recent years, some efforts have been made to consolidate out-of-field-teaching as research field in mathematics education. Taking teachers' professional knowledge as reference frame, out-of-field-teachers seem less qualified regarding CK and PCK in mathematics than teachers that were especially trained to teach their subject. In this paper, we approach the phenomenon of out-of-field-teaching by focusing on both teachers' beliefs and competencies while the latter is operationalized as skills in designing mathematical tasks for written exams. Data was collected throughout a qualification program particularly focusing on out-of-field-teachers' domain-specific professional knowledge. We discuss in detail design aspects of our on-going research and give first insights into the changes of out-of-field-teachers' beliefs reflecting their professional development in the course of the qualification program.

Keywords: Out-of-field-teaching, beliefs, competencies, CPD.

INTRODUCTION

Current developments in mathematics education show an interesting phenomenon: A considerable body of mathematics lessons is taught by teachers who have not been qualified as mathematics teachers through certified courses of studies at university (cf. Törner & Törner, 2010). In the following, we refer to this heterogeneous group of teachers by using the term "out-of-field". Due to a thin research base, there are barely information about the range of this phenomenon which strongly varies across different countries, educational systems and school types. In the U.S. for instance some efforts by Ingersoll (1999) have been made to estimate the amount of out-of-field-teaching in a large-scale study which reveals that up to one third of all high school mathematics teachers do not hold a teaching certificate in mathematics. For Germany, Törner and Törner (2010) state that almost 80% of primary mathematics lessons are taught by teachers who have not taken any mathematics courses during their professional education. Although, in higher education the average percentage of out-of-field-teachers in mathematics education decreases, it still remains on an estimated level of 15% referring to lower secondary grades (cf. Bosse & Törner, 2013). Against this backdrop, the issue of out-of-field-teaching is *undertheorized* and *underresearched* in reference to crucial aspects that characterize out-of-field-teachers' professional knowledge and practices. Within this contribution, we discuss an initial approach to address this lack of research.

Considering out-of-field-teaching in the light of models of teachers' professional competencies (cf. Blömeke, Suhl, & Döhrmann, 2012) we encounter a twofold problematic scenario: Due to missing courses of studies it can be assumed that – put carefully – out-of-field-teachers are faced with considerable knowledge-gaps concerning different facets of their professional competence. These gaps in turn affect their capability to act effectively in the classroom, to provide high quality mathematics lessons, and to support sustainably students' performances (Richter, Kuhl, Haag, & Pant, 2013). In our study, we thus put emphasis on key-aspects of out-of-field teachers' professional knowledge to understand which categories are decisive for describing their specific situation. Reflecting on research about out-of-field-teachers' professional identity we additionally focus on teachers' beliefs about mathematics and the teaching and learning of mathematics, as these are crucial parameters considering teachers' decision-making and lesson practice (cf. Bosse & Törner, 2012; Hobbs, 2012). In our study we address this aspect by focusing on out-of-filed-teachers' skills to design mathematical tasks for written exams. In sum, we will consider the issue of out-of-field-teaching from two perspectives:

(1) First, we outline some key aspects of the *theoretical* foundations, in particular on mathematics-related beliefs and on designing mathematical tasks as one crucial aspect of teachers' competencies.

(2) Second, we discuss in detail our *methodological* approach using mixed methods to capture teachers' beliefs and their competencies, displayed by designing mathematical tasks. As core element of this perspective we describe and validate a category system which has been developed to characterize out-of-field-teachers' decisions while choosing mathematical tasks for class assessment.

In this paper, we give preliminary results of our on-going research while accompanying a group of out-of-field teachers throughout a one-year qualification program.

THE ROLE OF BELIEFS AND COMPETENCIES

Teachers' professional knowledge has been researched in depth and from various perspectives considering both cognitive and affective-motivational aspects (cf., Shulman, 1986). Drawing on key findings of these studies teachers' professional knowledge is conceptualised as interplay of *Content Knowledge (CK), Pedagogical Content Knowledge (PCK), General Pedagogical Knowledge (PK), professional motivation, beliefs* and *self-regulation*. Large-scale empirical studies capture the interplay of these cognitive facets and affective-motivational characteristics and underpin efforts to capture classroom practices with regard to both dimensions (cf. Blömeke et al., 2012). Ball and Bass (2000) further work on a domain-specific conceptualization when focussing on mathematical knowledge needed for teaching. They elaborate on teachers' knowledge of mathematics as the decisive parameter for improving their instructional quality. One promising attempt to conceptualize teachers' professional knowledge from a more situative perspective is provided by Lindmeier and colleagues (2013). In particular, she stresses that a subject-specific model for teacher cognition encompasses three components: *basic knowledge* (CK, PCK), and the two complementary components *reflective competences* and *action-related competences*. In particular, the action-related competencies comprise the abilities needed to perform in the classroom. What is more, Lindmeier and colleagues (2013) stress that for both reflective and action-related components, basic knowledge plays a key role for enactment.

Considering in addition the role of teachers' beliefs, one can conclude that these play a key role for decision-making in the classroom (Törner, Rolka, Roesken, & Sriraman, 2010). Our research refers to the dimensions of beliefs presented by Grigutsch, Raatz, & Törner (1988): *beliefs about the nature of mathematics*, *the teaching and learning of mathematics* and *students' mathematics achievement*. Beliefs are often robust and therefore difficult to change or as Sowder (2007) puts it "many of teachers' core beliefs need to be challenged before change can occur" (p. 160). What this quotation stresses is that any change or development in teachers' beliefs is a long-term process. Accordingly, Toerner, Rolka, Roesken and Schoenfeld (2006) analyse the teaching practice of an experienced teacher after having participated in an in-service training course on using open-ended task in mathematics teaching. Since it was not the focus of the study to examine the effectiveness of the professional development event, it turned out that the teacher's beliefs built a hindrance to successfully implementing new ideas. Nevertheless, other studies report about quick changes in beliefs while teachers participated in a professional development program.

DZLM QUALIFICATION PROGRAM: PROFFUNT

The German Centre of Mathematics Education offers various qualification programs and training courses for multipliers and out-of-field-teachers in order to foster their *Continuous Professional Development* (CPD). The project ProFFunt[1] is a certification course especially designed to support out-of-field teachers in lower secondary school, and is a collaborative project of the Universität des Saarlandes, of the Landesinstitut für Pädagogik und Medien of the Federal State of Saarland and the DZLM. The course has lasted one year, and addressed teaching in grade five and six, and will be extended to grades seven and eight in the next year. ProFFunt draws on the success-

1 ProFFunt = „Professionalisierung fachfremd Unterrichtender"

ful KOSINUS program2 that so far has reached more than half of the respective schools in the Saarland. Support for teachers for probing issues in practice is provided by the chair of mathematics and its didactics.

The ProFFunt project especially focuses on the development of out-of-field teachers' competencies which are considered as being decisive in regards of teachers' professional knowledge (cf. Blömeke et al., 2012). The content was developed throughout analyses based on *Stoffdidaktik* including relevant mathematics topics in grade five and six such as algebra, geometry and basic ideas in stochastics. Throughout the ProFFunt project teachers additionally received a profound overview on PCK issues and were supported in implementing these aspects into their teaching, ranging from task design to planning of teaching sequences. Teachers were required to participate in teams of two (tandem) in order to foster collaboration in their respective school. Furthermore, the course consisted of six modules with a total workload of 200 hours. ProFFunt follows a so-called *sandwich-structure* with alternating theoretical and practical phases in which the participants experience a combination of learning, implementation and reflecting phases, comprising also e-learning and working on a portfolio. The intention of this structural design is to foster long-term changes in teachers' views on mathematics learning and to enable sustainable competence development. In this regard, it appears promising to stress teachers' beliefs and action related-competencies, as these are long-term developing aspects of teachers' professional knowledge.

2 http://didaktik-der-mathematik.de/pdf/gdm-mitteilungen-90.pdf, http://www.saarland.de/114409.htm

RESEARCH ON PROFFUNT

A broad research plan has been developed in order to cover various aspects of out-of-field teachers' professional knowledge. The research design acknowledges the researchers are not involved in conducting the course and that out-of-field teachers' development should not be bothered by intensive testing. An overview on the research design is given in Figure 1 and is briefly explained hereafter.

Our approach is twofold and addresses teachers' beliefs and competencies: On the first level we focus on out-of-field-teachers' development of beliefs throughout the qualification program. On the second level we scrutinize teachers' competencies in designing mathematical tasks for written exams. Both constructs are investigated in a pre-post-comparison design referring to the period before and after the ProFFunt course. In addition, we evaluate our results in view of a control group of teachers who studied mathematics as a school subject at university so that we are able to describe and understand crucial differences between *in-field* and out-of-field-teachers. Finally, the connection between out-of-field teachers' beliefs and the way of designing mathematical tasks before and after the qualification program is considered in order to check for correlation between these two categories (Figure 1). In particular we focus on the following research questions:

a) What categories do out-of-field-teachers consider in designing mathematical tasks for written exams? What influence has a one-year qualification program on the development out-of-field of teachers' competencies in designing tasks?

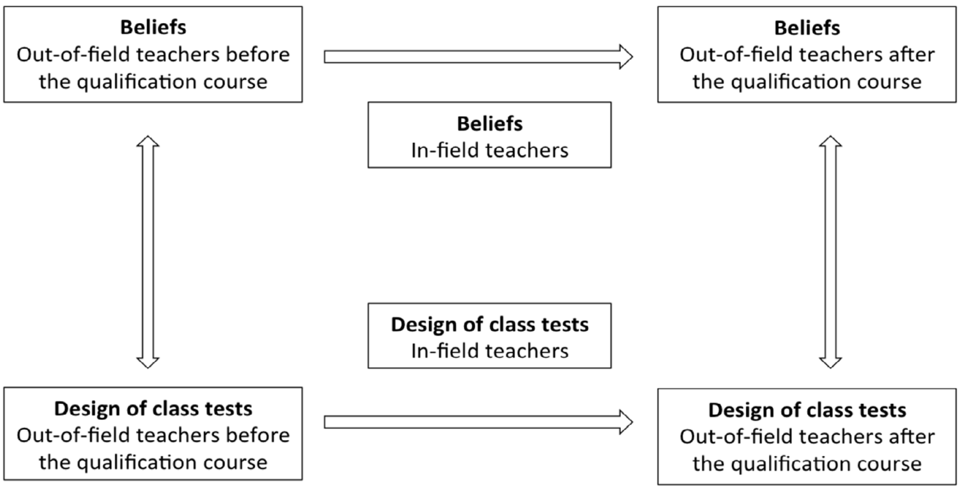

Figure 1: Overview on the research plan

b) What mathematics-related beliefs do out-of-field-teachers possess? What influence has a one-year qualification program on the development of out-of-field teachers' beliefs?

c) What differences between out-of-field-teachers and in-field teachers could be detected with reference to both constructs beliefs and action-related competencies?

In this paper, we do not fully answer our research questions, but outline our research design, carefully describe our instruments and present preliminary results.

METHOD

In the one-year ProFFunt course, 13 out-of-field-teachers participated, representing a highly heterogeneous group of teachers which non-mathematics educational practice varies from less than 5 years to up to 15 years. Working as a teacher within the educational system in Germany requires studies in CK and PCK in at least two different subjects complemented by pedagogical courses. Hence, participants of our study have a considerable body of previous knowledge concerning pedagogical aspects like for example learning theories and implementation strategies, but established in a different context than mathematics. Combined with the fact that the participants' knowledge differs in the specific subjects they have studied, these circumstances increase the heterogeneity of the sample and makes working with the group of out-of-field-teachers in terms of CPD-training and research challenging. As a control group we refer to data raised within the scope of the MT21 research program (Blömeke, Kaiser, & Lehmann, 2008), where prospective and in-service teachers (N=139) who studied mathematics at university participated.

Data was collected through combining qualitative and quantitative methods to capture the complexity of out-of-field-teachers' specific situation. Their mathematics-related beliefs have been revealed by using a questionnaire which has been developed and validated in the scope of the TEDS-M study (cf. Blömeke, Suhl, & Kaiser, 2011). This instrument displays a shortened and slightly modified version of the items originally developed by Grigutsch and colleagues (1988). The questionnaire was distributed to the participants before and after the DZLM qualification program. The questionnaire consists of 33 items that are rated on a six-point Likert-Scale, ranging from strongly disagree to strongly agree. Considering various aspects of teachers' beliefs the instrument encompasses the following five subscales: *nature of mathematics as rules and procedures* (6 items), *nature of mathematics as process of inquiry* (5 items), *learning mathematics through teacher direction* (8 items), *learning mathematics through active learning* (6 items) and *mathematic achievement as fixed ability* (8 items). A short excerpt from this questionnaire is shown in Figure 2. Applying the TEDS-M belief questionnaire ensures a stable instrument with sufficient scale reliability.

Focusing on designing mathematical tasks for written exams as a core aspect of teachers' action-related competencies, we developed a category system which refers to key findings from current research in mathematics education. Throughout the qualification program, participants of our study were asked to allocate and provide the written exams they delivered in grade five. Concerning content, the written exams mainly focussed on arithmetic and geometry in grade 5. As a result of this procedure we are able to gain theory-based insights into crucial aspects of out-of-field-teachers' professional knowledge. In the developmental process of the category system we brought together various findings from research

	Strongly disagree	Disagree	Slightly disagree	Slightly agree	Agree	Strongly agree
Mathematics is a collection of rules and procedures that prescribe how to solve a problem.	☐	☐	☐	☐	☐	☐
In mathematics many things can be discovered and tried out by oneself.	☐	☐	☐	☐	☐	☐

Figure 2: Exemplary items of the belief questionnaire

on mathematics education and pedagogical theory concerning learning and achievement through tasks. The instrument consists of four domains reflecting key-aspects of teachers' professional knowledge:

(OCS): *Orientation toward task-related competencies and educational standards,*

(OPK): *Orientation towards task-related PCK*

(OCP): *Orientation towards task-related cognitive processes*

(ODC): *Orientation towards task-related difficulty and complexity*

For the sake of brevity, we do not discuss every domain in detail, and limit ourselves to shortly describing our methodical procedure for analysing mathematical tasks. In the first domain (OCS) we focus on three categories referring to the learning standards in mathematics education in Germany (cf. Blum, 2006). In particular, we regard the design of a task from the perspective of content-related and process-related competences. An example of categories used in the domain OCS is shown in Figure 3. Categories located in OPK deal with key aspects of pedagogical content knowledge in mathematics. In the context of our research, we choose 4 categories in this domain: First, we take a look on how the structure of the task motivates students' learning processes (cf. Bruder et al., 2008), second we concentrate on which strategies in designing a task are used to foster students' active engagement in a comprehensive learning process (cf. Büchter & Leuders, 2005). While the third category in this domain emphasises the form of representation, the fourth category considers linguistic aspects of the task (cf. Meyer & Prediger, 2012). Reflecting on students' cognitive processes working on a mathematical task in the domain OCP, we include categories dealing with Blooms taxonomy of learning domains (cf. Bloom, 1976) to cover a more pedagogical perspective. The second category in this domain deals with the question what level of mathematics-related cognitive processes like generalising or using formal and abstract expressions are required for solving the task (cf. Cohors-Fresenborg, Sjuts, & Sommer, 2004). In the last domain ODC we include several complementary categories covering key aspects of difficulty and complexity of a mathematical task (cf. Bruder et al., 2008). In particular we focus on students' required time to solve the task, amount of steps to provide a correct solution, the response format, and finally mathematical correctness.

Three experts in mathematics education from our research team independently used the developed category system to analyse 10 exams (60 tasks) in order to check the quality and sensitivity of our instrument. The degree of agreement was calculated by estimating the inter-rater reliability in terms of Cohen's kappa for each category. In regard of the constructed instrument we estimated Cohen's kappa between 0.72 and 0.95 for tasks which shows satisfying agreement (Hallgren, 2012).

PRELIMINARY RESULTS AND DISCUSSION

Reflecting on our broad research plan, we will limit ourselves to preliminary results respecting the development of out-of-field-teachers' beliefs before and after participating in the ProFFunt course, in comparison to a group of "in-field" mathematics teachers. In addition, we would like to stress several aspects of our research questions, to draw some conclusions concerning our methodical approach and to anticipate expectable results of our on-going research process.

First, we concentrate on the results derived by the TEDS-M beliefs questionnaire. An overview showing means and standard deviation of out-of-field teachers' beliefs on the five subcategories before and after

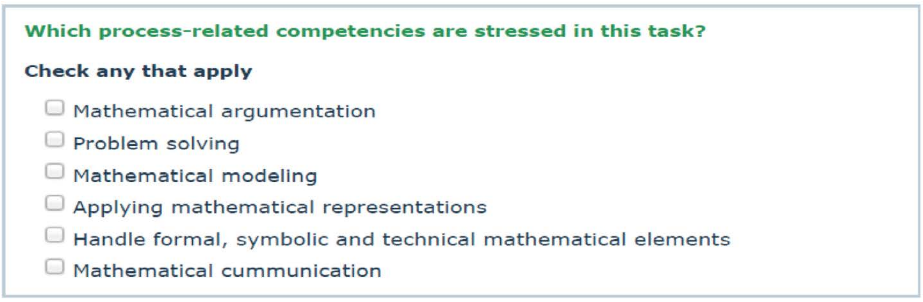

Figure 3: Exemplary item of the domain OCS of the category system

participating in the qualification program is given in Table 1. Considering beliefs about the nature of mathematics it becomes apparent that the participants are strongly oriented towards the *nature of mathematics as process of inquiry* at the beginning and even more at the end of the ProFFunt course, showing a small effect size. On the contrary the *nature of mathematics as rules and procedures* is rated less frequently before the course and decreases afterwards, showing a medium effect size. With reference to the beliefs about learning mathematics we can find that out-of-field-teachers in this study before and after the ProFFunt course lay a strong emphasis on *learning mathematics through active learning* whereas *learning mathematics through teacher direction* is considered minor relevant. In addition, *mathematics achievement as fixed ability* is rated low before and after the qualification program. Our results support the assumed changeability of out-of-field teachers' mathematics-related beliefs after participating in a qualification program.

In comparison to the sample of "in-field" teachers derived from the MT21 study, the group of out-of-field teachers in ProFFunt shows minor differences when it comes to their mathematics-related beliefs (Table 2). Out-of-field teachers consider *mathematics learning through teacher direction* as minor relevant compared to "in-field" teachers.

These findings show an interesting phenomenon: Despite the fact, that being out-of-field in most cases comes along with considerable knowledge gaps in CK and PCK, participants in this study strongly emphasize mathematics' dynamic character and its potential as source of effective and active learning opportunities at the start of the qualification program. These results are remarkable because they contradict the popular assumptions that out-of-field-teachers feel less competent in mathematics and teaching mathematics and therefore preferably refer to structures, rules and procedures. We take these surprising aspects in our findings to contemplate on one essential issue: Who are we talking about when labelling

Dimension of beliefs		pre	post	effect size
Nature of mathematics as rules and procedures	Mean StD	3.8 .81	3.4 .89	-0.29
Nature of mathematics as process of inquiry	Mean StD	5.0 .43	5.3 .61	0.11
Learning mathematics through teacher direction	Mean StD	2.4 .75	2.5 .68	0.05
Learning mathematics through active learning	Mean StD	5.1 .49	5.2 .60	0.01
Mathematic achievement as fixed ability	Mean StD	2.5 .70	2.5 .82	-0.08

Table 1: Out-of-field-teachers beliefs before and after the qualification program

Dimension of beliefs		Sample (n=11)	MT21-Germany (n=139)
Nature of mathematics as rules and procedures	Mean StD	4.0 .67	3.9 .98
Nature of mathematics as process of inquiry	Mean StD	5.1 .48	4.9 .87
Learning mathematics through teacher direction	Mean StD	2.4 .74	3.2 .88
Learning mathematics through active learning	Mean StD	5.2 .41	5.3 .71
Mathematic achievement as fixed ability	Mean StD	2.6 .68	2.2 .82

Table 2: Out-of-field-teachers and in-field-teachers mathematic-related beliefs

teachers as being "out-of-field"? What is missing is a concise definition who out-of-field teachers are and what out-of-field teaching is about. In the light of these thoughts our paper enables an innovative approach to characterize key aspects of out-of-field-teachers' professional knowledge on the basis of cognitive and affective competencies. In our further research we take up these findings and compare our results on out-of-field teachers' beliefs to their action-related competences. The category system described and validated in this paper provides a sustainable basis concerning the approach to characterize the issue of out-of-field-teaching more precise.

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Contradictions and shifts in teaching with a new curriculum: The role of mathematics

Konstantinos Stouraitis[1], Despina Potari[1] and Jeppe Skott[2, 3]

1 University of Athens, Athens, Greece, kstouraitis@math.uoa.gr

2 Linnaeus University, Kalmar, Sweden

3 Aarhus University, Aarhus, Denmark

In the framework of Activity Theory (AT), contradictions are sources of change and development. Borrowing concepts from AT, we attempt an interpretation of identified contradictions in a collaborative context where teachers plan and evaluate their teaching in the process of enacting a new curriculum. We examine the connection between contradictions and shifts in teaching activity, with a special focus on the mathematical character of these contradictions. We claim that dialectical oppositions lying in the background of these contradictions promote teachers to broaden their teaching activity by embedding into it new mathematical and pedagogical possibilities.

Keywords: Activity theory, contradictions, curriculum, teacher choices.

INTRODUCTION

A new, reform oriented curriculum was introduced and piloted in a small number of schools in Greece. During the year 2012–13 we collaborated with the teachers of three of these schools supporting them to enact the new curriculum in their classrooms. This collaboration was taking place in group meetings at the school where the teachers worked. During the meetings they planned and evaluated their teaching while the first author, who was also a member of the team that developed the curriculum, supported them by providing explanations about the rationale of the curriculum as well as teaching resources. Our main research goal is to understand the teachers' decisions and choices in relation to the curriculum documents and resources and the factors that framed them. Analysing the data we found that a variety of contradictions appeared to trigger discussions in most cases. Drawing on AT we explored further the possible role of these contradictions for teachers' choices and the development of the teaching activity. In general, the development of mathematics teaching has been studied in the context of planned interventions aiming to teachers' professional development. Few studies though investigate teachers' professional learning and the shift of their teaching activity in contexts where learning is not promoted by an expert. Some of them refer to teachers' learning through reflection on their own teaching (Chapman & Heater, 2010) or to joint reflection in collaborative contexts (Potari, Sakonidis, Chatzigoula, & Manaridis, 2010). In these contexts professional learning is a complex and long process framed by many different factors and conditions. Our study attempts to contribute in understanding of this complexity by using contradictions as a tool for our analysis.

In this paper, we refer to the emerged contradictions "as sources of change and development" (Engeström, 2001, p. 137). Especially, we focus on contradictions that seem to challenge teachers' choices and on possible shifts in their teaching activity.

THEORETICAL FRAMEWORK

Current research in mathematics education recognizes that in the context of reform teachers are not expected to implement a predefined set of methods in their classroom, because there are not such methods. On the contrary, teachers are required to play a substantial role as a link between the curricular and other reform priorities and their classroom (Skott, 2004). This requirement, which Skott calls "forced autonomy", brings the teacher at the center of the curriculum enactment but also creates new challenges and conflicts. Within this perspective, teachers are not considered as mere transmitters of a curriculum formulated by some experts outside their classroom, but as active agents and designers. Teachers' instructional actions are influenced by curricular materials but also shaped by their interactions with the students in the classroom (Remillard, 2005).

Analyzing data through a grounded theory approach we found ourselves trying to understand and interpret the emerged contradictions and we turned to AT for this purpose. AT is trying to capture the complexity of teaching, integrating dialectically the individual and the social – collective and focusing on the activity of the subject. The activity is driven by subject's motivation and directed towards an object (Leont'ev, 1978). The unit of analysis in this context is the activity system (AS) that incorporates social factors that frame the relations between the subject and the object with the mediation of tools. These factors are related to the communities in which the subject acts, the rules of these communities and the division of labor (Engeström, 2001).

In the present study we consider the activity of the participating teachers to be the teaching of mathematics in the context of introducing a new set of curricular materials. As the subject of this activity we see teachers as a group and as individuals. A main object and motive of the teachers' activity is their students' mathematical learning, in combination with other professional obligations such as implementing the mandates of the educational authorities. Their activity is mediated by tools such as curricular documents, school textbooks and other teaching-learning materials, instructional strategies, and lesson plans. They constantly balance school community, students' and parents' communities, mathematics teachers' community and other communities that influence the teaching activity. Teachers' activity is framed by rules such as institutional commitments (e.g., examinations, time constraints, timetables) or traits of mathematics as discipline and as a school subject. The division of labor refers to the teacher's role in the classroom or in the school and to the distribution of classrooms among the mathematics teachers in their school.

One of the fundamental characteristics of every AS is contradictions. They emerge when an AS adopts new elements from the outside, such as a new tool or a new rule, causing a conflict with the old elements. Contradictions are neither everyday solvable problems nor temporary conflicts that may easily be overcome. Moreover, the term "contradiction" in AT has not at all the meaning of a logical contradiction. "Contradictions are historically accumulating structural tensions within and between activity systems" (Engeström, 2001, p. 137). Roth and Radford (2011) refer to a special type of contradictions as "inner contradictions" to describe the often mutually exclusive aspects of the same phenomenon that coexist dialectically and "cannot be removed". Contradictions create learning opportunities for the subject and may broaden its activity to a wider horizon of possibilities (Engeström, 2001; Potari, 2013).

There is an increasing amount of research literature about contradictions in mathematics education. Some of them use the concepts of AT to identify, describe and interpret contradictions in teaching (for example, Barab, Barnett, Yamagata-Lynch, Squire, & Keating, 2002; Jaworski & Potari, 2009) and in teachers' professional development (Potari, 2013). In these studies contradictions refer mainly to pedagogical or professional issues, paying less attention to mathematical and epistemological ones. Another dimension of this research is related to the use of contradictions to stimulate expansive learning in developmental interventions with groups of teachers (Engeström, 1994; Jaworski & Goodchild, 2006). Our initial research goal although developmental, was not based on stimulating contradictions as the ground of expansive learning. So, our view in this paper related to expansive learning is restricted in a snapshot of what is called expansive cycle. However, we adopt Engeström's position that professional learning often is "something that is not stable, not even defined or understood ahead of time" and "there is no competent teacher" who knows what must be learned (Engeström, 2001, p. 137–138).

In this study we understand contradictions in two ways. First, as conflicting elements of the teaching-learning activity. Such is the contradiction between the tools and the rules of the activity (e.g., between the choice of computer based instructional tools and time restrictions). Second, we identify contradictions as conflicting opinions, practices or choices between two teachers or a teacher and some external agent (e.g., the curriculum, the students, etc.). An example of the latter is the contradiction between the use of tasks that require conceptual understanding of divisibility (promoted by the new curriculum) and, the use of tasks that can be solved using keywords such as "less than" or "at least" that indicate LCM or GCF.

Some of the emerging contradictions are characterized by a dialectical opposition. Here, as dialectical opposition we consider some opposing aspects of a mathematical concept or of how it is transformed in teaching. Often, these opposing aspects are complementary, they can't be separated, and both constitute the concept. For example, the distributive property encompasses two opposing but complementary uses: it can be used to transform a product to a sum or a sum to a product. In our analysis such oppositions appeared in some cases underlying a contradiction. Dialectical oppositions of this kind allow us to consider more deeply in our analysis the mathematical dimensions (e.g., content, processes) of teaching.

METHODOLOGY

The research was conducted during the pilot implementation of the new curriculum in three junior high schools (grades 7–9). The new curriculum emphasizes students' mathematical activity that promotes mathematical reasoning and argumentation, connections within and outside mathematics, communication through the use of tools and metacognitive awareness. It also attributes a central responsibility for the teacher in the process of designing teaching. In this study, the mathematics teachers at each participating school (school A, B and C) worked together to enact the new curriculum with the support of the first author. The main tasks undertaken in the groups were planning lessons and reflecting on their experiences with teaching some modules of the designed curriculum in the classroom.

For this paper data consist of transcriptions of audiotaped conversations and written documents (worksheets, lesson plans) from 8 meetings with the 5 mathematics teachers of school A. This school is an experimental and model school where participating teachers have long teaching experience and are familiar with educational innovations. In general, the school culture is characterized by an innovative spirit. In this paper, we refer to two teachers, Marina and Linda, with over 25 years of teaching experience, with extra qualifications (both have masters degree, Marina in mathematics and Linda in mathematics education) and both with experience of innovative teaching approaches in their classrooms. In the past, both had participated in teacher collaborative groups developing classroom materials or writing papers for mathematics teacher journals and conferences. In general, Marina was more informed than Linda about the research activities of the mathematics education community in Greece. Both had a critical stance to innovations in general, adopting some of them and rejecting others and had strong views about their teaching choices. Concerning the new curriculum, in an interview at the beginning of this research study they had said that it came as a legitimizing umbrella over their practice.

The transcribed conversations were analyzed with grounded theory methods (Charmaz, 2006). The written documents were used to exemplify the conversations. The initial open coding resulted in the identification of discussion themes for each meeting, forming thematic units. In each unit teachers' choices, their rationale and emerging contradictions were identified. As indicators of a contradiction were disagreements among the participants or between the participants and an external source. Each identified contradiction, was formulated as a dichotomy (e.g., the choice of tasks aimed at conceptual understanding or at procedural fluency). For every identified contradiction we used descriptive codes related to its content, the agents (e.g., a contradiction between participants and the curriculum) and teachers' awareness (whether or not they recognise the contradiction). Then, contradictions were categorized and traced through data for possible effects on teaching.

RESULTS

The content of the identified contradictions concerned issues such as: teaching planning and strategies, students' activity and difficulties, institutional constraints, teacher collaboration, classroom management and epistemological issues. Data analysis showed that in some cases contradictions led teachers to question their own teaching and start transforming it. Below we elaborate two of these cases as exemplars of shifts in teaching activity. The first exemplar underlies a dialectical opposition in the teaching of a mathematical property, while the second shows an epistemological opposition concerning the validation of school mathematical knowledge.

Contradicting goals and dialectical oppositions in teaching the distributive law

In the second meeting at school A (A2, turn 138) Marina described her teaching plans for algebraic transformations in grade 9 (operations with polynomials, identities and factorization). Starting from her observations on students' mathematical activity she said that in the expression $3 \cdot (a + b + c)$ "children see completely different things than us ... they see addition and multiplication [while] we see ... a product with two factors". Seeking to obtain "a common language" with the children about the structure of algebraic expressions Marina decided to emphasize this issue in her teaching. She designed to teach multiplication of polynomials in parallel to the factorization of a polynomial. In the meeting Marina presented the worksheet she used in the classroom and described that she divided the blackboard in two parts, with the expression $3 \cdot (a + b + c) =$ on the left and the $3a + 3b + 3c =$ on the right. With this approach she hoped to make clearer to the students that the use of distributive property depends on the structure of the expression we have and the structure we want to get.

At the next meeting (A3, turns 105–114) Marina said that she used this approach in the teaching of algebraic identities and she expressed her satisfaction commenting:

> I think ... they have understood better that this way or the other is in fact about polynomials' operations. Until now identities have been presented as something to be learned by heart, and it was a special thing, completely away from the other operations, as well as the factorization was ... say these things were not connected at all. And I think that their connection helped students to understand them and to use them in a flexible way. (A3, 113)

In this episode a contradiction comes to the fore: the teacher has the goal for the students to identify the structure of an algebraic expression as sum or product, while the students recognize only the operations that the expression calls them to do, a well-known problem in learning algebra (Sfard, 1991). In the background of this contradiction lays a dialectical opposition concerning the use of the mathematical object: distributive property may be used to transform a product to a sum or to factorize a sum to a product. Recognition of this contradiction by Marina was based on her past experience and observation of students' mathematical activity. However, it was triggered by the new curriculum, which called to emphasize the structure of algebraic expressions, and by the discussions in the group related to the new curriculum. Marina also recognized the dialectical opposition in the distributive property and based on this her teaching attempts to overcome the contradiction between her goals to emphasize the algebraic structure and her students' tendency to see the tasks operationally.

A "traditional" teaching approach leads to two separated readings of distributive property, attempting to plan teaching based once on the first (operations to get a sum) and then on the other (factorization to get a product). This is the structure of the school textbook, as there was not a new textbook in the philosophy of the new curriculum. This was also the approach of the other teachers in the group, as with different rationale everyone adopted the well known teaching sequence. Especially Linda explained her choice saying that she believed that students needed time to consolidate their knowledge in operations with polynomials. However, she also valued the recognition of the algebraic structure: "I also ask the students: is this a sum or a product?" (A3, 251)

So, Marina's instructional choice on this topic can be regarded as a change to what she was doing before and to what usually her colleagues and most mathematics teachers in Greece used to do. The new perspective that Marina adopted considers the two usages of distributive property as two dialectically opposite ways that need to become explicit to the students and, consequently, to highlight the structure of algebraic expression. This perspective, stemming from the recognition of the initial contradiction, can be considered as an indication of broadening the horizon of Marina's teaching activity, encompassing new possibilities to it. On the other hand, Linda recognized the same contradiction but she did not choose to change her teaching, following the mainstream approach.

Contradicting tools and dialectical oppositions in using geometrical transformations in teaching congruence

Although some elements of reflectional and rotational symmetry existed in the previous curriculum and in the textbook, the geometrical transformations, namely translation, reflection and rotation were introduced as a distinct topic in the new curriculum mainly in the 8th grade. The rationale of this introduction was connected with the development of students' spatial sense and with the value of transformations in tackling congruence and similarity. The topic emphasized the transformation of a figure as a whole supporting more intuitive and dynamic approaches to the geometric shapes and their properties. The focus was on the relationship between the two figures (original and image), highlighting the relation of congruence or similarity and attributing to the transformations the character of a proving tool (for further discussion on this relationship see Battista, 2007). Therefore, geometrical transformations constitute an alternative approach to the Euclidean perspective in school geometry indicating a different epistemology: the use of the moving figure as a proving tool is not compatible with the rigorous deductive rationale of Euclidean geometry.

In the discussion in the fourth meeting (A4), Marina referred to her introductory lesson on the congruence of triangles (grade 9) and she was pleased that in her question "how could we ascertain that these two triangles are congruent?" some students answered "if the triangles match after translation or reflection or rotation". She refers to Freudenthal's claim that Euclidean geometry is abused in school and she says that she is thinking to use tasks with geometrical transformations in teaching the congruence of triangles (A4, 132). However, she was questioning how this could be introduced in her teaching: "but there is a need of investigation and inquiry before doing so" (A4, 126); "I want them [the students] to understand that when we compare angles or segments or generally elements of polygons, we have two tools. One is the transformation and the other the criteria of triangle congruence" (A4, 136).

Linda listened to what Marina said and asked for clarifications. Finally, she commented that Marinas' thoughts were interesting but "every topic has its purpose". She did not criticize Marina's choices, but she claimed that "there is a purpose to learn to write, to observe the shape, to distinguish the given from the required, to make conclusions, and to prove ... [Congruence] has its meaning" (A4, 137).

In the next meeting (A5) Marina described the way that students of grade 9 worked with the congruence of triangles in parallel to geometrical transformations to prove the congruence of segments or angles. She argued that there are tasks that can show to the students when one approach is more appropriate than the other. For example, the task "the two triangles formed by three pairs of diametrically opposing points, are congruent" can be easily tackled by a 180° rotation, while the use of the criteria of triangle congruence is very complex (A5, turns 7, 9). On the basis of these special tasks, epistemological issues were also discussed in the meeting, about the rigor and the intuition of different approaches (A5, turns 11–15). Marina's descriptions show that her students used transformations as an alternative way to triangle congruence. This happened regularly in the classroom she had also taught in the previous year, but with more difficulty in a classroom she has been teaching only this year (A5, turn 7). Linda follows the discussion expressing positive opinions on Marina's strategies (turns 8, 10, 16, 20).

In the sixth meeting (A6) Marina said that in a test she asked her students to prove the congruence of two segments with two ways and many of them referred to rotation. Reflecting on her attempt to use transformations as an alternative to triangle congruence, Marina admitted:

> the introduction of transformations in the 8th grade gives you the opportunity to change the framework [of proving] in the 9th grade ... [for the students] to see that you can cope with the proof of geometrical properties with two strategies ... using transformations and the triangle congruence ... And it was done easily ... it came from the students. ... And I think it is very nice that for the first time it is given the possibility to get away from Euclidean geometry... (A6, turns 324–334)

In the 8th meeting (A8) Marina mentioned that some students used transformations in other topics, such as trigonometry, indicating that they used them as an operational tool to visualize and prove congruence. Reflecting on her favour for transformations, she mentioned a seminar on transformations she had attended three years ago and her experimental teaching in another school. Linda expressed her disagreement to such intertwining of different topics. She said: "I like transformations per se. I don't like overusing them later in congruence ... I don't find the reason to [do so]" (A8, turns 123,125)

In our interpretation, the discussions on transformations and triangle congruence reveal a contradiction between these two concepts as two different tools that a student can use to prove geometrical properties. In the background of this contradiction lies the epistemological difference between rigorous, deductive foundation of knowledge in Euclidean geometry and more intuitive, visual, dynamic aspects of geometrical transformations. We see this epistemological difference as a dialectical opposition, because the opposing aspects can be synthesized in a way that benefits students in grasping the concepts and properties of congruence. This is the intention of the new curriculum. Marina recognizes the contradiction between the two tools and their epistemological differences (as a dialectical opposition) although she does not use this terminology. This allows her to attempt a shift in her teaching by using the two tools in parallel and synthesizing them in the students' mathematical activity. This shift is a change in comparison with Marina's previous teaching and with the other teachers' teaching, for example, Linda's.

Any attempt to interpret Marina's shift, must incorporate social and cultural factors. Here we discuss some of them, trying to operationalize some concepts from AT. First of all is the obligation of implementing the mandated curriculum (rules) to which Marina's perspective was in accordance. The idea of using transformations in parallel to triangle congruence was triggered by students (community) on the basis of the curriculum philosophy (tools) and Marina's content knowledge and professional experiences (tools). The distribution of classrooms among the teachers (division of labor) led Marina to teach students she had taught transformations in the previous year. Marina's conversations in the group of teachers and with other mathematics teachers (community) helped her to clarify and identify her approach. The specific approach was consistent with the norms of classroom and the active role of students (division of labor). The aforementioned factors embed historical evolution, both in the teachers' biography and in the formation of the tools, rules and communities, but the space limits hinder any further reference on this.

Linda seemed to like the introduction of geometrical transformations by the new curriculum, and recognized them as a proving tool. However, she chose not to synthesize the two tools, pursuing the benefits of emphasizing the deductive approach of congruence in Euclidean geometry. Linda and Marina share similar perspectives about the new curriculum and similar experiences on teaching geometrical transformations. Both work in the same school with innovative culture and participate in the same collaborative group for planning and reflecting on teaching. The apparent differences can be possibly explained by the different tools they use (e.g., content knowledge on the topic of transformations) or the different communities they had participated). But what is making the difference in their activity are the different learning and teaching goals the two teachers set for their students concerning geometrical transformations.

DISCUSSION

Activity theory views contradictions as a prerequisite for the transformation of activity through an expansive cycle (Engeström, 2001). Here, we cannot follow the expansive cycle of the teaching activity but we observe some snapshots, some instances of "creative externalization ... in the form of discrete individual violations and innovations" (Cole & Engeström, 1993, p. 40). This is the way we see the shifts in Marina's teaching. In the two presented examples, dialectical oppositions of epistemological character underlie the identified contradictions. In the first example, the contradiction is between the teacher's goals and students' operational understandings while the dialectical opposition is in the use of distributive property. In the second, the two contradictory proving tools are underpinned by opposing epistemologies that can be dialectically synthesized.

Barab and colleagues (2001, p. 104) argue that "when systemic tensions are brought into a healthy balance they can facilitate a meaningful interplay that enriches and adds dynamism to the learning process". These claims highlight the dialectical dimension of tensions and contradictions that have emerged in our study. It appears from our analysis that recognizing the contradiction and deciding to incorporate both opposite aspects dialectically, has an effect on "broadening the horizon of the activity" (Engestrom, 2001; Potari, 2013) as in the case of Marina. The dialectical oppositions attribute mathematical and epistemological characteristics to the contradictions that can form the basis for a shift to the teaching activity. The above claims are in accordance with Chapman's and Heater's position (2010) that key issues on teacher change are: the experience of authentic tensions based on actual, personal classroom experiences, the willing to take ownership of the change and the acceptance of a degree of uncertainty. Although from the AT analysis we see commonalities in the social factors that frame both Linda's and Marina's teaching activity, we identified different teaching and learning goals for their students. In other words, their activity is motivated by different objects as "images of thought" (Leont'ev, 1978, p. 86), that is, they hold different anticipations about students' learning.

We don't know if the identified shifts in Marina's teaching will be sustained and if they can be expanded in the collective activity of mathematics teaching in Greek schools. Such an investigation requires long periods of time and different research methods. What we can claim from this study is that contradictions may be overcome in a dialectical way that challenges dichotomies between "effective" and "non effective" teaching towards a more dynamic view of teaching.

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Confluence of research and teaching: Case study of a mathematics teacher

Shikha Takker

TIFR, Homi Bhabha Centre for Science Education, Mumbai, India, shikha@hbcse.tifr.res.in

The complexity of research and teaching itself influences education in myriad ways. The insulation between research and teaching is counterproductive to the growth of mathematics education as a field in general and in developing a grounded understanding of teaching in particular. This study aims to understand teachers' knowledge about students' mathematical thinking in situ. Next, an attempt is made to enhance teachers' knowledge through contextually appropriate intervention using artefacts from teachers' practice. The collaboration between researcher and teacher to unpack students' thinking and make informed teaching decisions guided learning among students, teachers, and the researcher. This paper argues for the value of a situated approach to theories about teaching and also develops opportunities to support teacher learning from practice.

Keywords: Mathematics teacher learning, professional development, situated approach, teacher knowledge and practice, teacher-researcher collaboration.

INTRODUCTION

The gap between research and teaching has been widely acknowledged by researchers in mathematics education (Lerman, 1990). The insulation between the communities of teachers and researchers is unfortunate for there is a major overlap between their interests and focus. Concerns like understanding students' thinking, identifying and testing learning trajectories, trying out different teaching methodologies, focus on content and its connections, etc. are some of the research areas that are directly linked to teachers and teaching. In contrast to these, theorising about teacher knowledge, beliefs and practices has extensively been a domain of researchers or educationists. There has been lesser participation from teachers in unpacking their own knowledge and beliefs for that requires metacognitive awareness about their work. Teachers primarily focus on their *routines* of everyday classroom which leaves little time for them to reflect on what is being taught. Teachers make decisions about the content to be taught and ways in which this it can to be communicated to make it accessible for learners of specific age groups. This distinctive feature of teacher knowledge, choice of representations or mathematical tools along with the knowledge of students is described by Shulman (1986) as *pedagogical content knowledge* (PCK). In teacher education programmes in India, we are yet to think about ways in which PCK can be discussed and enhanced in teachers in ways that they become instrumental in their own learning from practice.

A framework that is particularly useful in identifying the relationship between teacher knowledge and practice for teacher learning is proposed by Cochran-Smith and Lytle (2000). They distinguish between knowledge *for* practice, knowledge *in* practice and knowledge *of* practice. *Knowledge for practice* separates the knowledge producers from knowledge users. Knowledge is generated by experts (researchers and teacher educators) and then selectively shared with teachers (practitioners) for implications in classroom. *Knowledge in practice* is the practical wisdom that teachers or practitioners gain from their practice of teaching over the years. Teacher deliberation and reflection on the experience of teaching is the source of this knowledge. *Knowledge of practice* is the knowledge generated in the field and connected with the existing theoretical knowledge. In a way, knowledge *of* practice combines knowledge *in* and *for* practice to suit the needs of classroom and support teacher reflection and learning.

In the Indian context, research geared towards identifying meaningful connections with the teacher community is scarce (Batra, 2005) and initiatives and interventions in the area of mathematics teaching have not been documented and analysed (Banerjee, 2012). More common approaches of working with in-service teachers are short term workshops and sessions where the knowledge from research is made accessible to teachers (Kumar, Dewan, & Subramaniam, 2012). The reforms in education are communicated to teachers in a top-down manner. Teachers are expected not to engage in matters of policy and theory (Kumar, 2008) but to implement the changes proposed by reforms in education. The agency of the teacher in actively informing curriculum change or policy is missing. Within this socio-political scenario, teachers perceive their role as 'teaching' (translated into telling) students to pass the examinations by completing the stated syllabi and assessments. Learning from the experience of teaching is assumed but not explicated or harnessed. More so, teachers actively taking part in their own professional development is not a part of this imagination. Practical difficulties of convincing authorities, teachers, and their sustained participation adds to the reluctance among researchers to work in collaboration with teachers or schools. The increasing insulation between research and teaching in India is mutual. Teachers criticise research for its apparent disconnectedness with the reality of classroom and the lack of its practical utility. A serious consideration of the gap between research and teaching has been identified by the recent policy documents (NCF, 2005; NCFTE, 2009) and efforts to bridge it in ways conducive to classroom teaching and learning are being explored. There is a need for models of teacher education where the community of teachers and researchers can work together to create opportunities for students as well as their own learning.

This study tries to identify and provide opportunities to teachers to generate *knowledge of practice* by reflecting on their practice using research based knowledge about students' thinking in a particular topic domain. Central to this research has been the collaboration of researchers and teachers to facilitate student learning in classroom. The process of generation of knowledge of practice gave researchers an opportunity to build a relationship with teachers and connection between research and teaching.

THE STUDY

The study was carried out in an English medium school in Mumbai, India. The school is a part of network of schools run by an autonomous body under the Government of India and caters to students and teachers from different locales of the country. Four elementary school mathematics teachers participated in the study which was carried out in 2012–2013.

Objectives and Context

The study aimed to understand the nature of teachers' knowledge of students' thinking gained from the experience of teaching mathematics and explore ways in which this knowledge can be supported and enhanced *in situ*. The observations of teaching practice and literature on students' thinking in the same topic domain, were utilised to design contextually embedded tasks to support teacher reflection and learning. Qualitative changes in teaching practice and ways in which teachers discussed about students and mathematics were noticed.

Knowledge of mathematics and about students' learning mathematics guides teachers in planning lessons as well as in taking in-the-moment decisions while teaching in the classroom. Knowing about students' mathematical thinking supports opportunities for asking questions linked to students' ideas, eliciting multiple strategies, drawing connections across strategies, and so on (Franke, Kazemi, & Battey, 2007). Unfortunately, knowledge of content and students' thinking are dealt separately in the teacher education programmes in India (NCFTE, 2009). The psychology courses deal with the components of students' thinking and learning. The concept-related discussions are confined to the subject related methods courses. It is believed that the experience of teaching would help teachers to integrate the two knowledge pieces together and blend them in their teaching. The nature of knowledge that teachers gain from experience remains hidden, unarticulated and mostly unchallenged. Discussions on concept-specific students' thinking and learning in teacher education needs exploration in the Indian context. Concept-specific students' thinking was chosen as an artefact for discussion with teachers because it is close to the *work of teaching*, provides opportunities for teachers to formulate hypothesis and creates opportunities in classroom to explore teaching possibilities (Takker & Subramaniam, 2012). The dynamic approach enables teachers to be more thoughtful and informed in their decision-making.

Methodology and data

The investigation adopted a case study methodology which included exploration and intervention. Four elementary school mathematics teachers, each with a teaching experience of more than ten years, participated in a two year long study. Two of these teachers teach mathematics and environment studies from Grades 1 to 5, and the other two teach mathematics and physics from Grades 6 to 10. The objectives of the study were explained to the teachers. The school time-table had 8 lessons per day. The participating teachers were assigned 6 (or more) lessons to teach everyday by the school. This made it difficult to maintain a follow up with each teacher immediately before and after a lesson. Data was collected through classroom observations, task based interviews before and after teaching a lesson, two long interviews with individual teachers, and after school meetings with all the teachers. The data is in the form of thick descriptions (Geertz, 1994), audio and/or video records which are in the process of being transcribed for the purpose of analysis.

Preliminary data analysis

The data from diverse sources is triangulated to make meaningful interpretation of the classroom events and then analysed for qualitative patterns. Thick descriptions allowed for a microscopic interpretation of the flow of discourse in classroom. Theoretical propositions that guided the study and description of each case (Yin, 2009) were used to analyse data from classroom observations and teacher interviews. In this paper, I discuss the case of one teacher, teaching division in a Grade 4 classroom.

Teaching division at Grade 4

Pallavi (pseudonym) is a primary school teacher. She has been teaching mathematics and environment studies to students from Grades 1 to 5 for over 15 years. In the years 2012–13, she was teaching mathematics to Grade 1, 4 and 5. Her class size varies between 30–38 students, including girls and boys. She appears confident about the content to be taught in a lesson. She does not plan her lessons because she thinks that with experience a teacher knows the content and its sequence. She is unhappy with the new textbooks introduced after the National Curriculum Framework 2005 (henceforth NCF 2005) because "they appear more like story books and less like mathematics textbooks". She likes the old textbooks as there were "sufficient number of problems for practice", concepts were neatly arranged with no overlaps, and "the teacher knew where to start and where to stop teaching [a topic]". In our initial interactions, Pallavi was reluctant to spend any time outside the classroom to talk about the teaching. Her classroom observations revealed a disciplined classroom where rules were laid down by the teacher. Over a period of time, the relationship between Pallavi and researcher changed. Pallavi became more reflective about her teaching and the researcher became a co-teacher.

Pallavi has been teaching division to fifth graders for almost as long as she started teaching. She has always 'taught' them long division beginning with division of a single digit number with another single digit number, double digit divided by a single digit, three digit by a single digit, two digit by a two digit and so on (see Year 2012 in Figure 2). She believes that children should be taught the algorithms as they are an important part of school mathematics. The revised textbooks (post NCF 2005) however do not provide a single method for division but suggests learners to think of different ways to solve division problems. These ways include repeated subtraction or subtraction using chunking of twos, repeated addition of the divisor, multiplication facts, etc. (NCERT, 2006a, 2006b). Pallavi has a clear opinion on these methods, which is captured in the following vignette from the interview transcript.

Pallavi: Different methods confuse a child. You know they are very young for understanding all this [methods]. Understanding comes later when their brain has grown a bit. In these [primary] classes, children should be clearly told what to do so that they can follow. And there are different methods to solve different problems. Like when I teach division, I don't teach anything else. I just teach long division and give lot of practice, sums [mathematical problems]. The more sums they do the more they learn. You can't expect them to learn so many methods like the new textbook gives. It says you teach this method also that method also. It is very confusing for students and then when you ask a question, which method do you want them to use? They should use long division. It is what we have been doing for ages, I did it

> when in school. And it is the systematic way. (P.I, 2012)

On several occasions, Pallavi was asked to think about why the long division algorithm works. In one of our (researcher and Pallavi) interactions, Pallavi was encouraged to unpack the algorithm by studying it in parts through questions like why take one digit at a time, what is the meaning of multiplying divisor with a number, what is the number being subtracted, etc. Pallavi is convinced about the digit-based approach (taking one digit in each step and dividing it by the divisor). She was taught the same algorithm as a student and once memorised it works for all numbers. Like many teachers, Pallavi 'operates in the field based on her old biases and prejudices formed through her schooling' (Mahapatra, 2004).

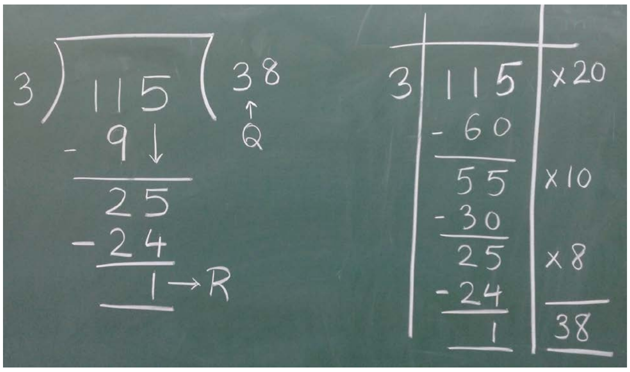

Figure 1: (a) Long division algorithm, (b) Grouping using convenient numbers

Apart from the discussions centred around her teaching, researcher had meetings with Pallavi and three of her fellow teachers teaching mathematics to middle grades. Tasks during teacher meetings involved thinking about students' responses and unpacking students' thinking in their (incorrect and correct) explanations. For instance, one of the tasks was to identify differences in the explanation of two students who both gave a different wrong answer for the same problem. The discussions around the task included talking about the errors, reasons for response, information they convey about student's knowledge or difficulty, and exploring connections between different topic areas of mathematics to identify possible thinking trajectories that students' might take with this kind of thinking. Teachers were engaged in the process of creating problems which would address different kinds of students' thinking.

Pallavi considers the new textbooks 'telling' her a number of methods to be 'taught' in the class. Looking at the methods from the perspective of something to be taught and ensure that it is learnt by all students, Pallavi was not confident on the rationale for using the grouping method. It was hard to convince her to try to use different methods for solving a division problem. She strongly held the belief that the alternative methods would be confusing for students and therefore should not be dealt with in the class.

> Pallavi: Now I have tried this method given in the book but see it is confusing... have always done long division only with children. So I am not sure how to introduce it, how to actually do it in class. I am comfortable in long division and it is shorter you know. It is a step by step process, take one digit at a time so they [students] can easily divide. Why don't you [to researcher] take this [division by grouping] in my class? Tell them what this method is. [After a pause] Yes we can see how they pick it and decide then only which method. I don't know if they will understand. I tried around 8 to 10 numbers, dividing them using that method. The bigger the number, the more confusing it was. I think it can confuse. But you try and let me see how they try to do it. (Pre-Class Interview, P119N10, 2013)

Several considerations seem to be guiding Pallavi's choice of a method for teaching division. These include her competence to use a method as a teacher, systematicity of the method i.e. one step follows from another, comprehensiveness and the cognitive demand it places on students while not being confusing, belief in the legitimacy of the established method, etc. At the same time it was clear that she is looking for support and some evidence of learning using different methods. As a result, she suggested an exchange of roles, where she becomes the observer and researcher becomes the teacher. The role reversal is an important event in this interaction as the teacher gives an opportunity to the researcher to collaborate in teaching which is at the core of the shared interest. As a researcher, there were two kinds of concerns emerging from this decision. Firstly, the teaching goal was to engage students with the grouping method as a way of solving division problems. This goal was partially shared with Pallavi who wanted to assess students' engagement with the method. Secondly, it was important to support Pallavi in teaching different methods as well as engage with students' struggles while they use the alternative routes in their attempt to solve a division problem. An important teaching decision was to think of a relevant context where students would find the need to use grouping in order to solve the problem.

Pallavi started teaching the lesson by revising long division using number problems (two-digit number divided by a one-digit number). Then she called upon the researcher to teach and explain a 'new' method to the students. The researcher started with the context of distributing money among three friends equally to find the share of each friend (Rupees 75 among 3 friends). Students were asked to think about the amount that can be safely given to each friend from the total amount to be shared. Students began by suggesting different combinations on how the money can be distributed "give them 10 each, 20 each or 15". When asked to justify their answers, students explained different combinations of numbers to distribute the money equally. In the second problem, the number of friends was increased to 5 while the amount to be shared remained the same. Even before solving the problem, students figured out that the share of each friend would be lesser and used grouping with convenient numbers to justify their answer. As the lesson was progressing, Pallavi gradually stepped in, helped in eliciting different combinations from students, making them think of efficient combinations and asking students for reasons for their choice. Later on she completely took over to use the method with different numbers (115 divided by 3 in Figure 1). It seemed that along with the learners, Pallavi was also testing whether this method works for different numbers, the appropriate choice of numbers (dividend and divisor), and its linkages with related concepts like multiplication. After the class, Pallavi reflected on the use of method and its affordances by comparing it with the long division algorithm.

> Pallavi: I think the method is good. They can use different ways to get it. Also it is very clear, this vertical arrangement of numbers. And grouping by tens they are aware also. Then slowly they can move to choosing bigger numbers. Actually you know the number of steps increases if you take small numbers [multiples]. But it doesn't matter because they anyway get it. They can use 8 directly or if not 4 and 4 or 5 and 3, it doesn't matter. This method is better and they [students] picked it up faster also. As a teacher, I can see how they are liking it. Taking it as a full number [number as a whole] is clear to them. They find it more easy. Easy only, no? They can make as many groups and how much they want. This also tells us about their multiplication knowledge. But you know one more difference is there. In long division I have to teach them for each increasing digit like dividing by one digit, then 2 [digit number] and 3, all are different. But in this they have to use the same method for big numbers, by themselves and they can do also. (Post-Class Interview, P120N10, 2013)

While reflecting on the lesson, Pallavi was considering the affordance of the grouping method mathematically as well as from the view point of students. Mathematically, she unpacks the idea of grouping using convenient numbers, flexibility in the number of steps, efficient choice of combinations, treating dividend as a whole, and workability of solution for other sets of numbers. She noticed that students came up with the method themselves and were using it flexibly without making a common error, skipping a digit while using long division algorithm without understanding. In the next lesson, Pallavi asked students to choose any method to solve the division problem and identify the similarities in long division and grouping. Further, Pallavi made a decision to include the grouping and other methods in her teaching of division henceforth.

In the course of intervention, Pallavi realised the importance of listening to students not just to evaluate what they do not know but to make it a part of her decision making in teaching. Pallavi engaged with the basic principles of analysing a given method of solving arithmetic problems. The structure that underlies the method and its connection with the algorithm was discovered by Pallavi along with the students. Thus, the choice of method(s) is not guided by the authority of the method given in the textbook but from an engagement with the structure of a method and its ra-

Year 2012	Year 2013 (Planned)	Year 2013 (Taught)
Multiplication tables	Multiplication tables and facts	Making combinations of a number in rows and columns*
Repeated subtraction	Repeated subtraction	Multiplication facts*
Long division (one-digit number by one-digit number)	Long division (one digit and two digit numbers)	Repeated subtraction (with and without chunking)*
Long division (two and three-digit number by one-digit number)	Word problems for long division	Long division
Long division (two and three-digit number by two-digit number)		Grouping with convenient numbers* for one, two, and three-digit numbers
		Link different methods and choose a method*

Figure 2: Teaching trajectory for division (*using both number and word problems)

tionale. Another evidence of change noticed in Pallavi is the difference in her teaching trajectory over these two years (refer Figure 2).

The episodes reported in this paper are used to exemplify one of the ways in which knowledge from teachers' classroom was utilised to mediate teacher reflection and learning around the content of mathematics. The role of researcher was to notice opportunities for communicating important mathematical ideas from teachers' teaching and bring these for discussion with the teacher(s). The discussions centred around the rationale of mathematical procedures, conceptual understanding, and identifying important mathematical ideas to be communicated.

DISCUSSION

The paper presents preliminary analysis of a case study where teacher's knowledge of students' thinking was utilised to support her reflection and learning. Although a more rigorous analysis is in process, some interesting patterns are reported in this paper. Evidences of students' thinking from teachers' classrooms as well as from research served as concrete artefacts to talk about mathematics, making intervention more grounded and connected to the classroom reality. Teachers' explorations in their classroom gave them an opportunity to test propositions emerging from meetings with other teachers and researchers. The change in the positioning of teacher and researcher marked an important learning turn in the collaboration. The methodology allowed for the researcher to engage with teaching and students' learning directly. The impact of such collaborations on learning needs further investigation.

Researcher's own experience as a teacher and work with teachers and teacher educators has indicated the need for building teacher communities. Teachers need spaces to articulate their struggles of teaching mathematics, their conjectures about students' learning and its interaction with the content, etc. A model for affording such an interactive space to discuss the work of teaching needs to be created in our country. Research which focuses on students' thinking is a beginning to an attempt towards building communities of teachers for sustained interactions. In this study, an exploration into knowledge required for teaching mathematics with a focus on practice created space for teachers and researchers to work together. More focused efforts are required to connect knowledge *in* and *for* practice to generate knowledge *of* practice through teacher-researcher collaboration. The voice and agency of teachers in programmes aimed at their professional development needs to be respected for making interventions meaningful.

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Teaching mathematics through storytelling: Engaging the 'being' of a student in mathematics

Amanjot Toor and Joyce Mgombelo

Brock University, Faculty of Education, St. Catharines, Canada, aman.toor@peelsb.com

The paper draws from a hermeneutic phenomenological research study, which aimed at exploring the meaning of teachable moments from the lived experiences of grades 6–8 mathematics teachers. A primary and important stage of hermeneutic phenomenological study is for researchers to begin their research with a personal story. This paper presents the results of this primary stage, which focused on one of author's experiences of storytelling in a grade 6 to 8 mathematics classes for the purpose of humanizing mathematics as a way of engaging both the 'being' of mathematics students and their cognitive faculties. Data was collected from personal teaching stories of one of the researcher/author. The paper argues that teacher's storytelling humanizes mathematics in ways that engages both the 'being' and cognitive faculties of a student.

Keywords: Humanizing mathematics, teaching, learning, storytelling.

INTRODUCTION

What does it mean to teach mathematics? Researchers in mathematics education address this question in various ways. On one side, mathematics education researchers tackle this question by focusing on the question of how one can be an effective mathematics teacher. With regard to the question of an effective mathematics teacher, research tends to focus on teachers' content knowledge and/or pedagogical knowledge, or an integration of the two. Foote, Smith and Gillert (2011) suggest that teachers seeking to communicate a subject effectively and comprehensively should be equipped with an in-depth understanding of content knowledge, whereby they are able to provide alternative explanations for the same concept or topic in order to fully explain the subject. However, having an in-depth understanding of content knowledge in mathematics does not necessarily equate to or result in teaching mathematics effectively. Researchers argue that in order to be a successful mathematics teacher, "[one] must develop multiple expertise in knowledge of mathematical content, pedagogy for teaching mathematics, and knowledge of students"(Foote et al., 2011, p. 71). Some researchers are concerned with the question of how the two categories of knowledge, mathematics (subject matter) and teaching (or pedagogy) can be integrated. An initial articulation of this integration comes from Shulman's (1986) work on pedagogical content knowledge. Building on Shulman's work, Ball and Bass (2000) elaborated on pedagogical content knowledge using the term 'mathematics knowledge for teaching' to capture the complex relationship between knowledge of mathematics content and knowledge of teaching. Even though researchers seem to agree on the importance of teachers' knowledge, there is little agreement over what teachers actually need to know in order to become effective mathematics teachers.

On the other side, mathematics educators and researchers suggest that to effectively teach mathematics means to humanize mathematics for its learners. While talking about the role of education, Doxiadis (2003) states that, "[it] should be, at its best – a process involving the complete human being" (p. 2). In this sense, humanizing mathematics requires teaching mathematics in a way that focuses on the 'being' of a student as a participant in mathematics, which is beyond delivering content of mathematics or teaching certain skill sets. From a humanistic perspective, Chapman (2008) reveals that, storytelling is "... a way of specifying experience, a mode of thought, a way of making sense of human actions or a way of knowing" (p. 16). Therefore, storytelling humanizes mathematics, where students are able to relate to mathematics at a personal level. Humanistic mathematics involves interdisciplinary connections between mathematics and other worlds of thought and methods of learning

(Tennant, 2014). According to Cernajeva (2012), it is important to humanize a study process, which means to approach educational content and its acquisition process using the principles of humanistic recognitions, while taking into consideration the interests and abilities of its learners. Consequently, humanistic approach to education may allows for one to develop human values, self-confidence, self-values, and gives room for self-reflections while at the same time it may increase awareness of others' need, which may result in sense of belonging in mathematics for students.

Doxiadis (2003) states that mathematics education will not change for its perspective learners unless what counts as mathematics changes. He adds that what counts as mathematics needs to include the stories of doing mathematics, where its participants are able to relate to the subject by talking about mathematics. Without stories, mathematics is striped off of its humanity. Students need to be involved in complex mathematics, where they are given the opportunities to think in depth, rather than to adopt easy to use mathematical procedures. Stories in mathematics provide such opportunities. In his presentation, *Mathematics through an Arts lens*, Gadanidis stated that students in mathematics crave to be surprised, where they can flex their imagination and gain new mathematical insights. Further, Gadanidis indicated that story is a biological necessity, an evolutionary adaptation that prepares and trains one to explore the possibility of mathematics in actuality, effortlessly and playfully, which in turn has the capacity to make all the difference for mathematics students. Stories in mathematics highlight one as a human 'being' and enhance one's experience of learning mathematics (Gadanidis, 2012). Storytelling provides opportunities to solve puzzles in artistic way where the storyteller creates situations for its audience to experience the pleasure of surprise and insight in mathematics (Gadanidis, 2012).

Our work aligns with this research which is premised on the idea that to teach mathematics effectively, means humanizing it for learners. By humanizing we mean making the subject more humane and relatable so that learners appreciate mathematics by developing a personal connection with it. This paper draws from a hermeneutic phenomenological research study, which addresses the question: what does it mean to teach mathematics? The purpose of the research is to explore the meaning of teachable moments from the lived experiences of grades 6 to 8 mathematics teachers. Here, the term teachable moment is defined as an opportunity that arises when connections have been made to advance learning by a learner and/or an educator. More specifically, the term moment is distinctly defined as an expected or unexpected occurrence that allows learners and/or educators to deepen their understanding. The paper presents findings from one of the author's experiences of storytelling in a grade 6 to 8 mathematics classes for the purpose of humanizing mathematics in ways that engage both students' 'being' and cognitive faculties in mathematics

STORYTELLING AS A WAY FOR HUMANIZING MATHEMATICS

From the beginning, storytelling has been used as a fundamental method for transferring knowledge. While discussing about the reason for many cultures to retain their storytelling legacy, Zazkis and Liljedahl (2009) state that storytelling, a traditional way of transferring knowledge, provides a unique way of looking at and understanding the world. On one side where mode of storytelling such as fairy tales, poems, nursery rhymes are entertaining, however, on the other side, they transfer essential information in a memorable way for its listeners (Zazkis & Liljedahl, 2009). Storytelling in mathematics is an approach for creating a safe-learning environment where a learner might openly appreciate, understand and enjoy mathematics (Modi, 2012). Storytelling in mathematics makes learning more accessible, where students are more engaged with their learning. Modi (2012) states that, "the value of story to teaching is precisely its power to engage the students' emotions and also, connecting their imaginations in material of curriculum (p. 31). Modi suggests that storytelling in mathematics classroom creates an environment of imagination, emotion, and thinking, which makes mathematics more enjoyable and more memorable for its participants. Storytelling provides ground for its participants to engage in a mathematical activity, which provides them with the opportunities to think, explore, and understand mathematical concepts and ideas (Modi, 2012). Further, storytelling creates a comfortable and supportive atmosphere in the classroom, and builds a bond between an educator and learners (Modi, 2012).

According to Schiro (2004), type of stories a society narrates, is a mirror of what information is considered important by that society, which contributes to the beliefs and the values it members adhere. Evidentially, type of stories that are narrated in mathematics classroom, might contributes to the beliefs and the values, which the learners may hold onto. For example, storytelling in mathematics classes that highlights the importance of failure as essential step to master a mathematical skill and/or to be successful in mathematics might influence one's perception of mathematics more so as a humanistic subject. According to Modi (2012), stories emerge in various places, where some are real while others are fictional, which may provoke one to think and wonder about the things which they may not have thought before. Though 'story' is defined at times as a 'sequence of events', it is not the events that are of interest here, rather the outcomes, in terms of impact, of storytelling on students in mathematics classroom (Modi, 2012). Modi believes that there are various types of stories that one might experience in mathematics. First type of story provides the background for a mathematical activity. For example, a historical content on mathematics, where the mathematician is given an image of a hero, while highlighting the importance of their initial struggle with a mathematical concept or idea. Second type of story provides explanation. For example, a story which may explain students about Pythagorean's triangle, division by zero, division by a fraction, and the manipulation of negative integers. Third type of story poses a question, for example, 'word problems such as 'does equal mean fair?'

RESEARCH FRAMEWORK

The epistemology of phenomenology centers on didactic meaning as opposed to arguing or developing abstract theory. In their discussion of the theoretical and conceptual framework for a phenomenological study, Savin-Baden and Major (2012) stated that "the essence experience is so central and is to be uncovered before it is categorized, researchers do not tend to use a theoretical or conceptual framework... [because] doing so could impose presuppositions on the meaning of the experiences" (p. 221). Additionally, Patton (2002) informs that phenomenology intends to acquire a deeper understanding of the nature or meaning of one's everyday experiences. Subsequently, the objective of the study here is not to make broad generalizations about experiences of all mathematics teachers, but instead to examine individual teachers' personal experiences associated with a very specific phenomenon, and to compare and contrast their experiences as well as lived realities. In order to understand what it means to teach mathematics, it is necessary to first gain insight into teachers' lived experiences of teaching mathematics.

Van Manen (1990) considers phenomenology as the most appropriate method to research the phenomena of pedagogical significance, which elaborates phenomenology as a response to how one orients to lived experience and questions the way one experiences the world. Van Manen states that " a phenomenological researcher cannot just have a question – he or she must live it" and that " ...lived experiences is the starting point and end point of phenomenological research. The aim of phenomenology is to transform lived experience into a textual expression of its essence – in such a way that the effect of the text is at once a reflexive re-living and a reflective appropriation of something meaningful: a notion by which a reader is powerfully animated in his or her own lived experience" (p. 36). Van Manen informs that "the questions of knowledge always refers us back to our world, our lives, to who we are, and to what makes us write, and read, and talk as educators: it is what stands ironically behind the words, the speaking and the language" (p. 46). A true phenomenological questioning is not possible until the researcher displays his or her interest in the phenomenon as lived. Additionally, Van Manen states that, lived human experiences cannot be captured in "deadening abstract concepts and in logical systems that flatten rather than deepen our understanding of human life" (p. 17).

METHODOLOGY

The question, *what does it means to teach mathematics,* is a highly subjective phenomenon, which calls for a hermeneutic phenomenological research design. A phenomenological research describes a "lived experience" of a phenomenon. Phenomenology becomes hermeneutical when its method is taken as interpretive manner compared to descriptive as in other forms of phenomenology such as transcendental phenomenology. Heidegger (Savin-Baden & Major, 2012) argues that all descriptions are based on interpretation and every form of human awareness is interpretive. Drawing from Heidegger, Van Manen (Savin-Baden & Major, 2012) offers a methodological structure encompassing six themes as a helpful guide to hermeneutic phenomenological research: (a) turning to the nature of lived experience in formulating a research question that concerns a phenomena of deep interest—in this case teachable moments in mathematics; (b) investigating lived experiences by collecting descriptions of lived experience of the phenomena under study; (c) reflecting in a hermeneutic phenomenological sense, by recovering themes constituting the essence of the phenomena (d) writing hermeneutically, describing phenomenological experience of the emerged themes; (e) preserving a strong and oriented relation by maintaining a true interest to the phenomenon; and (f) balancing the research context by considering parts and whole, by creating a structure and a planning work for carrying out the phenomenological research.

These six themes guided our primary data collection and data analysis while mindfully dealing with any ethical concern regarding the study. In "Researching Lived Experiences", Van Manen (1997) notes that to investigate the notion of teachable moments – in particular storytelling one needs to orient themselves to the question of meaning of teachable moments. And this meaning of the teachable moments needs to be found in the experience of teachable moments. "And so we need to search everywhere in the lifeword for lived experience material that, that upon reflective examination, might yield something of it fundamental nature" (p. 53). Van Manen offers a number of approaches to gathering or collecting lived experience materials of different forms such as personal experiences, interviews, observations and fictional texts. In our work in this paper we focused on personal experience of teaching mathematics as a lived experience material for reflection on teachable moments.

Van Manen (1990) recommends that phenomenological researchers to begin their research with a personal story as a starting point for a phenomenological study. Further, Mason (2002) states that, "in order to turn from professional development to research, you have to address the questions of convincing others of what you claim to have found out" (p. 176). Keeping this is in mind, as a focus of this paper, as well as a primary observation to the research question, we decided to explore one of the author's teaching experiences of storytelling in her teaching of grade 6 to 8 mathematics classes. Data in this sense was collected from the personal stories of the teacher/author. For the purpose of this paper we present two stories where the author utilized storytelling in her teaching for the purpose of humanizing mathematics. As in any phenomenological study, the author's personal experiences of teachable moments are immediately accessible to her in a way that others are not. However as Van Manen (1997) argues, the focus for a phenomenologist is not on private or autobiographical facts but rather "in drawing from personal descriptions of lived experiences, the phenomenologist knows that one's own experiences are also the possible experiences of others" (p. 54). The stories presented in this paper were selected by the author based on what she thought to be teachable moments and also in collaboration with the second author as what we thought could be possible experiences for others. In describing these stories we tried to align with the phenomenological tradition by trying to describe them in experiential terms focusing on classroom situation and events and refraining from abstractions.

First story

During my first year of teaching mathematics, I tried to make math more realistic and humanistic through storytelling. It was for the first Halloween as a teacher that I decided to talk about the invention of a concept of circle and the impact of such an invention on the world. Therefore, I dressed up like Cleopatra to bring the historical perspective and started talking about how the circle might have developed. First, I talked about how one day someone might have realized that there is a thing called line, which might have led to the idea of two lines intersecting and developing a corner. Next, I talked about the idea of four lines where two lines at a time, intersect, led to the development of four-sided figure like rectangles and squares. Further, I said, that there might have be one day that someone might have asked 'can a line intersect itself?' and in the process of examining that they might have connected the two ends of the line and might have made a circle, and this invention of a circle might have contributed to the invention of wheels. The moment I made the connection of circles and wheels, a lively class discussion ensued where one student said, "it may also be the reason why Christopher Columbus discovered the world because he might have realized that there are other shapes than four sided figures". Then the other student made a reference to spirituality and said, "It could also be when people realized the karma ...you know what goes around is what comes around". At this point, I was aware that my role was no longer didactic. The students have changed from being passive consumers of mathematics to participants in knowledge creation, a change may see as empowering. As their teacher, I was amazed with this class discussion, but did not know what to do with it other than to say, " That's interesting..." And what did it have to do with teaching mathematics? Even though my intention was to bring a humanistic aspect to mathematics, I knew then that what had just happened was bigger than my intention.

Second story

During one of my lesson on geometry to my grade 6's, I was showing a 3-dimensional objects to my students. The purpose of the lesson was to help student develop understand of the relationship between an object and its net. While using real life examples like chocolates boxes and lunch boxes, I demonstrated for my students (while holding a 3 dimensional object in my hand) that if we open up all the sides of this objects and lay it flat, it would result in a net. And (while demonstrating) that if we put together all the sides of the net together, it will result in a 3 dimensional object. Next, as a class, we discussed about the characteristics of a net, which upon folding can result into a 3 dimensional object. In addition, we talked about what type of net may not make a 3-dimensional object (i.e. missing face in net). After discussion, students were challenged to create various type of net and then to check what type of 3-dimension object their net would turns into.

While students were working, a student asked me, " Ms. Toor, what is a side?" I responded to her by saying, " it is one of the faces of an object." She asked again, " no Ms Toor, what is a definition of side...is a side always straight...because if it is straight, then why do we call a side of a road side when the road is curvy?" At this point I knew that she was thinking beyond what I intended to teach. I was aware that without knowing about different braches of mathematics, she was talking about the difference between Euclidian and non Euclidean geometry. Consequently I knew that had also become a participant in knowledge creation of mathematics. However, I did not know how to guide her thinking so that she would not be overloaded with difference of Euclidian and non Euclidean geometry, causing her to become a knowledge consumer of mathematics.

ANALYSIS, DISCUSSION AND CONCLUSION

The purpose of hermeneutic phenomenological reflection is to try to grasp the meaning of a phenomenon under study. In order to come to terms with the meaning of the phenomenon like teachable moments a phenomenologist has to engage in a reflective activity and think of the phenomenon described in terms of structures of meaning, meaning units or themes (Van Manen, 1997). Phenomenological themes may be understood as structures of experience and can be uncovered from any description of lived experience such the above stories. More importantly the meaning of phenomenon can only be communicated textually. So in order uncover the theme for this paper we reflected on the concrete situations in the above stories by asking questions such as what does this mean for this student, situation or action while focusing on the meaning of teachable moments. We then proceeded in describing the phenomenological experience of the theme.

Teacher's storytelling has its place in mathematics teaching. Storytelling as a method of humanizing mathematics, in ways that touches the 'being' of a student, was the common reoccurring theme in both of the stories mentioned earlier. In both of the stories there are three different personal experiences, from which students draw their humanistic connection to mathematics. These personal experiences are examples of the aspects that make these students human beings participating in mathematics. From close observation of the first story, we see that there are two personal experiences, historical curiosity and spiritual need, from which students make their human connection to mathematics. For example, in the first story, a student makes reference to Christopher Columbus by connecting it to the invention of circle. Here the student makes connection between mathematics with his historical curiosity. This connection allows for this student to be a human being, participating in mathematics, where he is able to talk about mathematics beyond the classroom boundaries. The spiritual need is another personal experience, an aspect of 'being' human, where a student talks about karma: "It could also be when people realized the karma ...you know what goes around is what comes around". Here, once again, the student draws connections between his personal experience and mathematics, which touches and deepens his connection with his 'being'. This connection, humanizes mathematics for this student where the student is able to explore his spiritual world by participating in mathematics.

Just as the first story, second story also highlights similar connections. In the second story, a student is curious about something that is beyond the curriculum expectation. Here the element that makes her 'being' in mathematics is the process where she question mathematical definitions. The student wonders about the definition of a side and relates it to her everyday life. Just like the other two students from the first story, this student also humanizes mathematics by making a connection to an element that makes her human. In all three cases students are engaged both cognitively and as human beings as they participate in mathematics.

From the researcher's storytelling experiences as a teacher, we have found that stories allow for one to teach and to learn about the things, which perhaps in the purposeful lesson plan would be lost. Near the end of the school year, the students from these classes where storytelling was utilized, often through voluntary verbal feedback inform their teacher that they (the students) viewed stories as a valuable learning tool, which enhanced their understanding of mathematics by acting as memory joggers and in some cases enticed them to further explore mathematics. All too often education becomes the consumption by students of defined knowledge presented by the teacher in a transmission model. Education should involve an exchange of knowledge between the learners and educators, where these experiences can enhance both teachers and students' mathematical understanding. Storytelling as a pedagogical practice is a method where both the teacher and students learn from one another, while engaging in mathematics. The stories that we have presented in this article show that students are able to relate to mathematics in ways that they are able to see themselves as mathematical 'beings' who have abilities to explore and to be curious about mathematics.

The findings for this study, as discussed above, have implications for both educators and further research. This study contributes to mathematics education research by focusing on lived experiences of teachers' storytelling to humanize mathematics, a certain element of teaching which most theories, models, and methods of mathematics teaching tend to ignore or to not address. This study brings awareness for mathematics education researchers and practitioners by highlighting the need for humanization of mathematics through utilizing storytelling as a teaching tool. Further, the findings from this study allows one to see mathematics teaching (and learning for students) possibilities in ordinary incidents, and to be able convert these seemingly unimportant incidents into significant mathematics teaching (and learning for students) moments – teachable moments.

As much as these stories had positive impact on the students, the teacher/author often found herself questioning her role as a teacher while utilizing storytelling in her classes. As a teacher, the author questioned her decision, and reason behind acting in a certain way especially when facing and dealing with the outcome of storytelling. She wondered if she was guiding her students in right direction, especially when students were engaged beyond the initial aim of story that the researcher intended. Van Manen (1990) states:

> In all of our interaction with children, we are constantly involved, whether we like it or not, in distinguishing between what is good and what is not good for them (in contrast, educational research is usually more interested in distinguishing between what is effective and what is in- effective). Yet even (or especially) the best educators temper their practice with the knowledge that we all often fall short and do not know what is best. (p. xii)

There is little research that focuses on the particularity and uniqueness of the day-to-day, moment-to-moment nature of teaching mathematics. It is clear from researcher's experiences of storytelling, that there is a need for further research where the focus needs to be on how decisions are made in a mathematics classes while focusing on, what it means to teach mathematics on day-to-day and moment-to-moment bases. This question will be explored in the bigger research study from which this paper was derived. The authors plan on conducting interviews with mathematics teachers in order to collect other experiences of teachable moments so as to deepen the understanding of the question, what does it mean to teach mathematics.

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The characterisation of the specialised knowledge of a university lecturer in linear algebra

Diana Vasco[1], Nuria Climent[2], Dinazar Escudero-Ávila[2] and Eric Flores-Medrano[2]

1 State Technical University of Quevedo, Quevedo, Ecuador, dianav350@yahoo.com

2 University of Huelva, Huelva, Spain

This paper analyses the classroom practice of a university lecturer in Linear Algebra under the model Mathematics Teacher's Specialised Knowledge (MTSK). The knowledge revealed, related to the topic of Matrices and Determinants, is found to pertain chiefly to the sub-domain Knowledge of Topics (KoT). The categories of classification which comprise this sub-domain are of particular utility in the identification of this knowledge. Evidence of knowledge associated with other sub-domains comprising Subject Matter Knowledge and Pedagogical Content Knowledge is also found.

Keywords: Specialised knowledge, linear algebra, university lecturer.

Studies into university lecturers' knowledge remain quite rare in the context of Mathematics Education, and thus it is that our study sets out to explore the specialised knowledge in evidence in the course of a lecturer's classroom practice while teaching Linear Algebra in the first year of a degree course. We use the model denominated *Mathematics Teacher's Specialised Knowledge* (MTSK) (Carrillo, Climent, Contreras, & Muñoz-Catalán, 2013) to analyse the lecturer's teaching sessions, with the aim of achieving a better understanding of the knowledge brought into play by topic *Matrices and Determinants*.

THEORETICAL BACKGROUND

In the field of research into teachers' content knowledge of a particular subject, Shulman's (1986) study has proved particularly influential for its manner of differentiating components. Various models have drawn on Shulman's groundwork to describe the kind of knowledge deployed by mathematics teachers (e.g., Fennema & Franke, 1992; Bromme, 1994; Rowland, Huckstep, & Thwaites, 2005; Ball, Thames, & Phelps, 2008), most notable amongst which has been the model developed by Ball and associates (*Mathematical Knowledge for Teaching* – MKT), which applies the basic distinctions made by Shulman (*Subject Matter Knowledge* – SMK – and *Pedagogical Content Knowledge* – PCK) to the specific demands of teaching this subject. Based on our experiences of using the MKT model, and specifically the difficulties encountered in specifying and demarking certain sub-domains when applying it to teacher research (Flores, Escudero, & Carrillo, 2013), the Research Group for Mathematics Education at the University of Huelva is currently developing an alternative model denominated *Mathematics Teacher's Specialised Knowledge* (MTSK) (Carrillo et al., 2013), the aim of which is to capture the specialised nature of the teacher's knowledge as a whole. This specialisation is intrinsic to the teaching of mathematics, and as such permeates both SMK and PCK.

The MTSK model consists of two principle domains. The first of these is *Mathematical Knowledge* (MK), which covers "the whole universe of mathematics, comprising concepts and procedures, structuring ideas, connections between concepts, the reason for, or origin of, procedures, means of testing and any form of proceeding in mathematics, along with mathematical language and its precision" (Carrillo et al., 2013, p. 2990). MK is divided into the following subdomains: *knowledge of topics* (KoT); *knowledge of the structure of mathematics* (KSM); and *knowledge of practices in mathematics* (KPM). The second, *Pedagogical Content Knowledge* (PCK) concerns the teacher's knowledge of teaching and learning mathematics. PCK includes the subdomains *knowledge of* *mathematics teaching* (KMT); *knowledge of features of learning mathematics* (KFLM); and *knowledge of mathematics learning standards* (KMLS). The MTSK model also contemplates teachers' conceptions about mathematics, its teaching and its learning (in line with Bromme, 1994).

KoT supposes a thoroughgoing, well founded knowledge of mathematical content. In order to facilitate the identification of this knowledge, five categories have been demarked. The first of these, *Procedures*, responds to the questions *How is something done?, Under what conditions can something be done?, Why is something done in this particular way?*, and *What are the key features of the result?* The second, *Phenomenology and applications*, concerns awareness of phenomena and contexts with which a particular topic is associated (Freudenthal, 1983) and relevant applications. As the name suggests, *Definitions*, the third category, refers to knowledge for describing and defining concepts, including appropriate examples and images. Fourthly, *Representations* comprise the knowledge of the different ways that a topic can be represented (Duval, 1995) including the associated notation and mathematical terminology. Finally, *Properties and fundamentals* encompasses knowledge of the properties inherent to a mathematical item or necessary for carrying out a particular procedure. This sub-domain receives a more detailed attention in comparison with the other sub-domains comprising MTSK for the reason that, as seen below, it has greatest relevance to the analysis of the work of the teacher participating in the study.

KSM consists of the connections between different contents. It implies being able to view content from different perspectives, that is, basic mathematics from a more advanced vantage point (*connections of increasing complexity*) and advanced mathematics from a more basic viewpoint (*connections of increasing simplicity*); it also includes connections between distinct concepts (Fernández, Figueiras, Deulofeu, & Martínez, 2010).

KPM corresponds to knowledge of the ways of proceeding in mathematics, such as the role of definitions, how to establish relations, correspondences and equivalences, the means of selecting representations, and forms of argumentation, generalisation and the examination of concepts (Carrillo et al., 2013).

KMT enables the teacher to select material appropriate to teaching a specific concept or procedure, and includes the categories of *theories of teaching*; *material and virtual resources*; *activities, tasks and examples for teaching*.

KFLM concerns the ways by which mathematical knowledge is acquired, with a clear focus on mathematical content as the object of learning. Categories in this sub-domain include *learning styles*, *areas of strengths and weaknesses associated with learning*, *students' forms of interacting with the content*, and *students' motivation with regard to mathematics*.

KMLS largely consists of knowledge of the curriculum and its varying demands, and the objectives and performance measures established by external agencies such as examination boards, professional associations and research groups.

A useful reference in the case of teachers of Linear Algebra is the study by McCrory, Floden, Ferrini-Mundy, Reckase, and Senk (2012), which places secondary teachers expertise on two planes, the first mapping their *Knowledge of Algebra for Teaching*, and the second describing their performance in terms of *Mathematical Uses of Knowledge in Teaching*. Some of the categories comprising the first of these dimensions overlap with corresponding categories within MTSK, for example *Knowledge of School Algebra*, detailing knowledge of the basics of the area, *Knowledge of Advanced Mathematics*, which establishes connections between contents, and *Mathematics for Teaching Knowledge*, in which elements of PCK and other teaching-specific content can be identified.

METHODOLOGY

The study followed a qualitative and interpretative methodology, with a case-study design (Yin, 1989), focussing on the specialised knowledge of a university lecturer teaching Linear Algebra to first year students. The research question was, 'What specialised knowledge does a Linear Algebra lecturer reveal in the course of teaching matrices and determinants?'

The lecturer was invited to participate for his disposition and willingness to collaborate. Also relevant was that he worked at the same university as the lead author of this paper, whose professional interests lie in exploring the knowledge of mathematics teachers working in university contexts, with a view to possible follow-up action.

The topic of *Matrices and Determinants* was chosen as it was the first topic in the programme for Linear Algebra and essential for subsequent topics. The lecturer had originally studied Educational Sciences (specialising in mathematics) at degree level, and could draw on 18 years' experience at secondary level and 10 years at the university. The main data collection instrument was non-participatory teaching observation. In total thirteen classes of around two hours each were video recorded. The observations were conducted in two blocks over two consecutive academic years (October 2011 to January 2012, and October 2012 to January 2013) to coincide with when the lecturer was teaching the topic of matrices and determinants. A semi-structured interview was also conducted to provide a degree of triangulation for the evidence garnered via the observation.

Data analysis followed the methodology of content analysis (Bardin, 1977), in which meant sifting the teacher's actions and utterances for evidence of the MTSK sub-domains. The theoretical basis for the analysis consisted of the categories outlined above in the background section. These categories, which we consider still somewhat in development, derive from the intersection between the theoretical readings and joint reflections of the research group and the practical application of the model to teachers' knowledge in various research projects in progress. Finally, it should be noted that although MTSK also contemplates teachers' *conceptions* about teaching and learning mathematics, this aspect of the model is not considered here.

ANALYSIS OF THE KNOWLEDGE OF A TEACHER OF LINEAR ALGEBRA

In terms of the definitions of the MTSK sub-domains and constituent categories, the analysis of the teaching sessions revealed a predominance of KoT, although evidence of KFLM, KMT and KPM were also found. By contrast, there was no evidence of knowledge which could be attributed to the KSM and KMLS sub-domains. Emphasis is thus given below to the consideration of KoT and its corresponding categories. The units of information which form the basis of the discussion are drawn directly from the analysis of the teacher's performance over various teaching sessions; in the case of KMT, these have been complemented by units of information taken from the interview so as to provide a degree of triangulation.

Knowledge of Topics (KoT)

Of the categories comprising this sub-domain, there is evidence of the teacher's knowledge in all five, albeit somewhat scant with regard to *Phenomenology and applications*. In terms of *Definitions*, those provided by the teacher generally tend towards the conventional. Examples include definitions of unitary matrices, equal matrices and orthogonal matrices. At the same time, he also shows knowledge of *Properties*, such as the fact that the determinant of a triangular matrix equals the product of the diagonal entries. The excerpts below illustrate these points:

> Teacher: The unitary matrix is the identity matrix. The matrices which are the same are those with the same dimension and the elements placed in the same location and with the same value. When the inverse coincides with the transpose, we get the orthogonal matrix.
>
> Teacher: This is a triangular matrix. What is the determinant of this matrix A? Well, we can compute the determinant using the properties; if we have a determinant with all its elements, we can make it triangular with the elementary transactions between rows and then we can calculate the determinant by multiplying the principle diagonal.

With regard to *Representations*, there is evidence of knowledge of the notation of mathematical objects, such as the notation for matrices and determinants.

> Teacher: An important point. The 2x2 matrix is like this [...] When we calculate the determinant we no longer write it this way. I'm going to calculate the determinant of the matrix A, so here instead of brackets we have to put vertical bars [...] The determinant is denoted like this <det (A)> or like this <|A|>.

The teacher uses *algebraic* and *arithmetic* registers (D'Amore, 2004) for representing basic operations between rows and for matrix algebra. In addition, and connected to the algebraic register, he reveals knowledge of the matrix register (Ramírez, Romero, & Oktaç, 2013) for different types of matrices, linear equation systems and determinants; and finally the verbal register for setting exercises where the students apply their understanding. His knowledge allows him to encourage different types of representation, although he is not observed to go into the significance of these. However, he does make his awareness of the conventions of notation explicit, an element of knowledge pertaining to the subdomain *Knowledge of Practices in Mathematics* (see the section on KPM below).

There is little evidence for the teacher's knowledge of *Phenomenology and applications* of matrices. He makes mention, albeit briefly, of applications outside mathematics, such as real situations or situations connected to the degree course he teaches. In terms of examples of his knowledge of applications within mathematics itself, we can mention the determinant for finding the inverse matrix or for solving systems of linear equations using Cramer's rule, basic operations between rows applied in calculating the determinant, obtaining stepped and canonical matrices, inverse matrices by the Gauss Jordan method, and the solution to systems of linear equations by the Gauss and Gauss Jordan methods.

With respect to knowledge of *Procedures*, he shows that he knows the procedures appropriate to the topic, and is able to give a full description of them (*How is something done?*). Likewise, he knows where these procedures have their limits, and what is required to be able use them (*Under what conditions can something be done?*), which can be exemplified with his knowledge of the necessary conditions for multiplying matrices:

> Teacher: We have matrix A. What are the dimensions of this matrix? [...] The dimension of this matrix is 2x3. In order to be able to multiply two matrices we need the number of columns of the first matrix to be the same as the number of rows of the second matrix. If A is like that, B needs to have three rows, it doesn't matter how many columns.

In some instances, he shows that he knows why certain procedures are used (especially in relation to ways of simplifying a procedure) (*Why is something done in this particular way?*). An example of this can be seen in his explanation for calculating the determinant of a matrix:

> Teacher: Likewise, you take the column or the row that you want, but here's an interesting thing. Here, there's a zero; this zero helps us to reduce the degree of difficulty because you are always going to multiply the determinant of the submatrix you've got by the value of the element you're focusing on, so it is always a good idea to take a row or a column which has a zero or lots of ones.

Regarding deeper knowledge relating to the mathematical foundations underpinning the procedures, no evidence came to light. It should be noted that the teacher emphasised the *features of the result* obtained by following certain procedures, as in the example below, summarising the characteristics of the product of matrices:

> Teacher: The product of matrices has the same number of rows as the first matrix and the same number of columns as the second.

In his teaching, a tendency to rely on exercises of a mechanical nature was observed. This was closely connected to the learning of algorithms and syntactic conventions for the representation of the content, before conceptual aspects (in the terms of Hiebert & Lefevre, 1986). That said, the teacher does call the students' attention to connections between properties, different registers for representation, certain features and reasons for the procedures and their results. As can be seen, analysis of actual instances of the teacher's practice reveals knowledge pertaining chiefly to categories pertaining to KoT. Nevertheless, evidence of other sub-domains was found and this shall be discussed in the following sections.

Knowledge of Features of Learning Mathematics (KFLM)

We also found evidence of the teacher's KFLM, in the category of *areas of strengths and weaknesses associated with learning* concerning unforeseen pitfalls associated with the topic. This knowledge prompts the teacher to warn his students about potential problems in putting certain procedures into effect. The unit of information below illustrates this knowledge with a warning about changing rows in a matrix and in a determinant:

> Teacher: So you need to be careful here. When we're working with matrices we can say that the matrices are equivalent if we change one row for another. But if we're working with determinants and we change one row for another, the determinant changes sign [...] Matrices and determinants are not the same thing. [...] I repeat, with determinants it's not the same as with matrices, with determinants a change in rows means a change in its sign.

There is abundant evidence of this category in the teacher's reiterated warnings about aspects of the topic which can easily lead to error.

Knowledge of Mathematics Teaching (KMT)

In this sub-domain, the teacher's *Knowledge of Mathematics Teaching* (KMT) is reflected in the *examples* he selects in his teaching. In this instance, we can take his introduction to row echelon forms of matrices, in which his approach aims to ensure that students receive a broad coverage of the features of this type of matrix (as expressed in interview):

> Teacher: Now, I'd like you to look this way at the blackboard. What do you notice about the first matrix? First, they are not square. [...] What do you notice about the zeros? [...] The zeros increase as you go down the rows. The important thing to note is that in this row there are various zeros before a non-zero element. [...] What do you notice in this second matrix? [...] What about the third matrix? Take good note that it is of no interest to us if there is a zero here, what is important is that in this row there are zeros before a non-zero element. This type of matrix is called the row echelon form. As you go down the rows, the number of zeroes increases until you get to the non-zero element or the whole row is zeroes.
>
> Teacher: My intention was to give them three different matrices, each with a different row echelon form. I gave them those so that the students would realise that there are different kinds of row echelon forms and realise what it is that fundamental to the row echelon form (interview excerpt).

The choice of these three examples constitutes evidence of the teacher's knowledge for teaching regarding the relevance and variability of examples, specifically three matrices of the same order, although with different row echelon forms.

Knowledge of Practices in Mathematics (KPM)

Finally, as mentioned above, we have the example of *Knowledge of Practices in Mathematics* (KPM), regarding mathematical notation, to which the teacher attaches great importance and mentions several times over the course of the observed sessions. Here is just one of the examples:

> Teacher: We need to be careful when we write on the blackboard, and not put signs and strokes in any old place, because there are signs in mathematics which mean something. Write the symbols properly. If you are going to calculate the determinant, don't put it in square brackets, you have to put it between vertical bars. This way, like that, is a matrix. But if you are going to calculate the determinant you need to put vertical bars instead of square brackets.

With respect to the two broad domains comprising MTSK, we have been able to note evidence of knowledge corresponding to both in the teacher's professional practice. In terms of MK, there is KoT and KPM (but not KSM, for which we failed to find evidence); and in terms of PCK, there is KFLM and KMT (but again, no KMLS).

CONCLUSIONS

The purpose of this study has been to get a picture of the specialised knowledge in evidence during the observation of a university lecturer's teaching of Linear Algebra. The purpose was not to make any evaluation of the teaching that took place in terms of what might be considered desirable practice. The final picture that emerged is essentially one predominated by KoT, with indications of KFLM, KMT and KPM. We failed to find evidence of knowledge related to the sub-domains KSM and KMLS, and it should be noted that the teacher's conceptions, whilst forming part of the model, were not made subject of the study. We can say that the teacher's practice is characterised by procedural knowledge, not only in terms of how to proceed, but also with regard to the conditions for proceeding, why to proceed and what characteristics can be expected as a result. In addition to this, he demonstrates knowledge of standard definitions, different representations of content, and applications to other areas within mathematics. His choice of examples, and his warnings to students of unforeseen pitfalls provide examples of his knowledge of the difficulties associated with learning certain content, and questions of exemplification.

The knowledge above was revealed through analysis of classroom observation (in conjunction with complementary data drawn from interview, and commenting on the former). To this extent, we cannot claim that the resultant snapshot corresponds to the teacher's knowledge of the topic, merely what was deployed in the course of his classroom practice. Evidence of knowledge pertaining to other sub-domains and categories of specialised knowledge might well be revealed through the use of other data collection tools. By the same token, the knowledge on display might be closely linked to teaching style and the teacher's beliefs and conceptions with regard to teaching and learning. To this end, the study aims to continue its investigation into better understanding the teacher's practice by bringing his conceptions into the analytical procedure.

Finally, we would highlight the utility of the MTSK model and its corresponding analytical categories, above all those relating to KoT, in enabling us to detail the knowledge in evidence during this teacher's classroom performance.

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To what extent can kindergarten teachers and primary school teachers initiate and foster learning mathematics in typical situations?

Gerald Wittmann[1], Stephanie Schuler[1] and Anne Levin[2]

1 University of Education Freiburg, Freiburg, Germany, gerald.wittmann@ph-freiburg.de

2 University Bremen, Bremen, Germany

Our study focuses on kindergarten teachers' and primary school teachers' professional competencies related to mathematics and didactics of mathematics, more precisely on the competencies which are absolutely necessary to support children in learning mathematics during their transition from kindergarten to primary school. To measure these competencies in a situated way, a computer based assessment using image vignettes and video vignettes was developed. The results of n = 123 kindergarten teachers and primary school teachers show a crucial aspect: the insufficient mathematics-related competencies of kindergarten teachers and primary school teachers who did not study mathematics but practise out-of-field-teaching in mathematics.

Keywords: Image vignettes, video vignettes, teachers' professional competence, mathematics, out-of-field teaching.

THEORETICAL BACKGROUND

The concept of *teachers' professional competence* was progressively differentiated in the past decades (e.g., Shulman, 1986; Weinert, 2001; Ball et al., 2008). In the following, we refer to a model developed in the context of COACTIV (cf. Figure 1, Baumert & Kunter, 2013). Thus teachers' professional competence comprises several interwoven aspects: professional knowledge (especially content knowledge, pedagogical content knowledge, pedagogical and psychological knowledge), beliefs, values and goals as well as motivational orientations and self-regulation. Regarding knowledge and beliefs, there is empirical evidence for their impact on the quality of teaching and students' achievement (cf. Staub & Stern, 2002; Hill et al., 2004; Richter et al., 2012; Kunter et al., 2013).

Overlooking current research, there can be found *two approaches to acquire teachers' professional compentecies*. On the one hand, questionnaires with multiple-choice-items aim at professional knowledge and beliefs in large populations, the comparison of partial groups of these populations and the correlation of different aspects of professional competence (e.g., Blömeke et al., 2010; Tatto et al., 2012). On the other hand, image vignettes and video vignettes are nowadays seen as the preferred method to reveal teachers' competencies in situations close to everyday professional life (e.g., Lindmeier, 2011). In contrast to a paper-and-pencil-test with multiple choice items the gain is obvious: Image vignettes and video vignettes do not only capture data regarding teachers' knowledge and beliefs but refer to teachers' acting in situations which are quite realistic and thus closer to their everyday practice.

To consider the *math-related professional competence of kindergarten teachers and primary school teachers*, there exist many differences in professional training of the two groups. Using the situation in Germany as an example, kindergarten teachers are mainly educated in vocational schools where mathematics and didactics of mathematics usually play a minor role. In contrast, primary school teachers study at university and therefore have an academic degree. Although primary school teachers study only two or three subjects, they usually teach more subjects. As a result, *out-of-field-teaching in mathematics* has been estimated as a key problem for mathematics education in primary school (for the situation in Germany cf. Richter et al., 2012; Törner & Törner, 2012; for an international report cf. Tatto, 2012).

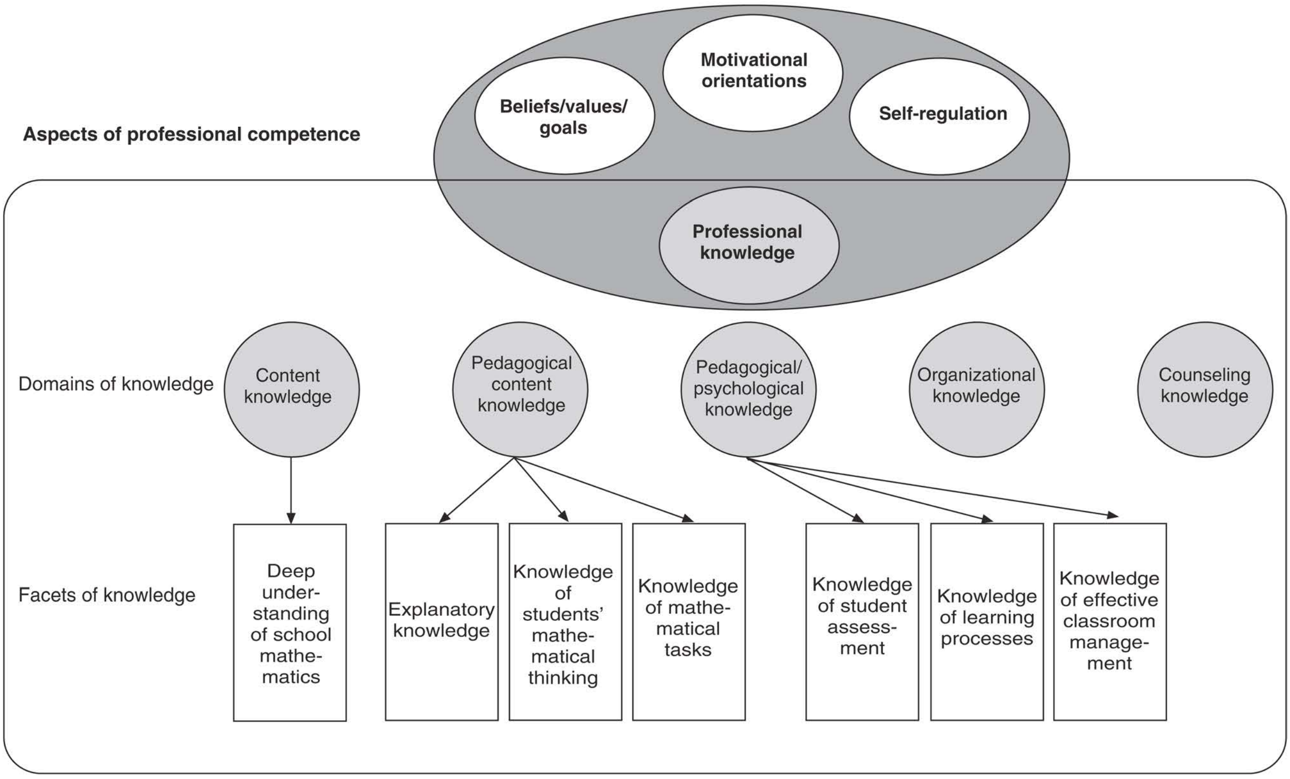


Figure 1: Teachers' professional competence (Baumert & Kunter 2013, p. 29)

Due to this contemporary situation in professional training, *it is an open question whether kindergarten and primary school teachers are able to initiate and foster mathematical learning processes in the transition from kindergarten to school*. We expect differences not only between kindergarten and primary school teachers but also between primary school teachers who studied mathematics and primary school teachers (without studies in mathematics) who practise out-of-field teaching.

RESEARCH QUESTIONS

With regard to the three professional groups aimed above – primary school teachers who studied mathematics, primary school teachers who did not study mathematics but practise out-of-field teaching in mathematics and kindergarten teachers – our *research questions* are the following:

1) To what extent do the three professional groups recognise the mathematical potential of typical situations in the last year of kindergarten or the beginning of the first year of primary school?

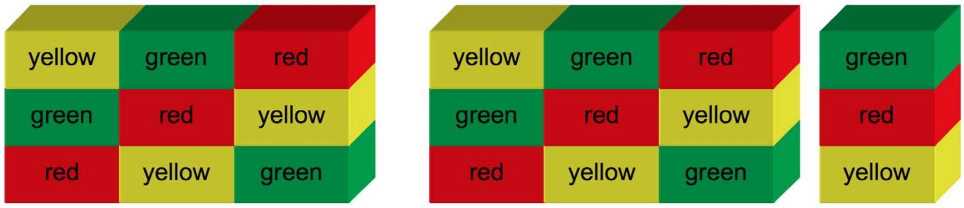
2) To what extent do the three professional groups make use of the range of the five main ideas (for example "number and operations" or "space and shape") in a reasonable way while analysing the mathematical potential of these situations?

3) To what extent are the three professional groups able to foster children's mathematical learning in these situations?

4) To what extent are the three professional groups able to foster process ideas (for example "problem solving" or "reasoning") in these situations?

As we are interested in the competencies of kindergarten teachers as well as primary school teachers, we need to focus on *the last year of kindergarten and the first year of primary school*. Of course, we are aware of the fact that our research objective only hits a small part of teachers' professional competencies, even with regard to mathematics education (e.g., we do not consider primary school teachers' knowledge in written arithmetic).

The reported study is part of a larger research project which aims at professional knowledge and beliefs of kindergarten and primary school teachers (Carle & Wittmann, 2015). Both knowledge and beliefs are seen as an indispensable prerequisite not only for their everyday practice but also for the cooperation of the two professions.



Initial situation — Situation with an additional fourth tower

Figure 2: Situations given in the described video vignette

DESIGN OF OUR STUDY

The *computer based assessment* includes *six image vignettes* and *four video vignettes*. All five main ideas (similar to curricular topics) "number and operations", "space and shape", "quantities and measurement", "pattern and structure", "data analysis, frequency and probability" are covered. Each of the ten situations is followed by two questions concerning one of the research questions. All in all, the assessment consisted of 20 questions for these ten situations.

An example of a video vignette might illustrate the setting. The introduction reads: "Three children (6 years old) got the task to build as many different towers composed of a red, a yellow and a green LEGO DUPLO brick as possible. The video shows a detail of one of the children dealing with this task." The video shows a boy who has already found three different towers (cf. Figure 2, left side) and additionally tries to build a fourth one (cf. Figure 2, right side). After a short moment he removes this tower again and says: "I'm done. I found them all."

Afterwards, the teachers were asked two questions: (1) "How would you react in this situation?" (2) "How can you prompt mathematical problem solving and reasoning in the given situation?" The first question aims at the teachers' reaction to the child's error and the second at an analysis of the situation regarding the learning of mathematics. To meet this situation in an adequate way, a teacher needs knowledge about basic enumerative combinatorics (there are six possible towers, not only three) and didactics of mathematics (e.g., how to give a stimulus) as well as the attitude that it does make sense to use the child's error as a fruitful learning chance.

The teachers typed their answers into the computer; the test took about 1.5 hours. From February to June 2013 in sum N = 161 assessments were led; for n = 123 detailed information about the teachers' professional training was given (especially whether a primary school teacher studied mathematics or not). The sample was drawn in two federal states in Germany (cf. Carle & Wittmann, 2015). It is a stratified sample (relevant criteria were sociodemographic aspects like the number of inhabitants of the community the kindergarten respectively school is located) and hence can be estimated as representative for both professions.

We rated the teachers' answers regarding the adequateness of the situational analysis concerning didactics of mathematics (research question 1). We coded which of the five main ideas the answers refer to (even in an indirect way) and counted how often each of the main ideas was allocated in total (research question 2). Furthermore we rated the quality of the intended action concerning didactics of mathematics (research questions 3) and to what extent they meet the criterion of fostering process competencies (research question 4).

Each answer was rated or coded by two researchers in an independent way; in the case of a divergence they yielded a common result in a consensual way. The interrater reliability was determined by the relative frequency (in percent) of the number of consensuses. It is 0.88 and can be estimated as satisfying.

RESULTS

(1) *To what extent do the three professions recognise the mathematical potential of typical situations?* Primary school teachers who studied mathematics recognise significantly more often the mathematical potential of a given situation than primary school teachers without studies of mathematics and kindergarten teachers (cf. Tables 1 and 2). The post-hoc test of Scheffé discloses two groups (primary school teachers who studied mathematics versus primary school teachers without studies of mathematics and kindergarten teachers; major effect: $\sigma^2 = .13$). With regard to the whole sample the quality of the given answers is quite low ($M = 0.88$;

Profession	M	SD
Kindergarten teachers (n = 82)	0.79	0.45
Primary school teachers without studies of mathematics (n = 20)	0.82	0.55
Primary school teachers who studied mathematics (n = 21)	1.23	0.55
Overall sample (n = 123)	0.88	0.53

Table 1: Recognition of the mathematical potential of typical situations

Source of variance	df	F[a]	Sig	η^2
Profession	2	8.65***	.00	.13
Error within the two groups	120	(0.24)		

[a]One-way analysis of variance, *p < .05; **p < .01; ***p < .001

Table 2: Effect of the profession on the recognition of the mathematical potential of typical situations

Profession	M	SD
Kindergarten teachers (n = 82)	2.26	0.87
Primary school teachers without studies of mathematics (n = 20)	2.95	1.23
Primary school teachers who studied mathematics (n = 21)	3.19	.75
Overall sample (n = 123)	2.53	1.00

Table 3: Number of reasonable allocations of typical situations to the main idea “patterns and structures”

Source of variance	df	F[a]	Sig	η^2
Profession	2	11.10***	.00	.16
Error within the two groups	120	(.85)		

[a]One-way analysis of variance, *p < .05; **p < .01; ***p <. 001

Table 4: Effect of the profession on the number of reasonable allocations to the main idea “patterns and structures”

thereby 0 signifies “no answer or no reasonable answer” and 3 signifies “reasonable answer considering the situation in a multi-perspective way”) with a broad variance (SD = 0.53). Thus, the sampled teachers only rudimentarily recognise the mathematical potential of the situations on average and the quality of answers differs considerably not only between but also within the professional groups.

(2) *To what extent do the three professions make use of the range of the five main ideas in a reasonable way while analysing the mathematical potential of these situations?* The three groups do not differ significantly in the number of reasonable interpretations related to the main ideas “number and operations”, “space and shape” as well as “quantities and measurement”. Yet regarding the main idea “patterns and structure”, primary school teachers (regardless whether they studied mathematics or did not) show significantly higher correct allotments than kindergarten teachers (cf. Tables 3 and 4). The post-hoc test of Scheffé distinguishes two groups (primary school teachers who studied mathematics versus primary school teachers without studies of mathematics and kindergarten teachers; major effect: η^2 = .16). Regarding the main idea “data analysis, frequency and probability”, primary school teachers who studied mathematics allocate the given situations significantly more often in a correct way than primary school teachers without studies of mathematics and kindergarten teachers (cf. Tables 5 and 6). Again the post-hoc test of Scheffé shows two groups which are different though (primary school teachers versus kindergarten teachers; major effect: η^2 = .17).

(3) *To what extent are the three professions able to foster children’s mathematical learning in typical situations?* As above, major effects are found between

Profession	M	SD
Kindergarten teachers (n = 82)	1.68	0.97
Primary school teachers without studies of mathematics (n = 20)	2.10	1.07
Primary school teachers who studied mathematics (n = 21)	2.81	.87
Overall sample (n = 123)	1.94	1.05

Table 5: Number of reasonable allocations of typical situations to the main idea "data analysis, frequency and probability"

Source of variance	df	$F^{a, b}$	Sig	η^2
Profession	2	11.84**	.00	.17
Error within the two groups	120	(.94)		

[a]One-way analysis of variance, *p < .05; **p < .01; ***p < .001; [b]Adjustment of the significance level as the variances are not homogeneus, *p < .01; **p < .001

Table 6: Effect of the profession on the number of reasonable allocations to the main idea "data analysis, frequency and probability"

Profession	M	SD
Kindergarten teachers (n = 82)	0.57	0.37
Primary school teachers without studies of mathematics (n = 20)	0.58	0.42
Primary school teachers who studied mathematics (n = 21)	1.15	0.49
Overall sample (n = 123)	0.68	0.46

Table 7: Teachers' mathematics-related action in typical situations

Source of variance	df	F^{a}	Sig	η^2
Profession	2	19.69**	.00	.25
Error within the two groups	120	(0.16)		

[a]One-way analysis of variance, *p < .05; **p < .01; ***p < .001

Table 8: Effect of the profession on the teachers' mathematics-related action in typical situations

the professions: The quality of the intended action is significantly higher for primary school teachers who studied mathematics than for primary school teachers without studies of mathematics and kindergarten teachers (cf. Tables 7 and 8). The post-hoc test of Scheffé shows two groups (primary school teachers who studied mathematics versus primary school teachers without studies of mathematics and kindergarten teachers; large major effect: $\eta^2 = .25$). Again, the standard derivation indicates differences within the professional groups as well.

(4) *To what extent are the three professions able to foster process ideas?* Based on four video vignettes, we could prove to some extent whether teachers are able to foster process-oriented ideas: "mathematical communication" and "reasoning" (video vignette 1), "reasoning" (video vignette 2), "reasoning" and "problem solving" (video vignette 3), "using mathematical representations" (video vignette 4). Regarding our previous results, it was not in all cases possible to concatenate each process idea to a specific video vignette as process ideas often are interwoven. Due to this fact and several other impediments (normal distribution of the data was not given, floor effects occurred, the number of situations in total was quite small), the comparison of the three groups could not be done in the same way as above. Instead, nonparametric tests were used: The comparison of the three groups (primary school teachers with studies of mathematics vs. primary school teachers without studies of mathematics vs. kindergarten teachers) is done by the H-test of Kruskal and Wallis whereas the U-Test of Mann and Whitney is used to analyse which of the groups differ significantly from each other.

The results of kindergarten teachers show a floor effect as many answers do not comprise any reference to the designated process ideas. Similarly, the perception and support of process ideas of primary school teachers is quite low. Nevertheless, there can be found differences between the professional groups. Regarding the two video vignettes "using mathematical representations" ($\chi^2 = 12.03$, df = 2; $p < .01$) and "reasoning and problem solving" ($\chi^2 = 9.08$, df = 2; $p < .05$), primary school teachers who studied mathematics act significantly better than primary school teachers without studies of mathematics and kindergarten teachers. The following U-test confirms these two groups ($U = 126.00$, $z = -2.45$, $p < .05$; $U = 636.00$, $z = -2.79$, $p < .01$).

DISCUSSION

The documented results are rather disillusioning. All three professionals groups are *barely able to initiate and foster mathematical learning processes of children* in the laboratory situation of the computer-based assessment. Indeed, we can assume that the acquired math related competencies are even overestimated as the situations presented by the image vignettes and video vignettes are characterised by a reduced complexity in comparison to real teaching practice (e.g., interaction with just one child or a small group of children instead of a whole class). On the other hand, we cannot exclude that the given situations do not really hit the teachers' professional experiences with the consequence that they might act considerably better in real classroom situations.

The differences between primary school teachers who studied mathematics and who did not study mathematics are in line with well-known research results. Whereas TEDS-M shows this differences for future primary school teachers who are still in teacher training (Blömeke et al., 2010), we can approve them for primary school teachers being in service. Hence our study emphasizes that even teachers' professional practice (in many cases for a number of years) cannot solve the reported problems of out-of-field teaching in mathematics. Obviously, the combination of everyday life mathematics, general didactics and pedagogical knowledge (which teachers who practice out-of-field teaching in mathematics for some years can refer to) is not fruitful enough for adequate mathematics instruction und does not lead to mathematics related competencies. The inclusion of kindergarten teachers into our sample furthermore shows that this profession has substantial deficits, too. This fact confirms the problem of out-of-field teaching in a drastic way because primary school teachers show only a gradual better performance than their colleagues from kindergarten.

Subsequently, two *demands regarding educational policy* seem to be reasonable: First, both the education and the in-service training of kindergarten and primary school teachers must be improved: Every kindergarten teacher should be educated and continuously trained in initiating and fostering mathematical learning processes for the age from three to six and the transition from kindergarten to primary school. Every primary school teacher should attend lectures and in-service training in elementary mathematics and didactics of mathematics to decrease the frequency of out-of-field teaching. Notably, we cannot say which elements of mathematics teacher education (for example which lectures of mathematics and didactics of mathematics or rather the practical training at school) and in which combination are really essential to improve the quality of teachers' acting. Second, the support of children while the transition from kindergarten to primary school should not be dedicated to individuals but needs professional teams (language, mathematics, physical development, ...) who are able to foster children in their special field.

ACKNOWLEDGEMENT

The project this report is based on is supported by the *Bundesministerium für Bildung und Forschung* and by the *Europäischer Sozialfonds für Deutschland* under the promotion code 01NV1025/1026 and 01NV1027/1028. The authors take the responsibilities for the content of this publication.

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Secondary mathematics teachers' content knowledge for teaching in two contexts: Interpreting versus managing didactically students' understandings

Sotirios Zoitsakos[1], Theodossios Zachariades[1] and Charalambos Sakonidis[2]

1 University of Athens, Greece, s-zoitsakos@math.uoa.gr

2 Democritus University of Thrace, Xanthi, Greece

Teachers' content knowledge for teaching mathematics has been the focus of considerable research that aims to identify this knowledge and to explore the ways in which it is related and affects teachers' initial and professional learning. The paper presents a study which examines secondary teachers' mathematical knowledge emerging in their interpretations of and instructional management suggestions for four hypothetical students' misconceptions regarding the decimal representation '0.3999…'. The results show that teachers' mathematically correct conceptions do not guarantee instructional approaches free of erroneous mathematical ideas.

Keywords: Procept, teacher knowledge, decimal number.

INTRODUCTION

Teachers' knowledge for teaching has attracted a lot of research attention in mathematics education, giving rise to finer specifications of the initial categorization suggested by Shulman (1986). For example, Ball, Thames and Phelps (2008) subdivided Subject Matter Knowledge (SMK) into Common Content Knowledge (CCK), Specialized Content Knowledge (SCK) and Horizon Content Knowledge (HCK); and Pedagogical Content Knowledge (PCK) into Knowledge of Content and Students (KCS), Knowledge of Content and Teaching (KCT) and Knowledge of Content and Curriculum (KCC). Some researchers (e.g., Huillet, 2009) question the appropriateness of the distinction between content knowledge and pedagogical content knowledge to help us understand the diversity of the mathematical knowledge for teaching.

Most studies dealing with the question of 'what mathematics knowledge is needed for teaching' focused on primary teachers (e.g., Ball et al., 2008; Rowland, Turner, Thwaites, & Huckstep, 2009; Ma, 1999). However, there are few relevant studies that concentrated on secondary mathematics teachers. For example, Zazkis and Lekin (2010) call for a more articulated relationship between Advanced Mathematical Knowledge (AMK) and mathematical knowledge for teaching. Hossain, Mendick and Adler (2013) adopt a poststructuralist approach to explore what positions of "understanding mathematics in-depth" makes available to prospective secondary mathematics teachers their participation in a training course.

The present study is part of a broader study that examines secondary teachers' knowledge for teaching decimal numbers of infinite digits and especially of period 9. The choice of focus is due to the importance of the idea for both primary and secondary mathematics curriculum and also to the difficulty of its understanding by students of all ages (e.g., Dubinsky, Weller, McDonald, & Brown, 2005; Giannakoulias, Sougioul, & Zachariades, 2007).

In particular, teachers' knowledge related to decimal representation of numbers with infinite digits of period 9, due to its special relationship with infinity, brings to the forth broader cognitive, epistemological and didactical issues. With respect to the former, these representations, their comparison rules, sequences, series and their limit, as well as important properties of the real number set, like density, are all relevant and cognitively challenging ideas for students and teachers alike. Epistemologically speaking, these representations raise questions connected to their nature (process – number), their multiple meaning and to potential and actual infinity. Finally, in terms of teaching, the focus on these representations permit a close examination of fine mathematical ideas widely used, like this of the infinity, for which strict definitions cannot be employed due to pedagogical considerations, unavoidably allowing for a range of interpretations. To this direction, our research aims at identifying teachers' broader mathematical knowledge required for a careful and deep classroom negotiation of these matters. In specific, we examine the involved teachers' conceptions of these numbers studied in a previous work (Zoitsakos, Zachariades, & Sakonidis, 2013) in relation to the mathematical content knowledge emerging through their instructional proposals concerning the effective teaching management of the relevant idea within the classroom.

THEORETICAL FRAMEWORK

Despite the significance attributed to teachers' content knowledge for teaching, its categorization suggested to date has been criticized in many ways. Huillet (2009) advocates that "the distinction between 'purely mathematical' knowledge (SMK or CCK) and mathematical knowledge adapted for teaching (PCK or SCK) is not appropriate" (p. 10). Davis and Simmt (2006) argue against the distinction between formal disciplinary knowledge and instructional knowledge as "mathematics – for – teaching is neither a matter of 'more of' nor 'to a greater depth than' the knowledge expected of students. It is qualitatively different … (it) might properly be regarded as a distinct branch of mathematics" (p. 294). We adopt this latter view in the present study and attempt to gain some insights into this 'distinct branch of mathematics' by concentrating on teachers' mathematical knowledge 'adapted' to the needs and the limitations of the learning and teaching processes.

Research has identified that one of the main difficulties in making sense of representations like '0.999…' is that are seen as referring to a process-sequence which tends simultaneously to a limit and to the value of the limit itself (Gray & Tall, 1994). This dual nature of mathematical notions operating as processes and objects at the same time has been recognized as a source of power but also of difficulties.

Sfard (1989) claims that the ability to conceive notions as processes and objects at the same time, although ostensibly incompatible, it is, in fact, complementary. Much along the same line, Gray and Tall (1994) argue that "[all those who are successful in mathematics] employ the simple device of using the same notation to represent both process and the product of the process" (p. 118). They proposed "the portmanteau word 'procept' to refer to this amalgam of concept and process represented by the same symbol. […] An *elementary procept* is the amalgam of three components: a *process* which produces a mathematical *object* and a *symbol* which is used to represent either process or object" (Gray & Tall, 1994, p. 120).

Gray and colleagues (1999) suggested a model for the performance spectrum of different individuals in different contexts, when using mathematics procedures, processes and procepts. Gray and Tall (2001) expanded this model, providing four possible outcomes of progressively higher level of sophistication: pre-procedure (no solution or partial solution), procedure (step-by-step solution for a routine problem), process (flexible solution with conceptual alternatives) and procept (ability to think about mathematics symbolically).

In Zoitsakos and colleagues (2013), adopting the above theoretical framework, we found that, despite their strong mathematical background, 106 secondary teachers of mathematics, all mathematics graduates, discussing four hypothetical students' answers as to the meaning of the representation '0.3999…' provided a rather disappointing picture as to their own understanding of the representation: almost 3 to 10 appeared to view the representation '0.3999…' only as a process, about 4 to 10 argued for a combination of process and concept (number) with deficiency(ies) somewhere, while nearly 2 to 10 considered the given representation correctly both as a process and as a concept (procept). Embarking from this, we wanted to examine how this outcome, related to these teachers' content knowledge for teaching, might differ when looked at in another context, which comes closer to the teaching practice itself: in teachers' instructional suggestions for dealing with the above hypothetical students' thinking. This, we believe, may provide some access to the qualitatively distinct content knowledge for teaching held by the teachers of the sample.

METHODOLOGY

We developed the following hypothetical classroom/teaching scenario for research purposes. The questions raised in the scenario were answered by 106 practicing secondary school mathematics teachers in writing (36 male and 70 female, with teaching experience ranging from 0 to more than 15 years). The answers were provided in the context of an entry examination paper for a Master's degree in Didactics and Methodology of Mathematics. The scenario had as follows:

> A final year secondary school teacher gave the following question to his students: 'What is the meaning of the representation 0.3999… (infinite number of 9)'?
>
> Four students gave the following answers: (i) *Student A:* The representation 0.3999… means a process that tends to 0.4, (ii) *Student B:* 0.3999… is a number that tends to 0.4, (iii) *Student C:* 0.3999… is the number just before the 0.4 and (iv) *Student D:* The representation 0.3999… is the sum of 0.3+0.09+0.009+… but, as it continuously increases, it cannot be equal to a number. (a) What could be the teacher's goal in asking this question? (b) Comment each student's answer according to the thought process articulated, the positive and the negative points imprinted in his/her view and his possible misconceptions (if there are any) and (c) If you were a teacher in this class, how would you help these students to overcome the misconceptions, you identified?

Teachers' responses to a scenario such as the above can serve as a means a) to study their understandings related to subtle differences of meaning through students' (mis) conceptions and b) to explore their teaching practices of helping students overcome their possible misunderstandings (Biza, Nardi, & Zachariades, 2007).

In a previous study (Zoitsakos et al., 2013) we examined teachers' own conceptions of the representation '0.3999…' through their written reaction to the hypothetical students' relevant misconceptions. The analysis of the data gave rise to five categories of conceptions: a) No response/irrelevant (10 teachers gave no or irrelevant answers); b) Process (30 teachers understood '0.3999…' mainly as a process); c) Process & Concept (number), with at least one of the two false (31 teachers viewed '0.3999…' as a process and the result of a process, but there was somewhere an error); d) Process & Concept (number), wavering between the two (12 teachers held a correct view of the notation '0.3999…' (procept) together with incorrect ones) and e) Procept (23 teachers expressed a correct view of the notation '0.3999…').

In the present study, we attempt to relate teachers' understanding of the representation '0.3999…' with their teaching suggestions for helping students to make sense of it. To this purpose, we analyze these suggestions with regard to two criteria, *mathematical correctness* and *level of formalism* adopted, following a grounded theory technique (repeated cycles of reading, coding, recoding and grouping and re-grouping of the data leading to the identification of categories of answers for each criterion) (Corbin & Strauss, 2008). For each criterion, three types of instructional suggestions were identified, forming a spectrum extending from no to full agreement to the feature indicated by the criterion.

Specifically, with respect to the mathematical correctness, the teaching proposals were distinguished in two types: a) mathematically *correct*, b) *mixed* (a combination of apparently correct and incorrect ideas or ideas that could be seen either as correct or as incorrect because of dubious articulations) and c) *problematic* (clearly mathematically mistaken ideas or thinking). As far as the criterion of formalism is concerned, the teaching propositions were characterized as a) *formal* (when adopting mainly purely formal mathematical methods), b) *combined* (when employing both formal and informal methods, directly or indirectly) and c) *informal* (when resorting to mainly empirical, descriptive and even intuitive approaches).

RESULTS

Table 1 below presents the results of the combined analysis of the teachers' conceptions emerging in their instructional suggestions for helping students understand the representation '0.3999…' with their conceptions of it according to the spectrum suggested by Gray and Tall (2001). There are ten teachers who provided no answer to any of the scenario questions and have been excluded.

		Process 30	Process & concept number, at least one of the two false 31	Process & concept number, wavering between the two 12	Procept 23	Sum 96
Correct 23	Formal			4	12	16
	Combined			1	6	7
	Informal					
Mixed 24	Formal				1	1
	Combined	2	6	4	3	15
	Informal	3	4	1		8
Problematic 28	Formal					
	Combined	1	2			3
	Informal	12	13			25
Incoherent		7	4			11
No-suggestions		5	2	2	1	10

Table 1: Classification of teaching proposals to the correctness and formalism

As it can be seen from Table 1, ten teachers offered no teaching suggestion, whereas eleven teachers' responses were impossible to classify in terms of mathematical correctness and also with respect to the formalism level employed.

The 75 teaching suggestions that were possible to classify are almost equally distributed between the three types of correctness (23 correct, 24 mixed and 28 problematic). However, this is less the case for the criterion of formalism, where informal teaching practices appear to dominate and formal to be the least popular (17 formal, 33 informal and 25 combined).

It noticeable that as the teachers' conceptions of the representation '0.3999…' range from 'process' to 'procept', their instructional suggestions shift from mathematically problematic to correct and from employing informal to drawing on formal mathematical methods. Furthermore, in their attempt to transform their mathematical knowledge to teaching practice, some of the teachers with a complete understanding of the representation '0.3999…' (i.e., among the 23 teachers in the category 'procept') move to mathematically partially correct teaching suggestions (18 correct and 4 mixed). Finally, the majority of the 23 teachers who articulate mathematically correct instructional proposals (16 teachers) employ a formalistic approach, while the remaining incorporate also informal methods of dealing didactically with the particular idea.

In the following, some characteristic excerpts from the data are presented in order to substantiate each category of answers, to clarify its specific features and to highlight the complexity of the teaching practices suggested by the teachers of the sample.

T37 (male, limited teaching experience, PhD student in Mathematics) is one of the 23 teachers who were found to hold a procept conception for the representation '0.3999…'. The passage that follows comes is his actual instructional proposal and constitutes a typical example of a proposition characterized as mathematically correct and formal.

> I would argue (using the knowledge related to the geometric progression studied in the second year of the Lyceum) that, since:
>
> $$0.09 + 0.009 + \ldots + 0.\underbrace{00\ldots0}_{n+1}9 = \frac{9}{10^2} + \frac{9}{10^3} + \ldots + \frac{9}{10^{n+2}} =$$
>
> $$= \frac{9}{10^2}\left(1 + \frac{1}{10} + \ldots + \frac{1}{10^n}\right) = \frac{9}{10^2} \cdot \frac{1-\left(\frac{1}{10}\right)^{n+1}}{1-\frac{1}{10}} = \frac{1}{10} \cdot \left(1-\left(\frac{1}{10}\right)^{n+1}\right)$$
>
> and $\left(\frac{1}{10}\right)^{n+1} = \frac{1}{10^{n+1}} \to 0$ as $n \to +\infty$ then:
>
> $0.3\overline{9} = \lim_{n} (0.3 + 0.09 + 0.009 + \ldots + 0.\underbrace{00\ldots0}_{n+1}9) = 0.3 + 0.1 = 0.4$. That is, that the number 0.4 is the number $0.3\overline{9}$ which is the limit of the sum $0.3 + 0.09 + 0.009 + \ldots + 0.\underbrace{00\ldots0}_{n+1}9$

We concentrate more on the case of teacher T4 (male, about 10 years of teaching experience), whose answer was classified in the 'procept' category and his teaching suggestion as mathematically correct and combining formal and informal methods. The teacher begins to present (a little hesitantly) a (formalistic) proof of the equality 0.3999… = 0.4, being aware though of the limited persuasion power this approach might have for the students. This can be seen as an element of the Knowledge of Content and Students (KCS) suggested by Ball and colleagues (2008). In particular, he writes:

> I would probably present to them the following process initially: $x = 0.3999…$ [then] $10x = 3.999…$, [so] $10x - x = 3.999… - 0,3999…$ [consequently] $9x = 3.6$ [thus] $x = 0.4$ [that is] 0.3999… = 0.4. This would surely seem strange to them. They usually react. They would say that there has been a mistake ….

He then goes on arguing that "I would use the strengths of the students' views in order to help them understand what is happening" and stating specifically:

> Student B says that 0.3999… is a number in contrast with students A and D. The exchange of views between the students will help everybody understand that 0.3999 … is one notation for a number.

At this point T4 highlights the question of the nature of the representation 0.3999… (number-process) given rise by the hypothetical students' views and proposes the development of a discussion among them about this issue guided appropriately by himself. He carries on attempting to respond to the students' misconceptions, restating them on a more theoretical form and using appropriate intuitive arguments. In particular, he argues:

> We must address the following misconceptions: a) A sum of infinite terms may tend to a number. This is done by way of example: cutting a paper in half and the half again in half etc. … ". Of course, this particular suggestion refers to a theoretical possibility rather than to a real application; b) The notion of succession does not exist in real numbers. 0.4 is not next to the number 0.3999… This can be addressed by asking students which number is next to 0.4. They may say that we put 1 *after* infinite zeros. However, there exists the contradiction after infinite ….

In the last part of his answer, after highlighting a key element of the students' misunderstanding, T4 provides a descriptive argument for the equality 0.3999 … = 0.4. He advocates:

> …c) Finally, students ought to understand that 0.3999… *is* the limit of the process they have in mind (and not the process itself). Once we have concluded that there is no successiveness in real numbers and that there is no number between 0.3999 … and 0.4, the numbers are identical, therefore 0.3999 … = 0.4. That is, the limit is equal to 0.4 "(his own emphasis).

The analysis of the responses to the questions of the scenario provided by teacher T6 (male, 12 years of teaching experience) showed that he conceptualizes the representation '0.3999…' as a procept. However, his teaching proposal included mathematically correct and incorrect elements and employed a formalistic approach, showing that formalism and mathematical correctness do not always go together. In particular, he states in his instructional suggestion:

> I would use the reduction ad absurdum. Let 0.3999… ≠ 0.4. Then given the completeness of R, we should be able to find a number between them. If such a number exists, then the number of its [decimal] digits would have been finite. Absurd.

Teacher's T13 (male, more than 15 years of teaching experience) conception was categorized as indicating 'process and concept number, with at least one of the two false'. His teaching practices, as exposed in the following excerpt, were identified as mixed mathematically and as informal methodologically:

> A suggestion would be to introduce students to the paradox of Zenon with Achilles and the tortoise [draws a line with the indicative positions of Achilles and the tortoise]. In this way they could understand the convergence process and accept that an infinite process can lead to a finite result, since experience tells that Achilles will definitely reach the tortoise".

Essentially, teacher T13 cites an empirical-informal argument (Achilles will definitely reach the turtle) to substantiate a correct mathematical idea (a sum of infinite terms leads to a finite result) for which he does not appear to be clear about (hence its classification as above).

Teacher T23 (female, almost no teaching experience) was classified as holding a 'process' conception and her teaching suggestions were seen as problematic as far as mathematical correctness is concerned and as using informal mathematical methods. In her instructional recommendations she states:

> I would explain to the students that this representation has infinite terms (exactly like a sequence has infinitely many terms tending to a limit, and there is always a previous and a subsequent term). Thus, 0.3999 ... is not immediately previous number of 0.4 since these two numbers are separated by thousands digits.

Here, there is a correct idea (0.3999... is not the previous of 0.4) but the justification is incorrect. Then, trying to explain to the students the representation '0.3999...', the teacher uses only intuitive approaches and glides to fundamental errors concerning the nature of numbers.

> This number can be continuously increased, but this is why it never becomes 0.4. In addition, we are not talking about a number, as student B states, approaching another number, but for a representation that gives us new terms 'tending' → [connection to limit] to a number. I would also show to student D that the terms of this representation, if seen as corresponding to images of a function, certainly it would not be a continuous function, so no need to confuse them. A good idea is to show to the students functions as the one with the following graph [has drawn the graph of a function of the type $f(x) = \frac{a}{b}$, which tend, have a limit but do not coincide with a value, they have infinite terms.

DISCUSSION AND CONCLUSIONS

It is clear from the preceding analysis that a significant part of the teachers with an accurate understanding of the representation '0.3999...' (12 teachers: procept, correct, formal) resort to purely formal approaches in articulating a teaching strategy for dealing with pupils' relevant misunderstandings. To this direction, they overdraw their ontological concerns and the mental paradoxes that emerge from the hypothetical students about whether the representation is '0.3999 ...' is a process or a number as well as if the infinite sum of positive numbers can be a number rather than infinity. This phenomenon of teaching mathematics decontextualized from the problems that gave rise to the corresponding ideas has been identified by Chevallard (1991) some years ago.

There is, however, a smaller number of teachers (6 teachers: procept, correct, combined) who, like teacher T4, recognize limited persuasiveness for the students of purely formalistic techniques and accompany them by appropriate intuitive arguments and narrative explanations.

It is also noticeable that of the 23 teachers who have good understanding of the specific representation, only 18 maintain accuracy in their teaching proposals. This means that there is a slippage of mathematical correctness when transforming knowledge for didactical purposes. The subtlety of the idea under consideration (the mathematical meaning of the representation '0.3999...') appears to contribute to this slippage, making clear the special attention needed during its didactical transformation. Furthermore, highlights the qualitatively distinct character of the content knowledge for teaching mathematics (Davis & Simmt, 2006).

On the other hand, the majority of the teachers were unable to both conceptualize and suggest instructional practices to the issue at hand that are mathematically correct. They adrift in informal and vague approaches which fail to respond effectively to the hypothetical students' relevant misconceptions. This is the reverse problem of the one reported in the beginning of this section: we have here a strong effort of embedding mathematical knowledge within an empirical-intuitive context, however, stripped of its mathematical content.

The above findings show clearly the specificity of the mathematical knowledge needed for teaching, especially at the advanced level. Also, highlight the importance of further searching for those features that make it distinct as well as for the reasons that mathematically correct conceptions do not guarantee instructional approaches free of erroneous mathematical ideas.

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TWG20

Posters

A proposal for the study of mathematics teacher reflection

Mayra A. S. Báez Melendres and Rosa María Farfán Márquez

Center for Research and Advanced Studies of the National Polytechnic Institute, Mathematics Education Department, Ciudad de Mexico, Mexico, mbaez@cinvestav.mx

The purpose of this poster is to present a proposal for the study of the mathematics teacher reflection. For that, we consider reflection as that which contributes to a change in relation to mathematics knowledge. In this text, we present some theoretical ideas that support the study and an outline of the methodology to be used.

Keywords: Reflection, social development of thought, school mathematical knowledge.

INTRODUCTION

Our problematic centres its attention on mathematics teacher reflection. It is recognized that in the research carried out reflection has been related with the didactic and pedagogic aspects; however, mathematics itself has not been the object of reflection for the discipline (Jaworski, 2014). Given the universality of mathematics, it does not seem to have sense to avoiding reflecting about it, focusing on the issues around it, such as teaching. What is the meaning about reflecting on mathematics? With this question, we do not pretend to question the knowledge of the teacher, but to understand how the thought is developed when the focus of reflection is mathematics itself.

That is how we make the general objective: to typify reflection of mathematics teacher about school mathematics. How to provoke this reflection? We consider a scenario of meanings confrontation is appropriate. Therefore, we highlight two research questions we intend to answer: (a) What reflections features about school mathematical knowledge are configured in a scenario of meanings confrontation? (b) What role does school mathematics play in the generation and depth of reflection?

CONCEPTUAL FRAMEWORK

As the basis of this paper, we have considered taking a social stance of the thought development, and therefore, of reflection. For example, for Radford (2006) thinking is more than a mental activity, where cultural *artefacts* are an integral part of it, so thinking has a close relationship with *objects*. Similarly, awareness is considered fundamental in the reflective process. For Freire (1982), reflection gives way to *awareness*, i.e. to a process of cultural action that seeks the liberation of individuals from the limitations and alienations which they are subjected to, affirming themselves as conscious subjects and co-creators of their historical future: a break of boundaries to be participants of the creation of new knowledge.

Regarding mathematics, we have also considered a stance of social construction, the Socioepistemology (Cantoral, 2013). This theoretical perspective is centered on studying the problematization of mathematical knowledge, taking into account the historical and contextual aspects which give sense and meaning to mathematics itself. Thus, the development of reflective thinking towards mathematics will deal with the relationship with it when meanings are constructed or reconstructed. The analysis of instances of reflections from this perspective will allow to identify elements to understand and explain the relationship between teachers regarding school mathematics and how reflections occurs.

THE STUDY

With qualitative style, we have considered making a case study (Merriam, 1998). Thus, the consideration of context is critical because it particularizes the *how* of the case study. The participant population is high school mathematics teachers in Mexico, with whom there will be working sessions with meanings confrontations: their own ones and those which were shed in the socioepistemology investigation, in the issue of proportionality (Reyes, 2014). We will also apply a semi-structured interview that allows us to know with acuity other aspects that influence in the teacher's thinking. We are interested in studying the moment of confrontation to identify the teacher reflection about the proportionality regarding their conceptions, their uses and the contexts or situations where the teacher is present. For this purpose, we construct categories of the pilot test.

CONCLUDING REMARKS

From our point of view, studying of reflection from the planned perspective will allow the teacher to question the school meanings and concepts in order to create conditions (arguments, instruments, strategies, etc.) that enable them, firstly, to modify their actions on the teaching of proportionality; and secondly, deepen their learning of mathematics.

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Teachers' subjective theories on algebra

Julia Meinke

Georg August University Göttingen, Göttingen, Germany, julia.meinke@mathematik.uni-goettingen.de

This project concerns the impact of the belief system of teachers on their algebra teaching practices in secondary education. The first step is to reconstruct this belief system. The methodological framework is provided by the Research Project Subjective Theories (RPST). The research design is described here.

Keywords: Algebra, teachers' beliefs, subjective theories.

INTRODUCTION AND THEORETICAL FRAMEWORK

The main prerequisite for research on subjective theories of teachers are the assumptions of teachers as crucial factors for learning success (Nye et al., 2004) as well as the necessity to consider teachers' beliefs to understand their actions, planning decisions and behaviour in class (Ernest, 1989). Typical questions that correspond to these actions are: Why is a special content chosen, emphasized or denied? What are the reasons why a teacher decides to use a special method? Stein and colleagues (2007) have shown that planning decisions and the realisation of teaching depend highly on teachers' beliefs. The following figure illustrates the model of Stein and colleagues (2007) that describes the different phases of curriculum use (Figure 1).

Ontological beliefs of teachers and their impact on teaching have been studied in detail for the areas of elementary geometry (Girnat, 2011) as well as of statistics and probability (Eichler, 2004). There are already some hints that subjective theories of teachers differ in dependence of specific disciplines in mathematics (Eichler, 2011). This is why the project focuses on the research concerning the impact of subjective theories of teachers on the teaching of algebra.

Algebra is regarded as one of the most difficult and at the same time most important topics in school mathematics. It can be considered as the language of mathematics. Algebra is used in every case of generalization or in case of simplifying a complex situation (Usiskin, 1995). For this process it has to be clear what the initial situation, what the target is and which role the variable is playing in it. In this context it is most important for students to understand the use of variables and their meaning (Arcavi, 1994).

METHODOLOGY

To reconstruct the structure of the intended curriculum the Research Project Subjective Theories (RPST) is used. Subjective Theories are defined as „complex cognition aggregates of the research object, in which their cognitions relating to the self and the world become manifest and which show an at least implicit argumentational structure" (Groeben & Scheele, 2001, p. 2). The outcome of this is the dialogue-hermeneutic method. The researcher has to communicate with his research object to understand the cognitions behind his actions. The study was conducted by interviewing nine teachers by using half-structured interview-guidelines, based on an interview-study that was conducted before. Afterwards teachers were visited twice in their algebra classes unheralded. To remain as impartial as possible in the observation of the classroom situation, the visits took place before the interviews were analysed. The observation records will be used to investigate how the intended and enacted curriculum relate.

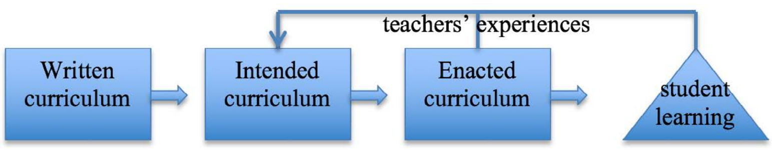

Figure 1: Four phases of curriculum use according to Stein and colleagues (2007)

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Mathematical knowledge for teaching geometric proof: Learning from teachers' practices

Lisnet Mwadzaangati

University of Malawi, Faculty of Education, Zomba, Malawi, lmwadzaangati@cc.ac.mw

The aim of mathematics education courses in both undergraduate and post graduate programmes in Malawi is to improve teacher's mathematical knowledge for teaching. Although the number of colleges and universities offering secondary mathematics education courses is increasing in Malawi, the quality of teaching has not improved to a greater extent (Government of Malawi, 2009). The capacity of the teachers to teach geometric proof in particular is still inadequate. The purpose of the study in progress, is to explore the knowledge that teachers require in order to teach geometric proof well in Malawian context. The initial findings of document analysis and the pilot study results will be presented in the poster.

Keywords: Geometry, geometric proof, teacher knowledge, teacher education.

THEORETICAL FRAMEWORK

Many studies have been conducted in different areas in reaction to Shulman (1986) notion of pedagogical content knowledge (PCK). Such studies aimed at developing content knowledge (CK) for specific subjects. In mathematics education, most of the studies aimed at developing models of mathematical knowledge for teaching (MKT) and developing instruments for measuring its domains (Ball, Thames, & Phelps, 2008). Professional Knowledge for Secondary School Mathematics Teachers (PKSSMT) is one of the recent models that build on Shulman's and Ball's models of teacher knowledge. PKSSMT was developed by Baumert and Kunter (2013) from the COACTIV project. It shares a common theoretical approach with Ball's mathematical knowledge for teaching (MKT) model, to the extent that its focus is on mathematical knowledge needed for understanding of instruction. Unlike MKT, PKSSMT regards CK as including secondary mathematics only. It views CK as a prerequisite for PCK. There are basically three categories of PKSSMT model which are CK, PCK and pedagogical/psychological knowledge (PPK). My research is guided by this model and it only concentrates on CK and PCK because PPK is mainly about individual and class management. In my view, PPK is common knowledge needed by every teacher hence not in the interest of this study. PCK emphasizes on three aspects; knowledge of cognitive activating tasks and their sequence, knowledge of student's cognitions and ways of assessing them, and knowledge of explanations and multiple representations. This shows that PCK mainly emphasise on tasks, student cognitions, representations and explanations (Baumert & Kunter, 2013). CK and PCK imply that research on the teaching of mathematics has to focus on teacher's conceptual understanding, tasks used for explaining and assessing teaching, how the teacher evaluate the tasks, and how the teacher explains and represent concepts to students. Based on this view, I developed these questions for my research; 1. How is secondary geometric proof conceptualised by the teachers? 2. What is the nature of the problems that are selected and used to enhance students' geometric thinking? 3. How do teachers interpret student productions/solutions of geometric proofs? 4. How are geometric proof concepts represented and explained to secondary school students?

METHOD

Data collection is being done in phases. During the first phase which was a baseline survey, I analysed the Malawi National Examinations Board (MANEB) chief examiners mathematics reports of both junior and senior secondary school for the past five years (2008 to 2013). The reports show that students fail mathematics mainly due to poor performance in geometric proof questions. The chief examiners attribute students' poor performance in geometry to teachers' lack of knowledge to teach this particular branch of mathematics. Students do not understand geometric proof because they are not taught properly how to construct and apply a proof. The Government of Malawi (2009) highlights that one of the challenges facing the education sector is limited human capacity and material resource. The challenge of limited human capacity is in both quantity and capability and is worse in mathematics and science. Challenges in terms of capability entail that teacher education is not equipping the pre-service teachers with proper skills that can enable them teach mathematics, in this case geometry proof, very well. That is why there is need to study MKT-geometry proof. The second phase of the study will use qualitative case studies on in-service teachers. Best and worst teachers will be involved in the study to ensure variety in data collection.

During the CERME9 conference, I will present the results of pilot findings which will be done using tests, individual interviews and lesson observations. The poster will be composed of the introduction, problem statement, conceptual framework, methodology, data analysis and results.

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